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AUTHOR Morrow, Richard; Beard, Earl
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ABSTRACT

This document contains various science problems which require a mathematical solution. The problems are arranged under two general areas. The first (algebra I) contains biology, chemistry, and physics problems which require solutions related to linear equations, exponentials, and nonlinear equations. The second (algebra II) contains physics problems which require solutions related to linear equations, quadratic equations, exponential functions, logarithms, nonlinear equations, conic sections (parabola) and conic sections (hyperbola). Each problem includes the mathematics and science content area (and concept) fostered, statement of the problem, and the solution. Among the concept areas included are velocity, Hooke's law, speed of sound, temperature, Ohm's Law, isotopes, molecular weight, heat transfer, population growth, mechanics, half-life, motion, kinematics, conservation of energy, acceleration, carbon-14 dating, radioactivity, sound, pH, electric circuits, and linear expansion. (JN)

Problems Relating Mathematics and Science in the High School

Richard Morrow, Department of Physics and Astronomy
Earl Beard, Department of Mathematics

University of Maine
at Orono
Orono, ME 04469

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Index of Problems

Topic

Math: Alg. I: Linear Equations Science: Physics: Rates

Background

The relation between distance travelled, d, at a constant speed, v, in a time, t, is $v = d/t$. If the speed is not constant during the trip then the average speed, $\bar{v}$, is defined by $\bar{v} = d/t$.

Problem

A student drives from Orono, ME to Bangor, ME at a constant speed of 40 mph. The return trip is made over the same route at a constant speed of 60 mph. Can an average speed be found even though neither the distance nor time are known?

Solution

Let d = distance between Orono and Bangor

Let t_1 = time going and t_2 = time returning

Let v_1 = speed going and v_2 = speed returning

The average speed $\bar{v} = \dfrac{2d}{t_1 + t_2}$

But $t_1 = \dfrac{d}{v_1}$ and $t_2 = \dfrac{d}{v_2}$

$$\therefore\ t_1 + t_2 = d\left(\frac{1}{40} + \frac{1}{60}\right) = d\left(\frac{1}{40} + \frac{1}{60}\right) = d\,\frac{40 + 60}{40 \times 60} = d \times 0.042$$

$$\bar{v} = \frac{2d}{d \times 0.042} = \frac{2}{0.042} = 48 \text{ mph}$$

Math: Alg. I: Linear Equations Graphing | Science: Physics: Hooke's Law

Background

When a spring is stretched or compressed by an amount, x, it pulls or pushes back with a force, F, given by $F = kx$ where k is known as the force constant of the spring.

Problem

A student hangs a series of weights from a spring and records the amount of stretch, obtaining the following data

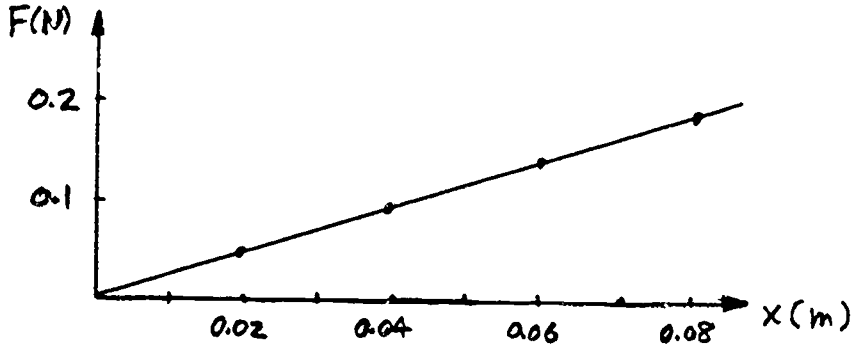
F (Newtons)	X (meters)
0.05	0.02
0.10	0.04
0.15	0.06
0.20	0.08

(a) Graph F as a function of x

(b) Write the equation for the line through the points and determine the spring constant.

Solution

(a)



(b) $y = mx + b$ or $F = kx + b$ with $b = 0$ and $\underline{k = 2.5}$

Math: Alg. I: Linear Equations Graphing

Science: Physics: Speed of Sound

Background

The velocity of sound in air increases as the temperature rises. The standard velocity is 331 m/s taken at 0 degrees Celcius ($0^\circ C$). For every one Celcius degree rise in temperature the velocity of sound increases by 0.6 m/s.

Problem

(a) Write an equation for the actual velocity of sound, v, at the temperature T.

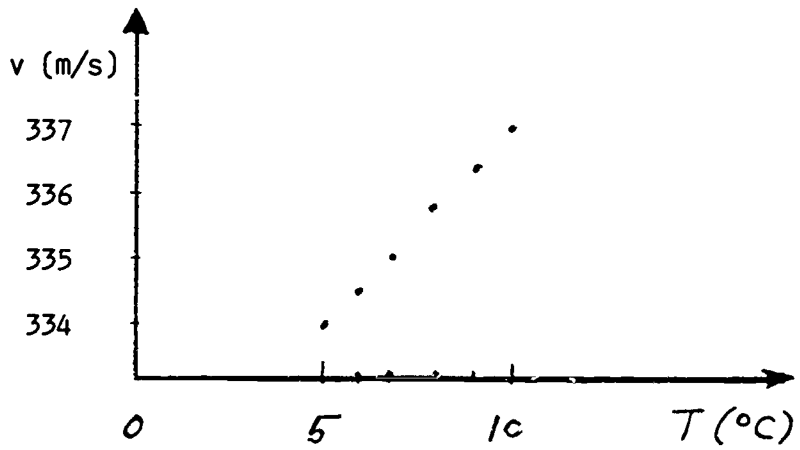
(b) Given a domain of $5^\circ C$ to $10^\circ C$ in intervals of $1C^\circ$ find the velocity of sound for these temperatures, and graph your results.

Solution

(a) $v = 0.6\,T + 331$

(b)

T(°C)	v(m/s)
5	334.0
6	334.6
7	335.2
8	335.8
9	336.4
10	337.0



Math: Alg. I: Linear Equation   Science: Physics: Temperature

Background

The Celcius and Fahrenheit temperature scales are defined in terms of the freezing and boiling points of water. The freezing point of water is 0°C or 32°F and the boiling point of water is 100°C or 212°F.

Problem

(a) Room temperature is 68°F. What reading does this correspond to on the Celcius scale?

(b) Find a linear relation between a temperature reading, F, on the Fahrenheit scale and the temperature reading, C, on the Celcius scale.

Solution

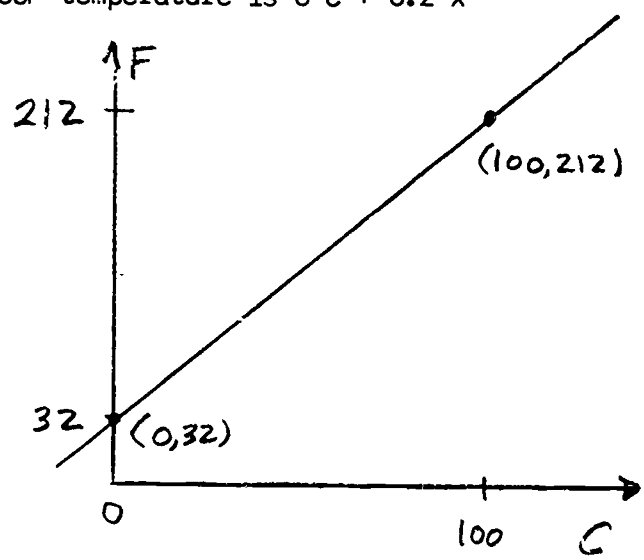
(a) The interval 32°F to 212°F includes 180°. The reading of 68°F is a fraction $\frac{68-32}{180} = \frac{36}{180} = 0.2$ along this interval.

The interval 0°C to 100°C includes 100°. Room temperature is a fraction 0.2 along this interval. So room temperature is 0°C + 0.2 x 100°C = 20°C.

(b) Set F = mC + b. This is a straight line. We know two points on this line: the freezing point and the boiling point of water. From the graph, the F(y) intercept b = 32 and the slope

$$m = \frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$$

$$\therefore F = \frac{9}{5}C + 32$$



Math: Alg. I: Linear Equations | Science: Physics: Ohm's Law, Power

Background

For an electrical appliance there are several relations between the applied voltage, V, the current, I, the resistance, R, the power drawn, P, and the energy, E, consumed in time, t:

$V = RI$ $\qquad P = VI$ $\qquad E = Pt$

Problem

A hair dryer has a power rating of 900 watts when used in a 120 volt circuit.

(a) Find the current passing through the hair dryer
(b) Find the resistance of the hair dryer
(c) Find the energy consumed by the dryer in 5 hours
(d) Find the cost of operating the dryer for 5 hours if energy costs 7¢ per kilowatt hour.

Solution

(a) $I = P/V = 900/120 = 7.5$ amps

(b) $R = V/I = 120/7.5 = 16$ ohms

(c) $W = Pt = 900 \times 5 = 4500$ watt-hrs = 4.5 kw-hrs

(d) Cost = 7¢ x 4.5 = $31\frac{1}{2}$¢

Math: Alg. I: Linear Equations Science: Chemistry: Isotopes

Background

Isotopes are atoms of the same element, eg. chlorine, but differ from each other in the number of neutrons their nuclei contain. They are therefore chemically identical but differ in mass from each other. Isotopes are distinguished from each other by a superscript to the chemical symbol. This denotes the isotopic mass, eg. ^{35}Cl, ^{37}Cl.

Problem

Chlorine has two naturally occuring isotopes - ^{35}Cl with a mass of 34.9689 atomic units and ^{37}Cl with a mass of 36. 9659 atomic units. If the mass of natural chlorine is 35.4527 atomic units what are the percent abundances of ^{35}Cl and ^{37}Cl?

Solution

Let x = percent of ^{37}Cl

Let 100 - x = percent of ^{35}Cl

$$\frac{36.9659\ x + 34.9689\ (100 - x)}{100} = 35.4527$$

$$36.9659 + 3496.89 - 34.9689\ x = 3545.27$$

$$1.997\ x = 48.33$$

$$x = 24.33$$

Hence, the abundance of ^{37}Cl is 24.53% and

the abundance of ^{35}Cl is 75.77%

Math: Alg. I: Linear Equations Science: Chemistry: Molecular Weight

Background

A general formula for a chemical compound is $X_mY_nZ_p\cdots$ where X, Y, Z, ··· are the chemical symbols of the elements occurring in the compound and m, n, p, ... are positive integers specifying the number of atoms of each element occurring in the compound. The molecular weight of the compound is then determined by

Molecular Weight of Compound = m (Atomic Weight of X) + n (Atomic Weight of Y) + p (Atomic Weight of Z) + ...

Problem

Given the atomic weights (in atomic mass units): C (12), O (16), P (31), K (39), N (14), calculate the molecular weight (in atomic mass units) of (a) CO_2 (b) P_2O_5 (c) K_3PO_4 (d) N_2O_4.

Solution

(a) Mol. Wt. of CO_2 = 1 (12) + 2 (16) = 44

(b) Mol. Wt. of P_2O_5 = 2 (31) + 5 (16) = 142

(c) Mol. Wt. of K_3PO_4 = 3 (39) + 1 (31) + 4 (16) = 212

(d) Mol. Wt. of N_2O_4 = 2 (14) + 4(16) = 76

Math: Alg. I: Linear Equations

Science: Chemistry: Specific Heat

Background

The amount of heat (Q) needed to raise the temperature of a substance is the product of the mass (m) of the substance, its specific heat (c) and the temperature change ΔT). The specific heats of four common substances are 1 cal/g-C^0 for water, 0.1 cal/g-C^0 for iron, 0.3 cal/g-C^0 for lead and 0.05 cal/g-C^0 for silver (cal = calorie, g = gram, C^0 = Celcius degrees).

Problem

(a) Write down the equation relating Q, m, c and ΔT.

(b) If it takes 180 cal of heat to raise 120 g of a substance 15^0C, what is the substance?

(c) Given a certain amount of a substance identify the variables and the constants in the equation of (a).

(d) Find the amount of heat needed to raise the substance of (b) from 25^0C to 45^0C, 55^0C, 65^0C. Express your results on a graph.

Solution

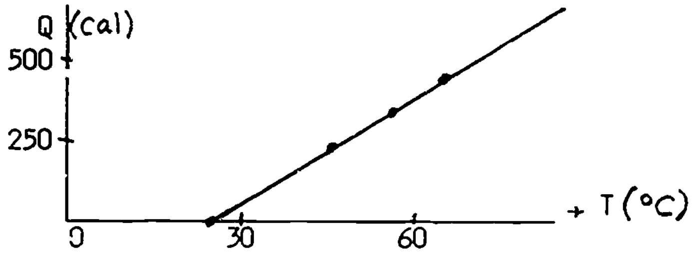
(a) $Q = mc\Delta T$

(b) $180 \text{ cal} = 120 \text{ g} \times c \times 15^0C \quad \therefore c = \frac{180 \text{ cal}}{120 \text{ g} \times 15C^0} = 0.1 \text{ cal/g-}C^0$ Lead

(c) Variables: Q, ΔT; constants m, c

(d)

T	ΔT	Q(cal)
25^0	0	0
45^0	20	240
55^0	30	360
65^0	40	480



Math: Alg. I: Linear Equations Science: Physics: Heat Transfer

Background

Heat flows between objects in contact, from the hotter to the colder object. The amount of heat gained (or lost) by an object of mass, m, undergoing a temperature change, ΔT, is given by $Q = mc\Delta T$ where c is the material-dependent specific heat capacity. If a phase change occurs, eg. melting of ice or boiling of water, then heat is gained (or lost) without a change in temperature in the amount $Q = mL$ where L is the material-dependent latent heat. Overall, heat is neither created nor destroyed so that the amount lost by some objects equals the amount gained by other objects.

Problems:

I. Harry dropped his 50 gram gold piece at 120°C into 80 grams of water at 0°C. What was the final common temperature of the gold and water? The specific heat capacities are 0.03 cal/gm-C° for gold and 1.0 cal/gm-C° for water.

II. A calorimeter of mass 100.0 grams contains 400.0 grams of water at 60°C when 200.0 grams of water at 0°C is poured into it. What is the final common temperature? The specific heat capacities are 0.09 cal/gm-C° for the calorimeter and 1.0 cal/gm-C° for the water.

III. An ice cube of mass 50 grams at 0°C is dropped into 500 grams of water at 80°C. What is the final common temperature? The specific heat capacity of water is 1.0 cal/gm-C° and the latent heat of ice is 80 cal/gm-C°.

Solutions

I. Heat gained by water = Heat lost by gold

$80 \times 1 \times (T - 0) = 50 \times 0.03 \times (120 - T)$

$80\ T = 180 - 1.5\ T$

$81.5\ T = 180 \quad \therefore T = 2.2\ ^{\circ}C$

II. Heat gained by 200 grams of water = Heat lost by calorimeter and 400 grams of water

$200 \times 1 \times (T - 0) = 100 \times 0.09 \times (60 - T) + 400 \times 1 \times (60 - T)$

$200\ T = 540 - 9\ T + 24{,}000 - 400\ T$

$609\ T = 24{,}540 \quad \therefore T = 40.3^{\circ}C$

III. Heat gained by ice = Heat lost by water

$50 \times 80 + 50 \times 1 \times (T - 0) = 500 \times 1 \times (80 - T)$

$4{,}000 + 50\ T = 40{,}000 - 500\ T$

$550\ T = 36{,}000 \quad \therefore T = 65^{\circ}C$

Math: Alg I: Exponentials Science: Biology: Population Growth

Problem

A scientist developed a bacterium that undergoes mitosis and doubles in size every minute. He places one bacterium in a 1 liter jar and one hour later finds the jar full. How long will it take to fill the 1 liter jar if he starts with 2 bacteria?

Solution

In 59 minutes the jar with initially one bacterium in it will be half full. Therefore in 59 minutes the jar with initially two bacteria in it will be full. Answer: 59 minutes.

Math: Alg. I: Nonlinear Equations　　　　Science: Physics: Mechanics

Background

Newton's Law of Universal Gravitation states that the force of attraction between two bodies, that is the force each experiences due to the other, is inversely proportional to the square of the distance between them and is directly proportional to the product of their masses.

Problem

(a) Write down an equation for the force (F) of attraction between two bodies of masses M and m a distance d apart.

(b) How would the force change if M were doubled?

(c) How would the force change if d were doubled?

(d) How would the force change if M were doubled and m were halved?

Solution

(a) $F = KMm/d^2$ where K is an unknown constant

(b) Let the new force be F^1. Then $F^1 = K(2M)m/d^2 = 2(KMm/d^2) = 2F$

$\therefore$ the force is doubled

(c) Let the new force be F^1. Then $F^1 = KMm/(2d)^2 = KMm/4d^2 = \frac{1}{4}(KMm/d^2) = F/4$

$\therefore$ the force is reduced to $\frac{1}{4}$ what it was

(d) Let the new force be F^1. Then $F^1 = K(2M)(m/2)/d^2 = KMm/d^2 = F$

$\therefore$ the force is unchanged

Math: Alg. I: Nonlinear Equation   Science: Chemistry: Kinetic Theory

Background

The average kinetic energy of the molecules of gases is dependent only on the temperature of the gases. That is, in a mixture of gases, such as air, the average kinetic energies of CO_2, N_2, O_2, etc. are equal. The average kinetic energy of a molecule can also be expressed in terms of the mass (m) of the molecule and the "average" speed (v) of the molecule as $K.E. = \frac{1}{2}mv^2$.

Problem

The mass of oxygen (O_2) is 32 amu, of nitrogen (N_2) is 28 amu, and of carbon dioxide (CO_2) is 44 amu. Find the average speed of an O_2 molecule and a N_2 molecule in terms of the average speed of a CO_2 molecule.

Solution

$$KE = \frac{1}{2}m_{O_2}v_{O_2}^2 = \frac{1}{2}m_{CO_2}\, v^2_{CO_2}$$

$$m_{O_2}v_{O_2}^2 = m_{CO_2}\, v^2_{CO_2}$$

$$32v_{O_2}^2 = 44\, v^2_{CO_2}$$

$$v_{O_2}^2 = 44/32\, v^2_{CO_2}$$

$$v_{O_2} = 1.17\, v_{CO_2}$$

Similarly, using $KE = \frac{1}{2}m_{N_2}v_{N_2}^2 = \frac{1}{2}m_{CO_2}\, v^2_{CO_2}$

we find $v_{N_2} = 1.25\, v_{CO_2}$

Math: Alg. I: Nonlinear Equations Science: Biology: Half-Life

Background

The physical half-life, T_P, of a radionuclide is defined as the time it takes for $\frac{1}{2}$ of the sample to decay. The biological half-life, T_B, is defined as the time it takes for $\frac{1}{2}$ of the sample to leave the body. The effective half-life, T_{EFF} - the time it takes for $\frac{1}{2}$ of the radioactivity to be released - is given by

$$1/T_{EFF} = 1/T_P + 1/T_B$$

Problem

(a) Determine T_{EFF} for ^{131}I if its physical half-life is 8.1 days and its biological half-life is 180 days.

(b) Determine T_{EFF} for ^{45}Ca if its physical half-life is 152 days and its biological half-life is 18,000 days.

Solutions

(a) $1/T_{EFF} = 1/8.1 + 1/180 = 0.123 + 0.0055 = 0.1285$

$$T_{EFF} = 7.8 \text{ days}$$

(b) $1/T_{EFF} = 1/T_P + 1/T_B = \dfrac{T_B + T_P}{T_P \times T_B}$

$$T_{EFF} = \frac{T_P \times T_B}{T_B + T_P} = \frac{152 \times 18,000}{18,000 + 152} = 152 \text{ days}$$

Math: Alg. II: Linear Equations

Sciences: Physics: Force
Motion

Background

If a force, F, is applied to an object of mass, m, along its line of motion then the object's speed will change from v_0 to v in the time, t. The relation between these quantities is

$$F = m \frac{v - v_0}{t}$$

Problem

A 1500 kg automobile moving with a speed of 25 m/s is braked and comes to rest in 6.3 s. Find the braking force applied to the car.

Solution

$m = 1500$ kg $\quad v_0 = 25$ m/s $\quad v = 0 \quad t = 6.3$ s

$$F = 1500 \text{ kg} \times \frac{0 - 25 \text{ m/s}}{6.3 \text{ s}} = -\frac{1500 \times 25}{6.3} \text{ kg m/s}^2$$

$$= -5952 \text{ kg m/s}^2 \text{ or } -5952 \text{ Newtons}$$

The - sign indicates that F is directed opposite to the motion.

Math: Alg. II: Quadratic Equations

Science: Physics: Force Kinematics

Background

An object of mass, m, acted upon by a force, F, experiences an acceleration, a, given by F = ma.

Problem

A 5 kg object initially at rest is accelerated by a constant force of 12 N. (a) What is the acceleration of the object? (b) How long will it take for the object to travel 15 m?

Solution

(a) $F = ma$ with $F = 12$ N, $m = 5$ kg

$a = F/m = 12/5 = 2.5$ m/s

(b) $s = v_0 t + \frac{1}{2}at^2$ with $s = 15$ m, $v_0 = 0$, $a = 2.5$ m/s

$15 = \frac{1}{2} \times 2.5\ t^2$, $\therefore t = \sqrt{\frac{2 \times 15}{2.5}} = \underline{3.5\ s}$

Math: Alg. II: Quadratic Equations   Science: Physics: Kinematics

Background

For an object undergoing constant acceleration along a line there are several useful relations between its acceleration, a, its initial velocity, v_0, its final velocity, v, its distance of travel s, and the elapsed time, t.

$$s = v_0 t + \frac{1}{2}at^2 \qquad v = v_0 + at \qquad v^2 = v_0^2 + 2as$$

A common example of this type of motion is that of an object moving vertically above the earth's surface where the only force acting is the gravitational attraction of the earth. If the positive coordinate direction is vertically upwards then for all objects $a = -9.8\ m/s^2$.

Problems

I. An object of mass 6 kg is dropped from a balloon at an altitude of 20,000 m. Neglecting air resistance calculate (a) the time elapsed before the object strikes the earth and (b) the speed of the object at the moment of impact.

II. A model rocket is shot vertically upwards from the ground with an initial speed of 24.5 m/s. Neglecting air resistance (a) when will the rocket first pass the top of a flagpole 19.6 m high? (b) When will the rocket again be 19.6 m above the ground? (c) When will the rocket strike the ground?

Solutions

I. Choose the + axis as vertically downwards; then $a = 9.8\ m/s^2$

(a) $s = v_0 t + \frac{1}{2}at^2$ with $s = 20{,}000$ m and $v_0 = 0$, $a = 9.8\ m/s^2$

$$20{,}000 = \frac{1}{2} \times 9.8\ t^2, \therefore t = \sqrt{\frac{40{,}000}{9.8}} = 64\ s$$

(b) $v = v_0 + at$ with $v_0 = 0$, and $t = 64$ s

$$v = at = 9.8 \times 64 = 626\ m/s$$

II. Choose the + axis vertically upwards; then $a = -9.8\ m/s^2$

(a) $s = v_0 t + \frac{1}{2}at^2$ with $s = 19.6$ m and $v_0 = 24.5$ m/s

$19.6 = 24.5\ t - \frac{1}{2}\ 9.8\ t^2$ $\therefore t = 1s$ or $4s$

or $t^2 - 5t + 4 = 0$ Answer: 1 s

or $(t - 4)(t - 1) = 0$

(b) Answer: 4s

(c) $s = v_0 t + \frac{1}{2}at^2$ with $s = 0$ and $v_0 = 24.5$ m/s, $a = -9.8\ m/s^2$

$0 = 24.5\ t - \frac{1}{2} \times 9.8\ t^2$ $\therefore t = 0$ or 5

$0 = 4.9\ t\ (5 - t)$ Answer: 5s

Math: Alg. II: Quadratic Equations   Science: Physics: Relativity

Background

If an object at rest has a mass m_0 then when it is moving at a speed v it appears to have a mass m given by

$$m = m_0 \, (1 - v^2/c^2)^{-\frac{1}{2}}$$

where $c = 3 \times 10^8$ m/s is the speed of light.

Problem

At what speed must an electron travel in order for its relativistic mass to be twice its rest mass?

Solution

$$m = 2\, m_0, \; c = 3 \times 10^8 \text{ m/s}$$

$$2\, m_0 = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$\sqrt{1 - \frac{v^2}{c^2}} = 0.5$$

$$\frac{v^2}{c^2} = 1 - 0.5^2 = 0.75$$

$$v = \sqrt{0.75\ c}$$

$$= 0.867 \times 3 \times 10^8$$

$$= 2.6 \times 10^8 \text{ m/s}$$

Math: Alg. II: Quadratic Equations   Science: Physics: Conservation of Energy

Background

For an object of mass m moving vertically above the earth, and with the gravitational force the only force acting on it, conservation of energy takes the form

$$\frac{1}{2} mv^2 + mgh = \text{constant, unchanging in time}$$

where v is its instantaneous speed, h is its instantaneous height above, say, the earth's surface and $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity.

Problem

A tennis ball of mass 0.060 kg is dropped from a height of 1 m and rebounds to a height of 0.8 m.

(a) What is the speed of the ball just before it strikes the floor?

(b) What is the speed of the ball just after it rebounds from the floor?

(c) How much energy was lost in the collision with the floor?

Solution

(a) $\frac{1}{2} \times 0.06 \times 0^2 + 0.06 \times 9.8 \times 1 = \frac{1}{2} \times 0.06\, v^2 + 0.6 \times 9.8 \times 0$

or $0.588 = 0.03\, v^2 \qquad \therefore\; v = 4.4 \text{ m/s}$

(b) $\frac{1}{2} \times 0.06 \times v^2 + 0.06 \times 9.8 \times 0 = \frac{1}{2} \times 0.06 \times 0^2 + 0.06 \times 9.8 \times 0.8$

or $0.03\, v^2 = 0.47 \qquad \therefore\; v = 4.0 \text{ m/s}$

(c) energy lost $= \frac{1}{2} \times 0.06 \times 4.4^2 - \frac{1}{2} \times 0.06 \times 4.0^2 = 0.1 \text{ J}$

Math: Alg. II: Linear Equations
Graphing

Science: Physics: Velocity
Acceleration

Background

If an object's speed changes from v_0 to v in time, t, then it is said to have an acceleration, a, given by $a = (v - v_0)/t$.

Problem

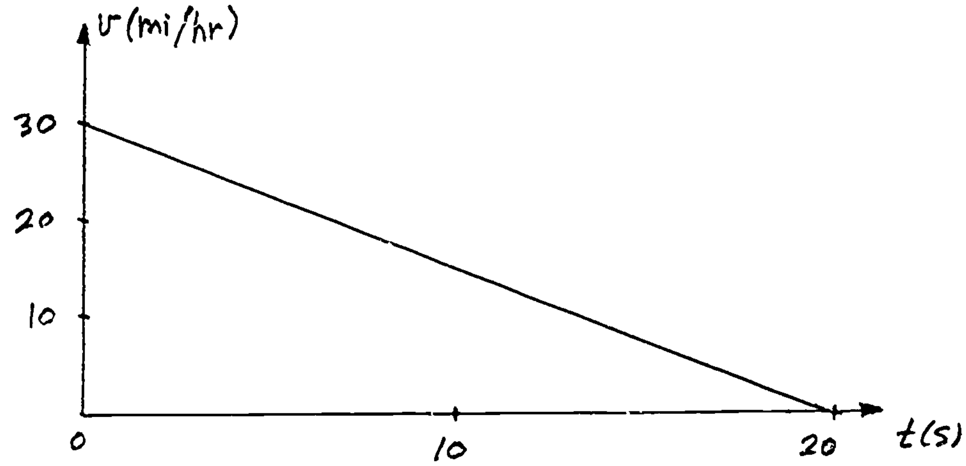
A car moving at 30 mi/hr coasts uniformly to a stop, slowing down at the rate of 1.5 mi/hr per second. (a) Find the acceleration of the car. (b) Graph the speed of the car as a function of time. (c) Find the time necessary for the car to come to a complete stop.

Solution

(a) In a time t = 1s the car's speed changes by 1.5 mi/hr. For example if v_0 = 30 mi/hr then v = 28.5 mi/hr

$$a = \frac{28.5 - 30}{1} = \frac{-1.5}{1} = -1.5 \text{ mi/hr/s}$$

(b)



(c) From the graph v = 0 when t = 20s.

From the equation, using v_0 = 30 mi/hr, v = 0 and a = -1.5 mi/hr/s

$$\text{we get } t = \frac{v - v_0}{a} = \frac{0 - 30}{-1.5} = 20\text{s}$$

Math: Alg. I.: Linear Equations
Graphing

Science: Chemistry: Gas Law
Absolute Zero

Background

The volume, V, of an ideal gas kept at constant pressure varies linearly with the temperature, T.

Problem

A volume of a sample of gas at constant pressure is observed to change with temperature:

T (°C)	V (Liter)
0	1.50
10	1.54
30	1.62
60	1.74

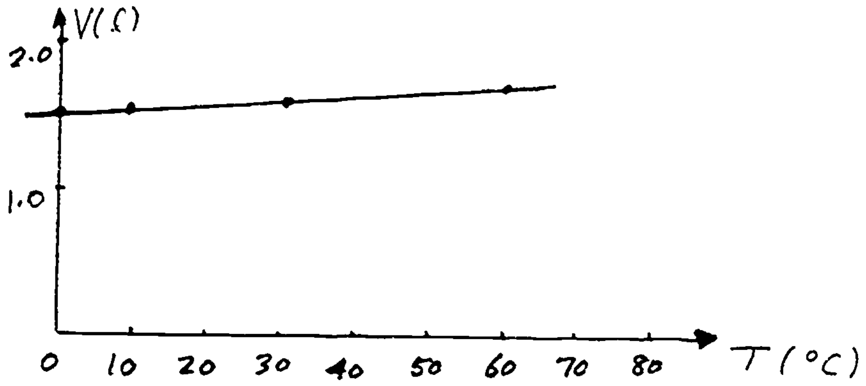
(a) Graph the data showing V as a function of T.

(b) Write the equation giving V as a function of T

(c) From your equation determine the absolute zero of temperature - the temperature at which V = 0

Solution

(a)



(b) $y = mx + b$ or $V = mT + b$ $b = 1.50$ $m = \frac{0.04}{10} = 0.004$

$V = 0.004\ T + 1.5$

(c) When $V = 0$ $0 = 0.004\ T + 1.5$

$T = -\frac{1.5}{0.004} = -375\ ^{o}C$ is absolute zero

Math: Alg II: Exponential Function
Advanced Math

Science: Physics: ^{14}C dating
Radioactive Decay

Background

Carbon dating makes use of the radioactive isotope ^{14}C. The method assumes that ^{14}C in living plants and animals is continually replenished. When death occurs this replenishment ceases and the amount of ^{14}C in the organism decreases due to radioactive decay. The amount of non-radioactive ^{12}C does not change with the passage of time. Consequently, if we know the "normal" ratio of ^{14}C to ^{12}C in living organisms we can, by measuring this ratio in dead tissue, determine how long the organism has been dead. The formula used is

$$R(t) = R(o)(\tfrac{1}{2})^{t/\tau}$$

where R(o) is the ^{14}C radioactivity per gram of carbon in the organism at the time of death, R(t) is the ^{14}C radioactivity per gram of carbon at a later time, t, and τ is the half-life of ^{14}C (5750 years).

Problem

a) After 2000 years, what ^{14}C radioactivity per gram remains in a (dead) tree?

b) If R(o) = 7.5 picocuries per gram and R, in a piece of wood from a dead tree, is presently measured to be 4.5 picocuries per gram, how long ago did the tree die?

Solution

a) $R = 7.5(\tfrac{1}{2})^{2000/5750} = 2.6$ pC/gm

b) $R = R_0(\tfrac{1}{2})^{t/\tau}$, $\log R/R_0 = (t/\tau) \log \tfrac{1}{2}$, $\log 4.5/7.5 = (t/5750) \log \tfrac{1}{2}$

$$t = 5750 \times \frac{\log(4.5/7.5)}{\log \frac{1}{2}} = 5750 \times (-0.22/-0.3) = 4217 \text{ years}$$

Math: Alg II: Exponentials Science: Physics: Electrical Circuits

Background

A capacitor is an electrical device constructed from two conductors (plates) placed close to but not touching each other. It is used to store electric charge: Q amount of charge on one conductor, -Q on the other. If a resistor is connected between the plates of a charged capacitor then the + charge on one plate drains off through the resistor and neutralizes the - charge on the other plate. This doesn't happen instantaneously but takes time. The formula expressing the size of Q at time t after the resistor is connected, in terms of the initial charge Q_0, the "size" of the capacitor C and the "size" of the resistor R is

$$Q = Q_0 e^{-t/RC}$$

Problem

If $R = 10^7$ ohms and $C = 10^{-6}$ farads find how long it takes the capacitor to lose one half of its original charge.

Solution

$$Q = Q_0/2, \; RC = 10$$

$$\tfrac{1}{2} = e^{-t/10}$$

$$\ln\tfrac{1}{2} = -t/10$$

$$t = -10(\ln 1 - \ln 2) = 10 \ln 2 = \underline{6.9 \text{ s}}$$

Math: Alg II: Adv. Math
Exponentials

Science: Physics: Linear Expansion

Background

If the length of a metal rod at temperature $0^{o}C$ is L_0 then its length at temperature T is given by $L = L_0 e^{\alpha T}$ where α is the coefficient of linear expansions and e = 2.71828.

Problem

Find α if a 10 meter rod expands by 0.025 m when it is warmed from $0^{o}C$ to $100^{o}C$.

Solution

$10.025 = 10\ e^{\alpha 100}$

$1.0025 = e^{100\alpha}$

$\ln 1.0025 = 100\alpha$

$\alpha = 1/100 \times 0.0025 = 2.5 \times 10^{-5}/C^{0}$

Math: Alg II: Logarithms

Science: Chemistry: pH

Background

The strength of an acid-base solution is measured on the pH scale. The pH is expressed in terms of the H^+ ion concentration, $[H^+]$(moles of H^+ per liter), by the formula

$$pH = \log 1/[H^+]$$

Low pH (large $[H^+]$) corresponds to an acidic solution while high pH (low $[H^+]$) corresponds to a base solution:

pH	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	Strong		Moderate			Weak/			\Weak		Moderate			Strong	
			Acid					Neutral			Base				

Problem

a) Calculate the pH of a solution containing 0.00048 moles of H^+ per liter

b) Calculate the concentration of H^+ in a liter of solution that has a pH of 4.6

Solution

a) $pH = \log 1/0.00048 = \log 10^4/4.8 = \log 10^4 - \log 4.8 = 4-0.68 = 3.32$

b) $10^{pH} = 1/[H^+]$ or $[H^+] = 10^{-pH} = 10^{-4.6} = 2.5 \times 10^{-5}$ H^+/liter

Math: Alg II: Logarithms Science: Physics: Sound

Background

Loudness of a sound is measured in decibels (db) by a logarithmic function $\beta = 10 \log I/I_0$ where I is the intensity (in watts/m^2) of the sound and $I_0 = 10^{-12}$ watts/m^2 is the standard intensity (the softest sound that can normally be heard).

Problem

Find the intensity of sound if (a) $\beta = 45$ db, (b) $\beta = 120$ db (pain threshold)

Solution

First solve the equation for I:

$\beta/10 = \log I/I_0$

$10^{\beta/10} = I/I_0$

$I \quad I_0 10^{\beta/10}$

a) $\beta = 45 \quad I = 10^{-12} \times 10^{4.5} = 10^{-7.5} = 3.2 \times 10^{-8}$ watts/m^2

b) $\beta = 120 \quad I = 10^{-12} \times 10^{12} = 10^0 = 1$ watt/m^2

Math: Alg II: Logarithms | Science: Physics: Radioactivity

Background

A sample of matter containing N_0 radioactive nuclei, e.g. radon, will be found, after a time t has elapsed, to have N radioactive nuclei remaining where $N = N_0 e^{-\lambda t}$ with λ known as the decay constant. The half-life of a radioactive sample is the time for one half of the radioactive nuclei to decay.

Problems

Given a sample of radon with a half-life of 3.8 days

a) How long will it take until only 1/8 of the sample remains?

b) Find the decay constant, λ, for radon.

c) Sketch a graph of the fraction of radioactive nuclei present as a function of time.

Solution

a) After 3.8 days the fraction of radon present is $\frac{1}{2}$.

After 2x3.8 days the fraction of radon present is $(\frac{1}{2})^2=1/4$

After 3x3.8 days the fraction of radon present is $(\frac{1}{2})^2=1/8$

Answer: 3 x 3.8 days = 11.4 days

b) $N = N_0/2$ for $t = \tau = 3.8$ days

$$\frac{1}{2} = e^{-\lambda\tau}$$

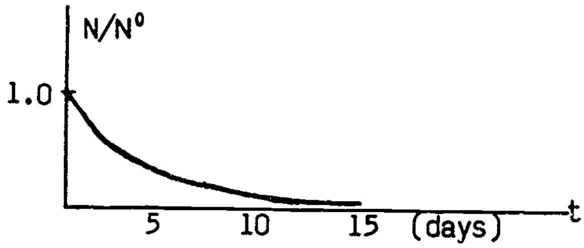
$$\ln\frac{1}{2} = -\lambda\tau$$

$$\lambda = -\frac{\ln 1 - \ln 2}{\tau}$$

$$= \frac{\ln 2}{\tau}$$

$$\lambda = 0.69/3.8 = 0.18 \text{ days}^{-1}$$

c)



Math: Alg.II: Nonlinear Equations Science: Physics: Doppler Effect
Senior Math

Background

If a source of sound of frequency, f_S, is in motion relative to you then the sound frequency, f_L, that you, the listener, hear is given by

$$f_L = f_S \frac{v + v_L}{v - v_S}$$

where v = speed of sound = 331 m/s in air

v_S = speed of the source relative to the air. It is +(-) if the source is moving towards (away from) you

v_L = speed of the listener relative to the air. It is +(-) if you are moving towards (away from) the source

For the formula to be correct both source and listener must be moving along the direction of a line joining them.

Problem

The frequency of a train whistle is 1050 Hz. What frequency do you hear if:

(a) you are stationary and the train is approaching at 40 m/s?

(b) you are stationary and the train is receding at 40 m/s?

(c) the train is stationary and you are approaching it at 40 m/s?

(d) the train is stationary and you are receding from it at 40 m/s?

Solution

(a) $f_S = 1050$ Hz $v = 331$ m/s $v_L = 0$ $v_S = +40$ m/s $f_L = 1050 \times \frac{331 + 0}{331 - 40} = 1194$ Hz

(b) $f_S = 1050$ Hz $v = 331$ m/s $v_L = 0$ $v_S = -40$ m/s $f_L = 1050 \times \frac{331 + 0}{331 + 40} = 937$ Hz

(c) $f_S = 1050$ Hz $v = 331$ m/s $v_L = 40$ m/s $v_S = 0$ $f_L = 1050 \times \frac{331 + 40}{331 - 0} = 1177$ Hz

(d) $f_S = 1050$ Hz $v = 331$ m/s $v_L = -40$ m/s $v_S = 0$ $f_L = 1050 \times \frac{331 - 40}{331 - 0} = 923$ Hz

Math: Alg. II: Nonlinear Equations   Science: Physics: Light Intensity

Background

According to the Hertzsprung-Russell diagram there is a direct relation between color and luminosity for many stars. That is, blue super giant stars are very bright (emit a great deal of light) while red dwarf stars are dimmest. The apparent brightness of a star depends on how far away from the star you are when viewing. We may write a relation between luminosity (L), apparent brightness (I) and distance (R) as

$$I = L/R^2$$

Problem

The starship Enterprise was disabled between two green stars (which means they have the same luminosity). The astronauts determined that star B, which lay directly ahead, appeared to be 36 times dimmer than star A, which lay directly behind. A simple check of star charts showed that stars A and B were 70 light-years apart. How far was the Enterprise from star B?

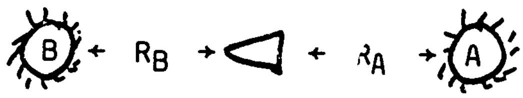
Solution

$I_A = L_A/R_A^2$, $I_B = L_B/R_B^2$

$L_A = L_B$ since both stars were green

$R_A + R_B = 70$, $I_A = 36\ I_B$

$L_A = I_A R_A^2$, $L_B = I_B R_B^2$



$I_A R_A^2 = I_B R_B^2$

$36\ I_B R_A^2 = I_B R_B^2$

$36\ R_A^2 = R_B^2$

$36\ (70 - R_B)^2 = R_B^2$

Solve for $R_B = 60$ or 84

Must have $R_B < 70$, hence $R_B = 60$ light years

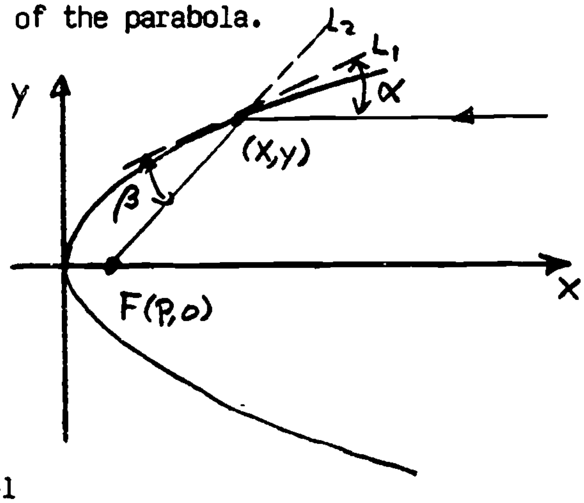
Math: Adv. Math: Parabola   Science: Physics: Mirror

Background

When light is reflected from a surface the angle between the incoming ray and the tangent to the surface at the point of reflection is equal to the angle between the outgoing ray and the same tangent.

Problem

Show that all light rays coming into a parabolic reflector parallel to the axes of symmetry are reflected through the focus of the parabola.



Solution

Choose a coordinate system in which the vertex of the parabola is at (0,0) and the focus is at (p,0). The equation of the parabola is $y^2 = 4px$. We will assume that the light passes through F and will prove that $\alpha = \beta$.

$\tan\alpha$ = slope of L_1 at reflection point = M_{L_1}

From $y^2 = 4px$: $2y\frac{dy}{dx} = 4p$ or $\frac{dy}{dx} = \frac{2p}{y}$ or $M_{L_1} = \frac{2p}{y}$ or $\tan\alpha = \frac{2p}{y}$

$$\tan\beta = \frac{M_{L_2} - M_{L_1}}{1 + M_{L_2}M_{L_1}} = \frac{\frac{y}{x-p} - \frac{2p}{y}}{1 + \frac{y}{x-p} \times \frac{2p}{y}}$$ which simplifies to $\tan\beta = \frac{2p}{y}$

Hence $\tan\beta = \tan\alpha$ and so $\alpha = \beta$, the angle of incidence = the angle of reflection.

Math: Alg. II: Conic Sections

Science: Physics: Motion

Background

If an object is launched in one dimension from the origin with an initial velocity, v_0, and an acceleration, a, then its distance from the origin at a later time, t, is given by $x(t) = v_0t + \frac{1}{2}at^2$.

Problem

(a) Rewrite the equation relating x and t by completing the square.

(b) Sketch x vs t on a graph. Identify the physically significant part of the curve and the significance of v_0.

(c) Does your curve imply that the object moves on a parabolic path in space?

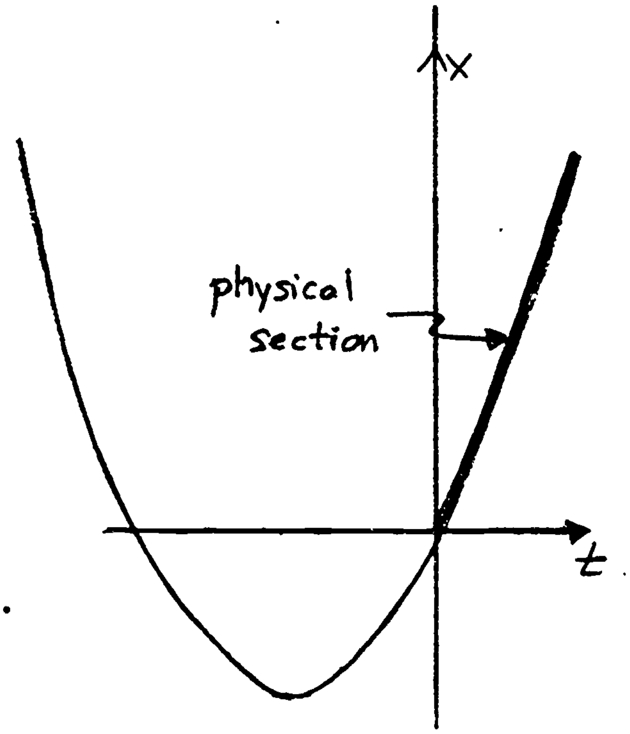
Solution

(a) $x = \frac{1}{2}at^2 + v_0t$

$$\frac{2}{a}x = t^2 + \frac{2v_0}{a}t$$

$$\frac{2}{a}x + (\frac{v_0}{a})^2 = t^2 + \frac{2v_0}{a}t + \frac{v_0^2}{a^2}$$

$$4(\frac{1}{2a})(x + \frac{v_0^2}{2a}) = (t + \frac{v_0}{a})^2$$



(b) This is the equation of a parabola with vertex at $(t,x) = (-\frac{v_0}{a}, -\frac{v_0^2}{2a})$. The physical portion of the curve is where $t \geq 0$. The slope of the curve at the origin is v_0.

(c) No! The motion is one-dimensional, i.e., the object moves along the x-axis only.

Math: Alg. II: Conic Sections Science: Physics: Orbits

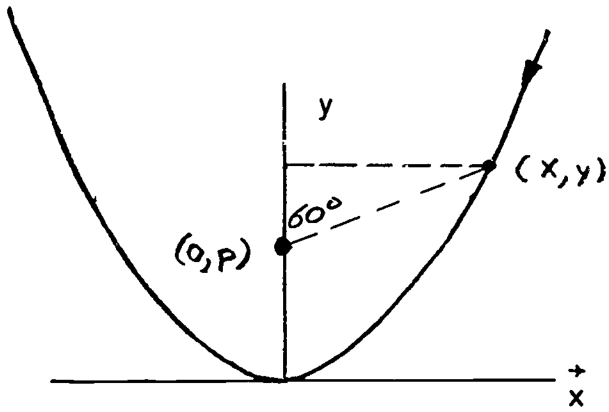
Background

Heavenly bodies can move about the sun in circular, elliptical, parabolic or hyperbolic orbits with the sun at the focus of the conic section. Once in one of these orbits the body remains in the orbit. Elliptical orbits (and the special case of circular orbits) are closed; hyperbolic orbits (and the special case of parabolic orbits) are open. Thus bodies in the latter orbits have one encounter with the sun and then never return.

Problem

A comet moves in a parabolic orbit. When the comet is 4×10^7 km from the sun its position with respect to the sun makes an angle of 60^0 with the axis of symmetry of the orbit (drawn in the direction in which the orbit opens). Find how close the comet comes to the sun.

Solution



Orient the coordinate system as shown. We are given $d = 4\times10^7$ km and wish to find p.

$x = d \sin 60^0 = 3.46\times10^7$

$y = d \cos 60^0 + p = 2\times10^7 + p$

$x^2 = 4py$

$(3.46\times10^7)^2 = 4p(2\times10^7 + p)$

$p^2 + 2\times10^7 p - 3\times10^{14} = 0$

$p = 10^7$ or $- 10^8$ $\therefore$ closest distance $= 10^7$ km

Math: Geometry: Hyperbola Science: Physics: Sound

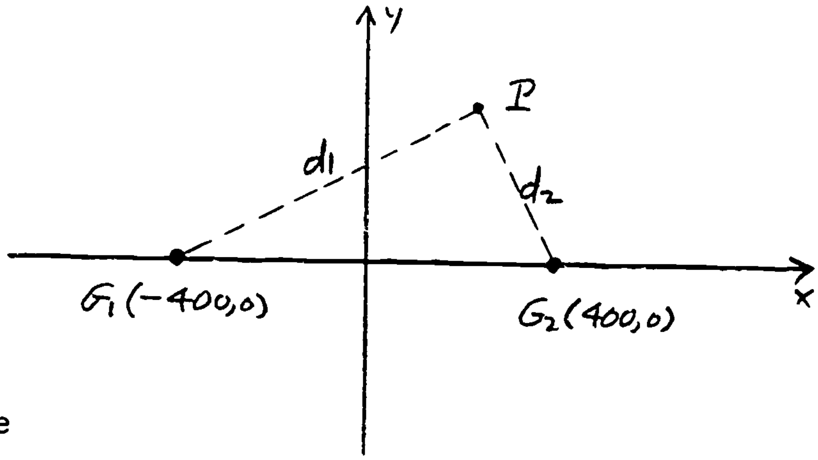
Background

The speed of sound, v, in air is approximately 300 m/s. The distance, d, sound travels in a time, t, is given by $d = vt$.

Problem

Guns are fired simultaneously from two positions 800 m apart. A person hears the shots 2 seconds apart. Where could the person be located with respect to the guns?

Solution



Set up a coordinate system where the guns are located on the x-axis, equidistance from the origin.

Let t_1, t_2 be the (unknown) times for the sound to reach the person at P from G_1, G_2. Then $t_1 = d_1/v$ and $t_2 = d_2/v$. But $t_1 - t_2 = 2$ seconds.

$$\therefore\ 2 = \frac{d_1}{300} - \frac{d_2}{300} \quad \text{or} \quad d_1 - d_2 = 600$$

Hence P lies on a hyperbola centered at the origin. The standard equation for a hyperbola is $x^2/a^2 - y^2/b^2 = 1$ where $2a = d_1 - d_2 = 600$ and c = distance from origin to G_1 or G_2 = 400. But $b^2 = c^2 - a^2 = (400)^2 - (300)^2 = 70{,}000$.

$$\therefore\ \text{P lies on the hyperbola } \frac{x^2}{90{,}000} - \frac{y^2}{70{,}000} = 1$$