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ABSTRACT

This Teacher Education and Mathematics (TEAM) content module focuses on probability. It consists of: (1) an instructor's text; (2) an instructor's guide and solutions to student exercises; (3) student materials and exercises; and (4) student summary and review. The instructor's text provides specific directions for guiding lessons and commentary on mathematics content and mathematics attitudes. This is accomplished by a "facing pages" format whereby the right-hand page provides step-by-step teaching directives while the left-hand page provides teaching insights, other options of instruction, and psychological or attitudinal strategies, when appropriate. The instructor's text also contains content objectives, specified to indicate the scope and structure of the module, and student evaluation materials. The instructor's guide and solutions to exercises provides approaches and solutions to problems. Student materials and exercises provide such items as diagrams, charts, and centimeter-squared paper to be used by students. Exercises include problems that apply the concepts and problem-solving strategies developed in the module; they may be used as part of the instructional activities, as content for small-group activities, as homework assignments, or as review materials. The student summary and review summarizes the content of the module, focusing on formulas, terminology, key concepts, problem-solving strategies, and examples of techniques used. (JN)

Choice and Chance

A Course to Reduce Math Anxiety and Sex-Role Stereotyping in Elementary Education

TEAM

TEACHER EDUCATION AND MATHEMATICS

Queens College of the City University of New York
Women's Educational Equity Act Program/U.S. Department of Education

TEAM

TEACHER EDUCATION AND MATHEMATICS

A Course to Reduce
Math Anxiety and Sex-Role Stereotyping
in Elementary Education

CHOICE AND CHANCE

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5

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CONTENTS

INTRODUCTION

The Choice and Chance module consists of an Instructor's Text, Instructor's Guide and Solutions to Student Exercises, Student Materials and Exercises, and Student Summary and Review.

The Instructor's Text provides the instructor with (1) specific directions for guiding lessons and (2) commentary on the math content and on math attitudes. This is accomplished by a "facing pages" format. The right-hand page gives directions that tell the instructor what steps to take in presenting the lessons, while the left-hand page, "Commentary and Notes," provides explanations, additional instructional options, attitudinal interventions, and organizational alternatives. Space for the instructor to add her or his own notes (for reference and future use) is also provided. When there is no commentary, the entire left-hand page is allotted to "Notes." The Instructor's Text includes a set of content objectives, specified to indicate the scope and structure of the module, and student evaluation materials, which contain several questions for each objective so that the instructor can select items for quizzes.

The Instructor's Guide and Solutions to Student Exercises gives the instructor approaches to the exercise materials and solutions for the problems.

Student Materials and Exercises provides such things as diagrams and charts to be used by the students. Instructors should plan to make transparencies of these materials for use on overhead projectors during class time. The exercises include problems that apply the concepts and problem-solving strategies developed in the module. These problems can be used as part of the instructional activities, as content for small-group activities, as homework assignments, or as review materials. Appropriate exercises (A.E.) are noted after each topic in the Instructor's Text.

Student Summary and Review summarizes the content of the modules (formulas, terminology, key concepts, problem-solving strategies, and examples of techniques used). These notes are to be given to students after they have participated in the learning activities of the module.

TEAM

CHOICE AND CHANCE

I

INSTRUCTOR'S TEXT

OBJECTIVES

The objectives of the Choice and Chance module are:

1. List the outcomes for an experiment, using a tree diagram.

2. Calculate the number of outcomes of an experiment, using the multiplication principle.

3. Calculate the number of permutations for a set of objects, using all of the objects.

4. Calculate the number of permutations for a set of objects, using any number of objects.

5. Perform computations with factorials.

6. Determine the total number of subsets that can be formed from a given set of elements, using the formula 2^n.

7. Use the Pascal Triangle to determine the number of subsets of a specific number of elements that can be formed from a given set.

8. Given the total number of outcomes and the number of favorable outcomes, determine the probability of an event.

9. Given the probability of an event, determine the probability that an event will not occur.

10. Given the probability of an event, predict the number of times the event can be expected to occur in a given number of trials.

These objectives are provided here to indicate the scope and structure of the module to you. They should be distributed to students with the Student Summary and Review so that students can use them to organize their study and preparation for a quiz.

Sample items for the objectives are included at the end of this section of the module.

Discuss with students the significance of such predictions for them in their lives.

The measure that is often called an average is actually the mean. The mean of three scores such as 67, 73, and 70 is computed as follows:

$$\frac{67 + 73 + 70}{3} = \frac{210}{3}, \text{ or } 70$$

Note whether students react with tension to the use of the mathematical term ratio. If they do, raise the question of how they are feeling. Proceed with the content after students have expressed their concerns. You might also reassure students that they are in a better position for learning mathematical terms now than they were as children. Their broader experience and greater vocabulary provide this current advantage.

Ask students to predict the number of heads they would expect in 50 tosses of a fair coin. Then ask them to experiment by actually tossing a coin 50 times and tallying the results in a table like the one below.

No. Heads Predicted	Heads	Tails

Make it clear that the experimental results will usually differ from what is predicted. Pool students' results to see whether the ratio of the number of heads to the number of tosses gets closer to 1:2.

The chances of obtaining a head on any particular toss are always the same: 50%, or $\frac{1}{2}$.

BEGINNING THE PROGRAM

Introduce the topic of probability with an informal discussion of its usefulness in making predictions. Ask students to speculate about how data might be used in the following areas:

1. Quality control in manufacturing
2. Genetic research
3. Testing drugs used in medicine
4. Weather forecasting
5. Baseball

Show how a "batting average" is computed. Point out that the number obtained is a measure of a player's past performance and that it is used as a predictive measure. Compare it with a simple average (mean) to show that "batting average" is not an accurate name.

Say: "Suppose a player has had 6 hits out of 20 times at bat. The ratio is expressed in thousandths, and we say the player is batting three hundred."

$$\frac{6}{20} = \frac{3}{10}, \text{ or } .3$$

$$\frac{3}{10} = \frac{300}{1000}, \text{ or } .300$$

Based upon the player's past performance, we predict that this player has a better chance of getting a hit the next time at bat than someone who's batting, say, .250. It would be correct to say that the probability that the first player will get a hit is .3 (or .30 or .300).

Next, point out that in order to understand some of the ideas of probability, we experiment with such devices as dice, coins, standard playing cards, and spinners.

Equally Likely Events

Ask the following questions and have students answer them intuitively. Do not make any formal calculations at this time.

1. How many times should I flip a coin to decide if it is a fair coin?

   This discussion should include some indication that with a fair coin we expect that it is equally likely that either heads or tails will come up. The common expression "50-50 chance" conveys

Help students see that whatever is substituted for a coin must be able to perform so that:

- There are exactly two outcomes
- The two outcomes are equally likely to occur

A spinner of two colors, one color on each side, would be acceptable.

You may wish to have students experiment with the tossing of thumbtacks. Toss 10 thumbtacks (of the same kind) 20 times and record the results: point up or point down.

1. Tally the results in a table similar to the one below.

2. Determine the total number of tacks (out of $20 \cdot 10 = 200$) that land point up.

3. Form the ratio of the number of favorable outcomes to the total number of trials. In this case, let

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of trials}} = \frac{\text{Number landing point up}}{\text{Total number of tacks}}$$

4. Show how the ratio obtained above can be used as a probability measure--to predict the number of favorable outcomes that can be expected in the future.

Toss Number	Point Up	Point Down	Total Up

the idea that each is expected about 50% (or half) of the time. It should be further pointed out that this is a prediction and that the actual results will more closely approximate the predicted number as the number of trials (flips of the coin) increases.

2. If you don't have a coin to flip, what could you use?

   Suggest a thumbtack. Students should intuitively find this unacceptable. The discussion should revolve around the idea of equally likely events or outcomes.

3. Which is more likely with a fair coin--two heads in two flips or three heads in three flips?

   Elicit from the students the understanding that they have to know which outcome is expected to occur more frequently. The answer is two heads in two flips. However, to be in a position to answer the question, we must first develop a systematic approach to counting outcomes.

4. Two dice are rolled. If you had a choice, would you bet on 6 or 7 as the sum of the dots on the two dice?

   Point out that the answer depends on which sum can be expected to occur more often. The numbers that have the sum 7 occur more often than those that have the sum 6, but we need methods of counting in order to verify this result.

Multiplication Principle

Present the following problem: A die is rolled at the same time as a coin is tossed. How many different outcomes are possible? What are they? Allow a few minutes for speculation and discussion.

Ask: "When a die is rolled, how many outcomes are possible? What are they? Are they equally likely?"

Ask: "When a coin is tossed, how many outcomes are possible? What are they? Are they equally likely?"

Point out that a systematic method is needed to list the outcomes.

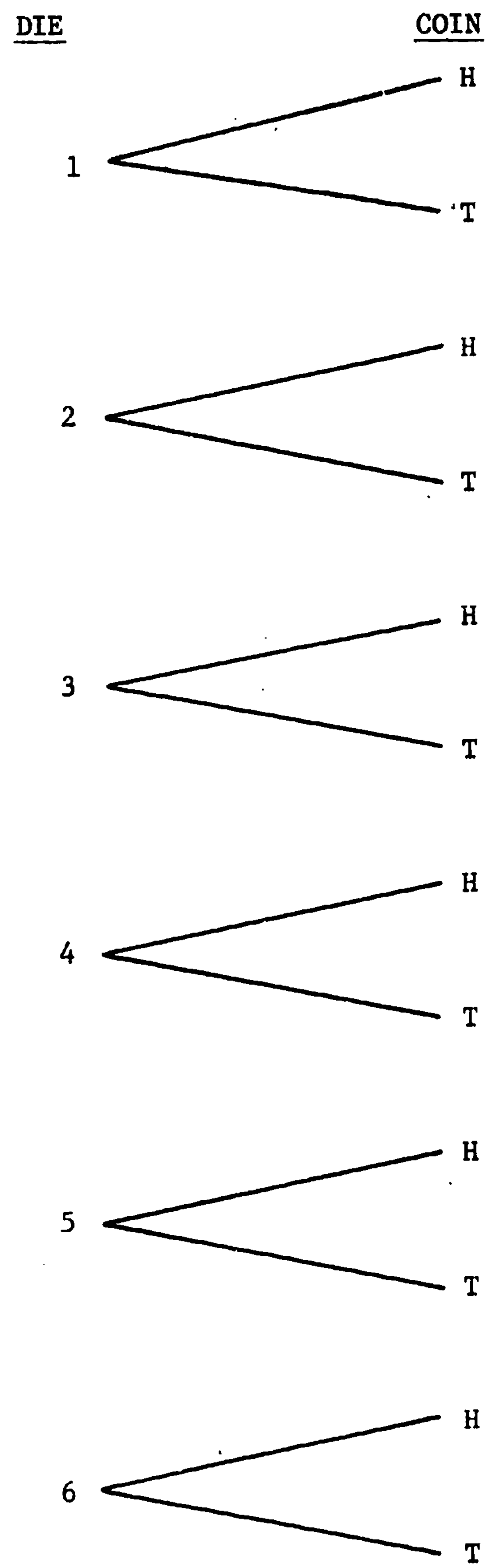
COMMENTARY AND NOTES

Diagrams and models of problems are an important strategy for problem solving. Use of the tree diagram for generating and counting expected outcomes will be helpful throughout this module.

Systematic recording of data should be demonstrated by the instructor and discussed at opportune moments. Students' ways of recording data should be noted and suggestions given when appropriate.

Tree Diagram

Develop the <u>tree diagram</u> with the students, step by step.



COMMENTARY AND NOTES

In dealing with uncertain outcomes in this module, the student can learn to feel more comfortable with outcomes that cannot be controlled. An individual can gain some sense of power from being able to recognize possible outcomes and to make predictions about what may happen in the future.

Small numbers are used throughout to minimize calculation problems. When students have solved these simpler problems, they can then transfer the process or method to problems with larger numbers, using a calculator if necessary.

At some point in the discussion help students understand that tossing two coins once or one coin twice yields the same set of outcomes.

From the tree diagram, we can see that:

$$6 \cdot 2 = 12$$

There are 12 outcomes.

Help students see that multiplication should be used since there are 6 outcomes for the die, with 2 branches (representing the 2 possible outcomes for the coin) from each.

Ask: "If you rolled a die and tossed a coin twelve times, would you see all the possibilities?" (Not necessarily)

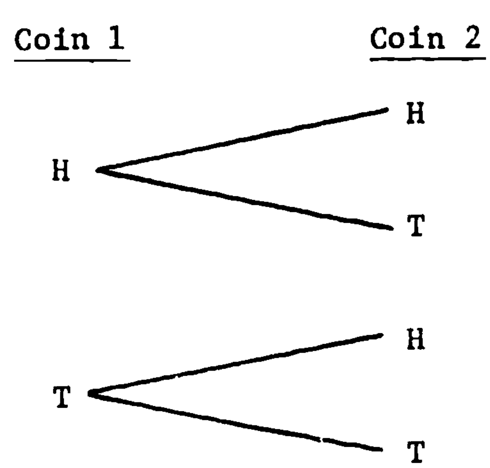
A.E. 3, 12, 34, 36, 51a, b, c

Outcomes and Events

Discuss with students some of the vocabulary that is customarily used in talking about probability. We speak of probability activities as <u>experiments</u>. Each of the possible occurrences is called an <u>outcome</u>. One or more outcomes is called an <u>event</u>. For example, in the die/coin experiment just performed, an event might be the die coming up an odd number and the coin coming up heads. (The tree diagram shows that this event consists of 3 outcomes.)

Continue to develop the multiplication principle by presenting each of the following problems and assisting the students with the construction of a tree diagram for each.

1. Toss two coins simultaneously. How many possible outcomes are there?



$2 \cdot 2 = 4$ outcomes, or $2^2 = 4$

COMMENTARY AND NOTES

This section of problems can also be used to review students' understanding of exponents. Students who are weak in this concept should work with peer tutors. Another teaching strategy is to check understanding of exponents before beginning the section. The class can be divided into two groups. Those who understand exponents can be given a set of exercises—for group problem solving while you work with those who need help with exponents.

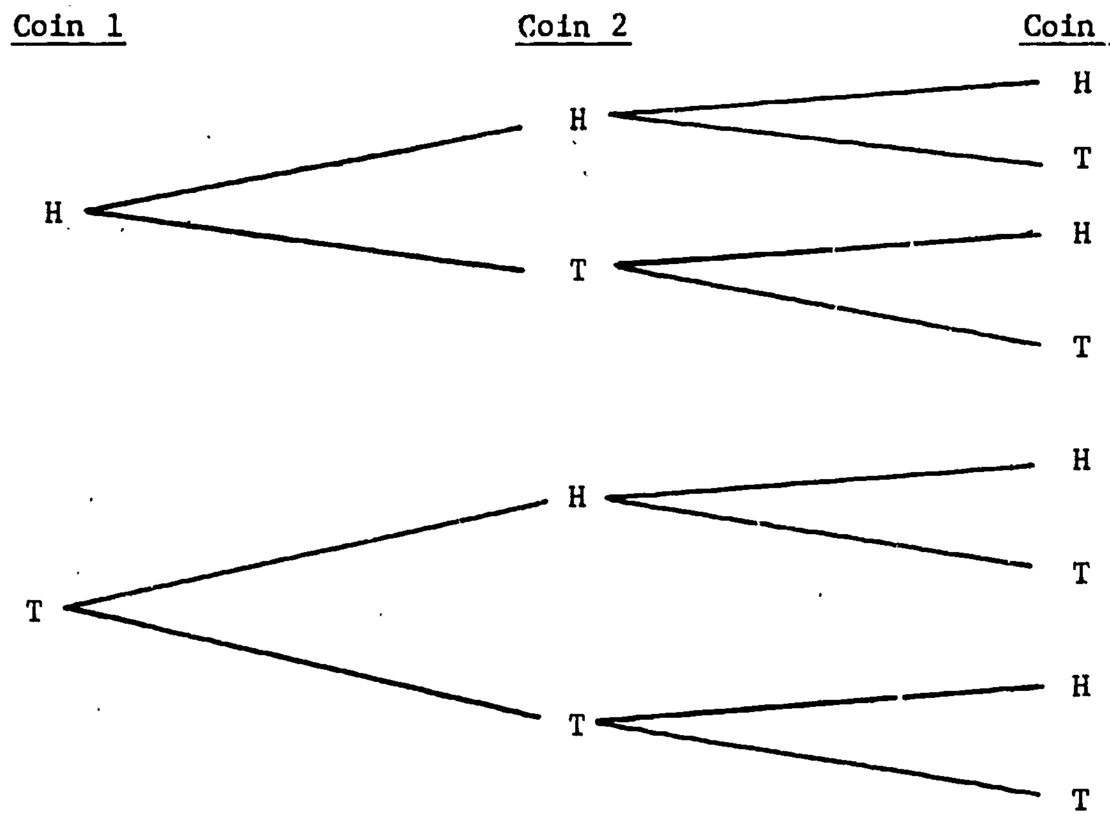
The presentation of a problem that has no immediate solution is expected to produce tension. Students should be aware that this is part of the problem-solving process. Excitement should be high as they move toward a solution.

Whenever n coins are tossed, there will be a total of 2^n outcomes. Thus, with 4 coins there are 2^4, or 16, possible outcomes. Students may need help with the evaluation of the powers of 2. It may be helpful to show them the way that parentheses operate. For example:

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2, \text{ or } (2 \cdot 2)(2 \cdot 2) = 4 \cdot 4$$

It will be interesting for students to calculate powers of 2 on calculators so that they can see how rapidly the numbers increase.

2. Toss three coins simultaneously. How many outcomes are possible?



$2 \cdot 2 \cdot 2 = 8$ outcomes, or $2^3 = 8$

Help students see the following pattern in the coin-tossing experiments:

2 coins: $2^2 = 4$

3 coins: $2^3 = 8$

The exponent is always the same as the number of coins.

3. There are 5 kinds of pie (peach, apple, cherry, blueberry, lemon) and 3 flavors of ice cream (vanilla, chocolate, strawberry). How many different pie-a-la-mode desserts are possible?

   Ask: "How many different kinds of pie can be put on your plate? What are they?"

   Ask students to list the kinds of pie vertically (in any order):

   peach

   apple

   cherry

   blueberry

   lemon

COMMENTARY AND NOTES

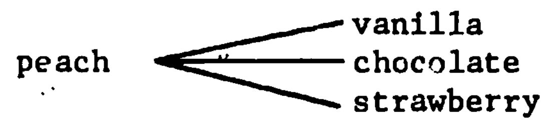
Help students see that a systematic way of noting data makes it possible for patterns to be discerned. It is easier to keep track of where you are in your progress toward a solution when data are organized.

Point out to students that the inductive approach is basic to these materials. In the last few pages, we have discerned a pattern by examining specific cases. Whenever we use specific cases to generalize, we are using inductive reasoning. Discuss with students the cautions that are needed whenever a person makes a generalization from a limited number of instances.

The use of m and n may evoke some tension on the part of students who experience math anxiety. It may be helpful to show how the numbers used in the previous examples can be substituted for m and n. Keep in mind, however, that the statement of the multiplication principle is not as important as the ability of students to understand and apply it.

Encourage students to ask questions that will clarify their understanding. Remember that students often are afraid to ask "dumb" questions unless they feel very comfortable.

Next ask: "How many different choices of ice cream are there to place on the peach pie?" Show the possibilities by means of three branches:



In the same way, construct three branches from each of the other kinds of pie. Finally, show the indicated product (5•3) that leads to the answer, 15 desserts.

As another example of the multiplication principle, present the following problem:

4. One can travel from City A to City B (or from B to A) via train, plane, or car. If one travels from A to B by one mode of transportation and returns by a different mode of transportation, how many different trips can one take? (3•2) How many trips can be made if one can use any mode in either direction? (3•3)

Review the preceding four problems to show that in each case, the total number of outcomes (whether trips, desserts, or the ways coins fall) is calculated by multiplication. State the multiplication principle and show that it can be extended to include any number of operations.

Multiplication Principle: If an operation can be performed in m ways, and if after it is performed in any one of these ways, a second operation can be performed in n ways, then the two successive operations can be performed in mn ways.

A.E. 1, 2, 4, 5, 6, 7, 31

To explore the multiplication principle further, ask students to read the following article, "Licentious Plates" (see also in Student Materials and Exercises).

LICENTIOUS PLATES*

There are no BUMS, CADS or DUDS on the road in Iowa, not even APES, HAGS or HAMS. These prefixes were all banned from the state's automobile license plates on grounds of taste by the Iowa Department of Transportation. But since new plates were issued last month, 130 irate motorists in Scott County have returned the plates because they bore the prefix GAY. One woman wrote: "I cannot be a single teacher and sport those

*"Licentious Plates," reprinted with permission from TIME, The Weekly Newsmagazine, 22 January 1979. Copyright © Time, Inc., 1979.

COMMENTARY AND NOTES

Point out that most calculators allow for a maximum of eight places on the display. Thus, we calculate 26^3 on the calculator and multiply the result by 10^3 by annexing three zeros to the right of 17 576.

This material uses the National Bureau of Standards notation for large numbers. The comma is no longer used to separate groups of three digits. Instead, a space is used. Thus 3,487,521 is written 3 487 521. In the case of four-digit numbers, neither a space nor a comma is used. Thus, 9438 is the correct form for a four-digit number.

Students should be aware of the reduction of tension when they grasp a problem or arrive at a solution. It is useful to explore briefly how students experience this aspect of the problem-solving process.

plates." A traveling salesman complained that while he was in Chicago, his car doors were kicked in because of the plates.

State officials blame the foul-up on their decision to use California's list of three-letter, three-digit combinations rather than prepare their own. The officials tried to eliminate prefixes that might be offensive to Iowans but overlooked GAY. Says Scott County Treasurer William Cusack: "Out in California I'm sure there is a waiting line for GAY plates. But not in Iowa." He is offering to exchange the GAY plates--1,000 were issued--on payment of a $4 fee.

Say: "First let us calculate the total number of plates that is possible under such a system. Each license plate is made up of 3 letters followed by 3 numbers."

Discuss the solution:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^3, \text{ or}$$

$$17\ 576 \cdot 10^3 = 17\ 576\ 000$$

In discussing the solution, observe that 26^3 allows for the repetition of letters and that 10^3 allows for the use of any of the digits 0 through 9 (10 digits) in any of the three places. For example, one possible license plate might be:

QQQ 007

Next, ask for several examples of license plates that would be withdrawn from circulation and ask what they have in common. (Answer: Those with GAY in the first three places: GAY 930, GAY 004, etc.)

Help students see that with GAY established as occupying the first three places in the withdrawn plates, they can go on to verify the final statement in the article, "1,000 were issued . . ." The GAY plates must then include 000, 001, 002, ..., 999--a total of $10 \cdot 10 \cdot 10$, or 1000, plates all together.

A.E. 8, 9, 10

COMMENTARY AND NOTES

Rewording a problem can often help students find the key to its solution.

Remember to leave time here for an airing of students' feelings. Knowing that they are not the only ones who lack confidence can often help students make significant progress.

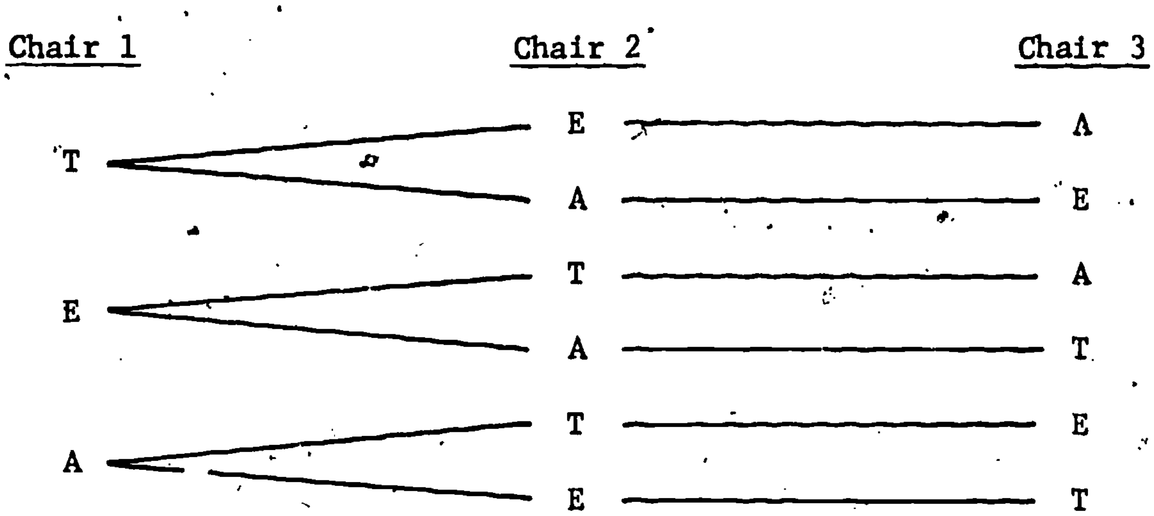
Permutations

Present the following problem:

Tom, Evelyn, and Alice are to sit in a row to have pictures taken. How many different arrangements are possible?

Help students develop a tree diagram and arrive at the solution.

$3 \cdot 2 \cdot 1$, or 6



Point out that once a person is seated, she or he cannot be seated again. Hence, the number of choices available decreases by one each time.

Reword the problem:

How many ways can you arrange the letters in the word TEA?

Refer to the tree diagram to show that there are $3 \cdot 2 \cdot 1$, or 6, arrangements of the letters in the word TEA. Point out that for each arrangement we assume that the letters are being moved and cannot be repeated.

Explain that the term <u>permutation</u> is used to mean an arrangement. Ask students to consider the following examples of permutation, expressing the solution to each as an indicated product.

1. Four names are written on separate sheets of paper. In how many different orders can they be drawn out of a hat, one at a time?

2. In how many different ways can we arrange four people in a row for a picture?

3. In how many different ways can we arrange five different colored flags in a column for a signal?

Suggest that students look at the factorial table in Student Materials and Exercises. Students should know that such tables are available and can be consulted. It might be reassuring for students to know, too, that factorials need not be evaluated from scratch. Show them such equalities as $5! = 5\cdot4!$

Discuss the use of formulas and the advantage of expressing a generalization in this form. Practice using the formula should help give students confidence to handle many problems of the same kind.

It is desirable to return to the tree diagram to show the selection process pictorially. Using the tree diagram for counting will very often help students see what's happening before the diagram has been completed.

Students should be given some practice with the factorial symbol to understand its meaning better. For example, they should actually evaluate such expressions as

$$\frac{7!}{5!} = \frac{7\cdot6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}, \text{ or } 42$$

and

$$\frac{7!}{7} = \frac{\cancel{7}\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{\cancel{7}}, \text{ or } 6!$$

to understand how the symbol operates.

Ask students to consult the table of factorials to see how rapidly the answers grow. For example,

$$4! = 24;\ 5! = 5\cdot4!, \text{ or } 120; \text{ and } 6! = 6\cdot5!, \text{ or } 720$$

Summarize the results by observing again that each problem is an example of a permutation. Make sure students are aware of the following points in the solution of these problems:

- Objects are being rearranged and their order is important.

- Each indicated product starts with the number of objects and includes each smaller whole number down to 1. That is, the number of permutations of 4 objects is $4 \cdot 3 \cdot 2 \cdot 1$. The number of permutations of 5 objects is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Tell students that there are tables that enable us to determine such products quickly and that there is a short way to write the products. For example,

$4!$ (read: "4 factorial <u>or</u> factorial 4") means $4 \cdot 3 \cdot 2 \cdot 1$, and

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Help students see that the number of permutations of _n_ objects is always n! Tell them that we shall use the symbol ${}_nP_n$ to stand for "the number of permutations of _n_ objects."

Thus the formula for the number of permutations of _n_ objects is: ${}_nP_n = n!$

Now present and discuss the following examples of permutation, in which only some of the available objects are used. Show how the multiplication principle applies in each case.

1. There are seven horses in a race. How many arrangements of first-, second-, and third-place winners can there be?

$$ {}_7P_3 = 7 \cdot 6 \cdot 5, \text{ or } 210 $$

2. In a game of musical chairs, there are 4 chairs and 5 people. In how many ways can the 4 people be seated when the music stops?

$$ {}_5P_4 = 5 \cdot 4 \cdot 3 \cdot 2, \text{ or } 120 $$

Point out that in the first example, ${}_7P_3$ stands for the number of permutations of 7 objects, 3 at a time. Note that there are 3 objects being arranged and 3 factors in the product.

A.E. 11-19, 27, 29, 32, 33, 35

COMMENTARY AND NOTES

Review sets as needed. Remind students that members of a set may be listed in any order. Thus set A can be written {a,b,d,c}. The braces are read as "the set whose members are . . ."

Review the idea of a subset. Set B is said to be a subset of set A if each object in set B is also in set A. Thus, if B = {a,c}, then set B is a subset of set A.

The empty set is symbolized by { } or ∅. By agreement the empty set is a subset of every set.

If students have had previous experience with sets, their reactions may vary. Some may react positively because they already understand sets. Others may feel that they've never understood the concept and thus may begin reacting to their fears.

Review the idea of the complement of a set. Every subset, B, that is formed creates another subset of those elements that are not in B. We call this subset B' (B prime). For example:

If U = {a,e,i,o,u} and B = {i,o,u}, then B' = {a,e}.

Subsets

Present the following problem:

> Given a set of 4 people, how many 3-person committees can be formed?

List the committees with the students by representing the 4 people as Set A: {a,b,c,d}

Say: "Let us list all the possible subsets of set A."

Ask: "Is set A a subset of itself?"

Point out that every set is a subset of itself. Then ask students to record all the subsets with you.

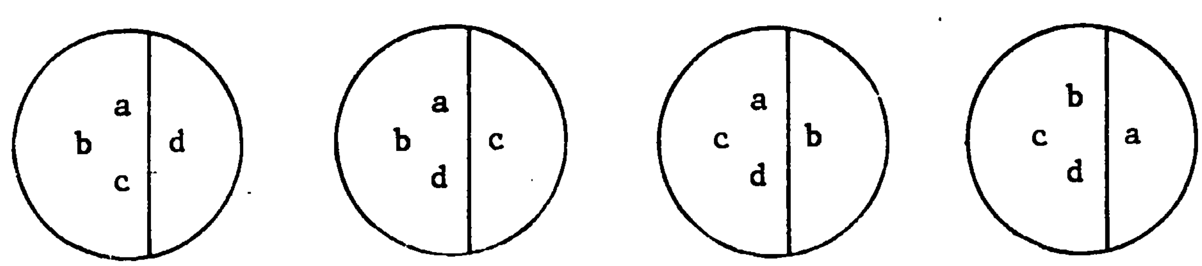
Subsets of {a,b,c,d}	Number
{a,b,c,d}	1
{a,b,c} {a,b,d} {a,c,d} {b,c,d}	4
{a,b} {a,c} {a,d} {b,c} {b,d} {c,d}	6
{d} {c} {b} {a}	4
{ } is the empty set	1

The 3-person committees (subsets of set A) are as follows:

{a,b,c} {a,b,d} {a,c,d} {b,c,d}

Explain the dichotomy (splitting) that is created by forming a subset of 3 elements from a set of 4 elements. For each set of 3 elements, a set of 1 element is created. Use a diagram like the following one to illustrate:

{a,b,c} {a,b,d} {a,c,d} {b,c,d}



creates {d} creates {c} creates {b} creates {a}

The complement of each set of 3 is a set of 1.

COMMENTARY AND NOTES

The French mathematician Blaise Pascal (1623-1662) arranged these numbers in this fashion for observing the patterns among the numbers in the triangle.

As students proceed with their work, help them feel the joy of creative activity and take pleasure in their sense of increased power.

Ask students to trace aloud the mental steps they followed to solve these problems. Acknowledge that there are various routes to a solution. Math is not a field in which answers are memorized.

Ask students to list and count all the possible subsets of a set of 2 elements. If necessary, work with them through the following steps:

Let A = {a,b}

Then	{a,b}	1
	{a}{b}	2
	{ }	1

Then ask students to do the same for a set of 3 elements.

Ask students to look at the triangle on page III-3 of Student Materials and Exercises. Project a transparency of the triangle and ask students to work with you to place the numbers in position.

Point out that this triangular array of numbers, obtained by counting subsets, is called the Pasçal Triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Show students how the top two rows of the Pascal triangle are derived from sets with 0 and 1 element.

Now ask students to count all the possible subsets for a set of 5 elements. Some students will be more comfortable writing out all the subsets. Others should be encouraged to study the patterns of the Pascal Triangle and to use the patterns to predict the next row, which represents a set of 5 elements. Guide students to observe and use the following patterns of the Pascal Triangle:

1. The 1's are written first and last in each row.

2. There is a sequence of consecutive numbers (1, 2, 3, ...) written diagonally in two directions. Thus the row for a set of 5 elements has 1, 5, __, __, 5, 1.

3. The sequence 1, 3, 6, ... appears diagonally in two directions. Row 5 now becomes 1, 5, 10, 10, 5, 1 to continue the sequence 1, 3(1 + 2), 6(1 + 2 + 3) with 10(1 + 2 + 3 + 4).

COMMENTARY AND NOTES

Many students may understand this material with an ease they usually attribute to the fact that the material is "easy," rather than to any insight or intellectual application on their part. It is an important point to have students consider whether the material is really easy or just seems so because their understanding and ability have increased.

4. There is a symmetry in each line that reflects the dichotomy observed earlier.

   If in a set of 5 elements there are

   5 subsets of ① element each,

   then there must be

   5 subsets of ④ elements each:

   $(1 + 4 = 5)$

   If there are

   10 subsets of ② elements each,

   then there must be

   10 subsets of ③ elements each

5. There is always one more entry on a line than the number of elements in the given set.

6. The following patterns of sums make it possible to generate subsequent lines of the triangle.

Number of Elements (n)	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1

COMMENTARY AND NOTES

Help students understand the advantage of expressing each number in the right-hand column as a power of 2. Each exponent is the same as the number of elements in the set.

35

Number of Elements (n)										Total Number of Subsets
0					1					$1 = 2^0$
1				1		1				$2 = 2^1$
2			1		2		1			$4 = 2^2$
3		1		3		3		1		$8 = 2^3$
4	1		4		6		4		1	$16 = 2^4$

Ask students to calculate the total number of subsets for each number of elements by adding across each row.

Assist students with the renaming of the sums in exponential form. Show how the definition ($2^0 = 1$) fits into the established pattern.

$$1 = 2^0$$
$$2 = 2^1$$
$$4 = 2^2$$
$$8 = 2^3$$
$$16 = 2^4$$

Ask students to observe the pattern of the total number of subsets. Elicit from students the understanding that the number of elements is the exponent. Thus with 5 elements, there is a total of 2^5 subsets. Ask students to follow the pattern and express the formula for the total number of subsets of a set of <u>n</u> elements: 2^n. A set of <u>n</u> elements has 2^n subsets.

Work with students to calculate the row for 6 elements (1, 6, 15, 20, 15, 6, 1). Show how the various patterns just explored (including the sums) can be used.

Now ask students to record the next several rows of the Pascal Triangle and to check the sum in each case.

Ask students questions that are to be answered by referring to the Pascal Triangle. Use the following questions as models:

1. How many subsets of 5 elements can one make from a set of 7 elements? (21)

2. How many subsets can be made from a set of 5 elements? (2^5, or 32)

COMMENTARY AND NOTES

Ask students to generate their own problems and solve them. Then ask: "Why is it important for you as learners to be able to make up questions and problems of this sort?"

This is a good time to vary the classroom organization. Working in small groups or pairs is appropriate for this material.

If an incorrect solution is presented aloud, have the student describe the process used to arrive at the solution. Identify those parts of the process that were correct. Help students understand that an incorrect solution can be a good learning experience for everyone.

The language of probability should be introduced whenever it seems appropriate and useful.

3. If a child has 3 friends, in how many ways can he or she invite one or more of them to dinner? [($2^3 - 1 = 7$) The empty set is not included in this case.]

Select an entry from the Pascal Triangle and ask students to pose a question for which that number would be the answer.

A.E. 20-26, 28, 30, 34, 56

Reinforce the distinction between permutations and combinations (subsets). Stress the importance of arrangement and order for permutations. Point out that a combination (or subset) is used when order is not part of the problem. Using pairs of examples such as the following may be helpful in making this distinction.

1. In how many different ways is it possible to choose a president, secretary, and treasurer from a club with 7 members?

2. In how many different ways is it possible to choose a committee of three from a club with 7 members?

Point out that in the first example, there are many permutations just for one group of 3 members, since each one can be president, secretary, or treasurer.

Order is of no importance in the second example. Any set or combination of 3 constitutes a single committee, regardless of the order in which they are chosen.

A.E. 27-30

Probability

Begin a discussion of the intuitive ideas we have about probability by asking: "Why is there a 50-50 chance that a coin will come up heads when it is flipped?"

The following points should be made:

1. The 50 refers to 50%, and $50\% = \frac{1}{2}$.

2. We expect the coin to come up heads roughly one-half the time, even though we cannot predict how it will land on a particular toss.

3. The 50-50 prediction assumes that heads and tails are equally likely and that the coin will be tossed many times.

4. We say that the probability of tossing a head is $\frac{1}{2}$.

The term experiment is used to describe any activity that one engages in for the purpose of making numerical predictions about what can be expected to occur. Tossing a coin, rolling a die, and selecting a card are all experiments.

Any listing of all possible outcomes of an experiment is called a sample space. Each outcome is called a simple event. Any event is a set of simple events.

Help students see that we are making the assumption that the outcomes in each sample space are equally likely. Evidence of this assumption is our intuitive recognition that a checker can be substituted for a coin in a coin-tossing experiment, but a thumbtack cannot be expected to yield equally likely outcomes.

Whatever event is specified, that event is called success. If you specify rolling a 5 on a die, then 5 is success. If you specify not getting a 5, then any number that is not a 5 is success.

Panic may start when so much new material is presented. Try to pace the presentation at a rhythm that is comfortable for your class.

Refer to the question posed earlier: Which is more likely with a fair coin--two heads in two flips or three heads in three flips (see No. 3, page I-7)? Show students how they can use the tree diagrams constructed previously. In the toss of two coins there is 1 favorable outcome out of 4 possible outcomes, whereas in the toss of three coins there is 1 favorable outcome out of 8 possible outcomes.

Propose the following probability experiment:

You roll a die. What is the probability that 5 will come up?

It is possible that students will intuitively offer the answer $\frac{1}{6}$. Help students arrive at this answer by guiding them through the following series of steps:

1. List the outcomes of the experiment: 1, 2, 3, 4, 5, 6

2. Since there are six equally likely outcomes, each outcome is assigned the probability $\frac{1}{6}$. The probability of getting a 5 is shown in the following way:

$$P(\text{five}) = \frac{1}{6}$$

3. There is just one way of getting a 5, but there are 6 possible outcomes. Consider the following probabilities:

$$P(\text{four or five}) = \frac{2}{6}, \text{ or } \frac{1}{3}$$

$$P(\text{not-five}) = \frac{5}{6}$$

$$P(\text{five or not-five}) = \frac{6}{6}, \text{ or } 1$$

$$P(\text{seven}) = \frac{0}{6}, \text{ or } 0$$

Help students derive the following definition for a probability measure. Let E represent any event.

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}}$$

Verify that the previous probability measures are consistent with this definition. Help students come to the following conclusions from the definition:

1. The probability of an impossible event is 0 (the numerator is always 0).

It may be necessary to remind students that a standard deck of playing cards contains 4 suits (spades, hearts, diamonds, and clubs), with 13 cards in each suit, making a total of 52 cards.

Review percent as needed. Point out that $\frac{1}{4} = \frac{25}{100}$, and 25% means 25 per hundred.

$$\frac{1}{13} = \begin{array}{r} .076 \\ 13\overline{)1.000} \\ \underline{91} \\ 90 \\ \underline{78} \\ 12 \end{array}$$

$$.076 \approx 8\%$$

Thus, $\frac{1}{13} \approx 8\%$

41

2. The probability of a certain event is 1 (the numerator is equal to the denominator).

3. A probability measure is always a number between 0 and 1, including 0 and 1.

Present the following experiment and some relatively simple probability questions:

> Suppose we draw a card from a standard deck of playing cards that has been shuffled.

Calculate the probability that the card is one of the following:

1. The ace of spades $(\frac{1}{52})$

2. An ace $(\frac{4}{52} = \frac{1}{13})$

3. A spade $(\frac{13}{52} = \frac{1}{4})$

Now discuss with students the meaning of each probability measure. For example, if you draw a card from a standard deck of 52 cards, replace the card, and then repeat the process many times (always drawing from a full deck), you can expect to draw a spade roughly $\frac{1}{4}$, or 25%, of the time. Thus, if you draw 100 cards, you can expect to draw about 25 spades.

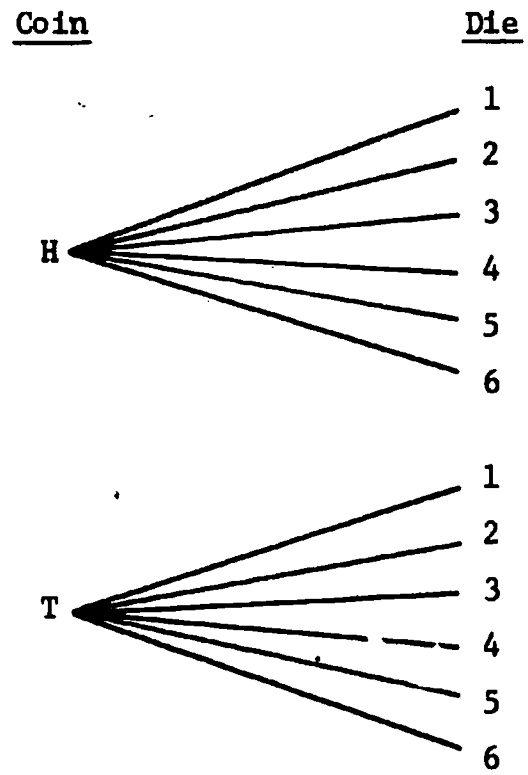
A.E. 36-42, 43a, b, c, 44, 45, 53, 54

Events and Ordered Pairs

Work with students to construct a tree diagram (following) of an experiment in which one tosses a coin and rolls a die. Verify that there is a branch for each possible outcome. The outcomes can be listed as ordered pairs. For example, (T,3) means tails and 3.

COMMENTARY AND NOTES

Continue to use the term <u>success</u> for the desired outcome.



(H,1)*
(H,2)
(H,3)*
(H,4)
(H,5)*
(H,6)

(T,1)
(T,2)
(T,3)
(T,4)
(T,5)
(T,6)

Help students arrive at the conclusion that there are 2·6, or 12, possible outcomes.

Ask: "What is the probability of getting tails and 3?" $(\frac{1}{12})$

Ask: "What is the probability of getting heads and an odd number?" Work through the following steps with students:

Let this event be F. There are 3 favorable outcomes, or successes, shown by the asterisk (*) in the tree diagram. Thus:

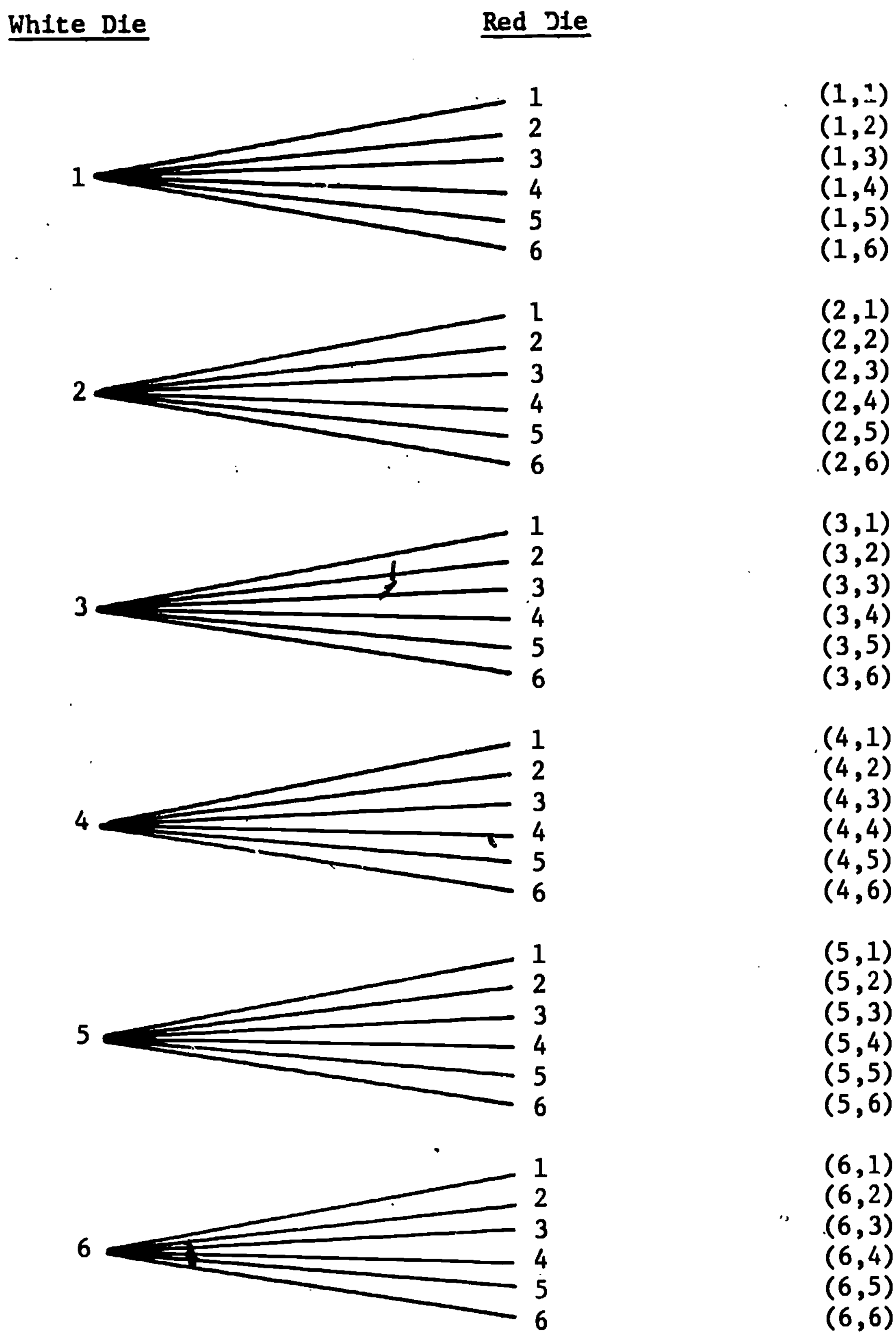
$$P(F) = \frac{3}{12}, \text{ or } \frac{1}{4}$$

Ask: "What does [$P(F) = \frac{1}{4}$] mean?" For the answer, refer to the preceding discussion of the probability of obtaining a spade (page I-35).

COMMENTARY AND NOTES

It is important to help students recognize that the dice must be distinguishable from each other (different colors or sizes) so that white 1, red 2 is a different outcome from white 2, red 1.

Next, consider the sample space for two-dice experiments. Work with students to construct a tree diagram like the one below for an experiment in which two dice are rolled--a white die and a red die.



COMMENTARY AND NOTES

The lattice is another example of a physical model that makes it easier to see the relationships between numbers.

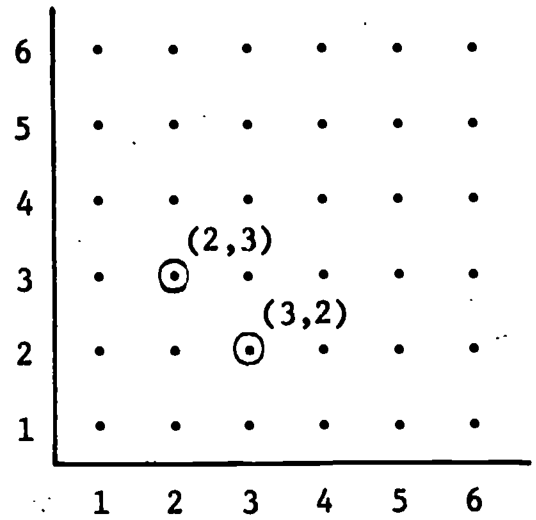
This lattice is a geometric model. For students who don't readily use a geometric approach, this can be a useful introduction to coordinate geometry.

47

Go over the various outcomes and show how, by agreement we can use ordered pairs of numbers to represent individual outcomes. Note that the order of the numbers is significant. Pairs such as (2,3) and (3,2) represent different outcomes.

Review with students the system for assigning ordered pairs to points in a plane. Work with them to construct a 6 x 6 lattice of 36 points, like the one below. This is another systematic way of writing the sample space for a two-dice experiment.

Lattice of Points



When you discuss the lattice, be sure to use appropriate language. For example, say: "The coordinates of the point (2,3) are 2 and 3."

Now present probability questions that can be answered by counting outcomes--branches of the tree diagram or points of the lattice. The following questions are provided as models.

1. What is the probability that the dice will come up 2 and 3 (or 4 and 1, etc.)?

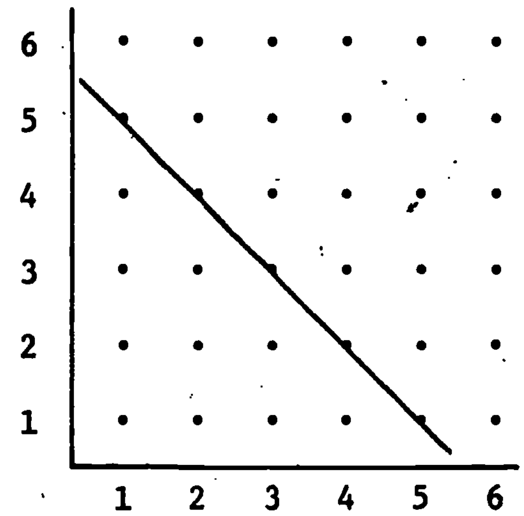
$$P[(2,3)(3,2)] = \frac{2}{36}, \text{ or } \frac{1}{18}$$

2. What is the probability that you will obtain a sum of 6 on the two dice? A sum of 8? A sum of 7?

   For the above questions, it is helpful to locate the points for any specified sum. (See lattice on page I-43.) Tell students that points whose coordinates have a sum of 6 lie along a line that passes through each axis at 6. There are 5 points: (1,5), (2,4), (3,3), (4,2), and (5,1).

COMMENTARY AND NOTES

Point out the symmetry of the 36-point lattice. A line through the 6 points representing the sum 7 is a line of symmetry for the square array of points. The sum 6 and the sum 8 occur the same number of times (5) as do the sums 2 and 12, 3 and 11, 4 and 10, and 5 and 9. Ask students to derive the probability of each sum and to pair equally likely events.

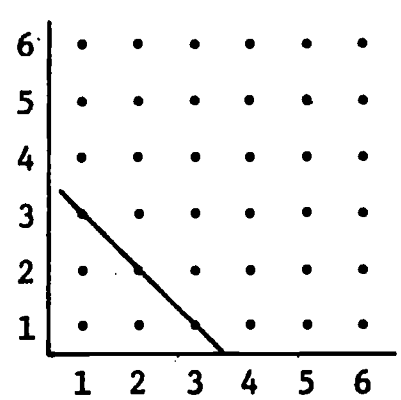


Let E represent success. Then calculate the probability of obtaining a sum of 6 in the following way:

$$P(E) = \frac{5}{36}$$

3. Show how the points line up for each of the following sums. Compute the probability of each sum.

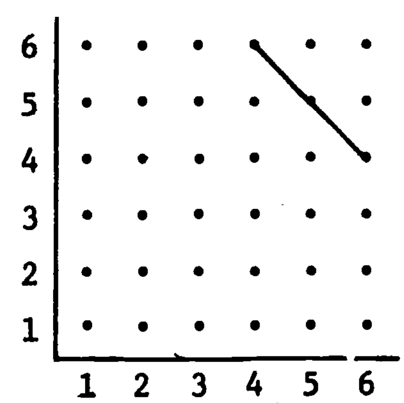
<u>Sum of 4</u>



{(1,3), (2,2), (3,1)}

$P(\text{sum of } 4) = \frac{3}{36}$, or $\frac{1}{12}$

<u>Sum of 10</u>



{(4,6), (5,5), (6,4)}

$P(\text{sum of } 10) = \frac{3}{36}$, or $\frac{1}{12}$

The sum of 7 occurs most often:

{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

$P(\text{sum of } 7) = \frac{6}{36}$, or $\frac{1}{6}$

NOTES

Mutually Exclusive Events

Present the following problem:

What is the probability of obtaining either a sum of 7 or a sum of 11?

$$P(\text{sum of } 7) = \frac{6}{36}$$

$$P(\text{sum of } 11) = \frac{2}{36}$$

By adding the separate probabilities for 7 and 11, we find that there are 8 favorable outcomes (outcomes in which the sum is 11 or 7). Thus:

$$P(\text{sum of 7 or 11}) = \frac{8}{36}, \text{ or } \frac{2}{9}$$

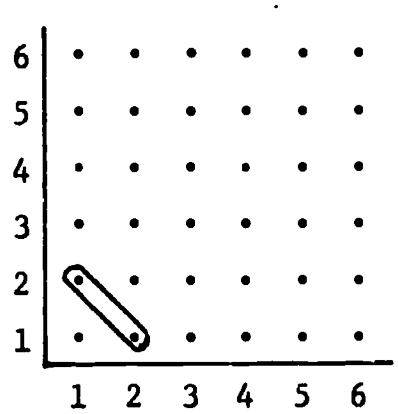
$$\frac{6}{36} + \frac{2}{36} = \frac{8}{36}, \text{ or } \frac{2}{9}$$

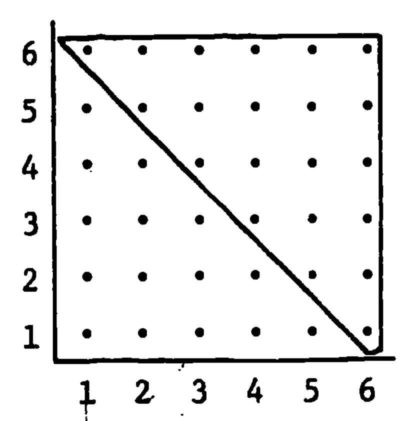
Use this example to illustrate the meaning of the phrase <u>mutually exclusive</u>. The occurrence of either one of these events (a 7 or an 11) excludes the possibility that the other will occur.

Point out to students that two events must be mutually exclusive for their probabilities to be added.

Ask which pairs of events in a roll of two dice are mutually exclusive. Have students draw a line around two sets of points on a lattice for each of the following pairs of events. Show that the demarcated areas are disjoint (do not intersect) when the events are mutually exclusive.

1. A sum of 3 or a sum of at least 7.





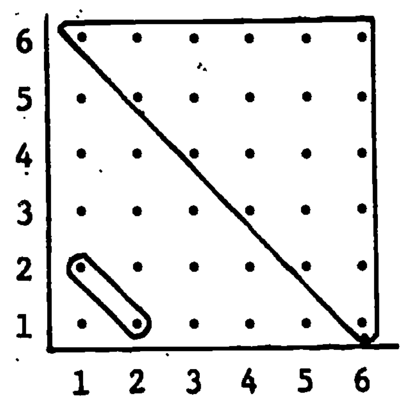
$P(\text{sum of } 3) = \frac{2}{36}$

$P(\text{sum of at least } 7) = \frac{21}{36}$

COMMENTARY AND NOTES

It may be helpful to review the meaning of <u>at least</u>. In this context it means 7, 8, 9, 10, 11, or 12.

Ask students to use the 36-point sample space to describe other pairs of mutually exclusive events.

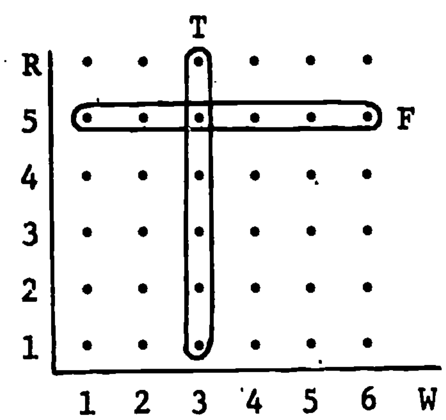


Count the dots/points with students to show that there are 23 in all.

$$P(\text{sum of 3 or a sum of at least 7}) = \frac{23}{36}$$

$$\frac{2}{36} + \frac{21}{36} = \frac{23}{36}.$$

2. The white die comes up 3 or the red die comes up 5.



T: White die, comes up 3

F: Red die, comes up 5

$$P(T) = \frac{6}{36}$$

$$P(F) = \frac{6}{36}$$

But $$P(T \text{ or } F) = \frac{11}{36}$$

These events are <u>not</u> mutually exclusive. Both can happen at the same time. Count the enclosed points with students to show that there are 11, not 12. There is one point that T and F have in common. This is the point (3,5) representing the outcome in which the white die comes up 3 and the red die comes up 5.

A.E. 49a-j, 50

COMMENTARY AND NOTES

Review the idea of the complement of a set. For every subset of outcomes (event) another subset is created, E', that contains all the outcomes that are not in E. E and E' are disjoint sets and together they contain all of the possible outcomes.

Complementary Events

Remind students that they have previously computed the probability of obtaining a sum of 6 on the roll of two dice.

$$P(\text{sum of } 6) = \frac{5}{36}$$

Now ask them to compute the probability of not getting a sum of 6. (Answer: $\frac{31}{36}$)

Then consider various other probabilities. In each case, compute as well the probability that the event will <u>not</u> occur. Juxtapose these measures so that their pattern becomes evident:

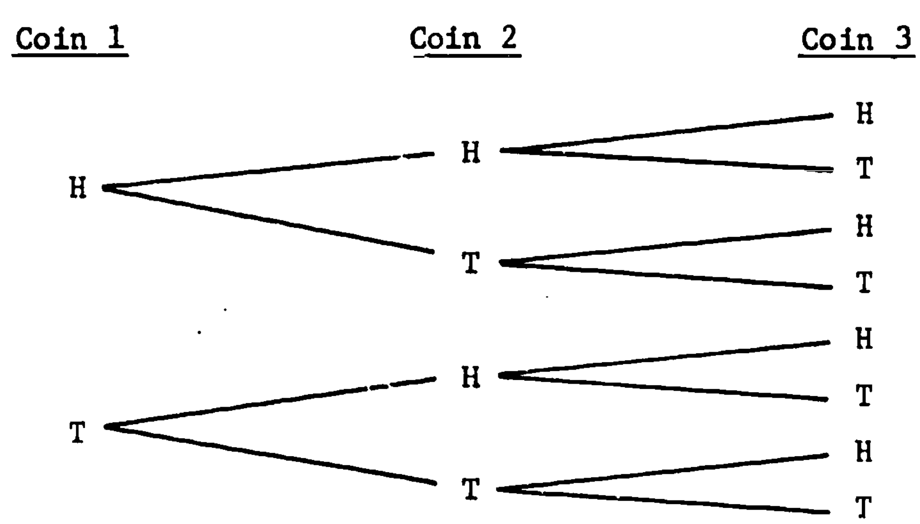
$P(\text{sum of } 6) = \frac{5}{36}$	$P(\text{sum not } 6) = \frac{31}{36}$
$P(\text{white is } 3) = \frac{6}{36}$	$P(\text{white not } 3) = \frac{30}{36}$

Assist students with the observation that for any event, E, if E' is the nonoccurrence of event E, then

$$P(E) + P(E') = 1$$

For additional practice have students use tree diagrams to construct sample spaces for the toss of 2, 3, 4, ... coins. Help them to note that the number of outcomes for <u>n</u> coins is 2^n.

SAMPLE SPACE FOR TOSSING THREE COINS



$$2^3 = 8$$

COMMENTARY AND NOTES

Raise a question about the relationship of these to the Pascal Triangle. Help students see why the numerators (1, 3, 3, 1) represent the number of combinations of the three coins that result in 3 heads, 2 heads, 1 head, 0 heads.

Use the sample space on page I-49 to compute the following probabilities in an experiment with the tossing of 3 coins.

1. The probability of getting 3 heads ($\frac{1}{8}$)

2. The probability of getting 2 heads ($\frac{3}{8}$)

3. The probability of getting 1 head ($\frac{3}{8}$)

4. The probability of getting 0 heads ($\frac{1}{8}$)

Have students extend the tree diagram on page I-49 to 4 coins and 2^4, or 16, outcomes. Ask them to calculate the probabilities of obtaining 4, 3, 2, 1, and 0 heads.

Help students count the branches of the tree diagram to show that:

1. There is 1 way of obtaining 4 heads

2. There are 4 ways of obtaining 3 heads

3. There are 6 ways of obtaining 2 heads

4. There are 4 ways of obtaining 1 head

5. There is 1 way of obtaining 0 heads

Relate these numbers (1, 4, 6, 4, 1) to the appropriate row of the Pascal Triangle and determine the probabilities by using the denominator 16.

A.E. 45, 46

COMMENTARY AND NOTES

Review examples of theoretical probability and experimental probability.

It is appropriate to summarize the methods developed in this module so that students can better appreciate the basic ideas of probability.

Experimental Probability

Present the following problem:

> There are some marbles in a container. You shake the container, withdraw a marble without looking, and record its color. The marble is replaced, and the procedure is repeated a total of 100 times.

Suppose the results recorded are 22 reds, 26 whites, 11 blues, and 41 greens.

1. What can you say about your chances of picking a white marble on the very next pick? A blue marble? A red one? A green one?

2. What is the probability of picking a yellow marble from the same container?

3. There are 20 marbles in the container. Estimate the number of each color.

Discuss the distinction between theoretical probability and experimental probability. In the case of coins, spinners, dice, etc., we make a theoretical determination beforehand as to what is to be expected. In the case of the marbles, weather, defective items on an assembly line, etc., we can make a prediction only after performance data have been collected.

A.E. 47, 48, 51, 52, 55

STUDENT EVALUATION

An approach to evaluation is provided in the section on student evaluation in the <u>Instructor's Handbook</u>, along with ideas for creating a classroom climate that encourages and supports students' achievement. Suggestions are offered for helping students to prepare for quizzes, for providing feedback on performance, and for reassessing when necessary.

The <u>Choice and Chance</u> objectives, below, are accompanied by sample items. Select items according to the content objectives you have covered in the module.

Objective 1. List the outcomes for an experiment, using a tree diagram.

a. There are 3 marbles in a bag: 1 blue, 1 yellow, and 1 red. In a certain experiment one is to pick a marble out of the bag <u>and</u> flip a coin. Draw a tree diagram to show all the possible outcomes of this experiment.

b. A restaurant offers 3 selections of pie (apple, peach, and blueberry) and 3 choices of ice cream (vanilla, chocolate, and strawberry). Draw a tree diagram to show all possible pie-a-la-mode desserts (one choice of ice cream and one choice of pie).

Objective 2. Calculate the number of outcomes of an experiment, using the multiplication principle.

a. One can travel from City A to City B by plane, bus, or train and from City B to City C by bus or automobile. In how many different ways can one travel to City C after a stopover in City B?

b. There are 3 flavors of ice cream: vanilla (v), chocolate (c), and strawberry (s). There are 4 toppings: fudge (f), butterscotch (b), marshmallow (m), and nuts (n). How many different kinds of sundaes are possible?

c. Bag A contains 3 marbles: a red one, a blue one, and a green one. Bag B contains 2 marbles: a yellow one and a red one. Without looking, a person draws 2 marbles, 1 from each bag. How many outcomes are possible?

Objective 3. Calculate the number of permutations for a set of objects, using all of the objects.

a. Calculate the number of permutations for a set of 5 objects.

b. In how many ways can 4 people be arranged in a line for a picture?

Objective 4. Calculate the number of permutations for a set of objects, using any number of the objects.

a. Six horses race. In how many ways can they come in first, second, and third?

b. How many four-digit numbers can be formed from 1, 2, 3, 4, 5, 6, and 7 if each digit is used exactly once in a number?

Objective 5. Perform computations with factorials.

a. Evaluate 5!

b. If 6! = 720, calculate 7!

Objective 6. Determine the total number of subsets that can be formed from a given set of elements, using the formula 2^n.

a. What is the total number of subsets in a set of 5 elements?

b. Determine the total number of subsets that can be made from a set of 7 elements. Express the answer in exponential form.

Objective 7. Use the Pascal Triangle (see following page) to determine the number of subsets of a specific number of elements that can be formed from a given set.

a. Write the next line of the Pascal Triangle (for 7 elements).

Number of Elements (n)							
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

b. Determine the number of subsets of 3 elements that can be made from a set of 6 elements.

c. How many committees of 3 can be made from a set of 7 people?

d. Given a set of 21 elements, the number of subsets of 3 elements is equal to the number of subsets of how many elements?

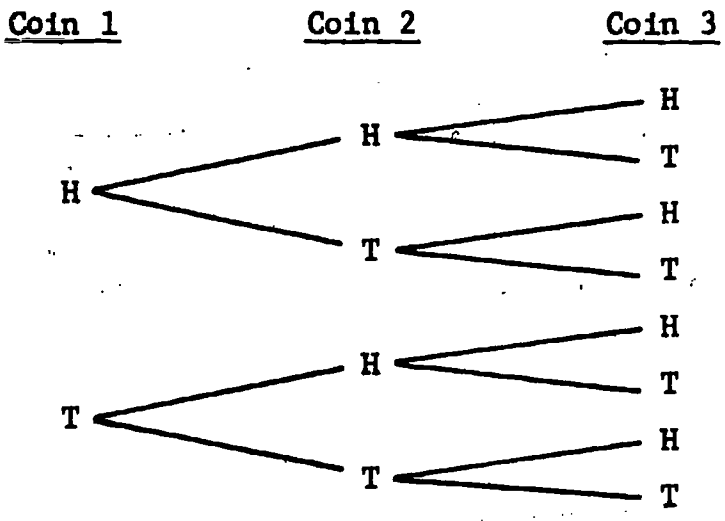
Objective 8. Given the total number of outcomes and the number of favorable outcomes, determine the probability of an event.

a. A bag contains 3 red marbles, 2 blue marbles, and 5 green marbles. One marble is drawn from the bag. What is the probability that it is either red or green?

b. Each month of the year is written on a slip of paper. Suppose you choose one at random. What is the probability that it will begin with the letter J?

c. What is the probability of drawing a yellow marble from a bag containing 3 red marbles, 2 blue marbles, and 5 green marbles?

d. Refer to the tree diagram below to answer this question. In this experiment, 3 coins are tossed. What is the probability that in the toss of 3 coins <u>exactly 2</u> coins will come up heads?



Objective 9. Given the probability of an event, determine the probability that the event will not occur.

a. The probability that an event will occur is $\frac{2}{3}$. Determine the probability that the event will <u>not</u> occur.

b. Suppose that a standard deck of cards is shuffled and one card is drawn. The probability that it is a heart is $\frac{13}{52}$. What is the probability that it is <u>not</u> a heart?

Objective 10. Given the probability of an event predict the number of times the event can be expected to occur in a given number of trials.

a. The probability of obtaining 2 heads in the toss of 2 coins is $\frac{1}{4}$. If 2 coins are tossed 100 times, in approximately how many cases should one expect to see 2 heads?

b. The probability of drawing a red block from a bag of colored blocks is $\frac{1}{2}$. If a block is drawn from this bag (and replaced) 50 times, how many times should one expect to draw a red block?

TEAM

CHOICE AND CHANCE

II

INSTRUCTOR'S GUIDE AND SOLUTIONS TO STUDENT EXERCISES

This section of Choice and Chance contains solutions for exercises presented in Student Materials and Exercises. The solutions are accompanied by some explanations and suggestions.

The student exercises include problems that apply the concepts and problem-solving strategies developed in this module. The exercises can be used as part of instructional activities, as in-class activities for individuals or small groups, as assignments, or as review materials. The asterisk (*) denotes a more challenging problem.

1. At a party there are 14 women and 11 men. In how many ways can mixed couples be paired?

Use the multiplication principle or a tree diagram.

$$14 \cdot 11 = \boxed{154}$$

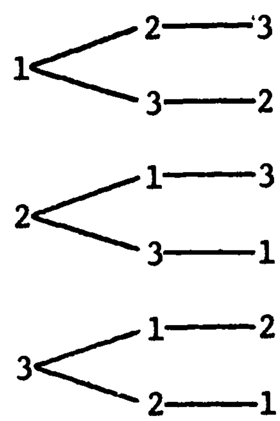
Encourage students to return to the tree diagram for help as needed.

2. Dapper Dan has 12 sports jackets, 2 reversible vests, and 6 pairs of slacks. How many different 3-piece outfits can he arrange for himself?

Since the vests are reversible, there are 4 possible vests to work with, 12 possible jackets, and 6 choices for slacks.

$$4 \cdot 12 \cdot 6 = \boxed{288}$$

3. How many three-digit numbers can be formed using the digits 1, 2, and 3 without repeating digits? Use a tree diagram.



The last branch of this diagram is needed to form the third digit of the number, but it will not change the number of outcomes. The total number of outcomes is developed in the first two branches and can be shown as

$$3 \cdot 2 \cdot 1 = \boxed{6}$$

4. How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 without repeating digits?

Since a tree diagram would be too cumbersome, show instead the number of possible ways to fill the spaces in the three-digit number.

$9 \cdot 8 \cdot 7 = \boxed{504}$

There are 9 ways to fill the first space. (→ 9)

Only 8 choices remain for the second space. (→ 8)

Only 7 choices remain for the third digit, since two numbers have already been used for the first two digits and repetition of digits is not permitted here. (→ 7)

5. How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9?

   Since repetition <u>is</u> permitted here, there are 9 ways to fill each space.

   9 choices for the hundreds digit

   9 choices for the tens digit

   9 choices for the units digit

   $9 \cdot 9 \cdot 9 = 9^3$, or $\boxed{729}$

   Compare this with question 4, in which repetition is <u>not</u> permitted.

6. How many four-digit even numbers can be formed from 1, 3, 2, 8, and 9 if each digit is used exactly once in a number?

   For the number to be even, the units digit must be an even number. Therefore, the units place can be filled by only two digits, 2 or 8.

   According to the multiplication principle, and given the number of ways to fill the other three places, the answer is

   $$4 \cdot 3 \cdot 2 \cdot 2 = \boxed{48}$$

7. How many different three-digit numbers greater than 500 can be made using the digits 1, 3, 5, 6, and 7 if no digits are repeated?

   A restriction is placed on the leftmost digit (the hundreds place) since the three-digit number must be greater than 500. Therefore, there are only three possibilities for the first digit: 5, 6, or 7. Then four possibilities remain for the second digit, and three remain for the last digit (the units place).

   $$3 \cdot 4 \cdot 3 = \boxed{36}$$

8. How many possible license plates are there that consist of a letter followed by three digits?

Since repetition is not mentioned, we can assume that there isn't any restriction and that repetition is permitted. There are 26 possible choices for the letters (A–Z) and 10 choices for each of the three-digit places (0, 1, 2, 3, 4, 5, 6, 7, 8, 9):

$$26 \cdot 10 \cdot 10 \cdot 10 = \boxed{26\ 000}$$

9. If seven digits are used for each telephone number for a given area code, how many different telephone numbers can be accommodated for that area code?

There are ten possible ways to fill each of the seven-digit places:

$$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^7, \text{ or } \boxed{10\ 000\ 000}$$

10. A wheel of fortune contains the numbers from 1 to 5. If we spin the wheel three times for each turn, how many different sequences of numbers can we obtain?

$$5 \cdot 5 \cdot 5 = 5^3, \text{ or } \boxed{125}$$

11. Calculate:

a. $2! = 2 \cdot 1$, or $\boxed{2}$

b. $3! = 3 \cdot 2 \cdot 1$, or $\boxed{6}$

c. $(4!)(2!) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1$, or $\boxed{48}$

d. $\frac{4!}{2!} = \frac{4 \cdot 3 \cdot \not{2} \cdot \not{1}}{\not{2} \cdot \not{1}}$, or $\boxed{12}$

The (2·1) is eliminated by dividing both the numerator and the denominator by (2·1).

e. $\frac{10!}{5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \not{5} \cdot \not{4} \cdot \not{3} \cdot \not{2} \cdot \not{1}}{\not{5} \cdot \not{4} \cdot \not{3} \cdot \not{2} \cdot \not{1}}$, or $\boxed{30\ 240}$

The answer is not 2. Cancellation cannot be performed until the numerator and the denominator are expanded.

f. $\frac{5!2!}{3!} = \frac{5 \cdot 4 \cdot (\not{3} \cdot \not{2} \cdot \not{1}) \cdot 2 \cdot 1}{(\not{3} \cdot \not{2} \cdot \not{1})}$, or $\boxed{40}$

g. $\frac{32!}{31!} = \frac{32\cdot(31\cdot30\cdot29\cdot\ldots\cdot2\cdot1)}{(31\cdot30\cdot29\cdot\ldots\cdot2\cdot1)}$, or $\boxed{32}$

h. $\frac{15!}{13!} = \frac{15\cdot14\cdot(13\cdot12\cdot11\cdot\ldots\cdot2\cdot1)}{(13\cdot12\cdot11\cdot\ldots\cdot2\cdot1)} = 1\ \cdot14$, or $\boxed{210}$

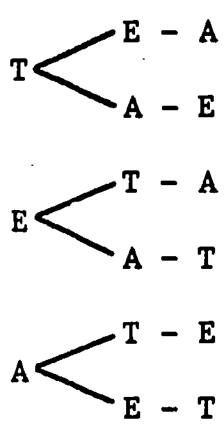
The following questions (12-19) involve finding the number of permutations.

12. How many arrangements of the letters in the word TEA are possible? How many form real words?

This question is similar to question 3, except that it involves three letters instead of three numbers.

The problem is also the same as that of arranging Tom, Evelyn, and Alice in their chairs, side by side, for a photograph (see page I-19). You can have three students assume the different positions for these arrangements. Have them move around to demonstrate that the letters T, E, and A can each be used only once in any arrangement. No repetition is permitted.

$$3\cdot2\cdot1 = \boxed{6}$$



List taken from tree diagram:

TEA
TAE
ETA
EAT
ATE
AET

Select from the list those arrangements which form English words (TEA, EAT, ATE).

13. How many three-digit numbers can be formed using the digits 1, 2, and 3 without repetition?

$${}_3P_3 = 3! \text{ and } 3! = 3\cdot2\cdot1, \text{ or } \boxed{6}$$

This is a question about <u>arranging</u> digits. Because the order of the digits is important, it is a question about permutations. Three items are being arranged in three positions, so all of the items are being used each time. Compare this with question 4, in which nine digits are used to fill the three positions.

14. In how many ways can 5 books be arranged on a shelf?

The word <u>arrange</u> tells us that <u>order is important</u>.

$$_5P_5 = 5! \text{ and } 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{120}$$

15. If a student has 9 books and wishes to arrange any 5 of them on a shelf, in how many ways can it be done?

Compare this with question 14, in which 5 books are arranged in 5 spaces. Here we have 9 books to arrange in 5 positions.

$$_9P_5 = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5, \text{ or } \boxed{15\ 120}$$

9 ways to fill the first position → 9
8 ways to fill the second position → 8
7 ways to fill the third position → 7
6 ways to fill the fourth position → 6
5 ways to fill the fifth position → 5

16. a. In how many ways can 7 people be lined up for a group picture?

b. In how many ways can they be lined up if Mike is to sit in the middle?

a. $_7P_7 = 7! \text{ and } 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{5040}$

b. One position is absolutely determined--Mike is in the middle. Therefore, only 6 people remain to be arranged, and only 6 positions remain to be filled:

$$6! = {_6P_6} \text{ and } 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{720}$$

17. In how many different orders is it possible to arrange the letters of the following words?

a. MANY

b. CARDS

a. $_4P_4 = 4! \text{ and } 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$

b. $_5P_5 = 5! \text{ and } 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{120}$

18. In how many different orders can 5 slips of paper be drawn from a hat?

$_5P_5 = 5!$ and $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{120}$

19. In how many different ways can 6 seats be filled by 10 people?

$$_{10}P_6 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5, \text{ or } \boxed{151\ 200}$$

First seat → 10
Second seat → 9
Third seat → 8
Fourth seat → 7
Fifth seat → 6
Sixth seat → 5

If you are using a table of factorials, then

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{10!}{4!}$$

$$= \frac{3\ 628\ 800}{24}$$

$$= 151\ 200$$

20. a. Enter the numbers in the Pascal Triangle* through line $\underline{n} = 12$.

b. Check: Does that line read 1, 12, 66, 220, 495, 792, and so on?

c. Pose six questions for which the numbers 1, 12, 66, 220, 495, 792, and 924 would be the respective answers.

c. 1: Given 12 people, how many committees of 12 people can be formed? <u>or</u>

Given 12 people, how many committees of 0 people can be formed?

12: Given 12 people, how many committees of 11 people can be formed? <u>or</u>

Given 12 people, how many committees of 1 person can be formed?

66: Given 12 people, how many committees of 10 people can be formed? <u>or</u>

Given 12 people, how many committees of 2 people can be formed?

*The Pascal Triangle is found on page II-11.

220: Given 12 people, how many committees of 9 people can be formed? <u>or</u>

Given 12 people, how many committees of 3 people can be formed?

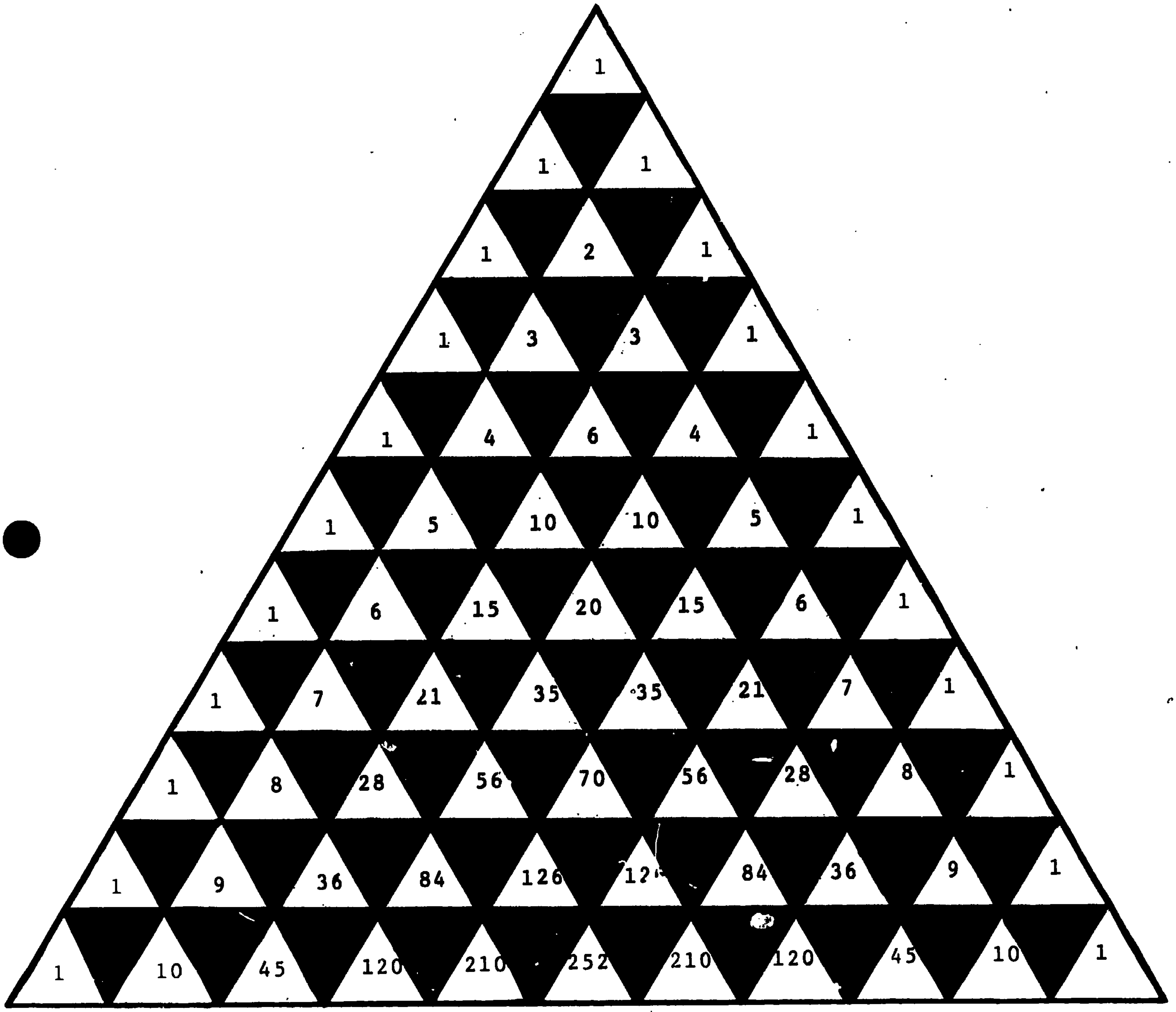
495: Given 12 people, how many committees of 8 people can be be formed? <u>or</u>

Given 12 people, how many committees of 4 people can be formed?

792: Given 12 people, how many committees of 7 people can be formed? <u>or</u>

Given 12 people, how many committees of 5 people can be formed?

924: Given 12 people, how many committees of 6 people can be formed?


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 12 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

The following questions (21-26) involve finding the number of possible combinations.

21. How many different amounts of money can be obtained from a nickel, a quarter, and a half-dollar, using one, two, or all three of these coins at a time?

Different sums of money are obtained by grouping the coins three at a time, two at a time, one at a time, or by using no coins at all. Solve the problem by showing the number of different subsets that can be formed from a set of three objects by taking none at all, one at a time, two at a time, or three at a time. The number of objects is small enough so that all the different groups can be listed.

n = nickel, d = dime, q = quarter

Groupings	Subsets Formed	Number of Subsets
0 at a time	{ }	
1 at a time	{n} {d} {q}	3
2 at a time	{n,d} {n,q} {d,q}	3
3 at a time	{n,d,q}	1
		8

The same information is shown in the third row of the Pascal Triangle*:

1 3 3 1

- Number of different sets that can be formed from a given set of 3 objects, using 0 at a time → 1
- Number of different sets that can be formed from a given set of 3 objects, using 1 at a time → 3
- Number of different sets that can be formed from a given set of 3 objects, using 2 at a time → 3
- Number of different sets that can be formed from a given set of 3 objects, using 3 at a time → 1

*The Pascal Triangle is found on page II-11.

22. There are 10 people on a school board. In how many ways can committees of 5 people be selected?

Use the Pascal Triangle.* Find the appropriate line and position that answer the related question, "Given 10 people, how many committees of 5 people can be formed?"

Row for n = 10, sixth position (middle) = **252**

Note that for a question like this one, order is not important. A committee composed of Marc, Alice, Kate, Jonathan, and Saul and one composed of Saul, Alice, Kate, Marc, and Jonathan would be exactly the same committee. Contrast this with question 12, for which a different arrangement of EAT, ATE yields a different result.

23. How many different clubs of 2 boys can be selected from a class of 6 boys?

Use the Pascal Triangle.*

Row for n = 6, third number from the left = **15**

24. If 8 people play a round-robin tennis tournament, how many matches are played?

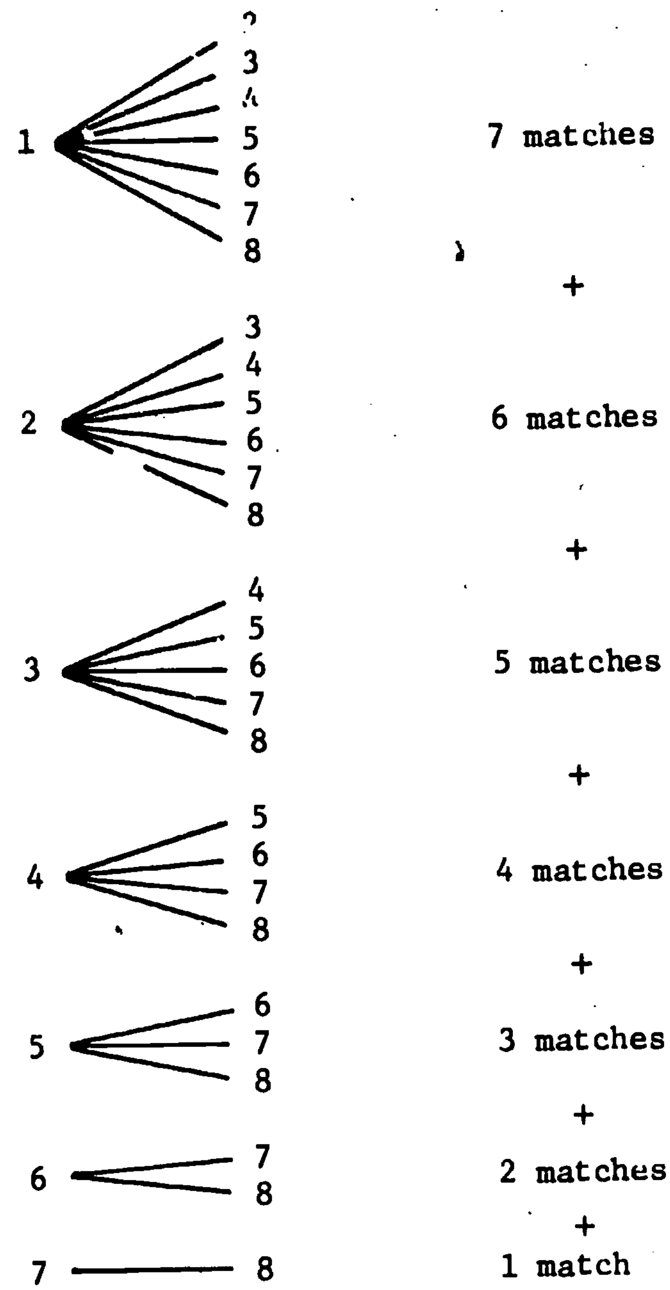
Use the Pascal Triangle* to find how many different combinations of 2 can be formed from a set of 8. Remember that order doesn't matter. Austin versus Shriver and Shriver versus Austin are the same match.

Row for n = 8, third number from the left = **28**

(Solution for question 24 continues on the next page.)

*The Pascal Triangle is found on page II-11.

An alternative method of solving this problem is to use a tree diagram:



For player number 7 there is only one branch, since his or her matches with each of the other players have already been listed in the branches for players 1 through 6.

The sum $1 + 2 + 3 + 4 + \ldots + 7$ can be solved by using the formula for the sum of the first <u>n</u> natural numbers by substituting 7 for <u>n</u> in

$$S = \frac{1}{2}n(n + 1)$$

Thus $1 + 2 + 3 + \ldots + 7 = \frac{1}{2}\cdot 7\cdot 8$, or 28

Other methods of solving for the sum of the first seven natural numbers may be used.

25. A committee of 5 people is to be chosen from a set of 10 people.

a. If Jones and Smith both serve on the same committee, how many committees can be formed?

b. Suppose Jones and Smith are too incompatible to serve on the same committee. In how many ways can the committees be formed?

- Since 2 people, Jones and Smith, are definitely included in the committee, the problem is one of choosing a committee of 3 people from a given set of 8 people.

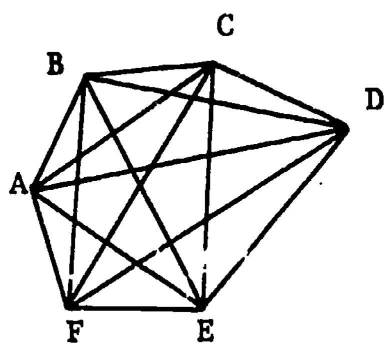
- Refer to the fourth position from the left in the eighth row of the Pascal Triangle.* **56**

The total number of possible committees of 5, given a set of 10 people, is 252. From this total, subtract all committees that have Jones and Smith together.

In answer (a), we found that there are 56 committees in which Jones and Smith are together. Thus:

$$252 - 56 = \boxed{196}$$

26. How many diagonals can be drawn in a hexagon?



$\overline{AB}$ $\overline{BC}$ $\overline{CD}$ $\overline{DE}$ $\overline{EF}$

$\overline{AC}$ $\overline{BD}$ $\overline{CE}$ $\overline{DF}$

$\overline{AD}$ $\overline{BE}$ $\overline{CF}$

$\overline{AE}$ $\overline{BF}$

$\overline{AF}$

$$5 + 4 + 3 + 2 + 1 = 15$$

A hexagon has six points. Two points determine a line. Therefore, solve for the number of pairs of points that can be formed. From this total, subtract 6 because the lines forming the six original sides of the hexagon have already been included in the total number of sets of two points. These original six lines are not diagonals.

Given a set of six points, 15 lines are determined by two points, and

$$15 - 6 = \boxed{9}$$

*The Pascal Triangle is found on page II-11.

From this point on, students must decide whether to find the number of combinations or the number of permutations. Questions 27 and 28 and questions 29 and 30 can be paired to point out these important differences between permutations and combinations:

- In permutations, order is important; in combinations, order is not important.
- The number of combinations will be fewer than the number of permutations.

27. In how many different ways is it possible to choose a president, secretary, and treasurer from a club with 10 members?

Order is important. "Pam is president, Fran is secretary, and Nan is treasurer" is different from "Fran is president, Nan is secretary, and Pam is treasurer." Use permutations.

$$\frac{10}{\text{president}} \cdot \frac{9}{\text{secretary}} \cdot \frac{8}{\text{treasurer}} = \boxed{720}$$

28. In how many ways is it possible to choose a committee of 3 from a club with 10 members? $\boxed{120}$

This differs from question 27 because it is only the number of groups that is of interest here, not the order in which the members of the group are arranged. Look at the fourth position from the left in the row of the Pascal Triangle* for n = 10. Compare this result (120) with the solution to question 27. When order is important, the result is larger because there are more ways each group can be arranged. The 120 different groups solved for here, for example, could each be rearranged 6 different ways, if order were important.

29. There are 10 qualified applicants for all of 3 available teaching positions. In how many ways can the positions be filled?

$$10 \cdot 9 \cdot 8 = \boxed{720}$$

30. How many sets of 3 lucky applicants can be selected from the 10? $\boxed{120}$

31. In how many ways can a string quartet (1 first violinist, 1 second violinist, 1 cellist, 1 violist) be formed from 4 first violinists, 2 second violinists, 3 cellists, and 3 violists?

*The Pascal Triangle is found on page II-11

Use the multiplication principle. There are 4 ways to choose a first violinist, 2 ways to choose a second violinist, 3 ways to choose a cellist, and 3 ways to choose a violist.

$$\frac{4}{\text{first violinist}} \cdot \frac{2}{\text{second violinist}} \cdot \frac{3}{\text{cellist}} \cdot \frac{3}{\text{violist}} = \boxed{72}$$

32. A librarian has 6 history books. Two are Volumes I and II of the same book. In how many ways can the 6 books be arranged on the shelf, keeping the 2 volumes together in the correct order?

Consider Volumes I and II as pasted together and moved together. You are really moving just 5 books. Use permutations.

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \text{ or } \boxed{120}$$

33. Here you have your favorite three-flavored ice cream cone.

a. In how many different ways can you arrange 3 scoops? (Use permutations.)

$$3 \cdot 2 = \boxed{6}$$

b. In how many ways can you arrange 2 scoops?

$$2 \cdot 1 = \boxed{2}$$

c. In how many ways can you arrange 1 scoop?

$$\boxed{1}$$

d. Suppose a store had 10 flavors of ice cream. How many different double-dip ice cream cones could be made?

$$10 \cdot 9 = \boxed{90}$$

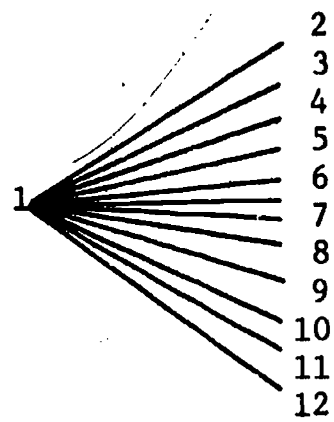
34. There are 12 people in a room. Each is to greet each of the others with a handshake. What is the total number of handshakes?

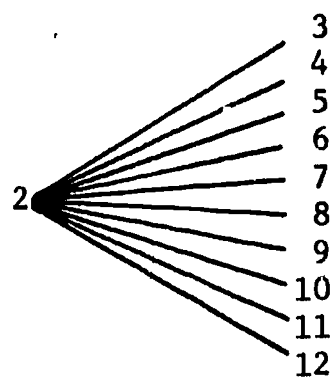
This can be reworded: How many distinct groups of 2 (since it takes 2 people to make a handshake) can be formed, given 12 people? Use Pascal's Triangle,* third position from the left in the row for n = 12. $\boxed{66}$

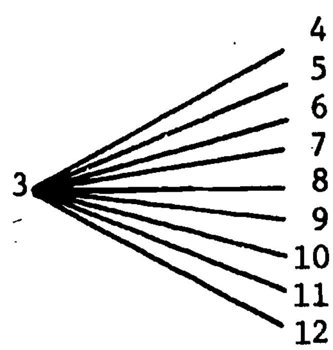
One can also make a tree diagram to see which people the first person will shake hands with (11 other people), which people the second person

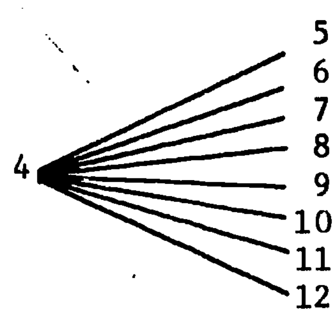
*The Pascal Triangle is found on page II-11.

will shake hands with (10 other people, since the second is already listed with the first), etc.

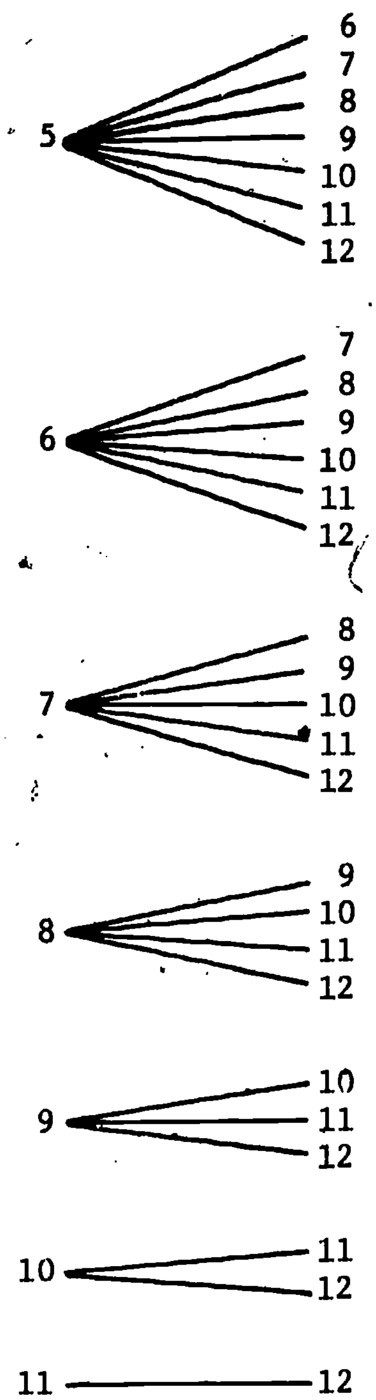








(Solution to question 34 continues on the next page.)



The number of handshakes is then

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11$$

Some students may be familiar with various methods for solving for the sum of the first 11 natural numbers. For example:

$$S = \frac{1}{2}n(n + 1)$$

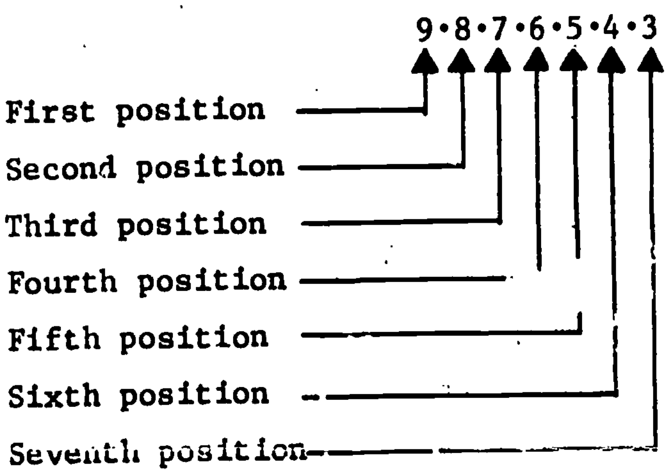
$$\frac{1}{2} \cdot 11 \cdot 12 = \boxed{66}$$

35. a. In how many ways can a librarian arrange 9 books on a shelf?

b. Suppose the shelf holds only 7 books. How many different arrangements can be made then?

a. $9! = 9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$, or **362 880**

b.



To calculate (b), divide the answer to (a) by $2\cdot1$.

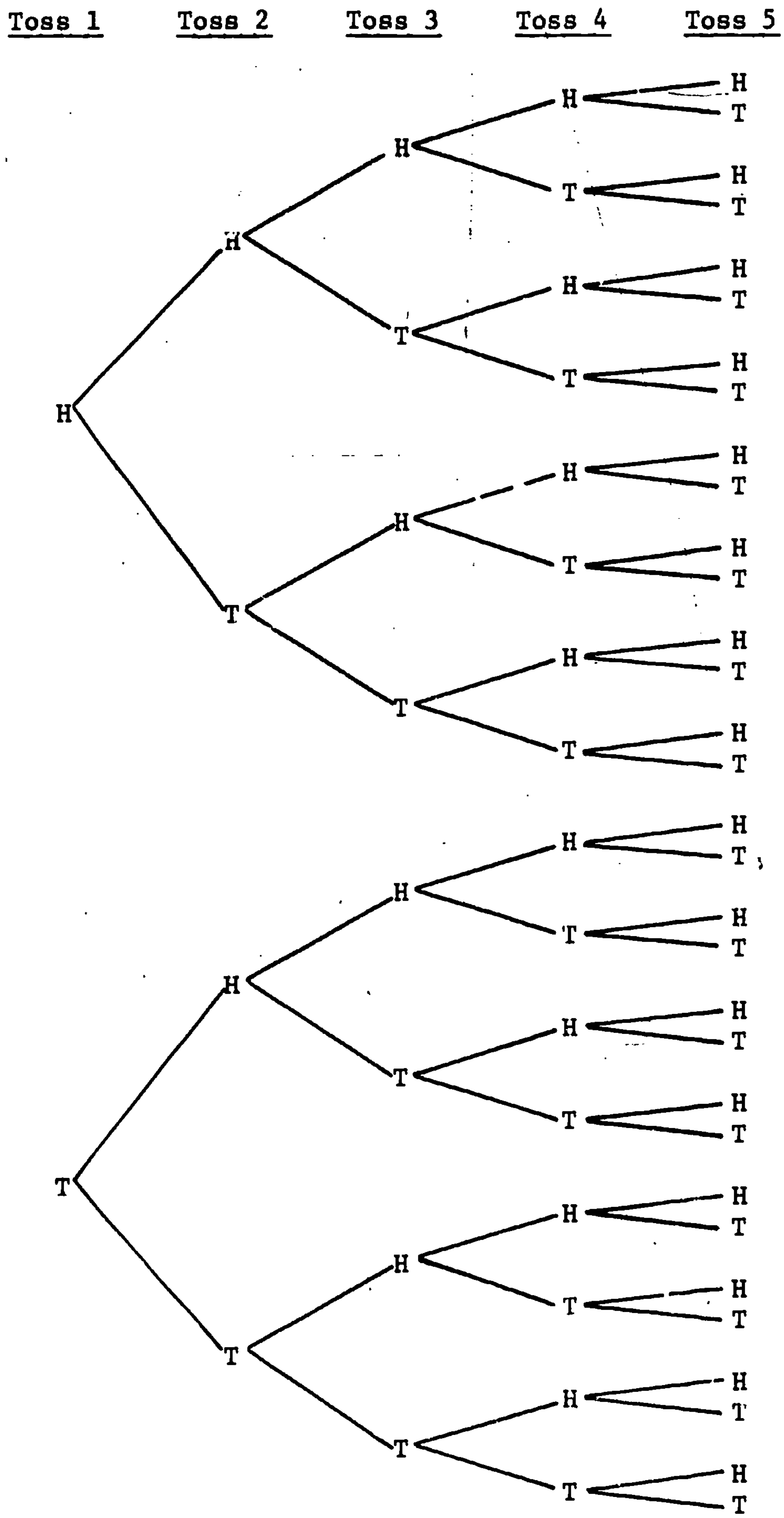
$$\frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot\cancel{2}\cdot\cancel{1}}{2\cdot1} = \frac{362\ 880}{2},$$

or **181 440**

36. a. How many different sequences of heads (H) and tails (T) can be obtained if a coin is tossed 5 times?

$2^5 =$ **32**

b. Draw a tree diagram to show these outcomes. (See tree diagram on following page.)



H H H H H
H H H H T
H H H T H
H H H T T
H H T H H
H H T H T
H H T T H
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H T H H T
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c. What is the probability that if you toss a coin 5 times, you will obtain:

(1) 5 heads?

(2) Exactly 3 heads?

(3) 4 heads and 1 tail <u>or</u> 1 head and 4 tails?

(4) At most 2 heads?

(5) At least 2 heads?

The answers for (c) can be taken directly from the tree diagram by counting the number of favorable outcomes and the total number of possible outcomes. (See page II-21.)

Notice that our previous methods of counting outcomes will come in handy to determine the denominator. For example, the possible outcomes when tossing a coin 5 times are 2^5, or 32. This result is also shown in the tree diagram.

(1) $\boxed{\frac{1}{32}}$ Only one of the possible 32 branches is H H H H H.

(2) $\boxed{\frac{10}{32}}$ These are the only possible outcomes, of the possible 32, that have exactly 3 H's:

H H H T T
H H T H T
H H T T H
H T H H T
H T H T H
H T T H H
T H H H T
T H H T H
T H T H H
T T H H H

(3) $\boxed{\frac{10}{32}}$

(4) $\boxed{\frac{15}{32}}$

(5) $\boxed{\frac{26}{32}}$ All the branches left over have either 0 heads or 1 head. Thus, the answer to (5) also tells you the probability of 0 or 1 head. It will be the complement of 2 heads, or $\frac{6}{32}$.

37. If I have 3 sets of keys on my desk, what is the probability that without looking I will take the right set for my car?

$$\text{Probability (P)} = \frac{\text{Number of favorable outcomes}}{\text{Number of total possible outcomes}}$$

Only 1 set is the correct one. Therefore, there is only 1 favorable outcome. 1 is the numerator. There are 3 different sets to choose from, thus a total of 3 possible outcomes. 3 is the denominator.

$$P = \boxed{\frac{1}{3}}$$

38. What is the probability of correctly guessing the month in which your instructor was born?

There is only 1 favorable outcome--the month in which the instructor was indeed born. There are 12 possible choices--the 12 different months of the year.

$$P = \boxed{\frac{1}{12}}$$

39. If 5 coins are tossed, what is the probability of tossing 5 tails?

Since there are 2 ways the coin can come up each time it is tossed, there are 32 possible outcomes:

$$2^5 = 32$$

There is only one favorable occurrence out of a possible 32. Therefore:

$$P = \boxed{\frac{1}{32}}$$

40. A bag contains 15 balls numbered from 1 to 15. If 3 balls are drawn from the bag, what is the probability that the numbers on all balls will be even?

There are 15•14•13 possible outcomes. Therefore, the denominator is 15•14•13. To determine the numerator, determine the number of favorable outcomes. There are only 7 even numbers between 1 and 15. Thus, there are 7•6•5 favorable draws from the bag.

$$P = \frac{7 \cdot 6 \cdot 5}{15 \cdot 14 \cdot 13}, \text{ and } \frac{210}{2730} = \boxed{0.076923}$$

41. If a student randomly answers a five-question true-or-false test, what is the probability of getting a perfect score?

First calculate the total number of possible arrangements of true and false answers for the test. There are 2 choices for the first answer:—T or F. Likewise, there are 2 choices for the second, third, fourth, and fifth answers. Using the multiplication principle, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$, or 32, possible answers. Only one of these is the correct sequence of answers. Therefore, the probability is

$$\frac{1}{2^5} = \boxed{\frac{1}{32}}$$

Note the similarity between this question and question 39. They both involve 5 events, with 2 possible outcomes for each event.

42. If a student randomly answers a four-question multiple-choice quiz, with 3 choices per question, what is the probability of obtaining a perfect score?

For the first question, there are 3 possible responses; likewise, for the second, third, and fourth questions. There are $3 \cdot 3 \cdot 3 \cdot 3 = 3^4$, or 81, possible arrangements of answers. However, only one sequence is correct. Therefore, the probability is

$$\frac{1}{3^4} = \boxed{\frac{1}{81}}$$

43. There are 6 red candies and 2 green candies in an opaque jar.

a. What is the probability of reaching in without looking and pulling out a red candy?

b. What is the probability of pulling out a green candy?

*c. If you do not replace the first candy drawn, what is the probability of drawing first a red and then a green candy?

a. There is a total of 8 candies to choose from (6 red candies + 2 green candies = 8 candies). However, in this problem red is the favorable outcome, and there are only 6 red candies. Therefore:

$$P = \frac{6}{8}, \text{ or } \boxed{\frac{3}{4}}$$

*The asterisk denotes a more challenging problem.

b. Again, the total number of candies is 8, and 8 will be the denominator. However, now green is the favorable outcome. Therefore:

$$P = \frac{2}{8}, \text{ or } \boxed{\frac{1}{4}}$$

c. There are 8 ways to choose the first candy and 7 ways to choose the second one. Thus, there is a total of 8•7 possible wavs to draw the candies. For the numerator, note that there are 6 possibilities for drawing a red candy and 2 possibilities for drawing a green candy. Therefore, there are 6•2 favorable outcomes. Thus:

$$P = \frac{6 \cdot 2}{8 \cdot 7}, \text{ or } \boxed{\frac{12}{56}}$$

44. Two parents and five children in a family take turns walking their dog. At the beginning of each week, they write the days of the week on seven slips of paper. Then each one draws a slip of paper from a hat.

a. If Dad draws first, what is the probability that he will be responsible for walking the dog on Wednesday?

$$\boxed{\frac{1}{7}}$$

b. If Dad does get Wednesday, what is the probability that Mom wil draw a day beginning with the letter S?

$$\frac{2}{6} = \boxed{\frac{1}{3}}$$

c. Then if Mom draws Saturday, what is the probability that Robin will draw a day beginning with the letter T?

$$\boxed{\frac{2}{5}}$$

d. Then if Robin draws Friday, what is the probability that Marc will draw a day beginning with a T?

$$\frac{2}{4} = \boxed{\frac{1}{2}}$$

e. Then if Marc draws Tuesday, what is the probability that Janet will draw a day beginning with the letter T?

$$\boxed{\frac{1}{3}}$$

f. Then if Janet draws Monday, what is the probability that Daniel will draw Sunday?

$$\boxed{\frac{1}{2}}$$

g. Then if Daniel draws Sunday, what is the probability that Matthew will draw a day beginning with the letter S? [0]

h. What is the probability that Matthew will draw Thursday? [1]

a. There is a total of 7 possible days. Wednesday is the only favorable outcome here. Therefore:

$$P = \frac{1}{7}$$

b. Since Wednesday has been drawn and not replaced in the hat, there are only 6 remaining slips of paper for the 6 remaining days of the week. Since 2 of these days satisfy the favorable condition of beginning with the letter S (Saturday and Sunday):

$$P = \frac{2}{6}, \text{ or } \frac{1}{3}$$

c. There are 5 remaining days. Tuesday and Thursday are the 2 favorable outcomes. Therefore:

$$P = \frac{2}{5}$$

d. There are still 2 days remaining that begin with the letter T. However, only 4 total possibilities remain. Therefore:

$$P = \frac{2}{4}, \text{ or } \frac{1}{2}$$

e. If Marc draws Tuesday, there is only 1 day remaining that begins with the letter T. At this point a total of only 3 possibilities remains. Thus:

$$P = \frac{1}{3}$$

f. Sunday is just 1 possibility out of the remaining 2 days. Therefore:

$$P = \frac{1}{2}$$

g. An impossible event has a probability equal to 0. If Daniel draws Sunday and, as we know from (c), Mom has drawn Saturday, then there aren't any more days that begin with the letter S. Therefore, the probability of drawing a day beginning with the letter S is equal to 0.

h. Since Thursday is the only day left, Matthew must draw Thursday. An event that must occur has a probability of 1.

45. Each letter of the word MATHEMATICS is written on a separate slip of paper and placed in a box. Without looking, you reach into the box and pick one slip of paper. What is the probability of the following:

a. Picking the letter I? $\boxed{\frac{1}{11}}$

b. Picking the letter A? $\boxed{\frac{2}{11}}$

c. Picking a letter in the word TIME? $\frac{6}{11}$

d. Picking a vowel? $\boxed{\frac{4}{11}}$

e. Picking a consonant? $\boxed{\frac{7}{11}}$

There is a total of 11 letters in the word MATHEMATICS. Therefore, all the denominators will be 11.

Note that (e) is the complement of (d). Any letter must be either a consonant or a vowel. Thus, P (picking a vowel) + P (picking a consonant) = 1.

46. A bag contains several marbles. Some are red, some white, and the rest blue. If you pick one marble without looking, the probability of picking a red one is $\frac{1}{3}$, and $P(w) = \frac{1}{3}$.

Compute the probability of picking a blue marble.

The sum of all probabilities is equal to 1. One of 3 distinct events must occur when a marble is picked.

$$P(r) + P(w) + P(b) = 1$$

$$\frac{1}{3} + \frac{1}{3} + ? = 1$$

$$P(b) = \boxed{\frac{1}{3}}$$

47. A bag contains 1 red marble, 2 white marbles, and 3 blue marbles. If you pick 1 marble without looking, what is

a. P(r)? $\boxed{\frac{1}{6}}$

b. P(w)? $\frac{2}{6}$, or $\boxed{\frac{1}{3}}$

c. P(b)? $\frac{3}{6}$, or $\boxed{\frac{1}{2}}$

d. How many white marbles must be added to the bag so that the following is true?

$$P(w) = \frac{1}{2} \quad \boxed{2}$$

There is a total of 6 marbles. Therefore, 6 is the denominator in all of the probabilities.

Note that when white marbles are added, the total number of marbles in the bag also increases. Begin with the probability of picking a white marble:

$$P(w) = \frac{2}{6}$$

If one white marble is added,

$$P(w) = \frac{3}{7}$$

If two white marbles are added,

$$P(w) = \frac{4}{8}, \text{ or } \frac{1}{2}$$

Thus, two white marbles must be added to the bag so that

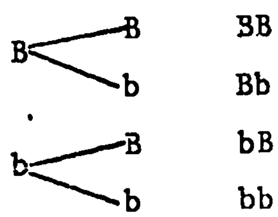
$$P(w) = \frac{1}{2}$$

48. According to the Mendelian theory of heredity, eye color is transmitted from two parents to a child by the transmission of one of two kinds of genes: B (brown) or b (blue). Brown is dominant and blue is recessive. That is, a contribution of a B from each parent (BB) will give a child brown eyes, and a hybrid contribution of brown from one parent and blue from the other (Bb) will also result in brown eyes. Only the combination (bb) will result in blue eyes.

a. Make a table or a tree diagram listing all of the possible outcomes from two "hybrid" parents (Bb and Bb). What is the probability that a blue-eyed child will be born?

Each parent will contribute either a B gene or a b gene to the child.

	B	b
B	BB	Bb
b	Bb	bb

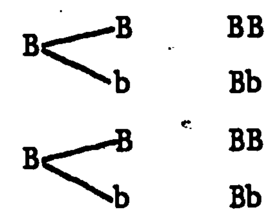


There is a total of 4 possible outcomes, but only 1 outcome (bb) will be blue eyes. Therefore:

$$P(bb) = \boxed{\frac{1}{4}}$$

b. Make a table or a tree diagram of the offspring of one "hybrid" parent (Bb) and one BB parent. What is the probability that a blue-eyed child will result from their union?

	B	b
B	BB	Bb
B	BB	Bb

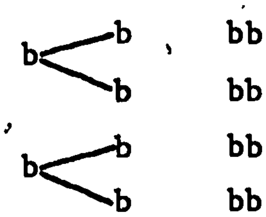


The outcome bb is not a possibility from this union. Therefore:

$$P(bb) = \boxed{0}$$

c. Make a table or a tree diagram of the offspring of two blue-eyed parents. How many outcomes are possible? What is the probability of their having a blue-eyed child?

	b	b
b	bb	bb
b	bb	bb



The only possibility is blue-eyed children. Therefore:

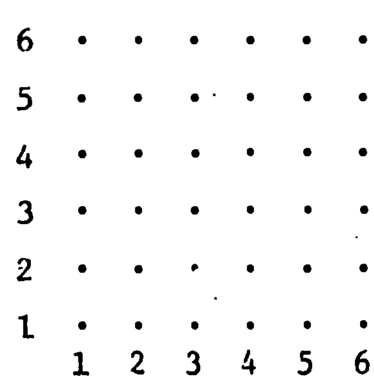
$$P(bb) = \boxed{1}$$

49. A red die and a green die are rolled simultaneously.

a. List the possible outcomes.

b. How many possible outcomes are there?

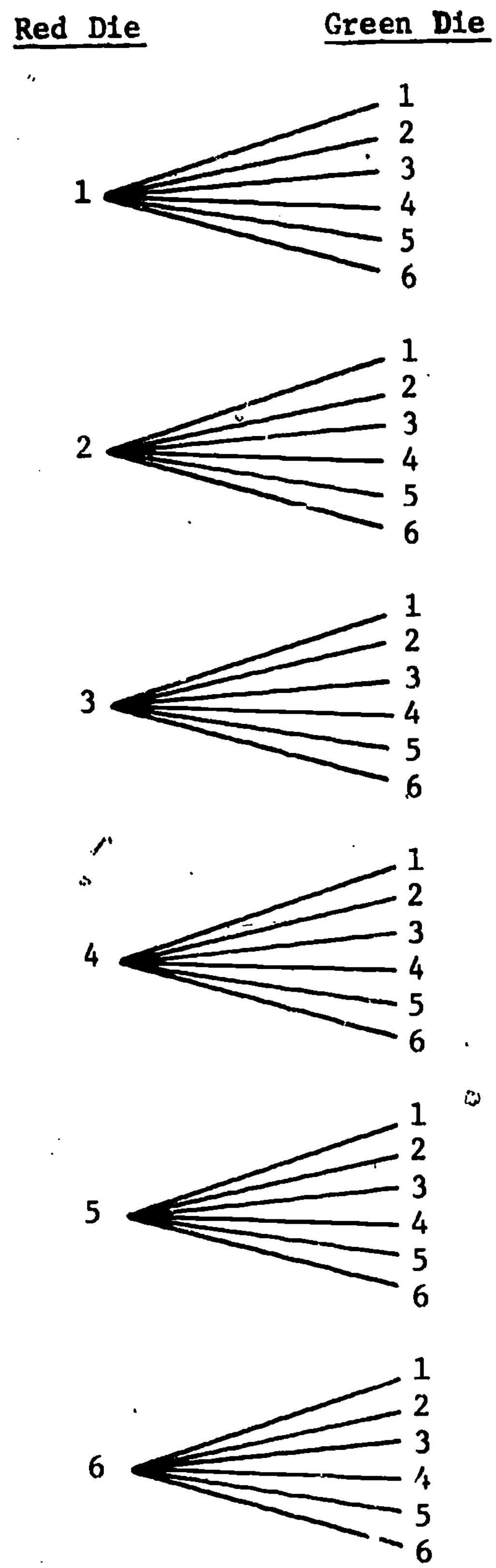
For (a) and (b), the 36 possible outcomes can be listed by using a tree diagram or a 6 x 6 lattice.



c. Describe the outcome in which the sum of the faces showing is equal to 2; equal to 12.

The sum of the faces showing is equal to 2 at point (1,1), or when there is 1 on the red die and 1 on the green die.

The sum of the faces showing is equal to 12 at point (6,6), or when there is 6 on the red die and 6 on the green die.



Red, Green

(1,1)
(1,2)
(1,3)
(1,4)
(1,5)
(1,6)

(2,1)
(2,2)
(2,3)
(2,4)
(2,5)
(2,6)

(3,1)
(3,2)
(3,3)
(3,4)
(3,5)
(3,6)

(4,1)
(4,2)
(4,3)
(4,4)
(4,5)
(4,6)

(5,1)
(5,2)
(5,3)
(5,4)
(5,5)
(5,6)

(6,1)
(6,2)
(6,3)
(6,4)
(6,5)
(6,6)

d. What is the probability that the sum of the faces will be 2? 3? 4? 5? 6? 7? 8? 9? 10? 11? 12?

P(2) = $\boxed{\frac{1}{36}}$

P(3) = $\boxed{\frac{2}{36}}$

P(4) = $\boxed{\frac{3}{36}}$

P(5) = $\boxed{\frac{4}{36}}$

P(6) = $\boxed{\frac{5}{36}}$

P(7) = $\boxed{\frac{6}{36}}$

P(8) = $\boxed{\frac{5}{36}}$

P(9) = $\boxed{\frac{4}{36}}$

P(10) = $\boxed{\frac{3}{36}}$

P(11) = $\boxed{\frac{2}{36}}$

P(12) = $\boxed{\frac{1}{36}}$

The sum of the probabilities = $\frac{36}{36}$, or 1

(One of these sums is a certain outcome when two dice are rolled.)

e. Which event has the greatest probability?

A sum of 7

f. What is the probability of rolling an odd sum? An even sum?

P(odd sum) = P(3) + P(5) + P(7) + P(9) + P(11)

$= \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36}$

$= \boxed{\frac{18}{36}}$

94

$$P(\text{even sum}) = P(2) + P(4) + P(6) + P(8) + P(10) + P(12)$$

$$= \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36}$$

$$= \boxed{\frac{18}{36}}$$

g. What is the probability of rolling a 7? An 11? A double? What is the probability of rolling a 7, an 11, or a double?

$P(7) = \boxed{\frac{6}{36}}$

$P(11) = \boxed{\frac{2}{36}}$

A double is rolled when the outcome on the red die is equal to the outcome on the green die: (1,1), (2,2), (3,3), (4,4), (5,5), (6,6)

$P(\text{double}) = \boxed{\frac{6}{36}}$

$P(7, 11, \text{or double}) = \boxed{\frac{14}{36}}$

Note that for P(7, 11, or double), any of the outcomes--a 7, an 11, or a double--is a favorable outcome satisfying the given condition. Furthermore, the events are mutually exclusive. Thus, the <u>or</u> tells us to add the separate probabilities.

*h. What is the probability that one die will be a 3 and the other will be a 4?

(3,4) or (4,3) $= \boxed{\frac{2}{36}}$

*i. What is the probability that at least one of the dice will be a 6?

Either die can be a 6 {(1,6), (2,6), (3,6), (4,6), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}.

$P = \boxed{\frac{11}{36}}$

*The asterisk denotes a more challenging problem.

*j. What is the probability of rolling a minimum of 8?

If 8 is the minimum, then

$$P(8) + P(9) + P(10) + P(11) + P(12)$$

$$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}, \text{ or } \boxed{\frac{15}{36}}$$

Again, you can count points on the lattice to find that 15 out of the 36 points satisfy this condition.

*50. Consider the outcomes for a roll of two dice (1 red and 1 green). Compute the following probabilities:

a. The sum is more than 3 but less than 7.

Favorable outcomes: $3 < \text{sum} < 7$, that is, sum = 4, 5, or 6.

$$P(4) + P(5) + P(6)$$

$$= \frac{3}{36} + \frac{4}{36} + \frac{5}{36}, \text{ or } \frac{12}{36} = \boxed{\frac{1}{3}}$$

b. The sum is either 3 or 7.

$$P(3) + P(7)$$

$$= \frac{2}{36} + \frac{6}{36}, \text{ or } \frac{8}{36} = \boxed{\frac{2}{9}}$$

c. The sum is neither 3 nor 7.

"The sum is neither 3 nor 7" is the complement of "the sum is 3 or 7." Therefore, refer to (b) and note that:

$$1 - \frac{2}{9} = \boxed{\frac{7}{9}}$$

*The asterisk denotes a more challenging problem.

Alternatively, take the sum of all the probabilities except P(3) and P(7):

$$P(2) + P(4) + P(5) + P(6) + P(8) + P(9) + P(10) + P(11) + P(12)$$

$$= \frac{1}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$

$$= \frac{28}{36}, \text{ or } \boxed{\frac{7}{9}}$$

d. The red die is less than 3, and the green die is greater than 3.

Both conditions must be satisfied. The red die must be less than 3, and the green die must be greater than 3. That is:

(1,4), (1,5), (1,6), (2,4), (2,5), (2,6)

Thus $P = \frac{6}{36}, \text{ or } \boxed{\frac{1}{6}}$

e. The sum is 3.

$$P(3) = \boxed{\frac{2}{36}}$$

f. The sum is not 3.

$$P(\text{not } 3) = 1 - P(3)$$

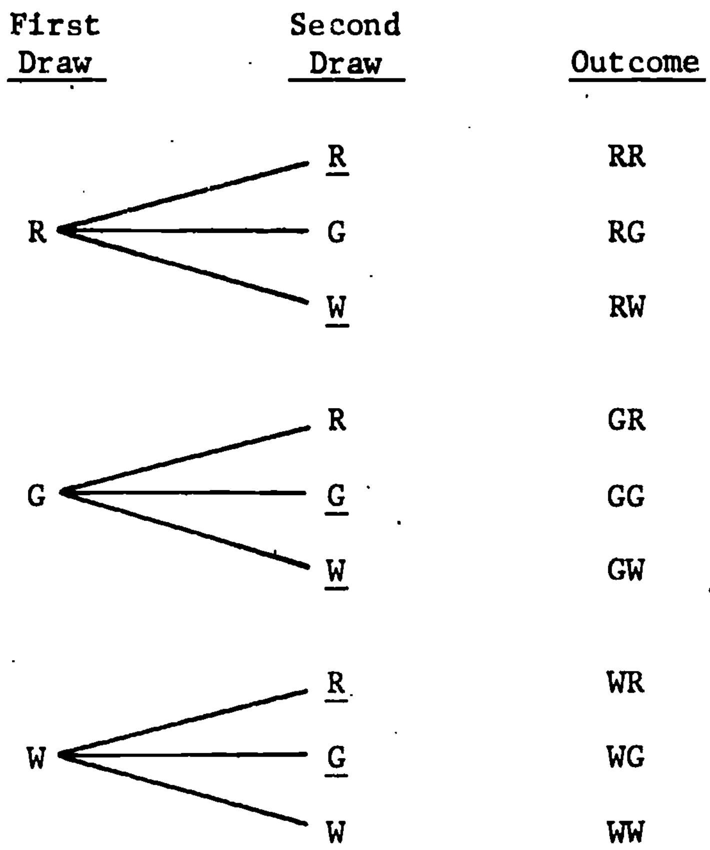
$$= 1 - \frac{2}{36}$$

$$= \boxed{\frac{34}{36}}$$

51. A bag contains 3 marbles--one red, one green, and one white. Suppose you draw a marble, note its color, and return it to the bag. Then you shake the bag and draw again.

a. Complete the tree diagram of the possible outcomes. (Tree diagram is on the following page.)

97



Use your tree diagram to answer the following questions.

b. Which of the outcomes have red on the first draw?

RR, RG, RW

c. Which of the outcomes have green on the second draw?

RG, GG, WG

d. Which outcome has red on the first draw <u>and</u> green on the second?

RG

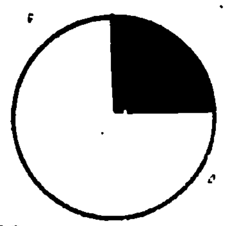
e. Which outcomes, then, are <u>either</u> red on the first draw or green on the second, <u>or</u> both red on the first and green on the second?

RR, RG, RW, GG, WG

f. What is the probability of event E--<u>either</u> red on the first draw or green on the second, <u>or</u> both red on the first and green on the second?

$\frac{5}{36}$

52. A spinner has a red and white dial that looks like the accompanying drawing.



a. If you spin the spinner 20 times, are you just as likely to spin a red as a white?

No

b. Are you more likely to spin a white or a red? Why?

You are more likely to spin a white. There is a greater white area than a red area. Specifically, the white area is three times that of the red. Therefore, you are more likely to spin a white.

c. What are the chances of spinning a white? A red?

P(spinning a white) = $\frac{3}{4}$

P(spinning a red) = $\frac{1}{4}$

d. Are you certain of getting at least 1 red in 20 spins?

It is probable; however, it is not necessarily certain.

e. Is it very likely that you will not spin any reds in 20 spins?

It is not very likely, but it is possible.

f. Is it possible never to spin a red in 20 spins?

This is also a possibility, though in theory, 5 out of 20 spins should turn up red.

53. List three occurrences whose probabilities are each 1 (that is, they are certain to happen).

Answers will vary. Two possibilities are that the sun will rise tomorrow and that when a coin is tossed, either a head or a tail will come up.

99

54. List three occurrences whose probabilities are each 0 (that is, they are impossible).

Answers will vary. Two possibilities are to roll a sum of 13 with a pair of dice and to toss 2 coins and have a result of H H H.

*55. A standard deck of cards is shuffled and 2 cards are dealt. What is the probability that:

a. The 2 cards are spades?

The total number of cards in a deck is 52. Therefore, the denominator must be 52•51--the total number of ways that 2 cards can be dealt from a deck. The number of spades in the deck is 13, so the numerator is 13•12--the total number of favorable outcomes.

$$P(\text{the 2 cards are spades}) = \boxed{\frac{13 \cdot 12}{52 \cdot 51}}$$

b. The 2 cards are of the same suit (any suit)?

The first card may be selected in any way from the deck of 52. The second card, however, must match the suit of the first card. After the first card has been drawn, there are only 12 cards of that same suit remaining. Therefore, the numerator is equal to 52•12.

$$P(\text{the 2 cards are of the same suit}) = \boxed{\frac{52 \cdot 12}{52 \cdot 51}}$$

c. The cards are 2 kings?

There are 4 kings in a deck. Thus:

$$P(\text{the cards are 2 kings}) = \boxed{\frac{4 \cdot 3}{52 \cdot 51}}$$

d. Neither card is a picture card?

There are 12 picture cards in a deck and 40 cards that are not picture cards. Thus:

$$P(\text{neither card is a picture card}) = \boxed{\frac{40 \cdot 39}{52 \cdot 51}}$$

*The asterisk denotes a more challenging problem.

You can also determine the probability that the 2 cards are picture cards--the complementary event--and subtract from 1.

$$P(\text{the two cards are picture cards}) = \frac{12 \cdot 11}{52 \cdot 51}$$

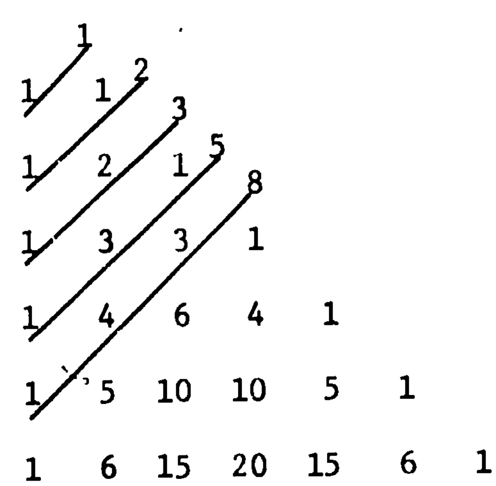
Therefore:

$$P(\text{neither card is a picture card}) = \boxed{1 - \frac{12 \cdot 11}{52 \cdot 51}}$$

56. a. Rewrite the Pascal Triangle in another triangular form. Part of it has been done below. Continue it for several rows.

1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

b. Now calculate the sums of the entries along the diagonals. Several are shown. For instance, 1 + 2 = 3, 1 + 3 + 1 = 5, and 1 + 4 + 3 = 8. Calculate the next several sums. What do you notice? Can you predict subsequent diagonal sums?



The sums form a sequence of numbers with a definite pattern. Each term is the sum of the previous two terms. For example:

1 + 2 = 3, 2 + 3 = 5, 5 + 3 = 8

To continue the pattern, the next several sums would have to be

$5 + 8 = \underline{13}$

$8 + 13 = \underline{21}$

$13 + 21 = \underline{34}$

The sequence 1, 2, 3, 5, 8 is also known as the Fibonacci sequence, and it possesses many other interesting properties for students to explore.

The Fibonacci sequence is explored in the Patterns module of the TEAM project materials.

TEAM

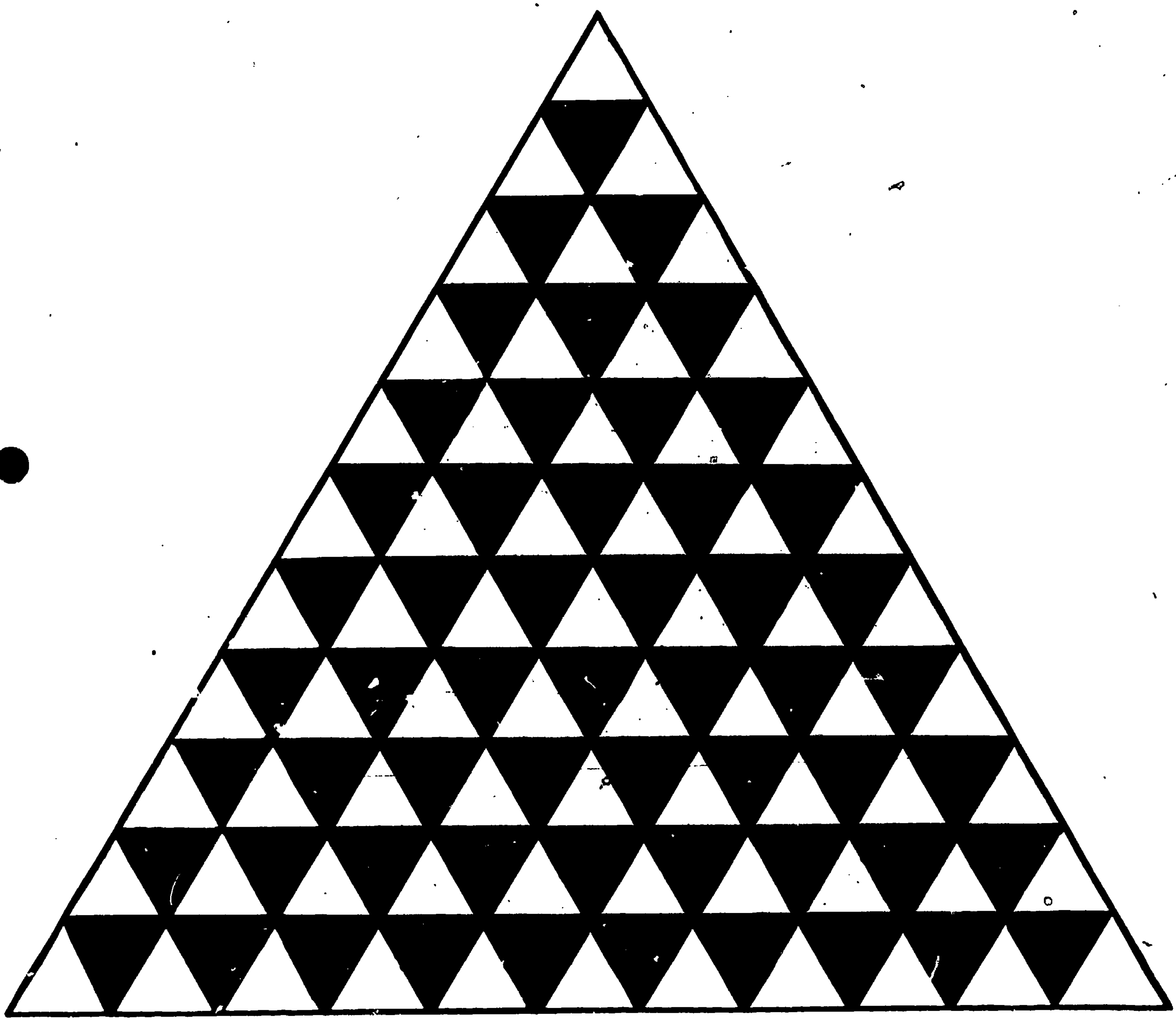
CHOICE AND CHANCE

III

STUDENT MATERIALS AND EXERCISES

STUDENT MATERIALS

Pascal Triangle



LICENTIOUS PLATES*

There are no BUMS, CADS or DUDS on the road in Iowa, not even APES, HAGS, or HAMS. These prefixes were all banned from the state's automobile license plates on grounds of taste by the Iowa Department of Transportation. But since new plates were issued last month, 130 irate motorists in Scott County have returned the plates because they bore the prefix GAY. One woman wrote: "I cannot be a single teacher and sport those plates." A traveling salesman complained that while he was in Chicago, his car doors were kicked in because of the plates.

State officials blame the foul-up on their decision to use California's list of three-letter, three-digit combinations rather than prepare their own. The officials tried to eliminate prefixes that might be offensive to Iowans but overlooked GAY. Says Scott County Treasurer William Cusack: "Out in Californi- I'm sure there is a waiting line for GAY plates. But not in Iowa." He is offering to exchange the GAY plates--1,000 were issued--on payment of a $4 fee.

*"Licentious Plates," reprinted with permission from TIME, The Weekly Newsmagazine, 22 January 1979.

TABLE OF FACTORIALS

1 FACTORIAL	=	1.
2 FACTORIAL	=	2.
3 FACTORIAL	=	6.
4 FACTORIAL	=	24.
5 FACTORIAL	=	120.
6 FACTORIAL	=	720.
7 FACTORIAL	=	5040.
8 FACTORIAL	=	40320.
9 FACTORIAL	=	362880.
10 FACTORIAL	=	3628800.
11 FACTORIAL	=	39916800.
12 FACTORIAL	=	479001600.
13 FACTORIAL	=	6227020800.
14 FACTORIAL	=	87178291200.
15 FACTORIAL	=	1307674368000.
16 FACTORIAL	=	20922789888000.
17 FACTORIAL	=	355687428096000.
18 FACTORIAL	=	6402373705728000.
19 FACTORIAL	=	121645100408832000.
20 FACTORIAL	=	2432902008176640000.
21 FACTORIAL	=	51090942171709440000.
22 FACTORIAL	=	1124000727777607680000.
23 FACTORIAL	=	25852016738884976100000.
24 FACTORIAL	=	620448401733239427000000.
25 FACTORIAL	=	15511210043330985600000000.
26 FACTORIAL	=	403291461126605582000000000.
27 FACTORIAL	=	10888869450418349800000000000.
28 FACTORIAL	=	304888344611713784000000000000.
29 FACTORIAL	=	8841761993739699765000000000000.
30 FACTORIAL	=	265252859812190987000000000000000.
31 FACTORIAL	=	8222838654177920120000000000000000.

STUDENT EXERCISES

1. At a party there are 14 women and 11 men. In how many ways can mixed couples be paired?

2. Dapper Dan has 12 sports jackets, 2 reversible vests, and 6 pairs of slacks. How many different 3-piece outfits can he arrange for himself?

3. How many three-digit numbers can be formed using the digits 1, 2, and 3 without repeating digits? Use a tree diagram.

4. How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 without repeating digits?

5. How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9?

6. How many four-digit even numbers can be formed from 1, 3, 2, 8, and 9 if each digit is used exactly once in a number?

7. How many different three-digit numbers greater than 500 can be made using the digits 1, 3, 5, 6, and 7 if no digits are repeated?

8. How many possible license plates are there that consist of a letter followed by three digits?

9. If seven digits are used for each telephone number for a given area code, how many different telephone numbers can be accommodated for that area code?

10. A wheel of fortune contains the numbers from 1 to 5. If we spin the wheel three times for each turn, how many different sequences of numbers can we obtain?

11. Calculate:

    a. 2!

    b. 3!

c. (4!)(2!)

d. $\frac{4!}{2!}$

e. $\frac{10!}{5!}$

f. $\frac{5!2!}{3!}$

g. $\frac{32!}{31!}$

h. $\frac{15!}{13!}$

12. How many arrangements of the letters in the word TEA are possible? How many form real words?

13. How many three-digit numbers can be formed using the digits 1, 2, and 3 without repetition?

14. In how many ways can 5 books be arranged on a shelf?

15. If a student has 9 books and wishes to arrange any 5 of them on a shelf, in how many ways can it be done?

16. a. In how many ways can 7 people be lined up for a group picture?

    b. In how many ways can they be lined up if Mike is to sit in the middle?

17. In how many different orders is it possible to arrange the letters of the following words?

    a. MANY

    b. CARDS

18. In how many different orders can 5 slips of paper be drawn from a hat?

19. In how many different ways can 6 seats be filled by 10 people?

20. a. Enter the numbers in the Pascal Triangle (see page III-3) through line n = 12.

    b. Check: Does that line read 1, 12, 66, 220, 495, 792, and so on?

    c. Pose six questions for which the above numbers 1, 12, 66, 220, 495, 792, and 924 would be the respective answers.

21. How many different amounts of money can be obtained from a nickel, a quarter, and a half-dollar, using one, two, or all three of these coins at a time?

22. There are 10 people on a school board. In how many ways can committees of 5 people be selected?

23. How many different clubs of 2 boys can be selected from a class of 6 boys?

24. If 8 people play a round-robin tennis tournament, how many matches are played?

25. A committee of 5 people is to be chosen from a set of 10 people.

    a. If Jones and Smith both serve on the same committee, how many committees can be formed?

    b. Suppose Jones and Smith are too incompatible to serve on the same committee. In how many ways can the commitees be formed?

26. How many diagonals can be drawn in a hexagon?

27. In how many different ways is it possible to choose a president, secretary, and treasurer from a club with 10 members?

28. In how many ways is it possible to choose a committee of 3 from a club with 10 members?

29. There are 10 qualified applicants for all of 3 available teaching positions. In how many ways can the positions be filled?

30. How many sets of 3 lucky applicants can be selected from the 10?

31. In how many ways can a string quartet (1 first violinist, 1 second violinist, 1 cellist, 1 violist) be formed from 4 first violinists, 2 second violinists, 3 cellists, and 3 violists?

32. A librarian has 6 history books. Two are Volumes I and II of the same book. In how many ways can the 6 books be arranged on the shelf, keeping the 2 volumes together in the correct order?

33. Here you have your favorite three-flavored ice cream cone.

a. In how many different ways can you arrange the 3 scoops?

b. In how many ways can you arrange 2 scoops?

c. In how many ways can you arrange 1 scoop?

d. Suppose a store had 10 flavors of ice cream. How many different double-dip ice cream cones could be made?

34. There are 12 people in a room. Each is to greet each of the others with a handshake. What is the total number of handshakes?

35. a. In how many ways can a librarian arrange 9 books on a shelf?

b. Suppose the shelf holds only 7 books. How many different arrangements can be made then?

36. a. How many different sequences of heads (H) and tails (T) can be obtained if a coin is tossed 5 times?

b. Draw a tree diagram to show these outcomes.

c. What is the probability that if you toss a coin 5 times, you will obtain:

(1) 5 heads?

(2) Exactly 3 heads?

(3) 4 heads and 1 tail <u>or</u> 1 head and 4 tails?

(4) At most 2 heads?

(5) At least 2 heads?

37. If I have 3 sets of keys on my desk, what is the probability that without looking I will take the right set for my car?

38. What is the probability of correctly guessing the month in which your instructor was born?

39. If 5 coins are tossed, what is the probability of tossing 5 tails?

40. A bag contains 15 balls numbered from 1 to 15. If 3 balls are drawn from the bag, what is the probability that the numbers on all 3 balls will be even?

41. If a student randomly answers a five-question true-or-false test, what is the probability of getting a perfect score?

42. If a student randomly answers a four-question multiple-choice quiz, with 3 choices per question, what is the probability of obtaining a perfect score?

43. There are 6 red candies and 2 green candies in an opaque jar.

   a. What is the probability of reaching in without looking and pulling out a red candy?

   b. What is the probability of pulling out a green candy?

   *c. If you do not replace the first candy drawn, what is the probability of drawing first a red and then a green candy?

44. 2 parents and 5 children in a family take turns walking their dog. At the beginning of each week, they write the days of the week on 7 slips of paper. Then each one draws a slip of paper from a hat.

   a. If Dad draws first, what is the probability that he will be responsible for walking the dog on Wednesday?

   b. If Dad does get Wednesday, what is the probability that Mom will draw a day beginning with the letter S?

   c. Then if Mom draws Saturday, what is the probability that Robin will draw a day beginning with the letter T?

---

*The asterisk represents a more challenging problem.

d. Then if Robin draws Friday, what is the probability that Marc will draw a day beginning with the letter T?

e. Then if Marc draws Tuesday, what is the probability that Janet will draw a day beginning with the letter T?

f. Then if Janet draws Monday, what is the probability that Daniel will draw Sunday?

g. Then if Daniel draws Sunday, what is the probability that Matthew will draw a day beginning with the letter S?

h. What is the probability that Matthew will draw Thursday?

45. Each letter of the word MATHEMATICS is written on a separate slip of paper and placed in a box. Without looking, you reach into the box and pick one slip of paper. What is the probability of the following:

a. Picking the letter I?

b. Picking the letter A?

c. Picking a letter in the word TIME?

d. Picking a vowel?

e. Picking a consonant?

46. A bag contains several marbles. Some are red, some white, and the rest blue. If you pick one marble without looking, the probability of picking a red one is $\frac{1}{3}$, and $P(w) = \frac{1}{3}$.

Compute the probability of picking a blue marble.

47. A bag contains 1 red marble, 2 white marbles, and 3 blue marbles. If you pick 1 marble without looking, what is

a. $P(r)$?

b. $P(w)$?

c. $P(b)$?

d. How many white marbles must be added to the bag so that the following is true?

$$P(w) = \frac{1}{2}$$

48. According to the Mendelian theory of heredity, eye color is transmitted from two parents to a child by the transmission of one of two kinds of genes: B (brown) or b (blue). Brown is dominant and blue is recessive. That is, a contribution of a B from each parent (BB) will give a child brown eyes, and a hybrid contribution of brown from one parent and blue from the other (Bb) will also result in brown eyes. Only the combination (bb) will result in blue eyes.

a. Make a table or a tree diagram listing all of the possible outcomes from two "hybrid" parents (Bb and Bb). What is the probability that a blue-eyed (bb) child will be born?

b. Make a table or a tree diagram of the offspring of one "hybrid" parent (Bb) and one BB parent. What is the probability that a blue-eyed child will result from their union?

c. Make a table or a tree diagram of the offspring of two blue-eyed parents. How many outcomes are possible? What is the probability of their having a blue-eyed child?

49. A red die and a green die are rolled simultaneously.

a. List the possible outcomes.

b. How many possible outcomes are there?

c. Describe the outcome in which the sum of the faces showing is equal to 2; equal to 12.

d. What is the probability that the sum of the faces will be 2? 3? 4? 5? 6? 7? 8? 9? 10? 11? 12?

e. Which event has the greatest probability?

f. What is the probability of rolling an odd sum? An even sum?

g. What is the probability of rolling a 7? An 11? A double? What is the probability of rolling a 7, an 11, or a double?

*h. What is the probability that one die will be a 3 and the other a 4?

*i. What is the probability that at least one of the dice will be a 6?

*j. What is the probability of rolling a minimum of 8?

*The asterisk denotes a more challenging problem.

*50. Consider the outcomes for a roll of two dice (1 red and 1 green). Compute the following probabilities:

a. The sum is more than 3 but less than 7.

b. The sum is either 3 or 7.

c. The sum is neither 3 nor 7.

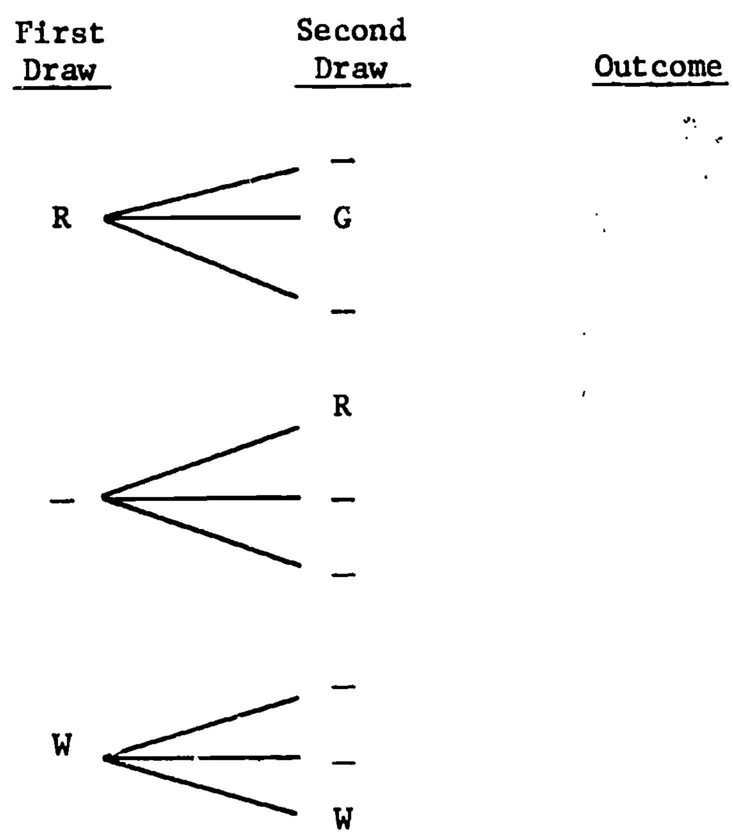
d. The red die is less than 3, and the green die is greater than 3.

e. The sum is 3.

f. The sum is <u>not</u> 3.

51. A bag contains 3 marbles--one red, one green, and one white. Suppose you draw a marble, note its color, and return it to the bag. Then you shake the bag and draw again.

a. Complete the tree diagram of the possible outcomes.



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*The asterisk denotes a more challenging problem.

Use your tree diagram to answer the following questions:

b. Which of the outcomes have red on the first draw?

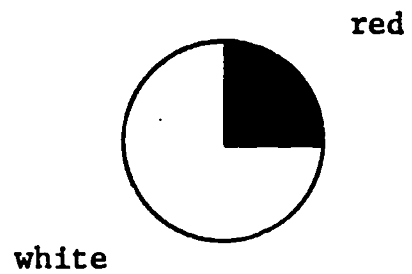
c. Which of the outcomes have green on the second draw?

d. Which outcome has red on the first draw <u>and</u> green on the second?

e. Which outcomes, then, are <u>either</u> red on the first draw or green on the second <u>or</u> both red on the first and green on the second?

f. What is the probability of event E--<u>either</u> red on the first draw or green on the second <u>or</u> both red on the first and green on the second?

52. A spinner has a red and white dial that looks like the accompanying drawing:



a. If you spin the spinner 20 times, are you just as likely to spin a red as a white?

b. Are you more likely to spin a white or a red? Why?

c. What are the chances of spinning a white? A red?

d. Are you certain of getting at least 1 red in 20 spins?

e. Is it very likely that you will not spin any reds in 20 spins?

f. Is it possible never to spin a red in 20 spins?

53. List three occurrences whose probabilities are each 1 that is, they are certain to happen).

54. List three occurrences whose probabilities are each 0 (that is, they are impossible).

*55. A standard deck of cards is shuffled and 2 cards are dealt. What is the probability that:

a. The 2 cards are spades?

b. The 2 cards are of the same suit (any suit)?

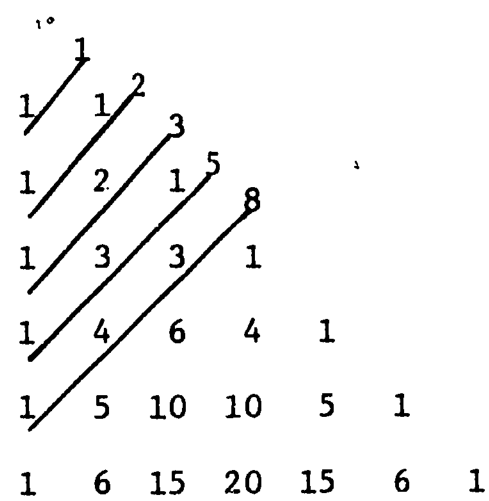
c. The cards are 2 kings?

d. Neither card is a picture card?

56. a. Rewrite the Pascal Triangle in another triangular form. Part of it has been done below. Continue it for several additional rows.

1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1
1					
1					
1					

b. Now calculate the sums of the entries along the diagonals. Several are shown. For instance, 1 + 2 = 3, 1 + 3 + 1 = 5, and 1 + 4 + 3 = 8. Calculate the next several sums. What do you notice. Can you predict subsequent diagonal sums?



*The asterisk represents a more challenging problem.

TEAM

CHOICE AND CHANCE

IV

STUDENT SUMMARY AND REVIEW

<u>To the Student</u>: This summary and review includes the key ideas discussed and presented throughout the <u>Choice and Chance</u> module. You will probably find it useful to compare your notes from class sessions with these.

I. MULTIPLICATION PRINCIPLE

A. Sometimes it is obvious and easy to list all of the possible outcomes or events for an experiment.

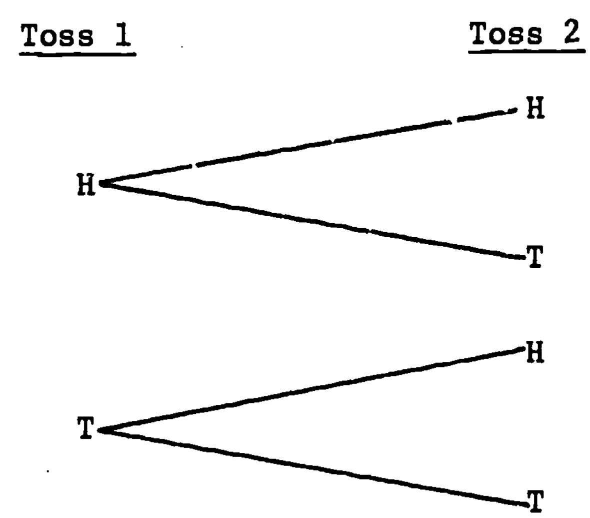
Example: When you toss a coin, there are 2 different possible outcomes--heads or tails.

When you roll a die, there are 6 different possible outcomes--1, 2, 3, 4, 5, or 6.

Try to list the outcomes in a systematic way.

B. <u>Tree diagrams</u> assist us in more difficult experiments.

Example: Toss a coin 2 times. What are the possible outcomes?

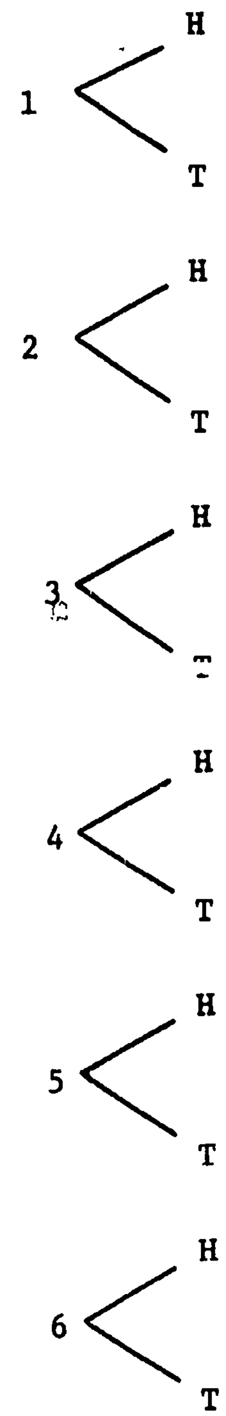


There are 4 possible outcomes: HH, HT, TH, and TT. There are 2 possible outcomes (for the first toss, either H or T) x 2 (for the second toss, either H or T) = 4.

$$2 \times 2 = 4$$

118

Example: Roll 1 die and then toss 1 coin. What are the possible outcomes?



1,H
1,T

2,H
2,T

3,H
3,T

4,H
4,T

5,H
5,T

6,H
6,T

There are $6 \cdot 2 = 12$ possible outcomes.

Question: If the only possible outcomes for the sex of a child are F (female) or M (male), list all of the possible combinations in a 2-child family. Draw a tree diagram to show the combinations.

How many different combinations of one F child and one M child can a family have?

Complete the tree diagram that has been started below to show the combinations of F and M in a 3-child family. Then list the combinations. One is MFM.

<u>3-child Family</u>

<u>Oldest</u>	<u>Middle</u>	<u>Youngest</u>
M		
F		

C. When the number of outcomes becomes too cumbersome to list, use the <u>multiplication principle</u>:

> If an operation can be performed in <u>m</u> ways, and if after it is performed in any one of these ways, a second operation can be performed in <u>n</u> ways, then the two successive operations can be performed in <u>mn</u> ways.

Example: How many different kinds of cars can you buy if you have a choice of 2 different body styles, 3 interior colors, and 10 exterior colors?

$$2 \cdot 3 \cdot 10 = 60$$

Number of choices for body style → 2

Number of choices for interior color → 3

Number of choices for exterior color → 10

Question: How many integers between 400 and 500 can be formed using the digits 1, 2, 3, 4, and 5 if no digit is repeated in any integers?

Question: How many mixed dancing couples can be formed from a group of 17 men and 8 women?

II. PERMUTATIONS AND FACTORIALS

A. A permutation is an arrangement of a group of things in a definite order.

Example: If Ted, Jan, and Amy are candidates for the position of president, vice-president, and treasurer, how many different tickets can be formed? Let P, V-P, and T stand for the respective offices.

Note that:

$\frac{\text{Ted}}{\text{P}}, \frac{\text{Jan}}{\text{V-P}}, \frac{\text{Amy}}{\text{T}}$ is different from $\frac{\text{Jan}}{\text{P}}, \frac{\text{Ted}}{\text{V-P}}, \frac{\text{Amy}}{\text{T}}$

3 x 2 x 1

Choices to fill position of P — 3

Once P slot is filled, there are 2 people left to fill position of V-P — 2

After P and V-P slots are filled, 1 person is left — 1

In permutation problems, order is important.

B. A factorial is the product of all the positive integers from one to n--symbol n!

$_2P_2$ = total number of different arrangements of 2 things = $2 \cdot 1$ = 2! (2 factorial, or factorial 2)

$_3P_3$ = total number of different arrangements of 3 things = $3 \cdot 2 \cdot 1$ = 3! (3 factorial, or factorial 3)

$_5P_5$ = $\frac{5}{\text{ways to fill first space}} \cdot \frac{4}{\text{4 ways left to fill second space}} \cdot \ldots \cdot \frac{1}{\text{only 1 choice left for last space}}$ = $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ = 5!

$_nP_n$ = total number of different arrangements of n things = $n(n-1)(n-2) \ldots 2 \cdot 1$ = n!

Example: $\frac{6!}{4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$

$= 6 \cdot 5$, or 30

Question: $(3!)(2!) =$

$\frac{9!}{3!} =$

$7(6!) =$

Question: In how many different ways can 4 books be arranged on a shelf?

Question: How many different arrangements of the letters in the word MONEY are possible?

C. Given n things, what is the total number of possible arrangements of fewer than n things? (Note that not all of the objects are being used at one time in these arrangements.)

Example: $_8P_2 = 8 \cdot 7$ (8 things, 2 at a time)

$_5P_2 = 5 \cdot 4$ (5 things, 2 at a time)

$_5P_3 = 5 \cdot 4 \cdot 3$ (5 things, 3 at a time)

Question: There are 10 candidates for the positions of president and vice-president. How many different tickets can be formed?

$_5P_4 =$

$_4P_3 =$

III. THE PASCAL TRIANGLE: COUNTING SUBSETS

A. List and count the number of subsets that can be made from a set of three elements.

Given: the set {a,b,c}

1 subset of 3 elements: {a,b,c}

3 subsets of 2 elements:

{b,c}, which creates its complement subset {a}

{a,c}, which creates its complement subset {b}

{a,b}, which creates its complement subset {c}

3 subsets of 1 element: {a}, {b}, {c}

1 subset of 0 elements: { } (the empty set)

1. Whenever a subset is made, its complement is created from the remaining elements. This symmetry (dichotomy) provides a strategy for writing subsets.

   Example: In a set of 5 elements, there are 5 subsets of 1 element each and 5 subsets of (5-1), or 4, elements each.

   Question: How many committees of 4 can be formed from a group of 5 students?

   Answer: There are 5 subsets of 4 elements each; thus, the answer is 5 committees.

2. Note that in forming subsets, order doesn't count. We are interested only in sets of objects.

   {a,b,c,d,e} is the same set as {a,c,d,b,e}

B. The Pascal Triangle is formed when the total number of subsets for the sets 0, 1, 2, ... elements is arranged in triangular form.

Given Number of Elements														Total Number of Subsets
0							1							1
1						1		1						2
2					1		2		1					4
3				1		3		3		1				8
4			1		4		6		4		1			16
5		1		5		10		10		5		1		32
6	1		6		15		20		15		6		1	64
.							.							.
.							.							.
.							.							.

Question: Write the next line for this triangle. What should the sum of the numbers in this line equal?

Observations about the Pascal Triangle:

1. The empty set is always included in the number of subsets that can be formed. That is, given n objects, 2^n subsets can be formed.

   Question: How many subsets can be formed from a set of 3 elements? 6 elements? 105 elements?

   Question: Can you pose a Pascal Triangle question for which 16 would be the answer?

2. The diagonals form patterns: (1, 2, 3, 4, ...) (1, 3, 6, 10, ...)

3. There is always one more entry on a line than the number of elements in the given set.

4. The sum of two adjacent numbers in a row is equal to the number below these two in the next row:

   1 2 1

   1 3 3 1

   Question: Use the numbers in this row--1, 3, 3, 1--to answer the following questions:

   Given 3 people, how many committees of 3 people can be formed?

   Given 3 people, how many committees of 2 people can be formed?

   Given 3 people, how many committees of 1 person can be formed?

   Given 3 people, how many committees of 0 people can be formed?

   Question: Using line 8 of the Pascal Triangle, pose a question for which the number 70 would be the correct answer.

   Question: Using the same line, pose a question for which the number 56 would be the correct answer. (Note that there are two types of possible questions.)

   Question: Given 6 elements, how many subsets of 3 elements can be formed?

Question: How many different teams of 9 members each can be formed from a group of 10 people?

Question: In how many ways can you choose a committee of 4 from a club with 8 members?

Question: What is the total number of handshakes in a room of 10 people if each person is to greet each of the other people?

Question: In a round-robin tennis tournament, each player must play each of the other players just once. What is the total number of matches played if there are 10 participants?

IV. PROBABILITY

A. A sample space is any systematic listing (including a tree diagram) of the outcomes of an experiment.

Experiment	Sample Space
Flip of a coin	{H,T}
Roll of a die	{1,2,3,4,5,6}
Flip of 2 coins	{HH, HT, TH, TT}

B. An event (E) is any subset of a sample space.

The probability (P) of an event is shown in the following way:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Example: What is the probability of each of these events?

Event E: Obtaining an odd number on a single roll of a die

Event F: Obtaining two heads on the flip of two coins

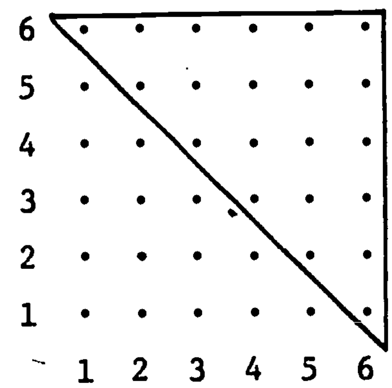
Count the outcomes in the sample spaces shown in A, above.

$$P(E) = \frac{3}{6}, \text{ or } \frac{1}{2}$$

$$P(F) = \frac{1}{4}$$

Example: What is the probability of event G--obtaining a sum of at least 7 on the roll of 2 dice?

The sample space for two-dice experiments is most easily shown with a 6 x 6 lattice of points:



Event G is shown as the set of points with sums of 7, 8, 9, 10, 11, and 12 (that is, sums of at least 7).

$$P(\text{sum of } 7) = \frac{6}{36}$$

$$P(\text{sum of } 8) = \frac{5}{36}$$

$$P(\text{sum of } 9) = \frac{4}{36}$$

$$P(\text{sum of } 10) = \frac{3}{36}$$

$$P(\text{sum of } 11) = \frac{2}{36}$$

$$P(\text{sum of } 12) = \frac{1}{36}$$

$$\text{Therefore, } P(G) = \frac{21}{36}, \text{ or } \frac{7}{12}$$

Question: A spinner has 5 numbers, equally spaced. In this experiment, you spin the spinner and flip a coin.

List the 10 possible outcomes of the experiment. Compute the following probabilities:

- The spinner points to 5 and the coin comes up heads.
- The spinner points to an odd number or the coin comes comes up heads.

Question: Two dice are tossed.

How many outcomes are possible? Compute the following probabilities:

- You roll doubles (that is, the same number on both dice).
- You do not obtain a sum of 7.

C. Observations about probability measures

1. A probability measure is a number between 0 and 1, including 0 and 1.
2. The probability of an impossible event is 0.
3. The probability of a certain event is 1.
4. The probability that an event either will occur or will not occur is 1.

$$P(E) + P(E') = 1$$

Example: On a roll of a die:

$$P(6) = \frac{1}{6}$$

$$P(\text{not } 6) = \frac{5}{6}$$

$$\text{Therefore, } P(6 \text{ or not } 6) = \frac{6}{6}, \text{ or } 1$$

Question: Roll two dice.

- Compute the probability of rolling a sum of 12.
- Compute the probability of rolling a sum that is not 12.

D. Probability measures can be used to predict. Look again at events E, F, and G in B, above.

1. $P(E) = \frac{1}{2}$

This means that event E is expected to occur about $\frac{1}{2}$ (or 50%) of the time in the future.

Question: A die is rolled 100 times. About how many times can we expect to roll an odd number?

$$P(E) = \frac{1}{2}$$

$$\frac{1}{2} \times 100 = 50$$

2. $P(F) = \frac{1}{4}$

Question: Two coins are tossed 120 times. How many times should we expect to obtain 2 heads?

$$P(F) = \frac{1}{4}$$

$$\frac{1}{4} \times 120 = 30$$

3. $P(G) = \frac{7}{12}$

Question: Two dice are rolled 24 times. In how many of these rolls can we expect to obtain a sum of <u>at least</u> 7?

$$P(G) = \frac{7}{12}$$

$$\frac{7}{\not{12}} \times \not{24}^{2} = 14$$