DOCUMENT RESUME

ED 259 921 SE 045 892

TITLE	Approximation and Estimation. A Course to Reduce Math Anxiety and Sex-Role Stereotyping in Elementary Education.
INSTITUTION	City Univ. of New York, Flushing, N.Y. Queens Coll.
SPONS AGENCY	Women's Educational Equity Act Program (ED), Washington, DC.
PUB DATE	84
NOTE	117p.; A product of the Teacher Education and Mathematics Project. For related documents, see SE 045 888-896.
AVAILABLE FROM	WEEA Publishing Center, 1984 Education Development Center, Inc., 55 Chapel St., Newton, MA 02160.
PUB TYPE	Guides - Classroom Use - Guides (For Teachers) (052)
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Attitude Change; Elementary Education; *Elementary School Mathematics; *Estimation (Mathematics); Females; Higher Education; *Mathematics Anxiety; Mathematics Education; *Mathematics Instruction; Sex Role; *Sex Stereotypes; Student Attitudes; *Teacher Education; Teaching Methods
IDENTIFIERS	*Teacher Education and Mathematics Project

ABSTRACT

This Teacher Education and Mathematics (TEAM) content module focuses on approximation and estimation. It consists of: (1) an instructor's text; (2) an instructor's guide and solutions to student exercises; (3) student materials and exercises; and (4) student summary and review. The instructor's text provides specific directions for guiding lessons and commentary on mathematics content and mathematics attitudes. This is accomplished by a "facing pages" format whereby the right-hand page provides step-by-step teaching directives while the left-hand page provides teaching insights, other options of instruction, and psychological or attitudinal strategies, when appropriate. The instructor's text also contains content objectives, specified to indicate the scope and structure of the module, and student evaluation materials. The instructor's guide and solutions to exercises provides teaching approaches and solutions to problems. Student materials and exercises provide such items as diagrams, charts, and centimeter-squared paper to be used by students. Exercises include problems that apply the concepts and problem-solving strategies developed in the module; they may be used as part of the instructional activities, as content for small-group activities, as homework assignments, or as review materials. The student summary and review summarizes the content of the module, focusing on formulas, terminology, key concepts, problem-solving strategies, and examples of techniques used. (JN)

Approximation and Estimation

A Course to Reduce Math Anxiety and Sex-Role Stereotyping in Elementary Education

TEAM

TEACHER EDUCATION AND MATHEMATICS

Queens College of the City University of New York
Women's Educational Equity Act Program/U.S. Department of Education

TEAM

TEACHER EDUCATION AND MATHEMATICS

A Course to Reduce
Math Anxiety and Sex-Role Stereotyping
in Elementary Education

APPROXIMATION AND ESTIMATION

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The activity which is the subject of this report was produced under a grant from the U.S. Department of Education, under the auspices of the Women's Educational Equity Act. Opinions expressed herein do not necessarily reflect the position or policy of the Department, and no official endorsement should be inferred.

Printed and distributed by WEEA Publishing Center, 1984
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Newton, Massachusetts 02160

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ACKNOWLEDGMENTS

The Teacher Education and Mathematics project was a comprehensive effort that required the cooperation of many people. It is impossible to identify all the Queens College faculty and staff whose assistance, support and encouragement helped bring the project to fruition.

The consultants listed below were most helpful in their advice, comments and suggestions.

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CONTENTS

INTRODUCTION

The Approximation and Estimation module consists of an Instructor's Text (including content objectives and evaluation materials), Instructor's Guide and Solutions to Student Exercises, Student Materials and Exercises, and Student Summary and Review.

The Instructor's Text provides the instructor with (1) specific directions for guiding lessons and (2) commentary on the math content and on math attitudes. This is accomplished by a special "facing pages" format. The right-hand page provides the instructor with the presentation modes of problems and teaching directives, while the left-hand page, "Commentary and Notes," provides teaching insights, other options of instruction, and, often, psychological strategies. Space for the instructor to add her or his own notes about a particular point in the lesson or about teaching experiences with the class (for future reference and use) is also provided on the left-hand page. In other words, the directions on the right-hand page clearly point out to the instructor what steps to take in presenting the lesson, while the commentary/notes page on the left supplements the instruction with explanations, additional instructional options, attitudinal interventions, and organizational alternatives to the teaching presentation. The Instructor's Text includes a set of content objectives, specified to indicate the scope and structure of the module, and student evaluation materials that contain several questions for each objective so that the instructor can select items for quizzes.

The Instructor's Guide and Solutions to Student Exercises provides teaching approaches to the exercise materials and solutions for the problems.

Student Materials and Exercises provides such things as diagrams and charts to be used by students. Instructors should plan to make transparencies of these materials for use on overhead projectors during class time. The exercises include problems that apply the concepts and problem-solving strategies developed in the module. These problems can be used as part of the instructional activities, as content for small-group activities, as homework assignments, or as review materials. Appropriate Exercises (A.E.) are noted after each topic in the Instructor's Text. Calculators are needed for this module, and an announcement to students about bringing one to class should be made at the outset.

Student Summary and Review summarizes the content of the module (formulas, terminology, key concepts, problem-solving strategies, and examples of techniques used). These notes are to be given to students *after* they have participated in the learning activities of the module.

TEAM

APPROXIMATION AND ESTIMATION

I

INSTRUCTOR'S TEXT

OBJECTIVES

The objectives of the Approximation and Estimation module are:

1. Given a set of numbers, indicate those that express approximate numbers and those that express exact numbers.
2. Place parentheses within a given mathematical expression to indicate a specified order of operations.
3. Cite references for various measurements and other numbers.
4. Round a given number to a specified degree of accuracy.
5. Compute using the square brackets.
6. Use approximations to compare numbers.
7. Translate written words into decimal numerals and conversely.
8. Express a given number in scientific notation and conversely.
9. Analyze a calculation for reasonableness of results.
10. Demonstrate the use of various computation shortcuts.

These objectives are provided here to indicate to you the scope and structure of this module. They should be distributed to students with the Student Summary and Review so that students can use them to organize their study and preparation for quizzes.

Sample items for the objectives are included at the end of this section of the module.

COMMENTARY AND NOTES

This material uses the National Bureau of Standards notation for large numbers. The comma is no longer used to separate periods of three digits. Instead, a space is used. Thus 3,487,521 is written 3 487 521. In the case of four-digit numbers, neither a space nor a comma is used. Thus, 9438 is the correct form for a four-digit number.

Point out that when they are faced with a problem using large numbers, students may find it easier to solve a related problem using smaller values. When they have solved the simpler problem, the process or method can be transferred to the original problem. Students should be encouraged to use this strategy in all problem solving.

For example, the product of 87 and 39 will have a 3 in the units place. The problem under discussion will have a 3 in the units place for the same reason.

Note whether students react with tension to the use of large numbers and to mathematical vocabulary. If so, ask students how they are feeling, and proceed with the content only after students have had a chance to express their concerns. You might also reassure students that because of their broader experience and larger vocabulary, they are in a better position now to learn a specific vocabulary than they were as children.

There are no fixed rules about the degree of accuracy required at the time a number is rounded. Students should be given the opportunity to decide what makes good sense for the required purpose. Sometimes it makes more sense to truncate (chop or lop off, round down) a number; sometimes it makes more sense to round up; and sometimes it is helpful to have a procedure for rounding up or down. All these methods are considered in this module. However, the appropriate method to use must be decided separately for each case.

Present the following problem:

One of the digits of the left-hand number in the multiplication problem below has been smudged. It is known that the number is in the thousands. What is the only number that the product can be?

6 87 x 39

(a) 24 993 (b) 24 996 (c) 249 093 (d) 249 096 (e) 249 083

The discussion should include the following observations:

- The units digit should be 3, since the product of the units digits 7 and 9 is 63. Therefore, (b) and (d) must be eliminated.
- Because the left-hand number is known to be in the thousands, the product must be at least 6000 x 39, or 234 000, and very close to 6000 x 40, or 240 000. Because the product is in the hundred thousands, (a) and (b) are eliminated.
- The product of 87 (the tens and units digits of the left-hand number) and 39 is 93. Therefore, (e) is eliminated, and the answer is (c).

Point out that the missing digit can now be calculated by division:

249 093 ÷ 39 = 6387, and 6387 x 39 = 249 093

Approximate Numbers

Ask students to read the following number. Then ask them to suggest ways in which the number can be rounded:

2 365 567

(Read: Two million, three hundred sixty-five thousand, five hundred sixty-seven.)

Do not spend too much time on the rounding process. You will return to this matter shortly.

Round to the nearest		
	ten:	2 365 570
	hundred:	2 365 600
	thousand:	2 366 000
	ten thousand:	2 370 000
	hundred thousand:	2 400 000
	million:	2 000 000

COMMENTARY AND NOTES

The major theme of this module is the effective use of approximate numbers in appropriate contexts.

Tell students that they will ne calculators for this module.

Individuals with high levels of anxiety often find it difficult to deal with the distinction between exact and approximate numbers because they feel there must be a single correct answer to every mathematics question. Encourage students to express their feelings on this matter. Point out the important role that context and purpose play. It is the problem solver who decides whether an exact or an approximate answer is most useful for the task at hand.

If the distinction between exact and approximate numbers continues to be a source of tension, help the group to generate more examples of each from their own life experiences.

Say: "Suppose the given number represents the population of a city. Which approximate number is close enough to the truth?" The discussion should lead to the conclusion that figures like 2 365 567 or 2 365 600 only give the impression of accuracy. Since population is constantly changing, it is more sensible to use approximate figures such as 2 370 000, 2 400 000, or even 2 000 000.

Say: "Suppose that the number 23 376.4 is the number of miles indicated on the odometer of your car. How many miles do you have on your car?"

- Ask: "When is 'about 23 000 miles' a sensible answer?" (During ordinary conversation about the car.)

- Ask: "When might a number like 23 300 miles be necessary or appropriate?" (When, for example, the sale of a used car depended on the mileage--when mileage of up to but not including 23 400 would bring a better price than mileage of 23 400 or more.)

Observe that in the case of the car sale, 23 300 would be the information needed. In ordinary conversation about the car, 23 000 would be sufficient. Each serves a different purpose.

Continue the discussion of approximate numbers and when to use them by pointing out that an exact number is one obtained by counting. Point out that an approximate number can be used in place of an exact number when an exact number is not needed.

Consider the following examples and observations with the students:

1. You inherit $74 526.37.

   - You might say that you inherited about 75 thousand dollars, or $75 000. This is an approximate figure.

   - If you were to deposit the money in an interest-bearing account, what amount would draw interest? (Possibly all of it; possibly just $74 526.)

   - Suppose the Internal Revenue Service were to tax your inheritance. What amount would they probably tax?

2. A spinal tap is performed on a patient suspected of having meningitis. After the spinal tap is performed, the fluid is placed in a counting chamber. If 6 white blood cells are counted in this culture chamber, the patient is suspected of having meningitis. If, after looking in the chamber, the doctor sees numerous white blood cells, it is obvious that the patient is infected, and the total number of white blood cells is estimated as follows:

People who are comfortable with numbers use approximate numbers extensively.

It is important to point out that the word <u>exact</u> should never be used in connection with a measurement. Although the counting process is used in measuring, the result is approximate. There is always an error that can be caused by the tools used, by one's eye, judgment, etc. In this case, "error" is not a mistake; it is an occurrence that is expected.

A brief review of fraction-decimal equivalents is desirable. A fraction like $\frac{33}{100}$ will yield .33 on the display of a calculator when 33 ÷ 100 is entered. Show students that each of the fractions below is equivalent to $\frac{1}{3}$ by actually dividing the denominator by 3:

$$\frac{1\frac{1}{3}}{4},\ \frac{2}{6},\ \frac{3}{9},\ \frac{3\frac{1}{3}}{10},\ \frac{33\frac{1}{3}}{100}$$

That is: $4 \div 3 = 1\frac{1}{3}$; $6 \div 3 = 2$; $9 \div 3 = 3$;

$10 \div 3 = 3\frac{1}{3}$; $100 \div 3 = 33\frac{1}{3}$

The doctor counts the cells in only one section of the counting chamber and multiplies this number by 9, the total number of sections in the counting chamber. If the doctor were to count 512 cells in one section, he would obtain exactly 4608 as the product of 9 and 512. The number 4608 sounds exact. However, the result is an approximation. The information it conveys is approximate, but sufficient to determine that the patient is in trouble.

3. One hundred twenty-three people attended a certain party.

   - This is an exact number and one that a caterer would use for food preparation and charges.

   - "There were about 120 people at the party" is a statement that might be made in conversation when an exact number is not needed.

   Ask students to give similar examples from their own experience.

4. A population figure like 1 428 838 might be given as 1 429 000 (rounded to the nearest thousand) or 1 400 000 (rounded to the nearest hundred thousand) or even "almost a million and a half" (1 500 000).

   Ask students to give examples of approximate numbers. A daily newspaper is a good source.

5. All numbers obtained by measuring are approximate. For example:

   - Height of 162 cm
   - Distance of 200 m
   - Mass of 55 kg

6. Since fractions like $\frac{1}{3}$ and $\frac{2}{11}$ are equivalent to unending decimals, the decimals .33 and .18 are only approximate values of these fractions.

   Ask students to demonstrate this on calculators. Entering 1 ÷ 3 produces repeating threes, .3333333, while 2 ÷ 11 produces .1818181.

   Thus, $\frac{1}{3} \approx .33$ and $\frac{2}{11} \approx .18$

   (We use the symbol $\approx$ to denote "approximately equal to.")

A.E. 15, 411

COMMENTARY AND NOTES

The left-to-right process in many calculators yields a result that is mathematically incorrect. In order for the calculator's results to be correct, it is necessary to revise the expression by inserting parentheses. Point out that each calculator comes with a set of directions for its use. It is important to study the directions and work the calculator in order to become familiar with it.

Students may be confused by the variety of results here. It must be emphasized that there is only one correct result for the original problem. The variation in answers results from the different ways in which the numbers are grouped.

Order of Operations: The Calculator

Students should be encouraged to use calculators whenever calculation is extensive. It is important, therefore, for them to know how particular calculators process information. Calculators that process information in the order in which it is received may yield mathematically incorrect results unless the user sets up the calculations in such a way that operations are carried out in the correct order.

Consider the expression: $2 \cdot 3 + 4 \cdot 5$

The correct result is: $6 + 20$, or 26

However, most inexpensive calculators will show the operations in this order:

- First: 2×3, or 6
- Next: $6 + 4$, or 10
- Then: 10×5, or 50

The calculator has grouped the expression $2 \cdot 3 + 4 \cdot 5$ so that:

$$(2 \cdot 3 + 4) \cdot 5 = 50$$

Ask students to process the following expression the way such a calculator would, in the order given (left to right) and to insert parentheses to indicate the way the calculator processes the data.

$$2 + 4 \times 5 - 9 \div 3$$

This expression in the calculator becomes:

$$[(2 + 4) \times 5 - 9] \div 3 = (6 \times 5 - 9) \div 3$$

$$= (30 - 9) \div 3$$

$$= 21 \div 3, \text{ or } 7$$

However, the value of the expression is:

$$2 + (4 \times 5) - (9 \div 3)$$

$$2 + 20 - 3, \text{ or } 19$$

since multiplication and division take precedence over addition and subtraction.

COMMENTARY AND NOTES

Anticipate that students will work at different speeds. This variation in pace may be related to such factors as experience, comfort, and manual dexterity.

Students may work individually or in groups. Group practice has the advantage of enabling members of the group to share ideas. In this way, each individual may be stimulated to reason in ways that might not have been possible otherwise.

The calculator method to produce 19 depends upon whether or not the machine has a memory. First, consider 2•3 + 4•5:

Without a memory (boxed operating sign indicates signal calculator keys):

Enter 2, [x], 3, [=]

Write down the 6 and clear (press [C])

Enter 4, [x], 5, [=] (20 in display)

Next press [+], 6 (26 in display)

Of course, 20 [+] 6 = 6 [+] 20 (commutative property of addition)

With a memory:

Enter 2, [x], 3, [=], [M+] (6 in memory)

Enter 4, [x], 5, [=], [M+] (20 in memory)

Press [MR] (26 is recalled from memory)

Next, consider 2 + 4 x 5 - 9 ÷ 3. Remind students that parentheses are not required in such an expression. It is implicit that the expression should be evaluated in the following way:

$$2 + (4 \times 5) - (9 \div 3)$$

Without a memory:

Press 4, [x], 5, [=] (Write down 20, then clear.)

Press 9, [÷], 3, [=] (Write down 3, then clear.)

Now enter 2, [+], 20, [-], 3, [=]

With a memory:

Enter 4, [x], 5, [=], [M+] (20 in memory)

Enter 9, [÷], 3, [=], [M-] (-3 in memory)

Enter 2, [+], [MR], [=] (2 + 17 = 19)

Students should be given the opportunity to use a calculator to evaluate various expressions. When they enter numbers in the order given, they should insert parentheses to show what is taking place. For example, ask students to enter the following expression in the calculator in the order given and to insert parentheses that indicate the order in which the calculator processes the information. Next ask students to calculate intermediate answers or to rearrange terms in order to obtain the mathematically correct answer.

COMMENTARY AND NOTES

Note that the pacing of a session is up to the instructor. Timing will vary widely, depending upon students' attitudes and experience. The activities given here should be used to review fundamental operations and concepts.

Consider and evaluate the expression: $5 + 8 - 4 \times 3 + 6 \div 2$

Answer: The expression is processed by the calculator as:

$$[(5 + 8 - 4) \times 3 + 6] \div 2 = 16.5$$

<u>Without a memory</u>:

$4 \times 3 = 12$

$6 \div 2 = 3$

$5 + 8 - 12 + 3 = 4$

<u>With a memory</u>:

4, [x], 3, [=], [M-]
6, [÷], 2, [=], [M+]
5, [+], 8, [+], [MR], [=]

Note that [MR] = +4. Pressing [M-] places -12 in the memory. Pressing [M+] places 3 in the memory and adds it to the -12, making -9:

$$13 - 9 = 4$$

Other expressions should be considered and evaluated, so that the significance of parentheses and the correct use of the calculator can be understood.

Consider the expression $2 \cdot 3 + 4^2$. Its value is $6 + 16 = 22$.

However, the expression can be grouped in a number of ways. The following is the way that a left-to-right calculator calculates:

$2 \times 3 = 6$

$6 + 4 = 10$

$10^2 = 10 \times 10$, or 100

This calculation is represented by the parenthesized expression:

$$(2 \cdot 3 + 4)^2 = 100$$

Ask: "How do we use this calculator to obtain 22?"

Answer: Calculate $2 \cdot 3$ and $4 \cdot 4$ separately.

This can be done by placing $2 \cdot 3$, or 6, in the memory and pressing [+], then [MR] to add it to $4 \cdot 4$, or 16.

COMMENTARY AND NOTES

In many real-life applications it isn't necessary to have an exact number. All that's necessary is a familiar quantity as a reference point. Reference numbers can make approximation and estimation less threatening because they give the students a familiar base to start from. Students can easily visualize a fingernail or a sheet of paper. Be sure to discuss this observation with your students.

Ask: "How can we obtain 40 from $2 \cdot 3 + 4^2$?"

Answer: $(2 \cdot 3 + 4) \times 4 = 10 \times 4$, or 40

Ask: "How can we obtain 54?"

Answer: $2 \times 3 + 4^2 = 4^2 + 2 \times 3$

$= (4^2 + 2) \times 3$

$= 18 \times 3$, or 54

Evaluate: $2 \times (3 + 4)^2$

Answer: $2 \times 7^2 = 2 \times 49$, or 98

A.E. 6, 7, 8, 39

In each case, ask the students why a particular rearrangement is needed in order to obtain a particular result with the calculator they are using.

Reference Numbers

Daily living demands an ability to make estimates. In many types of activities, the need to obtain reasonable estimates arises more frequently than the need to calculate exact answers. Everyone should work out a list of reference numbers that can be used as an aid in making estimates. These should be taken from familiar objects or situations and may be different for different individuals. Making quantitative comparisons with numbers (including measures) that we understand and have experienced increases our ability to approximate.

Following are some examples that may be cited or discussed. Others should be supplied by students.

1. There are about 25 students in a class.
2. There are about 50 cards in a standard deck of playing cards.
3. There are 100 squares in a hundreds chart.
4. A ream of paper contains 500 sheets.
5. Most rooms have a standard 8' ceiling.
6. It's about one mile from ________ to ________.
7. A standard sheet of paper measures $8\frac{1}{2}$" x 11", or about 22 cm x 28 cm.
8. A small index card measures 3" x 5".
9. A brand-new nickel weighs 5 grams.
10. The width of the fingernail of the index finger is about 1 cm.

COMMENTARY AND NOTES

If students are comfortable with the metric system, use a meter stick as a number line. The relationship of the meter and its subdivisions can be extremely helpful in visualizing the rounding process.

For some students, this may be a review of previous work. You may find, however, that some students have not used this geometric approach before. The physical model can clarify the rounding procedure--a procedure that some students have learned without understanding. At this point in particular, be aware of the many levels of confidence and competence that may exist in your class. Grouping can permit confident students to serve as peer tutors.

11. I am about _______ cm tall.

12. A minute is the time it takes a second hand to go around the clock.

13. Room temperature is about 20°C.

14. If a person's heart beats about 70 beats per minute, it would beat about a million beats in 10 days.

15. There are 1 000 000 square millimeters in a square meter.

16. Sometime during the year 2739, there will have been 1 000 000 days during the Christian era.

17. A novel of average length contains about 125 000 words. Thus, eight such novels would contain about a million words.

18. The sun is about 100 million miles away (93 000 000 miles).

A.E. 9, 39e, f, g

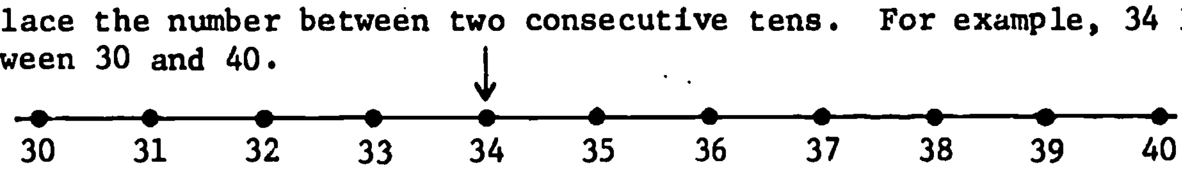
Rounding Numbers

Review some of the cases (population, mileage, money, etc.) cited earlier in the module in which approximate numbers were obtained by rounding.

Clarify the rounding process by using a number line. (Number lines have been provided in the student materials so that students can work along with you.) Note that each segment is divided into 10 equal parts.

To round to the nearest ten:

Place the number between two consecutive tens. For example, 34 lies between 30 and 40.

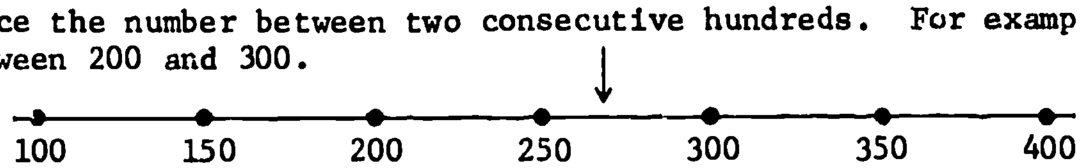


The number 34 is closer to 30 [illegible] to 40. Therefore, 34 rounds to 30 when rounded to the nearest ten.

Ask students to name other numbers that round to 30 when rounded to the nearest ten. Cite a sufficient number of examples (including numbers like 32.6, 34.99, and 25.1) so that it becomes possible for students to accept 25 and all numbers that are greater than 25 but less than 35 as numbers that round to 30.

To round to the nearest hundred:

Place the number between two consecutive hundreds. For example, 268 lies between 200 and 300.



COMMENTARY AND NOTES

It is appropriate and necessary to help students express their feelings about decimals and fractions. Giving them this opportunity typically results in a sense of relief among students. Helping students to see that others have the same feelings is part of the strategy of lessening anxiety about math. Having stated their concerns, they may feel more ready to work on the task at hand.

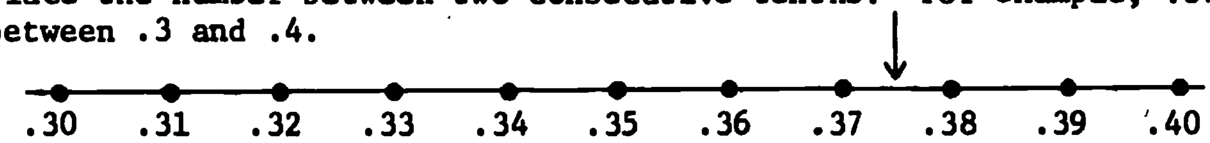
The zero is written to the left of the decimal point to show that the whole-number part of the decimal is zero.

Students may need some help in understanding the meaning of precision in measurement. The word precise does not mean exact. Precision depends upon the smallest unit used in the measurement. Thus, 5.0 is more precise than 5 because the 0 indicates that a tenth of a gram was the smallest unit used. Precise means sharply defined or minutely exact.

Show that 268 is closer to 300 than to 200 because it is greater than 250 (the number halfway between 200 and 300). It is the 6 in 268 that tells us to round to 300.

To round to the nearest tenth:

Place the number between two consecutive tenths. For example, .3749 lies between .3 and .4.



The number .3749 lies to the right of .35 and thus is closer to .4 than to .3. It is the 7 that tells us to round to .4.

The convention is to examine the digit in the first place to the right of the one wanted. If this digit is less than 5, it and all others to its right are dropped. If it is 5 or more, the digit preceding it is increased by 1 and the rest are dropped.

Thus, rounding 0.374921 to four places gives 0.3749

to three places gives 0.375

to two places gives 0.37

Rounding 0.2398 to the nearest hundredth gives 0.24

to the nearest thousandth gives 0.240

Be careful to distinguish between such symbols as 0.2 and 0.20:

- 0.2 is assumed to be correct to the nearest tenth. The true number could be 0.15 or any number greater than 0.15 but less than 0.25.

- 0.20 is assumed to be correct to the nearest hundredth. All numbers greater than or equal to 0.195 but not as great as 0.205 will round to 0.20 when rounded to the nearest hundredth.

As a further example, consider that a new nickel weighs 5.0 grams.

Ask: "How does this number differ from 5 grams?"

Help students understand that 5.0 grams indicates correctness to the nearest tenth of a gram, while 5 grams indicates correctness to the nearest gram. Thus, 5.0 is more precise than 5.

- The smallest number that rounds to 5.0 is 4.95. The largest is a number smaller than 5.05.

- The smallest number that rounds to 5 is 4.5. The largest is a number smaller than 5.5.

COMMENTARY AND NOTES

An effective way to present the concept of square brackets is as a game or a new way to estimate. Students do not need to know that the use of these brackets is the <u>Greatest Integer Function</u>. In fact, the label may engender anxiety and impede learning.

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- If a number such as 0.2 is exact, it would be wiser to give the answer as $\frac{1}{5}$ since this gives no suggestion of rounding.

Next consider such figures as the $75 000 inheritance, the 120 people at the party, and the population figure of 1 500 000.

- $75 000 is assumed to be a number correct to the nearest thousand.
- 120 is assumed to be correct to the nearest ten. Thus the exact number of people might be 115, 116, 117, 118, 119, 120, 121, 122, 123, or 124.
- The number 1 500 000 without additional information is assumed to be correct to the nearest hundred thousand.

A.E. 4, 10-17, 37

Chopping Numbers

Ask students to think about instances when rounding down is appropriate. One example is the way we are in the habit of giving our age. If you are 20 years, 10 months, you give your age as 20, not 21. The fractional part of the year is chopped off rather than rounded up.

As another example, ask: "If glasses cost 89¢ each, how many can you buy for $10?" Division (1000 ÷ 89) yields 11 and a remainder of 21. You can buy 11 glasses. The remaining 21¢ is insufficient to purchase another glass.

Explain that the symbol $\left[\ \ \right]$ will denote rounding down (chopping or truncating) and that $\left[1\frac{1}{3}\right]$ is read "square brackets of one and one-third."

Then write these examples on the board:

$$\left[1\frac{1}{3}\right] = 1$$

$$[2] = 2$$

$$\left[\frac{7}{3}\right] = 2$$

$$[5.9] = 5$$

$$\left[\frac{7}{8}\right] = 0$$

As students proceed with their work, help them feel the joy of creative activity and take pleasure in their sense of increased power over numbers and the enthusiasm it engenders.

This is another good opportunity for individual or small-group activity. Another technique is to have students work individually on the chalkboard, then check and discuss each other's work.

From these examples the students should understand that the addition of square brackets chops off or truncates whatever was added to the whole number. Some calculators <u>truncate</u> or <u>chop off</u> digits that do not fit on the display.

Show how each of the introductory examples on the previous page can be written with square brackets to indicate that the number is to be chopped.

$$[20 \text{ years, } 10 \text{ months}] = 20$$

and

$$\left[\frac{1000}{89}\right] = 11$$

Do not formalize the definition of square brackets. Rather, present the next set of exercises to check students' understanding of what the addition of square brackets does to a number.

(1) $\left[2\frac{2}{3}\right] =$ (4) $\left[\frac{9+2}{3}\right] =$ (7) $\left[1\frac{1}{2}\right] + \left[1\frac{1}{2}\right] =$

(2) $\left[\frac{11}{3}\right] =$ (5) $\left[1\frac{1}{2} + 5\right] =$ (8) $\left[\frac{7\frac{1}{3}}{4}\right] =$

(3) $\left[1\frac{1}{2}\right] =$ (6) $\left[1\frac{1}{2} + 1\frac{1}{2}\right] =$

Answers: (1) 2, (2) 3, (3) 1, (4) 3, (5) 6, (6) 3, (7) 2, (8) 1

Help the students make the following observations:

- In (6) and (7), establishing an order of operations is important. In (6) you may not drop the fractional parts. The addition must be carried out and then the square brackets applied. However, in (7) you chop first and then add.
- (8) is not as complex as it may seem. All that you need to know is that $4 \div 4 = 1$ and $8 \div 4 = 2$. Because $7\frac{1}{3}$ is less than 8 but greater than 4, the answer is 1.

Ask the students to estimate the answer to 2.4617 + 3.5120 by using $[2.4617 + 3.5120]$

Ask: "Why is 6.0737 an unreasonable answer?" Students should see that chopping $[2.4617 + 3.5120] = [5.9737]$, or 5, so that having a 6 in the units digit is impossible.

COMMENTARY AND NOTES

It may be necessary to review the idea that when the denominators of fractions are equal in value, it is sufficient to compare the values of the numerators to determine which fraction has the greatest value. Use fraction diagrams such as the one in the text to to enhance your review. Move slowly from the concrete examples to the abstract ideas.

The diagram in the text can be used to illustrate the following:

$$\frac{2}{3} > \frac{1\frac{1}{2}}{3} \quad \text{and} \quad \frac{1}{3} < \frac{1\frac{1}{2}}{3}$$

Similarly: $\frac{2\frac{1}{2}}{5} < \frac{3}{5}$

And: $\frac{5\frac{1}{2}}{11} > \frac{5}{11}$

Ask the students to share their methods of working out (5) and (6). If a student is in error, consider clarifying how the thinking proceeded rather than assessing its accuracy. The object here is to do as little computation as possible and instead to find ingenious ways of manipulating the numbers effectively.

Ask the students to write down the answers to the following problems rapidly, without doing any calculations.

(1) $\left[\frac{1347}{1346}\right] =$ (4) $\left[\frac{63}{10}\right] =$ (7) $\left[2\frac{3}{5} + 1\frac{1}{2}\right] =$

(2) $\left[\frac{186}{187}\right] =$ (5) $\left[\frac{1}{2} + \frac{1}{3}\right] =$ (8) $\left[4\frac{5}{11} + 1\frac{1}{2}\right] =$

(3) $\left[\frac{59}{10}\right] =$ (6) $\left[\frac{1}{2} + \frac{2}{3}\right] =$

Answers: (1) 1, (2) 0, (3) 5, (4) 6, (5) 0, (6) 1, (7) 4, (8) 5

Point out that an approximation, not an exact answer, is all that is needed in (5) and (6).

In (5) we need to know if $\frac{1}{3}$ is greater than $\frac{1}{2}$.

If $\frac{1}{3} > \frac{1}{2}$; then $\left[\frac{1}{2} \text{ plus a value greater than } \frac{1}{2}\right] = 1$

But $\left[\frac{1}{2} \text{ plus a value less than } \frac{1}{2}\right] = 0$

Look at fractions that are equal to $\frac{1}{2}$ so that the students can see that each numerator is equal to one-half the denominator:

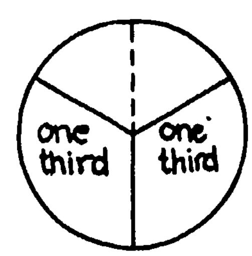
$$\frac{2}{4}, \quad \frac{2\frac{1}{2}}{5}, \quad \frac{3}{6}, \quad \frac{3\frac{1}{2}}{7}$$

The accompanying diagram shows that $1\frac{1}{2}$ thirds is equivalent to one-half,

or: $\frac{1\frac{1}{2}}{3} = \frac{1}{2}$

Thus in (5): $\frac{1}{3} < \frac{1\frac{1}{2}}{3}$, so $\left[\frac{1}{2} + \frac{1}{3}\right] = 0$

But in (6): $\frac{2}{3} > \frac{1\frac{1}{2}}{3}$, so $\left[\frac{1}{2} + \frac{2}{3}\right] = 1$



A.E. 21

NOTES

Rounding Up

Now ask students to consider cases when <u>rounding up</u> is appropriate. For example, if an item in the supermarket is priced at 6 for $1.87, the price for one is $.32, <u>not</u> $.31. Even though

$$1.87 \div 6 = .311...$$

the practice is to charge the buyer an additional cent.

Show how one can estimate beverage needs for a group. Ask: "If each person consumes about 5 ounces of soda and soda is purchased in 12-ounce cans, how many six-packs should you buy to accommodate 30 people?"

<u>Estimate</u>:

2 people to a can

12 people to a six-pack

3 x 12, or 36 people to 3 six-packs

Thus, 18 cans should be <u>more</u> than enough.

Actually, 150 ounces (5 ounces x 30 people) using 12-ounce cans yields:

$$150 \div 12 = 12 \text{ cans} + 6 \text{ ounces, or } 13 \text{ cans}$$

Ask: "Is a $10 bill sufficient to purchase items costing $.36, $1.19, $.79, $3.27, $1.69, and 4 cans at $.49 each?" In this case, students should recognize the appropriateness of rounding up to leave a safe margin. Sometimes we round up to the nearest ten cents; sometimes, to the nearest dollar.

To be on the safe side, the students should estimate as follows:

$$\underbrace{.40 + 1.20 + .80}_{2.40} + 3.50 + 2.00 + 2.00$$

$$\underbrace{2.40 + 3.50}_{6.00}$$

$$\underbrace{6.00 + 2.00 + 2.00}_{10.00}$$

A $10 bill should certainly be sufficient. The purchases actually total $9.26.

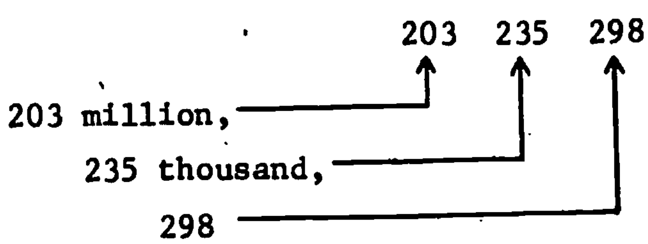
A.E. 22, 23

36

Give a brief explanation of what is meant by rounding to two figures, or to three figures. The reference is to the number of places at the extreme left that are filled by numerals other than 0 (zero).

Encourage students to consider multiple approaches to problem solving so that they will be freer to think about problems.

A step-by-step, place-by-place review of place value may be necessary here. Students may also wish to review the reading of very large numbers, such as:



Comparing Large Numbers

Point out to students that it is customary to round very large numbers by replacing some digits with zeros. For example:

- The circumference of the earth is about 25 000 miles. (It is actually about 24 875 miles. This number has been rounded to two figures, or to the nearest thousand.)
- The velocity of light is 186 000 miles per second. (This is assumed to be rounded to the nearest thousand, or to three figures.)
- The area of Rhode Island is about 1000 square miles.

These numbers have all been rounded up, but there are times when it may be convenient to round down. Make it clear that this is a personal decision. There is no single correct way. The decision to round up or down depends on personal preference and the way in which a number is to be used.

Discuss several examples to help clarify these ideas, as outlined below.

1. If Rhode Island is 1058 square miles and Texas is 267 000 square miles, how do these states compare in size?

   A ~~good~~ way to proceed is to use what may be called <u>front-end approximation</u>. With each number rounded to the place at the extreme left, the area of Rhode Island becomes 1000 square miles, and the area of Texas becomes 300 000 square miles.

   Comparison by means of a ratio yields:

$$\frac{300\ \cancel{000}}{1\ \cancel{000}}$$

   Thus, Texas is about 300 times the size of Rhode Island.

2. Ask the students to make additional comparisons using the following population data for the year 1980:

a. Compare the population of each country below with that of the United States.

U.S.	226 504 825
People's Republic of China	970 000 000
France	53 950 000
Japan	117 650 000
Monaco	30 000
U.S.S.R.	267 600 000
Vatican	1 000

b. To compare the population of China with that of the U.S., use a ratio:

Round 970 000 000 down to 900 000 000
Round 226 504 825 down to 200 000 000

$$\frac{9\cancel{00\ 000\ 000}}{2\cancel{00\ 000\ 000}} \text{ becomes } \frac{9}{2}, \text{ or } 4\frac{1}{2}$$

Thus, China has about $4\frac{1}{2}$ times the population of the U.S.

If the China figure is rounded up to 1 000 000 000, then

$$\frac{1\ \cancel{000\ 000\ 000}}{2\cancel{00\ 000\ 000}} = 5$$

and we would say that China has about 5 times the population of the U.S.

c. For France, use the figure 50 000 000 and compare it with 200 000 000 for the U.S. Ask students to simplify the ratio below.

$$50\ 000\ 000 \text{ is } \frac{5\cancel{0\ 000\ 000}}{2\cancel{00\ 000\ 000}}, \text{ or about } \frac{1}{4} \text{ of}$$

the U.S. figure.

d. For Japan, use the figure 100 000 000 and compare it with 200 000 000.

Ask students what the ratio is. (Japan's population is about $\frac{1}{2}$ the U.S. population figure.)

COMMENTARY AND NOTES

If interest wanes, consider briefly with the students the purposes being served by their mastery of these concepts.

Ask: "Why is approximation important?" Following are some of the answers that might be elicited by this discussion.

- Newspaper articles report numbers and the reader needs to have a rough idea of what they mean.
- It is often necessary to judge the number of people in a group or to compare populations or areas.

e. The figure for Monaco of 30 000 can most easily be compared with the figure 210 000 000 for the U.S., since

$$\frac{30\ \sout{000}}{210\ 000\ \sout{000}} = \frac{1}{7000}$$

In this case, approximating the U.S. figure at 210 000 000 is convenient, even though the usual rounding rule yields 200 000 000.

f. For the Vatican, compare:

$$\frac{1\ \sout{000}}{200\ 000\ \sout{000}}$$

The U.S. has 200 thousand times the population of the Vatican.

g. In the case of the U.S.S.R., since both numbers have the same number of digits and agree in the figure at the extreme left, it may be best to round each to two figures and calculate their difference. Thus,

U.S.S.R.	270 000 000
U.S.	230 000 000

yields about 40 000 000, or 40 million, more people in the U.S.S.R. than in the U.S.

Student Exercises designated below are appropriate for additional practice.

S.E. 18-20

Numbers in Words

Point out that newspapers usually report very large numbers in a shortened form that uses words instead of zeros. Thus, 4 000 000 is written 4 million, and 183 000 000 is written 183 million.

Give students practice in using both standard decimal notation and word form. Provide them with the opportunity to read and write such numbers, using the examples below.

1. Ask students to write each of the following as they would be read:

   2 000 000 000   919 000 000

2. Ask students to write each of the following as standard decimal numerals:

   34 billion   211 million

3. Explain to students that this method of recording numbers also produces $2\frac{1}{2}$ million and 1.4 billion. Then use the following procedure to help the students develop a method of expressing each of these numbers as standard numerals (that is, in standard decimal notation).

$$2\frac{1}{2} \text{ million} = 2\frac{1}{2} \times 1\,000\,000$$

$$= 2 \times 1\,000\,000 + \frac{1}{2} \times 1\,000\,000$$

$$= 2\,000\,000 + 500\,000$$

$$= 2\,500\,000$$

Help students observe that half a million is equivalent to 500 thousand.

Also, since $2\frac{1}{2} = 2.5$,

$$2.5 \times 1\,000\,000 = 2\,500\,000$$

or $$2.5 \times 10^6 = 2\,500\,000$$

$$1.4 \text{ billion} = 1.4 \times 1\,000\,000\,000$$

$$= 1\,400\,000\,000$$

or $$1.4 \times 10^9 = 1\,400\,000\,000$$

Point out that the following device can help students remember how many zeros are needed in standard decimal notation.

1 million = 10^6 (6 zeros)

1 billion = 10^9 (9 zeros)

A.E. 17-20, 41

Scientific Notation

Give the students examples of the way very large and very small numbers are written compactly.

The distance from the sun to the earth in miles is 93 000 000, or 9.3×10^7.

Having a systematic way of noting data makes it possible to discern patterns. When their data are organized, students can keep track of where they are in their progress toward a solution to a problem.

Encourage the students to develop this on their own. Suggest that they multiply 2.34 by successive powers of 10, using a calculator and tabulating the results on a data chart until they perceive a pattern.

The term algorithm may be unfamiliar. Indicate that it simply refers to a set of rules for doing a computation. You can use the word procedure instead, if you wish.

In one year, light travels about 6 000 000 000 000, or 6×10^{12}, miles.

The width of a human hair in inches is about .003, or 3×10^{-3}.

The thickness of a red corpuscle in inches is 0.00008, or 8×10^{-5}.

The number of words in a novel of average length is 125 000, or 1.25×10^{5}.

Point out that each number has been written in the following form:

(a number between 1 and 10) x (a power of 10)

It is common practice to write very large and very small numbers this way in scientific notation.

Review the meaning of various powers of ten. Ask students to follow the pattern below. Help them see that it is the pattern that leads to the following definitions:

$$10^{0} = 1$$

$$10^{-1} = .1$$

10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
10 000	1000	100	10	1	.1	.01	.001	.0001

Ask students to develop an algorithm for multiplying a number by 10 and for multiplying by 10^{-1}.

Multiplying by 10 (or 10^1) results in a number with the decimal point one place to the right of the original position.

Given: 2.34

$$10 \times 2.34 = 23.4$$

Multiplying by .1 (or 10^{-1}) results in a number with the decimal point one place to the left of the original position.

Given: 2.34

$$10^{-1} \times 2.34 = 0.234$$

Provide practice in multiplying numbers by various powers of 10.

COMMENTARY AND NOTES

Emphasize the importance of knowing whether or not an answer is reasonable. For instance, errors can arise during calculator computation. The calculator user should be able to know when errors have occurred, because the answers will seem unreasonable.

Ask students to express each product below as a standard numeral.

Example: $10^5 \times 2.3 = 230\ 000$

1. $10^2 \times .0057$
2. $10^{-3} \times 0.68$
3. 186×10^3
4. 27.365×10^{-2}

Answers: 1. 0.57, 2. 0.00068, 3. 186 000, 4. 0.27365

Work with students to express numbers in scientific notation. Use the following models:

	Given Number	Answers
(1)	25 000	(2.5×10^4)
(2)	3.1416	(3.1416×10^0)
(3)	5.0	(5.0×10^0)
(4)	0.0038	(3.8×10^{-3})
(5)	0.060	(6.0×10^{-2})

Note that in (2) and (3) the 10^0 is needed to write the numbers in the correct form. In (5) the zero in 6.0 is needed so that the precision of the given number (0.06<u>0</u> correct to the nearest thousandth) is preserved.

A.E. 24-28, 39f, g

Reasonable Results

Remind students of the introductory computation problem, in which the product of the units digits was checked and the rounding process was used to check the correct number of digits in the answer.

Discuss some of the following techniques that can be used to decide whether the results obtained are reasonable.

Rounding to the nearest thousand

Say: "Round these numbers to the nearest thousand to discover whether the total shown below is a reasonable one."

6 753	7 000*
4 806	5 000*
2 374	2 000
7 541	8 000*
9 738	10 000*
1 462	1 000
5 699	6 000*
38 373	39 000*

*These numbers were rounded up, so 39 000 is probably high.

COMMENTARY AND NOTES

Explain to students that multiply is a commonly misunderstood word. Multiplying does not always produce a larger result, as this case demonstrates.

Students may wish to discuss their recollections about the meaning of the word multiply. They may also wish to cite other terms that have created misunderstanding.

For a closer approximation, add the thousands column: 34

Check the hundreds column for sums of 10 or greater than 10:

$$(7 + 8),\ (5 + 7),\ (4 + 6)$$

This adds about 4 to the thousands column:

$$34 + 4 = 38$$

Placement of the decimal point

To decide whether

$$14.8 \times 3.26 = 482.48$$

is reasonable, place each number between two consecutive tens or units.

Round down		Round up
14	14.8	15
x 3	x 3.26	x 4
42		60
Lower limit		Upper limit

The result must be a two-digit number between 42 and 60. Therefore, 482.48 is incorrect. The result should be 48.248.

Ask the students to try this technique again, with 1.48 x 32.6 = 48.248

1	1.48	2
x 32	x 32.6	x 33
32		66

The result is between 32 and 66. Therefore, 1.48 x 32.6 = 48.248 is a reasonable result.

Computation with fractions

1. To decide whether

$$\frac{7}{8} \times \frac{2}{7} = 1.2 \quad \text{and} \quad \frac{7}{8} \times 464 = 532$$

are reasonable results, you must note that if you take $\frac{7}{8}$ of a number, the result must be smaller than the number. Since $1.2 > \frac{2}{7}$ and $532 > 464$, each of these results is unreasonable.

2. For $\frac{7}{8} \times 466 = 413$

note that 466 is not divisible by 8. Therefore, the product cannot be a whole number.

COMMENTARY AND NOTES

The use of the symbol $\sqrt{\quad}$ may evoke some tension on the part of students who are anxious about math. Play with some numbers that work out evenly. This can help students lessen their anxiety by coping successfully with sample problems. For example, take three numbers and square them. Then discuss the square root of each product.

If $2^2 = 4$, then $\sqrt{4} = 2$

If $5^2 = 25$, then $\sqrt{25} = 5$

If $8^2 = 64$, then $\sqrt{64} = 8$

Also:

If $\sqrt{4} = 2$, then $2^2 = 4$

If $\sqrt{25} = 5$, then $5^2 = 25$

If $\sqrt{64} = 8$, then $8^2 = 64$

Thus, squaring and taking the square root are inverse operations: what one does the other undoes.

3. For $12 \div \frac{7}{8} = 11\frac{5}{7}$

note that if $12 \div \frac{7}{8} = \square$, then $\frac{7}{8} \times \square = 12$

The result must be larger than 12 because $\frac{7}{8}$ of the unknown number is 12.

Computation and estimation

1. Approximating the square root of a number, using the calculator

With students, work through the process of finding the square root of 2.

Start by observing that:

$1 < \sqrt{2} < 2$ ($\sqrt{2}$ lies between 1 and 2)

Try 1.5:

$(1.5)^2 = 2.25$ (1.5 is too large)

Try 1.4:

$(1.4)^2 = 1.96$ (1.4 is too small)

Point out here that $\sqrt{2}$ must therefore be somewhere between 1.4 and 1.5.

$\sqrt{2} = 1.4...$

Try 1.41: $(1.41)^2 = 1.9881$

Try 1.42: $(1.42)^2 = 2.0164$

Try 1.411: $(1.411)^2 = 1.990921$

Try 1.415: $(1.415)^2 = 2.002225$

Thus the thousandth's place must be smaller than 5, and

$\sqrt{2} = 1.41$, correct to the nearest hundredth.

Give students additional practice in using this method to apply the square root.

2. Estimating the quotient in division

Show students how the number of places in the quotient can be estimated. Consider the following division problem:

$72 \overline{)58\ 732}$

$10 \times 72 = 720$

$100 \times 72 = 7\ 200$

$1\ 000 \times 72 = 72\ 000$

Therefore, the quotient lies between 100 and 1000, closer to 1000. The quotient is a three-digit number.

As another example, discuss 1 403.73 ÷ 528.4. The quotient lies between 2 and 3, since:

$2 \times 500 = 1000$

and $3 \times 500 = 1500$

3. Multiplication shortcuts

a. $25 \times 36 = (\frac{1}{4} \times 100) \times 36$

$= \frac{1}{4} \times 3600$

$= 900$

$50 \times 36 = (\frac{1}{2} \times 100) \times 36$

$= \frac{1}{2} \times 3600$

$= 1800$

b. Approximating 15% of a number

$15\% = 10\% + 5\%$, or $10\% + \frac{1}{2} \times 10\%$

Round 15% of \$3.96 to 15% of 4.00:

$10\% \text{ of } 4 = .1 \times 4.00$, or $\frac{1}{10} \times 4.00 = .40$

$\frac{1}{2} \text{ of } 10\% \text{ of } 4 = \frac{1}{2} \times .40$, or $.20$

$10\% \text{ of } 4 + \frac{1}{2} \text{ of } 10\% \text{ of } 4 = .40 + .20 = .60$

Therefore, 15% of \$3.96 is about \$.60.

COMMENTARY AND NOTES

It is appropriate to help students summarize the ideas developed in this module so that they can better appreciate the importance of approximation and estimation.

4. **Fractions**

a. $8\frac{3}{4} \div 2\frac{7}{8}$

<u>Estimation</u>

$9 \div 3 = 3$

<u>Computation</u>

$\frac{35}{4} \div \frac{23}{8} =$

$\frac{35}{{}_{1}4} \times \frac{8^{2}}{23} = \frac{70}{23}$

$= 3\frac{1}{23}$

b. $2\frac{3}{8} \times 4\frac{5}{6}$

<u>Estimation</u>

Lower limit	Upper limit
$2 \times 4 = 8$	$3 \times 5 = 15$

$2 \times 4 < 2\frac{3}{8} \times 4\frac{5}{6} < 3 \times 5$

$8 < 2\frac{3}{8} \times 4\frac{5}{6} < 15$

<u>Computation</u>

$\frac{19}{8} \times \frac{29}{6} = \frac{551}{48}$

$= 11\frac{23}{48}$

c. $.24\overline{)7.92}$

<u>Estimation</u>

.24 is close to .25, and $.25 = \frac{1}{4}$

7.92 rounds to 8

$8 \div \frac{1}{4} = 8 \times 4$, or 32

<u>Computation</u>

```
       33.
.24 )7.92
     7 20
       72
       72
```

A.E. 29-39, 40

STUDENT EVALUATION

An approach to evaluation is provided in the student evaluation section of the Instructor's Handbook, along with ideas for creating a classroom climate that encourages and supports students' achievement. Suggestions are offered for helping students to prepare for quizzes, for providing feedback on performance and for reassessing when necessary.

The ten Approximation and Estimation objectives, below, are accompanied by sample items. Select items according to the content objectives you have covered in the module.

Objective 1. Given a set of numbers, indicate those that express approximate numbers and those that express exact numbers.

a. Which number is an exact number? Answer (1), (2), or (3).

(1) I have 3 children in my family.
(2) There are 10 000 students at Queens College.
(3) I spend $75 a week in the supermarket.

b. In each case indicate whether you interpret the given number as an exact number or an approximate one.

(1) The newspaper indicated that 15 000 Cubans have entered the United States in recent weeks.
(2) A survey was made of the records of 2238 patients.
(3) It would cost industry $1 billion to comply with the new federal environmental protection regulations.

c. Which number is an exact number? Answer (1), (2), or (3).

(1) I have 2 television sets.
(2) My car has gone 17 000 miles.
(3) I spend $100 a semester on books.

d. Which is an exact number? Answer (1), (2), or (3).

(1) I expect 10 people for dinner.
(2) Her waist measures 75 cm.
(3) There are 8 000 000 people in the city of New York.

Objective 2. Place parentheses within a given mathematical expression to indicate a specified order of operations.

a. Insert parentheses so that the resulting statement is true:

$$3 + 4 \cdot 5 = 35$$

b. Insert parentheses to indicate the order in which the expression should be evaluated to produce the required result:

$$8^2 - 37 \div 3 + 6 = 15$$

c. Which is a true statement?

(1) $2 \cdot 3 + 4^2 = 100$

(2) $2 \cdot (3 + 4)^2 = 100$

(3) $(2 \cdot 3 + 4)^2 = 100$

(4) $(2 \cdot 3) + 4^2 = 100$

Objective 3. Cite references for various measurements and other numbers.

a. Cite a reference (from your own experience) for <u>one hour</u>. How long does it take for an hour to pass?

b. Name a part of the human body which can be used as a reference for the measure <u>one centimeter</u>.

Objective 4. Round a given number to a specified degree of accuracy.

a. Round 1497 correct to the nearest hundred.

b. What is the smallest number that rounds to 2030 correct to the nearest ten?

c. What is the smallest number that can be rounded to 1760 to the nearest ten?

d. Which number has a value closer to 8 than to 9?

(1) 8.2 (2) 8.5 (3) 8.7 (4) 8.8

e. Round 63.54 to the nearest tenth.

f. Round 53.461 to the nearest tenth.

g. Round 53.46 to the nearest tenth.

h. Round 71.437 to the nearest tenth.

i. About how many hundreds are there in 53 710?

Objective 5. Compute using the <u>square brackets</u>.

a. Calculate the value of:

$$\left[\frac{5\frac{1}{3}}{2}\right]$$

b. Evaluate:

$$\left[3\frac{1}{2}\right] \times \left[4\frac{1}{3}\right].$$

c. Calculate the value of:

$$\left[\frac{34}{7}\right]$$

Objective 6. Use approximations to compare numbers.

a. If records cost $3.97 each, and I have $20 to spend, <u>about</u> how many records can I buy?

b. If Rhode Island is 1058 square miles and Texas is 267 000 square miles, how do these states compare in size?

c. According to National Geographic Society statistics, the area of Australia is 2 968 000 square miles and the area of North America is 9 390 000 square miles. Use any convenient approximations of these numbers to compare the sizes of these two continents. Show your work and thinking.

Objective 7. Translate written words into decimal numerals and conversely.

a. Express the number 5.3 million as a standard numeral.

b. Write 2.5 million as a standard decimal numeral.

c. Complete the statement: 1 300 000 = _______ million.

Objective 8. Express a given number in scientific notation and conversely.

a. Express the number 53 000 in scientific notation.

b. Express the number 432 000 in scientific notation.

c. Express the number .000678 in scientific notation.

d. Multiply 8.36×10^5 and express the product as a standard numeral.

e. Multiply 36.8×10^6 and express the product as a standard numeral.

f. Multiply 23.8×10^{-4} and express the product as a standard numeral.

g. Which is the correct answer for $.0058 = 5.8 \times$ _____?

(1) 10^2 (2) 10^0 (3) 10^{-2} (4) 10^{-3}

Objective 9. Analyze a calculation for reasonableness of results.

a. Which of the following fractions is greater than $\frac{1}{3}$? Answer (1), (2), or (3).

(1) $\frac{1}{5}$

(2) $\frac{4}{9}$

(3) $\frac{4}{13}$

b. Which of the following fractions is less than $\frac{1}{4}$? Answer (1), (2), or (3).

(1) $\frac{5}{12}$

(2) $\frac{5}{15}$

(3) $\frac{1}{5}$

c. Which of the following numbers cannot be a square? Choose (1), (2), or (3).

(1) 2311

(2) 5766

(3) 1132

d. Explain how you can tell without multiplying that 2444 is not the correct product for 62 x 43.

e. The quotient 1493.8 ÷ 573.26 is between which of the following?

   (1) .02 and .03
   (2) .2 and .3
   (3) 2 and 3
   (4) 20 and 30

Objective 10. Demonstrate the use of various computation shortcuts.

a. The tax included in restaurant charges is 8% of the cost of the food. Estimate the tip if the tip is figured at 15% and the tax on a meal is $.83.

b. Demonstrate a shortcut for multiplying 4 x 37 x 25.

c. If 10% of a number is 74, how much is 5% of the same number?

TEAM

APPROXIMATION AND ESTIMATION

II

INSTRUCTOR'S GUIDE AND SOLUTIONS TO STUDENT EXERCISES

This section of Approximation and Estimation contains solutions for exercises presented in Student Materials and Exercises. The solutions are accompanied by some explanations and suggestions.

The exercises include problems that apply the concepts and problem-solving strategies developed in this module. They can be used as part of instructional activities, as in-class activities for individuals or small groups, as assignments, or as review materials. The asterisk (*) denotes a more challenging problem.

1. Which of the following implies an exact number? Which implies an approximate number?

(Note: Any measurement must be an approximate number. Countable sets usually yield exact numbers, but the context determines whether you assume that the number is approximate or exact. For example, age is approximate, since people usually round it to a whole number, in most cases that of the last birthday.)

(A) = Approximate number
(E) = Exact number

a. Weight of your desk (A)

b. Number of letters in the word floccipaucinihilipilification (evaluating as worthless) (E)

c. Number of chocolates in a box of candy (E)

d. Cost of a bridge toll (E)

e. Your age (A)

f. Speed of Arthur Ashe's tennis serve (A)

g. Number of angles of a triangle (E)

h. Sides to a snowflake (E)

2. A record 420 784 people were in attendance at the World Series in 1959 between the Los Angeles Dodgers and the Chicago White Sox. Is this an exact number? Discuss.

Try to elicit some of the following points:

- People try to give an exact number to impress you.
- It is foolish to try to get an exact figure here.
- Some people could be missed, others counted more than once.
- A figure such as 400 000 or "almost a half million" would convey the information.

61

3. In the local school election, 1563 people voted.

In what circumstances would it be sufficient to approximate this number--that is, to say "about 1500"? When is the exact number needed? Discuss.

In ordinary conversation or for publicity purposes, it would be sufficient to report that "about 15 or 16 hundred" people voted in the school election. However, election results are exact numbers, obtained by automatic tallying on voting machines.

4. Put a check mark in the column that corresponds to the number that is most applicable to each situation.

	More than billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths	Millionths	Billionths	Less than a billionth
Average height of an adult human (in meters) (1 meter ≈ 1 yard)											✓						
Estimated worth of the Du Pont family (in dollars)		(3.5) ✓															
Total U.S. population			✓														
Olympic-record discus throw by Faina Melnik of the U.S.S.R. in 1976 (in feet)									(231'4) ✓								
Total 1979 motor fuel consumption in the U.S. (in gallons)	✓	(122 114 932)															
Number of homes with at least one TV set as of 1981				✓	(77525840)												
Paid circulation of Ms. magazine in 1981						✓											
Speed of a garden snail (in miles per hour)															(.03) ✓		

Here we are not interested in the answers per se but in the "ballpark figures." In other words, we want to know approximately how large or how small the number is.

5. For each fraction/decimal pair, decide whether the decimal representation is an exact value of the fraction or whether it is an approximation. Check with a calculator.

a. $\frac{2}{9}$, .2 (A)

b. $\frac{1}{5}$, .2 (E)

c. $\frac{1}{9}$, .111 (A)

d. $\frac{1}{10}$, .1 (E)

e. $\frac{1}{16}$, .0625 (F)

f. $\frac{1}{4}$, .25 (E)

g. $\frac{1}{11}$, .09 (A)

h. $\frac{1}{6}$, .16 (A)

i. $\frac{1}{7}$, .142857 (A)

j. $\frac{1}{8}$, .125 (E)

6. Evaluate each of the following expressions.

a. $2 + 3^2 =$ 11

b. $(2 + 3)^2 =$ 25

c. $2 \cdot 3^2 =$ 18

d. $(2 \cdot 3)^2 =$ 36

63

7. a. Evaluate the expression $1 + 4 \cdot 3^2$. [37]

   b. Insert parentheses in the expression $1 + 4 \cdot 3^2$ to show how a left-to-right calculator will evaluate the expression.

   $[(1 + 4)3]^2$, and $15^2 =$ [225]

   c. Describe the sequence of keys pressed on that type of calculator in order to obtain the correct value for $1 + 4 \cdot 3^2$.

   3, [x], 3, [x], 4, [+], 1, [=]   Answer: [37]

   d. Show how to use a calculator with a memory in order to obtain the correct value for $\frac{72}{4} - 4 + 42 \times 6$.

   42, [x], 6, [=], [M+], [C], 72, [÷], 4, [=], [M+], [C], [MR], [-] 4, [=]   Answer: [266]

8. a. Insert parentheses as needed to make each of the following statements true.

   (1) $1 + 3 \cdot 2^2 = 16$ $(1 + 3) \cdot (2^2) = 4 \cdot 4$, or 16

   (2) $1 + 3 \cdot 2^2 = 49$ $[1 + (3 \cdot 2)]^2$

   (3) $1 + 3 \cdot 2^2 = 13$ $1 + 3 \cdot 2^2$

   (4) $1 + 3 \cdot 2^2 = 37$ $1 + (3 \cdot 2)^2 = 1 + 6^2$, or 37

   b. Use a calculator to obtain each result in (a) above.

9. Think of a familiar measurement, number, or situation that will help you better visualize the following facts:

   a. The paint job on the Queensborough Bridge will use 60 000 gallons of paint.

   b. The first orchestra record to sell a million copies was the 1935 Boston Pops disc "Jalousie."

   c. The circumference of the earth is 40 million meters.

   d. Water for New York City comes from reservoirs 125 miles upstate.

   e. The temperature is 20°C.

   f. I expect about 25 people to show up.

g. There are 206 bones in the human body.

h. The United States paid Spain $5 000 000 for Florida in 1819.

i. Billie Jean King defeated Bobby Riggs before 30 000 spectators in the tennis match titled "The Battle of the Sexes."

Students will provide reference numbers that work for them. Some answers might include the following:

a. A gallon jug of milk or a one-gallon can of paint.

b. A cubic meter is equal to 1 million cubic centimeters.

c. One meter is a little longer than one yard.

d. It's about 100 miles from __________ to __________.

e. Room temperature is about 20°C.
Body temperature is about 37°C.

f. The average school class has 25 students.

g. There are about 200 pages in a text workbook.

h. Many homes in Beverly Hills now cost one-half million to two million dollars.

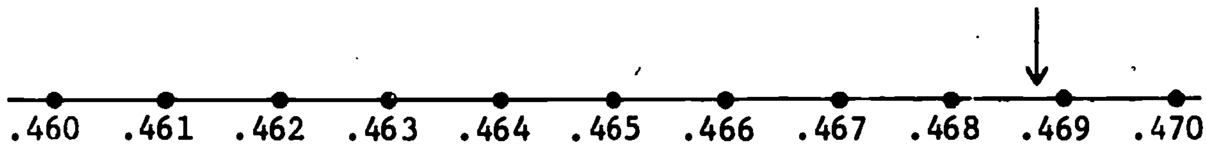
i. The full seating capacity of Yankee Stadium is about 60 000 (57 145).

10. Round each of the following to the nearest:

		Whole Number	Tenth	Hundredth	Thousandth
a.	.5485	1	.5	.55	.549
b.	.4689	0	.5	.47	.469
c.	.6150	1	.6	.62	.615
d.	.5706	1	.6	.57	.571
e.	.2051	0	.2	.21	.205
f.	.9193	1	.9	.92	.919
g.	.9893	1	1.0	.99	.989
h.	6.9445	7	6.9	6.94	6.945
i.	7.4851	7	7.5	7.49	7.485
j.	4.0472	4	4.0	4.05	4.047
k.	3.14187	3	3.1	3.14	3.142

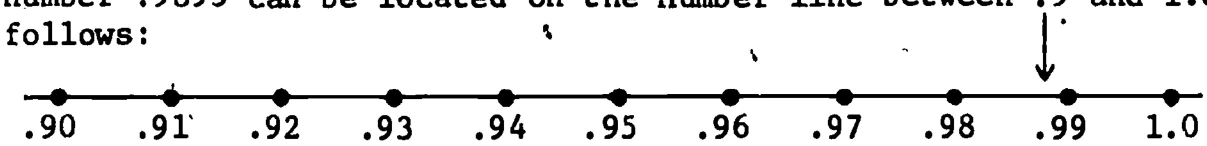
It would be helpful to set up these numbers on a number line. Use major divisions equal to the place value to which the number is to be rounded and smaller divisions equal to one-tenth of the place to which it is to be rounded. Then determine which major division is physically closer to the given point.

For example, here is (b), .4689, rounded to the nearest hundredth:



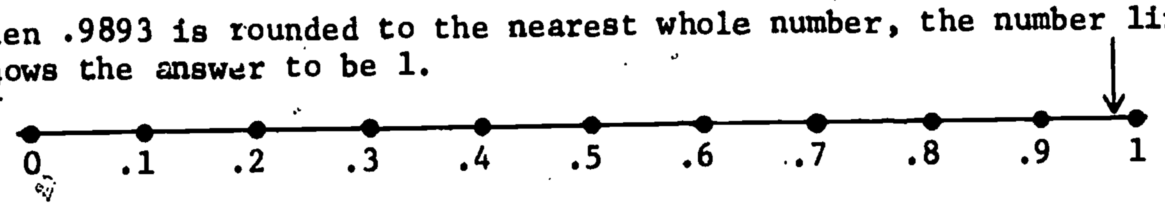
The number .4689 lies somewhere in the neighborhood of the upper arrow between .468 and .469. When the number has been located on the number line, it is obvious that it falls closer to .47 than to .46. Thus the answer, to the nearest hundredth, is .47.

The next example shows (g), .9893, rounded to the nearest tenth. The number .9893 can be located on the number line between .9 and 1.0 as follows:



Rounded to the nearest tenth, .9893 is closer to 1.0 than to .90. The zero in 1.0 indicates that it has been rounded to the nearest tenth rather than to the nearest whole number.

When .9893 is rounded to the nearest whole number, the number line shows the answer to be 1.



11. Round these numbers to the nearest whole number:

(a) 0.8	(b) 2.54	(c) 12.49	(d) 12.999	(e) 499.711
1	3	12	13	500

12. Round these numbers to the nearest 10:

(a) 8	(b) 25	(c) 98	(d) 175	(e) 5897
10	30	100	180	5900

13. Round these numbers to the nearest 100:

(a) [illegible]	(b) 55	(c) 350	(d) 849	(e) 8914
100	100	400	800	8900

14. Round these numbers to the nearest 1000:

(a) 499	(b) 500	(c) 6487	(d) 7092	(e) 7600
0	1000	6000	7000	8000

15. For each of the following rounded numbers, give the range of the original number. (Using a number line may be helpful.)

a. 20 to the nearest ten
b. 120 to the nearest ten
c. 42 650 to the nearest ten
d. 4200 to the nearest ten
e. 2500 to the nearest hundred
f. 4200 to the nearest hundred
g. 0.042 to the nearest thousandth

a. From 15 up to, but not including, 25:

All points on $\overline{AB}$ would round to 20 when rounded to the nearest 10. Point 25 is not included, hence the open circle.

	From	Up to, but not including:
b.	115	125
c.	42 645	42 655
d.	4 195	4 205
e.	2 450	2 550
f.	4 150	4 250
g.	.0415	.0425

16. It helps to use a number line for the questions below.

a. What is the smallest number which, when rounded to the nearest thousand, is 1000?

Find first all the numbers that would round to 1000; then see which is the smallest. To test the reasonableness of your results in these exercises, round your answer to see if it does indeed round to the given number.

500

b. What is the smallest odd number which, when rounded to the nearest thousand, is 1000?

If 500 is the smallest number which, when rounded to the nearest thousand, is 1000, and it is an even number, then 501 is the smallest odd number.

It is a good method first to solve these problems for the largest or smallest values and then to adjust for even or odd.

c. What is the smallest even number which, when rounded to the nearest ten, is 1000?

996

d. What is the largest <u>even</u> whole number which, when rounded to the nearest <u>ten</u>, is 100?

The largest whole number which, when rounded to the nearest <u>ten</u>, is 100 (10 tens), is **104** . It is also even.

e. What is the smallest <u>even</u> number which, when rounded to the nearest <u>ten</u>, is 100?

The smallest whole number which, when rounded to the nearest <u>ten</u>, is 100, is 95. Since we need the smallest even number that satisfies these conditions, the answer is **96** .

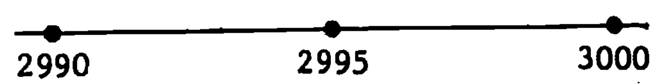
f. What is the smallest <u>even</u> number which, when rounded to the nearest <u>thousand</u>, is 3000?

2500

g. What is the smallest <u>odd</u> number which, when rounded to the nearest <u>ten</u>, is 3000?

2995

In the number 3000, we are looking at 300 tens on a number line. The number we are looking for is 299 tens plus 5 more.



17. From the chart below, give answers to (a), (b), (c), and (d).

a. To what place does it seem that the population figures were rounded? The area figures?

b. Would it be better to have listed these figures rounded to the nearest 100?

c. Round the figure for each area to the nearest 100 000 miles; to the nearest million miles.

d. Round the figure for the highest point on each continent to the nearest 1000 feet.

CONTINENTAL STATISTICS

CONTINENTS	AREA: SQ. MI.	% OF EARTH	POPULATION	% WORLD TOTAL	HIGHEST POINT IN FEET	LOWEST POINT IN FEET
Asia	16 988 000	29.5	2 391 200 000	58.6	Everest 29 028	Dead Sea 1302
Africa	11 506 000	20.0	423 000 000	10.4	Kilimanjaro 19 340	Lake Assal 512
North America	9 390 000	16.3	353 000 000	8.6	McKinley 20 320	Death Valley 282
South America	6 795 000	11.8	223 000 000	5.5	Aconcagua 22 834	Valdes Peninsula 131
Europe	3 745 000	6.5	670 800 000	16.4	Elbrus 18 510	Caspian Sea 92
Australia	2 968 000	5.2	13 900 000	0.3	Kosciusko 7 310	Lake Eyre 52
Antarctica	5 500 000	9.6	-----	---	Vinson Massif 16 860	Unknown

Source: National Geographic Society, Washington, D.C.

(Questions for 17 are repeated, with answers, on following page.)

a. To what place does it seem that the population figures were rounded? The area figures?

To the nearest hundred thousand. To the nearest thousand.

b. Would it be better to have listed these figures rounded to the nearest 100?

For general purposes, all necessary information can be accurately conveyed by these approximations. Especially in the case of population, which is always changing, it is usually not appropriate to round to the nearest hundred.

c. Round the figure for each area to the nearest 100 000 miles; to the nearest million miles.

d. Round the figure for the highest point on each continent to the nearest 1000 feet.

	Area rounded to the nearest 100 000 miles	Area rounded to the nearest million miles	Highest point rounded to the nearest 1000 feet
Asia	17 000 000	17	29
Africa	11 500 000	12	19
North America	9 400 000	9	20
South America	6 800 000	7	23
Europe	3 700 000	4	19
Australia	3 000 000	3	7
Antarctica	5 500 000	6	17

18. The area of the state of Alaska is 566 432 square miles of land, and the area of Rhode Island is 1049 square miles.

How do the areas of the following countries compare with that of Alaska? With that of Rhode Island?

Country	Approximate Area
Egypt	400 thousand square miles
Peru	500 thousand square miles
France	200 thousand square miles
Jamaica	4 thousand square miles
Liechtenstein	Sixty-one square miles
Luxembourg	1 thousand square miles
Qatar	4 thousand square miles
Samoa	1 thousand square miles

First, students must decide how to round the areas of Alaska and Rhode Island to make comparisons. When each area is rounded to one significant digit, comparisons are easy to make. Make certain that your students can write all of the approximate numbers in standard decimal notation.

The comparisons are approximately as follows:

Country	Alaska	Rhode Island
Egypt	$\frac{2}{3}$ as large	400 times as large
Peru	Almost the same	500 times as large
France	$\frac{1}{3}$ as large	200 times as large
Jamaica	Less than $\frac{1}{100}$ as large	4 times as large
Liechtenstein	(Very small!)	$\frac{1}{20}$ as large
Luxembourg	$\frac{1}{600}$ as large	The same
Qatar	$\frac{1}{150}$ as large	4 times as large
Samoa	$\frac{1}{600}$ as large	The same

19. The contiguous United States contains 3 million square miles. India contains 1 221 880 square miles. Round the area of India and compare its size with that of the United States.

The area of India rounds to 1 million; thus, the United States is about 3 times as large as India.

20. Round the population of Burma (19 856 000) and compare it with that of Texas (8 million).

The population of Burma rounds to 20 million; thus, Burma's population is a little more than double the population of Texas.

21. Square brackets: Find the values of (a) through (p) in the exercises below.

Examples: [2] = 2; [2.1] = 2; [2.94] = 2; and [2.999] = 2, but [3] = 3

a. $\left[\frac{101}{100}\right] = 1$

b. $\left[\frac{100}{101}\right] = 0$

c. $\left[\frac{898}{300}\right] = 2$

This can be likened to a division example: $300\overline{)898}$

Ask: "What is the largest whole number that we can place in the quotient?" Point out that the procedure is like estimating the first (leftmost) place in long division.

d. $\left[\frac{558}{100}\right] = 5$

e. $\left[\frac{22}{7}\right] = 3$

f. $\left[\frac{1}{2} + \frac{3}{5}\right] = 1$

Compare $\frac{1}{2}$ and $\frac{3}{5}$. Since $\frac{2\frac{1}{2}}{5} = \frac{1}{2}$, $\frac{3}{5} > \frac{1}{2}$. Therefore, $\frac{1}{2} + \frac{3}{5} > 1$.

g. $[1.5 + 2.7] = 4$

Since .7 > .5, .5 + .7 > 1.

h. $[1.5] + [2.7] = 3$

i. $[7.2 \times 3] = 21$

7.2 x 3 = (7 + .2) x 3, or 21 + 3 x .2. Since 3 x .2 < 1, 3 x .2 contributes 0 to the result in the brackets.

j. $\left[7.2\right] \times \left[3\right] = 21$

k. $\left[2\frac{1}{2}\right] \times \left[5\frac{1}{2}\right] = 13$

l. $\left[2\frac{1}{2}\right] \times \left[5\frac{1}{2}\right] = 10$

m. $\left[\frac{12}{6} \times \frac{3}{4}\right] = 1$

n. $\left[\frac{12}{6}\right] \times \left[\frac{3}{4}\right] = 0$

o. $\left[\frac{36\ 712}{105}\right] \times \left[\frac{7}{8}\right] = 0$

p. $\left[\frac{[6.123 + 4.321]}{[0.986] + [1.105]}\right] = 10$

22. The following items are being dropped into my shopping basket. With only $10.00 in my wallet (and my checkbook at home), can I cover it all?

	Approximation
2 jars apple juice at 99¢ each	$ 2.00
1 head of lettuce, 69¢	.70
2 half-gallon containers of milk at $1.07 each	2.00
4 lb. bananas, $1.00	1.00
1 box diapers, $1.69	1.70
1 salami, $2.39	2.50
1 white bread, 45¢	.50
4 cans tuna fish at 99¢ each	4.00
1/2 gallon ice cream, 99¢	1.00
	$15.40

It is a good idea to round up to be certain that you have enough money. In most cases, rounding to ten cents will work well. In the given example, we have rounded 45¢ to the nearest half-dollar.

Even by rounding as shown, the total exceeds $10.00. I probably don't have enough money. The actual sum is $15.29.

23. For each of the following situations, decide whether it is appropriate to round up or to round down. Determine the result.

a. If a can of tennis balls costs \$3.00, how many cans can be purchased with a ten-dollar bill?

$$\text{Round down: } [10 \div 3] = 3$$

b. If first-class mail costs 20¢ for the first ounce and 17¢ for each additional ounce (or fraction thereof), how much would it cost to send a letter weighing 7.2 ounces?

Round 7.2 up to 8 because any fraction of an ounce is charged the same as a whole ounce.

$$20 + 7 \cdot 17 = 20 + 119, \text{ or } \$1.39$$

24. Examine the following approximate distances of planets from Earth.

	Maximum	Minimum
Mercury	136 000 000	50 000 000
Venus	161 000 000	25 000 000
Mars	248 000 000	35 000 000
Jupiter	600 000 000	368 000 000
Saturn	1 031 000 000	745 000 000
Uranus	1 953 000 000	1 606 000 000
Neptune	2 915 000 000	2 667 000 000
Pluto	4 644 000 000	2 663 000 000

a. To which place is each figure rounded?

These numbers seem to be rounded to the nearest million. Ask the students when more precise figures might be needed.

b. Express each of these distances in scientific notation.

Remind students that scientific notation is a compact way of writing numbers and that it aids in discovering relationships between them.

	Maximum	Minimum
Mercury	1.36×10^8	5.0×10^7
Venus	1.61×10^8	2.5×10^7
Mars	2.48×10^8	3.5×10^7
Jupiter	6.00×10^8	3.68×10^8
Saturn	1.031×10^9	7.45×10^8
Uranus	1.953×10^9	1.606×10^9
Neptune	2.915×10^9	2.667×10^9
Pluto	4.644×10^9	2.663×10^9

25. Express each of the following in scientific notation.

a. 0.000123 = 1.23×10^{-4}

b. 0.00123 = 1.23×10^{-3}

c. 0.01230 = 1.230×10^{-2}

d. 0.1230 = 1.230×10^{-1}

e. 1.230 = 1.230×10^0

f. 12.3 = 1.23×10^1

g. 123.0 = 1.230×10^2

h. 1230.0 = 1.2300×10^3

i. 12 300.000 = 1.2300000×10^4

j. 123 000. = 1.23000×10^5

k. 1 230 000 = 1.23×10^6

l. 12 300 000 = 1.23×10^7

m. 1 230 000 000 000 = 1.23×10^{12}

Ask: "What do all of these answers have in common?"

In each case, the three non-zero digits (1, 2, 3) are the same. Note therefore that all of the answers will take the form:

$$1.23 \times 10^?$$

26. Which of the following are not expressed in scientific notation? If a number is not expressed in scientific notation, express it in that form.

a. 78.5×10^4 = 7.85×10^5

b. 78×10^4 = 7.8×10^5

c. 3.5×10

d. 3×10

e. $.03 \times 10^4$ = 3.0×10^2

f. 4.25 = 4.25×10^0

27. Write each of the following in standard decimal notation.

a. 1.26×10^3 = [1260]

b. 1×10^{-4} = [.0001]

c. 9.9×10^{-3} = [.0099]

d. 1.23×10^5 = [123 000]

e. 1.0×10^2 = [100]

28. Express the following numbers in scientific notation, showing all significant digits.

a. Mass of the earth: 6 588 000 000 000 000 000 000 tons.

[6.588×10^{21}]

b. Proxima Centauri, the star nearest to us (excluding our sun), is 25 000 000 000 000 miles away.

[2.5×10^{13}]

c. The remotest heavenly body clearly visible to the naked eye is the Great Galaxy in Andromeda--13 000 000 000 000 000 000 miles from earth.

[1.3×10^{19}]

d. The largest viruses, the pox viruses, measure 0.0003 of a millimeter.

[$3. \times 10^{-4}$]

29. The restaurant bill for two people looked like this:

2 tropical fruit salads	at $3.90	$7.80
2 coffees	at .50	1.00
		$8.80

Mentally estimate the tax at 7% and a 15% tip. How much, approximately, is the total? Everything is rounded <u>up</u> to be on the safe side.

Total of $8.80 becomes: $ 9.00

Tip of 15%: (15% = 10% + 5%)

10% of 9.00 = .90

5% of 9.00 = .45 ($\frac{1}{2}$ of .90, since

1.35 5% = $\frac{1}{2}$ of 10%)

Tip is about: 1.40

Tax: 7% is about $\frac{1}{2}$ of 15%

$\frac{1}{2}$ of $1.40 .70

Total bill: $11.10, or $11.00

This kind of calculation is very <u>rough</u> and leads to paying more than is necessary.

If the bill includes the tax

$ 8.80
.62 (tax at 7%)
$ 9.42

then double the tax for the tip, since 15% 2 x 7%

$ 8.80
.62 tax
1.24 tip
$10.66

30. Part of the left-hand number is missing. It is known to be in the <u>hundreds</u>. Which of the following is the only number that could possibly be the correct product?

1[redacted]8 x 4 = ?

(a) 457 (b) 472 (c) 4072 (d) 102 (e) 476

Give the students time to discover as many clues as possible. Ask: "How many digits are really covered up?" Since the left-hand number is known to be in the hundreds, there is just one digit missing between the 1 and the 8. Therefore, the product should be a number in the hundreds. This eliminates (c).

The units digit of the result must be 2, since the product of the units digits, 8 and 4, is 32. This eliminates (a) and (e).

When multiplying a given positive number by another positive number, the product must be greater in value than the original number (4 x 100 = 400). This eliminates (d).

Thus, (b) is the only possible product.

You can divide by 4 to determine the value of the missing digit.

31. In the figure given, a card is covering part of the left-hand number, which is known to be in the hundred thousands. Which of the following is the only number that could possibly be the product?

5 63 x 21 728 = ?

(a) 1 107 130 464

(b) 1 107 130 466

(c) 11 076 130 444

(d) 11 076 130 464

(e) 11 076 130 466

The units digit must be 4, since the product of the units digits, 3 and 8, is 24. This eliminates (b) and (e).

Because the product is known to be in the hundred thousands, it must be at least 500 000 x 21 728, or 10 864 000 000. This eliminates (a).

Since 21 728 x 63 (the last two digits given in the covered number) yields 64 as the last two digits in the product, (c) is eliminated and the answer is (d).

32. Are the following products reasonable? If not, explain why not.

a. 308 x 4 = 1232 — Yes.

b. 308 x 4 = 232 — No. 4 x 300 = 1200, so 232 is too small.

c. 215 x 10 = 21 500 — No. When multiplying by 10, one zero is annexed to the right of the original number. The number 21 500 has two zeros.

d. 24 x 24 = 625 — No. Since the units digit of each factor is 4, the units digit of the product must be 6 (4 x 4 = 16).

33. Which of the following is a good approximation of 29.95 x .6666666? Why?

a. 18 000 000 000

b. 20

c. 2000

d. 20 000 000

e. 180

Answer: b

First, .6666666 is approximately equal to $\frac{2}{3}$.

Second, 29.95 can be approximated as 30; thus, the approximate result is $30 \times \frac{2}{3} = 20$.

Answers can be judged individually for reasonableness. The number .6666666 is a value less than 1 and greater than 0. When we multiply any whole number by a value less than 1 and greater than 0, the result must be smaller than the number we multiplied. Therefore, the result must be less than 29.95.

34. If 16 x 82 = 1312, find, without multiplying, the value of the following:

a. 15 x 82

b. 17 x 82

a. Note that 15 x 82 represents one less 82 than 16 x 82. In other words, (16 x 82) - (1 x 82) = 15 x 82. Therefore, if 16 x 82 = 1312, then 15 x 82 = 1312 - 82, or 1230

b. Similarly, 17 x 82 represents one more 82 than 16 x 82. Therefore, if 16 x 82 = 1312, then 17 x 82 = 1312 + 82, or 1394

35. Explain why each of the following calculations is "obviously" wrong.

a. $25.111 \times 4.92 = 1235.4612$

b. $62 \times 43 = 2444$

c. $56 \times 36 = 2019$

d. $4235 + 2628 + 3361 = 9424$

e. $\frac{15}{16} \times 35 = 70$

f. $32.3 \times 0.3 = 94.9$

g. $(4 + 3\frac{1}{7}) \div 5 = 2\frac{6}{7}$

h. $(3\frac{1}{3} + 2) \times 4\frac{1}{2} = 19\frac{1}{2}$

i. $1.52 + 1.67 = 2.19$

j. $1.07 + 1.92 = 2$

k. 723 is a square number

Solutions

a. $25 \times 5 = 125$, so 1235.4612 is too large.

b. The product of the units digits, 3 and 2, is 6, not 4.

c. The units digit of the product must be 6, not 9, since $6 \times 6 = 36$.

d. Even when all the figures are rounded down, the sum is greater than 9424 ($4200 + 2600 + 3300 = 10\ 100$).

e. If a whole number is multiplied by a positive fraction, the product must be less than the original whole number. But 70 is greater than 35.

f. $0.3 < 1$, so the product cannot be greater than 32.3.

g. The value in parentheses must have a sum of at least 10 if it is to yield a quotient of 2 when divided by 5.

h. The number $19\frac{1}{2}$ is too small for the product:

$(3\frac{1}{3} + 2) > 5$

$4\frac{1}{2} > 4$

$4 \times 5 = 20$, so $(3\frac{1}{3} + 2) \times 4\frac{1}{2}$ is definitely greater than 20.

i. The number 2.19 is too small:

1.52 > 1.5

1.67 > 1.5

Therefore, 1.52 + 1.67 > 1.5 + 1.5, and 1.5 + 1.5 = 3. The sum must therefore be greater than 3.

j. The number 2 is too small:

1.07 > 1

1.92 > 1

Therefore, 1.07 + 1.92 > 1 + 1. The sum must therefore be greater than 2.

k. No number exists that, when multiplied by itself, will yield a 3 in the units column.

36. A student multiplied 567 x 426. Then she multiplied 567 x 427 and subtracted the smaller product from the larger product. She was absolutely certain about the accuracy of her multiplication. Without multiplying, find the difference in the products.

a.
```
  567
x 426
-----
```

b.
```
  567
x 427
-----
```

The difference in the products can be found this way:

(567 x 427) - (567 x 426) = 567 x (427 - 426), by the distributive property of multiplication

= 567 x 1

= **567**

In example (b) we are multiplying 567 one more time than in example (a). Therefore, the product in (b) will be 567 more than the product in (a).

37. Use rounding to estimate each answer.

(a) 237 + 425 = **662**

(b) 15 720 + 20 590 = **36 310**

(c) 68 x 7 = **476**

(d) 12.3 x 4.78 = **58.794**

(e) 298 - 60 = **238**

(f) 5762 - 3124 = **2638**

(g) $\frac{328}{115} \approx$ **2.8521739**

(h) $\frac{420}{25} =$ **16.8**

The answers above are those determined on a calculator. The rounded estimates should be within a reasonable range of the answers obtained.

For example, in (d), the rounding of

$$\begin{array}{r} 12.3 \\ \times\ 4.78 \\ \hline \end{array} \quad \text{to} \quad \begin{array}{r} 12 \\ \times\ 5 \\ \hline \end{array}$$

yields an approximate product of 60.

In (g), the rounding of the numerator and denominator

$$\frac{328}{115} \quad \text{to} \quad \frac{300}{100}$$

gives an approximate value of 3.

38. a. If 20 + 60 = 80, then 23 + 60 is (greater, less) than 80. **Greater**

b. Since 90 - 30 = 60, 90 - 27 is (greater, less) than 60. **Greater**

c. Since 300 + 500 = 800, which of the following are less than 800?

315 + 500	**295 + 500**
275 + 495	395 + 505

Each addend is compared with 300 and 500, respectively.

d. Since 5 x 80 = 400, which of the following is greater than 400?

5 x 8	**[5 x 89]**
5 x 80	5 x 60

e. Since 3500 ÷ 70 = 50, is 3500 ÷ 75 greater or less than 50?
[Less than 50]

39. First estimate. Then use your calculator to perform the calculation.

a. 5812 + 4406

b. 864 - 231

c. 479 x 24

d. 8118 ÷ 41

e. In what year will there have been 1 000 000 days in the Christian era?

f. Determine the distance in miles that light travels in one year (a light-year) if it travels at 186 000 miles per second. Use scientific notation in your computation.

g. How many years does it take to live 1 000 000 000 seconds?

Exact answers:

a. 10 218

b. 633

c. 11 496

d. 198

The purpose of questions (e), (f), and (g) is to give students references so that they will have a better sense of the magnitude of these very large numbers.

e. 2739

f. 5 865 696 000 000

(It will be necessary to use scientific notation in this calculation since most calculators will display a maximum of eight digits.)

g. There are 31 536 000 seconds in one year.

Therefore, 1 000 000 000 seconds is approximately equal to 30 years because:

$$\frac{1\ 000\ 000\ 000}{31\ 536\ 000} \approx 31.70979$$

*40. In the following crypto-numbers, each letter represents a different digit, from 0 to 9. Use your knowledge of computation to figure them out.

a.	SEND	9 567
	+ MORE	+ 1 085
	MONEY	10 652
b.	DEFER	10 806
	- DUTY	- 1 574
	NOGO	9 232

Help students get started by pointing out that in (a), the M must be 1, since S + M yield a 2-digit number. Now, S has to be large enough to yield a 2-digit number when it is added to 1, or 2 if 1 is carried from the preceding column. Thus, S must be 8 or 9. Ask students to continue in the same manner and to keep track of their reasoning.

41. The data given below appeared in recent issues of a large city newspaper.

a. Decide whether each number represents an exact number or an approximate one. Support your viewpoint.

b. If a number is expressed in words, rewrite it in standard decimal notation.

c. Study an issue of any newspaper. List five numbers that are exact, five that are approximate, and five that are expressed in words.

(1) A $120 million oceanfront hotel is planned for Atlantic City. A 1000-room project is planned.

(2) West Germany reported a $919 million trade surplus in June.

(3) The House approved an amendment by a 232 to 187 vote.

---

*The asterisk signifies a more challenging exercise.

(4) One corporation reported earnings of $86.5 million last year.

(5) An oil company reported it sold the equivalent of 18 billion gallons of petroleum products.

(6) The announcement of new layoffs raises to 44 100 the number of auto workers laid off in recent weeks.

(7) The company had originally predicted that sales would reach the $11.5 million-unit mark.

(8) The total strength of the Army, Navy, Air Force, and Marines stood at 2 018 299 at the end of the month.

(9) Nicaragua has allowed 199 people to leave.

(10) The company has 5000 accounts in the metropolitan area.

(11) The number of seniors who failed to receive diplomas is 13 032.

(12) The huge spill ran 45 miles past the Arizona border.

(13) Only 18 000 trash buckets remain on the streets.

(14) Between 1975 and 1977, New York City bought 1 429 500 tons of asphalt.

(15) The corporation announced that 552 of its products might have defective transistors.

a. In general, large sums of money are approximations, as in (1), (2), and (4).

When the number is quite large and the unit is actually written out, the number was probably rounded to that unit, as in (5) and (7).

In (1), 1000 rooms is probably an approximation, since the hotel has not yet been built.

Exact numbers are probably given in (3), (8), (9), (11), and (15).

A case can be made for both sides in (6), (10), (12), (13), and (14), depending upon the source of the information and its purpose. For example, in (10), 5000 accounts might have been rounded to the nearest thousand, since the number would convey enough information rounded this way. On the other hand, if the IRS had asked for the information, 5000 would probably be the <u>exact</u> number of accounts reported.

b. Numbers rewritten in standard decimal notation are as follows:

(1) $120 000 000

(2) $919 000 000

(4) $86 500 000

(5) 18 000 000 000

(7) 11 500 000

Notice that in (4), .5 million is equivalent to $\frac{1}{2}$ million, which is equivalent to 500 000.

c. Answers will vary. Give students an opportunity to share them.

42. a. Find the value of each of the following.

(1) [7.4 + .5] = 7

(2) [7.5 + .5] = 8

(3) [7.6 + .5] = 8

(4) [8.2 + .5] = 8

(5) [3.9 + .5] = 4

b. Round the following numbers to the nearest whole number.

(1) 7.4 = 7

(2) 7.5 = 8

(3) 7.6 = 8

(4) 8.2 = 8

(5) 3.9 = 4

c. How do your answers in (a) compare with those in (b)?

They are the same.

d. Explain why the answers in (a) and (b) are so related.

They are the same because of the definition of rounding to the nearest whole number:

> If the decimal part of the number is greater than or equal to .5, the number rounds up to the next highest whole number.
>
> If the decimal part of the number is less than .5, the number rounds down to the whole-number part, and the decimal part is truncated (cut off).

The effect of (a) is to compare the decimal part of the number with .5 and to perform the truncation that results in the rounded number.

TEAM

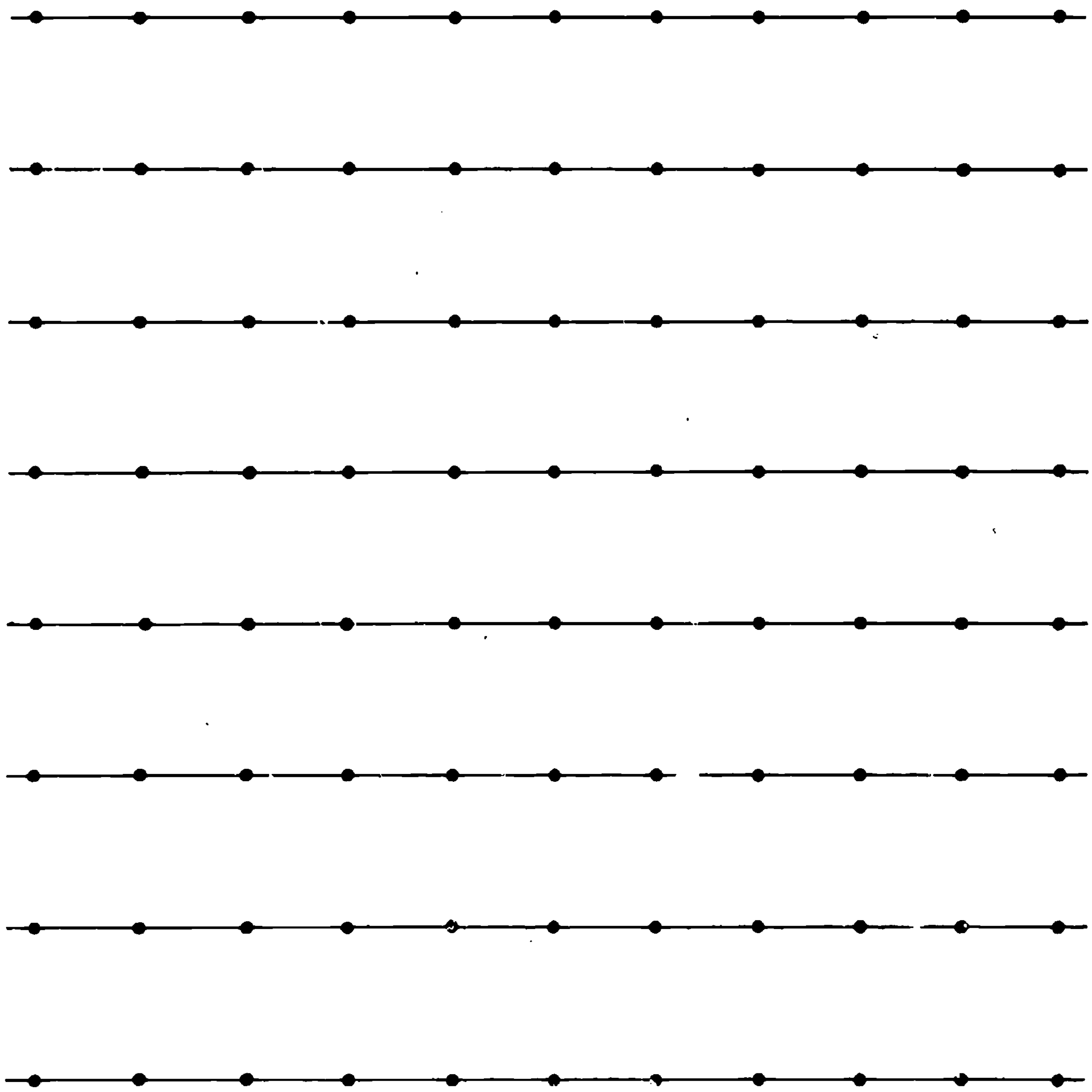
APPROXIMATION AND ESTIMATION

III

STUDENT MATERIALS AND EXERCISES

STUDENT MATERIALS

These number lines are provided for use in class activities and student exercises.



STUDENT EXERCISES

1. Which of the following implies an <u>exact</u> number? Which implies an <u>approximate</u> number?

(A) = Approximate number
(E) = Exact number

   a. Weight of your desk

   b. Number of letters in the word <u>floccipaucinihilipilification</u> (evaluating as worthless)

   c. Number of chocolates in a box of candy

   d. Cost of a bridge toll

   e. Your age

   f. Speed of Arthur Ashe's tennis serve

   g. Number of angles of a triangle

   h. Sides to a snowflake

2. A record 420 784 people were in attendance at the World Series in 1959 between the Los Ang les Dodgers and the Chicago White Sox. Is this an exact number? Discuss.

3. In the local school election, 1563 people voted.

   In what circumstances would it be sufficient to approximate this number--that is, to say "about 1500"? When is the exact number needed? Discuss.

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4. Put a check mark in the column that corresponds to the number that is most applicable to each situation.

	More than billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths	Millionths	Billionths	Less than a billionth
Average height of an adult human (in meters) (1 meter ≈ 1 yard)																	
Estimated worth of the Du Pont family (in dollars)																	
Total U.S. population																	
Olympic-record discus throw by Faina Meinik of the U.S.S.R. in 1976 (in feet)																	
Total 1979 motor fuel consumption in the U.S. (in gallons)																	
Number of homes with at least one TV set as of 1981																	
Paid circulation of Ms. magazine in 1981																	
Speed of a garden snail (in miles per hour)																	

Here we are not interested in the answers per se but in the "ballpark figures." In other words, we want to know approximately how large or how small the number is.

5. For each fraction/decimal pair, decide whether the decimal representation is an exact value of the fraction or whether it is an approximation. Check with a calculator.

a. $\frac{2}{9}$, .2

b. $\frac{1}{5}$, .2

c. $\frac{1}{9}$, .111

d. $\frac{1}{10}$, .1

e. $\frac{1}{16}$, .0625

f. $\frac{1}{4}$, .25

g. $\frac{1}{11}$, .09

h. $\frac{1}{6}$, .16

i. $\frac{1}{7}$, .142857

j. $\frac{1}{8}$, .125

6. Evaluate each of the following expressions.

a. $2 + 3^2$

b. $(2 + 3)^2$

c. $2 \cdot 3^2$

d. $(2 \cdot 3)^2$

7. a. Evaluate the expression $1 + 4 \cdot 3^2$.

b. Insert parentheses in the expression $1 + 4 \cdot 3^2$ to show how a left-to-right calculator will evaluate the expression.

c. Describe the sequence of keys pressed on that type of calculator in order to obtain the correct value for $1 + 4 \cdot 3^2$.

d. Show how to use a calculator with a memory in order to obtain the correct value for $\frac{72}{4} - 4 + 42 \times 6$.

8. a. Insert parentheses as needed to make each of the following statements true.

(1) $1 + 3 \cdot 2^2 = 16$

(2) $1 + 3 \cdot 2^2 = 49$

(3) $1 + 3 \cdot 2^2 = 13$

(4) $1 + 3 \cdot 2^2 = 37$

b. Use a calculator to obtain each result in (a) above.

9. Think of a familiar measurement, number, or situation that will help you better visualize the following facts:

a. The paint job on the Queensborough Bridge will use 60 000 gallons of paint.

b. The first orchestra record to sell a million copies was the 1935 Boston Pops disc "Jalousie."

c. The circumference of the earth is 40 million meters.

d. Water for New York City comes from reservoirs 125 miles upstate.

e. The temperature is 20°C.

f. I expect about 25 people to show up.

g. There are 206 bones in the human body.

h. The United States paid Spain $5 000 000 for Florida in 1819.

i. Billie Jean King defeated Bobby Riggs before 30 000 spectators in the tennis match titled "The Battle of the Sexes."

10. Round each of the following to the nearest:

	Whole Number	Tenth	Hundredth	Thousandth
a. .5485				
b. .4689				
c. .6150				
d. .5706				
e. .2051				
f. .9193				
g. .9893				
h. 6.9445				
i. 7.4851				
j. 4.0472				
k. 3.14187				

11. Round these numbers to the nearest whole number:

(a) 0.8 (b) 2.54 (c) 12.49 (d) 12.999 (e) 499.711

12. Round these numbers to the nearest 10:

(a) 8 (b) 25 (c) 98 (d) 175 (e) 5897

13. Round these numbers to the nearest 100:

(a) 98 (b) 55 (c) 350 (d) 849 (e) 8914

14. Round these numbers to the nearest 1000:

(a) 499 (b) 500 (c) 6487 (d) 7092 (e) 7600

15. For each of the following rounded numbers, give the range of the original number. (Using a number line may be helpful.)

   a. 20 to the nearest ten

   b. 120 to the nearest ten

   c. 42 650 to the nearest ten

   d. 4200 to the nearest ten

   e. 2500 to the nearest hundred

   f. 4200 to the nearest hundred

   g. 0.042 to the nearest thousandth

16. It helps to use a number line for the questions below.

   a. What is the smallest number which, when rounded to the nearest <u>thousand</u>, is 1000?

   b. What is the smallest <u>odd</u> number which, when rounded to the nearest <u>thousand</u>, is 1000?

   c. What is the smallest <u>even</u> number which, when rounded to the nearest <u>ten</u>, is 1000?

   d. What is the largest <u>even</u> whole number which, when rounded to the nearest <u>ten</u>, is 100?

   e. What is the smallest <u>even</u> number which, when rounded to the nearest <u>ten</u>, is 100?

   f. What is the smallest <u>even</u> number which, when rounded to the nearest <u>thousand</u>, is 3000?

   g. What is the smallest <u>odd</u> number which, when rounded to the nearest <u>ten</u>, is 3000?

17. From the chart below, give answers to (a), (b), (c), and (d).

a. To what place does it seem that the population figures were rounded? The area figures?

b. Would it be better to have listed these figures rounded to the nearest 100?

c. Round the figure for each area to the nearest 100 000 miles; to the nearest million miles.

d. Round the figure for the highest point on each continent to the nearest 1000 feet.

CONTINENTAL STATISTICS

CONTINENTS	AREA: SQ. MI.	% OF EARTH	POPULATION	% WORLD TOTAL	HIGHEST POINT IN FEET	LOWEST POINT IN FEET
Asia	16 988 000	29.5	2 391 200 000	58.6	Everest 29 028	Dead Sea 1302
Africa	11 506 000	20.0	423 000 000	10.4	Kilimanjaro 19 340	Lake Assal 512
North America	9 390 000	16.3	353 000 000	8.6	McKinley 20 320	Death Valley 282
South America	6 795 000	11.8	223 000 000	5.5	Aconcagua 22 834	Valdes Peninsula 131
Europe	3 745 000	6.5	670 800 000	16.4	Elbrus 18 510	Caspian Sea 92
Australia	2 968 000	5.2	13 900 000	0.3	Kosciusko 7 310	Lake Eyre 52
Antarctica	5 500 000	9.6	-----	---	Vinson Massif 16 860	Unknown

Source: National Geographic Society, Washington, D.C.

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18. The area of the state of Alaska is 566 432 square miles of land, and the area of Rhode Island is 1049 square miles.

How do the areas of the following countries compare with that of Alaska? With that of Rhode Island?

Country	Approximate Area
Egypt	400 thousand square miles
Peru	500 thousand square miles
France	200 thousand square miles
Jamaica	4 thousand square miles
Liechtenstein	Sixty-one square miles
Luxembourg	1 thousand square miles
Qatar	4 thousand square miles
Samoa	1 thousand square miles

19. The contiguous United States contains 3 million square miles. India contains 1 221 880 square miles. Round the area of India and compare its size with that of the United States.

20. Round the population of Burma (19 856 000) and compare it with that of Texas (8 million).

21. Square brackets: Find the values of (a) through (p) in the exercises below.

Examples: [2] = 2; [2.1] = 2; [2.94] = 2; and [2.999] = 2, but [3] = 3

a. $\left[\frac{101}{100}\right]$

b. $\left[\frac{100}{101}\right]$

c. $\left[\frac{898}{300}\right]$

d. $\left[\frac{558}{100}\right]$

e. $\left[\frac{22}{7}\right]$

f. $\left[\frac{1}{2} + \frac{3}{5}\right]$

g. $\left[1.5 + 2.7\right]$

h. $\left[1.5 \quad + \quad 2.7\right]$

i. $\left[7.2 \times 3\right]$

j. $\left[7.2 \quad \times \quad 3\right]$

k. $\left[2\frac{1}{2} \times 5\frac{1}{2}\right]$

l. $\left[2\frac{1}{2} \quad \times \quad 5\frac{1}{2}\right]$

m. $\left[\frac{12}{6} \times \frac{3}{4}\right]$

n. $\left[\frac{12}{6} \quad \times \quad \frac{3}{4}\right]$

o. $\left[\frac{36\ 712}{105} \quad \times \quad \frac{7}{8}\right]$

p. $\left[\frac{[6.123 \quad + \quad 4.321]}{[0.986] + [1.105]}\right]$

22. The following items are being dropped into my shopping basket. With only $10.00 in my wallet (and my checkbook at home), can I cover it all?

<u>Approximation</u>

2 jars apple juice at 99¢ each

1 head of lettuce, 69¢

2 half-gallon containers of milk at $1.07 each

4 lb. bananas, $1.00

1 box diapers, $1.69

1 salami, $2.39

1 white bread, 45¢

4 cans tuna fish at 99¢ each

1/2 gallon ice cream, 99¢

23. For each of the following situations, decide whether it is appropriate to round up or to round down. Determine the result.

a. If a can of tennis balls costs $3.00, how many cans can be purchased with a ten-dollar bill?

b. If first-class mail costs 20¢ for the first ounce and 17¢ for each additional ounce (or fraction thereof), how much would it cost to send a letter weighing 7.2 ounces?

24. Examine the following approximate distances of planets from Earth.

	Maximum	Minimum
Mercury	136 000 000	50 000 000
Venus	161 000 000	25 000 000
Mars	248 000 000	35 000 000
Jupiter	600 000 000	368 000 000
Saturn	1 031 000 000	745 000 000
Uranus	1 953 000 000	1 606 000 000
Neptune	2 915 000 000	2 667 000 000
Pluto	4 644 000 000	2 663 000 000

a. To which place is each figure rounded?

b. Express each of these distances in scientific notation.

25. Express each of the following in scientific notation.

a. 0.000123

b. 0.00123

c. 0.01230

d. 0.1230

e. 1.230

f. 12.3

g. 123.0

h. 1230.0

i. 12 300.000

j. 123 000

k. 1 230 000

l. 12 300 000

m. 1 230 000 000 000

26. Which of the following are not expressed in scientific notation? If a number is not expressed in scientific notation, express it in that form.

a. 78.5×10^4

b. 78×10^4

c. 3.5×10

d. 3×10

e. $.03 \times 10^4$

f. 4.25

27. Write each of the following in standard decimal notation.

a. 1.26×10^3

b. 1×10^{-4}

c. 9.9×10^{-3}

d. 1.23×10^5

e. 1.0×10^2

28. Express the following numbers in scientific notation, showing all significant digits.

a. Mass of the earth: 6 588 000 000 000 000 000 000 tons.

b. Proxima Centauri, the star nearest to us (excluding our sun), is 25 000 000 000 000 miles away.

c. The remotest heavenly body clearly visible to the naked eye is the Great Galaxy in Andromeda--13 000 000 000 000 000 000 miles from earth.

d. The largest viruses, the pox viruses, measure 0.0003 of a millimeter.

29. The restaurant bill for two people looked like this:

2 tropical fruit salads	at $3.90	$7.80
2 coffees	at .50	1.00
		$8.80

Mentally estimate the tax at 7% and a 15% tip. How much, approximately, is the total?

30. Part of the left-hand number is missing. It is known to be in the hundreds. Which of the following is the only number that could possibly be the correct product?

1 8 x 4 = ?

(a) 457 (b) 472 (c) 4072 (d) 102 (e) 476

31. In the figure given, a card is covering part of the left-hand number, which is known to be in the <u>hundred thousands</u>. Which of the following is the only number that could possibly be the product?

563 x 21 728 = ?

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(b) 1 107 130 466

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j. 1.07 + 1.92 = 2

k. 723 is a square number

36. A student multiplied 567 x 426. Then she multiplied 567 x 427 and subtracted the smaller product from the larger product. She was absolutely certain about the accuracy of her multiplication. Without multiplying, find the difference in the products.

a. $\begin{array}{r} 567 \\ \times\ 426 \\ \hline \end{array}$

b. $\begin{array}{r} 567 \\ \times\ 427 \\ \hline \end{array}$

37. Use rounding to estimate each answer.

(a) $\begin{array}{r} 237 \\ +\ 425 \\ \hline \end{array}$

(b) $\begin{array}{r} 15\ 720 \\ +\ 20\ 590 \\ \hline \end{array}$

(c) $\begin{array}{r} 68 \\ \times\ 7 \\ \hline \end{array}$

(d) $\begin{array}{r} 12.3 \\ \times\ 4.78 \\ \hline \end{array}$

(e) $\begin{array}{r} 298 \\ -\ 60 \\ \hline \end{array}$

(f) $\begin{array}{r} 5762 \\ -\ 3124 \\ \hline \end{array}$

(g) $\frac{328}{115}$

(h) $\frac{420}{25}$

38. a. If 20 + 60 = 80, then 23 + 60 is (greater, less) than 80.

b. Since 90 - 30 = 60, 90 - 27 is (greater, less) than 60.

c. Since 300 + 500 = 800, which of the following are less than 800?

315 + 500	295 + 500
275 + 495	395 + 505

d. Since 5 x 80 = 400, which of the following is greater than 400?

5 x 8	5 x 89
5 x 80	5 x 60

e. Since 3500 ÷ 70 = 50, is 3500 ÷ 75 greater or less than 50?

39. First estimate. Then use your calculator to perform the calculation.

a. 5812 + 4406

b. 864 - 231

c. 479 x 24

d. 8118 ÷ 41

e. In what year will there have been 1 000 000 days in the Christian era?

f. Determine the distance in miles that light travels in one year (a light-year) if it travels at 186 000 miles per second. Use scientific notation in your computation.

g. How many years does it take to live 1 000 000 000 seconds?

*40. In the following crypto-numbers, each letter represents a different digit, from 0 to 9. Use your knowledge of computation to figure them out.

a.
$$\begin{array}{r} \text{SEND} \\ +\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

b.
$$\begin{array}{r} \text{DEFER} \\ -\ \text{DUTY} \\ \hline \text{NOGO} \end{array}$$

*The asterisk signifies a more challenging exercise.

41. The data given below appeared in recent issues of a large city newspaper.

a. Decide whether each number represents an exact number or an approximate one. Support your viewpoint.

b. If a number is expressed in words, rewrite it in standard decimal notation.

c. Study an issue of any newspaper. List five numbers that are exact, five that are approximate, and five that are expressed in words.

(1) A $120 million oceanfront hotel is planned for Atlantic City. A 1000-room project is planned.

(2) West Germany reported a $919 million trade surplus in June.

(3) The House approved an amendment by a 232 to 187 vote.

(4) One corporation reported earnings of $86.5 million last year.

(5) An oil company reported it sold the equivalent of 18 billion gallons of petroleum products.

(6) The announcement of new layoffs raises to 44 100 the number of auto workers laid off in recent weeks.

(7) The company had originally predicted that sales would reach the $11.5 million-unit mark.

(8) The total strength of the Army, Navy, Air Force, and Marines stood at 2 018 299 at the end of the month.

(9) Nicaragua has allowed 199 people to leave.

(10) The company has 5000 accounts in the metropolitan area.

(11) The number of seniors who failed to receive diplomas is 13 032.

(12) The huge spill ran 45 miles past the Arizona border.

(13) Only 18 000 trash buckets remain on the streets.

(14) Between 1975 and 1977, New York City bought 1 429 500 tons of asphalt.

(15) The corporation announced that 552 of its products might have defective transistors.

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42. a. Find the value of each of the following.

(1) [7.4 + .5]

(2) [7.5 + .5]

(3) [7.6 + .5]

(4) [8.2 + .5]

(5) [3.9 + .5]

b. Round the following numbers to the nearest whole number.

(1) 7.4

(2) 7.5

(3) 7.6

(4) 8.2

(5) 3.9

c. How do your answers in (a) compare with those in (b)?

d. Explain why the answers in (a) and (b) are so related.

TEAM

APPROXIMATION AND ESTIMATION

IV

STUDENT SUMMARY AND REVIEW

To the Student: This summary and review includes the key ideas discussed and presented throughout the Approximation and Estimation module. You will probably find it useful to compare your notes from class sessions with these.

I. APPROXIMATE AND EXACT NUMBERS

A. An exact number is obtained by counting.

Example: I have 5 fingers on my right hand.

B. Approximate numbers

1. An approximate number is obtained by rounding an exact number when an exact number is not needed.

   Example: About 200 people attended the meeting.

2. A measurement is always an approximate number.

   Example: The table is about 14 dm long.

3. Any portion of a repeating decimal is an approximation of the fraction it represents.

   Example: $\frac{2}{3}$ .67, because $\frac{2}{3} = .666...$

4. Other numbers, such as π and $\sqrt{3}$, can be expressed as approximate decimals.

   Examples: π 3.14 $\sqrt{3}$ 1.73

II. LEFT-TO-RIGHT CALCULATORS

A. The accepted hierarchy in the order of operations in a mathematical expression is:

1. Parentheses
2. Exponents (powers and roots)
3. Multiplication and division
4. Addition and subtraction

To evaluate $1\cdot2 + 5\cdot6$, first multiply $1\cdot2 + 5\cdot6 = 2 + 30$; then add $2 + 30 = 32$

Example: $6 + (8 \div 2)^2 = 6 + (4)^2$
$= 6 + 16$
$= 22$

Practice:

(a) $2 \times 6 - 5$

(b) $2 \times (6 - 5)$

(c) $12 \div 1 + 3$

(d) $12 \div (1 + 3)$

(e) $8 + 4 \div 2$

(f) $8 \div 2^2 + 5$

(g) $4 - 3 + 6 \cdot 2$

(h) $(7 - 2 \times 3)^2$

Answers: (a) 7, (b) 2, (c) 15, (d) 3, (e) 10, (f) 7, (g) 13, (h) 1

B. Inexpensive calculators process information in the order in which it is received, from left to right.

Example: $1 \cdot 2 + 5 \cdot 6$

Calculator: $[(1 \cdot 2) + 5] \cdot 6 = (2 + 5) \cdot 6 = 7 \cdot 6$, or 42

Example: $6 + (8 \div 2)^2$

Calculator: $[(6 + 8) \div 2]^2 = (14 \div 2)^2$

$= 7^2$

$= 49$

Parentheses can be used to order operations.

Practice:

Insert parentheses to show how a left-to-right calculator will calculate the following expressions.

(a) $7 - 5 \times 2$

(b) $6 + 8 \div 4$

(c) $6 - 1 + 5 \times 3$

(d) $12 \div 2^2 + 5$

Answers: (a) $(7 - 5) \times 2$, (b) $(6 + 8) \div 4$, (c) $[(6 - 1) + 5] \times 3$, (d) $(12 \div 2)^2 + 5$

C. A calculator can be manipulated in any or all of the following ways so that it yields the correct answer:

1. By rearranging terms

2. By calculating intermediate answers

3. By using memory keys

Insert parentheses to show how a left-to-right calculator would evaluate the expression

$$5 \cdot 2 + 3^2$$

What result would be given by the calculator?

Practice:

Insert parentheses in the expression given above to show how each of the following answers was obtained.

(a) 169 $[(5 \cdot 2) + 3]^2$

(b) 55 $5 \cdot (2 + 3^2)$

(c) 125 $5 \cdot (2 + 3)^2$

(d) 19 $(5 \cdot 2) + 3^2$

III. REFERENCE NUMBERS

A. In relating new experiences to old, you develop references for making comparisons. Reference numbers taken from your own experience provide a good way to visualize a given number.

Example: The Sears Tower in Chicago, Illinois, has 110 stories.

By referring to the fact that the Empire State Building has 102 stories or that your apartment house has 6 stories, you can better appreciate what 110 stories represents.

B. Here are some useful reference numbers:

- 1 meter--the approximate distance of a doorknob from the floor
- 50--the approximate number of cards in a standard deck of playing cards
- 300--the approximate number of words on a double-spaced typewritten page

Cite other reference numbers within your experience.

C. Complete these statements in order to increase your own supply of reference numbers.

1. A centimeter is about as big as ____________________________.
   (Choose an appropriate part of your body.)

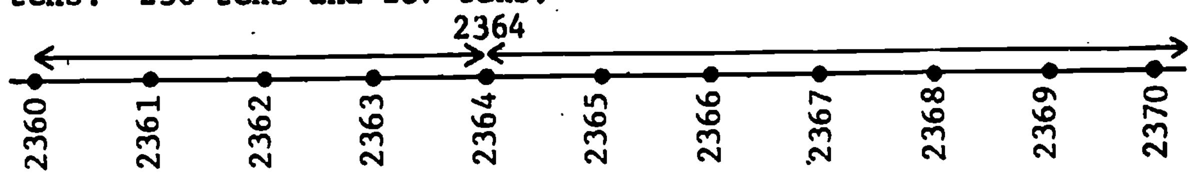
2. A square bridge table is about ____ inches by _____ inches.

3. The distance from my home to the college is about ________ kilometers.

4. I can comfortably seat _____ people around my dining room table.

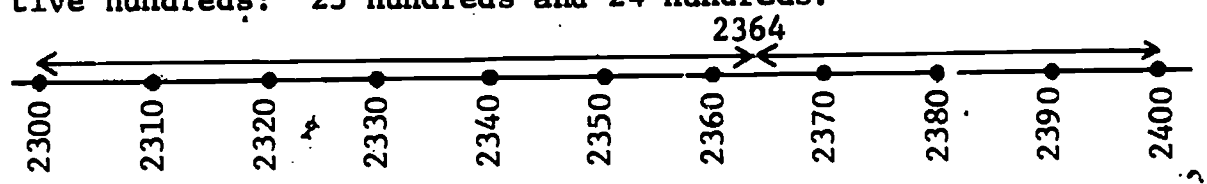
IV. ROUNDING NUMBERS

A. Round 2364 to the <u>nearest ten</u>. Locate between two consecutive tens: 236 tens and 237 tens.



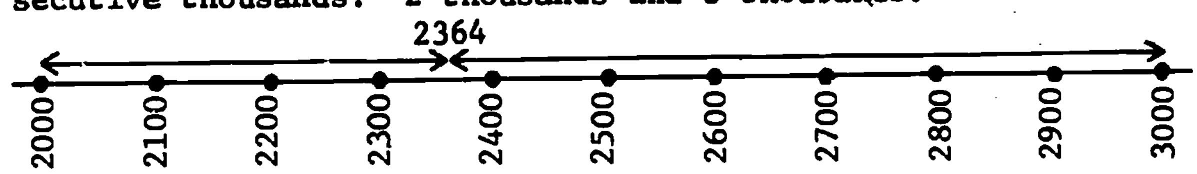
Answer: 2360 (④ rounds tens place to 6)

B. Round 2364 to the <u>nearest hundred</u>. Locate between two consecutive hundreds: 23 hundreds and 24 hundreds.



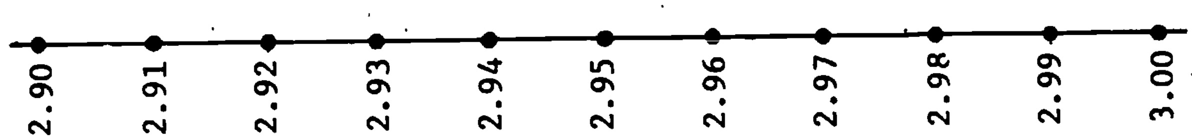
Answer: 2400 (⑥ rounds hundreds place to 4)

C. Round 2364 to the <u>nearest thousand</u>. Locate between two consecutive thousands: 2 thousands and 3 thousands.



Answer: 2000 (③ rounds thousands place to 2)

D. Round 2.96 to the <u>nearest tenth</u>. Locate between two consecutive tenths: 2.9 and 3.0.

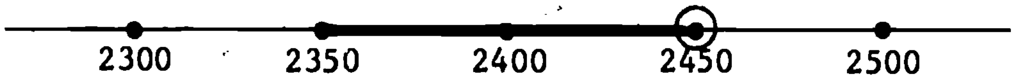


Answer: 3.0

E. Range of numbers

Rounding 2360 to the nearest ten:

Rounding 2400 to the nearest hundred:



Rounding 2000 to the nearest thousand:

F. Examine the digit in the first place to the right of the one wanted. For example, if you want to round to the nearest hundred, examine the digit in the tens column.

If the digit is less than 5, it and all others to its right are dropped.

If the digit is 5 or more, the previous digit is increased by one and all the digits to the right of it are dropped.

<u>Practice</u>: Round 7834.217 to the nearest ten

hundred

thousand

tenth

hundredth

Answers: 7830, 7800, 8000, 7834.2, 7834.22

G. What is the smallest number that rounds to:

7830 to the nearest ten

7800 to the nearest hundred

8000 to the nearest thousand

Answers: 7825, 7750, 7500

H. What is the smallest number that, when rounded to the nearest thousand, is 3000?

What is the largest even number that, when rounded to the nearest thousand, is 3000?

What is the largest even number that, when rounded to the nearest hundred, is 3000?

What is the largest even number that, when rounded to the nearest ten, is 3000?

Answers: 2500, 3498, 3048, 3004

V. CHOPPING NUMBERS, OR ROUNDING DOWN

Square brackets are used to show that the fractional part of the number included in them is to be chopped off.

Examples:

$[4.2] = 4$ $\quad [4\frac{7}{8}] = 4$ $\quad [4.998] = 4$

$[\frac{8}{2}] = 4$ $\quad \left[\frac{9\frac{1}{4}}{2}\right] = 4$ $\quad \left[\frac{2\frac{1}{2} + 5\frac{1}{2}}{2}\right] = 4$

$[\frac{15}{16}] = 0$ $\quad [\frac{1}{2} + \frac{7}{16}] = 0$

In the last example, since $\frac{7}{16}$ is less than $\frac{1}{2}$ (or $\frac{8}{16}$), the sum of $\frac{1}{2}$ and $\frac{7}{16}$ would be less than 1.

<u>Practice</u>:

(a) $[\frac{10}{11} + \frac{1}{10}] =$

(b) $[\frac{10}{11}] + [\frac{1}{10}] =$

(c) $[\frac{238}{25}] =$

Answers: (a) 1, (b) 0, (c) 9

VI. ROUNDING UP

Sometimes it is more appropriate to round up than to round down.

Examples:

In estimating supermarket purchases, you round up for safety. Thus, you figure that a $1.89 item costs about $2.00.

Assume that fabric is sold by the yard and no fractional parts of a yard are sold. If a pattern calls for $9\frac{1}{4}$ yards, 10 yards would be the minimum purchase.

If a supermarket sells white bread at 3 for $1.00, one loaf would cost 34¢.

VII. NUMBERS IN WORDS

A. Remember that:

1 million = 1 000 000, or 10^6

1 billion = 1 000 000 000, or 10^9

B. A number like 120 000 000 can be read and written as 120 million. Similarly, the number that is read and written as 81 billion is 81 000 000 000 in standard decimal notation.

<u>Practice</u>:

(a) Write each of the following as standard decimal numerals:

28 billion, 3 million, 128 thousand

(Answers: 28 000 000 000, 3 000 000, 128 000)

(b) Write each of the following as it would read, using words in place of the zeros:

5 000 000 000, 916 000 000, 79 000

(Answers: 5 billion, 916 million, 79 thousand)

C. Decimals can be used when long numbers are written in words: 1.5 million, 91.4 billion. These numbers can be rewritten as standard decimal numerals.

Examples:

1.5 million = 1.5 x 1 000 000

= 1 500 000

(The number .5 million, or a half-million, is 500 000, or five hundred thousand.)

91.4 billion = 91.4 x 1 000 000 000

= 91 400 000 000

(The number .4 billion = 400 000 000, or four hundred million.)

Practice:

(a) Write each of the following as standard decimal numerals:

83.5 billion, 9.3 million, 2.5 thousand

(Answers: 83 500 000 000, 9 300 000, 2500)

(b) Write each of the following as it would be read, using words in place of the zeros:

1 100 000 000, 11 300 000, 74 300

(Answers: 1.1 billion, 11.3 million, 74.3 thousand)

VIII. SCIENTIFIC NOTATION

A. Pattern of exponents

10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
10 000	1000	100	10	1	.1	.01	.001

Thus, $.00001 = 10^{-5}$, and $10^6 = 1\ 000\ 000$.

B. In scientific notation a number is represented as the product of a number between 1 and 10 and a power of 10.

Examples: $234\ 000 = 2.34 \times 10^5$

$.0234 = 2.34 \times 10^{-2}$

To multiply by 10^1, move the decimal point one place to the <u>right</u>.

$2.34 \times 10^1 = 23.4$

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To multiply by 10^{-1}, move the decimal point one place to the left.

$$2.34 \times 10^{-1} = .234$$

Examples: $4.6 = 4.6 \times 10^{0}$

$.046 = 4.6 \times 10^{-2}$

$46\ 000\ 000 = 4.6 \times 10^{7}$

Practice:

(a) Express each number in scientific notation.

.63, 3, 4 680 000, 385.76

(Answers: 6.3×10^{-1}, 3×10^{0}, 4.68×10^{6}, 3.8576×10^{2})

(b) Verify that there are about 3.2×10^{7} seconds in a year.

IX. CHECKING FOR UNREASONABLE RESULTS

A. Checking the units digit in multiplication

1. For 38 x 59, since 8 x 9 = 72, the product must have 2 in the units place.

2. Square numbers may have only 0, 1, 4, 5, 6, or 9 in the units place, so 2368 cannot be a square, but 3136 might be a square.

B. Estimating

1. Show that 154 is an unreasonable result for 38 x 59.

30	38	40
x 50	x 59	x 60
1500		2400

The product of 38 x 59 lies between 1500 and 2400.

2. Show that the product of 56 348 and 348 327 cannot be 194 257 796.

Round numbers down: 50 000 x 300 000 = 15 000 000 000

The product must contain at least 11 places.

C. Using multiplication shortcuts

1. $25 \times 28 = (\frac{1}{4} \times 100) \times 28$ $\quad 25 = \frac{1}{4} \times 100$

$= \frac{1}{4} \times 2800$

$= 700$

2. $75 \times 28 = (\frac{3}{4} \times 100) \times 28$ $\quad 75 = \frac{3}{4} \times 100$

$= \frac{3}{4} \times 2800$

$= 2100$

3. Approximating 15% of a number

15% = 10% + 5%, or 10% + $\frac{1}{2}$ (10%)

Example: 15% of \$8.00 = 10% of \$8.00 + 5% of \$8.00

10% of \$8.00 = .80

5% of \$8.00 = $\frac{1}{2}$ (10% of \$8.00); $\frac{1}{2}$(.80) = .40

Therefore 15% of \$8.00 = .80 + .40, or \$1.20