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ABSTRACT

This Teacher Education and Mathematics (TEAM) content module focuses on patterns. It consists of: (1) an instructor's text; (2) an instructor's guide and solutions to student exercises; (3) student materials and exercises; and (4) student summary and review. The instructor's text provides specific directions for guiding lessons and commentary on mathematics content and mathematics attitudes. This is accomplished by a "facing pages" format whereby the right-hand page provides step-by-step teaching directives while the left-hand page provides teaching insights, other options of instruction, and psychological or attitudinal strategies, when appropriate. The instructor's text also contains content objectives, specified to indicate the scope and structure of the module, and student evaluation materials. The instructor's guide and solutions to exercises recommends teaching approaches to the materials and provides answers to problems. diagrams, charts, and centimeter-squared paper to be used by students. Exercises include problems that apply the concepts and problem-solving strategies developed in the module; they may be used as part of the instructional activities, as content for small-group activities, as homework assignments, or as review materials. The student summary and review summarizes the content of the module, focusing on formulas, terminology, key concepts, problem-solving strategies, and examples of techniques used. (JN)

Patterns

A Course to Reduce Math Anxiety and Sex-Role Stereotyping in Elementary Education

TEAM

TEACHER EDUCATION AND MATHEMATICS

Queens College of the City University of New York
Women's Educational Equity Act Program/U.S. Department of Education

TEACHER EDUCATION AND MATHEMATICS

A Course to Reduce
Math Anxiety and Sex-Role Stereotyping
in Elementary Education

PATTERNS

Elaine B. Chapline
and
Claire M. Newman
Project Codirectors

Assisted by:
Francine Sicklick
and
Elenor Rubin Denker

Queens College
of the City University of New York

Women's Educational Equity Act Program
U.S. Department of Education

T. H. Bell, Secretary

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PROJECT STAFF

Elaine B. Chapline
Project Codirector
Psychologist

Claire M. Newman
Project Codirector
Mathematics Educator

Carol Kehr Tittle
Evaluator

Elenor Rubin Denker
Evaluation Associate

Francine Sicklick
Mathematics Associate

Annette Berson
Project Associate

Sarah Knecht
Research Assistant

Dorothy Watkins
Research Assistant

Katherine Flanagan
Administrative Assistant

Lucille Joseph
Secretarial Assistant

Kim Longo
Secretarial Assistant

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James Bruni
Mathematics Educator
Lehman College of CUNY

Hilda Cooper
Mathematician
Queens College of CUNY

Sheila Crowell
Editor
<u>Caribou</u>

Elizabeth Fennema
Mathematics Researcher
University of Wisconsin

Stanley Kogelman
Mathematician, Author
<u>Mind Over Math</u>

Ellen Kolba
Editor
<u>Caribou</u>

Mitchell Lazarus
Mathematics Educator
Independent Consultant

Edna Meyers
Psychologist
City College of CUNY

Frederic Paul
Mathematics Educator
State Education Department, New York

Anne Peskin
Mathematics Educator
City College of New York

Lila Swell
Psychologist
Queens College of CUNY

CONTENTS

INTRODUCTION

The Patterns module consists of an Instructor's Text, Instructor's and Solutions to Student Exercises, Student Materials and Exercises, Student Summary and Review.

The Instructor's Text provides the instructor with (1) specific directions for guiding lessons and (2) commentary on the math content and on math attitudes. This is accomplished by a special "facing pages" format. The right-hand page provides the instructor with the presentation modes of problems and teaching directives, while the left-hand page, "Commentary and Notes," provides teaching insights, other options of instruction, and, often, psychological strategies. Space for the instructor to add her or his own notes about a particular point in the lesson or about teaching experiences with the class (for future reference and use) is also provided on the left-hand page. In other words, the directions on the right-hand page clearly point out to the instructor what steps to take in presenting the lesson, while the commentary/notes page on the left supplements the instruction with explanations, additional instructional options, attitudinal interventions, and organizational alternatives to the teaching presentation. The Instructor's Text includes a set of content objectives, specified to indicate the scope and structure of the module, and student evaluation materials that contain several questions for each objective so that the instructor can select items for quizzes.

The Instructor's Guide and Solutions to Student Exercises recommends teaching approaches to the materials and provides answers to the math problems.

Student Materials and Exercises provides such things as diagrams and charts to be used by the students. Instructors should plan to make transparencies of these materials for use on overhead projectors during class time. The exercises include problems that apply the concepts and problem-solving strategies developed in the module. These problems can be used as part of the instructional activities, as content for small-group activities, as homework assignments, or as review materials.

Student Summary and Review summarizes the content of the module (formulas, terminology, key concepts, problem-solving strategies, and examples of techniques used). These notes are to be given to students *after* they have participated in the learning activities of the module.

TEAM

PATTERNS

I

INSTRUCTOR'S TEXT

OBJECTIVES

The objectives of the Patterns module are:

1. Identify and write triangular numbers.
2. Apply the formula n^2 to calculate the sum of the first $\underline{n}$ odd numbers.
3. Show, through diagram, that $1 + 3 + 5 + \ldots + (2n-1) = n^2$.
4. Apply the formula $\frac{1}{2}n(n+1)$ to calculate the sum of the first $\underline{n}$ natural numbers.
5. Apply the formula $n(n+1)$ to calculate the sum of the first $\underline{n}$ even numbers.
6. Demonstrate knowledge of the sums of even and odd numbers.
7. Identify and write a specified term of the binary sequence.
8. Express the sum of any number of terms, $\underline{n}$, of the binary sequence in the form 2^n-1.
9. Write a specified number of terms of a Fibonacci-type sequence.
10. Follow and extend a given number sequence.

Th se objectives are provided to you at the outset to indicate the scope and structure of the module. They should be distributed to students with the Student Summary and Review so that they can use them to organize their study and preparation for a quiz.

Sample it ms for the objectives are included at the end of this section of the module.

COMMENTARY AND NOTES

As students proceed with the work, help them feel the joy of creative activity and take pleasure in their sense of increased power.

The key skill of <u>perceiving relationships</u> is introduced here. Students will be helped to develop knowledge about patterns and skill in perceiving them.

Review the meaning of "..." as "in the same manner." Note that the sequence continues indefinitely--infinitely. Ask such questions as:

1. Are you comfortable with the feeling that the sequence goes on indefinitely?

2. Will you always know what the next picture or number is? How?

BEGINNING PATTERNS

Introduce a discussion of patterns. Students should observe that patterns occur in the world around us--in art, nature, behavior, architecture, mathematics, language development, and the motor and intellectual development of children.

Ask: "Why do we use a term such as behavior pattern? What is a pattern? What are its characteristics?" Discuss the ideas of symmetry, motion, repetition, beauty, form, recurrence, and predictability as terms we associate with patterns.

Patterns in nature are mathematically based. Try to circulate pictures of patterns. Your collection might include:

bees' honeycombs
spiderwebs
pinecones
soap bubbles
snowflakes
flowers
crystals
spirals
butterflies
examples from art
architectural designs

Help students understand the importance of perceiving relationships, of seeing how various parts are mentally related to the whole, and of obtaining intuitional glimpses of what comes next.

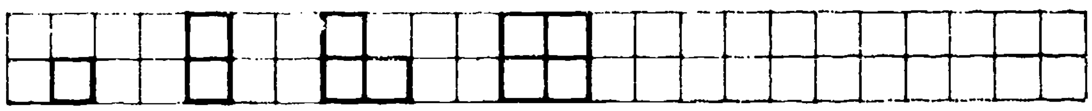
Natural Numbers

Ask students to turn to Diagrams 1.1, 1.2, and 1.3 (see Student Materials and Exercises, page III-3). Ask them to study each pattern and to continue the diagram in the same manner. Allow a few minutes for students to talk about what they have done.

Ask students to study Diagram 2.1, to continue the pattern of squares, and to assign a number to each figure.

Diagram 2.1 leads to the following sequence of natural numbers:

1, 2, 3, 4, ...



COMMENTARY AND NOTES

Note whether students react with tension to the use of a mathematical vocabulary. If so, raise the question of how they are feeling, and proceed with the content after students have expressed their concerns. You might also reassure students that they are in a better position for learning a specific vocabulary now than they were as children. Their broader experience and larger vocabulary provide this current advantage.

When you are faced with a problem using large numbers, it is somewhat easier to solve a related problem using smaller values. When you have solved this simpler problem, you can then transfer the process or method to the original problem. Students should be encouraged to use this key strategy, which can be used in all problem solving.

This material uses the National Bureau of Standards notation for large numbers. The comma is no longer used to separate groups of three digits. A space is used, instead. Thus, 3,487,521 is written 3 487 521. In the case of four-digit numbers, neither a space nor a comma is used. Thus, 9438 is the correct form for a four-digit number.

At this point, clarify the use of the word <u>term</u>. Consider with the class the sequence 1, 3, 5, 7, ...

Say: "To facilitate reference to these numbers, we call the first number of the sequence (1) the first term, the second number (3) is the second term, the third term is 5, and so forth."

Sequence:	1,	3,	5,	7,	...
Number of term:	1st	2nd	3rd	4th	

Ask students to name the <u>fifth term</u> and to determine which term of the sequence 13 would be.

Ask students to consider the sequence 2, 4, 6, ..., and ask them such questions as "What is the first term? The third term? Which term of the sequence is 12?"

Sums of Natural Numbers

Relate the story about Frederic Gauss (1777-1855). Gauss's elementary school teacher asked him to calculate the sum of the first 100 natural numbers. The teacher hoped the task would keep Freddy busy so that the other children could receive attention. Gauss came up with the answer (5050) quickly. How did he do it?

Write: 1 + 2 + 3 + ... + 100

Encourage suggestions for approaches from students.

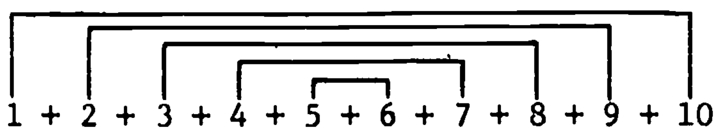
Recommend that the group work with a shorter, easier sum, such as

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10

Suggest that by working this way they may be able to discover some pattern that can be used in other longer examples.

Ask students to begin by studying the new sum.

Ask: "Is there a useful way to pair these numbers?" Note that some students may pair as follows:



COMMENTARY AND NOTES

Encourage students to consider multiple approaches to problem solving. If a student is in error, consider clarifying how the thinking proceeded rather than assessing its accuracy. Accepting students' efforts can contribute to a positive learning climate.

There are 5 pairs of numbers whose sum is 11.

$$11 + 11 + 11 + 11 + 11 = 55$$

$$\text{or } 5 \times 11 = 55$$

Allow the students to practice using this method to calculate the sums of an even number of natural numbers. For example, ask students to calculate the sum of the first 12 natural numbers.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12$$

$$6 \times 13 = 78$$

Next ask students to calculate the sum of the first 20 natural numbers (10 x 21 = 210) and the first 100 natural numbers (50 pairs; each sum is 101; 50 x 101 = 5050).

Now present the following sum:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15$$

Students who proceed to solve the problem using the previous <u>method of pairing</u> should observe that:

- There are 7 pairs.
- Each pair has the sum 16.
- The middle number, 8, is not paired.

These observations yield

$$7 \times 16 + 8 = 112 + 8, \text{ or } 120$$

Ask for an explanation that compares this example with those solved previously.

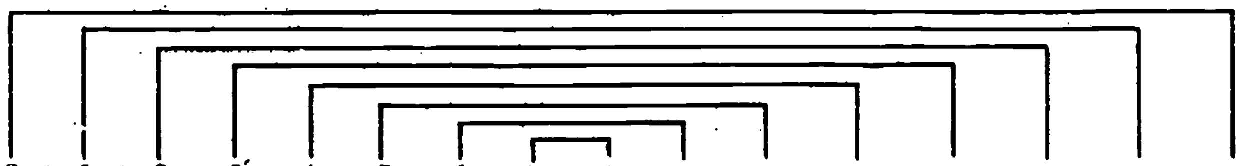
COMMENTARY AND NOTES

Note that in 8 x 15 = 15 x 8, only the order is changed. This is an instance of the commutative property of multiplication.

Calculation methods should be reviewed as necessary. However, a calculator should be available for those who wish to use it.

In each of the other examples, students were asked to add an even number of terms (1-10, 1-12, 1-100). Ask students to suggest other methods for 1 + 2 + 3 + ... + 13 + 14 + 15.

One possibility is the method of adding 0. A zero can be added to the indicated sum so that there are sixteen numbers to pair. Of course, the sum will remain the same.

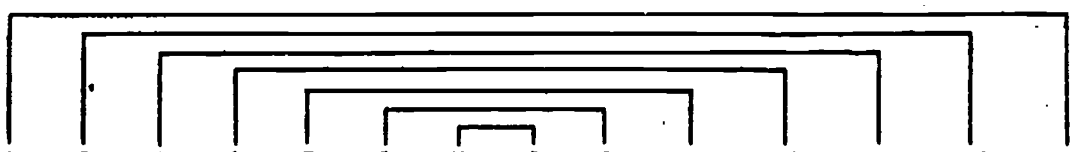


0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15

This yields 8 pairs whose sums are 15.

8 x 15 = 120

Another possibility:



1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15

The first 14 numbers can be summed in one of the ways just mentioned, and the 15 can be added to the result. This gives 7 pairs whose sums are each 15, plus one more 15, or eight 15s.

7 x 15 + 15 = 8 x 15, or 120

Point out that 8 x 15 = 15 x 8 and that 15 x 8 means we are using 8 in place of each of the 15 terms. Thus:

1 + 2 + 3 + 4 + 5 + ... + 13 + 14 + 15

becomes

8 + 8 + 8 + 8 + 8 + ... + 8 + 8 + 8

Review the procedure for multiplying a decimal by 10, 100, 1000, etc. Use a calculator as an aid. Note that:

$$10 \times 5\frac{1}{2} = 10 \times (5 + \frac{1}{2})$$

$$= 10 \times 5 + 10 \times \frac{1}{2}$$

$$= 50 + 5$$

Students can work individually or in groups. Group practice has the advantage of enabling members of the group to share ideas. In this way, each individual receives the stimulation to reason in different ways. Another technique is for students to work examples individually on the board, checking and discussing each other's work.

It is possible to use averaging to calculate the sum.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15$$

The average of each pair of numbers is one-half of 16, or 8. Will this method of the average work in all cases?

Try: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

Average: $\frac{1}{2}(10 + 1) = 5\frac{1}{2}$

Sum: $10 \times 5\frac{1}{2}$, or $10 \times 5.5 = 55$

Verify: $1 + 2 + 3 + \ldots + 10 + 11 + 12$

Average: $\frac{1}{2}(12 + 1) = \frac{1}{2} \times 13$, or $6\frac{1}{2}$

Sum: $12 \times 6\frac{1}{2} = (12 \times 6) + (12 \times \frac{1}{2})$

$= 72 + 6$, or 78

Help students see that:

- The average $\frac{1}{2}(15 + 1) = \frac{1}{2} \times 16$, or 8, is not fractional because 15 is odd and $1 + 15$ is even.
- The average $\frac{1}{2}(12 + 1) = \frac{1}{2} \times 13$, or $6\frac{1}{2}$ is fractional because 12 is even and $1 + 12$ is odd.

Give students practice with the average method and with the method of adding 0.

Calculate: $1 + 2 + 3 + \ldots + 13 + 14 + 15 + 16 + 17$

Average: $\frac{1}{2}(17 + 1) = 9$

Sum: $17 \times 9 = 153$

Also: $0 + 1 + 2 + \ldots + 17$ yields 18 terms, 9 pairs--each sum is 17

$9 \times 17 = 153$

COMMENTARY AND NOTES

This may be an opportune moment to point out that there are often multiple strategies that can be used in the solution of a problem.

Calculate: $1 + 2 + 3 + \ldots + 7 + 8 + 9$

Average: $\frac{1}{2}(9 + 1) = 5$

Sum: $9 \times 5 = 45$

Also: $0 + 1 + 2 + \ldots + 7 + 8 + 9$ yields 10 terms, 5 pairs--each sum is 9

$5 \times 9 = 45$

Calculate: $1 + 2 + 3 + \ldots + 18 + 19 + 20$

Average: $\frac{1}{2}(20 + 1) = \frac{1}{2} \times 21$, or $10\frac{1}{2}$

Sum: $20 \times 10\frac{1}{2} = 20 \times 10 + 20 \times \frac{1}{2}$

$= 200 + 10$, or 210

Note that the last sum is more easily calculated by the pairing method:

20 numbers (10 pairs)--each sum is 21

$10 \times 21 = 210$

Introduce <u>Gauss's Strategy</u>. Show how Gauss's strategy functions.

Add:

$$\begin{array}{r} 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ \underline{10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1} \end{array}$$

Sum: $11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 = 10 \times 11$

Thus: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{1}{2} \times (10 \times 11)$

$(\frac{1}{2} \times 10) \times 11 = 5 \times 11$

$= 55$

Note that the <u>average method</u> leads to $10 \times 5\frac{1}{2} = 55$

Help students derive $10 \times 5\frac{1}{2}$ from $\frac{1}{2} \times (10 \times 11)$:

$$\frac{1}{2} \times 10 \times 11 = (\frac{1}{2} \times 11) \times 10$$

$$= 5\frac{1}{2} \times 10$$

$$= 10 \times 5\frac{1}{2}$$

COMMENTARY AND NOTES

Note that the pacing of a session is the instructor's decision. Timing will vary widely among groups, depending upon their attitudes and experiences. These activities with patterns should be used to review fundamental operations and concepts.

Form is a key idea in the search for patterns. The way that numbers are written is sometimes more important than the answers themselves. Very often a pattern is revealed by numbers expressed in the form of indicated products or sums. Note the form in each case: 1 + 2 + 3 is an indicated sum; 6 is the sum expressed as a standard numeral (the usual decimal form); $2 \cdot 3$ is an indicated product and the dot indicates multiplication. (The dot will be used hereafter in place of the x. This may be a new symbol for some students.) Reinforce these forms throughout all subsequent work.

Don't assume that all students clearly understand the meanings of sum and product. Be aware of those students who need clarification and practice.

Observe that two different approaches lead to the same result. Allow time for students to practice Gauss's Strategy. Calculate the sum of the first 4, 6, 12, 15, 100 natural numbers.

$1 + 2 + 3 + 4 = \frac{1}{2} \times 4 \times 5$, or 10

$1 + 2 + 3 + 4 + 5 + 6 = \frac{1}{2} \times 6 \times 7$, or 21

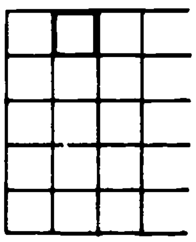
$1 + 2 + 3 + \ldots + 10 + 11 + 12 = \frac{1}{2} \times 12 \times 13$, or 78

$1 + 2 + 3 + \ldots + 12 + 13 + 14 + 15 = \frac{1}{2} \times 15 \times 16$, or 120

$1 + 2 + 3 + \ldots + 98 + 99 + 100 = \frac{1}{2} \times 100 \times 101$, or 5050

Geometric Strategy

Ask students to refer to Diagram 2.2 in Student Materials and Exercises, page III-4.



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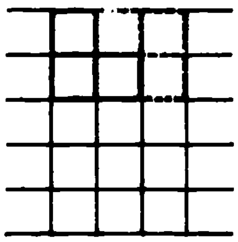
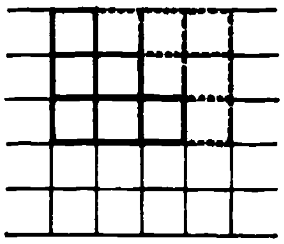


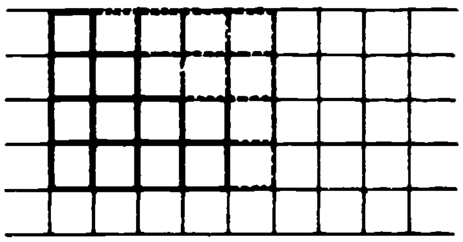
$1 + 2 = \frac{1}{2}(2 \cdot 3)$

$= 3$



$1 + 2 + 3 = \frac{1}{2}(3 \cdot 4)$

$= 6$



A prediction is more than a guess. Data and a pattern tend to make us more certain than we would be otherwise.

A systematic way of noting data makes it possible for patterns to be discerned. Problem solvers can keep track of where they are in their progress toward a solution when data are organized.

Systematic recording of data should be modeled by the instructor and discussed at opportune moments. Students' ways of recording data should be noted and suggestions given when appropriate.

Underlining the numeral representing the number of terms will call attention to the form of the indicated product. That is, to obtain the sum of the first 4 terms, one writes:

$$\frac{1}{2}\,(\underline{4}\cdot 5)$$

Ask students to continue the picture pattern and verify that:

$$1 + 2 + 3 + 4 = \frac{1}{2}(4 \cdot 5)$$

Predict:

$$1 + 2 + 3 + 4 + 5$$

$$1 + 2 + 3 + 4 + 5 + 6$$

$$1 + 2 + 3 + \ldots + 8 + 9 + 10$$

$$1 + 2 + 3 + \ldots + 12 + 13 + 14 + 15$$

Stress the importance of the systematic recording of data.

Have students follow this format:

Number of Terms	Indicated Sum	Sum
$\underline{1}$	1	1
$\underline{2}$	$1 + 2$	$3 = \frac{1}{2}(\underline{2} \cdot 3)$
$\underline{3}$	$1 + 2 + 3$	$6 = \frac{1}{2}(\underline{3} \cdot 4)$
$\underline{4}$	$1 + 2 + 3 + 4$	$10 = \frac{1}{2}(\underline{4} \cdot 5)$
$\underline{5}$	$1 + 2 + 3 + 4 + 5$	$15 = \frac{1}{2}(\underline{5} \cdot 6)$
$\underline{6}$	$1 + 2 + 3 + 4 + 5 + 6$	$21 = \frac{1}{2}(\underline{6} \cdot 7)$
$\underline{10}$	$1 + 2 + 3 + \ldots + 10$	$55 = \frac{1}{2}(\underline{10} \cdot 11)$
$\underline{15}$	$1 + 2 + 3 + \ldots + 15$	$120 = \frac{1}{2}(\underline{15} \cdot 16)$

COMMENTARY AND NOTES

The use of n may evoke some tension on the part of students who experience math anxiety. Clarify the meaning and use of n. Note that n and n + 1 always represent consecutive numbers (6 and 7, 15 and 16, etc.). Give examples to show that if one is an odd number, the other is even.

Discuss the use of formulas and the advantage of expressing a generalization in this form. Practice in using the formula should help students experience the importance of n and sense the power it gives them to control many cases of the same kind. Note the use of S to represent the sum.

It is appropriate and necessary to encourage students to express their feelings about using symbols and formulas. Having stated their concerns, they may feel more ready to work on the task at hand.

Giving them an opportunity to express their concerns typically results in a sense of relief among students. To know that others have the same feelings is part of the strategy of lessening anxiety about math.

You will probably want to point out to students that generalizations do not constitute proof in mathematics. You may want to take this opportunity to introduce formal proofs.

Ask students to continue the picture pattern and verify that:

$$1 + 2 + 3 + 4 = \frac{1}{2}(4 \cdot 5)$$

Predict:

$$1 + 2 + 3 + 4 + 5$$

$$1 + 2 + 3 + 4 + 5 + 6$$

$$1 + 2 + 3 + \ldots + 8 + 9 + 10$$

$$1 + 2 + 3 + \ldots + 12 + 13 + 14 + 15$$

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Have students follow this format:

Number of Terms	Indicated Sum	Sum
1	1	1
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3	1 + 2 + 3	$6 = \frac{1}{2}(\underline{3} \cdot 4)$
4	1 + 2 + 3 + 4	$10 = \frac{1}{2}(\underline{4} \cdot 5)$
5	1 + 2 + 3 + 4 + 5	$15 = \frac{1}{2}(\underline{5} \cdot 6)$
6	1 + 2 + 3 + 4 + 5 + 6	$21 = \frac{1}{2}(\underline{6} \cdot 7)$
10	1 + 2 + 3 + ... + 10	$55 = \frac{1}{2}(\underline{10} \cdot 11)$
15	1 + 2 + 3 + ... + 15	$120 = \frac{1}{2}(\underline{15} \cdot 16)$

COMMENTARY AND NOTES

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Predict:

$$1 + 2 + 3 + 4 + 5$$

$$1 + 2 + 3 + 4 + 5 + 6$$

$$1 + 2 + 3 + \ldots + 8 + 9 + 10$$

$$1 + 2 + 3 + \ldots + 12 + 13 + 14 + 15$$

Stress the importance of the systematic recording of data.

Have students follow this format:

Number of Terms	Indicated Sum	Sum
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2	1 + 2	$3 = \frac{1}{2}(\underline{2} \cdot 3)$
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4	1 + 2 + 3 + 4	$10 = \frac{1}{2}(\underline{4} \cdot 5)$
5	1 + 2 + 3 + 4 + 5	$15 = \frac{1}{2}(\underline{5} \cdot 6)$
6	1 + 2 + 3 + 4 + 5 + 6	$21 = \frac{1}{2}(\underline{6} \cdot 7)$
10	1 + 2 + 3 + ... + 10	$55 = \frac{1}{2}(\underline{10} \cdot 11)$
15	1 + 2 + 3 + ... + 15	$120 = \frac{1}{2}(\underline{15} \cdot 16)$

COMMENTARY AND NOTES

The use of n may evoke some tension on the part of students who experience math anxiety. Clarify the meaning and use of n. Note that n and n + 1 always represent consecutive numbers (6 and 7, 15 and 16, etc.). Give examples to show that if one is an odd number, the other is even.

Discuss the use of formulas and the advantage of expressing a generalization in this form. Practice in using the formula should help students experience the importance of n and sense the power it gives them to control many cases of the same kind. Note the use of S to represent the sum.

It is appropriate and necessary to encourage students to express their feelings about using symbols and formulas. Having stated their concerns, they may feel more ready to work on the task at hand.

Giving them an opportunity to express their concerns typically results in a sense of relief among students. To know that others have the same feelings is part of the strategy of lessening anxiety about math.

You will probably want to point out to students that generalizations do not constitute proof in mathematics. You may want to take this opportunity to introduce formal proofs.

After recording the results for 6 terms, ask for data for 10 terms without referring to previous computations. Follow the same procedure for 12, 15, and 100 terms.

Assist students with the discernment of the pattern leading to the following generalization: The sum of the first n natural numbers is:

$$\frac{1}{2} n (n + 1)$$

That is, in each case, observe that the number of terms added is the first number of the indicated product, and the second number is the next consecutive number.

Finally, ask for the sum of the first n terms, including in the table:

$$1 + 2 + 3 + \ldots + n$$

$$\frac{1}{2} (\quad) (\qquad) .$$

Ask: "What belongs here?" (first parentheses)

(n)

Ask: "What belongs here?" (second parentheses)

(n + 1)

Work through several examples, showing specific substitutions. Practice using the formula:

$$S = \frac{1}{2} n (n + 1)$$

Remind students that

$$\frac{1}{2} \cdot 15 \cdot 16 = \frac{1}{\cancel{2}_{\,1}} \cdot \cancel{16}^{\,8} \cdot 15$$

$$= 8 \cdot 15$$

Since either n or n + 1 is even, the even number may be divided by 2 before multiplying.

Have students use the formula to calculate the sum of the first 30 natural numbers, the first 41, and the first 52. Show how each can be calculated by using one or more of the nonformula methods developed earlier.

Triangular Numbers

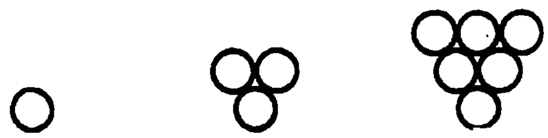
The sums of natural numbers generate a sequence of their own:

1, 3, 6, 10, ...

Refer to Diagram 2.3 (see Student Materials and Exercises, page III-4) to observe the reason these sums are referred to as the triangular numbers. Mention that they are sometimes called bowling pin numbers, too, because of the configuration of the circles. Ask students to continue the pattern and draw the fourth triangular number.

Ask: "What is the fourth triangular number?"

Triangular numbers:



Order	First	Second	Third	Fourth
n	1	2	3	4
Sum of natural numbers	1	1 + 2	1 + 2 + 3	

Ask: "How can we calculate the fifth triangular number?"

Answer: $1 + 2 + 3 + 4 + 5 = \frac{1}{\cancel{2}_{1}} \cdot 5 \cdot \cancel{6}^{3}$, or 15

"The tenth?" $\frac{1}{\cancel{2}_{1}} \cdot \cancel{10}^{5} \cdot 11 = 55$

"The nth?" $\frac{1}{2} n (n + 1)$

COMMENTARY AND NOTES

If interest wanes, consider briefly with the students the purposes being served by their mastery of these concepts and strategies.

Ask: "Why are number patterns important? What have you learned about the problem-solving process? Suppose you are presented with a new problem for which you have no immediate solution--what are some behaviors that might be productive?"

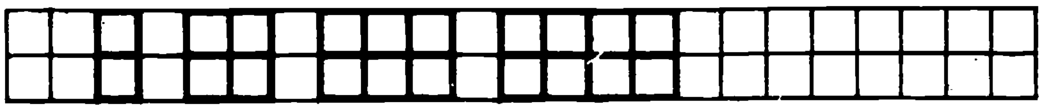
The numerical expression $n \cdot 2$ denotes n "blocks of two." However, it is conventional to write this product as 2n. Of course, $2n = n \cdot 2$ because multiplication is commutative.

Even Numbers

Refer to Diagram 3.1 in Student Materials and Exercises, page III-5. Ask students to extend the pattern and to assign numbers to the pictures.

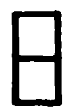
Diagram 3.1 leads to the following sequence of even numbers:

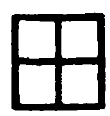
2, 4, 6, 8, ...

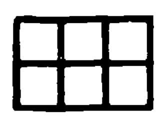


Ask students to define or describe an even number. The following observations should lead to the definition of an even number as a number that can be written in the form 2n.

1. Help students observe that each figure of Diagram 3.1 is made up of "blocks of two."

   The first is one block of two. 

   The second is two blocks of two. 

   The third is three blocks of two. 

2. This should then be translated into $1 \cdot 2$, $2 \cdot 2$, $3 \cdot 2$, $4 \cdot 2$, ...

   Record even numbers in table form.

NOTES

3. Observe the form of each indicated product and its relationship to the number of the term.

Term Number	Indicated Product	Even Number
1	1•2	2
2	2•2	4
3	3•2	6
4	4•2	8
.		
.		
.		
n	n•2	2n

Ask questions such as the following:

1. Is 10 an even number? Why? (Yes, because 10 = 2•5)

2. Is 11 an even number? (No, it cannot be written in the required form.)

Calculate the fifteenth even number, the twenty-seventh, the hundredth, and the nth.

Review the meaning of the nth term (or even number). In the sequence of even numbers

2, 4, 6, 8, ..., 2n,

n represents the number of the term. 2n represents the even number itself.

The fifth even number is 2•5, or 10.

The twenty-seventh even number is 2•27, or 54.

Definition: An even number is one that can be written in the form 2n, where n is a natural number.

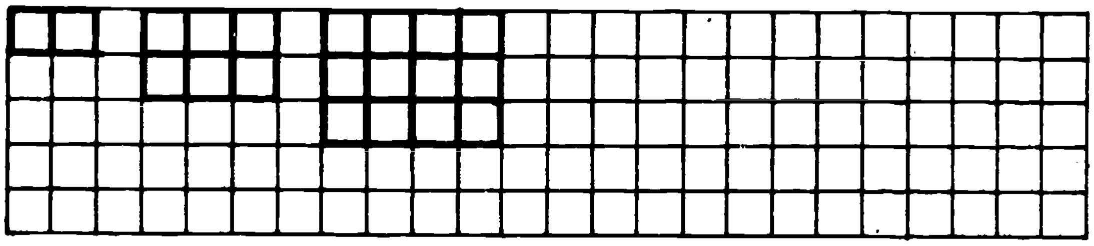
A logical equivalent of this definition is the following: If a number cannot be written in the form 2n (where n is a natural number), that number is not even (e.g , 11).

COMMENTARY AND NOTES

Squares have been shaded in order to distinguish the consecutive even numbers included in each sum.

Sums of Even Numbers

Refer students to Diagrams 4.1 and 4.2 in Student Materials and Exercises, page III-6. Ask them to extend the pattern of squares in 4.1. Ask them to express the number of squares in each figure of Diagram 4.1 as an indicated product.



1•2 2•3 3•4 ...

It is possible that some students will be able to extend the pattern of squares for the above pattern only after observing the pattern of the products.

Indicated products:	1•2,	2•3,	3•4,	4•5, ...
Standard numerals	2	6	12	20

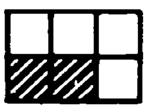
Now ask students to study Diagram 4.2 and to extend the pattern of shaded and unshaded squares. Assist by displaying transparency overlays to build each figure from the previous one. Develop the sequence of even numbers for the pattern of shaded and unshaded squares:

2, 4, 6, 8, ...

Use a line of step-by-step questions that will elicit indicated sums and indicated products.

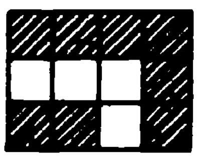
1. 

Start with 2 = 1•2

2. 

Overlay 1

How many unshaded squares have been added?
Indicated sum? 2 + 4
Indicated product? 2•3
Conclusion: 2 + 4 = 2•3

3. 

Overlay 2

How many shaded squares have been added?
Indicated sum? 2 + 4 + 6
Indicated product? 3•4
Conclusion: 2 + 4 + 6 = 3•4

Now ask students to build the next rectangle and to express the next sum and the equivalent product.

Record the data in table form in order to look for a pattern.

Number of Terms (n)	Indicated Sum	Sum (Indicated Product)	Sum (Standard Numeral)
2	2 + 4	2·3	6
3	2 + 4 + 6	3·4	12
4	2 + 4 + 6 + 8	4·5	20

Ask students to predict the sum of the first five even numbers:

$$2 + 4 + 6 + 8 + 10$$

Indicated product: 5·6 = 30 Check by actually adding 2 + 4 + 6 + 8 + 10 or by using a pairing strategy.

$$2 + 4 + 6 + 8 + 10$$

Sum = 2·12 + 6, or 30

Write these data in the table. In each case compare the number of terms with the indicated product. Assist students in observing that expressing the sums as indicated products produces a more helpful pattern than the sums themselves. Use the table to reinforce the meaning of n as the number of the term or the number of terms. (Note underlined numerals!) Develop the following generalization: The sum of the first n even numbers is

$$n\,(n + 1)$$

$$2 + 4 + 6 + \ldots + 2n = n\,(n + 1)$$

Remember: The nth even number is 2n.

Practice using the generalization.

Ask: "What is the sum of the first 10 even numbers?" (10·11 = 110)
"What is the 10th even number?" (2·10 = 20)

Calculate the sum of the first 12, 17, 25 even numbers (12·13 = 156, 17·18 = 306, 25·26 = 650)

Ask: "What are the 12th, 17th, 25th even numbers?"

Some students may be interested in seeing that the distributive property justifies the conclusion that the sum of n even numbers is twice the sum of n natural numbers. That is:

$$2\cdot(1 + 2 + 3 + 4 + 5 + 6) = 2\cdot(\frac{1}{2}\cdot 6\cdot 7)$$

$$2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5 + 2\cdot 6 = 6\cdot 7$$

or

$$2 + 4 + 6 + 8 + 10 + 12 = 6\cdot 7$$

At this point show the relation between the sum of n natural numbers, $\frac{1}{2}n(n + 1)$, and the sum of n even numbers, $n(n + 1)$. Use an example such as this:

Add 6 natural numbers: $\frac{1}{2} \cdot 6 \cdot 7 = 21$

Add 6 even numbers: $6 \cdot 7 = 42$

$$1 + 2 + 3 + 4 + 5 + 6 = \frac{1}{2} \cdot 6 \cdot 7$$
$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$
$$2 + 4 + 6 + 8 + 10 + 12 = 6 \cdot 7$$

Show that each even number is twice the corresponding natural number. Therefore, the sum of the first n even numbers is twice the sum of the first n natural numbers.

Odd Numbers

Refer to Diagram 3.2 in Student Materials and Exercises, page III-5. Ask students to write the sequence of numbers suggested by the unshaded squares (1, 3, 5, 7).

Ask them to extend the pattern of squares. Help students build the pattern by comparing each picture with the corresponding picture of the even numbers of Diagram 3.1, term by term.

$1 \cdot 2 - 1$	$2 \cdot 2 - 1$	$3 \cdot 2 - 1$
1	3	5

Ask: "How do you get the odd number from the corresponding even number? What does the shaded box tell you to do?"

After extending the picture pattern, record the data in a table.

Term	Even Number	Odd Number
1	$1 \cdot 2 = 2$	$1 \cdot 2 - 1 = 1$
2	$2 \cdot 2 = 4$	$2 \cdot 2 - 1 = 3$
3	$3 \cdot 2 = 6$	$3 \cdot 2 - 1 = 5$
4	$4 \cdot 2 = 8$	$4 \cdot 2 - 1 = 7$

COMMENTARY AND NOTES

The presentation of a problem that has no immediate solution is expected to produce tension. Students should be aware that this is part of the problem-solving process. Excitement is high as the solution proceeds.

This section of problems can also be used to review students' understanding of exponents. Students whose grasp of this concept is weak should work with peer tutors. Another teaching strategy is to check students' understanding of exponents before beginning the section on sums of odd numbers. The class can be divided into two groups. Those who understand exponents can be given a set of exercises for group problem solving while you work with the remedial group.

Students should be aware of the reduction of tension when they "see through" a problem or arrive at a solution. It is useful to point this out and to explore briefly how students experience this aspect of the problem-solving process.

Note that Diagram 5.1 (Student Materials and Exercises, page III-7) portrays the sequence of squared numbers and is the same as Diagram 5.2--without shadings.

At this point show the relation between the sum of n natural numbers, $\frac{1}{2}n(n + 1)$, and the sum of n even numbers, $n(n + 1)$. Use an example such as this:

$$\text{Add 6 natural numbers: } \frac{1}{2}\cdot 6\cdot 7 = 21$$

$$\text{Add 6 even numbers: } 6\cdot 7 = 42$$

$$\begin{array}{cccccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & 5 & + & 6 & = \frac{1}{2}\cdot 6\cdot 7 \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\ 2 & + & 4 & + & 6 & + & 8 & + & 10 & + & 12 & = 6\cdot 7 \end{array}$$

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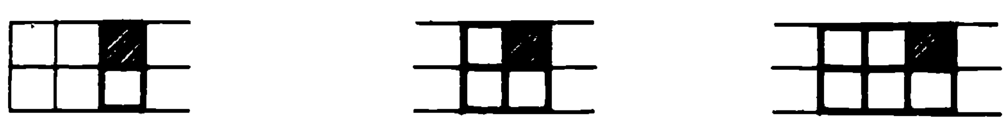

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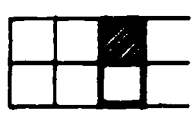
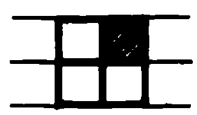


$1 \cdot 2 - 1$
1



$2 \cdot 2 - 1$
3

$3 \cdot 2 - 1$
5

Ask: "How do you get the odd number from the corresponding even number? What does the shaded box tell you to do?"

After extending the picture pattern, record the data in a table.

Term	Even Number	Odd Number
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2	$2 \cdot 2 = 4$	$2 \cdot 2 - 1 = 3$
3	$3 \cdot 2 = 6$	$3 \cdot 2 - 1 = 5$
4	$4 \cdot 2 = 8$	$4 \cdot 2 - 1 = 7$

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Note that Diagram 5.1 (<u>Student Materials and Exercises</u>, page III-7) portrays the sequence of squared numbers and is the same as Diagram 5.2--without shadings.

Ask:	"What is the fifth odd number?"	$2 \cdot 5 - 1 = 9$
	"The tenth?"	$2 \cdot 10 - 1 = 19$
	"The eleventh?"	$2 \cdot 11 - 1 = 21$
	"The nth?"	$2n - 1$

Definition: An odd number is a number that can be written in the form $2n - 1$, where n is a natural number. The nth odd number is $2n - 1$.

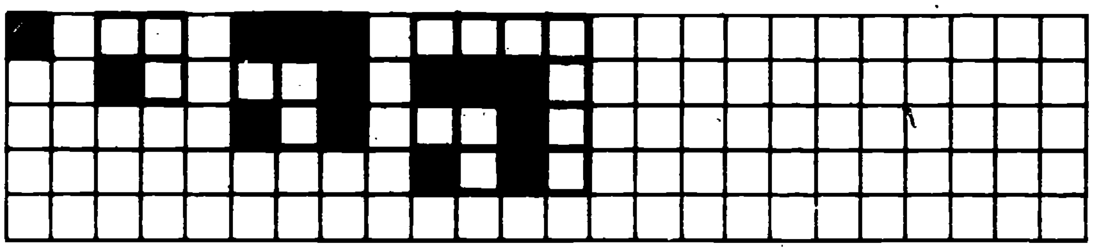
Sums of Odd Numbers

Present the problem: Calculate the sum of the first 25 odd numbers.

Ask questions that will help students see that they should start by solving easier, smaller problems of the same kind, and that the focus is on finding a method for calculating the sum of any number of odd numbers.

Strategy: Calculate the sums of 2, 3, 4 odd numbers. Record the results in a table. Study the data for a pattern.

Refer to Diagram 5.2 (Student Materials and Exercises, page III-7) to check each sum against the sums of the shaded and unshaded squares. Ask students to extend the diagram.



Number of Terms	Indicated Sum	Sum
1	1	$1 = 1^2$
2	1 + 3	$4 = 2^1$
3	1 + 3 + 5	$9 = 3^2$
4	1 + 3 + 5 + 7	$16 = 4^2$
.	.	.
.	.	.
.	.	.

The inductive approach, basic to these materials, is being made explicit at this point to help students begin to develop an awareness of their learning experiences and of teaching strategies.

Say: "Do you notice a pattern? Predict the number of odd numbers that must be added to obtain a 25-unit square; a 36-unit square?"

"What is the sum of the first three odd numbers? The first four? The first ten? The first twenty?"

Ask: "Do you think this pattern will persist? Why?"

The sum of the first n odd numbers can be expressed n^2. It is only because the pattern persists that the sum is predictable and can be expressed in the form n^2.

Remind students that n tells how many odd numbers to add, while n^2 tells their sum.

Square Numbers

Ask: "Why are the numbers 1, 4, 9, 16, 25, 36, ... called square numbers?" Ask students to write the next six squares in the sequence.

Ask: "If the sum of a certain number of odd numbers is 36, how many are there?" (There are 6, because $6^2 = 36$.)

Inductive Reasoning

Review the strategy used to develop the generalization concerning the sums of even and odd numbers.

Discuss the use of formulas and the compactness and advantage of expressing generalizations in this form. Out of many cases comes one formula that includes all of them and many more.

Say: "We have been discerning patterns by examining specific cases. Whenever we use specific cases to generalize, we are using inductive reasoning."

Discuss the cautions that are needed in making generalizations from a limited number of cases, terms, or instances.

Say: "Given 1, 3, 5, ..., what should the next several terms be? Justify each."

(a) 1, 3, 5, 3, 9, 3, 13, ... (3, 17, 3, 21)

(b) 1, 3, 5, 7, 9, 11, 13, ... (15, 17, 19, 21)

Binary Sequence

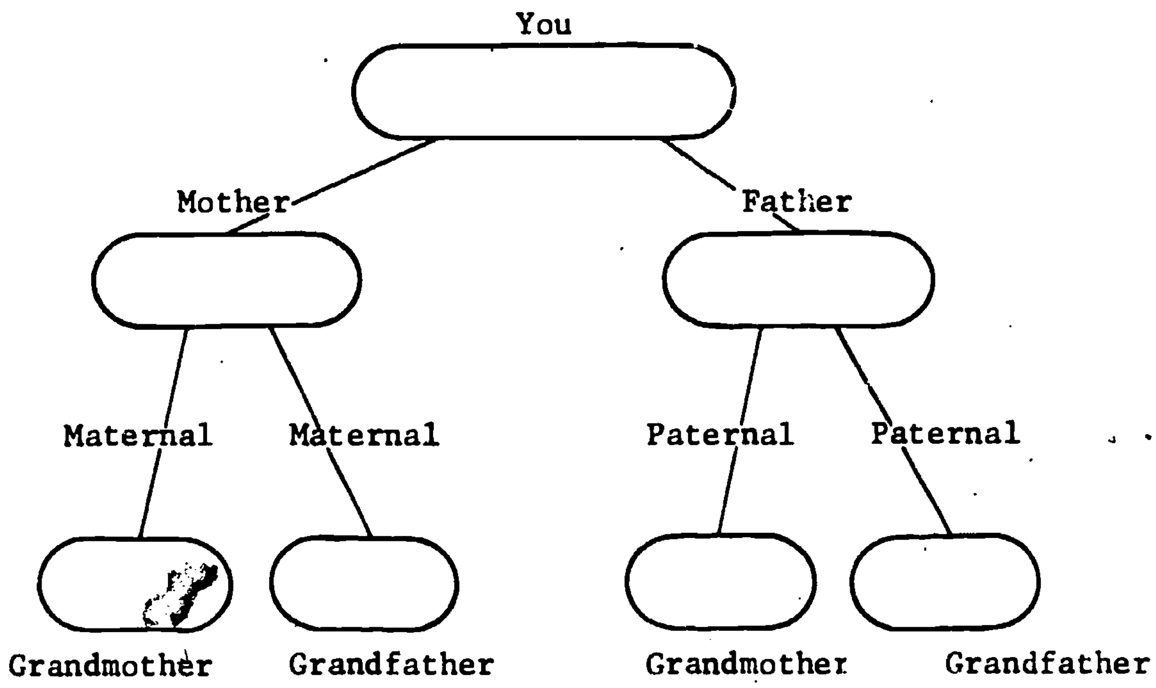
Refer to the pictures of the nautilus shell and the spirals (Diagrams 6.1, 6.2, 6.3, and 7 in Student Materials and Exercises, pages III-8 and III-9).

Ask students to count the length of the side of each square. This leads to the binary sequence:

1, 2, 4, 8, 16, ...

Ask: "What should be the length of each side of the next square?" (32)

As another example of the binary sequence, discuss the Family Tree Exercise (see Student Materials and Exercises). After students have filled in the first few lines of the tree and have answered questions concerning the numbers of grandparents and great-grandparents, work with them to summarize their findings in a table.



Have students fill in the names of their grandparents. Ask them how many great-grandparents they had.

Say: "Suppose you are the first generation, your parents are the second generation, your grandparents the third, and so on. How many ancestors of yours could have been living ten generations ago? How many ancestors would you have to trace altogether?"

Say: "Once again, in order to deal with this problem effectively, it is desirable to record the data systematically. Then we can study the data for any patterns that may exist."

Review exponential notation: Clarify the meaning of 2^{n-1}, $2^n - 1$, and $2n - 1$. Use the calculator to compute powers of 2. Help students see how rapidly the value of 2^n grows.

Here is an opportunity for students to pose problems to be solved or to provide examples. After students generate their own problems and solve them, ask them how they feel. Ask: "Why is the development of questions important to you as learners?"

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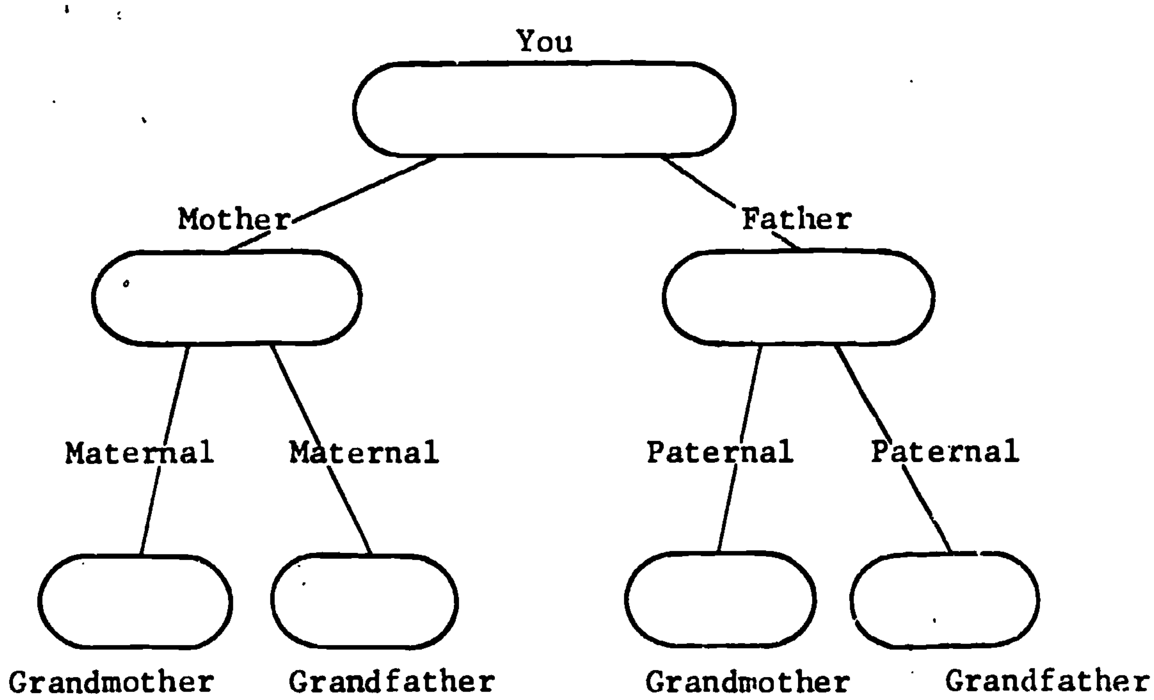

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Here is an opportunity for students to pose problems to be solved or to provide examples. After students generate their own problems and solve them, ask them how they feel. Ask: "Why is the development of questions important to you as learners?"

Generation (Term Number) (n)	Number in Generation (Term)	Total Number of People (Sum of n Terms)
1	1	1
2	$2^1 = 2$	3 (1 + 2)
3	$2^2 = 4$	7 (1 + 2 + 4)
4	$2^3 = 8$	15 (1 + 2 + 4 + 8)
.		
.		
.		
n	2^{n-1}	$2^n - 1$

Work with students to fill in the first two columns of the above table.

Develop the definition $2^0 = 1$ by observing the pattern of exponents.

Ask: "If the pattern is to persist, what power of 2 shall we call 1? It fits into the pattern nicely."

Ask for the fifth term of the binary sequence ($2^4 = 16$) and for the tenth term ($2^9 = 512$) and so forth. In the case of the number in a particular generation, help students observe that he power of 2 is always one less than the number of the term. This leads to the generalization that there were 2^{n-1} ancestors in the nth generation, or:

The nth term of the binary sequence is 2^{n-1}

Since $2^{20} = 1\ 048\ 576$, over a million ancestors may have been living 21 generations ago.

Next, work with students to calculate the sum of any number of terms of the binary sequence. Make the comparisons in the table that lead to the sum as:

$$2^n - 1$$

COMMENTARY AND NOTES

This particular problem has a quality of fun and humor that makes it interesting and appealing. Student-to-student conversation should be expected and encouraged.

Fibonacci Sequence

The Rabbit Problem

The Rabbit Problem is said to have been posed by the mathematician Leonardo Fibonacci (son of Bonacci) in 1202. Fibonacci was also known as Leonardo of Pisa.

Say:

> Suppose there is one pair of newly born rabbits that produce another pair in the second month following birth and each month thereafter. Each additional pair also becomes productive in the second month and breeds in the same way. Assuming that all rabbits live, how many pairs will there be at the end of a year? Let us record what is taking place and actually count the number of pairs during the first several months.

Ask students to read the Rabbit Problem (see Student Materials and Exercises). Then work with them to record what is taking place. Count the number of pairs of rabbits during the first several months. Work through line 5 and let the class continue with line 6.

Ask students to predict the number of pairs for month number 7.

Month Number	Rabbit Pairs	Number of Pairs
1	R_1*	1
2	R_1	1
3	R_1 R_2	2
4	R_1 R_3 R_2	3
5	R_1 R_4 R_3 R_2 R_5	5
6	R_1 R_6 R_4 R_3 R_7 R_2 R_8 R_5	8

*R_1 represents the first pair of rabbits.

COMMENTARY AND NOTES

As with other teaching suggestions, several alternate strategies can be used here. Students can work in pairs or small groups; students can work at the board; students can work individually. Students who grasp mathematical concepts more readily can work with one or two students who have more difficulty.

Students may wish to state their feelings about this mathematical discovery.

The calculator should be used by any student who wishes to use it.

Provide the opportunity for students to discuss some of the ideas, skills and techniques they have learned from this module. This will give them the opportunity to review and summarize.

Refer to the Instructor's Handbook for ideas on quiz preparation and student evaluation.

Allow students time to discover the pattern in the Fibonacci sequence:

1, 1, 2, 3, 5, 8, ...   (2 = 1 + 1, 3 = 2 + 1, 5 = 3 + 2, 8 = 5 + 3)

Many students will be able to predict the next term of the sequence (13). They should verify it by using the pattern and write additional terms (21, 34, 55, 89, 144, 233).

The numbers of the Fibonacci sequence exhibit many surprising properties.

Ask students to select any two small natural numbers and generate their own Fibonacci-type sequence. Record one such sequence on the board (perhaps 1, 3, 4, 7, 11, 18, ...) and have one or more students use that sequence to calculate the sum of the first ten terms.

Before students have recorded the sum, write the sum on the board:

11 x (7th term)

For instance:

1 + 1 + 2 + 3 + 5 + 8 + (13) + 21 + 34 + 55 = 143

and

11 x (13) = 143

or:

1 + 3 + 4 + 7 + 11 + 18 + (29) + 47 + 76 + 123 = 319

and

11 x (29) = 319

This "trick" is a property of all Fibonacci-type sequences.

STUDENT EVALUATION

An approach to evaluation is provided in the student evaluation section of the Instructor's Handbook, along with ideas for creating a classroom climate that encourages and supports students' achievement. Suggestions are offered for helping students to prepare for quizzes, for providing feedback on performance, and for reassessing when necessary.

The ten Patterns objectives below are accompanied by sample items. Select items according to the content objectives you have covered in the module.

Objective 1. Identify and write triangular numbers

a. Write the twelfth triangular number.

b. Write the ninth triangular number.

c. Write the tenth triangular number.

Objective 2. Apply the formula n^2 to calculate the sum of the first n odd numbers.

a. Calculate the sum of the first 12 odd numbers.

b. The sum of a certain number of terms of the sequence 1, 3, 5, 7, ... is 81. How many terms are there?

Objective 3. Show by diagram that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

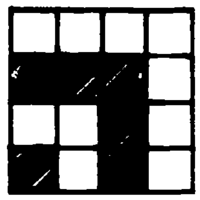
a. Fill in the missing number. The accompanying diagram is shaded to show that:

$1 + 3 + 5 + 7 + 9 = \square^2$

60

b. What is the missing number? The accompanying diagram has been shaded to show that:

$1 + 3 + 5 + 7 = \square^2$



Objective 4. Apply the formula $\frac{1}{2}n(n + 1)$ to calculate the sum of the first $\underline{n}$ natural numbers.

a. Use a pattern or the formula to calculate the sum of the first 17 natural numbers.

b. Use a pattern or the formula to calculate the sum of the first 19 natural numbers.

c. Use a pattern or the formula to calculate the sum of the first 21 natural numbers.

Objective 5. Apply the formula $n(n + 1)$ to calculate the sum of the first $\underline{n}$ even numbers.

a. Express the sum of the first 31 even numbers as an <u>indicated product</u>.

b. Express the sum of the first 43 even numbers as an <u>indicated product</u>.

c. Express the sum of the first 27 even numbers as an <u>indicated product</u>.

d. The sum of a certain number of natural numbers is 300. What is the sum of the same number of even numbers?

e. The sum of a certain number of natural numbers is 110. What is the sum of the same number of even numbers?

f. The sum of a certain number of natural numbers is 91. What is the sum of the same number of even numbers?

Objective 6. Demonstrate knowledge of the sums of even and odd numbers.

a. Show, by specific example, that the following statement is <u>false</u>:

The sum of two odd numbers is an odd number.

b. The sum of one odd number and one even number is

(1) sometimes odd and sometimes even.

(2) always odd.

(3) even, if the even number is greater.

(4) always even.

Objective 7. Identify and write a specified term of the binary sequence.

a. A certain term of the binary sequence is 2^9. Which term is it?

b. Write the eleventh term of the binary sequence.

Objective 8. Express the sum of any number of terms, <u>n</u>, of the binary sequence in the form $2^n - 1$.

a. Express the sum of the first <u>10</u> terms of the sequence 1, 2, 4, 8, ... <u>in exponential form</u>.

b. Express the sum of the first <u>12</u> terms of the sequence 1, 2, 4, 8, ... <u>in exponential form</u>.

c. Express the sum of the first <u>8</u> terms of the sequence 1, 2, 4, 8, ... <u>in exponential form</u>.

Objective 9. Write a specified number of terms of a Fibonacci-type sequence.

a. Write the next three terms of the sequence 1, 3, 4, 7, 11, ...

b. The first four terms of the Fibonacci sequence are 1, 1, 2, 3. Write the next four terms.

Objective 10. Follow and extend a given number sequence.

a. Write the next <u>three</u> terms of the sequence .3, .03, .003, .0003, ...

b. Write the next <u>three</u> terms of the sequence .6, .06, .006, .0006, ...

c. Write the next three terms of the sequence
.1, .01, .001, .0001, ...

d. Write the next three numbers in the sequence
2, 3, 5, 8, _____, _____, _____

TEAM

PATTERNS

II

INSTRUCTOR'S GUIDE AND SOLUTIONS TO STUDENT EXERCISES

This section of _Patterns_ contains solutions for exercises presented in _Student Materials and Exercises_. The solutions are accompanied by some explanations and suggestions.

The exercises include problems that apply the concepts and problem-solving strategies developed in this module. They can be used as part of instructional activities, as in-class activities for individuals or small groups, as assignments, or as review materials. The asterisk (*) denotes a more challenging problem.

1. Find a pattern and extend each sequence by five more numbers.

a. 3, 6, 12, 24, ...

b. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

c. 1, 2·1, 3·2·1, 4·3·2·1, ...

d. $3^0, 3^1, 3^2, 3^3, \ldots$

e. $1^3, 2^3,$ ... $4^3, 5^3, \ldots$

In these examples, even if the mathematical definitions are not familiar--for example, factorial in c or exponential in d--it is the pattern that we are looking for.

a. 48, 96, 192, 384, 768
(Each number is double the previous number.)

b. $\frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}$

c. 5·4·3·2·1, 6·5·4·3·2·1, 7·6·5·4·3·2·1, 8·7·6·5·4·3·2·1,
9·8·7·6·5·4·3·2·1

d. $3^4, 3^5, 3^6, 3^7, 3^8$

e. $6^3, 7^3, 8^3, 9^3, 10^3$

2.

$$1^3 = 1^2$$
$$1^3 + 2^3 = 3^2$$
$$1^3 + 2^3 + 3^3 = 6^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = \boxed{10}^2$$

a. Verify each of the above statements. Use a calculator for verification.

$$1 + 8 = 9$$
$$1 + 8 + 27 = 36$$
$$1 + 8 + 27 + 64 = 100, \text{ or } 10^2$$

b. Do you see a pattern? What should the missing number be? $\boxed{10}$
Check to see whether you are correct.

c. Write the next two lines of the pattern. Verify each.

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 15^2$$
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = 21^2$$

3. Calculate the sum of

   a. The first 25 natural numbers.
   b. The first 31 natural numbers.
   c. The first 40 natural numbers.
   d. The first 50 natural numbers.
   e. The first 99 natural numbers.
   f. The first 450 natural numbers.

   Answer: Various methods may be used.

   Using the formula $S = \frac{1}{2}n(n + 1)$:

   $S = \frac{1}{2}(25)(26) = \boxed{325}$

   $S = \frac{1}{2}(31)(32) = \boxed{496}$

   $S = \frac{1}{2}(40)(41) = \boxed{820}$

   $S = \frac{1}{2}(50)(51) = \boxed{1\ 275}$

   $S = \frac{1}{2}(99)(100) = \boxed{4\ 950}$

   $S = \frac{1}{2}(450)(451) = \boxed{101\ 475}$

4. Look at the picture of a pile of logs. How many logs would there be in such a pile if there are 12 in the bottom layer, 11 in the next highest layer, and so forth, until there is 1 at the top?

   $12 + 11 + \ldots + 2 + 1 = ?$

   Use the formula for the sum of the first twelve natural numbers:

   $$\frac{1}{2}n\,(n + 1)$$

   where $n = 12$

   $$\frac{1}{2}(12)(13) = 6(13), \text{ or } \boxed{78}$$

   Note that other methods can also be used.

5. Calculate the total number of gifts given in "The Twelve Days of Christmas," given that:

> On the first day of Christmas my true love gave to me a partridge in a pear tree. On the second day of Christmas my true love sent to me two turtle doves and a partridge in a pear tree. . . . On the twelfth day of Christmas, my true love sent to me:
>
> Twelve drummers drumming,
>
> Eleven pipers piping,
>
> Ten lords-a-leaping,
>
> Nine ladies dancing,
>
> Eight maids-a-milking,
>
> Seven swans-a-swimming,
>
> Six geese-a-laying,
>
> Five golden rings,
>
> Four calling birds,
>
> Three French hens,
>
> Two turtle doves,
>
> A partridge in a pear tree.

Note that the solution to Exercise 4 is just one part of this problem: the number of gifts received on Day 12. But here we really must solve twelve separate problems and then add the results.

Use of a calculator might be in order.

$$\text{Total gifts for Day } 12 = \frac{1}{2}(12)(13) = 78$$

$$\text{Total gifts for Day } 11 = \frac{1}{2}(11)(12) = 66$$

$$\text{Total gifts for Day } 10 = \frac{1}{2}(10)(11) = 55$$

$$\text{Total gifts for Day } 9 = \frac{1}{2}(9)(10) = 45$$

$$\text{Total gifts for Day } 8 = \frac{1}{2}(8)(9) = 36$$

$$\text{Total gifts for Day } 7 = \frac{1}{2}(7)(8) = 28$$

$$\text{Total gifts for Day } 6 = \frac{1}{2}(6)(7) = 21$$

$$\text{Total gifts for Day } 5 = \frac{1}{2}(5)(6) = 15$$

$$\text{Total gifts for Day } 4 = \frac{1}{2}(4)(5) = 10$$

$$\text{Total gifts for Day } 3 = \frac{1}{2}(3)(4) = 6$$

$$\text{Total gifts for Day } 2 = \frac{1}{2}(2)(3) = 3$$

$$\text{Total gifts for Day } 1 = \frac{1}{2}(1)(2) = 1$$

$$\text{Total} = \boxed{364}$$

Note: These are the triangular numbers.

6. If, on the first day of Chanukah, you light 2 candles; on the second day, 3; ...; on the eighth day, 9: How many candles must a box contain to suffice for the eight days?

$$2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$$

You can use Gauss's method and pair off to get 4 pairs of 11:

$$4(11) = 44$$

or solve for $\underline{1} + 2 + 3 + 4 + \ldots + 8 + 9$ using $\frac{1}{2}n(n + 1)$ where $n = 9$

$$\frac{1}{2}\cdot 9\cdot 10 = 45$$

and subtract 1, since the smallest number of candles used is 2.

7. Calculate each sum:

a. $7 + 8 + 9 + 10 + 11 + 12 + 13$

$$7 + 8 + 9 + 10 + 11 + 12 + 13$$

These can be paired off, as indicated, yielding

3 pairs of 20	= 60
plus one 10	+ 10
	70

You can also determine the sum of the first 13 natural numbers ($\frac{1}{2}\cdot 13\cdot 14 = 91$) and subtract from this the sum of the first 6 natural numbers ($\frac{1}{2}\cdot 6\cdot 7 = 21$) since the sum above begins at 7.

b. $32 + 33 + 34 + 35 + \ldots + 46 + 47 + 48$

$$32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 + 41 + \ldots + 45 + 46 + 47 + 48$$

8 pairs of 80 = 640
plus one 40 + 40

680

or

$$\frac{1}{2}\cdot 48\cdot 49 = 1176$$

$$-\frac{1}{2}\cdot 31\cdot 32 = -496$$

680

8. If a clock strikes only the hours--once for one o'clock, two times for two o'clock, etc.--how many times does it strike in a 24-hour day?

Solve for the number of gongs in 12 hours and double the result.

Note that

$$\underbrace{1 + 2 + 3 + \ldots + 12}_{} + \underbrace{13 + \ldots + 24}_{}$$

is not equal to

$$2 \times (1 + 2 + 3 + \ldots + 12)$$

since

$$(13 + 14 + \ldots + 24) \neq (1 + 2 + \ldots + 12)$$

$$1 + 2 + 3 + \ldots + 12 = \frac{1}{2}n(n + 1) \text{ where } n = 12$$

$$\frac{1}{2}\cdot 12\cdot 13 = 6\cdot 13, \text{ or } 78$$

$$2 \times 78 = \boxed{156}$$

9. Calculate:

a. The sixteenth triangular number.
b. The seventeenth triangular number.
c. The eighteenth triangular number.
d. The fiftieth triangular number.
e. The eleventh triangular number.

Answer:

Note that this is equivalent to the statement: Calculate the sum of the first 16 natural numbers, using the formula for the nth triangular number:

$$\frac{1}{2}n(n + 1)$$

a. $\frac{1}{2}(16)(17) = \boxed{136}$

b. $\frac{1}{2}(17)(18) = \boxed{153}$

c. $\frac{1}{2}(18)(19) = \boxed{171}$

d. $\frac{1}{2}(50)(51) = \boxed{1275}$

e. $\frac{1}{2}(11)(12) = \boxed{66}$

10. a. Add pairs of consecutive triangular numbers. Record the data in the table provided. Is there a pattern to the sums? (Use the first ten triangular numbers.)

Answer: They are square numbers.

b. Compute the differences of the consecutive triangular numbers. What do you notice?

Answer: The differences listed in order are the natural numbers beginning with 2.

c. How do the differences compare with the sums?

Answer: The differences are the square roots of the respective sums.

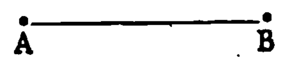
Triangular Numbers	Sums	Differences
1		
	4	2
3		
	9	3
6		
	16	4
10		
	25	5
15		
	36	6
21		
	49	7
28		
	64	8
36		
	81	9
45		
	100	10
55		

11. There are 15 people in a room. Each is to greet each of the other persons with a handshake. What is the total number of handshakes? (Hint: Begin with simpler situations--2 people, 3 people, etc. Record the results. Look for a pattern!)

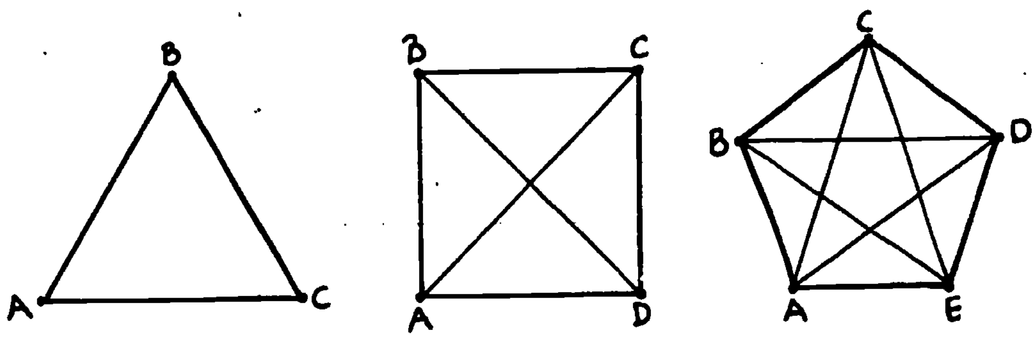
You can have the students actually do these handshakes in class and then count them.

A good way to picture the number of handshakes needed is to let each point represent a person and each line joining two points, a handshake. No three points may be collinear.

Then, in the case of two people, A and B,



you have one handshake.



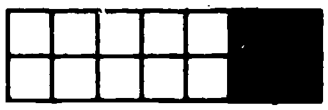
The results are triangular numbers.

Term Number	Number of People	Number of Handshakes
1	2	1
2	3	3
3	4	6
4	5	10
.	.	.
.	.	.
.	.	.

12. Add two even numbers. Is their sum even or odd? Try another pair. Will this be true for <u>any</u> two even numbers?

Answer: It is always even.

Use a pair of pictures of even numbers to help you justify your answers. You may find it helpful to cut them out and move them around. (See diagram below).



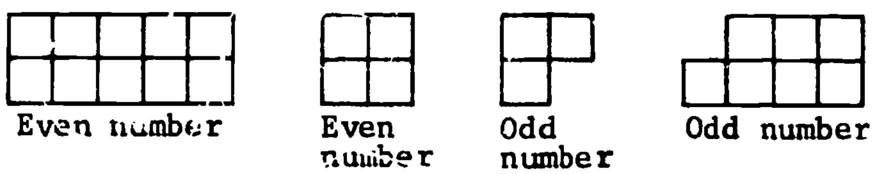
13. Repeat the above procedure for a pair of odd numbers. What can you say about the sum of any two odd numbers?

Answer: It is always even.

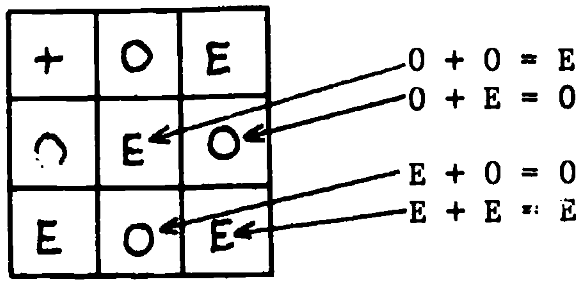
Use pictures to justify your answer.

14. If you add an even number and an odd number, what can you say about the sum? Why?

Answer: It is always odd. Odd means there is always one left over when you attempt to divide it in half.



15. Summarize your findings in the accompanying table. O stands for any odd number and E stands for any even number.



16. Calculate:

a. The tenth even number.
b. The twenty-fifth even number.
c. The thirtieth even number.

Answer: Recall that the nth even number = 2n. Therefore:

a. 2·10 = 20

b. 2·25 = 50

c. 2·30 = 60

17. What is the sum of

a. The first 10 even numbers?
b. The first 25 even numbers?
c. The first 30 even numbers?

Answer: The sum of the first n even numbers is n(n + 1). Therefore:

a. (10)(11) = 110

b. (25)(26) = 650

c. (30)(31) = 930

18. What is:

a. The ninth odd number?
b. The eleventh odd number?
c. The twenty-fifth odd number?

Answer: The nth odd number = $2n - 1$. Therefore:

a. $2(9) - 1 = \boxed{17}$

b. $2(11) - 1 = \boxed{21}$

c. $2(25) - 1 = \boxed{49}$

19. Calculate the sum of:

a. The first 10 odd numbers.
b. The first 30 odd numbers.
c. The first 12 odd numbers.

Answer: The sum of the first n odd numbers = n^2. Therefore:

a. $(10)^2 = \boxed{100}$

b. $(30)^2 = \boxed{900}$

c. $(12)^2 = \boxed{144}$

20. a. Find the sum of the first 3 even numbers.
b. Find the sum of the first 3 odd numbers
c. Find the sum of the first 6 natural numbers.
d. How does the sum of the results for a and b compare with your result for c?

Answer:

a. $n(n + 1) = 3(4)$, or $\boxed{12}$

b. $n^2 = 3^2$, or $\boxed{9}$

c. $\frac{1}{2}n(n + 1) = \frac{1}{2} 6 \cdot 7$, or $\boxed{21}$

d. $9 + 12 = 21$

An interesting result--they are equal. The sum of the first 3 even numbers plus the sum of the first 3 odd numbers is equivalent to the sum of the first 6 natural numbers.

21. a. What is the eighth even number? The twentieth?

$2 \cdot 8 = \boxed{16}$

$2 \cdot 20 = \boxed{40}$

b. Calculate the sum of the first 8 even numbers.

$\underline{n} = 8$

$n(n + 1) =$

$8 \cdot 9 = \boxed{72}$

c. Calculate the sum of the first 20 even numbers.

$\underline{n} = 20$

$n(n + 1) =$

$20(21) = \boxed{420}$

d. Use your answers to b and c to calculate the sum

$18 + 20 + 22 + \ldots + 40$

$420 - 72 = \boxed{348}$

22. Make up and solve a problem similar to Exercise 21, using odd numbers.

23. a. How many consecutive odd numbers, starting with 1, must a person add to yield a sum of 169?

Answer: $n^2 = 169$; $\underline{n} = 13$

You do not even need an algorithm to figure out the square root.

$10 \times 10 = 100$
$20 \times 20 = 400$

So the result must be between 10 and 20.

For determining the units digit, note that $3 \times 3 = 9$ and $7 \times 7 = 49$, so the results must be either 13×13 or 17×17.

b. Show <u>without adding</u> that it is impossible to add consecutive odd numbers starting with 1 and produce a sum of 1237.

Answer: 1237 is not a perfect square

$35^2 = 1225$

$\longleftarrow 1237$

$36^2 = 1296$

No number multiplied by itself gives a 7 in the units digit.

24. Suppose you fold a piece of paper in half. That fold will divide the sheet into two equal parts. Suppose, after folding the paper, you fold it in half again. The paper will then be divided into four equal parts. The table shows what takes place during the first several folds.

Number of Folds	Number of Parts
1	2
2	4
3	8

Suppose it is possible to continue the folding process indefinitely. How many equal parts will there be after 4 folds? 5 folds? _n_ folds?

Answer: $2^4 = \boxed{16}$

$2^5 = \boxed{32}$

$\boxed{2^n}$

Number of Folds	Number of Parts
0	1
1	2 (2)
2	4 (2 x 2)
3	8 (2 x 2 x 2)
4	16 (2 x 2 x 2 x 2)
5	32 (2 x 2 x 2 x 2 x 2)
.	
.	
.	
25	
.	
.	
.	
n	

Elicit from the students that, according to the pattern, the number of parts doubles each time. These questions might be asked:

"Can anyone see the relation between the number of folds and the number of parts?"

"Is there multiplication involved?"

"How many 2's would be multiplied to get 16?"

25. Write the twelfth term of the binary sequence (in exponential form). Now write the eightieth term.

Answer: The nth term of the binary sequence is 2^{n-1}. Therefore:

$$2^{12-1} = 2^{11}$$

$$2^{80-1} = 2^{79}$$

26. Write the sum of:

a. The first 12 terms of the binary sequence (in exponential form).
b. The first 80 terms of the binary sequence.

Answer: The sum of the first n terms of the binary sequence is $2^n - 1$. Therefore:

$$2^{12} - 1$$

$$2^{80} - 1$$

80

27. Investigate the differences between the squares of two successive odd numbers. Is there a pattern?

Number	Square of the Number	Difference
1	1	
		8
3	9	
		16
5	25	
		24
7	49	
		32
9	81	

Answer: The differences between the squares of two consecutive odd numbers is always a multiple of 8.

$8 = \underline{1} \cdot 8$

$16 = \underline{2} \cdot 8$

$24 = \underline{3} \cdot 8$

$32 = \underline{4} \cdot 8$

28. Under each term of the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... write O or E (determined by whether the number is odd or even).

a. Is there a pattern to the O's and E's?

Answer: Yes

b. Explain the answer to a on the basis of your knowledge of the sums of odd and even numbers.

1, 1, 2, 3, 5, 8, ...

O O E O O E

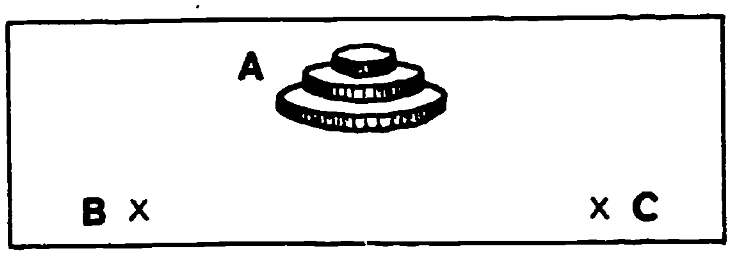
When adding two odd numbers, you will get an even number, as in 3 + 5 = 8. However, since the preceding number, 5, is odd, you will get an odd number, since O + E = O:

5 + 8 = 13 (odd)

Then E + O = O, e.g., 8 + 13 = 21

*29. Game

a. Choose three specific locations on a piece of paper. Place three disks or coins of three different sizes in one location, as shown.



b. Move the disks, one at a time, to another location, using the least number of moves, so that the disks are arranged in the same way according to size. Move only one disk at a time. Verify that the minimum number of moves required is 5.

*The asterisk denotes a more challenging problem.

c. Suppose the game were played with 2 disks, 4 disks, or 6 disks. What would happen? Record your results. Is there a pattern? Try to write a formula for the number of moves required for any number of disks.

Number of Disks	Number of Moves
1	1
2	3
3	5
4	7
5	9
6	11
.	.
.	.
.	.
n	2n - 1

*30. Tower of Hanoi puzzle

Refer to Exercise 29. This time use the following additional rule: A larger disk may never be placed on top of a smaller one. What is the minimum number of moves for 10 disks? (Hint: Begin with 1 disk, then 2 disks, 3 disks, and so on. Look for a pattern!)

The new restriction is that a larger disk may not be placed on top of a smaller one.

Number of Disks	Number of Moves
1	1
2	3
3	7
4	15
.	.
.	.
.	.
n	$2^n - 1$

Note the sequence formed by the differences of the terms in the right-hand column (number of moves). The sequence of differences is 2, 4, 8, 16, ...

*The asterisk denotes a more challenging problem.

TEAM

PATTERNS

III

STUDENT MATERIALS AND EXERCISES

STUDENT MATERIALS

Diagram 1.1

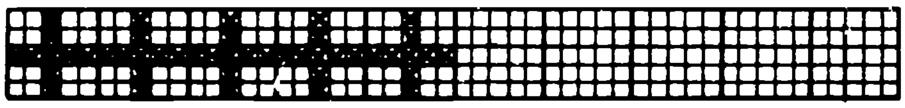


Diagram 1.2

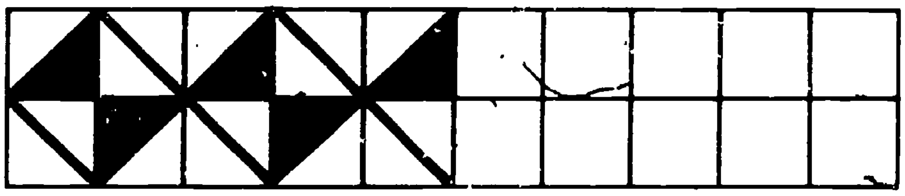


Diagram 1.3

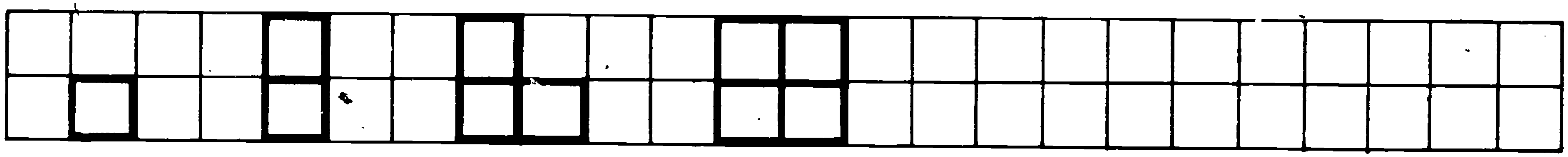


Diagram 2.1

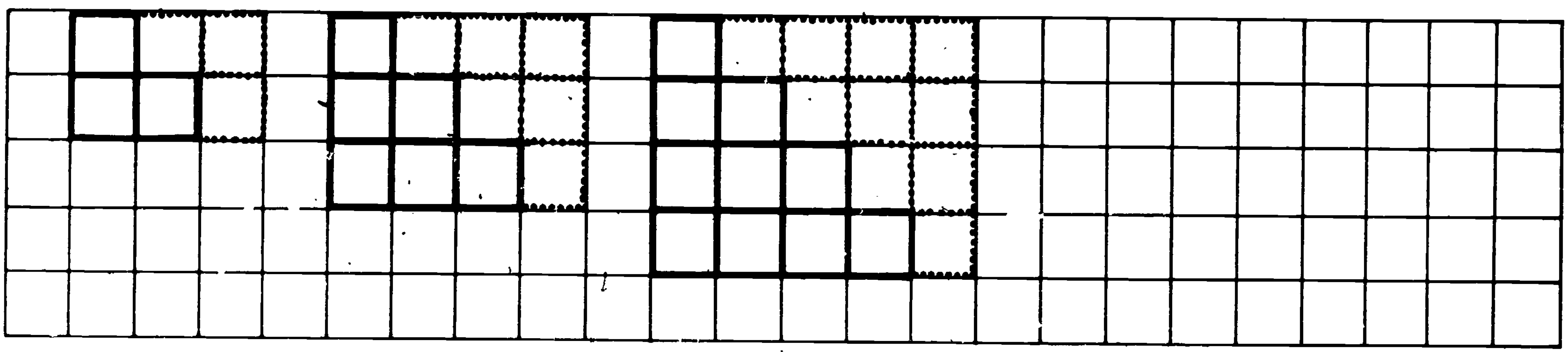


Diagram 2.2

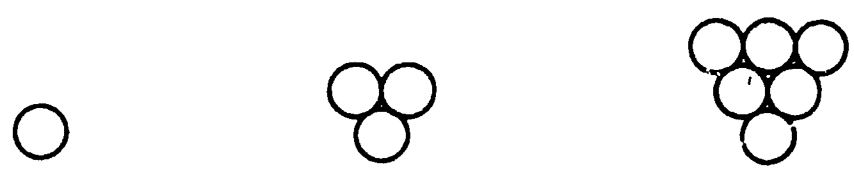


Diagram 2.3

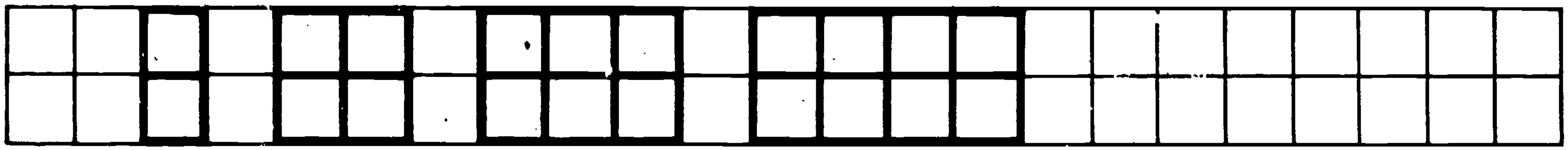


Diagram 3.1

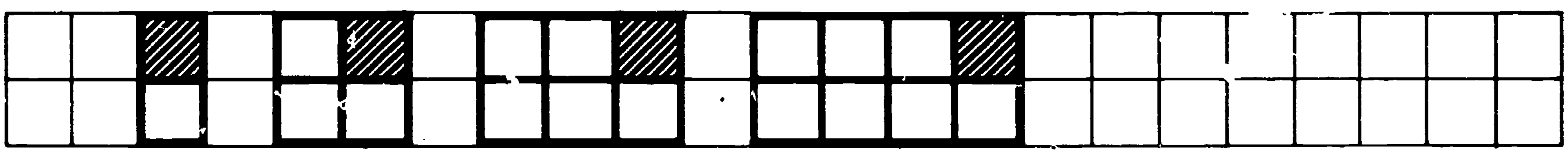


Diagram 3.2

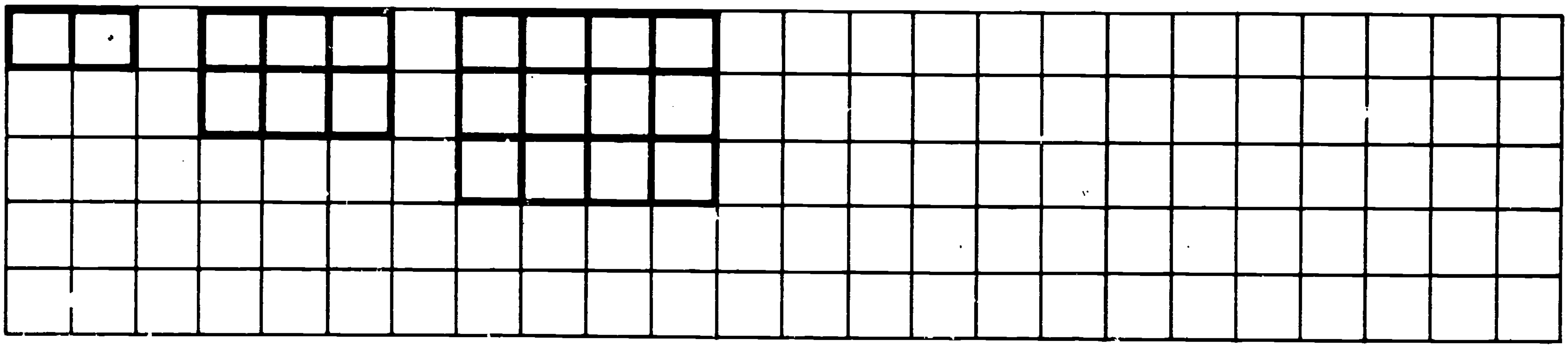


Diagram 4.1

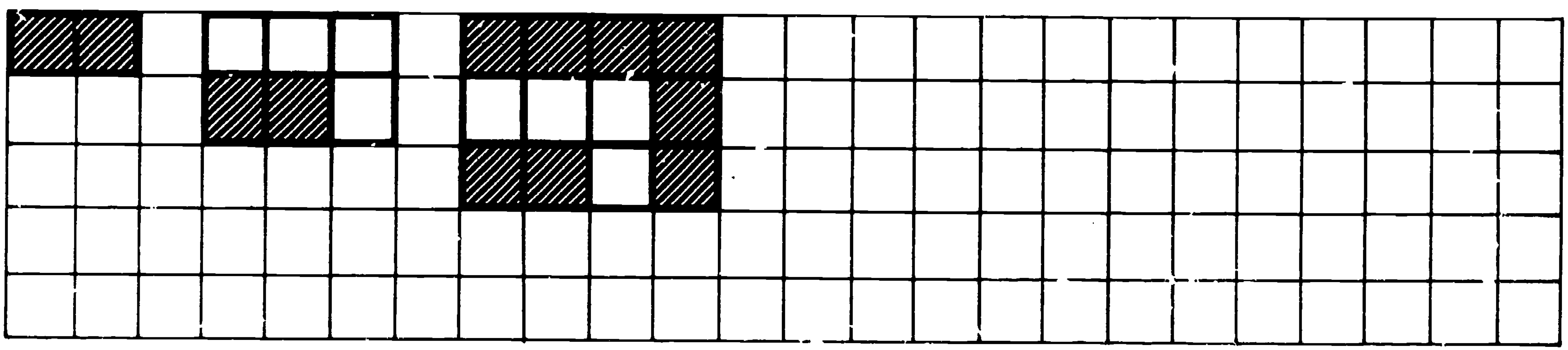


Diagram 4.2

III-7

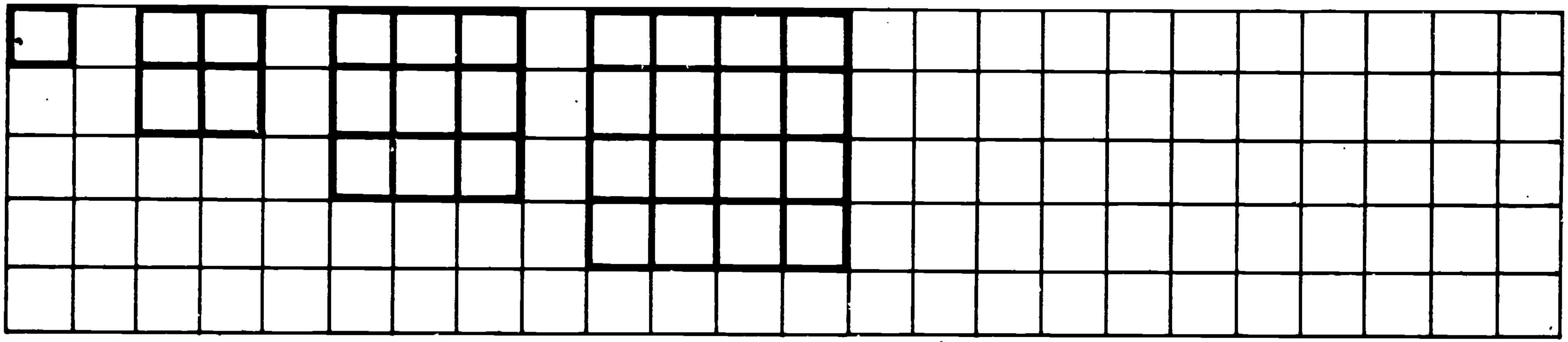
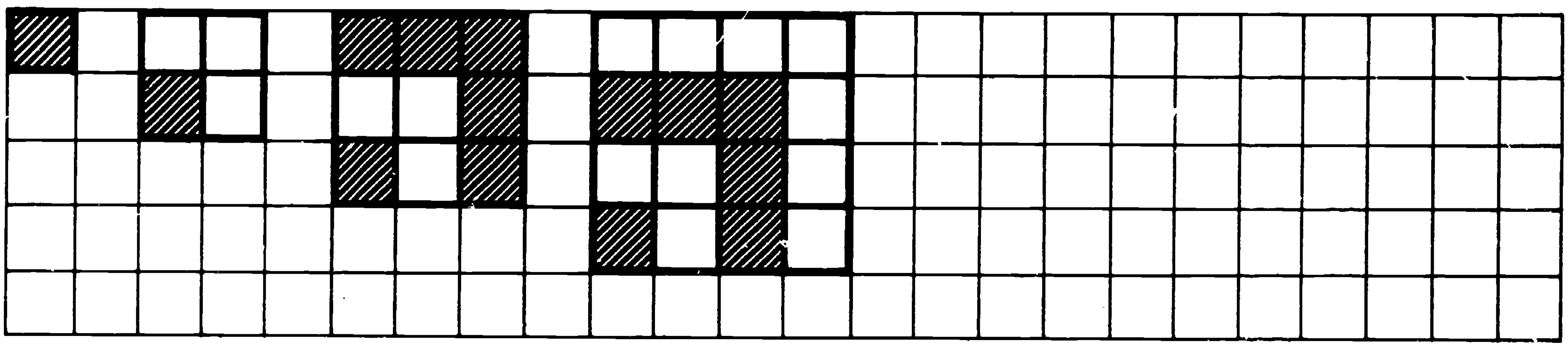


Diagram 5.1



Di·gram 5.2

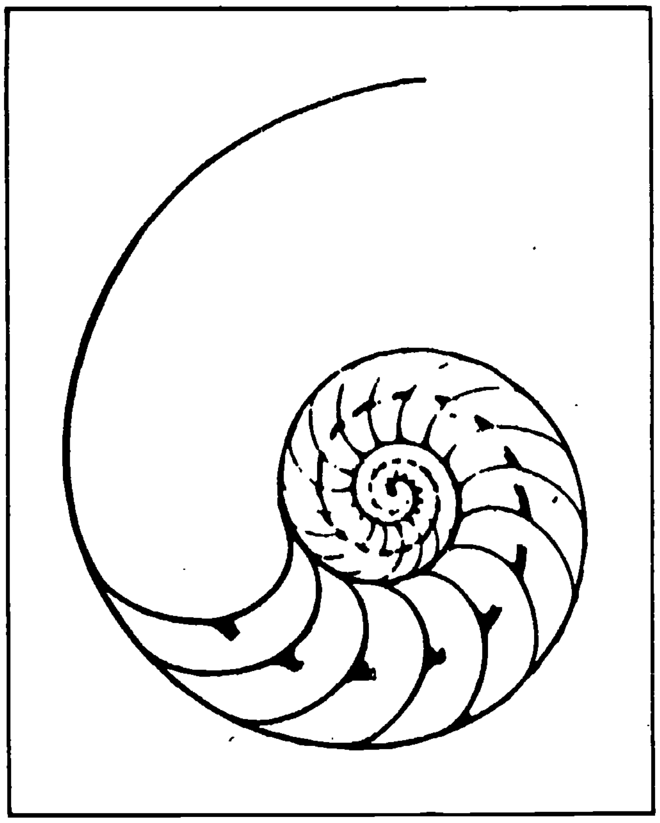


Diagram 6.1

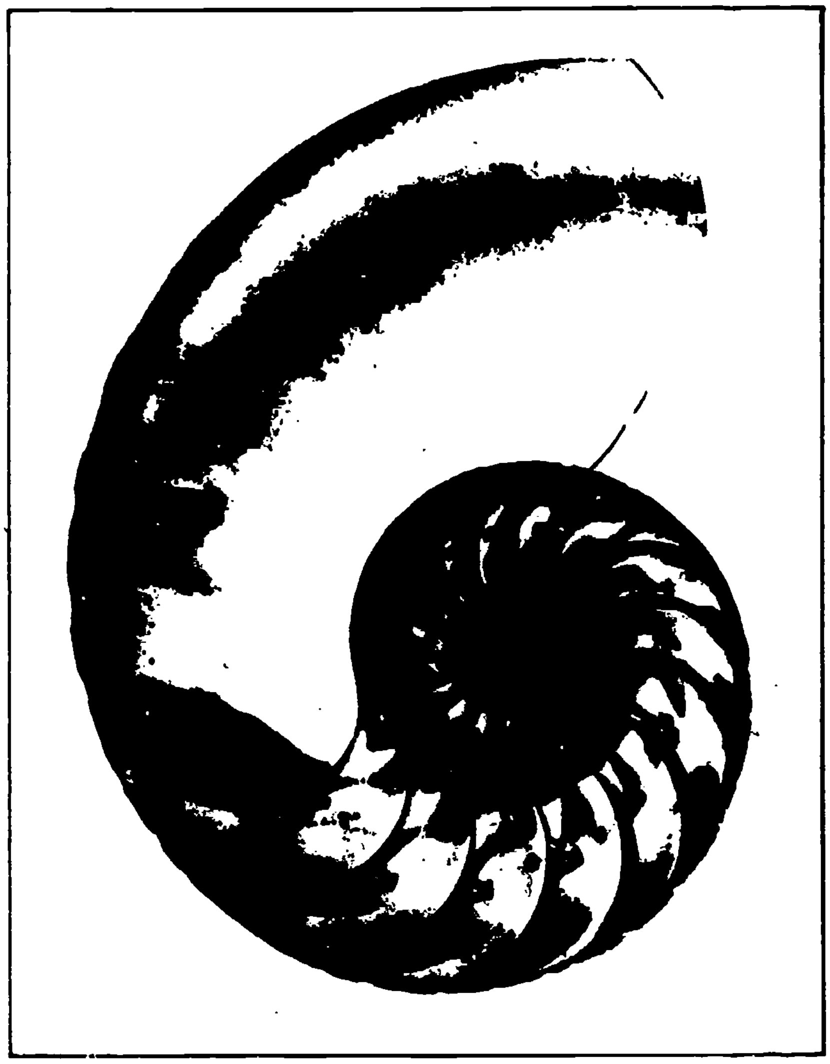


Diagram 6.2

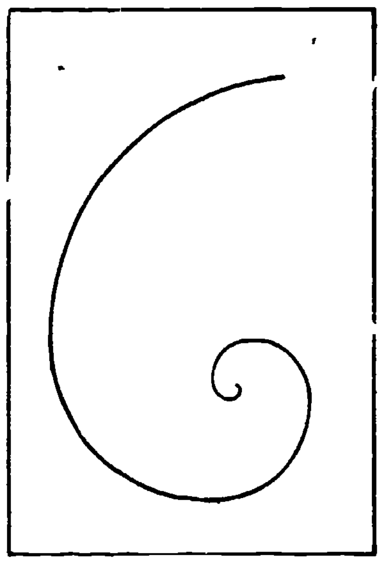


Diagram 6.3

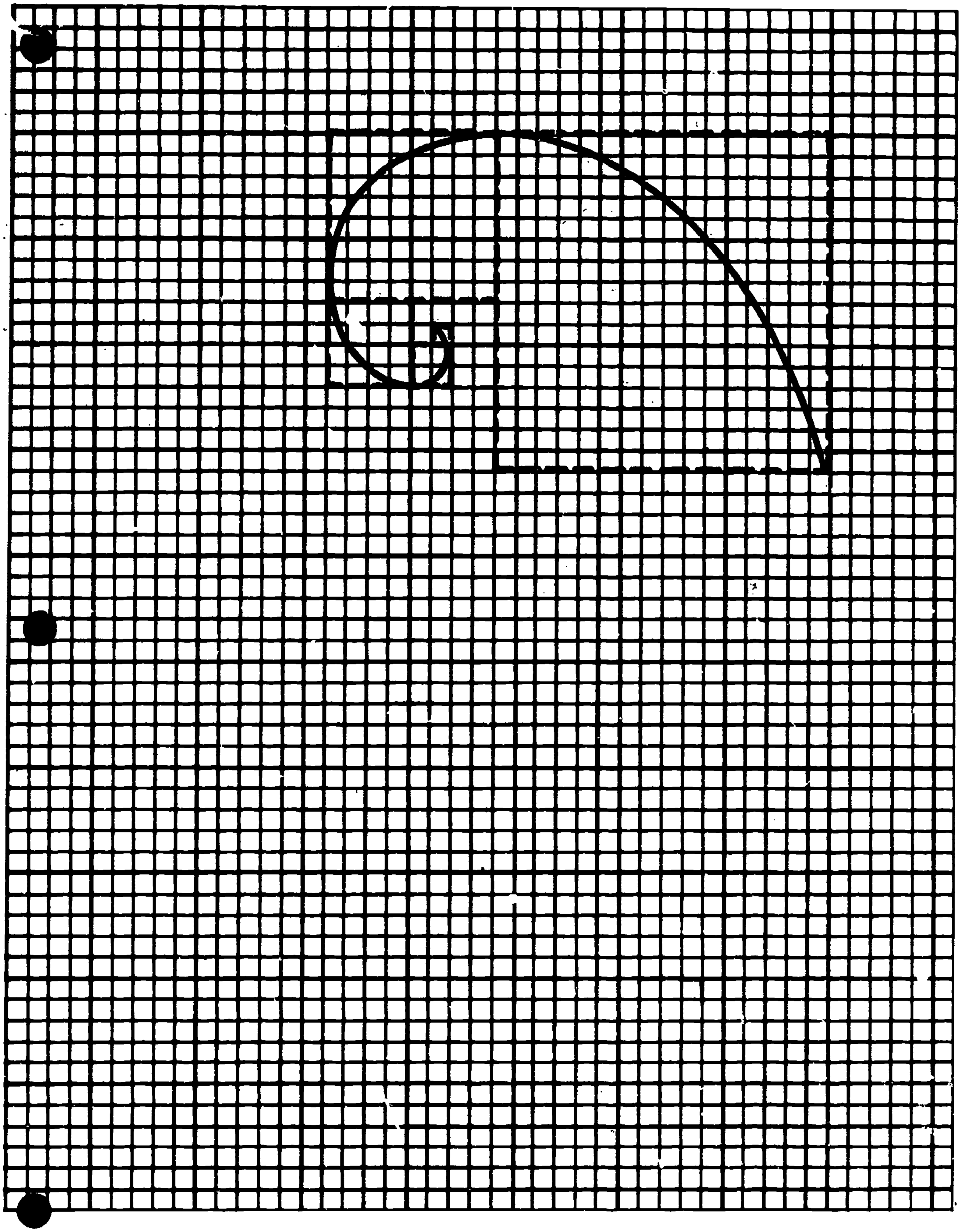
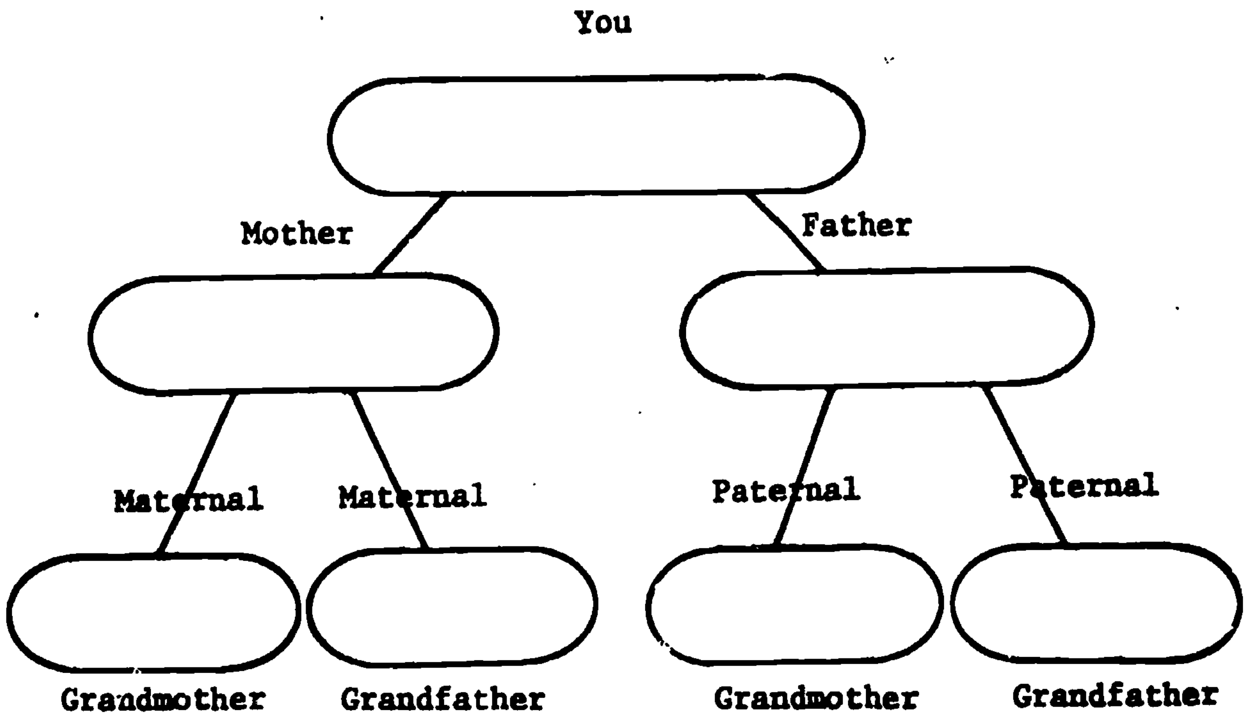


Diagram 7

THE FAMILY TREE EXERCISE

The construction of family trees has become a popular pastime. You might begin your family tree like this:

Family Tree



Can you fill in the names of your four grandparents? How about the next generation, your great-grandparents? How many great-grandparents did you have?

Generation (Term Number) (n)	Number in Generation (Term)	Total Number of People (Sum of n Terms)

THE RABBIT PROBLEM

The Rabbit Problem is said to have been posed by the mathematician Leonardo Fibonacci (son of Bonacci) in 1202. Fibonacci was also known as Leonardo of Pisa.

The Problem

Suppose there is one pair of newly born rabbits that produce another pair in the second month following birth and each month thereafter. Each additional pair also becomes productive in the second month and breeds in the same way. Assuming that all rabbits live, how many pairs will there be at the end of a year?

Month Number	Rabbit Pairs	Number of Pairs
1	R_1	
2	R_1	
3	R_1 R_2	
4		
5		
6		
7		

STUDENT EXERCISES

1. Find a pattern and extend each sequence by five more numbers.

   a. 3, 6, 12, 24, ...

   b. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

   c. 1, 2•1, 3•2•1, 4•3•2•1, ...

   d. $3^0, 3^1, 3^2, 3^3, \ldots$

   e. $1^3, 2^3, 3^3, 4^3, 5^3, \ldots$

2.

$$\begin{aligned} 1^3 &= 1^2 \\ 1^3 + 2^3 &= 3^2 \\ 1^3 + 2^3 + 3^3 &= 6^2 \\ 1^3 + 2^3 + 3^3 + 4^3 &= \square^2 \end{aligned}$$

   a. Verify each of the above statements. Use a calculator for verification.

   b. Do you see a pattern? What should the missing number be? Check to see whether you are correct.

   c. Write the next two lines of the pattern. Verify each.

3. Calculate the sum of

   a. The first 25 natural numbers.
   b. The first 31 natural numbers.
   c. The first 40 natural numbers.
   d. The first 50 natural numbers.
   e. The first 99 natural numbers.
   f. The first 450 natural numbers.

4. Look at the picture of a pile of logs. How many logs would there be in such a pile if there are 12 in the bottom layer, 11 in the next higher layer, and so forth, until there is 1 at the top?

5. Calculate the total number of gifts given in "The Twelve Days of Christmas," given that:

> On the first day of Christmas my true love gave to me a partridge in a pear tree. On the second day of Christmas my true love sent to me two turtle doves and a partridge in a pear tree. . . . On the twelfth day of Christmas, my true love sent to me:
>
> Twelve drummers drumming,
>
> Eleven pipers piping,
>
> Ten lords-a-leaping,
>
> Nine ladies dancing,
>
> Eight maids-a-milking,
>
> Seven swans-a-swimming,
>
> Six geese-a-laying,
>
> Five golden rings,
>
> Four calling birds,
>
> Three French hens,
>
> Two turtle doves,
>
> A partridge in a pear tree.

6. If, on the first day of Chanukah, you light 2 candles; on the second day, 3; ...; on the eighth day, 9: How many candles must a box contain to suffice for the eight days?

7. Calculate each sum:

   a. $7 + 8 + 9 + 10 + 11 + 12 + 13$

   b. $32 + 33 + 34 + 35 + \ldots + 46 + 47 + 48$

8. If a clock strikes only the hours--once for one o'clock, two times for two o'clock, etc.--how many times does it strike in a 24-hour day?

9. Calculate:

   a. The sixteenth triangular number.
   b. The seventeenth triangular number.
   c. The eighteenth triangular number.
   d. The fiftieth triangular number.
   e. The eleventh triangular number.

10. a. Add pairs of consecutive triangular numbers. Record the data in the table provided. Is there a pattern to the sums? (Use the first ten triangular numbers.)

    b. Compute the differences of the consecutive triangular numbers. What do you notice?

    c. How do the differences compare with the sums?

Triangular Numbers	Sums	Differences
1		
	4	2
3		
	9	3
6		
	16	
10		

11. There are 15 people in a room. Each is to greet each of the other persons with a handshake. What is the total number of handshakes? (Hint: Begin with simpler situations--2 people, 3 people, etc. Record the results. Look for a pattern!)

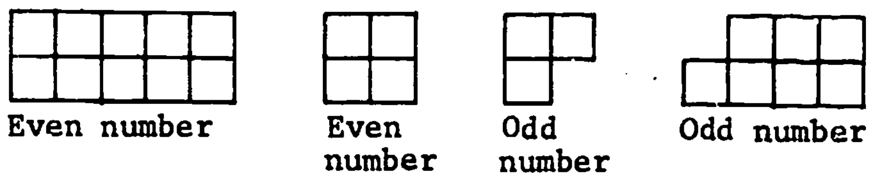
12. Add two even numbers. Is their sum even or odd? Try another pair. Will this be true for any two even numbers?

Use a pair of pictures of even numbers to help you justify your answers. You may find it helpful to cut them out and move them around. (See diagram below.)

13. Repeat the above procedure for a pair of odd numbers. What can you say about the sum of any two odd numbers?

Use pictures to justify your answer.

14. If you add an even number and an odd number, what can you say about the sum? Why?



15. Summarize your findings in the accompanying table. O stands for any odd number and E stands for any even number.

+	O	E
O		
E		

16. Calculate:

a. The tenth even number.
b. The twenty-fifth even number.
c. The thirtieth even number.

17. What is the sum of

a. The first 10 even numbers?
b. The first 25 even numbers?
c. The first 30 even numbers?

18. What is:

a. The ninth odd number?
b. The eleventh odd number?
c. The twenty-fifth odd number?

19. Calculate the sum of:

a. The first 10 odd numbers.
b. The first 30 odd numbers.
c. The first 12 odd numbers.

20. a. Find the sum of the first 3 even numbers.
b. Find the sum of the first 3 odd numbers.
c. Find the sum of the first 6 natural numbers.
d. How does the sum of the results for a and b compare with your result for c?

21. a. What is the eighth even number? The twentieth?
b. Calculate the sum of the first 8 even numbers.
c. Calculate the sum of the first 20 even numbers.
d. Use your answers to b and c to calculate the sum

$$18 + 20 + 22 + \ldots + 40$$

22. Make up and solve a problem similar to Exercise 21, using odd numbers.

23. a. How many consecutive odd numbers, starting with 1, must a person add to yield a sum of 169?
b. Show without adding that it is impossible to add consecutive odd numbers starting with 1 and produce a sum of 1237.

24. Suppose you fold a piece of paper in half. That fold will divide the sheet into two equal parts. Suppose, after folding the paper, you fold it in half again. The paper will then be divided into four equal parts. The table shows what takes place during the first several folds.

Number of Folds	Number of Parts
1	2
2	4
3	8

Suppose it is possible to continue the folding process indefinitely. How many parts will there be after 4 folds? 5 folds? $\underline{n}$ folds?

25. Write the twelfth term of the binary sequence (in exponential form). Now write the eightieth term.

26. Write the sum of:

a. The first 12 terms of the binary sequence (in exponential form).
b. The first 80 terms of the binary sequence.

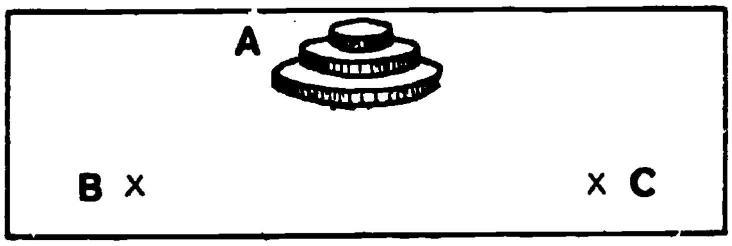
27. Investigate the differences between the squares of two successive odd numbers. Is there a pattern?

28. Under each term of the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... write O or E (determined by whether the number is odd or even).

a. Is there a pattern to the O's and E's?
b. Explain the answer to a on the basis of your knowledge of the sums of odd and even numbers.

*29. Game

a. Choose three specific locations on a piece of paper. Place ıree disks or coins of three different sizes in one location as .



b. Move the disks, one at a time, to another location, using the least number of moves, so that the disks are arranged in the same way according to size. Move only one disk at a time. Verify that the minimum number of moves required is 5.

c. Suppose the game were played with 2 disks, 4 disks, or 6 disks. What would happen? Record your results. Is there a pattern? Try to write a formula for the number of moves required for any number of disks.

*30. Tower of Hanoi puzzle

Refer to Exercise 29. This time use the following additional rule: A larger disk may <u>never</u> be placed on top of a smaller one. What is the minimum number of moves for 10 disks? (<u>Hint</u>: Begin with 1 disk, then 2 disks, 3 disks, and so on. Look for a pattern!)

*The asterisk denotes a more challenging problem.

TEAM

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IV

STUDENT SUMMARY AND REVIEW

To the Student: This summary and review includes the key ideas discussed and presented throughout the Patterns module. You will probably find it useful to compare your notes from class sessions with these.

I. NATURAL NUMBERS: 1, 2, 3, ..., n

A. Sum by Formula

1. $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$

   Examples: See #2 following.

2. The sum of the first 9 natural numbers is

   $(n = 9)\quad \frac{1}{\cancel{2}_1}\cdot 9\cdot \cancel{10}^5 = 45$

   The sum of the first 10 natural numbers is

   $(n = 10)\quad \frac{1}{\cancel{2}_1}\cdot \cancel{10}^5\cdot 11 = 55$

3. Note that n and n + 1 are consecutive numbers. If the first is odd, the second is even.

B. Other Approaches for Sum

1. If n is an odd number, the middle number of the sequence is the average of n and 1, or $\frac{1}{2}(n + 1)$

   $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$

   $5 = \frac{1}{2}(9 + 1)$

2. Calculate the sum by using n times the average, $\frac{1}{2}(n + 1)$:

   $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$

   $5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 9\cdot 5$, or 45

3. If n is even, use n times the average: $n\cdot\left[\frac{1}{2}(n + 1)\right]$, or $\frac{1}{2}n(n + 1)$: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$

   $\text{Average} = \frac{1}{2}(1 + 8)$

   $= \frac{1}{2}\cdot 9$

$$\text{Sum} = n \cdot \frac{1}{2}(n + 1)$$

$$= \overset{4}{\cancel{8}} \cdot \frac{1}{\underset{1}{\cancel{2}}} \cdot 9$$

$$= 36$$

Note that $n \cdot \frac{1}{2}(n + 1) = \frac{1}{2}n\,(n + 1)$

4. When n is even, use pairs:

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8

Four pairs $(\frac{1}{2} \cdot 8)$; each sum is 9; $4 \cdot 9 = 36$

5. When n is odd, add zero. Then use pairs:

0 + 1 + 2 + 3 + 4 + 5

Three pairs; each sum is 5; $3 \cdot 5 = 15$

C. Geometric Approach

Figure 1

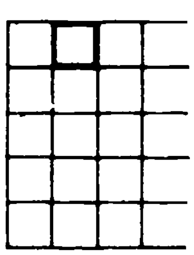


Figure 2

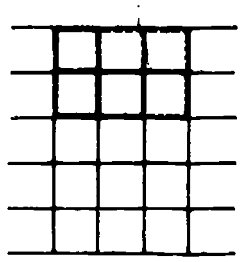


Figure 3

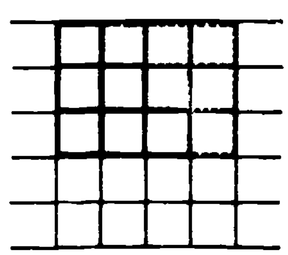
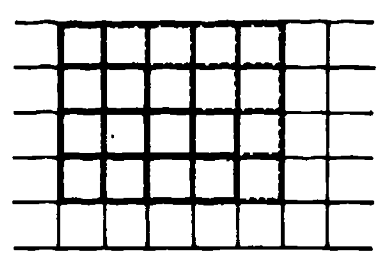


Figure 4



$1 = \frac{1}{2}(1 \cdot 2)$ $\quad 1 + 2 = \frac{1}{2}(2 \cdot 3)$ $\quad 1 + 2 + 3 = \frac{1}{2}(3 \cdot 4)$ $\quad 1 + 2 + 3 + 4 =$

1. Complete Figure 4 to show that $1 + 2 + 3 + 4 = \frac{1}{2}(4 \cdot 5)$.

2. Draw a picture of squares and use it to express the sum of the first 5 natural numbers as an indicated product.

Express the sum of the first 86 natural numbers as an indicated product.

Calculate the above sum using any nonformula approach. Justify what you have done.

D. Triangular Numbers: 1, 3, 6, 10, 15, ...

1. The sequence of triangular numbers comes from the sequence of natural numbers:

Number of Term ($\underline{n}$)	Sum of $\underline{n}$ Natural Numbers	Triangular Numbers
1	1	1
		2
2	1 + 2	3
		3
3	1 + 2 + 3	6
		4
4	1 + 2 + 3 + 4	10
.	.	
.	.	
.	.	
$\underline{n}$	1 + 2 + 3 + ... + $\underline{n}$	$\frac{1}{2}n(n + 1)$

2. The sixth triangular number is:

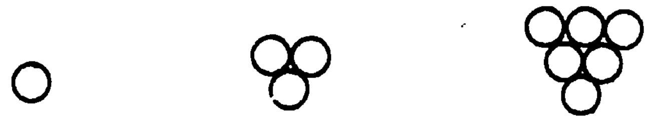
$$\frac{1}{\cancel{2}_1}\cdot\cancel{6}^{3}\cdot 7 = 21$$

The eleventh triangular number is:

$$\frac{1}{\cancel{2}_1}\cdot 11\cdot\cancel{12}^{6} = 66$$

Calculate the twelfth triangular number; the fifteenth.

3. Extend the picture pattern by two additional pictures.



II. EVEN NUMBERS: 2, 4, 6, ..., 2n

A. The <u>n</u>th even number is 2n.

Examples: The eleventh even number is 2·11, or 22

The thirtieth even number is 2·30, or 60

B. An even number is any natural number that can be written in the form 2n, where <u>n</u> is a natural number.

30 is even because 30 = 2·15

31 is <u>not</u> even because 2n = 31 does <u>not</u> yield an <u>n</u> that is a whole number

0 is even because 0 = 2·0

C. The sum of the first <u>n</u> even numbers is n(n + 1).

$2 + 4 + 6 + \ldots + 2n = n(n + 1)$

Examples: The sum of the first <u>9</u> even numbers is:

(<u>n</u> = 9) 9·10 = 90

The sum of the first <u>12</u> even numbers is:

(<u>n</u> = 12) 12·13 = 156

Express the sum of the first <u>18</u> even numbers as an indicated product.

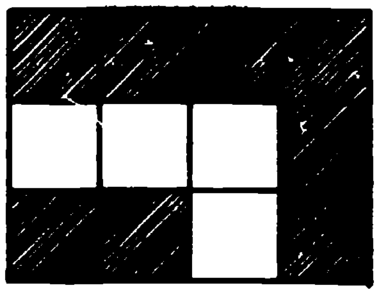
Calculate the sum. (Express it as a <u>standard numeral</u>.)

D. <u>Geometric Approach</u>

6 squares

4 squares

2 squares



The above rectangle has been shaded to show that:

$$2 + 4 + 6 = 3 \cdot 4$$

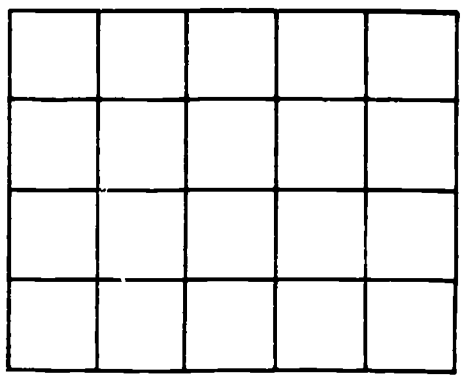
1. Shade the diagram below to show the sum

$$2 + 4 + 6 + 8$$

2. Use the shaded diagram to express the sum

$$2 + 4 + 6 + 8$$

as an indicated product.



III. ODD NUMBERS: 1, 3, 5, ..., 2n - 1

A. The <u>n</u>th odd number is 2n - 1.

Examples: The eleventh odd number is $2 \cdot 11 - 1 = 21$

The thirtieth odd number is $2 \cdot 30 - 1 = 59$

B. An odd number is any natural number that can be expressed in the form 2n - 1, where <u>n</u> is any natural number.

C. The sum of the first n odd numbers is n^2.

Examples: The sum of the first 9 odd numbers is $9^2 = 81$

The sum of the first 12 odd numbers is $12^2 = 144$

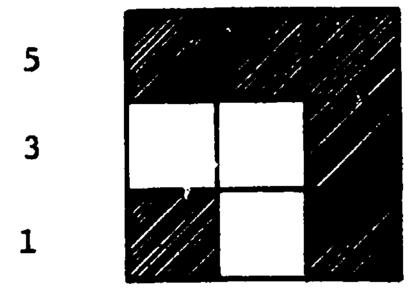
Express the sum of the first 18 odd numbers as an indicated product.

Calculate the above sum.

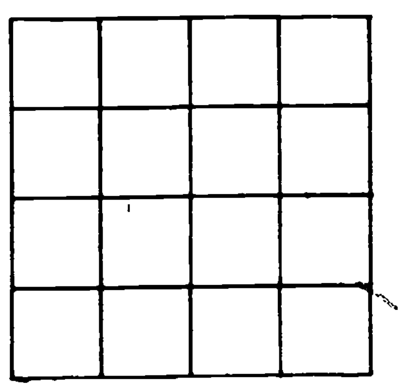
D. Geometric Approach

The accompanying square has been shaded to show that:

$1 + 3 + 5 = 3^2$



Shade the square below to show that the sum (1 + 3 + 5 + 7) is equal to a square number.



IV. SUMS OF ODD AND EVEN NUMBERS

A. The sum of two even numbers is even.

Join these two ⟶ to form this.

111

B. The sum of two odd numbers is even.

Join these two ————————→ to form this.

C. The sum of an even and an odd number is odd.

Join these two ————————→ to form this.

V. BINARY SEQUENCE: $1, 2, 4, 8, \ldots, 2^{n-1}$

A. The <u>n</u>th term of the binary sequence is 2^{n-1}.

B. The sum of the first <u>n</u> terms of the sequence $1, 2, 4, \ldots, 2^{n-1} = 2^n - 1$.

The <u>fourth</u> term of the binary sequence is $2^{4-1} = 2^3$, or 8.

The sum of the first <u>four</u> terms of the binary sequence is $2^4 - 1 = 16 - 1$, or 15.

Write the <u>twentieth</u> term of the binary sequence.

Write the sum of the first <u>twenty</u> terms of the binary sequence.

Express each number as the sum of terms of the binary sequence:

1. 17
2. 34
3. 42
4. 71

VI. FIBONACCI SEQUENCE

The Rabbit Problem

Month Number	Rabbit Pairs	Number of Pairs
1	R_1	1
2	R_1	1
3	R_1 R_2	2
4		
5		
6		
7		

A. Write the first ten terms of the Fibonacci sequence. The first three are shown below.

1, 1, 2, ...

B. Use the numbers 3 and 4 to generate your own Fibonacci-type sequence.