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ABSTRACT

A group of 24 14-year-old students were presented with sets of algebra tasks using the Leeds Modelling System (LMS), a computer-based modeling system. Four months later, these same students were given a paper-and-pencil test which covered comparable tasks set by LMS. In addition, certain students were given detailed diagnostic interviews. A comparison between the results obtained is presented. Results obtained on the paper-and-pencil test and the interviews were consistent, showing that students had some profound misunderstandings of algebraic notation. Furthermore, from the interviews it was possible to determine classes of strategies some students were using, including (1) searching for solutions (substituting) and (2) applying a "global" rule, such as collecting all numbers on one side, whether or not they were coefficients. The overall picture evolving from this and other studies seems to be: (1) that the difficulties of learning algebra have been greatly under-estimated; (2) that students have a great difficulty in inferring their own rules, or sometimes higher-level schema, and then using them consistently and often in appropriate situations; and (3) that few students have evolved mechanisms by which they can verify whether a proposed algorithm is feasible. (JN)

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Basic Algebra Revisited

A Study with 14 Year Olds

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BASIC ALGEBRA REVISITED: A STUDY WITH 14-YEAR-OLDS

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ABSTRACT

This paper reports the results obtained with a group of 24 14-year-old pupils when presented with sets of algebra tasks by the Leeds Modelling System, LMS. Four months later, these same pupils were given a comparable paper-and-pencil test and detailed interviews. A comparison between these sets of results is presented.

The results obtained on the paper-and-pencil test and the interviews were consistent, and show that the pupils had some profound misunderstandings of algebraic notation. Further, from the interviews it is possible to determine classes of strategies some pupils were using, namely:

-searching for solutions (ie substituting).

-applying a "global" rule, such as collecting ALL the numbers on one side, whether or not they were coefficients.

Moreover, this work further demonstrates the importance of interviews to interpret curious protocols.

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1. INTRODUCTION

In 1979 we implemented a Computer-based modelling system, LMS - Leeds Modelling System (Sleeman & Smith, 1981) and at the same time collected and analysed protocols for 15-year-old pupils solving basic algebra tasks. A set of rules were formulated which were sufficient to solve the tasks, and the errors which the pupils made were expressed as incorrect, or mal-rules. Mal-rules noted includes moving a number from one side of the equation and omitting to change the sign; multiplying some, but not all, of the terms of bracketed expressions, etc. (We have subsequently called these manipulative mal-rules as we believe such pupils know the rules but make an error in their execution). The domain rules, the associated mal-rules and sets of tasks form LMS's algebra database. See figure 1a for algebra domain rules, and figure 1b for some of the mal-rules noted.

[Figure 1 about here]

In March 1980, a small group of pupils, average age 14 years 10 months, used the modeller, and by and large it was able to spot their difficulties. Indeed, the same pupils were seen a few days later by some experimenters, whose assessment of the pupils' performance agreed very closely with those of the modeller (Sleeman, 1982). Some shortcomings were noted in the modeller during this experiment, namely that it was unable to spot difficulties with rules other than at the "level" at which they were first introduced. During the 1980 experiment, we saw ample evidence that when tasks became more complex, pupils would make errors with simpler rules too. For example, pupils would be able to successfully work tasks of the type:

M*X = N

[where M and N stand for integers]

but when the task involved expanding brackets we noted pupils who appeared to do that successfully, but inverted the final answer. Given the task:

12*X = 2 * (4*X + 5)

we have seen several times the answer:

X=4/10

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This shortcoming was removed from the modeller (Sleeman, 1983a) and in May 1981 another group of pupils used LMS with the same database. This group of 24 pupils, average age 14 years 3 months, were judged to be at or above the national average for their age-group in Mathematics; however the results were dramatically different from the earlier group's, [1]. Indeed many of their difficulties were not diagnosed by LMS and had to be analysed by the investigator. This task was made very difficult because it had been assumed that pupils would at most make one or two minor manipulative errors (e.g. changing side and not sign) and so LMS had been designed such that the pupil could merely input his final answer, and none of his intermediary steps. In figure 2 we give a sample of the protocols observed, together with the mal-rules which the investigator suggested were appropriate. In figure 3, we summarise the new mal-rules which the investigator considered explained the pupils' behaviour with LMS.

By stating that a protocol can be explained by a mal-rule of the form

M*X => M+X

we do not wish to imply that given a task of the type:

3*X + 4*X = 5

the pupil would produce the response:

3+X + 4+X = 5

Indeed, we have seen several pupils write

X+X = 5-3-4

and when asked to provide intermediary steps they have said categorically that there were none as the above was done in "one step" (figure 2a). Nevertheless, we are happy to accept that both forms are "explained" by the mal-rule; the first form however requires that several additional rules fire in order to get it into the state given by the "second" pupil. (It should be noted that the mal-rules given in figures 2d and 2e are more comprehensive and carry out several housekeeping steps). Note that the difference between "basic" and "comprehensive" mal-rules is significant when one tries to perform remedial instruction, as it is important to ensure that the grain of the instruction matches the pupil's.

[Figures 2 and 3 about here]

Further as the result of the analysis of these protocols, a number of questions were

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1. Most of these pupils had been introduced to algebra several years earlier in their several middle schools; furthermore the high school had retaught algebra - virtually from the beginning - in the year before the experiment took place.

raised, including:

- What is the essential difference between the task-sets which the pupil can and cannot solve?
- Does the pupil's perception of algebraic tasks vary from one task-type to another?

(See figure 4a for a list of task-types given to the pupils and (Sleeman, 1982 & 1983a) for a more detailed discussion of how this set was constructed).

Unfortunately, because this analysis took a while and the school vacation then intervened, it was not possible to meet with the pupils again until September (1981). Because of the time that had elapsed, the pupils were given a paper-and-pencil test which covered comparable tasks to those set by LMS. These tests were analysed in detail by the investigator, and as a result of this certain pupils were given detailed diagnostic interviews. The next sections give more details of these stages.

[Figure 4 about here]

2. THE PAPER-and-PENCIL TEST

Figure 4b gives a comparison between the performance of selected pupils in May and September. We have omitted those pupils who did well on both, and those whose only error was incorrect precedence in tasks of the type:

$$2 + 3X = 6$$

as this proved to be a virtually universal error. (Such pupils would return the answer $5X = 6$ for the above task. This was also a <u>very</u> common error in our earlier study, (Sleeman, 1982). Note we also classify this as a <u>parsing</u> error - many other parsing errors were noted in this experiment.)

From a review of the data we noted:

1. The performance was generally considerably better in September than in May. (Note <u>no</u> additional teaching in algebra had been given, however the pupils had presumably done some independent study in preparation for their end of year examinations).
2. A considerable number of tasks were <u>not</u> solved on the written test - (whereas LMS insisted on the pupil giving a response to each question).

3. Some pupils who appeared to have "wild" rules in May, seemed to solve this type of task correctly in September, eg AB5, or had less "serious" mal-rules.

4. Some pupils whose behaviour had been "random" or "wild" in May had now settled to use incorrect rules consistently, e.g. pupil AB18.

5. One pupil at least, AB7, gave multiple values in an equation where X occurred more than once.

6. Many of the pupils made the common precedence error noted above.

As a result of this comparison it was decided to interview all those who appeared, on the written test, still to have major difficulties and all those who had had major difficulties which appeared to have "cleared up". (But not those with the common precedence errors).

3. THE INTERVIEWS

These proved to be remarkably revealing and very rewarding as the pupils without exception were extremely articulate. The dialogues were taped, and the figures are a reconstruction from the tapes and the worksheets used.

After analysing the pupils' protocols obtained with LMS we conjectured that some pupils may actually "see" some of these tasks differently from the standard algebraic interpretation. That is a pupil like AB18, we conjectured, might actually see a task of the type:

$$MX + NX = P \text{ as } M + X + N + X = P$$

To test this hypothesis we started each interview by asking the pupil merely to <u>read</u> the list of algebra tasks given in figure 5. Each pupil without exception <u>read</u> them correctly but, some pupils like AB18, still processed them as indicated above (more details are given in section 3.4).

[Figure 5 about here]

In all cases the investigator then presented the pupil with a series of tasks and asked him to work each one explaining as he went along exactly what he was doing. In some cases the investigator asked the pupil to tell him which of two alternative forms were correct and to explain why. The tasks presented were different for each pupil, and were based on the difficulties noted in the individual's September test. The ewer thus started each session with a list of task-types to be explored, but often generated particular tasks as a result of answers given to questions. (For example, some pupils were initially asked to rework the tasks they had apparently done in "interesting" ways in the September test).

The following is a summary of the main features noted during the interviews:

1. Some pupils searched for solutions as they were unable to compute MX + NX and/or deal with MX = N, when M>N.

2. One pupil computed a separate value for <u>each</u> X given in the equation.

3. Several pupils admitted that there were a number of quite <u>distinct</u> ways of solving an equation (even when it is demonstrated that each different approach leads to different answers).

4. Some pupils have "hard", consistent, mal-rules.

5. Some pupils have the correct rules and can explain why it is not permissible to perform the illegal transformation (frequently the one the pupil appeared to use in May was selected).

Each of these points are discussed in the following sub-sections.

3.1 Searching for Solutions

Searching for a solution appears to be a *very* common way of solving equations with pupils beginning algebra, and presumably arises because the initial equations presented *could* be solved using this algorithm. (See figure 6.1 for details of pupil AB11's protocol, [2]). Teachers should thus be suspicious that a pupil is using the naive algorithm if he appears to be unable to solve tasks where the variable is a negative integer, large-integer or non-integer. The teacher should be concerned because the naive algorithm is only applicable to a sub-set of algebraic equations, and hence should be deemed a highly significant weakness to be remedied. It seems clear that the use of simplistic tasks leads to a naive algorithm which causes major conceptual difficulties on more advanced tasks, [3].

[Figure 6 about here]

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2. Indeed, in a more recent test with 100 13-year-olds, it appears that about 95% of them use this approach. And we argue that this leads to the type of errors noted below for tasks where this naive algorithm is inappropriate. For instance a significant proportion experience difficulties with tasks of the form

$$3X = 2$$

Many of them return the answer:

$X = -1$, explaining they had *subtracted* 3 from both sides.

These same pupils completely fail on equations which contain 2 Xs and attempt to guess a value for *each* X.

$$3*X+4*X=3$$

has been "solved" as:

$$3*1 + 4*0 = 3, \text{ making } X=1 \text{ and } X=0.$$

Similarly,

$$3X + 4X = 98$$

has been "solved" as:

$$3*22 + 4*8 = 66 + 32 = 98.$$

It is indeed intriguing to watch these pupils changing their approach when solving tasks of the form:

$$MX=N$$

depending on whether the task is solvable by search. They do not appear to notice this discrepancy. [Clearly this point should have been raised in an interview with these pupils].

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3. Some readers may prefer to think about the method discussed in section 3.1 as being a *consistent* substitution method, and the one discussed in section 3.2 as an *inconsistent* substitution method.

3.2 Multiple Values for X. [3]

In this section we report a pupil who has a very weird, but nevertheless very consistent algorithm for solving tasks involving 2 Xs. (The 13-year-old pupils quoted above displayed the same general phenomena but were generally arbitrary in their choices for X, [2]). When pupil AB7 was orignally working at the terminal she was heard to mutter:

"If this X was 2, then it would work if this second X was 4".

Moreover, in both the paper-and-pencil exercise and in the interview this pupil had been remarkably consistent. (See figure 6.2 for her protocol). That a pupil would follow such an algorithm was originally a great source of amazement to me, and I should add to the School's Maths teachers.

On the other hand, pupil AB7 was able to explain exactly what she was doing. Given the task:

3 * X + 2 * X = 12

ie task e) of figure 6.2 she gave the following explanation:

"What I do is take the 3 and I make the first X equal to 2, so I write: 3 * 2"

When asked by the interviewer why the first X is equal to 2 she explained that it's the next number along, and then added "I think that's wrong, but that's what I do".

She then continued " and then I write down the +2 making

3 * 2 + 2

I then work this out, this is equal to 8 and so the second X is

12 - 8, that is 4".

She then completed the solution and gave the 2 values for X, and so the final state of her worksheet was:

3 * 2 + 2 + 4 = 12

X = 2

X = 4

NB She worked tasks c to k using this "algorithm", but used a variant with tasks, a and b. Tasks 1 to o show she was clearly solving these tasks by "searching", c.f., figure 6.1.

3.3 Alternative Algorithms

Figure 6.3 gives the flavour of the dialogue between this pupil and the interviewer. Although for several task-types pupil AB17 was able to solve the tasks correctly, he was easily distracted and quite unable to tell the investigator why his alternatives were illegal. On some tasks he suggested several illegal solutions, and again was really unable to distinguish between them.

Moreover, this pupil gave, as an aside, a rationale for his "method", which is discussed in section 4c.

3.4 "Hard"/Consistent Mal-Rules

Many of the pupils were using consistent mal-rules. Just over half of the 24 pupils we saw mis-handled precedence in equations of the form:

$$2 + 3*X = 9$$

A section for one such pupil is given in figure 6.4.I. (Indeed, more recently we have discovered that 90% of a sample of 13-year-olds had precedence difficulties with arithmetic expressions involving the "+" and "*" operators). Figure 6.4.II is part of a protocol where pupil AB17 consistently applies a further intriguing transformation to a complete set of tasks. (A more detailed explanation for this protocol is given in section 4c).

Pupil AB18, figure 6.4.III, is remarkably consistent with his mal-rules over a whole range of task-types. NB, the application of his algorithm to task c which involves 3 X-terms. (To give him justice, he realised that he had got tasks d) - g) wrong as he noticed that the equations did not balance when he substituted his answers back in). Further, having worked task h) he noticed that when he moved the 4 across to the Right Hand Side, he changed the sign and suggested that when he move the X (associated with 2*X) to the LHS, he should also change its sign. He then verbalized that

$$X - X \text{ is } 0,$$

and so the LHS became 0 and the RHS did not, and so he realized that this proposed solution was impossible. However, for good measure he also worked task i) with the "revised" algorithm.

In the course of our discussion this pupil also gave the basis for his algorithm, which is discussed in more detail in Section 4.c.

3.5 "Saved Souls"

In figure 6.5 we see pupil AB5 working correctly tasks which she had got consistently wrong in May, namely task sets 7 and 8. For task set 8 she appeared to use mal-rule:

$$M*X = N*X + P ==> X + X = M + N + P.$$

Moreover, when presented with a fallacious alternative during the interview she was able to spot it and to say why it was wrong. ("Not able to add a number to an X term", "not able to separate a number from an X term" etc., see the figure for more details).

In May, this pupil showed a lack of understanding of basic algebraic notation which appeared to be remedied by September. To see whether this was the case I also presented tasks from task sets 12 and 13, namely tasks e to h, which she worked correctly and was able to verbalize the stages she went through. I also presented equation i) which contained an unusual variable, AA, and again this was worked correctly.

Similarly several other pupils, eg. AB4 showed substantial progress, and again it was associated with the ability to explain what they were doing. In the next section we give a summary of the points inferred from these various analyses.

4. SUMMARY OF THE EXPERIMENT

There appear to be five major, and not totally unrelated, points:

a) The apparent difficulty of tasks involving multiple Xs

In the introduction, we listed a number of questions which occurred to us following an analysis of the pupils' protocols with LMS. These included the observation that some pupils could solve certain earlier task-types, but appeared to fail on a particular set, and yet can subsequently go on to solve further sets. In particular, we had noticed pupils having difficulties with tasks of type:

2X + 3X = 10

yet be able to solve tasks of type:

2X + 4 = 16 and 4 + 2X = 18.

From talking to the pupils it became clear that they were searching for a solution, because they did not know how to compute the sum of MX and NX. The second and third tasks thus appear easier to them because it did not contain this "difficulty". Thus, the interviews very nicely resolved this issue.

b) Mal-Rules retained as genuine

As a result of the interviews we believe that many of the mal-rules reported in figure 3 are "phantom" and a result of the pupil having to give a response to LMS. Indeed, we now believe mal-rules sets 4 and 6 are spurious. (The modeller has since been changed so that it is now possible for the pupil to indic..e that he wishes to give up on a particular task). Al.o there are additional mal-rules that .. ld be added, namely those which can be generated by the schema discussed in the next section.

c) Schema for "Generating" Mal-Rules

In figure 6.4.III we gave a substantial section of pupil AB18's protocol. In the course of our discussion he explained that he was carrying out the teacher-given algorithm of: "Collecting all the Xs on the left hand side and collecting all the numbers on the right hand side", and added that he was not really sure what to do about the "extra multiply signs". Pupil AB17 gave a similar explanation for his action.

This gives us a schema for generating mal-rules. For example given the task-type:

M*X + N*X = P

This schema gives the following action sides for mal-rules:

X+X = P-M-N

X+X = P+M+N

where in the second case the X coefficients are treated "specially", ie the coefficients of the Xs were taken across to the RHS of the equation but the signs were NOT changed. And the form given by pupil AB17, and quoted in figure 6.4.II, namely:

*X*X=P-M-N

This form, I believe, was recognized as being non-standard, and so the pupil changed it into a more recognizable form, namely:

X*X=P-M-N

The "complementary" form being:

X*X=P+M+N

(For further examples of "normalization" see (Sleeman, 1982)).

Similarly, given the task-type:

M*X = N*X + P

This schema creates the following forms:

X = N + P - M

X = N + P + M

X + X = N + P - M

X + X = N + P + M

X - X = N + P - M

X - X = N + P + M

For example, on task h pupil AB18 suggested the use of both the third and the fifth forms (see figure 6.4.III).

Brown and Burton's earlier modelling work (Brown & Burton, 1978) assumed that pupil errors were due to _perturbations_ of the correct rules (c.f., the assumptions which we also made initially in this project, and which seems to be true for _some_ of this domain's rules). This however, is certainly _not_ true for the class of errors which we are discussing here. Indeed, pupils AB18 and AB17 do _not_ use the usual "task segmentation" and thus do not even _consider_ applying, at the first step, a correct rule (Sleeman, 1983b). On the other hand, it appears possible to generate all the _manipulative_ mal-rules by systematically removing one of the rule's sub-steps (Sleeman, 1983b).

d) Longitudinal Studies

From this set of pupils alone, I think it is possible to infer that behaviour progresses as:

UNPREDICTABLE/"WILD" -> CONSISTENT USE of MAL-RULES -> CORRECT

That is, pupils who were really unsure which method to use, c.f., pupil AB17 of figure 6.3, would apply different "methods" to different tasks apparently randomly. (Analogously, Greeno and his collaborators who are investigating pupils' performance whilst they are being taught algebra, have noticed very considerable variations over time for the individual pupils (Greeno, 1981)). Then there are the pupils, c.f., figure 6.4, who are using consistent mal-rules on particular task sets. Further there are some, like pupil AB18 of figure 6.4.III, who are using a schema which guides their action in a whole variety of situations. Moreover, once the correct algorithms are understood it appears that pupils are able to explain them quite articulately.

With experience or maturation pupils appear to move through the stages given above. I do not wish to suggest that every pupil acquires a consistent mal-rule before he is able to do the task correctly, but to indicate a trend. Similarly, I do not wish to suggest that no pupil regresses. Results obtained over a period of a year and a half, with various groups aged from 13 to 15 support this claim, as does the study reported in (Kuechemann, 1981).

e) Interpretation of the various types of Mal-Rules & the Remedial Teaching Experiment

As mentioned earlier, prior to meeting this group of pupils we had anticipated that most of the errors made would be omission of sub-steps in complex manipulations, as instanced earlier (changing the side but NOT the sign of a number, only multiplying out some of the elements in a bracket). In this experiment we encountered pupils who viewed the tasks totally differently from the way it was presented, and we have referred to this class of errors as PARSING errors.

This seems a useful distinction, and one which may be of particular importance when this information is used in remedial instruction. That is, different explanations may well be necessary to overcome these two classes of error. In the case of the manipulative mal-rules it would appear that the pupil basically knows the rule, but due to cognitive overload, or inattention omits sub-steps. The parsing errors appear to arise from a profound misunderstanding of algebraic notation.

It was important to determine whether, once a pupil's shortcomings have been diagnosed, it is possible to carry out remedial instruction which will lead to a long-term improvement in the pupil's solving of algebraic equations. As a high percentage of the group mishandled mixed expressions of the form 2+3*X = 6, this and related points were dealt with in a class lesson. Those pupils with very individualistic difficulties were seen separately by the investigator, who spent on average half an hour with each pupil. The post-test was administered two months after the completion of the remedial teaching. The results for the post-test are summarized in figures 7 and 8. In order to factor out effects of maturation and subsequent instruction the post-test was also given to pupils who had not been screened earlier; the mathematics teacher paired the pupils on the basis of their performance on an algebra test taken in January 1981.

[Figures 7 and 8 about here]

Figure 7 gives the performance on the post-test of the pupils who had had very individualistic difficulties and hence had been given personal remedial interviews. This figure also includes a summary of the matched pupils' performance. The results in figure 7 should be compared closely with their earlier results in figure 4b and the interviews in section 3. By and large the tutored pupils did much better than their matched partners. AB17 (figures 6.3 & 6.4.II) and AB18 (figure 6.4.III) were now virtually perfect, AB7 (figure 6.2) and AB11 (figure 6.1) were able to solve tasks involving more than one variable (nb AB7 was the pupil with the very idiosyncratic algorithm), and AB10 who had been misparsing equations was now only using manipulative mal-rules.

Further the untutored group displayed all the main classes of errors which had been encountered earlier, namely:

1. The SEARCH procedure was noted again with a sizeable number of pupils.

2. Given a task of the form MX=N and M>N then several pupils were unable to solve these.

3. Precedence and Parsing problems were noted.

4. Manipulative Mal-Rules [MSOLVE, MXTOLHS etc.].

5. A completely new set of Parse Mal-Rules were encountered with pupil AB13M. For example with task-set 5, the following was observed twice:

   M*X+N*X=P => M*(X+N)*X=P => M*X∧2+M*N*X=P => 2*M*X+M*N*X=P

   and once: M*X+N*X=P => M*(X+N)+N*X=P

   Similarly mis-parsings were given for task-sets 6, 7 and 8. These confusions appear to be caused by the pupil being initially exposed to the "flow-chart" methods at one school and then transferred to the more traditional rule-based approach. [This has been confirmed by the school].

Figure 8 gives an overall summary of the performance of the two groups of pupils. (This comparison would be even more dramatic had we given the total number of errors made - rather than an overall classification of each student's performance). From this it is very clear that the tutored pupils overall performance is very much better than the control group. Some of the observed improvement can probably be attributed to the fact that the tutored/remedial group were given additional exposure to algebra tasks. However, we suggest that the very substantial improvement noted cannot be totally attributed to this additional exposure, which was only 2 hours in total, and that it is reasonable to attribute the substantial effect to the (individualized) remediation process outline above. Although we have not demonstrated that this result is statistically significant, [4], we have in effect factored out the two major factors, namely additional instruction and natural maturation by using a matched group.

The crucial question now is whether this improvement will be maintained indefinitely; in this regard it is to be very much regretted that the planned 6 month's post-test was not carried out by the school.

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4. It would have been pleasing to have had this additional piece of supporting evidence, however this author believes that highly significant improvements are self-evident and do not need to be supported by statistical data. I believe that figures 7 and 8 show such an improvement.

5. FURTHER WORK

a) Enhance the modeller's rule set The modeller's rule set has been enhanced with the bona fide mal-rules of figure 3, and it is planned shortly to use the Modeller with a comparable group of pupils.

b) Remedial teaching subsystem. It is planned to analyse the remedial teaching dialogues discussed above with a view to enhancing the Modelling System such that it will perform remedial teaching, using the techniques noted in these dialogues, and using the models inferred by the Modeller.

c) The effects on conventional classroom teaching It would appear that knowing typical errors must have a major impact on the feed-back provided by the class-room teacher when he, or she, marks a set of exercises. (For instance, he should be immediately suspicious that the pupil has an inadequate algorithm whenever he finds a pupil who is unable to solve tasks whose answers are not small positive integer). Further, the type of dialogue which is reported with pupil AB17, figure 6.3, could well form the basis for some very valuable classroom lessons, where each pupil is asked to state the answer he got for a particular task and to explain how he obtained the result. (As noted above the pupils at this age are well able to articulate their methods, whether they are corract or incorrect. Indeed, this technique was used in the remedial lesson which I gave).

6. COMPARISON WITH OTHER STUDIES

There is a steadily growing body of data about how pupils and students solve algebra tasks. [5]. Some of the earliest work was done by Paige and Simon (Paige & Simon, 1966) who reported considerable differences in approaches on a set of physically contradictory tasks. Some of their subjects set up the equations and proceeded to solve them, others noted that the situation was an impossible one and did not go any further. They also noted that people used a lot of background knowledge to solve these tasks.

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5. For a more general review of the problems of teaching Mathematics and their relationship to Intelligent Teaching Systems (Lovell, 1980).

Lewis (Lewis, 1980) also studied University students solving algebra equations and he reports two main observations: some students had a super-operator, delete, which was used when either divide or subtract was appropriate, and secondly, most students had no mechanism by which possible algorithms were verified on known data. For example, if the proposed algorithm to add together algebraic expressions of the form:

a/b + c/d

did <u>not</u> give the anticipated result of 3/4, when applied to known data, namely:

1/2 + 1/4

then it should be discarded.

The Illinois group (Davis, Jockusch & McKnight, 1978) has collected a great deal of data for abler high school pupils solving a series of tasks in both algebra and geometry. In particular, with algebra pupils they have reported a variety of phenomena including: over-generalization from instances, using an old operator instead of a more recently introduced one, [6], and regression under cognitive load. (All of these have also been encountered in some form in this study). Further, the Illinois group have interpreted these observations within the framework(s) provided by Cognitive Science.

Matz (Matz, 1982) has taken this interpretation one stage further, and has suggested a number of high-level schema which explain series of observed errors, these include her "extrapolation principle" which explains why a pupil who has seen the legal transformation:

(A * B) 1/n ==> A 1/n * B 1/n

would then write:

(A + B) 1/n ==> A 1/n + B 1/n

(c.f., the schema given by pupils AB17 and AB18).

The study which seems to be most similar in age range and ability to ours, is the one just reported by the Chelsea Mathematics team (Kuechemann, 1981). This project sampled large numbers of pupils from many schools throughout the UK, and reports the percentage of various age groups which give different answers (including the correct ones). Kuechemann suggests that it is useful to categorize pupils into 4 conceptual levels, namely: below late Concrete, late Concrete, early Formal and late Formal, corresponding to the usual Piagetian stages. Further,

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6. + instead of *, * instead of Exponential.

he instances the types of tasks which he anticipates each group would be able to solve and the sorts of errors he expects each group to make on the more advanced tasks. The summary of competence for the several age groups is particularly interesting, as it suggests there is a sharp improvement between 13 and 14, but only a small change between 14 and 15.

Although the tasks we have used have been less varied than those used by Chelsea, we have also found a great improvement between the ages of 13 and 14, and a less but still substantial improvement between the ages of 14 and 15. Indeed the 13-year-olds' grasp of algebra seems to be very minimal, and clearly shows that, in this domain at least, pupils are unable to manipulate formal entities, whereas a proportion of the 14-year-olds appear to be able to do so. (The performance of the 13-year-olds is even more perplexing, in the light of a recent experiment which we have carried out with pre-school children (Sleeman, in preparation)).

7. CONCLUSIONS

Many of the observations noted during this experiment bear out the remarks made in (Sleeman&Brown, 1982):

"......Perhaps more immediately, it suggests that a Coach must pay attention to the sequence of worked examples, and encountered task states, from which the pupil is apt to abstract (invent) functional invariances. This suggests that no matter how carefully an instructional designer plans a sequence of examples, he can never know all the intermediate steps and abstracted structures that a pupil will generate while solving an exercise. Indeed, the pupil may well produce illegal steps in his solution and from these invent illegal (algebraic) "principles". Implementing a system with this level of sophistication still presents a major challenge to the ITS/Cognitive Science community... "

The overall picture which is evolving from this and the studies discussed in the previous section seems to be:

1. The difficulties of learning algebra have been greatly under-estimated.

2. Pupils have a great facility for inferring their own rules, or sometimes higher-level schema, and then using them consistently and often in inappropriate situations.

3. Few pupils have evolved mechanisms by which they can verify whether a proposed algorithm is feasible. (The findings of Lewis (Lewis, 1980) and Matz (Matz, 1982) also support this observation).

Presumably, these points are also true for many other areas within both school and University Mathematics. Fortunately, we now have some new techniques to help us understand the misunderstandings which have been pertetuated for generations.

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Figure 1

a) RULES for the ALGEBRA domain (slightly stylized).

RULE NAME	CONDITION	ACTION
FIN2	(X = M/N)	((M N)) or ((M))
SOLVE	(M * X = N)	(X = N/M) or (INFINITY)
SIMPLIFY	(X = M/N)	(X = M'/N')
ADDSUB	(lhs M +|- N rhs)	(lhs [evaluated] rhs)
MULT	(lhs M * N rhs)	(lhs [evaluated] rhs)
XADDSUB	(lhs M*X +|- N*X rhs)	(lhs (M +|- N) * X rhs)
NTORHS	(lhs +|- M = rhs)	(lhs = rhs -|+ M)
REARRANGE	(lhs +|-M +|- N*X rhs)	(lhs +|- N*X +|- M rhs)
XTOLHS	(lhs = +|- M*X rhs)	(lhs -|+ M*X = rhs)
BRA1	(lhs < N > rhs)	(lhs N rhs)
BRA2	(lhs M*<N*X +|-P> rhs)	(lhs M*N*X +|- M*P rhs)

Where M, N and P are integers and where lhs & rhs are general patterns (which may be null), where +|- means either + or - may occur, and where < and > represent standard algebraic brackets.

b) Some MAL-RULES for the Domain

RULE NAME	CONDITION	ACTION
MSOLVE	(M*X = N)	(X = M/N) or (INFINITY)
MNTORHS	(lhs +|- M = rhs)	(lhs = rhs +|- M)
MXTOLHS	(lhs = +|- M*X rhs)	(lhs +|- M*X = rhs)
M1BRA2	(lhs M *<N*X +|- P> rhs)	(lhs M*N*X +|- P rhs)
M2BRA2	(lhs M*< N*X +|- P> rhs)	(lhs M*N*X +|- M +|- P rhs)

Using the same conventions as above.
Note MSOLVE is the mal-rule associated with the domain rule, SOLVE etc.

Figure 2

Protocols from which new mal rules were inferred by the Investigator. NB the teacher specified the way in which the X-coefficient should be represented. NOTE too that some of the protocols are not totally consistent: the investigator has given the mal-rule which summarizes each pupil's behaviour on the majority of the tasks.

Task set 5
Task is (2 * X + 4 * X = 12) Pupil's solution was (1 * X = 3)
Task is (2 * X + 3 * X = 10) Pupil's solution was (2 * X = 10 - 2 - 3)
Task is (3 * X + 2 * X = 11) Pupil's solution was (1 * X = 6 // 2)
Task is (2 * X + 6 * X = 10) Pupil's solution was (1 * X = 1)
Task is (3 * X + 4 * X = 9) Pupil's solution was (1 * X = 1)
Task is (2 * X + 4 * X = 3) Pupil's solution was (1 * X = -3 // 2)
Task is (4 * X + 2 * X = 4) Pupil's solution was (8 * X = 4)

Figure 2a. Protocol apparently showing M * X => M + X (pupil AB17).

Task set 6
Task is (2 * X + 4 = 16) Pupil's solution was (1 * X = 8 // 3)
Task is (2 * X + 3 = 9) Pupil's solution was (1 * X = 9 // 5)
Task is (3 * X - 4 = 6) Pupil's solution was (1 * X = -6)
Task is (2 * X + 5 = 10) Pupil's solution was (1 * X = 10 // 7)
Task is (6 * X + 4 = 6) Pupil's solution was (1 * X = 3 // 5)
Task is (5 * X + 2 = 5) Pupil's solution was (1 * X = 5 // 7)

Figure 2b. Protocol apparently showing M * X + N => (M + N) * X (pupil AB20).

Task set 7
Task is (4 + 2 * X = 16) Pupil's solution was (1 * X = 8)
Task is (2 + 4 * X = 14) Pupil's solution was (1 * X = 6)
Task is (3 + 5 * X = 11) Pupil's solution was (1 * X = -4)
Task is (4 + 6 * X = 11) Pupil's solution was (1 * X = -13)
Task is (4 + 5 * X = 6) Pupil's solution was (1 * X = -14)
Task is (5 + 2 * X = 8) Pupil's solution was (1 * X = -2)

Figure 2c. Protocol apparently showing M + N * X => M * N + X (pupil AB3).

Task set 8
Task is (4 * X = 2 * X + 6) Pupil's solution was (1 * X = 6)
Task is (3 * X = 2 * X + 5) Pupil's solution was (1 * X = 5)
Task is (3 * X = -2 * X + 7) Pupil's solution was (1 * X = 4)
Task is (4 * X = 2 * X + 3) Pupil's solution was (1 * X = 9 // 2)
Task is (4 * X = -2 * X + 8) Pupil's solution was (1 * X = 5)
Task is (6 * X = 2 * X + 3) Pupil's solution was (1 * X = 11 // 2)

Figure 2d. Protocol apparently showing MX=NX+P => X+X=M+N+P (pupil AB1).

Task set 7
Task is (4 + 2 * X = 16) Pupil's solution was (1 * X = 2)
Task is (2 + 4 * X = 14) Pupil's solution was (1 * X = 4 // 2)
Task is (3 + 5 * X = 11) Pupil's solution was (1 * X = 5 // 3)
Task is (4 + 6 * X = 11) Pupil's solution was (1 * X = 6 // 4)
Task is (4 + 5 * X = 6) Pupil's solution was (1 * X = -5 // 4)
Task is (5 + 2 * X = 8) Pupil's solution was (1 * X = 8 // 2)

Figure 2e. Protocol apparently showing M+N*X=P => N*X=M (pupil AB7).

Figure 3. Summary of major new mal-rules encountered in the May 1981 experiment.

Where sets, 1 to 6, give "parsing" mal-rules and 7-9 additional manipulative mal-rules. "F" in rule 8 is a common factor of "M" and "N", and "<" and ">" are used to represent the usual algebraic brackets.

1a.	M • X + N • X	=> M • X • N => M • X + N => M + X + N + X
1b.	M • X	=> M + X
2.	M + N • X =	=> M • N + X = => M + N + X =
3.	M • X + N =	=> M + X + N = => M • X • N = => (M + N) • X =
4.	M • X = N•P	=> X = M
5.	M•X = N•X + P	=> X + X = M + N + P
6.	M • X = N + P	=> M • X = N => M • X = P
7.	M • X = N	=> X = N
8.	M • X = N	=> X = (N/F)/M => X = N/(M/F)
9.	M • <N • X + P>	=> M • X + M • P

Figure 4a. Typical task for each task set and which rule(s) are focussed on.

Task Set	Rules Focussed On	Typical Task
2	SOLVE	5 • X = 7
3	ADDSUB	3 • X = 5 + 3
4	MULT	5 • X = 2 • 2
5	XADDSUB	2 • X + 3 • X = 10
6	NTORHS	2 • X + 4 = 16
7	REARRANGE	4 + 2 • X = 16
8	XTOLHS	4 • X = 2 • X + 3
9	BRA1	2 • X = 5 • <3 + 1>
10	BRA2	6 • X = 2 • <2 • X + 3>
11	ADDSUB&MULT	2 • X = 2 + 4 • 6
12	ADDSUB&XADDSUB	2 + 3 • X + 4 • X = 16
13	ADDSUB&BRA2	15 • X = 2 + 4 • <2 • X + 3>
14	MULT&XADDSUB	2 • 4 • X + 2 • X = 12
15	MULT&BRA2	14 • X = 2 • 3 • <2 • X + 3>

Where "<" and ">" are the usual algebraic brackets.

Figure 4b

Comparative table showing the pupils' performance in May and September 1981. Where the figures in square brackets give the number of times the rule has been activated. So, for instance, pupil AB4 made 6 errors on task-set 7 in the May 1981 session, but was perfect on all tasks in September. However, AB13 made 6 errors on task-set 7 in May, and gave a very similar performance in September. (See figure 1 for details of the mal-rules quoted in this figure).

Pupil	Task Set	Summary of Performance in May	Summary of Performance in September
AB2	TS-7	M+N*X=P -> (M+N)*X=P [7]	M+N*X=P -> (M+N)*X=P [1]
AB3	TS-7	M + N * X -> M * N + X	M +N * X -> (M + N) * X
AB4	TS-7	M + N*X -> M*N + X [3] -> M + N + X [3]	Correct
AB5	TS-7 TS-8	M + N*X ->(M + N)*X M*X=N*X+P->X+X=M+N+P [6]	Correct Correct
AB6	TS-5 TS-6	M*X+N*X=P ->M*X*N=P [3] M*X+N=P->M*X+M+N=P [4] ->M+X+N=P [2]	2 OK, 1 Omitted. 2 OK,M*X+N=P->M*X=N [1]
AB7	TS-3 TS-5 TS-6 TS-7 TS-8	M*X=N+P -> X=P/M Wild 2 OK, 4 Wild M + N * X =P -> X = M/N [6]	OK Wild 1 Mis-Parse 3 Mis-Parses 2 Values for X.
AB10	TS-5 TS-6 TS-7 TS-8	4 OK, 3 Wild. 5 OK. M+N*X=P -> (M+N)*X=P [4] 3 OK, 2 Wild	1 Wild, M*X+N*X=P-> X+X+M+N=P [1] 1 Wild, 2 Parse. -> X=P-M-N [3] M*X=N*X+P->X+X=N+P-M [3]
AB11	TS-7	M+N*X=P -> (M+N)*X=P [5]	Unchanged.
AB12	TS-5	M*X+N*X=P->M+X+N+X=P [1] ->X+X+M*N=P [1]	OK
AB13	TS-7	M+N*X=P -> (M+N)*X=P [6]	Unchanged
AB14	TS-7	2 OK, 4 Mal (MSOLVE)	M+N*X=P->(M+N)*X=P [2] 1 Wild
AB17	TS-5	M*X+N*X -> M+X+N+X [6] -> M*X*N [1]	TS-5 to 8 consistent but very curious.
AB18	TS-5 TS-7 TS-8	7 [Pretty Wild]	M*X+N*X=P->M+X+N+X=P [3] M+N*X ->(M+N)*X [3] M*X=N*X+P -> X+X=N+P-M [3]
AB20	TS-3 TS-4 TS-5 TS-6 TS-7 TS-8	M*X=N+P -> N*X=M+P [1] M*X=N*P -> X=M [4] OK M+N*X=P -> (M+N)*X=P [6] M+N*X=P -> (M+N)*X=P [6] Mal (MXTOLHS[1], MSOLVE[3])	Mal (MSOLVE) Mal (MSOLVE) Mal (MSOLVE) Mal (MSOLVE) Mal (MSOLVE)
AB21	TS-6	M*X+N=P -> (M+N)*X=P [2] Wild [4]	OK

Figure 5

List of algebraic expressions which the pupils read:

$$4 + 3 = 7$$

$$3 \bullet X = 6 + 3$$

$$2 \bullet X + 3 \bullet X = 10$$

$$2 \bullet X + 4 = 16$$

$$4 + 2 \bullet X = 16$$

$$4 \bullet X = 2 \bullet X + 3$$

$$4 + 3 - 5 = 2$$

$$5 \bullet X = 5 + 3$$

$$6 \bullet X - 3 \bullet X = 12$$

$$4 \bullet X + 5 = 19$$

$$6 + 2 \bullet X = 20$$

$$2 \bullet X = -4 \bullet X + 6$$

Figure 6.1

Pupil AB11: Searching for solutions.

(NB these and all subsequent figures are tidied up forms of the work sheets used, together with comments added from the verbal interchanges which took place).

a) The task given was: 4 * X = 2 * X + 6

```
Pupil:  Try  X = 2     8 = 4 + 6
                         8 = 10

             Try  X = 3    12 = 6 + 6
                           12 = 12

                           So   X = 3
```

b) The task given was : 3 * X = 2 * X + 5

```
Pupil:  Try  X = 3     9 = 6 + 5
                         9 = 11

             Try  X = 4    12 = 8 + 5
                           12 = 13

             Try  X = 4.5  13.5 = 9 + 5
                           13.5 = 14

             GIVES UP
```

c) The task given was: 2 * X + 3 * X = 17

```
Pupil:  Try  X = 3     6 + 9 = 17
                         15 = 17

             Try  X = 4     8 + 12 = 17
                            20 = 17

             Try X = 3 1/2  7 + 10 1/2 = 17
                            17 1/2 = 17

             Try X = 3 1/4  6 1/2 + 9 3/4 = 17
                            16 1/4 = 17

             Try X = 3 1/3  6 2/3 + 10 = 17
                            16 2/3 = 17

                            GIVES UP.
```

Figure 6.2

Pupil AB7 on PS-5 solving equations involving two Xs. For comparison, we also give 4 equations, namely l-o, which involve only one X to show that she again solves equations by searching.
NB In this case we have included a representative part of the dialogue between the pupil and the investigator in the body of the text.

a) The task given was: 2 * X + 3 * X = 10
The pupil wrote: 2 * 2 + 3 + 3 = 10
X = 2
X = 3

b) The task given was: 2 * X + 4 * X = 12
The pupil wrote: 2 * 2 + 4 * 1/2 = 12
X = 2
X = 1/2

c) The task given was: 3 * X + 2 * X = 11
The pupil wrote: 3 * 2 + 2 + 3 = 11
X = 2
X = 3

d) The task given was: 3 * X + 2 * X = 10
The pupil wrote: 3 * 2 + 2 + 2 = 10
X = 2
X = 2

e) The task given was: 3 * X + 2 * X = 12
The pupil wrote: 3 * 2 + 2 + 4 = 12
X = 2
X = 4

f) The task given was: 3 * X + 2 * X = 13
The pupil wrote: 3 * 2 + 2 + 5 = 13
X = 2
X = 5

g) The task given was: 2 * X + 3 * X = 10
The pupil wrote: 2 * 3 + 3 + 1 = 10
X = 3
X = 1

h) The task given was: 2 * X + 4 * X = 15
The pupil wrote: 2 * 4 + 4 + 3 = 15
X = 4
X = 3

i) The task given was: 2 * X + 4 * X = 14
The pupil wrote: 2 * 4 + 4 + 2 = 14
X = 4
X = 2

j) The task given was: 2 * X + 5 * X = 20
The pupil wrote: 2 * 5 + 5 + 5 = 20
X = 5
X = 5

k) The task given was: 2 * X + 5 * X = 21
The pupil wrote: 2 * 5 + 5 + 6 = 21
X = 5
X = 6

l) The task given was: 2 * X + 4 = 16
The pupil wrote: 2 * 6 + 4 = 16
X = 6

m) The task given was: 2 * X + 10 = 2•
- Unable to do.

n) The task given was: 2 * X + 4 = 15
- Unable to do.

o) The task given was: 2 * X + 4 = 14
The pupil wrote: 2 * 5 + 4 = 14
X = 5.

Figure 6.3

Pupil AB17 on task set 6.

a) The task given was: 2 * X + 3 = 9

Pupil writes 1) 2X = 9 - 3
2) X = 3

Interviewer writes X = 9 - 3 + 2
Interviewer: says could you say whether your step 1) above or what I've just written is correct.

Pupil says he really could not.

b) The task given was: 2 * X + 4 = 16

Pupil writes 1) 2X = 16 - 4
2) 2X = 12
3) X = 6

Interviewer writes X = 16 - 4 - 2
Interviewer: says could you say whether your step 1) above or what I've just written is correct.

Pupil says his 1) probably is.

Interviewer says: can you say why?

Pupil: I'm afraid not.

Interviewer: Now look back at the last example, there I suggested a slightly different method there. Would that be possible here?

Pupil: That's right, it would.

Interviewer: Which of these do you think is correct?

Pupil: Really not sure. I often have a lot of methods to choose between, which makes it pretty confusing. I sometimes have as many as 5 or 6.

[And so this conversation continues. After this point the pupil voluntarily offers 2 or 3 solutions to each task, as in the next task.]

c) The task given was 4 * X = 2 * X + 6

Pupil writes 1) X = 2 - 4 + 6
2) X = 4

Then suggests the following reworking:

1) 4X = 2X + 6
2) 4X = 8X
Then Quits.

Interviewer: Which solution do you think is right?

Pupil: Oh, I'm not really sure.

Interviewer: If you were a betting man, which would you put your money on?

Pupil: Probably the first.

Figure 6.4

Examples of very consistently used MAL-RULES noted with three pupils.

I) Pupil AB11 on task set 7.

a) The task given was: 4 + 2 * X = 16
Pupil writes 1) 6X = 16
2) X = 2.6666

b) The task given was: 2 + 4 * X = 14
Pupil writes 1) 6 * X = 14
2) X = 2.333

c) The task given was: 3 + 5 * X = 11
Pupil writes 1) 8 * X = 11
(and is told she can leave it in that form)

d) The task given was: 5 - 3 * X = 11
Pupil writes 1) 2 * X = 11
(and is told she can leave it in that form)

II) Pupil AB17 on task set 5
a) The task given was: 2 * X + 4 * X = 12
Pupil writes 1) * X * X = 12 - 2 - 4
2) X ** 2 = 6
3) X = ROOT 6

b) The task given was: 2 * X + 3 * X = 10
Pupil writes 1) * X * X = 10 - 2 - 3
2) X ** 2 = 5
(and is told he can leave it in that form)

c) The task given was: 2 * X - 3 * X = 10
Pupil writes 1) * X * X = 10 - 2 + 3
2) X ** 2 = 11
(and is told he can leave it in that form)

III) Pupil AB18 on task sets 5, 6, 7 and 8.
a) The task given was: 2 * X + 3 * X = 10
Pupil writes 1) 2 * X = 10 - 2 - 3
2) 2 * X = 5
3) X = 2.5

b) The task given was: 3 * X + 5 * X = 24
Pupil writes 1) X + X = 24 - 3 - 5
2) 2 * X = 16
3) X = 8

c) The task given was: 3 * X + 4 * X + 5 * X = 24
Pupil writes 1) X + X + X = 24 - 3 - 4 - 5
2) 3 * X = 12
3) X = 4

d) The task given was: 2 * X + 4 = 20
Pupil writes 1) X = 20 - 2 - 4
2) X = 14

```
e) The task given was:  3 * X + 5 = 7
   Pupil writes        1) X = 7 - 3 - 5
                       2) X = -1

f) The task given was:  4 + 3 * X = 14
   Pupil writes        1) X = 14 - 3 - 4
                       2) X = 7

g) The task given was:  5 + 6 * X = 20
   Pupil writes        1) X = 20 - 5 - 6
                       2) X = 9

h) The task given was:  4 * X = 2 * X + 6
   Pupil writes        1) 2 * X = -4 + 2 + 6
                       2) 2 * X = 4
                       3) X = 2

  Pupil then wrote     1) X - X = 2 + 6 - 4
                       2) 0 = 4
                       and QUITS.

i) The task given was:  6 * X = 3 * X + 6
   Pupil writes        1) 0 = 4
                       and QUITS.
```

Figure 6.5

Pupil AB5 on task sets 7,8,12 and 13.

a) The task given was: 3 + 4 • X = 7

Pupil writes
1) 4 • X = 7 - 3
2) 4 • X = 4
3) X = 1

Interviewer writes: 7 • X = 7
Interviewer: says could you say whether 1) above or what I've just written is correct?

Pupil: says what I wrote.

Interviewer: Can you tell me why what I wrote is not possible.

Pupil: It's not possible to add numbers and Xs.

b) The task given was: 6 + 8 • X = 12

Pupil writes
1) 8 • X = 12 - 6
2) 8 • X = 6
3) X = 6/8 = 3/4

Interviewer writes: 14 • X = 12
Interviewer: says could you say whether 1) above or what I've just written is correct?

Pupil: says what I wrote.

Interviewer: Can you tell me why what I wrote is not possible.

Pupil: It's not possible to add numbers and Xs.

Interviewer writes: 14 + X = 12
Interviewer: says could you say whether 1) above or what I've just written is correct?

Pupil: says what I wrote.

Interviewer: Can you tell me why what I wrote is not possible.

Pupil: It's not possible to separate the number from the X.

c) The task given was: 4 • X = 2 • X + 5

Pupil writes
1) 4 • X - 2 • X = 5
2) 2 • X = 5
3) X = 5/2

Interviewer writes: X + X = 4 + 2 + 5
Interviewer: says could you say whether 1) above or what I've just written is correct?

Pupil: says what I wrote.

Interviewer: Can you tell me why what I wrote is not possible.

Pupil: You have again separated the number from the X.

d) The task given was: 6 * X = 3 * X + 9

Pupil writes
1) 6 * X - 3 * X = 9
2) 3 * X = 9
3) X = 3

Interviewer writes: X + X = 6 + 3 + 9
Interviewer: says could you say whether 1) above or what I've just written is correct?

Pupil: says what I wrote.

Interviewer: Can you tell me why what I wrote is not possible.

Pupil: This is again a splitting of the number and X which is not possible.

e) The task given was: 2 + 3 * X + 4 * X = 16

Pupil writes
1) 3 * X + 4 * X = 16 - 2
2) 7 * X = 14
3) X = 2

f) The task given was: 3 + 4 * X + 5 * X = 20

Pupil writes
1) 4 * X + 5 * X = 20 - 3
2) 9 * X = 17
3) X = 17/9

g) The task given was: 2 * 3 * X + 4 * X = 16

Pupil writes
1) 6 * X + 4 * X = 16
2) 10 * X = 16
3) X = 16/10

h) The task given was: 3 * 4 * X + 5 * X = 20

Pupil writes
1) 12 * X + 5 * X = 20
2) 17 * X = 20
3) X = 20/17

i) The task given was: 2 + 3 * AA = 6

Pupil writes
1) 3 * AA = 6 - 2
2) 3 * AA = 4
3) AA = 4/3

Figure 7

A comparison between a tutored and a matched control group on a post-test given 2 months after the remedial teaching had been completed. This table gives information only on those pupils who received personal remedial instruction. The numbers in square brackets give the number of occurrences of the phenomena, and TSn indicates the number of the task-set involved.

Group A.	Group B
Pupils who received personal Remedial Instruction.	"Matched" pupils.
AB7	AB7M*
TS6: MX+N*P=> MX = -P+N Several Arithmetic Errors NB. TS5: (MX+NX=P) now OK	TS3: (MX=N) unable to solve M>N TS7: Precedence Problems TS8: (MX=NX+P) unable to solve
AB10	AB10M
TS5: MX+NX=P=> NX+N=P[1] TS6: MNTORHS[1] TS7: Parse error[1] TS8: MXTOLHS[3]	SEARCHER?: unable to solve equations if the solution is not a small positive integer. TS8: MXTOLHS [1]
AB11	AB11M
TS6: MSOLVE[1] TS7: Precedence Problem [3] & MSOLVE[2] NB. TS5 & TS8 now OK.	TS2: (MX=N) unable to do if M>N TS7: Precedence Problems [3] TS8: unable to do Also 6X=16=>X=2 4/6=> X=2 1/4
AB17	AB17M
TS8: MSOLVE[1] else perfect	TS2: MSOLVE[1] TS6: MNTORHS[1] TS6: has been wrongly Parsed, but correctly reworked.
AB18	AB18M
TS5: 5X=12 => X = 2 5/12 except for this Perfect.	TS7: Precedence errors - but correct answers for tasks which have integer solutions (SEARCHER?).

*Where AB7M is the pupil who was matched with pupil AB7 etc.

Figure 8

An overall comparison between the performance of the group given remedial instruction and the control group.

Summary of Performance	The Group given remedial Assistance	The Control Group
Serious errors (eg. Parsing errors, Multiple Manipulative errors etc.)	1	9
Fairly consistent MANIPULATIVE Errors	7	5
Infrequent MANIPULATIVE/careless errors	4*	5
PERFECT	12	5

*3 Pupils placed here had only one manipulative mal-rule error, and hence could be considered to be perfect.