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ABSTRACT

Twenty-five outstanding middle grade mathematics teachers were selected to participate in a three-week workshop presented by the Middle Grades Mathematics Project (MGMP) at Michigan State University. A major focus was on the development of leadership; the workshop involved guest speakers, field trips, and an intensive immersion in the MGMP curriculum materials. Following a brief introduction, a list of participants is provided. Eleven lectures are included in the first part of the publication: (1) Maximizing the Surface Area of Children's Comprehension (Masterson); (2) Logo and Middle School Math (Winter); (3) Spotlight on Problem Solving (Lester); (4) Improving General Mathematics (Lanier and Madsen-Nason); (5) Developmental Levels in Geometry (Mitchell); (6) The LES Instructional Model: Launch-Explore-Summarize (Shroyer); (7) Problem Solving in the Transition from Arithmetic to Algebra (Lampert); (8) Research on Similarity and Proportional Reasoning (Friedlander); (9) Motivating Algebra Through Problem Solving (Phillips); (10) "Problems" (Wagner); and (11) A Suggested Outline for a Course in Teaching and Learning Probability and Statistics (Shaughnessy). Part 2 presents Avital's test of 20 historical problems, with solutions developed by the teachers. The third part focuses on an evaluation of the workshop. Finally, miscellaneous information is provided, and appendices contain evaluation forms and data, plus selection procedures. (MNS)


MSU
HONORS TEACHERS
WORKSHOP 1984

PROCEEDINGS OF THE
HONORS TEACHERS WORKSHOP OF
MIDDLE GRADE MATHEMATICS

NOVEMBER 27 TO DECEMBER 15, 1984

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824

William M. Fitzgerald
Director

Cover T-shirt design by Joan Hall

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Document prepared by Mary Reynolds.

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INTRODUCTION

Twenty five outstanding middle grade mathematics teachers were selected by the staff of the Middle Grades Mathematics Project (MGMP) at Michigan State University to participate in an Honors Teachers Workshop from November 27 to December 15, 1984. The participants came from sixteen different states and represented a cross section of the best mathematics teachers from throughout the nation.

Among the approximately one hundred applicants who were not selected were many very highly qualified persons. It was heartening indeed to learn of many of the commendable programs which are in practice and of the talented people who are responsible for those programs. Criteria for selection are found in Appendix D.

A major focus of our workshop was the development of instructional leadership. The potential for leadership weighed heavily in the selection process. The effects of the workshop will be assessed in a follow-up poll in May, 1985. Those results will be available from the director at that time.

The participants came knowing very little of what to expect, except that they had been asked to work hard. They did so in twelve hour days packed to the brim with guest speakers, field trips, personal interchanges and an intensive immersion in the MGMP curriculum materials and philosophy by Glenda Lappan.

In these proceedings, we have attempted to capture some of the excitement and enthusiasm which was ever present. This is, of course, impossible. We do hope, however, that the readings become useful to a wide audience.

William M. Fitzgerald, Director
Alex Friedlander, Editor

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PART I - LECTURES

1. Maximizing The Surface Area Of Children's Comprehension
   John Masterson

2. Logo and Middle School Math
   Mary Jean Winter

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   Frank Lester

4. Improving General Mathematics
   Perry E. Lanier and Anne Madsen-Nason

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10. "Problems"
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    Michael Shaughnessy

MAXIMIZING THE SURFACE AREA OF CHILDREN'S COMPREHENSION

John Masterson
Department of Mathematics
Michigan State University

A frontal lobotomy - if you are not familiar with it - is an operation on the brain which removes a number of the many dimensions of the functioning of the brain. It leaves an individual with the ability to perform basic mechanical functions but little else.

A mathematics textbook - this can be extended to any curriculum program - can have the same effect on a classroom environment as a frontal lobotomy does on the brain of a human being: it takes a complex organism and reduces its functioning to one or two dimensions.

Any talk given at this hour of the morning should begin with a gross overstatement of its major point. It accomplishes what several cups of coffee do without the detrimental effects of the caffeine.

Printed material does, however, have the effect of removing from the teacher, the need to (and hence, the responsibility to) confront the complex task of creating learning in the classroom.

My remarks, by the way, are addressed to teachers at all levels, so my focus - and hence criticism also - are not directed at the middle school scene in particular. In fact, the curriculum problem I just introduced is most acute at the college level: the presentation of material in logical order is the only item on the agenda and serious educational questions have not yet been raised.

My goal is not to lead a holy war against texts: we could not do the job we are required to do without them. But I would like to see us all focus on the problem of educating children seriously as educated people and decide how to use material rather than let it continue using us.

For one thing, there are two major and divergent entities operating when children are learning mathematics: children and mathematics. Each creates its own agenda in the classroom To complicate things even more, each of these can be focused on from several different and valuable perspectives.

My major point is that such a multiple perspective approach to teaching a concept broadens our potential for making meaningful contact with children well beyond what we see in a textbook or what we see in most structured curriculum programs.

I will consider three very specific perspectives: The perspective of psychology will concentrate on the comprehension of the child; the perspective of history will concentrate on the construction of the concept by our forebeares; the perspective of mathematical theory will concentrate on the foundation of the concept itself and its links to other mathematics. We summarize it in the following matrix:

BASIC COMPONENTS

PERSPECTIVES	Children	Math
psychological		
historical		
theoretical		

I suggest that focus on each one of these boxes opens up a different window on the concept in question. Matters appear even more complicated when we notice that what we see through one window interacts with what we see through another in quite complex ways. What I hope to exhibit is the rich fabric of the concept which arises, and leaves you with the notion that the best resource you have in the classroom is your own well-educated selves.

If this sounds like a philosophical cakewalk through the clouds, it is not. After briefly establishing a point of view about children and about mathematics, I will develop everything else within the framework of a single example: and that is the curriculum relating to factoring and solving polynomial equations.

So I'll begin with a brief examination of children and their mathematics.

We accept that the brain of the child is a complex developmental organism. We don't kid ourselves that the job of teaching is simply to front-load the material (lecturing) and then check it out on the other end (exams). In fact we won't accept retention as the primary goal. We seek, instead, comprehension, and, beyond that, learning. I give these words the meaning they have been given by Frank Smith of the Ontario Institute for Learning in his book entitled Comprehension and Learning. (By the way, he subtitled the work "A conceptual framework for teachers".)

Comprehension means relating new experience to the already known. Learning means something more. It involves changing or elaborating on what is already known. Comprehension, according to Smith, goes one giant step further than perception. It "locks our identification of objects into the network of all we know, into the cognitive interrelationships which make our experience meaningful". The emphasis of all this is on individual past experience.

A major problem confounding comprehension - and one which seems to have been taken into consideration recently as a serious epistomological problem - is the inherent subjectivity of comprehension: its great variation from individual to individual and its dependence upon each individual's entire past history of concept formation.

In his 1983 address to the International Group for the Psychology of Mathematics Education, Ernst von Glasersfield of the University of Georgia states the problem so:

> If you grant this inherent subjectivity of concepts and, therefore, of meaning, you are immediately up against a serious problem. If the meanings of words are, indeed, our own subjective construction, how can we possibly communicate? How could anyone be confident that the representations called up in the mind of the listener are at all <u>like</u> the representations the speaker had in mind when he or she uttered the particular words? This question goes to the very heart of the problem of communication. Unfortunately the general conception of communication was derived from and shaped by the notion of words as containers of meaning. If that notion is inadequate, so must be the general conception of communication. (p.53)

Of course, as a teacher, it would be impossible to track each student's individual subjective experience. Even if you do grant the possibility, it is impractical. Von Glasersfield goes on to suggest that this is not necessary. Reduced to simplest terms, he proposes that the language of the teacher, at least, not interfere with, or contradict, the subjective formations of the student and, at best, that the teacher know enough to be able to foster the reflective awareness of the student to connect subjective past experience with the words in which a new abstract concept is imbedded.

Von Glasersfield describes what the teacher needs to be doing within his framework of the teacher as experimenter in the classroom. I want to draw a very significant statement from this without implying that the single teacher - single student concentration here is at all practical.

> The teaching experiment, as I suggested before, is, however, something more than a clinical interview. Whereas the interview aims at establishing "where the child is", the experiment aims at ways and means of "getting the child on". Having generated a viable model of the child's present concepts and operations, the experimenter hypothesizes pathways to guide the child's conceptualizations towards adult competence. In order to formulate any such hypothetical path, let alone implement it, the experimenter/teacher must not only have a model of the student's present conceptual structures but also an analytical model of the adult conceptualizations towards which his guidance is to lead. (p.62)

When Von Glaserfield talks of the teacher's knowledge of the student's present conceptual structures, he presupposes a psychological perspective on the part of the teacher. When he talks of the teachers possessing an analytical model of the adult conceptualizations, he presupposes a theoretical point of view - an understanding of mathematics. I would add that is investigation of student conceptual structures is greatly aided by a sort of macro view of the formulation of these concepts: in other words, a historic point of view. This completes the generation of the three perspectives from which I approach children.

The phiolsopher, scientist and teacher David Hawkins, has, prior to Von Glaserfield, covered much of the same ground in a magnificant little rambling paper which suggested the title for this talk and which you have been given. I cite briefly from Finding the Maximum Surface Area in Education:

> The alternative, of course, which you discover as a teacher when you have to work with diverse human beings in diverse contexts, is that you yourself reorganize, in your own understanding, your knowledge of that subject matter. And you reorganize it according to a different principle than that of the textbook. You organize it for maximum surface area. Let me go back to the analogy of the contact between a child and "it". By spreading way out, by making many parts of the logically organized subject matter accessible to the already established means of knowing, to the interests and commitments of the learner, you greatly increase the rate of learning in that subject matter.

In another paper entitled Nature, Man and Mathematics he touches upon both the psychological and the theoretical demands from a teacher.

> For such a teacher a limiting condition in mapping a child's thought into his own is, of course, the amplitude of his own grasp of those relationships in which the child is involved. His mathematical domain must be ample enough, or amplifiable enough, to match the range of a child's wonder and curiosity, his operational skills, his unexpected ways of gaining insight... But a teacher of children, of the kind I postulate, must be a mathematician, what I would call an elementary mathematician, one who can at least

> sometimes sense when a child's interests and proposals - what I have called his trajectory - are taking him near to mathematical sacred ground. (pp.118-119)

Now that I have utterly complicated the task of assisting children in comprehending mathematics, I move on to do the same to mathematics itself, only much more briefly.

No other subject matter (except perhaps chemistry) is taught as if it were a disembodied collection of miscellaneous facts floating somewhere outside culture and reality - as is mathematics. Indeed, the problem gets worse as you move up the line from elementary school to college. We like to think of mathematics as a universal body of ultimate truth. This is nonsense. It is a collection of logically coherent structures pointing in different directions, constructed by mortal people in history for the purpose of dealing with real quantifiable problems. It is not immutable but continually restructuring itself to meet its own inadequacies.

If we are seeking comprehension in children, the subject matter itself must be comprehensible. It must have a chance of attaching itself to the subjectively variable mental schema of the child. Its ideas must be developed to meet the needs of specific people in history. The subject matter must reflect the idiosincracies of these people as much it does any kind of timeless truth. So, mathematical ideas have also historical and psychological ramifications which cannot be separated from seemingly universal and self-justifying concepts. And so we have muddied up the waters of mathematics itself. Let us get on to specifics.

And it is here, through examples, that we can see I would hope, that what might appear confused in the abstract unveils in the concrete. A lucid interwoven fabric of history, psychology and theoretical mathematics enriches both the comphrehension of the child and the substance of the mathematics itself, and makes great demands upon us as teachers: people educated sufficiently to look at concepts from all these many points of view.

Now on to the example! Let me emphasize from the beginning that this matrix, like a psychological theory of learning, is a tool. It is not meant to be pursued for its own sake, followed rigidly or chipped in stone: it's simply a vehicle for turning up useful ways of thinking about children's learning.

I mention this because when I first looked at it I noticed the demand for a psychological perspective of mathematics itself - seeming nonsense. But that turns out not be to be case.

Look at how we read a second or third order equation: we say "x-squared" or "x-cubed", not "x-upper-2" or "x-upper-3". There is a built-in geometric perception in the language itself. An examination of history will show us in fact, that the elementary theory of equations was generated within a geometric framework. From the point of view of the child, does the notation generate a geometric mental image? Might this not be useful in giving us a different approach to equations? Later!

The following is a basic outline of a textbook approach to solving quadratic equations:

$ax^2 + bx + c = 0$

(1) Divide by a: $x^2 + mx + k = 0$

(2) Look for r_1, r_2 so $m = r_1 + r_2$, $k = r_1 r_2$

(3) Write $(x - r_1)(x - r_2) = 0$,
then $(x - r_1) = 0$, and $(x - r_2) = 0$;
then $x = r_1$, $x = r_2$

(4) If this can't be done write $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(5) You obtain all solutions of $ax^2 + bx + c = 0$.

The questions we hope can be answered at this stage are:

(1) Solve $x^2 - 7x + 12 = 0$ or $x^2 + 14x - 20 = 0$

(2) Solve $3x^2 + 5x - 2 = 0$

(3) Solve $x^2 + 4x = 0$

(4) Solve $x^2 + x + 1 = 0$, $x^2 + 1 = 0$

(5) At what value of x does $y = x^2 + 7x + 12$ cross x-axis.

Children may ask different questions with regard to this topic:

(1) Can we solve any polynomial equation? How many solutions?

(2) Is $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ a legitimate factoring?

(3) Can we factor $x - 5$ or $x^2 + 1$; why not? or why?

(4) I remember factoring $12 = 3 \times 4$. Is this a different kind of factoring?

(5) FACTORING means breaking into parts (smaller parts?). Is $\frac{3}{4} \cdot \frac{4}{3}$ a factoring of 1?

Is $\frac{x^2 - 5x + 6}{2} \cdot 2$ a factoring of $x^2 - 5x + 6$?

(6) Does $5 = (2 + i)(2 - i)$ make sense?

Let's examine factoring from the point of view of the child. If we take a psychological approach we might begin, using David Hawkins' framework, by asking what is the expected trajectory of the student - where is the student pointed when first seeing factoring of algebraic expressions. Or in Von Glasersfield's words, what compound of experiential elements constitutes an individual middle school student's concept of factor and algebraic expression.

Previous to the introduction of algebraic factoring, students have certainly factored numbers. Do we build on this? In what way does it confound the new concepts we are trying to teach?

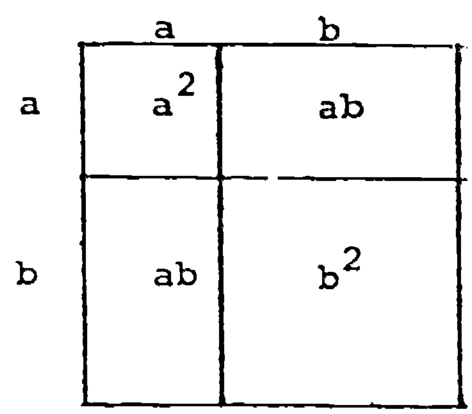
Again, using our knowledge of psychological theory, what does an approach along the lines of Piaget or Bruner tell us about the developmental level of the students? What concrete or representational ways of presenting factoring make connections with what is really being done? Are we teaching only symbol manipulation and not realizing it?

It is at this point that we can see some of the complex interaction which I spoke earlier. While a psychological perspective has raised some significant questions, it is to historical and mathematical perspectives that we turn to get suggestions.

Beginning with the second question about children's need for concrete or representational forms, history tells us that there were more concrete ways of looking at most mathematics. Abstract algebraic language did not exist even in the most sophisticated culture: that of Ancient Greece. Yet, the elementary - and some not so elementary - laws of algebra were well understood, not only in Greece but in Babylon. Let's look at an algebraic law formulated in Euclid's Book II:

<u>Proposition 4</u>: If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

In our notation: $(a + b)^2 = a^2 + 2ab + b^2$

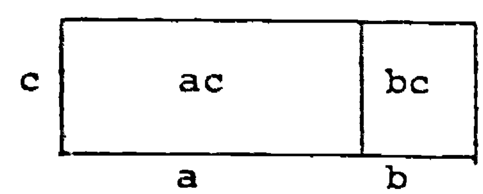


Some of the exercises that can be framed around this proposition are:

(1) $(a + b)c = ac + bc$

(2) $(a + b)(c + d) = ac + ad + bc + bd$

(3) $a^2 - b^2 = (a - b)(a + b)$



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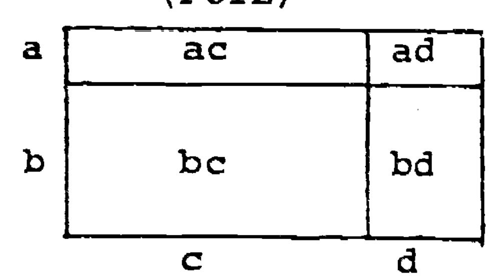


To the Babylonians, equations were cast not so much in a geometric form as in an arithmetic-verbal form. An Ancient Babylonian text sets the problem: "The product of two numbers is 1 and their sum is a given number b. What are the two numbers?"

The solution was given by telling what you would do to the number b to obtain the answer: much like you would do if you took a computer process approach to solving it.

Some other Babylonian problems are:

1. Find a root of the cubic equation

$$x^3 + 2x^2 - 3136 = 0$$

<u>Solution</u>: Construct tables of $n^3 + n^2$.

2. Solve the system

$$xyz + xy = \frac{7}{6}$$
$$y = \frac{2x}{3}$$
$$z = 12x.$$

3. Reduce any cubic $ax^3 + bx^2 + cx + d = 0$ to the form

$$x^3 + x^2 + c = 0.$$

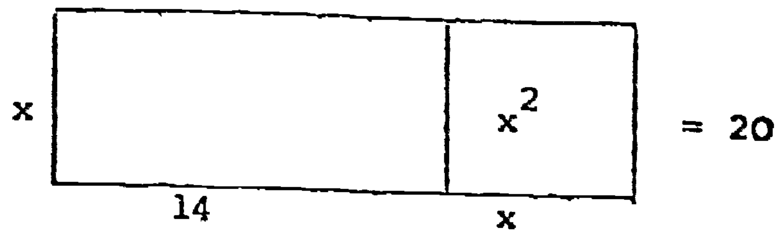
Returning to our problem of finding more representational ways of viewing quadratic equations, the Arabs of the 9th century and beyond raised the Greek geometric approach to a higher level by actually using geometry to solve quadratic equations. Consider the following problem:

A square of side x has added to it a rectangle of length 14 and width x. The answer is 20. What is x?

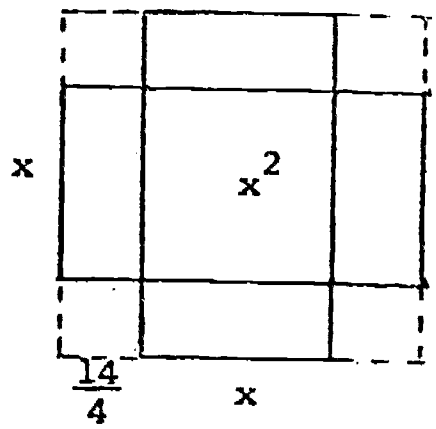
<u>To us</u>: Solve $x^2 + 14x - 20 = 0$.

Arabic Solution

(a) Divide the rectangle into 4 equal rectangles of length 14/4 and width x and attach as in 2nd picture.



(b) If you add the corner squares, you get a large square and you have added $4(\frac{14}{4})^2$ to area.



(c) So large square has area $20 + 4(\frac{14}{4})^2$. But side length of large square is $x + 2(\frac{14}{4})$.

(d) So $x + 2(\frac{14}{4}) = \sqrt{20 + 4(14/4)^2}$;

hence $x = -2(\frac{14}{4}) + \sqrt{20 + 4(14/4)^2}$.

We might ask ourselves - would a version of this approach be useful? It certainly gives us a geometric explanation of quadratic equations.

Let's take up now the second problem created by our approaching equations from the point of view of children's learning: namely, the natural relations which are drawn and ought to be drawn by children between factoring numbers and factoring polynomials. One obvious point to be made is that equations in variables are just equations in unknown numbers. But, in fact, this is a false and diverting relationship.

To understand the real and important relationship you actually need to know something about higher mathematics - where the real relationship can actually be seen. This brings us to the window of theoretical mathematics.

Listed below are the four basic numerical operations and six important sets in which some of these operations can be performed. In four of the six, all 4 operations are legitimate and produce like objects.

For two of them, however, division does not produce a like object. These two are $\mathbb{Z}$ and $\mathcal{P}[x]$. For example, in $\mathbb{Z}$, $5 \div 2$ does not produce an element of $\mathbb{Z}$. In $\mathcal{P}[x]$, $(x^2 + 1) \div x$ does not produce a polynomial. It is this inability to always divide which makes factorization meaningful.

<u>Operations</u>	(1)	(2)	(3)	(4)
	"+"	"×"	"−"	"÷"

$\mathbb{Z}$ = Integers − {...−3,−2,−1,0,1,2,3,4,...}

$\mathbb{Q}$ = Rational Nos. − eg: $-\frac{1}{2}, \frac{5}{7}, \frac{157}{183}$

$\mathbb{R}$ = Real Nos. − including Q, $\sqrt{2}$, e, $-\pi$, $\sqrt[-7]{19}$

$\mathbb{C}$ = Complex Nos. − {a + bi: a,b in $\mathbb{R}$} $i = \sqrt{-1}$

$\mathcal{P}[x]$ = Polynomials − eg: 3, x + 5, $x^2 + x + 1$, $x^9 - \sqrt{2}\, x^3 + 1$

$\mathcal{R}[x]$ = Rational Functions − including [x] and also $\frac{x^2+1}{x(x^6+5)}$

Moreover, the study of factorization in sets like $\mathbb{Z}$ and $\mathcal{P}[x]$ produces remarkable similarities. The important facts are very much the same. Below we have listed two of the foundations of the theory of factoring: the Unique Prime Factorization Theorem (UPFT) and the Division Algorithm (DA). Notice the similarity.

$\mathbb{Z}$	$\mathcal{P}[x]$
UPFT $a = p_1^{b_1} \cdots p_k^{b_k}$ $120 = 2^3 \cdot 3^1 \cdot 5^1$	UPFT $p(x) = \pi$(linear, quadratic over $\mathbb{R}$ $(x^3 - 1) = (x - 1)(x^3 + x + 1)$, $x^3 - 2 = (x - \sqrt{2})(x + \sqrt{2})$
DA $a = qb + r,\ 0 \leq r < b$	DA $p(x) = q(x)b(x) + r(x)$ $\deg r(x) < \deg b(x)$ <u>Sp Case Remainder Th</u> $p(x) = q(x)(x - a) + R$

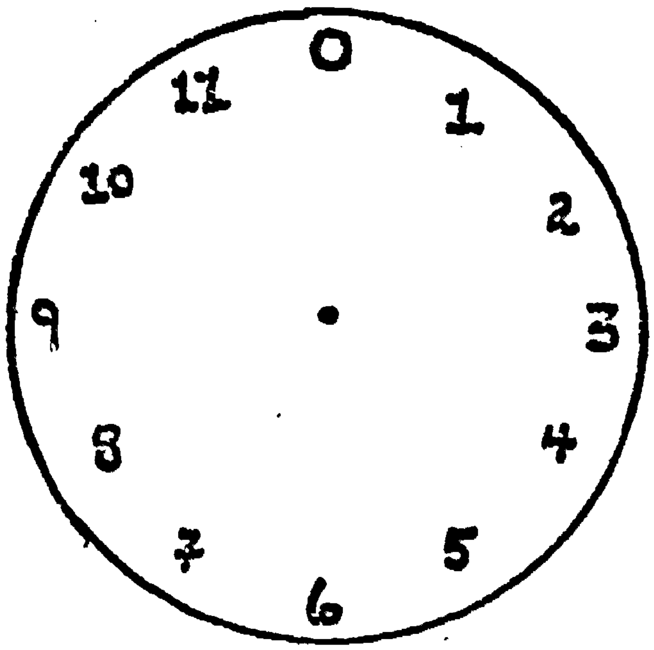
$(x - \sqrt{2})(x + \sqrt{2})$ is then seen as a legitimate factoring of $x^2 - 2$ while the factoring of 1 as $\frac{3}{4} \cdot \frac{4}{3}$ is seen to fall outside the realm in which factoring is of any interest. $5 = (2 + i)(2 - i)$ is seen as a legitimate and fascinating factoring if one allows "complex integers". Indeed, the complex integers behave like Z and [x], so the question of whether 5 is really prime is given a new twist.

By knowing this rather abstract topic, a teacher gains more power and may pose some fascinating problems as well. I give only one of them which fits nicely the Bruner's "torpedo" idea.

Suppose we apply our usual procedures to solving a quadratic equation in the situation where our operations are performed on a normal clock. We notice that something goes wrong. The usual method of factoring doesn't give all our solutions. Why not? See if you can find the property of real numbers which does not hold for clock numbers. But why then does it work in a clock with 7 numbers?

<u>Factoring in Clock Arithmatics</u>

<u>Solve</u> $x^2 - 1 = 0$



The solutions are

$$(x - 1)(x + 1) = 0$$
$$x = -1$$
$$x = 1$$

But -1 is really 11

The solutions are

{1, 11} - <u>Yes?</u>

Check by hand - <u>BIG SURPRISE</u>

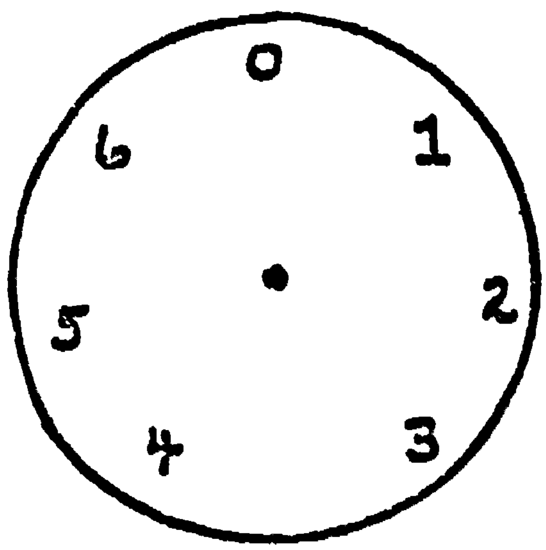
$(5)^2 - 1 = 0$

$(7)^2 - 1 = 0$

The solutions are

{1, 5, 7, 11}

However, the usual algorithm works perfectly here →



Some of the ideas may be a little deep, but every opportunity we take to learn a little more mathematics, a little more learning theory and a little more history gives us new power over curricula, new insight into the ways of making mathematics more meaningful to children - it increases the surface area of contact between the concept in question and the minds of children.

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LOGO AND MIDDLE SCHOOL MATH

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This session considered three uses for LOGO in the middle school math class:

1. LOGO's geometrical nature makes it an appropriate language for pre-written programs used in demonstration and explanation of concepts.
2. Students develop geometric intuition while writing LOGO procedures. This intuition is essential for success in the study of formal geometry.
3. LOGO can be used as a source of other mathematical investigations.

```
TO TESTD
 TEST :T = 90
 IFF PU SETXY :X :Y SETH :ANG PRINT [TRY AGAIN] TURN1
 PRINT [GOOD] TOPLEVEL
END

TO TURN1
 PRINT [ENTER TURN]
 MAKE "T FIRST RQ
 PD
 RT :T FD 30 REPEAT 500 [FD 1 IF ANYOF XCOR*XCOR + YCOR*YCOR >9990 YCOR < 0 TESTD]
 TESTD
END

TO SEMI
 PU SETXY - 100 0 PD
 REPEAT 60 [FD 200*3.14/120 RT 3]
 RT 90 FD 200
END

TO INSCRIBE
 SEMI
 RT 90
 MAKE "ANG ( RANDOM 60 ) + 10
 RT :ANG FD SQRT ( 2 * 10000 * ( 1 - COS 2 * :ANG ) ) - 2
 MAKE "X XCOR MAKE "Y YCOR
 TURN1
END
```

```
TO TURN
 PRINT [ENTER TURN]
 MAKE "ANG FIRST RQ
 RT :ANG FD 100
 IF ( XCOR ) * XCOR + ( YCOR + 50 ) * ( YCOR + 50 ) < 0.1 PRINT [GOOD] STOP
 PU SETXY 0 50 PD
 PRINT [TRY AGAIN]
 SETH 90
 TURN
END

TO DIAM
 TANGENT
 PRINT [HIT THE CENTER]
 TURN
END

TO TANGENT
 REPEAT 4 [FD 1 RT 90] PU
 FD 50 RT 90 PD
 REPEAT 90 [FD 100*3.14/90 RT 4]
 FD 40 BK 40
END

TO L
 DRAW
 PU SETXY - 150 ( - 20 )
 PD
END

TO POLY :ANGLE
 SETH 0
 MAKE "S 50 IF :ANGLE < 30 MAKE "S 30
 IF :ANGLE < 15 MAKE "S 20
 IF :ANGLE < 10 MAKE "S :ANGLE
 IF :ANGLE < 20 FULLSCREEN
 REPEAT 200 [FD :S RT :ANGLE IF HEADING = 0 THEN STOP]
END

TO TESTB
 IF ( 80 - XCOR ) * ( 80 - XCOR ) + ( :BP - YCOR ) * ( :BP - YCOR ) < 10 DONE
 PU SETXY 0 0 SETH 0 PRINT [TRY AGAIN]
 PD GUESS
END

TO C
 DRAW
END
```



```
TO DONE
 PRINT [GOOD]
 TOPLEVEL
END
```

```
TO TESTC
 IF ( :BP - XCOR ) * ( :BP - XCOR ) + ( - 60 - YCOR ) * ( - 60 - YCOR ) ( 18 DONE
 PU SETXY 0 0 SETH 0.PRINT [TRY AGAIN]
 PD GUESS2
END

TO GUESS2
 NOWRAP
 PRINT [ENTER ANGLE]
 MAKE "ANG FIRST RQ
 RT :ANG FD 80 / COS :ANG
 IF XCOR ) 80 TEST C
 MAKE "X XCOR
 RT 180 - :ANG * 2
 MAKE "H ( 80 - :X ) / SIN :ANG
 MAKE "NY :H * COS :ANG
 IF YCOR - :NY ( - 60 FD ( YCOR + 60 ) PRINT [MISSED THE SIDE] TESTC
 FD ( 80 - :X ) / SIN :ANG
 MAKE "Y YCOR
 IF :Y ( - 60 TESTC
 RT 2 * :ANG
 FD ( 60 + :Y ) / COS :ANG
 IF YCOR ( - 70 TESTC
 IF XCOR ( - 120 TESTC
 TESTC
END

TO BILLIARDS2
 DRAW
 PU
 SETXY - 120 80 PD SETXY 80 80 SETXY 80 ( - 60 ) SETXY - 120 ( - 60 )
 MAKE "BP ( RANDOM 159 ) - 79
 PU SETXY :BP - 2 ( - 60 )
 PD REPEAT 4 [FD 4 RT 90]
 PU SETXY 0 0 SETH 0 PD FD 40 BK 40
 GUESS2
END

TO B2
 BILLIARDS2
END

TO GUESS
 NOWRAP
 PRINT [ENTER ANGLE]
 MAKE "ANG FIRST RQ
 RT :ANG FD 80 / COS :ANG
 IF XCOR ) 80 TESTB
 MAKE "X XCOR
 RT 180 - :ANG * 2
 IF YCOR - ( 80 - :X ) / SIN :ANG ( - 70 FD 110 PRINT [OUT OF BOUNDS] TESTB
 FD ( 80 - :X ) / SIN :ANG
 TESTB
END
```

```
TO BILLIARDS1
 DRAW PU SETXY - 80 80 PD
 SETXY 80 80
 SETXY 80 ( - 20 )
 MAKE "BP RANDOM 50
 PU SETXY 80 :BP - 2
 PD REPEAT 4 [FD 4 RT 90]
 PU SETXY 0 0 SETH 0 PD FD 40 BK 40
 GUESS
END

TO B1
 BILLIARDS1
END

TO ANGLE :ANG
 PC 1
 FD 90 BK 90
 REPEAT :ANG / 4 [RT 4 FD 50 BK 50]
 FD 90 BK 90
END

TO EDGES :ANGLE
 SETH 0
 MAKE "S 50 IF :ANGLE < 30 MAKE "S 30
 IF :ANGLE < 15 MAKE "S 20
 IF :ANGLE < 10 MAKE "S :ANGLE
 IF :ANGLE < 20 FULLSCREEN
 REPEAT 200 [FD :S+15 BK 15 RT :ANGLE IF HEADING = 0 THEN STOP]
END
```

SPOTLIGHT ON PROBLEM SOLVING

Frank Lester
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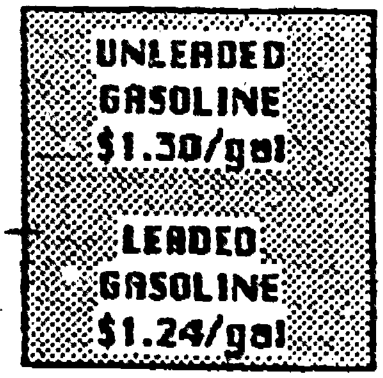
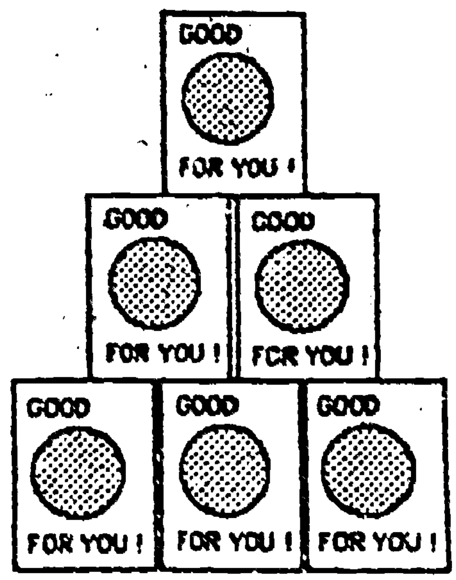
(Many of the items that follow can be pursued more fully in references numbered 3 and 4.)

WARM UP PROBLEMS



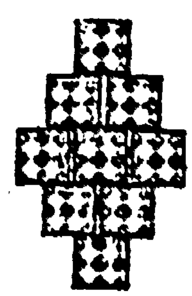
Problem #1: The "amount" part of a gasoline pump was broken so the manager had to find the amount of a sale on his own. How much is the bill for 14.2 gallons of unleaded gasoline?



Problem #2: A store clerk was told she had 45 cereal boxes to be stacked in the display window and all of the boxes had to be used. The manager told the clerk that the boxes had to be in a triangle like the one shown at the right. The sales clerk wondered how many boxes need to be placed on the bottom row to build the triangle using all boxes.

Problem 2 - Extension 1

A patio was to be laid in a design like the one shown. A man had 50 blocks to use. How many blocks should be placed in the middle row to use the most number of blocks?



Process Problems

5. Nancy's mother said she could order 1 sandwich and 1 drink for lunch. How many different lunches can Nancy order from this menu?

Sandwiches	Drinks
hamburger	juice
hot dog	milk
grilled cheese	soft drink

6. A tennis club was having a tournament for its 8 members. If each member was to play each other member one time, how many matches would be played?

7. Jeanette has 2 dogs and 2 cats. Each dog eats 1 can of dog food a day. The cats share 1 can of cat food each day. How many cans of dog and cat food do Jeanette's pets eat each week?

8. Janet and Vicki put up a rope to mark the starting line for a sack race. The rope was 10 m long. They put posts at the ends of the rope and every 2 m. How many posts did they use?

9. A recipe calls for 3 eggs and 4 cups of flour. A baker used 24 eggs. How many cups of flour did he use? Copy and complete the table.

EGGS	3	6	9	
FLOUR	4	8	12	

Applied Problems

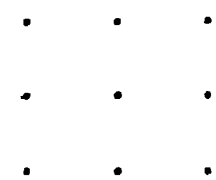
10. Suppose you and your family go on vacation for two weeks, and you must board your dog at a kennel. At which kennel will you board your dog?

11. You and a friend want to open a lemonade stand. You want to make a profit. You need to decide how much to charge for each glass of lemonade.

12. You want a new bicycle. Should you sell your old bicycle through a newspaper advertisement, or should you trade it in on a new one?

Puzzle Problems

13. A LINE SEGMENT PUZZLE - Draw 4 straight line segments to pass through all 9 dots. Each segment must be connected to an endpoint of at least one other segment.



PROBLEM TYPES - SUMMARY

1. <u>One-Step Problems</u> - These problems can be solved by choosing one of the operations addition, subtraction, multiplication, or division. These are the familiar "story problems" that have been a traditional part of school mathematics programs.

2. <u>Multiple-Step Problems</u> - These problems can be solved by choosing two or more of the operations addition, subtraction, multiplication, and division. Problems that require the repeated use of the same operation also fall in this category.

3. <u>Process Problems</u> - These problems can be solved using one or more strategies such as:

* guess and check
* make an organized list
* look for a pattern
* solve a simpler problem
* use logical reasoning
* draw a picture
* make a table
* work backwards
* write an equation

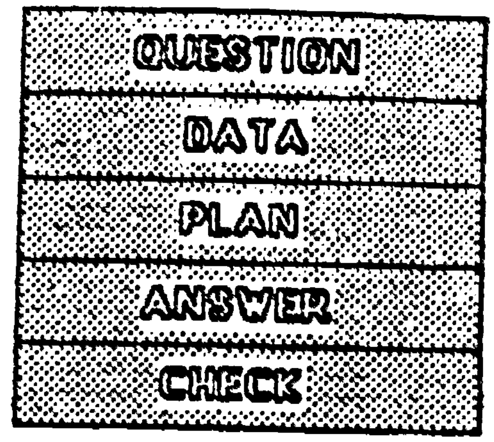
Process problems cannot be solved by simply adding, subtracting, multiplying or dividing.

4. <u>Applied Problems</u> - These require students to collect data and make a decision. The solution may require the use of one or more operations and/or one or more of the strategies given above for process problems. These problems reflect realistic situations and they often have no one correct answer.

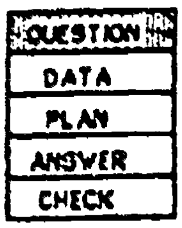
5. <u>Puzzle Problems</u> - These problems allow students an opportunity to engage in potentially enriching recreational activities. Solutions to puzzle problems often require students to look at the problem from a different perspective.

A 5-POINT CHECKLIST FOR PROBLEM SOLVING

1. Understand the QUESTION
2. Find the needed DATA
3. PLAN what to do
4. Find the ANSWER
5. CHECK back

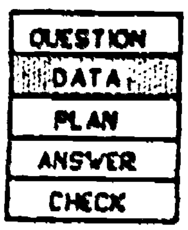


PROBLEM SOLVING SKILL ACTIVITIES



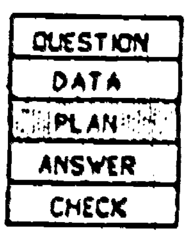
1. Write a question you can answer using data from this story.

   Mr. Timms makes $3 profit on the sale of each $15 travel book. He sold 8 of these books.



2. This problem has missing data. Make up needed data and solve.

   Betty and Hans put 2 356 beans in a jar for a math contest. Jerry's guess was the closest. By how much did Jerry miss the total?



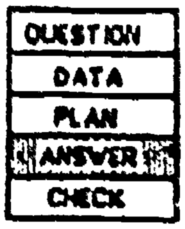
3. Tell which operations (+ − × ÷) you can use to solve this problem.

   The U.S. Congress has □ Senators and □ Representatives. There are □ more members of Congress than there are judges on the U.S. Supreme Court. How many judges are on the Supreme Court?

4. Copy and complete this table to help you solve this problem.

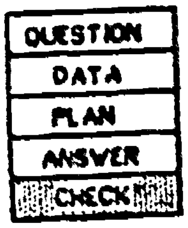
   Darrel needs 4 cups of pancake mix for every 3 cups of milk. How many cups of mix does he need for 15 cups of milk?

CUPS OF MIX	4	8	?	?	?
CUPS OF MILK	3	6	9	12	15



5. Estimate the answer by rounding to the nearest whole number.

   It takes Mars 1.888 years to revolve around the sun once. About how many years would it take Mars to revolve around the sun 5 times?



6. Decide if the answer to this problem makes sense. If it does not make sense, tell why.

   Problem: The desk is 45 cm wide and 92 cm long. How much greater is the length than the width?

   Answer: The length is 137 cm longer than the width.

DESIRABLE CHARACTERISTICS OF A TEACHING STRATEGY FOR PROBLEM SOLVING

* Useable by most teachers
* Requires minimal teacher training
* Research based
* Valued by teachers
* Teacher as a guide

TEACHING ACTIONS FOR PROBLEM SOLVING

<u>Before</u>

TA 1. Read the problem to the class or have a student read the problem. Discuss words or phrases students do not understand.

TA 2. Have a whole-class discussion about understanding the problem. Ask questions for understanding the problem - see "Understanding the Problem".

TA 3. Ask students to suggest strategies which might be helpful in finding a solution. Do not evaluate students' suggestions.

During

TA 4. Observe and question students about their work.

TA 5. Give hints for solving the problem as needed - see "Planning a Solution". If necessary, repeat questions from "Understanding the Problem".

TA 6. For students who obtain a solution, have them check their work and answer the problem.

TA 7. Give a problem extension to students who complete the original problem much sooner than others - see "Problem Extension".

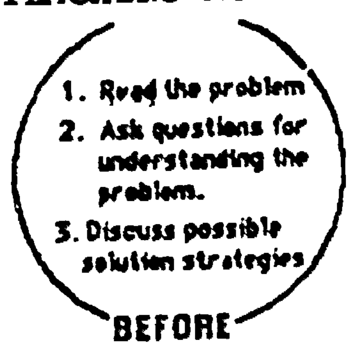
After

TA 8. Show and discuss students' solutions to the original problem. Have students name the strategies used.

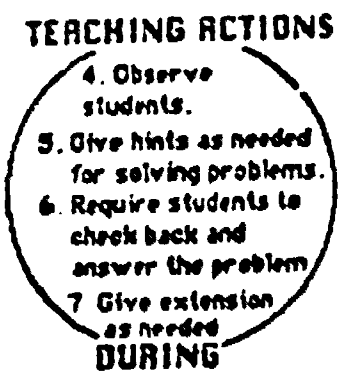
TA 9. Relate the problem to similar problems solved previously - see "Related Problems". Discuss the problem extension.

TA 10. Discuss special features of the problem (if any).

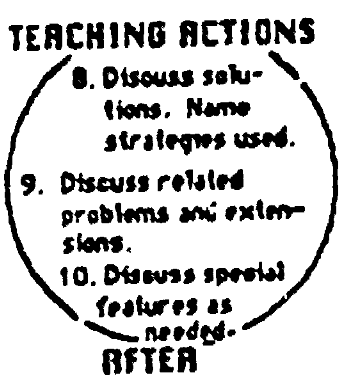
TEACHING ACTIONS



UNDERSTANDING THE PROBLEM



PLANNING A SOLUTION



FINDING THE ANSWER

Related Problems:

Problem Extension:

PROBLEM SOLVING GUIDE

QUESTION - DATA

Read the problem

Decide what you are trying to find

Find the important data

PLAN

Look for a pattern	Draw a picture
Guess and check	Make an organized list
Write an equation	Make a table
Use logical reasoning	Use objects or act out
Work backwards	Simplify the problem

ANSWER - CHECK

Be sure you used all the important information

Check your work

Decide whether the answer makes sense

Write the answer in a complete sentence

There were 9 people in a tennis tournament. Each person played one match with each other person. How many matches were played altogether?

TEACHING ACTIONS

1. Read the problem
2. Ask questions for understanding the problem.
3. Discuss possible solution strategies

BEFORE

UNDERSTANDING THE PROBLEM

* How many people were in the tournament? (8)
* Who did each person play? (Each of the other people).
* If A plays B, will they play again (no).
* If A plays B, is that 1 match of 2? (1).

TEACHING ACTIONS

4. Observe students.
5. Give hints as needed for solving problems.
6. Require students to check back and answer the problem
7. Give extension as needed

DURING

PLANNING A SOLUTION

* If player A starts, how many matches will A play? (7). If B goes next, how many matches will B play? (6) B has already played A).
* Write down the names of 8 people and show who each would play (see solution).
* Suppose there were only 2 people in the tournament. How many matches would be played? (1) 3 people? (2) 4 people? (6)...

TEACHING ACTIONS

8. Discuss solutions. Name strategies used.
9. Discuss related problems and extensions.
10. Discuss special features as needed.

AFTER

FINDING THE ANSWER

Make an organized list

<u>A B C D E F G H</u>
B C D E F G H
C D E F G H
D E F G H
E F G H
F G H
G H
H

7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 28

Simplify the problem. Make a table. Look for a pattern.

people	2	3	4	5	6	7	8
matches	1	3	6	10	15	21	28

28 matches were played

Related Problems:
Problem Extension:

Suppose the 8 people were 5 men and 3 women. If only boys and girls played each other, how many matches were played? (5 X 3 = 15).

TIPS FOR WORKING WITH SMALL GROUPS

1. Limit the group size to 3 or 4.
2. Accept a higher noise level in the classroom.
3. Do not interrupt a group that is working well. If a group appears to be floundering, however, ask a student what the group is discussing or which part of the problem is giving difficulty.
4. Try different grouping patterns.
5. Interact with the groups. Listen to their discussions.
6. Give students rules for group work. Some possible rules are:
   a. Require that students ask for you help only when everyone in the group has the same question.
   b. Require that everyone in a group agree on one answer.
   c. Interaction between groups in not permitted.
   d. All students should participate.
   e. Be considerate of others.
   f. Help any group member who asks.
7. Promote involvement of all students in group work.
   a. Identify a group captain for the day. This person is responsible for explaining the group's work.
   b. Identify a recorder to write all of the group's work.
   c. Question students who appear not to be involved in the group's work. Try to determine whether the students do not understand the problem or whether they are not participating.

GUIDELINES FOR WRITING HINTS AND QUESTIONS FOR THE PROBLEM-SOLVING LESSON PLAN

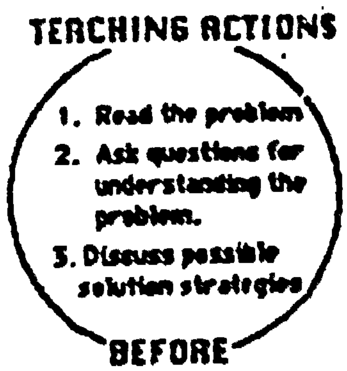
Understanding the Problem

1. Focus on the QUESTION and the DATA
2. Focus on both explicit and implicit information
3. Ask what the problem is asking them to find.
4. Do not tell students how to solve the problem.
5. Ask only questions that can be answered mentally.
6. Write only problem-specific questions.

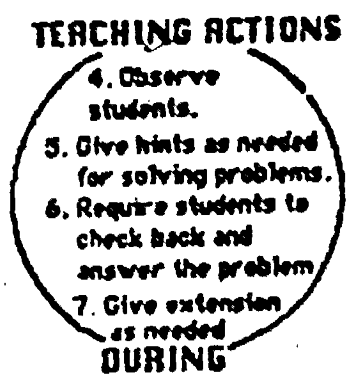
Planning a Solution

7. Focus on the PLAN.
8. Suggest a specific action the student might take to get started toward finding a solution.
9. Suggest specific strategies by name if other hints are not successful.
10. Write only problem-specific hints.

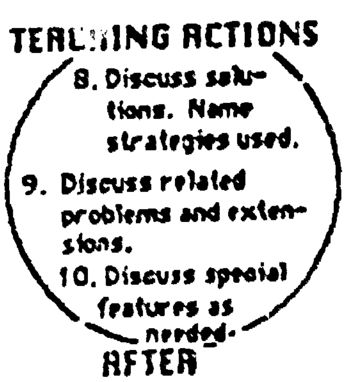
Stanley has $50 to buy tickets for the theatre. Box seats are $7 each and grandstand seats are $5 each. If he buys 8 tickets, some of each kind, how many are box seats and how many are grandstand seats?



UNDERSTANDING THE PROBLEM



PLANNING A SOLUTION



FINDING THE ANSWER

Make a table

$5 tickets

number	1	2	3	4	5	6	7	8
cost	5	10	15	20	25	30	35	40

$7 tickets

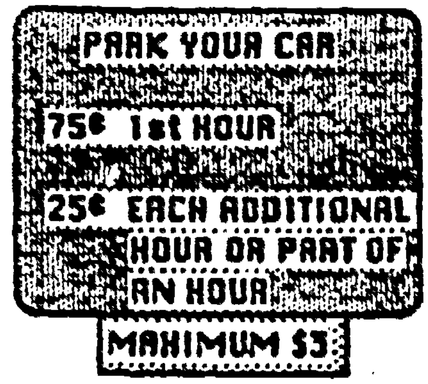
number	1	2	3	4	5	6	7	8
cost	7	14	21	28	35	42	49	56

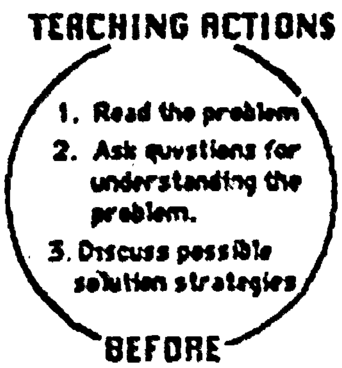
He will buy 3 $5 tickets and 5 $7 tickets.

Related Problems:

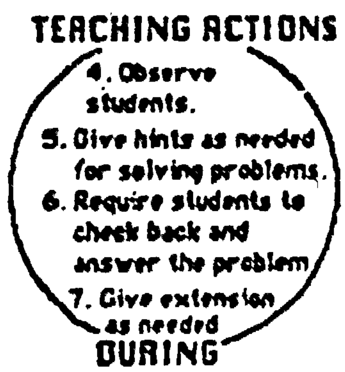
Problem Extension:

The Grimsley's parked in a car lot while they went to the circus. How much did they have to pay if they parked from 10:15 a.m. until 1:45 p.m.?

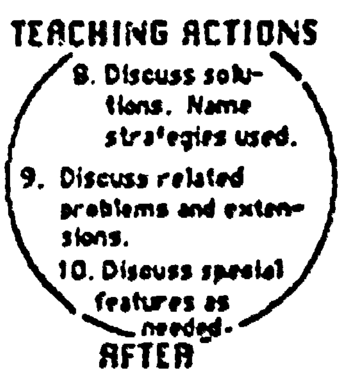




UNDERSTANDING THE PROBLEM



PLANNING A SOLUTION



FINDING THE ANSWER

Choose the operations

10:15-11:15 ⟶ 75¢

11:15-12:15
12:15-1:15 ⟶ 25¢ for each
1:15-1:45

$$\begin{array}{r} 25¢ \\ \times\ 3 \\ \hline 75¢ \end{array} \qquad \begin{array}{r} 75 \\ +75 \\ \hline 150¢ \end{array}$$

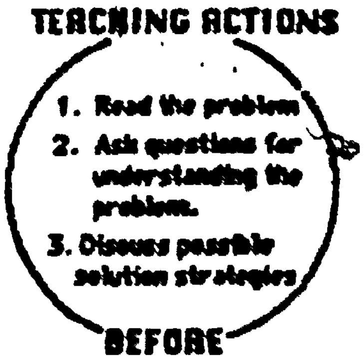
The Grimsley's would have to pay $1.50 for parking.

Related Problems:

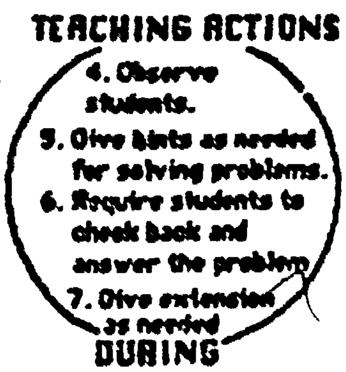
Problem Extension:

BEST COPY AVAILABLE

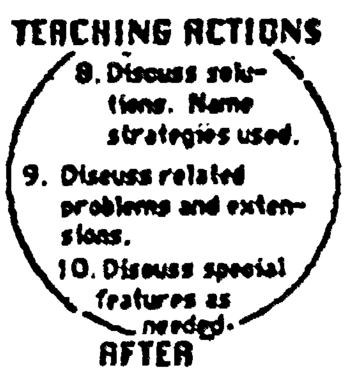
Max Miller has 58¢ consisting of 9 coins. He does not have a half dollar. What are the coins?



UNDERSTANDING THE PROBLEM



PLANNING A SOLUTION



FINDING THE ANSWER

Make an organized list:

Q	D	N	P	*coins
2	0	1	3	6
1	3	0	3	7
1	2	2	3	8
1	1	4	3	9 Yes

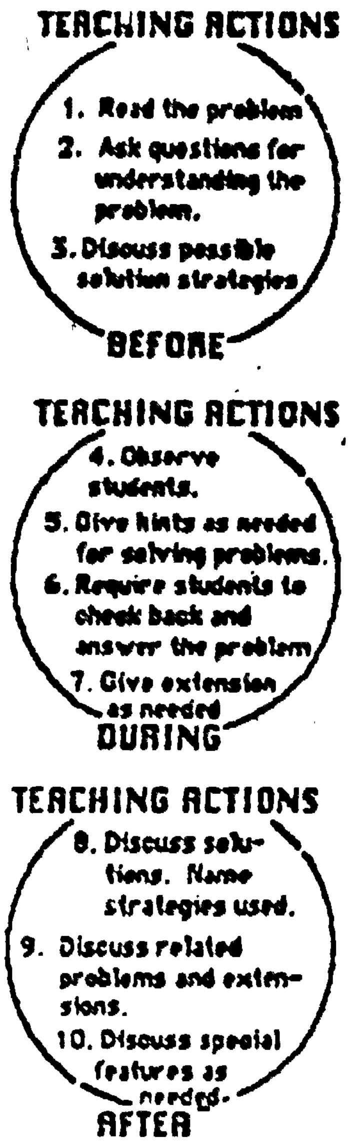
Guess and check: Try 2Q + 1N + 3P = 58¢, 6 coins No.
Try 1Q + 2D + 2N + 3P = 58¢, 8 coins No.
Try 1Q + 1D + 4N + 3P = 58¢, 9 coins Yes.

Max had 1 quarter, 1 dime, 4 nickels, and 3 pennies.

Related Problems:

Problem Extension:

Table tennis balls are sold in packages of 6, but they can be sold separately. A store has an inventory of 1,176 table tennis balls. How many packages could the store have?



UNDERSTANDING THE PROBLEM

*What are you trying to find out about the table tennis balls? (Number of packages the store could have).
*How many balls are in 1 package? (6)
*What does 1,176 tell us? (The number of balls)
*Can a package have less than 6 balls? (No)

PLANNING A SOLUTION

*Can you describe the action in the story?
*Suppose I had 24 balls and made packages of 6. How many packages would I make?
*You want to place the balls into same-size groups where each group has 6. Which operation is suggested by that action? (divisio

FINDING THE ANSWER

<u>Use division</u>

$$1{,}176 \div 6 = 196$$

There could be 196 packages in the store.

Related Problems:

Problem Extension: Suppose there were 1,408 balls. How many packages could be made? (1,408 ÷ 6 = 234r4 so 234 packages).

PROBLEM VARIATIONS

<u>Original Problem</u>: There are 8 people in a tennis club. How many matches would be played if each person played each other person one time?

Principle of Variation	Sample New Problem
1. Change the context or setting	There were 8 people at a party. How many handshakes would be exchanged if each person shook hands with each other person one time?
2. Change the numbers	There were 12 people in a tennis club. How many matches would be played if each person played each other person one time?
3. Reverse the wanted and given	At a certain tennis tournament, each player played each other player one time. At the end of the tournament, 66 matches had been played. How many players were in the tournament?
4. Change the conditions	There were 8 people in a tennis tournament. Five were from one club and 3 were from another club. How many matches were played if each person from one club played each person from the other club one time?
5. Combination of the above	All 20 students in Mr. Wilson's class decided to have a Ping Pong tournament. There are 10 boys and 10 girls in the class and each boy was to play each girl one time. How many games were played?

INSTRUCTIONAL PRACTICE

Problem 1: Grace had to number the 396 pages of her art notebook. How many digits would she have to write?

INSTRUCTIONAL PRACTICE

Problem 2: Some elves were having a convention. The first time there was a knock at the door. 1 elf entered. On each of the following knocks, a group of elves entered that had 2 more elves than the group that entered on the previous knock. On the tenth knock, all of the elves had entered the convention hall. How many elves were at the convention?

INSTRUCTIONAL PRACTICE

Problem 3: Allen and his friends are sitting at a large round table playing a card game. In this game there are 25 cards in the deck. The cards are passed around the table, and each player takes 1 until there are no cards left. Allen takes the first card and also ends up with the last card. He may have more than the first and last cards. How many people could be playing cards?

SOME GUIDELINES FOR GIVING HINTS

You should consider giving a hint:

1. To help students accurately implement a good solution strategy;
2. to avoid excessive frustration;
3. to help students from applying an inappropriate plan;
4. to help with misunderstood and misinterpreted information;
5. to encourage students to evaluate their thinking and work.

GOALS FOR THE TEACHING OF PROBLEM SOLVING*

1. Improve students' willingness to try problems and improve their perseverance when solving problems.
2. Improve students' self concepts with respect to their abilities to solve problems.
3. Make students aware of problem-solving strategies.

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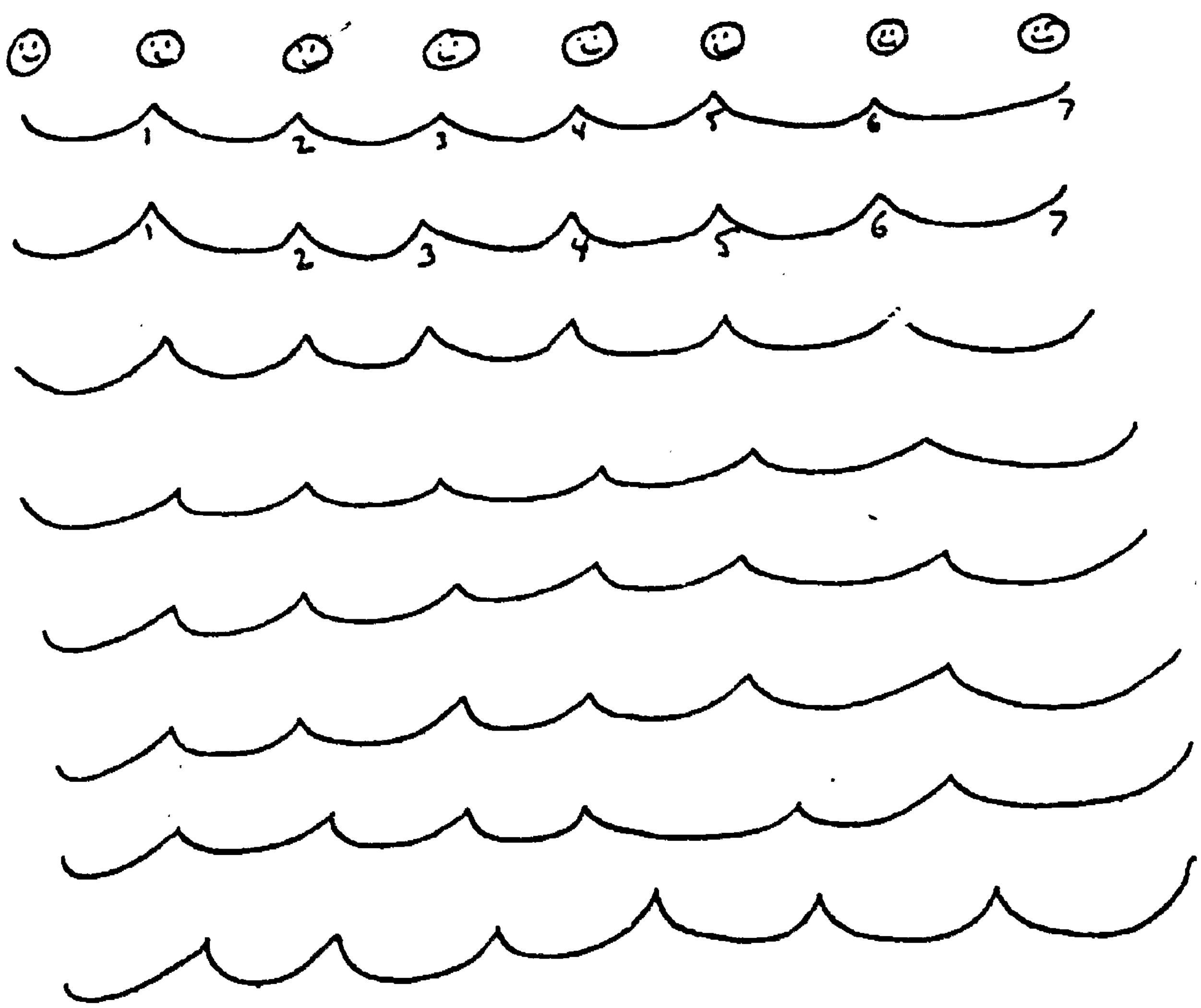
*From Charles, R.I. and others, Problem-solving experiences in mathematics, Grades 1-3 (8 books), Menlo Park, CA: Addison-Wesley Publishing Company, 1983.

4. Make students aware of the value of approaching problems in a systematic manner.
5. Make students aware that many problems can be solved in more than one way.
6. Improve students' abilities to select appropriate solution strategies.
7. Improve students' abilities to implement solution strategies accurately.
8. Improve students' abilities to monitor and evaluate their thinking while solving problems.
9. Improve student's abilities to get more correct answers.

TENNIS TOURNAMENT PROBLEM

Solution 1



TENNIS TOURNAMENT PROBLEM

Solution 2

8		16	
x 2	in a game	8	2nd round
16		+ 4	3rd round
		28	

28 games

A POINT SYSTEM FOR ASSESSING STUDENTS' WORK*

Understanding the Problem:

0 - Completely misinterprets the problem.

1 - Misinterprets part of the problem.

2 - Complete understanding of the problem.

Solving the Problem:

0 - No attempt or a totally inappropriate plan.

1 - Partly correct procedure based on part of the problem interpreted correctly.

2 - A plan that could lead to a correct solution with no arithmetic errors.

Answering the Problem:

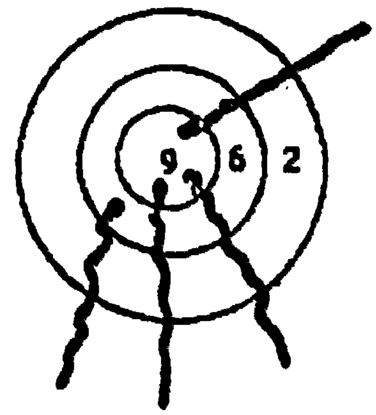
0 - No answer or wrong answer based on an inappropriate plan.

1 - Copying error; computational error; partial answer for problem with multiple answers; answer labeled incorrectly.

2 - Correct solution.

*Taken from R. Charles and F. Lester (1982). Teaching problem solving: What, why and how, Palo Alto, CA: Dale Seymour Publishing Company.

1. Don bet Paula that he could score a total of exactly 34 points on the dartboard. What is the least number of darts needed to score exactly 34?




```
  9
x 3
 27
+ 6
 33
```

2. How much more does it cost to buy 12 cans of soup at 21¢ a can than 7 containers of yogurt at 34¢ a can?

```
  24        34      42¢
 x12       x 7
  48       238
 240
 288
 238
  42
```

PROBLEM SOLVING OBSERVATION CHECKLIST

Student ____________________

Date ______________

	Frequently	Sometimes	Never
1. Select appropriate solution strategies	______	______	______
2. Accurately implements solution strategies	______	______	______
3. Tries different solution strategies when stuck (without aid from the teacher)	______	______	______
4. Approaches problems in a systematic manner	______	______	______
5. Shows a willingness to try problems	______	______	______
6. Demonstrates self confidence	______	______	______

SOME GUIDELINES FOR EVALUATING PROBLEM-SOLVING PERFORMANCE AND ATTITUDES

1. Evaluation is not synonymous with grading. All teachers should have a plan for evaluating problem-solving performance and attitudes.
2. Evaluate thinking processes as well as the correct answer.
3. Use observations of students' work.
4. Match evaluation to instructional content and emphases.
5. Assess attitudes and beliefs as well as performance.
6. Interview students individually if possible.
7. Every student does not have to be evaluated in every problem solving experience.
8. Inform students of your evaluation plan.

COMPONENTS OF AN ASSESSMENT PROGRAM FOR PROBLEM SOLVING

1. Performance
   A. Assess students' abilities to get correct answers for a variety of types of problems.
   B. Assess students' abilities to successfully perform the thinking processes involved in problem solving.
2. Attitudes and Beliefs
3. Related Knowledge and Abilities:
   Exs. computational skills, reading ability.
4. Processes Related to the Improvement of Future Problem-Solving Performance:
   Exs. to formulate problems, to create extensions of a solution to evaluate the elegance of a solution, to generalize a solution approach.

DIAGNOSING AND REMEDIATING PROBLEM-SOLVING PERFORMANCE

Problem Area: Understanding the Problem

1. Have discussions "before" students start work on a problem that focus on understanding.
   a. Ask questions that focus on what it is they are asked to find (i.e., the question), the conditions and variables in the problem, and the data (needed and unneeded).
   b. Have students explain problems in their own words.
   c. Remind students of similar problems.
   d. Have students use colored markers to highlight important phrases and data.
   e. Have students list the data given.

2. Use skill activities that focus on the QUESTION and DATA phases of problem solving.
   Examples:
   a. Given a story, write a question that can be answered using data in the story.
   b. Given a problem with unneeded data, identify the data needed to find a solution.
   c. Given a problem with missing data, make up appropriate data.

Problem Area: Developing a Plan

1. Suggest a solution strategy with the problem statement.
2. Discuss possible solution strategies before students start solving a problem.
   a. Have students suggest reasons why they believe particular strategies might work, being careful not to evaluate their ideas.
   b. For 1-step and multiple-step problems, have students tell what action is taking place that suggests a particular operation.
3. Remind students of similar problems.
4. Use skill activities that focus on the PLAN phase of problem solving.
   Examples:
   a. Given a 1-step or multiple-step problem, tell the operation or operations needed to find a solution (i.e., practice choosing the operation(s)).
   b. Given a 1-step or multiple-step problem without numbers, tell the operation or operations needed to find a solution.
   c. Given a completed solution (e.g., a number sentence, an organized list, a picture), tell a story problem that would be solved with the given solution.
5. Discuss solution strategies used in solving a problem after students have completed work on the problem.
   a. Have students tell why they selected particular strategies (cf. the discussion before students started solving the problem).
   b. Show different solutions (strategies), if possible.
   c. Evaluate the usefulness of different solution strategies after students have completed work on the problem.
   d. Show and discuss incorrect solution attempts (inappropriate strategies) used by students. Discuss which strategies were used and why those strategies were not appropriate.

Problem Area: Implementing a Plan

1. Have students evaluate the implementation of a solution strategy to determine if it was accurately done.
2. Give students the start of a solution (strategy) and have them complete the solution to find the answer.
3. Give a hint with the problem statement telling how to start a solution (strategy).
4. Give direct instruction and practice with particular solution strategies.
5. If possible, show solution strategies which, if properly implemented, would have led to the correct solution to the problem. Show where the error occurred in implementing the strategy.

Problem Area: Answering the Problem and Checking the Answer

1. Have students check to be sure they used all important information in the problem.
2. Have students check any arithmetic they might have used in finding the answer.
3. Have students tell answers to problems in complete sentences.
4. Use skill activities that focus on the ANSWER and CHECK phases of problem solving.
   Examples:
   a. Use estimation to find answers.
   b. Use estimation to check answers.
   c. Given the numerical part of an answer, tell the answer in a complete sentence.
   d. Given a problem and an answer, decide if the answer is reasonable.

SOME INSTRUCTIONAL MODELS FOR A PROBLEM SOLVING PROGRAM

1. Teach an Instructional Unit on Each Problem Solving Strategy

   This approach might be organized as follows:
   Week 1: Teach "guess and check"
   Week 2-3: Practice "guess and check"
   Week 4: Teach "drawing a picture"
   Weeks 5-6: Practice "drawing a picture"
   etc.

2. Practice Solving Problems Randomly Mixed by Solution Strategies

   This approach might be organized as follows:
   Week 1: Problem 1 - guess and check
   Problem 2 - make a table
   Week 2: Problem 3 - make an organized list
   Problem 4 - look for a pattern
   Week 3: Problem 5 - guess and check
   Problem 6 - work backward
   etc.

3. Teach Problem Solving Skills

   This approach might be organized as follows:
   Unit 1: Teach students how to "understand the question"
   Unit 2: Teach students how to "find the needed data"
   Unit 3: Teach students how to "plan what to do"
   etc.

4. Combination of the Above: A Weekly Cumulative Approach

   Weeks 1-3: (a) Teach "guess and check"
   (b) Practice "guess and check"
   (c) Teach problem solving skills
   Weeks 4-6: (a) Teach "drawing a picture"
   (b) Practice "guess and check" and "drawing a picture"
   (c) Teach problem solving skills
   Weeks 7-9: (a) Teach "make an organized list"
   (b) Practice "guess and check", "drawing a picture", and "make an organized list"
   (c) Teach problem solving skills

5. Combination of the Above: A Problem-of-the-Day Approach

   In this approach students would get 1 problem solving experience each day. Each week could be organized as follows:

   Monday: Teach a problem solving skill
   Tuesday: Solve 1 process problem
   Wednesday: Solve 1 multiple-step problem
   Thursday: Solve 1 process problem
   Friday: Solve 1 one-step problem

   The process problems could be organized as follows:

   Week 1: guess and check
   Week 2: draw a picture
   Week 3: make an organized list
   Week 4: make a table
   Week 5: look for a pattern
   Week 6: work backward
   Week 7: use logical reasoning

   The 2 Process problems each week would be related to the same strategy and a strategy hint would accompany the problem statement.

Week 8: guess and check
Week 9: draw a picture
Week 10: make an organized list
Week 11: make a table
Week 12: look for a pattern
Week 13: work backward
Week 14: use logical reasoning

The 2 process problems each week would be related to the same strategy. A strategy hint would not accompany the problem statement.

Week 15: on...

The 2 process problems each week would not be related to the same strategy and strategy hints would not accompany problem statements.

SOME WAYS TO MEET THE NEEDS OF HIGH ACHIEVERS AND LOW ACHIEVERS

High Achievers (Experienced Problem Solvers)

1. Use general hints rather than problem-specific hints.
2. Use challenging problem extensions.
3. Mix problems by probable solution strategies when sequencing problems.
4. Use a greater variety of problems (transfer is more difficult).
5. Have students evaluate the elegance of different solutions.
6. Use more abstract problems (not necessarily real-world settings).
7. Have students write their own problems.
8. Spend less time helping students understand problems.
9. Introduce challenging problems sooner when sequencing problems.
10. Use more individual problem solving.
11. Use flexible grouping (size, members).
12. Introduce more sophisticated problem-solving techniques (e.g., Rubrics).

Low Achievers (Inexperienced Problem-Solvers)

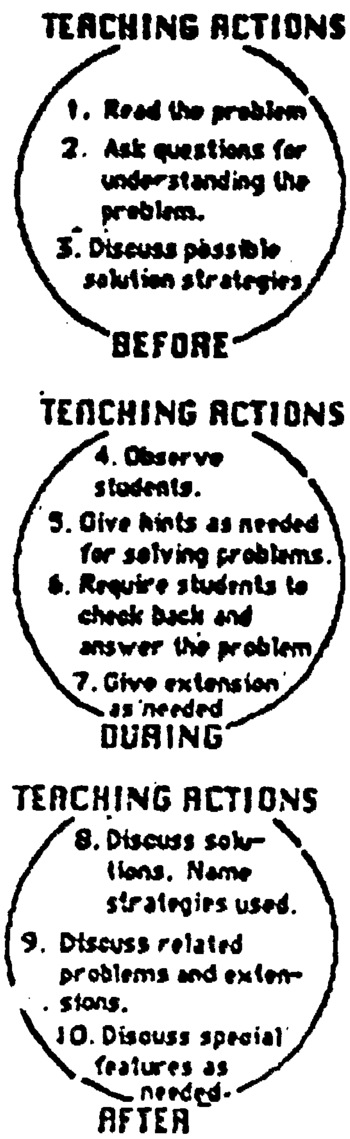
1. Use problem-specific and direct hints.
2. Use relatively easy problem extensions.
3. Teach specific strategies initially then mix problems by probable solution strategies.
4. Motivate problems.
5. Emphasize understanding problems.
6. Emphasize the 3-point checklist.
7. Use more concrete and real-world problems.

8. Teach specific problem-solving skills (with skill activities).
9. Consider using hints in the problem statement initially.
10. Encourage the use of a calculator.
11. Small group work should be the primary grouping pattern.
12. Pay more attention to developing a positive classroom environment.

FACTORS THAT AFFECT THE CLASSROOM CLIMATE

1. Content
2. Time commitment
3. Evaluation practices
4. Teacher's attitude and actions
   A. Demonstrate the importance of problem solving.
   B. Emphasize the processes used to solve problems, not only the correct answer.
   C. Provide the appropriate amount and kind of assistance.

Grace had to number the 396 pages of her art notebook. How many digits would she have to write?



UNDERSTANDING THE PROBLEM

*How many pages are in the notebook? (396)
*How many digits are needed to number page 9? (1)
*How many digits are needed to number page 23? (2)
*How many digits are needed to number page 300? (3)

PLANNING A SOLUTION

*What kind of numbers are needed for the first 9 pages? (single digit numbers)
*How many single digit numbered pages are there? (9)
*What kinds of numbers are needed for the rest of the pages? (2 and 3 digit numbers)
*How many 2-digit numbers are there? (90)
*How many total digits is that? ($2 \times 90 = 180$)
*How many 3-digit numbers are there? (297)
*How many total digits is that? ($3 \times 297 = 891$)

FINDING THE ANSWER

Make an organized list or table

Pg.No.	No. of pages	No. of digits	Total Nc of digit
1-9	9	9	9
10-99	90	180	189
100-199	100	300	489
200-299	100	300	789
300-396	97	291	1,080

Grace would have written 1,080 digits

Related Problems:

Problem Extension: Mickey wrote 690 digits. How many pages were in his notebook? (266)

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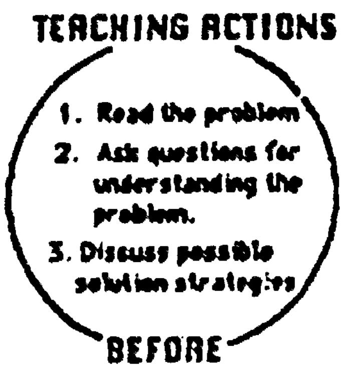
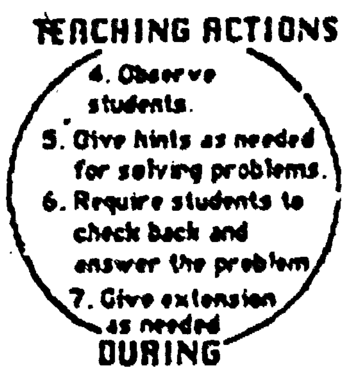
Some elves were having a convention. The first time there was a knock at the door, 1 elf entered. On each of the following knocks, a group of elves entered that had 2 more elves in it than the group that entered on the previous knock. On the tenth knock, all the elves had entered the convention hall. How many elves were at the convention?



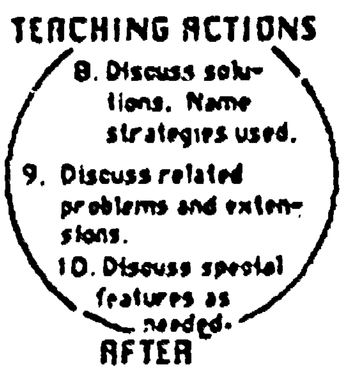
UNDERSTANDING THE PROBLEM

*How many times did a different group of elves knock at the door? (10)
*Did the same number of elves knock each time? (No)
*Suppose 3 elves entered on 1 knock. How many elves would have entered on the next knock? (5 + 2 = 7)



PLANNING A SOLUTION

*How many elves entered on the first knock? (1) The second knock? (2 + 1 = 3)
*After the first and second knocks, how many elves were present? (1 + 3 = 4)
*Could you make a list showing the knocks, the number of elves that entered on each knock, and the total number of elves present? Look for a pattern (see solution)



FINDING THE ANSWER

	Knock	No. Entering	Total Present
Make a Table	1	1	1 = 1 × 1
Look for a	2	3	4 = 2 × 2
Pattern	3	5	9 = 3 × 3
	4	7	16 = 4 × 4
	10	19	100 = 10 × 10

Patterns: The number entering each time increases by 2. The total number of elves present is the square of the number of the knock.

There were 100 elves present at the convention.

Related Problems:

Problem Extension: Suppose 2 elves entered on the first knock instead of 1. How many elves would be present after the 10th knock? (110)

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IMPROVING GENERAL MATHEMATICS

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"If they had their druthers, they wouldn't be there" (Pamela Kaye, General Math Teacher)

Ninth grade general mathematics has long been regarded by teachers and students as an unpleasant and unrewarding experience - it has been tolerated by some, and endured by others. General mathematics classes have been refered to by students and teachers alike as "The Zoo," where the difficulties involved in teaching and learning are numerous, and varied. In order to identify these complex problems, researchers on IRT's General Mathematics Project studied teachers, students, learning and instruction in targeted general mathematics classes.

The students who end up in general mathematics classes are generally in the academic lower half of the total population of ninth graders. For many of these students this class is the capstone of their formal mathematical education. Although the majority of general mathematics students are freshmen, some are sophomores, juniors and seniors who are repeaters from a previous failure. In addition to the range of student ages, there is also a wide range of mathematical achievement. In one of the Project's target classes, researchers found scores on the mathematical achievement test of the Iowa Test of Basic Skills varied from the first percentile (Grade equivalent: 5.1) to the eighty-fifth percentile (Grade equivalent: 10.4). Aside from the ranges in student age and mathematical achievement, project researchers identified characteristics typical of general mathematics students which acted as deterrents to their success in mathematics, these were:

1. a history of poor mathematical achievement and attitude;

2. a repertoire of fragmented mathematical concepts, algorithmic skills, and problem solving strategies;
3. student-teacher interaction problems;
4. the perception that mathematics is irrelevant to either the present or the future;
5. poor school habits (attendance, study, etc.);
6. resistance to instruction, particularly instruction on topics which were somewhat familiar; and
7. the clamor for seatwork (mundane worksheet assignments).

One of the most pervasive deterrents to success in mathematics was poo mathematical attitude; a characteristic of most students in the general mathematics classes. The following segment from a student interview captured this attitude:

Interviewer: Are you going to take math next year?
Sandy: Nope
Interviewer: You've had enough math this year, huh?
Sandy: Yeah, if I'm lucky to get through it!
Interviewer: What makes you think you won't get through it?
Sandy: Because I hate math. I think we ought to do away with the whole math system!
Interviewer: Why?
Sandy: Because it doesn't do any good. I hate math.
Interviewer: Is it because it's hard for you?
Sandy: No. I don't like it. I don't want it. I could do without it the rest of my life!

(Lanier Clinical Interview: 102:11:21-28)

Sandy, like many general mathematics students, did not intend to continue with mathematics beyond the ninth grade. She expressed her dislike for the subject, and went so far as to suggest that it should be abolished. She believed she could do without any kind of mathematics for the rest of her life - even though she applied mathematics daily in her paper route. Another deterrent to success, as problematic for teachers and students as poor math attitude, was the students' fragmented concepts of mathematics. The interview segment below depicts the fragmentation of one general mathematics student as she calculated the perimeter of a square:

Rhuta: (Reading the problem) A square has sides measuring six,...

Interviewer: (Reading the rest of the number)... and three-fifths.

Rhuta: Six and three-fifths. What is its perimeter? (Pause) Thirty-three.

Interviewer: How did you get that?

Rhuta: Well, five times six is thirty and three is thirty-three.

Interviewer: Okay. You changed this from a mixed number into an improper fraction, so it would be thirty-three fifths. What's the distance around the square?

Rhuta: Thirty-three-fifths is six and three-fifths. Six and three-fifths.

(Lanier Clinical Interview: 1C9:2:5-12)

Af'·r reading the problem, "A square has sides measuring 6 3/5 inches. What is it's perimeter?", Rhuta realized the problem contained only one number. She changed the mixed number into an improper fraction, since it was the one calculation that she knew was usually performed on a single mixed number. When the interviewer asked her if her answer was thirty-three-fifths, she changed it to six and three-fifths because she knew that improper fractions were <u>not accepted</u> by her teacher as final answers to word problems. Since Rhuta had completely covered her piece of scrap paper with many calculations in working this problem, and since her final calculation was a properly reduced mixed number, she was quite certain six and three-fifths was indeed the correct answer. Rhuta's fragmented mathematical concepts and Sandy's poor mathematical attitude were typical deterrents to success for general mathematics students.

The research staff of the General Mathematics Project, in collaboration with three ninth grade general mathematics teachers, sought an answer to the question: Can interventions be designed for ninth grade general mathematics students that concomitantly alleviate constraints and ameloriate learning opportunity and teaching conditions?

Classroom field data collected during the Project's Baseline Period portrayed the following about the students, teachers and content of general mathematics classes:

1. Communication between teacher and students concerning the content of mathematics was minimal.
2. Students communicated with the teacher to find out what they needed to know in order to complete daily assignments.
3. Teachers communicated with students to give directions on how to work the assigned problems and to give the students the answers to the assigned work at the end of the class period.
4. Teachers used less math-specific language with their general mathematics students than with their algebra students.
5. The large number of students prohibited successful individualized instruction and the wide range of ability levels of the students reduced the effectiveness of whole group instruction.
6. Although teachers expected students to work individually during the seatwork period, the students formed groups in order to socialize as they worked.
7. The teachers believed general mathematics students needed a lot of practice problems in order to remember basic arithmetic algorithms. Thus the core of the general mathematics class activity was drill and practice and more drill and practice.
8. General math students expected the content of their general mathematics class to be a review of the arithmetic they had been exposed to in previous classes. They did not expect to learn anything new in mathematics.
9. Both teachers and students did not think the content of general mathematics was either interesting or challenging; it was accepted as the work of students and teachers in general mathematics.

The classroom field data indicated teachers and students had definite expectations about their respective roles in the general mathematics classroom. In addition, they shared a common set of beliefs concerning the content and purpose of general mathematics. In the classrooms which were observed, these shared expectations, beliefs and behaviors were accepted and understood by students and teachers as early as the first day of class.

In collaboration with the research staff, the Project's participating teachers began to formulate a set of strategies to improve general mathematics. These were later called Instructional Improvement Strategies.

Strategies for Improvement

The General Mathematics Project staff studied the data obtained from classroom observations of targeted General Mathematics classes and interviews with the collaborating teachers in order to identify critical and problematic areas in the teaching and learning of general mathematics. Three areas emerged from the data which were believed to be central to the problems of general mathematics; these included problems related to communication, social organization and mathematical content. It seemed likely that improvements in these areas would lead to resolution of the problems inherent in general mathematics. Instructional Improvement Strategies, developed from the critical areas, served to guide the study's planning, implementation and evaluation of interventions designed to improve general mathematics. The Instructional Improvement Strategies were:

1. Increasing the quality and quantity of COMMUNICATION about the content;
2. Using the SOCIAL ORGANIZATION to facilitate learning and instruction; and
3. Modifying the MATHEMATICAL CONTENT of the curriculum to include new and revised instructional units.

General Math Project researchers and teachers collaboratively studied readings related to each Instructional Improvement Strategy. They discussed the ideas, concepts, and research results contained in the readings and examined their potential use for instructional interventions. Each teacher self-selected specific interventions he/she would systematically try in their general mathematics classes. The Project's Classroom/Consultant participated in the teachers' implementation of the instructional intervention. His feedback and recommendations were presented in terms of the Instructional Improvement Strategies (communication, social organization and mathematical content).

In the following sections each Instructional Improvement Strategy is examined. Each section contains a sample of the readings and a description of the instructional interventions which were derived from the readings and implemented by Pamela Kaye in her classroom.

Increasing the Quality and Quantity of Communication:

The baseline data showed that communication between teachers and students about the mathematical content of the lessons was minimal, directive, non-mathematical, and nearly void of any interactions dealing with conceptual understanding of the content. The literature selected for reading and discussion on increasing communication in general mathematics included articles by Rudnitsky (1981), Driscoll (1983a.), Rowe (1978), Jencks (1980), and Manning (1984).

The readings provided the basis for staff discussion and collaboration on the development of instructional interventions to be implemented in the general mathematics classrooms. Pamela Kaye, one of the project's teachers, selected the following instructional interventions to implement in order to improve communication in her general mathematics classroom:

(1) Engage the students in more talking about mathematics.
- Ask student to explain their thinking when they answer questions or solve problems.
- Engage students in error analysis discussions of wrong answers.

(2) Increase wait time for students to respond to or to ask questions.
- Use controlled practice during instruction to generate student questions and increase wait time.

(3) Give students more complete explanations and illustrations of mathematical content.
- Elicit student explanations of the mathematical content in concrete, pictorial and abstract terms.
- Discuss several different ways to solve problems.

(4) Give students more feedback by using pre-and posttest results.

(5) Increase students' responses to the mathematical content.
   - Use self-communication as a strategy to talk through problems.
   - Have students explain to one another how to solve problems.

The other teachers in the Project selected similar instructional improvement strategies to systematically try in their classes throughout the duration of the Project.

Using the Social Organization to Facilitate Learning and Instruction

Baseline observations revealed numerous and diverse problems of instructional organization in the classroom. The Project's teachers reported that extended periods of whole-class instruction proved ineffective in promoting learning. Therefore, they resorted to lengthy and mundane seatwork assignments which encouraged student socializing and frequent off-task behaviors. The literature selected for reading and discussion on facilitating learning and instruction by using the social organization, included such topics as: Cooperative learning strategies and effective methods of organizing for instruction. Articles or books by Kounin (1977), Fisher and Berliner (1981), Slavin (1978), Good and Grouws (1979), and Emmer and Evertson (1980) were used for this purpose.

Instructional interventions developed from the readings were intended to facilitate learning and teaching through the organization of students and instruction. Pamela Kaye selected the following strategies to improve the social organization in her classroom:

1. Plan more for lessons -- both short term and long term.
2. Keep a log of lessons that were taught and record the outcomes of each lesson.
3. Manage paperwork better.
4. Put a DAILY PLAN on the board for the students to see.
5. Structure mathematical tasks and activities around student groups.

6. <u>Engage the students in a short review of previously taught content at the start of class.</u>
   - Five or six problems with a high rate of success.
   - The review would take no longer than 5-6 minutes.

7. <u>Use the overhead to focus student attention on the lesson.</u>

Pamela implemented these organizational strategies throughout the Project.

<u>Modifying the Mathematical Content of General Mathematics</u>

The baseline data indicated that the content of the general mathematics curriculum was meager, consisting of many practice problems on arithmetic algorithms. Further, it was barren of either new or challenging content. The readings selected for staff study and discussion focused on presenting new mathematical content, instruction for conceptual understanding in mathematics, and effective methods for math instruction.

The Project's collaborating teachers unanimously agreed that one of the most problematic areas in general mathematics was the teaching and learning of fractional operations. Readings were selected which focused on learning of fractions and methods of teaching fractional concepts (Driscoll, 1983b; Berman and Friederwitzer, 1983).

A second troublesome area in the general mathematics curriculum identified by the Project's teachers was problem solving. Readings by Carpenter (1980), Driscoll (1983c), and Anderson (1980), served as starting points for discussions.

In addition to the selected readings on improving the content of general mathematics, the staff reviewed curriculum materials. These materials included the Dolan and Williamson text, <u>Teaching Problem Solving Strategies</u> and the instructional units developed by the Middle Grades Mathematics Project on <u>Similarity</u> <u>Probability</u>, <u>Factors and Multiples</u>, and <u>Spatial Visualization</u>. Pamela Kaye selected the following instructional improvement interventions to improve the mathematical content of her general mathematics classes:

(1) Incorporate new units into the math curriculum
- Probability
- Similarity
- Factors and Multiples
- Statistics
- Problem Solving

(2) Modify previously taught math units
- Focus on developing conceptual understanding through linkages between concrete manipulatives, pictorial representations, and numerical abstractions.
- Form conceptual links between fractions, decimals, and percent units.

(3) Drop whole number reviews and reduce drill and practice exercises

She implemented these content interventions along with the organizational and communication interventions throughout the Project.

Consequences for General Mathematics Students

Classroom observations conducted during three years indicated that the quality and quantity of instructional activity of the classroom had been improved. Interviews with the Project's teachers provided evidence that they had changed their perceptions and beliefs about instruction and learning in general mathematics. Yet, what evidence was there of student improvement in general mathematics? Researchers thought that improving both the instruction and the environment should promote math achievement and positive attitudes in the general mathematics students. Student achievement and attitudinal data was collected on the students in the target and non-target general mathematics classes in the schools of the Project's three teachers. This data was gathered throughout the Project's third year, the Final Intervention Period. The results of a sample of this student data collected in the general mathematics classes in Pamela Kaye's school will be reported.

There were four general mathematics classes in Pamela's high school. Only one of Pamela's two general mathematics classes was observed and supervised; this is referred to as the target class. Pamela's non-target class received the same instructional interventions which she implemented in the target class, however, this class was not observed or supervised by Project researchers. The remaining two general mathematics classes were taught by another mathematics teacher who, according to Pamela, spent the year teaching computation of whole numbers, fractions, decimals, and percents.

Two mathematical achievement tests were administered to all general math students in Pamela's school. The Stanford Diagnostic Mathematics Test (SDMT) measured achievement in mathematical numeration and concepts, computation, and application. This test was selected by the Project's teachers and researchers to be given to the general mathematics students during the 1983-1984 school year. Pamela's students took the SDMT in September, February, and May. The Shaw-Hiehle Computation Test measured student achievement in computation (and some application) of whole numbers, fractions, decimals, and percents. This test was chosen by Pamela and the other general mathematics teacher in her school and administered to only their general mathematics students in September, January, and June of the 1983-1984 school year.

The Shaw-Hiehle pretest and posttest results on the five subtests and the total test of the four general mathematics classes in Pamela's school are included in the following table:

PRETEST AND POSTTEST RESULTS ON THE
<u>SHAW-HIEHLE COMPUTATION TEST</u>
IN THE FOUR GENERAL MATHEMATICS CLASSES IN
PAMELA KAYE'S SCHOOL

CLASS		WHOLE NUMBERS	FRACTIONS	DECIMALS	PERCENTS	WORD PROBLEMS	TOTAL TEST
PAMELA'S TARGET CLASS	PRE	77.5%	26.8%	51.8%	16.8%	33.6%	46.5%
	POST	88.3%	63.9%	75.2%	63.0%	60.9%	73.3%
PAMELA'S NON-TARGET CLASS	PRE	67.1%	15.2%	45.2%	14.3%	25.2%	37.4%
	POST	81.1%	52.1%	62.1%	42.5%	36.1%	59.1%
NON-TARGET CLASS	PRE	79.3%	22.4%	58.1%	22.4%	45.7%	52.6%
	POST	80.5%	45.2%	67.6%	22.4%	41.4%	56.3%
NON-TARGET CLASS	PRE	79.3%	31.8%	55.5%	18.2%	40.0%	52.5%
	POST	79.1%	51.7%	65.2%	26.5%	41.3%	57.6%
AVERAGE FOR ALL CLASSES	PRE	76.0%	24.3%	52.6%	17.8%	36.0%	47.8%
	POST	82.5%	53.3%	67.5%	38.6%	45.0%	61.6%

Pamela's target and non-target classes had pretest-to-posttest gains on the TOTAL TEST of 58% each; while the pretest-to-posttest gains in the remaining two classes were 7% and 10% respectively. The TOTAL TEST pretest-to-posttest gain for the four classes was 29%. While Pamela's two general mathematics classes showed greater gains than the other two classes on the five subtests, the most striking gains were made by her classes on the PERCENT and APPLICATION subtests. On the PERCENT subtest, Pamela's classes had pretest-to-posttest gains of 275% and 197%; while the pretest-to-posttest gains of the other general math classes were 0% and 45%. On the APPLICATION subtest, Pamela's classes had pretest-to-posttest gains of 81% and 43%. Of the remaining general mathematics classes, one class showed a pretest-to-posttest gain of 3%, while the other class had a pretest-to-posttest loss of 9%.

A second analysis of the test items on the Shaw-Hiehle Computation Test provided the Project staff with a measure of STUDENT EFFORT. Each answer on the Shaw-Hiehle Computation Test had to be calculated by the students, therefore, a student either calculated the answer correctly, incorrectly, or did not attempt to work the problem at all -- leaving both the work space and answer blank empty. In obtaining a STUDENT EFFORT score, the correctness or incorrectness of the test items was not important. What was measured was whether the test item was attempted or not attempted. It was believed by the Project's researchers that one way to measure student improvement in math attitude from pretest-to-posttest would be to find out whether the student thought there was a chance for success on a test item, and would take the time to attempt to work the problem. It seemed likely that general mathematics students who tried to answer more items on the posttest as compared to the pretest, probably believed there was a chance of getting the problems correct. On the other hand, students who answered the same or fewer items on the posttest compared to the pretest, probably believed there was little chance of getting the items correct the second time. The following table shows the percent of items attempted on the pretest and posttest by the students in each of the four general mathematics classes:

THE PERCENT OF ITEMS ATTEMPTED ON THE PRETEST AND THE POSTTEST OF THE SHAW-HIEHLE COMPUTATION TEST 1983-1984

CLASS		WHOLE NUMBERS	FRACTIONS	DECIMALS	PERCENTS	APPLIC-ATIONS	TOTAL TEST
PAMELA'S TARGET CLASS	PRE	98.9%	96.4%	97.1%	46.4%	66.4%	84.0%
	POST	100%	98.7%	100%	94.8%	97.0%	98.4%
PAMELA'S NON-TARGET CLASS	PRE	92.6%	91.4%	92.4%	53.3%	74.8%	82.9%
	POST	97.0%	87.1%	92.9%	81.8%	78.2%	89.0%
NON-TARGET CLASS	PRE	95.5%	85.2%	91.9%	49.0%	76.7%	82.3%
	POST	93.3%	81.0%	90.0%	54.8%	71.9%	80.7%
NON-TARGET CLASS	PRE	98.0%	9[illegible].2%	95.0%	50.9%	65.5%	83.4%
	POST	99.3%	88.7%	94.3%	49.6%	71.3%	83.4%

The results indicated that Pamela's target class attempted more items on the posttest than they did on the pretest. Every item on the WHOLE NUMBER and DECIMAL subtests was tried by every student in the target class. The results showed that for all the subtests and the total test, the students in Pamela's target class answered more items than did the students in the three remaining classes. On the PERCENT subtest, the students in the target class tried 104% more items on the posttest than they tried on the pretest. The students in Pamela's non-target class tried 53% more items on the posttest than on the pretest. On the APPLICATIONS subtest, the students in Pamela's target class tried 46% more problems on the posttest than on the pretest. One assumption made from the data could be that the students in Pamela's target class attempted more problems on the posttest than on the pretest because they believed they had a chance to be successful on these items.

In addition to mathematical achievement data, the Project staff was interested in the outcomes of the interventions on student attitudes towards mathematics and the general mathematics class. Pamela asked the students in her target and non-target classes to respond to the following three questions at the end of the first semester and at the end of the school year:

(1) Tell me what (if anything) NEW you learned this year.

(2) Tell me if there was anything you had before but didn't UNDERSTAND -- and now you do.

(3) Please tell me any CHANGES you think should be made for next year, or any other suggestions you might have.

(Nason: Student Data Report: 6/1/84)

The questions were open-ended which allowed the students to respond in any way they wished. Pamela asked them not to write their names on their responses explaining that she thought they would feel freer to answer the questions more honestly. Some of the students responses (typed as written), were:

"I new most all of the things we did but neve understood it real well. You made things clear and helped me all of the time i learned more this semester than i did all last year
yours turley
guess"

"Almost everything. The math teacher last year spent 2 days on 1 thing that you would have spent 2 weeks on."

"I never really understood math. But all it really took was a good teacher. I know a lot more than befor I came in here."

(Nason. Student Data Report: 6/1/84)

The results indicated that although these students had been in mathematics classes for the past eight years, sixteen students still said there was A LOT that was new they learned this year. Of the new learning, thirty-two students said they learned about fractions and percents. There were thirty-two students who mentioned they now understood fractions and percents. When the students were asked to describe any changes they would make in the class, twenty-eight said that there should be no change or it was a good class and Pamela was a good teacher. Three students wanted more group work, one student wanted more fractions, one wanted more work on similarity, and one wanted more work on probability. Not all students responded to every question, but overall, the responses from the students indicated they were aware they had learned some new mathematics, now understood previously taught mathematics, and enjoyed the classroom, the instruction, and the teacher. Most of the students in Pamela's target and non-target general mathematics classes felt it was a good place to be. Pamela's views of the attitudes of her general math students at Baseline were, "If they had their druthers, they wouldn't be there. It has nothing to do with me, it's just that it's general math class." The attitude of one of Pamela's general math students at the end of the Final Intervention Year, was, "I think this course is just fine. In fact, if you could take it over for credit I would."

The student was pleased with the class and even wanted to take it over next year.

Conclusion

"...becous no class is perfact."

(Pamela Kaye's General Mathematics Student)

The classroom is a dynamic arena containing interrelated networks of complex interactions between students, teachers, the curriculum, learning, and instruction. Weinstein (1983) used Moos' characteristics of human environments to describe the setting of classrooms:

> "Moos (1974) has characterized all human environments as having relationship dimensions (in classrooms, the degree of involvement, affiliation, and teacher support), personal development dimensions (task orientation and competition), and a system maintenance and change dimensions (order and organization, rule clarity, teacher control, and innovation)." (p.299)

Project researchers found the problems of general mathematics to be embedded within the dimensions mentioned above. According to baseline data, these classes lacked the relationship dimensions of involvement, affiliation and teacher support. Little effort was made by either teachers or students to establish the positive personal development dimensions of task orientation and competition. The system maintenance and change dimensions of order and teacher control appeared to be the single environmental characteristic which governed the general mathematics classes.

Improving these classes necessitated the creation and establishment of both the relationship and personal development dimensions. It also required restructuring the system maintenance and change dimensions. Improving the quality and quantity of communication in the classroom created student and teacher involvement, affiliation and mutual support. Modifying the curriculum and mathematical tasks established a task orientation and content focus in the classes. Implementing organizational strategies to improve instruction and learning changed the order and organization within the classes. Through the efforts of the Project's researchers and teachers, the relationship and personal development dimensions of the targeted general mathematics classroom began to emerge. As a result of the establishment of these dimensions, the general mathematics classes improved for teachers and students. At the end of the Project's final year, one of Pamela Kaye's general mathematics students wrote, "Overall on a scale of 1 to 10 this class gets a 9 becous no class is perfact."

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DEVELOPMENTAL LEVELS IN GEOMETRY

Bruce Mitchell
Department of Teacher Education
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This talk marked a first in the career of Bruce Mitchell, Associate Professor of Mathematics Education and part time fashion consultant to the stars The speakers outfit was at least as nice as that worn by the average participants. It is only fair to mention that this was accomplished by the speaker wearing his best clothes and the participants wearing their seediest.

Next came an attempt on the part of the speaker to defend his wearing of tennis shoes as an usher at Frank Lester's wedding.

Feeling compelled to head in the direction of the agreed upon talk the following outline was offered as an attempt at organization:

- background related to the "Project" (Assessing Children's Development in Geometry)
- overview of the Van Hiele levels
- general comments about the levels
- overview of project
- activities developed for interviews
- examples of responses to the activities (from kids that were interviewed)
- miscellaneous comments
- some implications

Two articles were distributed: Spadework Prior to Deduction in Geometry by Bill Burger and Mike Shaughnessy and an Interim Report prepared by Bill Burger, project director. The latter contains all the activities developed by the project and contains a summary of the levels.

Our (Bill Burger, Mike Shaughnessy, Alan Hoffer and myself) interest began as a result of a talk by Izaak Wirszup at the closing general session of the 52nd Annual Meeting of the National Council of Teachers of Mathematics, Atlantic City, April 20, 1974. The talk was titled "Some Breakthrough's in the Psychology of Learning and Teaching Geometry". For me the content of the talk was consistent with my experiences teaching tenth grade geometry. Things happened, interests jelled and Bill wrote a proposal that was funded by the National Science Foundation (1979-1981). Bill, Mike, Alan and I all worked on the project.

The purpose of the project included the examination of relationships between characterizations of students reasoning abilities (on two concepts: triangle and quadrilateral) and the Van Hiele levels. In order to attempt this it was necessary to develop tasks and procedures.

In addition to referring to the interim report for an overview of the levels, the following comments were made:

Level 0 (Visualization)

Kids recognize, perhaps because of some classification type skills, figures when they see them in "standard" orientation. So given a square ("standard" orientation) a youngster will tell you "thats a square" same with "rectangle", "triangle" (isosceles, equilateral, "standard orientation") and circle. The child, however, is viewing these shapes as separate entities and not only doesn't see relationships between some shapes (square, rectangle) but also does not analyze the properties of the shape itself (square has four sides, sides equal, angles at corners equal, diagonals equal, opposite sides never "meet", etc.)

Comments: What often occurs in schools is that young children (K-1) are tested on shape recognition by being asked to mark, or color certain shapes. The triangles are usually equilateral or isosceles and in "standard" orientation. The kids mark the triangles correctly and get checked off on some objective that indicates the child "knows" "triangle". The fact is the child is still quite naive about "triangle" and its important their teachers recognize this. The depth of their level of understanding will be demonstrated when responses to activities are discussed.

Level 1 (Description/Analysis)

At this level kids begin to analyze the properties of figures. A square is seen as having 4 sides, 4 corners that are the "same" (although the word "angle" may not be used. If diagonals are drawn they might be perceived as being the same length. In addition to recognizing that a square has 4 sides the child when asked to draw a four sided figure can produce a square as an example. Comparisons between figures, set subset relations, and applying definitions are still out of reach.

Level 2 (Logical Ordering/Role of Definition)

As the student begins to understand the role of the definition, logical relationships between a set and its subsets become possible. Questions like: Are all squares parallelograms? Are some rhombuses rectangles? Are any kites squares? etc. can be answered correctly. Also, orientation does not influence decisions about shape type.

Comments: Concerning the levels - it seems clear to me that we shouldn't think of a child as a "level two" child, but rather as attaining level two reasoning in relation to some idea or concept. Something that has been helpful to me was pointed out by Alan Hoffer. Within a level, given a certain concept kids are at different stages of development, but when the level is "filled" we expect certain things. When I get confused about levels it helps me to think about what it would mean to "fill" the level. If this doesn't work, I have about 5 beers and forget all about the levels.

It also seems clear to me that one can fill a level in relation to one concept and be in level zero with relation to another - partly because of the complexity of the idea and partly because the words used to make definitions are not understood. At some point in high school (likely not prior to tenth grade geometry) students fill level two with regard to quadrilateral and triangle. Certainly these same students are not at that same stage with regard to continuity or perhaps even similarity.

Understanding the role of the definition is a powerful tool. Far too often teachers assume their students have this ability when they don't! Kids get an accurate definition from their teacher and as a result are suppose to "know" the term defined. It's more complicated than that! There are clear examples of this from our project and my experience - children in kindergarten or first grade will tell you a triangle "has three sides and three points" yet when asked to identify a cut out shape similar to the one in figure one,

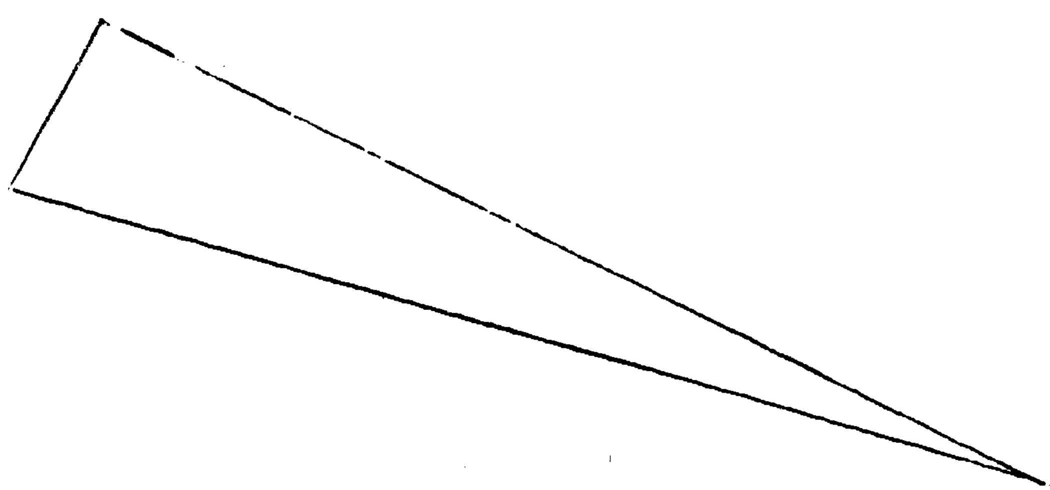


Figure 1

the children say it is not a triangle. Why? "It's much too long for a triangle". Does it have 3 sides? "Yes". Does it have three points? "Yes". It is a triangle? "No"!

Level 3 (Deduction)

Level 3 reasoning involves the ability to reason deductively and understand the role of axioms. In order to understand what's going on in tenth grade geometry one needs to be operating at this level.

Comment: It turns out that many students taking the high school geometry course are not reasoning at level three. This means they do not understand whats going on. (Consistent with conclusions in the Burger and Shaughnessy draft). Proof means putting all the given first the "to prove" last (if they can figure out what is to be proved) and try to head in the correct general direction in between. Combine this with some problems requiring only algebra or arithmetic skills, politeness, being socially acceptable, mentioning that you like some article of clothing the teacher is wearing, throw in a "have a good weekend" and we have a C+ to B geometry student!

Level 4 (Rigor)

At this level different geometries are studied (non-Euclidean) and the properties of postulate systems are examined.

Comment: The project did not include any work with this level.

One of the ideas I had back in the 60's, while searching for ways to do a better job teaching the geometry, was to do a two week unit on logic. The unit included standard stuff; truth tables, the law of detachment, indirect proof, etc. I was excited. Of course it never helped! No wonder the students didn't understand it. The problem was that the unit on logic was the same level of difficulty (level three) as the geometry. I was using level three material to help kids (who were operating at levels one and two) understand level three material. What a dope!

Here are some general comments about the levels that seem important and that I have found fascinating.

- Each level has its own language and network of ideas. Although the words may be the same their use and meanings vary. "Triangle" to a level 3 or 4 person means something different than it does to a level 0 person.

- Two people reasoning on different levels do not understand or at least aren't communicating with one another. A teacher who is presenting a well organized perfectly sequenced articulate argument of a proof to a group of geometry students -- many of whom are listening at level 2 is not communicating with those students. Understanding is not taking place. Also, the teacher is frustrated and often upset with the kids for not understanding such a careful, well thought out presentation.

Of course kids can memorize and find patterns but that's not understanding. A student figures out that congruent triangles often are needed in a proof -- especially in the "congruent triangle" chapter, but often the arguments leading to congruence are weak and/or faulty. It is not unusual for kids to put everything that's "given" first, write some steps that may or may not follow, then have as the last step exactly what was to be proved. A memorized pattern of procedure. Try writing ten parts in a "given" only two of which are needed for the proof and see how many write them all somewhere in their argument. Include some items that have no relation to the problem (i.e., in a problem involving only triangles put "the diagonals of a square are equal") in the given; it will appear in proofs. This is understanding? Tell your students that there is no congruence theorem for corresponding sides and the non-included angle because ASS spells a bad word - no problem! Try teaching why it isn't a congruence theorem - confusion. At the end of a geometry course I'll bet there won't be one kid in twenty that could demonstrate why "two sides and the non-included angle" can't be a congruence theorem.

There are many mathematics teachers (high school and college) that equate teaching to writing a theorem followed by a well organized proof and assignments of problems. Questions are answered with well organized explanations. One crucial missing element of consideration is the level discrepency between the speaker and the listener.

- Levels can't be skipped. In order to get to level three level two must be "filled".

It would seem apparent that anyone who has taught high school geometry has experienced this "level difference" frustration. If not, their students likely have.

The activities that were written for the project took a long time to develop. Activities needed to be designed that could separate up to three levels of responses and we would interview K-12. We decided upon two concepts: triangle and quadrilateral. Activities were developed, piloted and revised.

While activities were developed to which one could separate out level O, 1, and 2 answers, it seemed necessary to write separate questions for level 3. For instance, if one looks at Drawing Triangles; I do not know in my mind what response would lead one to imply the person being interviewed was operating at level 3. The maximum detectable level seemed to be 2. Of course, the more we interviewed and thought about the responses the better we got at analyzing answers and thinking of new activities and questions.

This next section is an overview of some of the responses to some of the activities. During the talk, the participants had the activities to refer to (Appendices of Interim Report). For this paper it seems beneficial to include those activities to which reference was made. It doesn't matter anyway because that's what I'm going to do.

Note: Since we were working one-on-one and sometimes with 5-6 year olds, we were able to make adjustments when necessary. If a child was struggling to make a "2", we could have them just put 11 in the triangle and 111 for "3", etc. Perhaps the greatest advantage of the interview was that we could ask "follow ups", "why did yous", and take time to do our best to try to understand how the child was thinking about the solution.

TRIANGLE ACTIVITIES

Activity 1: Drawing Triangles

Purpose: To discover what attributes (shape, size, proportion, orientation, etc.) the student attends to when drawing distinct triangles (student-generated triangles).

Script:

1. *Draw a triangle. Let's label that #1.*
2. *Draw another triangle that is different in some way from #1. Let's call it #2.*
3. *Draw another triangle that is different from #1 and #2 (call it number 3).*
4. *Can you draw another triangle different from #1, #2, and #3? (If so, draw it and call it #4.)*
5. *How many different triangles could you draw?*
6. *How is #2 different from #1?*
7. *How is #3 different from #1 and #2?*
8. *How is #4 different from #1, #2, and #3?*
9. *How many different triangles do you think you could draw? How would they all be different from each other?*

If the student focuses on only 1 attribute, say orientation, ask, "*Can you find some other way to make a different triangle, other than just turning it (or making it larger/smaller, etc.).*"

Note: Have students put their names on <u>all</u> pages they use during the interview, including sketches.

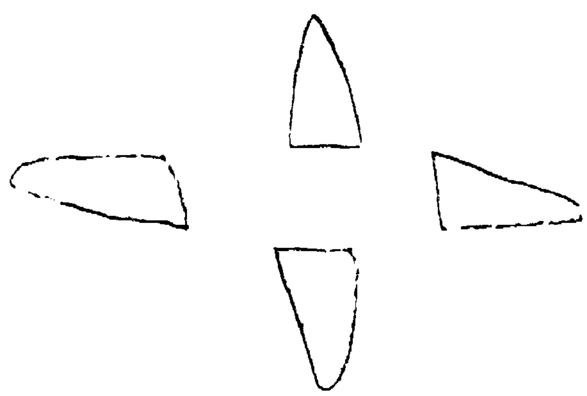
The little kids (5-6 yrs.) have a rough time drawing. They draw one, maybe two. Often the triangles look like:

and their second one like


and when asked how two is different from one they might say: "it's pointier". While some can draw another, some can't, and when asked, their expression "says", "another! you've got to be kidding!" Or, they say "that's very difficult". Then, when asked how many different triangles they think there are in the "whole world", they'll say "oh, six or seven", or "twenty". Like they realize their limitations and just figure there must be a few more floating around out there! "Can you draw any of those?" No that's much too difficult". They are so neat!

The responses that are given are generalizations and I do not mean to imply this occurs with every kid; they are just trends; responses we heard. This is the case for all the comments on student responses.

When kids get a little older (9-10 yrs.), they start giving responses like this:



These age kids might say that all together there are 40 or 50 (they always cover themselves), but they aren't sure how to draw them. Or, they knew at one time, but have forgotten.

Junior high kids give a variety of responses: "a lot", "as many as you want", "hundreds", "Eighty", etc. Many of the kids that indicate there are "a lot of triangles" have no organized way to describe how they could be generated. Some youngsters this age, as in younger children, answer according to orientation. Instead of only four triangles however, they might answer 60 or 360 (one per degree!). I found quite a few similar answers when I asked this question to my tenth grade geometry students on the first day of class. You know you might be in for a lot of work when 20% of your high school geometry students, soon to be exposed to deductive proof think there are only 360 triangles or less in existence and they only differ by orientation!

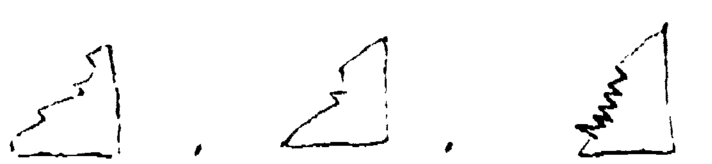
The 360 idea brought two stories to mind:

- One of the Chicago Bear quarterback who told a reporter when asked to what he attributed his improved playing -- since I discovered religion it turned my life around 365°.
- Although you might expect such a statement from a Bear quarterback, a recent all-american from Purdue mentioned that since he had been traded his attitude has taken a 360° turn. Purdue!

These stories prompted a comment from a member of the under dressed audience. Casey Stengel or Yogi Berra once said that baseball is 90% mental and the other half is physical.

Back to activity one. After or late in the geometry course students, for some reason, take "different in any way" to imply we are after classes of triangles. These students draw isosceles, equilateral, right, etc.

One younger child was neat. He said there were millions but unfortunately he was only able to produce four of those! After a little thought though he came up with more and explained how one could get to millions. They looked like this:



, etc.

Other results are summarized in the articles that were distributed.

On to Activity Two for triangles.

Activity 2: Identifying and Defining Triangles

Part A.

Purpose: To determine whether the student can identify certain triangles.

Script:	*Put a T on each triangle on this sheet.*

Part B.

Purpose: To determine the properties that the student focuses on when identifying triangles.

Script:
1. *Why did you put a T on* ____________ ? (Pick out at least 3/4 of those marked.) Be sure to include all "unusual" responses.
2. *Are there any triangles in #12?* If so, *"how many do you see?"*
3. *Are there any triangles in #10?* If so, *"how many do you see?"*
4. Pick out at least 4 (if possible) not marked as triangles. Ask, *Why did you not put a T on* ____________ ? (for each one)

Part C.

Purpose: To elicit properties the student perceives as necessary for a figure to be a triangle.

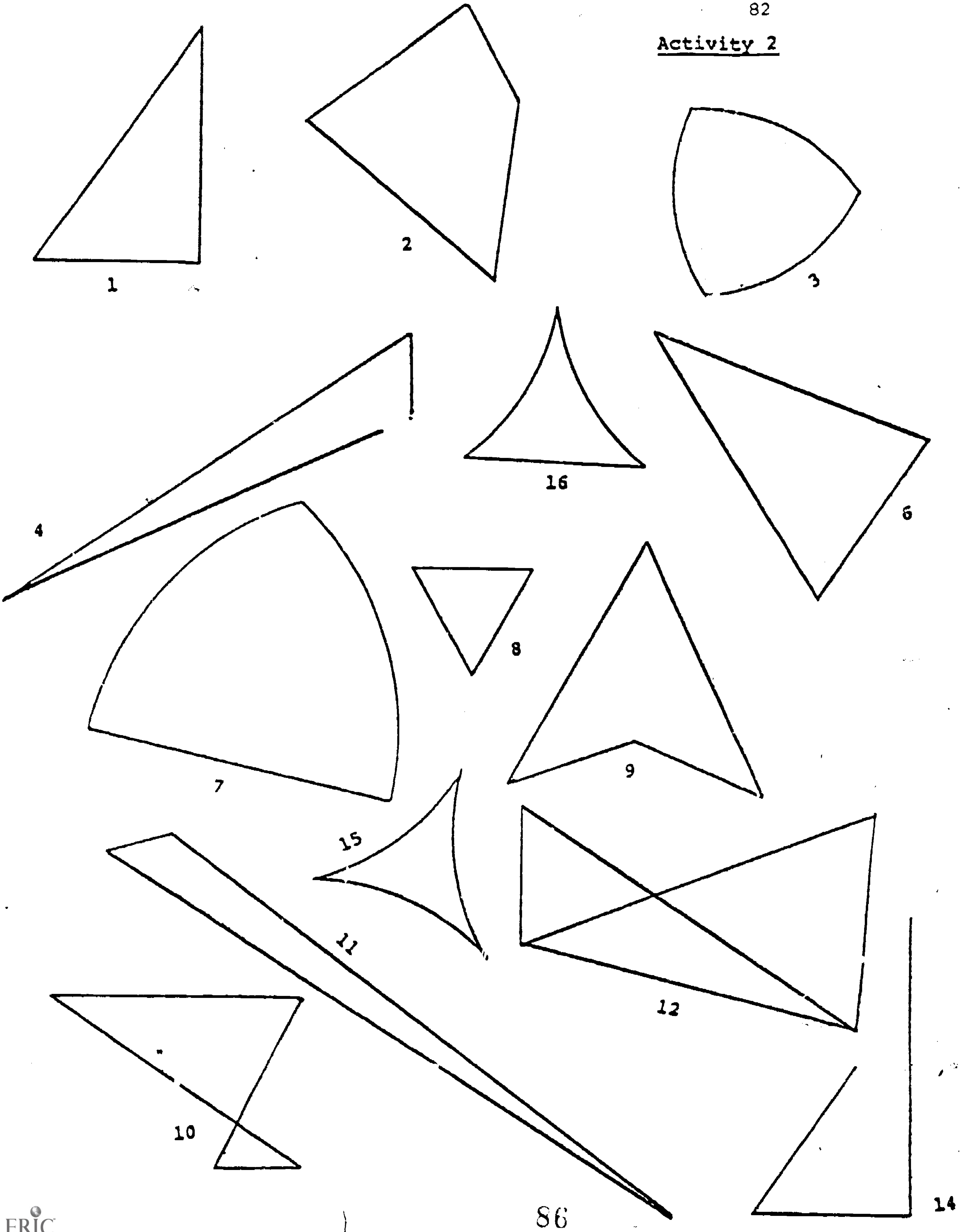
Script:	*What would you tell someone to look for to pick out all the triangles on a sheet of figures?*

Part D.

Purpose: To elicit properties the student perceives as nec· ssary and sufficient for a figure to be a triangle.

Script:	*What is the shortest list of things you could tell someone to look for to pick out all the triangles on a sheet of figures?*

Activity 2



The first day of my high school geometry class this activity was given (1981, 1982). About sixty percent got all correct; forty percent are still not sure what a triangle is!

There are a variety of misconceptions. Young children (K-2) tend not to mark #11 (it's too long for a triangle), put a "t" in #3, #16, #15, etc. (it looks like a triangle), put a "t" in #9 (it's a triangle with a side pushed in). Recently, a child said of #9, "I can't explain the bottom, but I know it's a triangle". Often children are puzzled by #8. It's an "upside down" triangle. Some turn the paper around, put a "t" in it and then turn the paper back.

Major misconceptions kids in grades 4 and above have are with the figures with curved sides.

Some kids were confused by #10 and #12. Some put 2 "t's" in #10 and 3 "t's" in #12. If nothing was placed we asked them if they saw any triangles in those figures, most did. Some saw five triangles in #12. (This was in response to a question from the audience).

Many of the young children (K-2) who did not put a "t" in #11 would comment that they put a "t" in #1 because it had 3 sides and 3 points. "Does #11 have 3 sides?" "Yes". "Does it have 3 points?" "Yes". "It's much too long for a triangle".

Even though these kids had memorized some sort of a definition, they were not ready to apply it. Also in their experience and mind, other conditions for triangles were necessary. Results from recent experiences with youngsters this age verify this.

"Sorting triangles" was briefly discussed but will not be described in this paper. It is likely though that a good sorting job would require one to be aware of the properties of the triangles or at least level one thinking. Young children sorted by their own rules which were, for the most part, unrelated to properties (i.e., angle size, length of sides, etc.); older kids looked for equal sides.

The next activity is "Drawing Quadrilaterals".

QUADRILATERAL ACTIVITIES

Activity 1: Drawing Quadrilaterals

Purpose: To discover what attributes (shape, size, proportion, orientation, etc.) the student attends to when drawing distinct quadrilaterals (student-generated quadrilaterals).

Script:

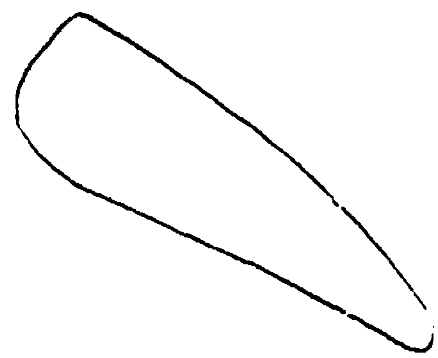
1. *Draw a 4 sided figure. Let's label that #1.*
2. *Draw another 4 sided figure that is different in some way from the first one (call it #2).*
3. *Draw another 4 sided figure that is different in some way from #1 and #2.*
4. *Draw another 4 sided figure that is different in some way from the others.*
5. *Draw another 4 sided figure that is different in some way from the others.*
6. *How many different four sided figures could you draw?*
7. *How is #2 different from #1?*
8. *How is #3 different from the first two?*
9. *How is #4 different from the first three?*
10. *How is #5 different from the first four?*
11. *How many different 4 sided figures could you draw? How would they all be different from each other?*

If the student focuses only on the attribute of orientation or the attribute of size, ask: "*Can you find some other way to make them different other than just turning them (or just making them bigger, smaller)?*"

Note: Have students write their names on <u>all</u> pages they use during the interview, including sketches they make.

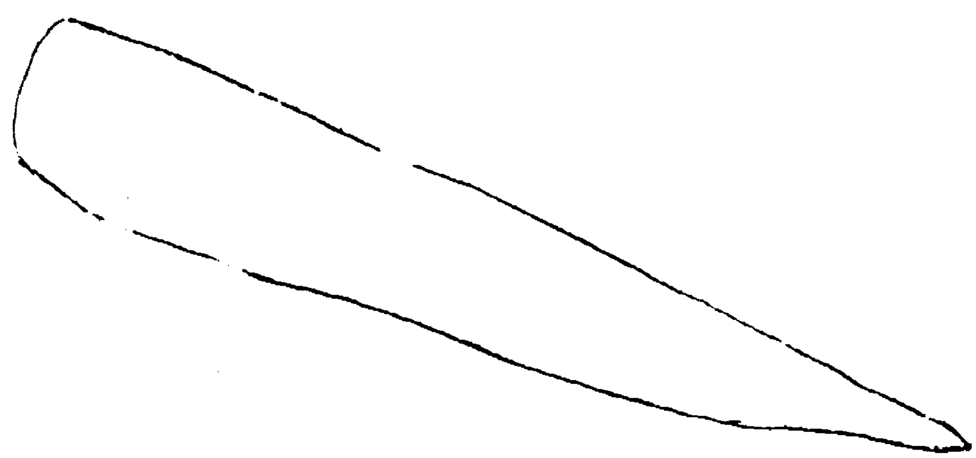
For K-2 kids this activity seemed to be more difficult than the "Drawing Triangles" activity. Drawing one four sided figure was hard enough but to produce more was not easy. Two sometimes three but the time it took was more than that for triangles.

A classic example of a level 0 childs response to this activity occurred in the pilot. A six year old drew the following:

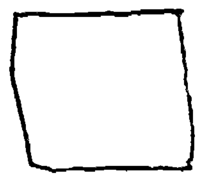


and her comment was "Nuts". "What's wrong?" "It only has 3 sides". "Want to try again?" "Yes".

This time she drew much slower and drew longer lines hoping perhaps that another side would pop in somehow. Despite her concentration she ended up with:



and another "nuts". I asked her to turn her paper over and draw a square. She drew without hesitation



"Can you draw a rectangle?" "Yes".

Here's another paper. "Can you draw a four sided figure?" "Much too difficult!"

She could visualize the square and rectangle so she drew them, but she hadn't analyzed those figures for any of their properties.

Older kids (11-12 yrs.) produced more shapes but viewed the number of different shapes as being finite. High school students tended to draw classes of shapes.

Activity 2: Identifying and Defining Quadrilaterals

Part A.

Purpose: To discover whether the students can identify certain 4-sided shapes.

Script:

Have you ever heard the word "square"? (If so:)
Put an S on each square.
(Stop and let the student proceed.)

Have you ever heard the word "rectangle"? (If so:)
Put an R on each rectangle.
(Stop again.)

Have you ever heard the word "parallelogram"? (If so:)
Put a P on each parallelogram.
(Stop again.)

Have you ever heard the word "rhombus"? (If so:)
Put a 8 on each rhombus.
(Stop again.)

Part B.

Purpose: To determine the properties of figures which the student focuses ç when identifying quadrilaterals.

Script:

Why did you put an S on ____________ *?*
(Pick out two or so.)

Repeat the same question for each shape in Part A familiar to the student.

Ask about any "unusual" responses that may have occurred in the lettering.

Activity 2 (continued)

Part C.

Purpose: To elicit properties that the student perceives as necessary for certain figures.

Script:

What would you tell someone to look for to pick out all the squares on a sheet of figures?

(Repeat for rectangles, if familiar.)

(Repeat for parallelograms, if familiar.)

(Repeat for rhombs, if familiar.)

Part D.

Purpose: To elicit properties that the student perceives as necessary and sufficient for determining certain quadrilaterals.

Script:

(If "square" if familiar:)

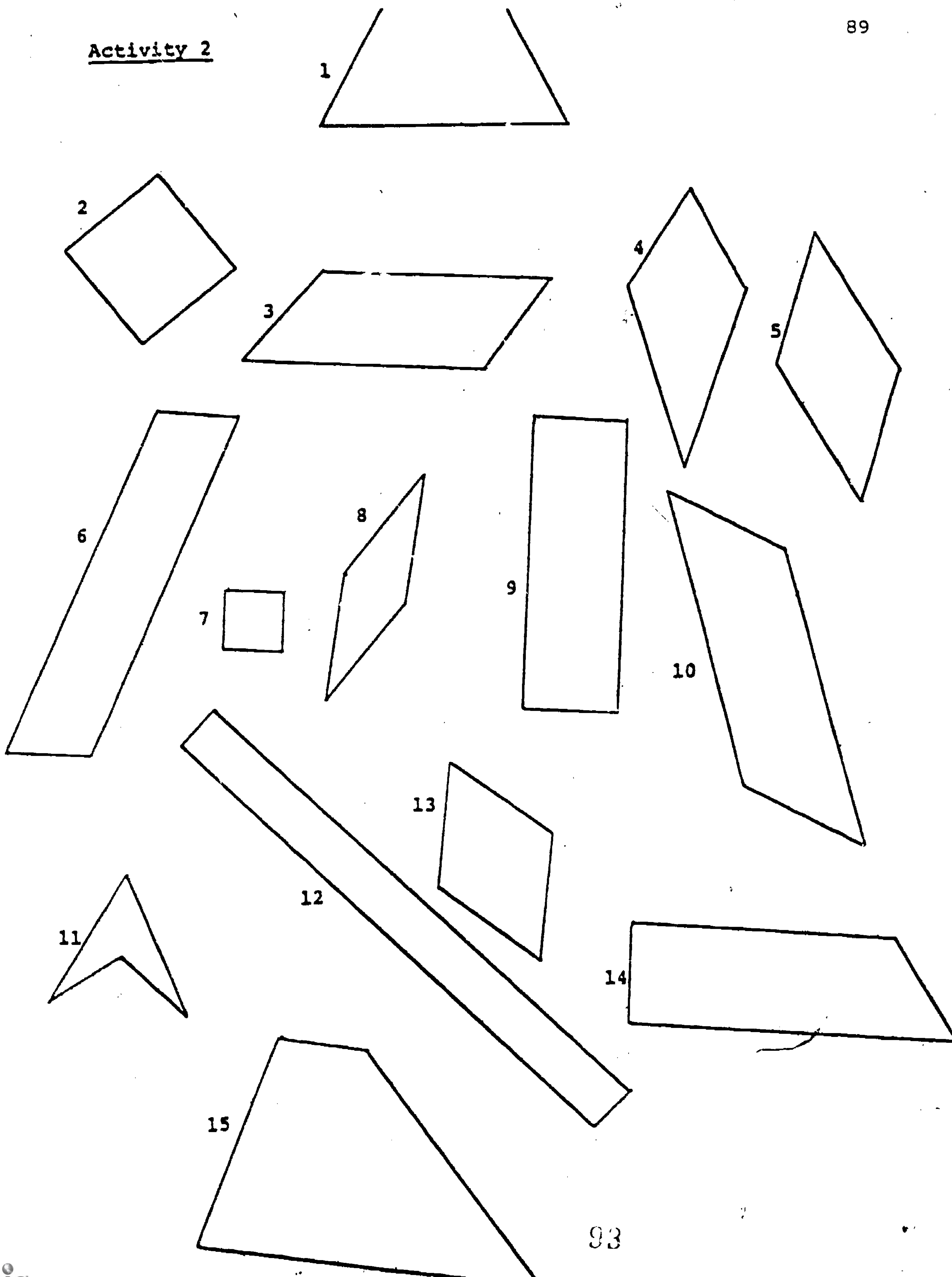
What is the shortest list of things you could tell someone to look for to pick out all the squares on a sheet of paper?

(Repeat for rectangles, if familiar.)

(Repeat for parallelograms, if familiar.)

(Repeat for rhombs, if familiar.)

Activity 2



Activity 2 (continued)

Part E.

Purpose: To examine whether the student applies his/her own properties of quadrilaterals consistently.

Script:

(If "rectangle" is familiar:)
Is #2 a rectangle? Why?

(If "parallelogram" is familiar:)
Is #9 a parallelogram? Why?

(If "rhombus" is familiar:)
Is #7 a rhombus? Why?

If the student changes his/her mind and decides that squares are rectangles, or rectangles are parallelograms, etc., then ask: *"If you went back and marked these shapes all over again, would you do it the same way?"* (If so, have the student reletter underneath the figures on the same sheet.)

(If there are inconsistencies in the relettering -- i.e., R & P under #9 but only R on #12 -- ask why he/she did not put the second/third letter on that figure.)

In activity two (Quadrilatorals) it was interesting t hear the sixth and seventh graders responses to our initial questions about: "Have you heard the word _____". Suppose the word was rhombus. The kids wouldn't admit they hadn't heard it (or didn't know it). They may have figured they should have and would say something like: "yeh, I've heard of it but I know kind of what it is but not exactly", or "I used to know about them but I don't anymore."

The K-2 kids picked out some squares and some rectangles. To many, #2 was not a square. It was a diamond. If you turn it, it's a square but in the position as shown it is not a square. The K-2 kids also marked figures like #6 as being a rectangle. I don't recall any 11-13 yr. olds double marking any figures. They could analyze some of the properties of the figure but did not see (or know) relationships between them.

Even those completing the geometry did not do well on seeing all the inclusions.

Some children viewed figures like #6 as rectangles because they viewed it as a "table top" or a 3D figure drawn in two dimensions.

Activity 3: Sorting Quadrilaterals

Part A.

Purpose: To determine what properties the student focuses on when comparing quadrilaterals.

Script: (Place cutouts on table.)

1. *Put some of these together that are alike in some way.* (Record the grouping.)
2. *How are they alike?*

   (Put them all together again.)
3. *Can you put some together so that they are alike in another way?*

   (Repeat as long as sortings appear useful. Remind students, if necessary, that they can reuse figures.)

Part B.

Purpose: To determine the student's ability to distinguish properties of preselected quadrilaterals.

Script:

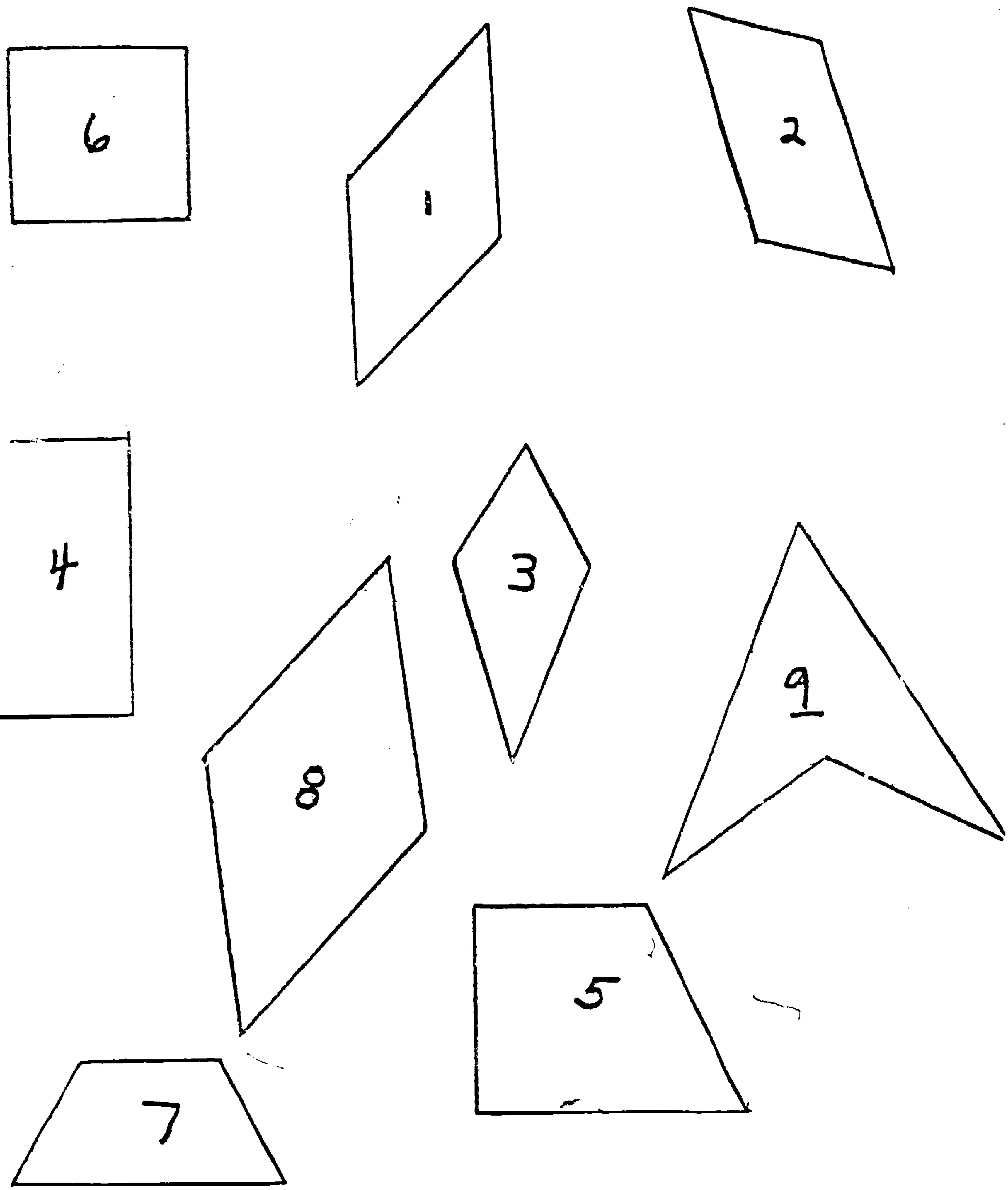
1. (Interviewer selects a set of quadrilaterals that have some common property: number of parallel sides, exactly 2 pairs of equal sides, a right angle, etc.)

   All of these shpaes are alike in some way. How are they alike?

   (The student may find a property that the shapes share, but which does not distinguish them from the others. If this happens, praise can be given and the student can be told, *"There is another way -- can you find it?"* Continue as long as the search seems fruitful.)
2. Repeat part 1 with a different sorting rule.

   (Make sure at least one of the sortings contains more than two shapes.)
3. Include at least one sorting using a group that the student had formed in part A.

Activity 3



Activity 4: What's My Shape?

Purpose: To determine the properties and their interrelationships that the student perceives as sufficient to determine a shape among types, i.e., rectangle, trapezoid, etc.

Script: (Carefully give the directions in Part A.)

Part A.

1. *I'm going to show you a sheet of paper with some clues about a certain shape. I will uncover the clues one at a time.*

2. *Stop me when you have just enough clues to know for sure what type of shape it is. Ask for another clue if you want one.*

3. *Make a drawing of the shape if you want to. Think out loud if you want to, and tell me what you are thinking about.*

Part B.

Begin the "game" by uncovering clue #1 for shape A. Continue uncovering clues as the student requests. Reassure the student, if necessary, that the clues are consistent.

If the student makes drawings, have him/her go from top to bottom of a sheet of paper and label the drawings, shape A, B, or C.

4. If after one clue he/she says "quadrilateral", say, *"Good, let us look at another clue or so, and see what kind."*

5. Once he/she has decided for sure, ask *"Why?"* Then ask *"Is there any other shape you could draw that fits these clues?"*

   If the student says no, ask *"If I show you another clue, could it change your mind?"*

6. Uncover another clue and ask, *"Does this one change your mind?"*

   If the student decides he/she made an incorrect decision, replay the game until you are through and repeat all the questions in Part B.

Shape A

1. It is a closed figure with 4 straight sides.
2. All the sides are the same length.
3. One of the angles is 60°.
4. One of the angles is 120°
5. Another angle is 60°.
6. Another angle is 120°.
7. Two sides are parallel.
8. Two other sides parallel.
9. The diagonals are perpendicular.
10. The diagonals bisect each other.

Shape B

1. It is a closed figure with 4 straight sides.
2. It has 2 long sides and 2 short sides.
3. It has a right angle.
4. The 2 long sides are parallel.
5. It has 2 right angles.
6. The 2 long sides are not the same length.
7. The 2 short sides are not the same length.
8. The 2 short sides are not parallel.
9. The 2 long sides make right angles with one of the short sides.
10. It has only 2 right angles.

Shape C

1. It is a closed figure with 4 straight sides.
2. It has 2 long sides and 2 short sides.
3. The 2 long sides are the same length.
4. The 2 short sides are the same length.
5. One of the angles is larger than one of the other angles.
6. Two of the angles are the same size.
7. The other 2 angles are the same size.
8. The 2 long sides are parallel.
9. The 2 short sides are parallel.

The "Sorting Quadrilaterals" activity was described. Results were consistent with Activity 2. Young children sorted using descriptors like "pointer", "tatter", etc., next came sortings that took properties into account, but even those in high school geometry didn't use relationships between figures to sort (i.e., these are all parallelograms, these are not). The younger the child the more difficult each sorting beyond the first.

Activity 4, What's My Shape, was an audience participation activity. Each of the clues for each shape was given and the jean clad, scuzzy shirted, slowly getting out of hand teachers worked through each shape.

Most pre high school kids at some stage in the clue giving gave square or rectangle as the answer. The younger ones (5-6 yrs.), knowing the figure was a square after the first clue. This is consistent with their developmental level (heirarchical classification). A square is closed with 4 straight sides therefore a closed figure with 4 straight sides must be a square.

Some 8-10 year olds would change their guess to rectangle when long and short were mentioned but other clues no matter what they were did not change their guess.

One ten year old after receiving a clue about a 60° angle looked puzzled. I asked if he knew what a 60° angle was and he said that he used to know but wasn't sure anymore. He thought he knew about right angles. I drew the figure below and asked him to point to what he thought would be a right angle.

He pointed to $\angle B$. Hum, I wonder. "And what kind of an angle would you call this (pointing to $\angle A$)". "That's a left angle" he confidently replied. Isn't that great!

Many high school kids I've talked with since the interviews tend to guess too early and don't seem to have only square, rectangle, parallelogram in mind.

The tape recorder did not pick up many of the audiences questions (inaudible). At this point some trouble causer asked why we do call it a right angle. How the hell would I know?! We woke up Bill and he mumbled something about Euclid which showed he didn't know either.

Activity 5. Equivalent Definitions of "Parallelogram"

Purpose: To determine if the student can establish the logical equivalence of two definitions of parallelogram.

Script:

A. *Suppose we have a quadrilateral in which the opposite sides are parallel. Can we be sure that the opposite sides are congruent?*
(If not, *draw one.* If so: *How can we show this?*)

B. *Suppose we have a quadrilateral with opposite sides congruent. Do they have to be parallel?*
(If not, *show an example.* If so: *How can we be sure?*)

C. 1. *Have you ever heard the word "theorem"?*
Have you ever heard the word "axiom"?
Have you ever heard the word "postulate"?

2. *Can you give an example for each of these terms that you have heard of?*

3. *What is the difference between a theorem and a postulate?* (or *an axiom and a theorem* if familiar, etc. Contrast all pairs of familiar terms.)

Activity five was given only to students who were taking or had taken high school geometry. Successful completion of the activity requires level three ability.

In order to answer this question high school students thought for awhile and might then say "yes". Their arguments were more intuitive than deductive. This side would have to be here, this one here and the only way for these to fit and be parallel would be for them to be equal. One girl started a proof but as Burger and Shaughnessy mentioned in their article she said that she just never did understand proofs so she wouldn't be able to do this one!

What does all this mean? Now that we know more about the way kids development in geometry what are the implications for schools, curriculum, and teaching? These of course are the crucial questions. Unfortunately, I don't think at this stage we know the answers. Research and exploration in this country is just beginning. We must explore alternatives and systematically record results.

From my experiences on the project, teaching a high school geometry class each day (1978-1983), and spending at least one period per week teaching in elementary schools over the last 12 years the following, to me, are legitimate implications:

- Student's entering the high school geometry course are not prepared to do the level of reasoning required by the content of that course. While this could improve, today students entering the course have not filled level two and the course requires level three ability. Although this does not apply to every student, it likely is true for 80% of them. This is an important point -- we need to either prepare students before the course or use the course for that preparation. Many students completing the course have not attained the level or reasoning required to begin it!
- Beginning in kindergarten children need more directed experiences with classification. Experiences with how things are alike and how they are different, grouping objects that are alike, guess my rule, sorting, etc. With regard to geometry these experiences should include a variety of shapes.

- Young children (K-2) do not learn much by memorizing definitions. To know a triangle is a closed figure with three straight sides does not mean children understand triangles. Especially when their experience is viewing only equilateral and isosceles triangles.
- Activities that explore relationships between figures and analysis of shapes should be included at all levels K-8.
- Tesselations (tiling), Symmetry (via mirrors, kaliedoscopes, etc.), Tangrams, Proportional reasoning, and topological ideas should all be explored K-8. Jacobs, Mathematics a Human Endeavor is a good source for this.

There are three more earth shaking implications but time ran out so they will have to wait.

At this point the audience rose giving the speaker a well deserved standing ovation. Shouts of "Bravo, Bravo", were plentiful, someone shouted "author" but that was some kitchen employee that didn't understand the situation. As a final touch the speaker was carried off on the shoulders of the participants to the dining area.

THE LES INSTRUCTIONAL MODEL: LAUNCH - EXPLORE - SUMMARIZE

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The LES Instructional Model used by the Middle Grade Mathematics Project reflects definite assumptions about content, learning and teaching. To effectively teach activities in accordance with this model requires an understanding of (1) the mathematical content and tasks for which the model is appropriate, (2) the process by which students are expected to learn the mathematics, (3) how an activity is to be structured and the content organized, and (4) the techniques recommended for presenting the mathematics and interacting with students as a class and individually.

Understanding the motivations behind the script of an MGMP activity enables a teacher to be more effective in following directions and making decisions to produce the desired learning environment. Success in implementing MGMP activities also contributes to the next step -- using the LES Model to design and teach activities.

The LES anacronism refers to the instructional phases of an activity -- the Launch, Exploration and Summary. Organizing a lesson into three phases is not the distinctive feature of this model since most mathematics is taught with three recognizable components. Class periods typically begin or end with the checking of answers. For each new task or page in the textbook, teachers introduce or review the material and assign problems to be done as seatwork or homework.

The important distinctions between common practice and teaching as characterized in the LES Model are found in the nature of the content goals, tasks and instructional techniques.

Differences are also apparent in the assumptions made about learners and learning. Descriptions of the important characteristics of the LES Model are given in the following sections on goals and assumptions, organization and structure, and teaching techniques.

Goals and Assumptions

The instructional goals of the LES Model are concerned with the mathematics to be learned and the type of learning environment to be created. Appropriate content goals seek some form of pattern recognition. Being a broad goal, pattern recognition refers to such varying behaviors as identifying the necessary attributes of a concept, a rule for computing some value, a relationship between numbers, measures, shapes or variables, or even the best strategy for playing some "game".

Pattern recognition occurs as the result of performing a task and carefully examining the results or reflecting on the process. An appropriate choice of task, therefore, is vital to an activity where pattern recognition is the goal. Unfortunately, teachers often select a task for a lesson without realizing its potential for pattern recognition or without knowing the task is inadequate for the mathematical goal.

Equally important is the way in which data and experience are examined. The sequence in which data is obtained or recorded during a task may not facilitate the detection process. Consequently, data often needs to be re-organized in some manner such as a table of numbers or an arrangement of a concrete models. While recognition of patterns is an obvious and necessary step in satisfying a content goal for patterns, it is not enough. Students also need to be able to apply the emerging concepts, rules, relationships and strategies.

Effective problem solvers need to understand concepts and patterns, not simply to know workable procedures. Students gain this understanding by actively constructing their own knowledge. The key words are "active" and "construct". For a learner to be active means to be engaged mentally and often concretely as well. For active, construction of knowledge to occur requires that students have opportunities to formulate and check their conjectures, opportunities which allow the risk of failure.

Since students necessarily impose their own interpretations of the mathematics being learned, it is only natural that the learning environment foster the process. An important assumption of the LES Model is that this style of learning is more effective than one in which students simply wait for someone or the textbook to tell them exactly what to do or how to do it.

The above premise is at variance with the underlying assumption of the most common practice for teaching mathematics. The familiar "show and tell" strategy assumes students need only listen, watch and practice to be become effective problem solvers. When evaluating or selecting an instructional strategy for teaching mathematics by the LES Model, an essential question to ask is what portion of a lesson each student is actively engaged and with what mathematics?

The view that learning occurs through active construction of knowledge, particularly in an environment which fosters these activities, is essential to the LES Instructional Model. Materials constructed in accordance with this model should actively engage students in performing, recognizing and inventing good mathematics.

Structure and Organization

To create a learning environment which satisfies the content goals and assumptions of the LES Instructional Model an activity is structured in three phases known as the Launch, Exploration and Summary. Each phase has distinctive purposes and characteristics. The Launch prepares students to work on the major challenge which is the task for the Exploration phase. Data and experiences from the major challenge are used to search for and apply patterns in the last phase, the Summary.

Both the Launch and Summary are taught in a total class mode with the teacher orchestrating the interactions. Most any style of teaching could be said to fit this structure, but the requirements of the LES Model extend far beyond the skeletal frame of an activity. A more complete characterization follows for each phase.

LAUNCH. During the Launch students are introduced to new concepts and ideas and reviewed on necessary but previously encountered ones. Whenever possible new concepts are expressed through stories which rely on more familiar language and settings. More formal mathematical terms and definitions are generally delayed until after the students have had an opportunity to grasp the general ideas. Along with the introduction or review of concepts students should be asked to demonstrate their understanding by performing a task or solving a problem.

The second and essential component of the Launch is the mini-challenge. The function of the mini-challenge is to model both the task and directions for the major challenge, the primary task of the Exploration phase. Verbal explanations and directions are simply not as effective in communicating this information as having students attempt the same or a simpler version of the major challenge. Inadequate attention to the mini-challenge inevitably results in numerous questions from individuals who are unsure of how to proceed. These are unnecessary, nuissance questions which result in wasted time, time students should be using to complete the major challenge. A verbal launching of the major challenge signals the end of this first phase.

EXPLORATION. Students work to solve the major challenge during this second phase. Interaction between students enriches the learning experience because of the help and challenge students are able to give one another. Grouping students helps facilitate such interchange. Experiences suggests students work well in groups of 3 or 4 unless the activity requires students to work in pairs.

Because of the normal diversity in students, they do not all finish a major challenge at the same time. Students who finish ahead of others are ready for an extra challenge. As the term suggests, an extra challenge seeks recognition and application of ideas that have not yet been formally examined in class but can be abstracted from the major challenge. Students are intrinsicly rewarded for working on and figuring out the extra challenges. For students capable of solving them, the extra challenges are preferred to the boredom of doing more of the same type of problem or doing nothing. The Exploration phase comes to an end when most all students have completed the major challenge.

SUMMARY. For the Summary phase students return to the total class mode to bring clarity to the mathematical ideas imbedded in the major challenge. Results are gathered and displayed to facilitate the search for patterns. Since incorrect information is quickly disputed by students, it is the search for patterns and not the checking of answers that receives the most emphasis. Conjectures and descriptions of patterns need to be verified or disputed with the available data. Once new rules, relationships or strategies have been established, students need to apply them in new situations. Thus, the learning experience is extended with additional problems and questions at the close of the activity.

Unlike the sequence in which the phases are taught, the design of an activity begins with the identification of (1) the mathematics goals -- the desired concepts, relationships, rules and strategies, and (2) the major challenge -- the task from which these mathematical patterns are to emerge. With the major challenge defining the Exploration phase, concern needs to be given to the way in which data is to be displayed and patterns are to be detected and elicited in the summary. The Launch is the last phase to be planned. Decisions on how concepts are to be presented and what constitutes the mini-challenge are equally important as the Launch sets the stage for the second phase. The dependancy of one phase on another continues as a successful Exploration is necessary for a successful Summary.

Teaching Techniques

Teaching techniques appropriate for the Launch, Explore and Summarize phases are techniques which seek to maximize student involvement. Both the time in which students are actively on task and the number of students involved are to be maximized. Many of the more commonly used teaching behaviors such as explaining, demonstrating and telling are less effective because students can more easily remain passive watchers and listeners. Although teaching techniques are described by instructional phases, many of them are general in nature and easily apply to more than one phase.

LAUNCH. When introducing new ideas or reviewing old ones during the launch, it is essential that students be asked to demonstrate their understanding as quickly as possible. Once a concept is communicated, through story form or by name, every student should be asked to apply it. Student efforts are readily apparent when concrete materials are in use, somewhat less apparent when responses are written, and even less so when only mental thought is required. Fortunately, another good indicator of student involvement is the eagerness with which they seek to respond.

Teachers' reactions to the responses students give to their questions are also important. For example, when a student answers a question and the teacher rewards, corrects or probes this same student, the interchange is only between the two individuals. More student involvement is generated when the teacher refrains from giving immediate feedback about correctness and, thereby, shifts more of the responsibility to the students.

Playing "dumb" for a few moments encourages more students to offer opinions and discourages their looking to the teacher for a judgment, whether verbal or nonverbal. When there is no obviou: concensus among the students or when different answers are not immediately forthcoming, ask if anyone has a different answer or if they agree. It is also unwise to identify the student who is to answer a question until after it has been asked. Give all the students a moment to think or try before selecting the respondent.

Since students benefit from verbalizing the mathematical ideas they are learning, questions which elicit verbal descriptions also need to be incorporated.

Both the Launch and the Summary need to be conducted at a relatively fast-pace. Since interchanges with individuals can interrupt this fast pace, personal contacts are best avoided or delayed. A common dilemma for teachers, particularly during the Launch, is the student who "doesn't get it". The natural tendency is to stop and help such a student. Unfortunately, interruptions of this type leave other students with nothing to do which, in turn, leads to off-task behaviors and disruption of the flow. The point is to notice how students are responding and encourage effort and contributions from all members of the class, not to engage in isolated interchanges with individuals. The appropriate time for personal attention to students is during the Exploration phase.

Individuals can often be helped without destroying the pace of the lesson. Seeing what someone else did or hearing how others verbalize a mathematical idea or explain a procedure may be all that is needed. Capitalize on incorrect responses as well as correct ones. These are opportunities for learning, not experiences to be prevented. And, if an individual does need help, a brief delay until the second phase will not make all that much difference.

Throughout the Launch a teacher needs to notice how students are responding. When a number of students in the class are obviously struggling with an idea or task, continue posing more problems or questions until there is a sense of success. Decisions concerning the exact number of problems for students to try must remain with the teacher. No script can anticipate and cover all possible situations. There must be some give and take with regard to the number of examples given and questions to ask as well as the amount of emphasis given to certain information. The key to flexibility, however, is sensitivity to student performance and knowledge of the mathematics to be learned.

EXPLORATION. During the Exploration phase the teacher needs to move about the class. Staying in any one place or with any particular students too long could be unwise as there are a number of roles to be assumed. First is the task of monitoring what students are doing, even if they are all successfully engaged on the major challenge. Unexpected or varied ways in which individuals perform a task is informative. Other tasks for the teacher during this phase require maintaining, helping and extending behaviors.

Maintaining behaviors are the things teachers do to keep students on task. Attention to students progress is the primary goal. Commenting on their progress, challenging particular responses, checking to see how many examples have been found, and the like are techniques which help keep students focused on the task.

Helping behaviors are used with students experiencing difficulties. Since it is important to find out what such students are thinking, simple questions such as, "Can you show me how you got that?" are often enough to determine where help is needed. Teachers need to avoid remaining so long with an individual that other students are neglected. They also need to avoid any propensity to simply tell or show a student how to do something rather than to find the source of difficulty by asking questions. As the nature of misunderstanding becomes clear, the information needed may also.

Teachers are not the only source of help during this phase; students in the same group are often quite helpful to one another. Because learning is a social as well as a private activity, verbalization and interaction about ideas is one of the purposes of grouping students for this second phase. Unfortunately, the danger of helping by telling applies to students as well as teachers.

SUMMARY. Many of the comments about the Launch apply to the Summary phase, as it is also conducted at a fast pace through total class instruction. Tasks for the teacher in this phase include gathering and displaying data, and seeking, verifying and extending patterns. Results of the major challenge should be elicited and displayed quickly and efficiently. Since the purpose is to seek patterns, the data also need to be arranged to best reveal the patterns of interest.

When conjectures are offered make an effort to see they are heard and understood so they can be verified. While some conjectures are best left to verbal description, others are best shared in symbolic form. Rules should be written in a manner which reflects what students are doing and saying. If, for example, they report finding perimeter by summing the bottom and side edges and then doubling them, the rule should be written as $(B + S) \times 2 = P$ and not the more conventional $P = 2(B + S)$.

Flexibility is important as students may offer unanticipated suggestions and may miss some critical conjectures. An honest examination of most all suggested patterns demonstrates a valuing of student ideas. Some patterns can immediately be recognized from the display while others need to be verified. Focusing students' attention on certain patterns is handled through verifying and extending moves. Asking students to verify a pattern from selected examples and obtaining agreement helps to identify it as important, as do questions which ask students to apply the new patterns to new problems. Since individuals offer conjectures, student involvement is increased by asking everyone to verify a pattern or to try a new problem.

Student diversity comes through in other ways besides the obvious differences in time and effort needed to complete the major challenge. One is in the recognition and reporting of patterns. Another is the propensity or lack of it to use patterns which emerge during a summary. Just because students have reported a rule does not mean they will use the rule the next day. Students often rely on a more primitive approach for quite a long time. Skills are acquired at varying rates.

Helping students make the transition from a concrete or more primitive approach to a more formal or rule governed behavior often takes time and additional effort. Although extra reviews are not generally written into the scripts of the MGMP units, teachers may find it necessary to incorporate an occasional review. Since it can never be the case that all students learn at the same rate, acceptance of the variability is essential. The goal is not to ensure that all students are performing at the same level, but that each student has some way to perform a task that makes sense to that student. Reviews provide additional opportunities for students to retain and possibly extend what they have learned, as do follow-up assignments from textbooks.

To teach successfully with the LES Instructional Model means to retain the integrity of the model's mathematical goals, assumptions about learners and learning, structural criteria, and valued teaching techniques. While the type of learning environment produced by this approach is needed in the mathematical experience of every student, not all lessons can or even should attempt to comply with the LES Model. Finding and recognizing appropriate content and tasks is but the first step to incorporating such activities into a teacher's normal routine. Teaching the MGMP Units is another valuable step in learning how to implement the LES Model.

PROBLEM SOLVING IN THE TRANSITION FROM ARITHMETIC TO ALGEBRA

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Children in the middle school years present mathematics teachers and curriculum developers with a special problem. Because they are betwixt and between in so many ways, our instructional materials and strategies often seem inappropriate for them. They have either learned to do computation or they are so bored with reviewing it that they are on the verge of rejecting mathematics entirely as a potential area of personal interest. They no longer need or want to count and sort concrete objects, yet they balk when a teacher starts to talk about multiplying "a" by 7 and getting "b" for an answer: "But what _is_ a?" they ask, somewhat confused by the notion that a can be "anything." Although they are capable of learning to apply the rules of algebra to expressions and equations, the algebraic forms themselves have little meaning.

By the time they finish sixth grade, most students have been taught to do computation with whole numbers, fractions and decimals. But very few 11-year olds have the intellectual sophistication to judge whether answers like .0096 or $\frac{24}{56}$ "make sense" unless these numbers are tied to concrete representations. They have also been taught to solve word problems which require one, or at the most two, arithmetical computations, but they have not developed efficient procedures for keeping track of the numbers and the steps in more complex problems. It is the more complex problems, however, like figuring out how long would it take to get to the sun travelling at 7000 miles per hour, that they are beginning to find interesting.

How do we help them make connections between the arithmetic of childhood and the algebra of adolescence? What sorts of mathematical curriculum and instruction fits the special needs of children in the middle years? The crucial importance of these questions was underscored in the 1982 yearbook of NCTM. More currently, we are reminded that our "agenda for action" needs to take account of the kinds of mathematical knowledge which will be most useful to students who have calculators and computers at their disposal. This article addresses those issues by exploring a kind of problem solving that is particularly appropriate to children in grades five through nine because it bridges the gap between arithmetic and algebra.

If we observe middle grade students, we see that they are getting ready to leave the fantasies and play world of early childhood behind and are anxious about acquiring the skills and knowledge -- if not the appearances -- of adulthood. They want to make real, useful things in woodshop rather than toy boats or houses. They begin to collect and categorize stamps and coins rather than stickers, and they like to take care of live rather than stuffed animals. They read historical novels and watch real adventure stories on television. In mathematics class it would seem appropriate, therefore, to present them with <u>real</u> problems of the sort that adults know how to solve.

If mathematics textbooks for the middle years can be considered as an indication of what is taught, it seems that our efforts to provide children with a sense that mathematics is a tool for solving practical problems are limited to the application of formulas. We teach them that one can find the area of a circle by doing certain calculations with the diameter, and then give them a page of exercises which require them to do nothing more than "plug in" values for the variables and do the indicated arithmetic operations. We do the same thing with simple and compound interest, using time and distance to find speed, and finding how much a discounted item costs "on sale". Considerations of why these formulas make sense are usually relagated to "enrichment" or the supplementary activities section of the textbook.

Very few adults actually calculate sale discounts or complex interest -- that is usually done for us by machines. And if we need to know how many tiles are needed to cover the kitchen floor, we leave the figuring up to the salesperson in the tile store. In any case, what we want children to learn is <u>why</u> certain rules turn up the correct number of tiles or the right sale price or the exact amount of interest. We want them to notice the patterns and relationships from which a formula is derived and to understand the concepts that underlie the application of that formula to a real-world problem. Such understanding does not follow simply from the <u>application</u> of memorized formulas; it depends rather on figuring out what information is required to solve particular kinds of problems, how one piece of information is related to another within the framework of the problem, and whether there are patterns in that relationship. In order for students in the middle grades to think about such relationships among different kinds of information, it is necessary for them to see that information in a meaningful <u>context</u>.

I would like to give one extended illustration of such a context and the mathematics that can be derived from it drawn from my work with a class of fifth graders. It differs from textbook examples of "practical" problems in two ways. First, it involves students in several days work of independently constructing a strategy that relates known quantities to find an "unknown" and then exploring the application of the resulting "formula" to a few sets of variables. Secondly, it focuses on numerical patterns and how they can be represented and used rather than on finding "answers". This illustration is drawn from a new set of classroom materials in mathematics and science developed at Bank Street College of Education which is built around a series of 13 videotaped dramatic episodes that take place on a sailing ship. That curriculum is particularly rich in open-ended contexts for exploring the strategies that adults use to solve problems, but it is not the only source of such contexts. A close look at the example I have drawn from it will suggest other resources from which teachers can extract the sort of mathematical problems that provide a setting for thinking about how numbers are related and how those relationships can be used to find an unknown quantity.

Aboard the "Mimi"

Imagine yourself on a 58-foot ketch sailing full speed ahead toward Georges Bank. There is no land in sight, nothing to see in any direction except blue sky and green water. The shoals at Georges Bank are very dangerous sailing territory, but that's exactly where we want to go -- that's where the humpback whales usually feed at this time of the year, and we're trying to figure out where they come from, where they go when they leave here and why.

Our ship, the "Mimi", has been chartered for a whale research expedition by two young scientists: Ramon, a marine biologist and Anne, an oceanographer. Their crew and research assistants are a deaf college student who has learned to deep sea dive because she can feel the sounds whales make under water, and two high school students who have been apprenticed to the two scientists on the voyage. "Mimi" is a vintage sailing vessle, but it is outfitted with sophisticated electronic devices to be used for collecting and recording information about the whales everyone is hoping to spot. Mimi's skipper, Captain Granville, is a real old salt, and he runs a tight ship. Captain Granville has brought along his 10-year old grandson from the midwest to spend the summer with him on the sea.

On the first day out, Captain Granville is charting the course, taking account of wind speed and direction, and hoping to get through the tricky shoals of Georges Bank before the sun gets too low in the sky. He is also trying to teach the younger members of the crew some of the rudiments of sailing. As everyone begins to get to know one another and becomes familiar with the quarters and responsibilities they will share for the summer, the Captain notices that one of the electronic navigational instruments seems to be faltering. He is troubled by this possible malfunction because of the difficult maneuvering he will have to do to avoid going around on the shoals. It will be important for him to know exactly where the ship is, how fast it is going, and in what direction they are headed so as to avoid a disaster.

He looks at the knot meter (a shipboard "speedometer" that tells how fast the boat is going in knots*) and says in a knowing voice: "I thought we were going faster than that." If the knot meter is misreading, he cannot accurately predict when to change course to avoid the shoals. Anne notices his concern and remarks that her computer has been acting strangely as well. They suspect that a partial electrical failure is causing these problems. If that is the case, they need to know quickly so that they can change course and get back into port safely before dark. Is the knot meter registering the speed of the boat accurately? Understanding the solution to Captain Granville's problem will require some knowledge of sailor's lore as well as some mathematical analysis. It is a real concrete problem of some importance to both the captain and the crew, since if their instruments are misreading, they will be unable to proceed with the summer's sail and their planned research. But this problem also opens up abstract mathematical questions about patterns in the relationships among time, speed, and distance and how they can be measured while traveling on water. It is not necessary to be on the boat to understand either the importance of solving this problem or the meaning of its elements; a sense of what information is needed and how to use it can be conveyed by the drama of the story.

In order to solve his problem, Captain Granville goes up on the deck and asks Rachel, one of the crew members, for help. He gives Rachel his stop watch and tells her to stand at a point about 10 feet forward of the stern of the boat. He walks up to the bow and tells Rachel to start timing when he drops a piece of bread into the water, and stop the watch when the place where she is standing on the boat passes the bread. Captain

*Nautical miles per hour.

Granville is measuring how many seconds it takes for the boat to pass by the floating bread -- an old seaman's technique for determining speed on water that is more reliable than electronic knot-meters! He announces that the boat passed the bread in 4.9 seconds and that Mimi's speed is therefore about 6 knots*. The knot meter was reading only 5 knots -- so they are closer to Georges Bank and its dangerous shoals than he thought!

Back in the Classroom

How did he figure that out? And how did he do it so fast? Now we are ready to do some mathematics at the middle school level that leads to connections between arithmetic and algebra. We have a concrete setting for thinking about time and distance: 48 feet of a 58-foot boat went by a piece of bread that was floating still on the water in 4.9 seconds. The amount of "teaching" that needs to be done to get students to figure out how Captain Granville used this information to determine the speed of the boat in knots will depend on the skill level and experience of the students. Under any circumstances, the story can provide a context for practicing computational procedures using fractions, decimals, and large whole numbers. But most important, the setting provides students with a meaningful reference which they can use independently to judge whether the answers to their computations "make sense".

The movement of the boat past the bread can be simulated in the classroom to give a clearer idea of what is happening. If students don't know where to begin, it will probably be enough to ask: "If the boat travels 58 feet in 4.9 seconds, how far does it travel in one second?" In the course of changing from seconds to minutes to hours and from feet to nautical miles, it is important to keep in mind that the question we want to be able to answer is: "Is the knot meter working?" In order to answer that question, we (and Captain Granville) need to know the speed of the boat in knots. Captain Granville's problem gives students a context for checking the accuracy of their own computation procedures. They can recognize that if they arrive at an answer which is not somewhere near 7 knots, they need to rethink their strategies. Thus they are called upon to think about what makes sense mathematically rather than simply asking the teacher: "Is this right?" Teachers can help by raising questions about the placement of the decimal point in a product or whether multiplication by 60 is a more appropriate procedure than division, but students are likely to go back and check their own work when they have a clear standard for evaluating the outcome of computations involving many steps.

Figuring out how to go from speed given in "feet per second" to speed given in "knots" offers an opportunity to talk about order of operations, as well as to emphasize that there are several different valid procedures one might follow to get to the answer. For example, one student might change feet per second to feet per minute by multiplying by 60 and then change to feet per hour by multiplying the result by 60 again. Another might multiply 60 by 60 to find the number of seconds in an hour, and then change directly from feet per second to feet per hour. Given the use of a calculator, some students will even begin by figuring out what fractional part of a nautical mile is 58 ft. The teacher can ask: "Why do all of these strategies work?" Focusing on the concrete problem makes associativity seem like a much more useful and meaningful concept that exercises in rearranging parentheses [e.g. $(6 \times 2) \times 8 = 6 \times (2 \times 3)$]. In classrooms where students are familiar with computer programming, they can even translate their strategies into programs which will tell the computer how to get the speed when they type in the time. Challenging students to reduce the number of steps in their program pushes them toward the idea of a single "constant" which can always be used to operate on the time to find the speed.

Moving from the question "How could Captain Granville have figured out the speed in knots?" to "How could he have figured it out so quickly?" takes us into even more interesting and complex mathematical ideas. The drama of the story is an instructional tool for moving middle grade students from concrete numerical relationships to the abstractions that are necessary to build a workable formula. In order to explore how Captain Granville knew the speed in knots so quickly, students will first need to do some suppositional reasoning: What if the boat went by the bread in 3 seconds? or 10 seconds? or 8? As they figure out what each of these times implies for the speed of the boat in knots, they will see that no matter how many seconds it takes for the boat to go by the bread, you can apply the same procedure to find the speed in knots. They are thus gaining valuable experience with the idea of a formula involving dependent and independent variables while still remaining tied to a concrete reference. A chart of ordered pairs can be generated by considering different hypothetical times for the bread to go by the boat and finding the related speed. If students use calculators at this stage, they will be more able to focus on the regularity of their procedures and the relationship between the known information and what they are trying to figure out.

When a few pairs of values are generated which relate the time it takes for the boat to pass the bread and the speed of the boat in knots, students can be asked to speculate about the patterns in that relationship. Graphing the points on the Cartesian plane can now be introduced as a mathematical tool for helping us to represent and understand that relationship. The relationship between variables in equations and their graphs is central to the problem solving process in much of higher mathematics. In the middle school years, we often teach children graphing terms and procedures, but give them very little context for understanding them as tools for examining the nature of real mathematical relationships.

As they plot a few points associated with ordered pairs for Mimi's time and speed, students observe that the points look like they might be on a line (Fig. 1). Some students

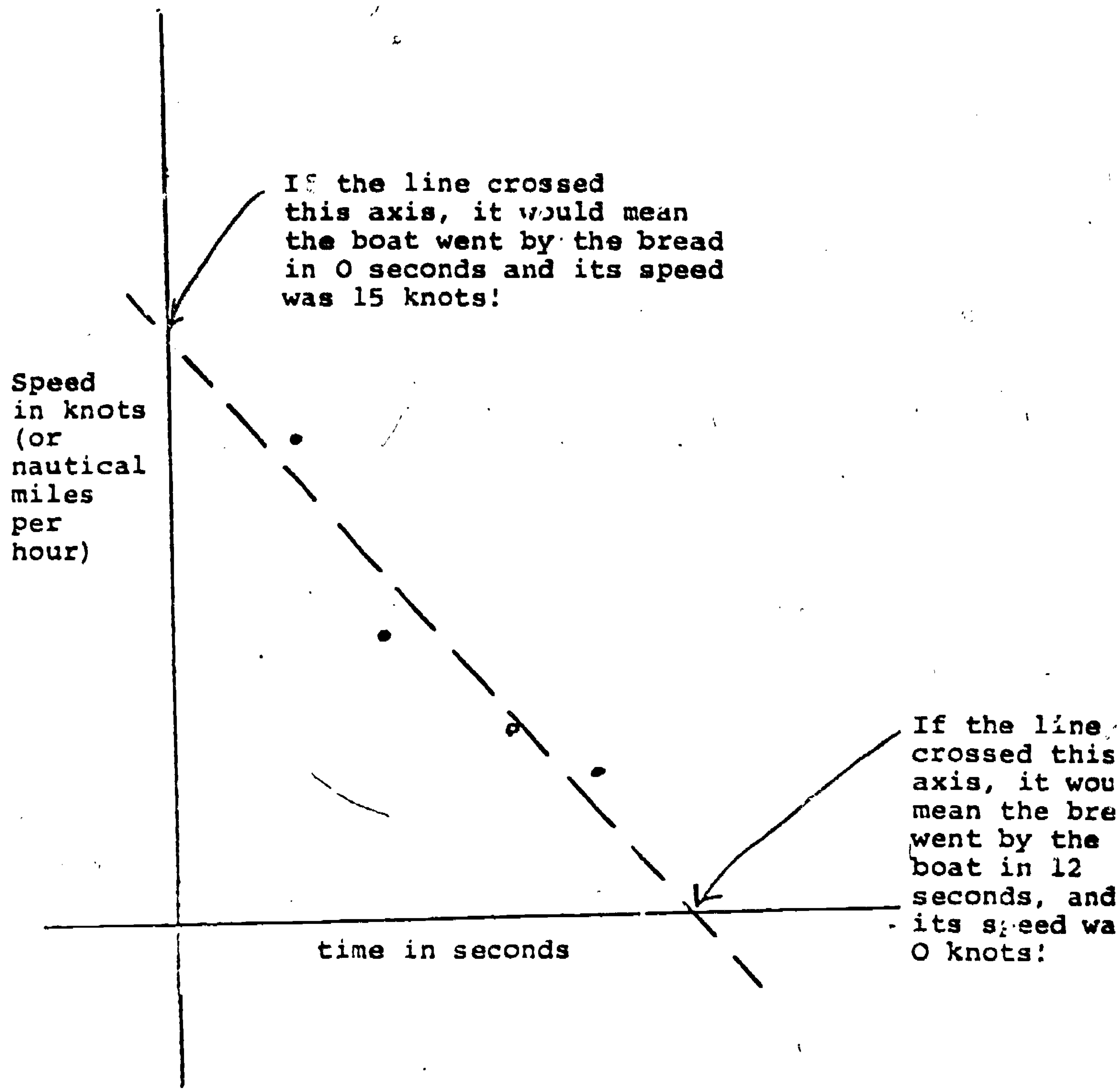


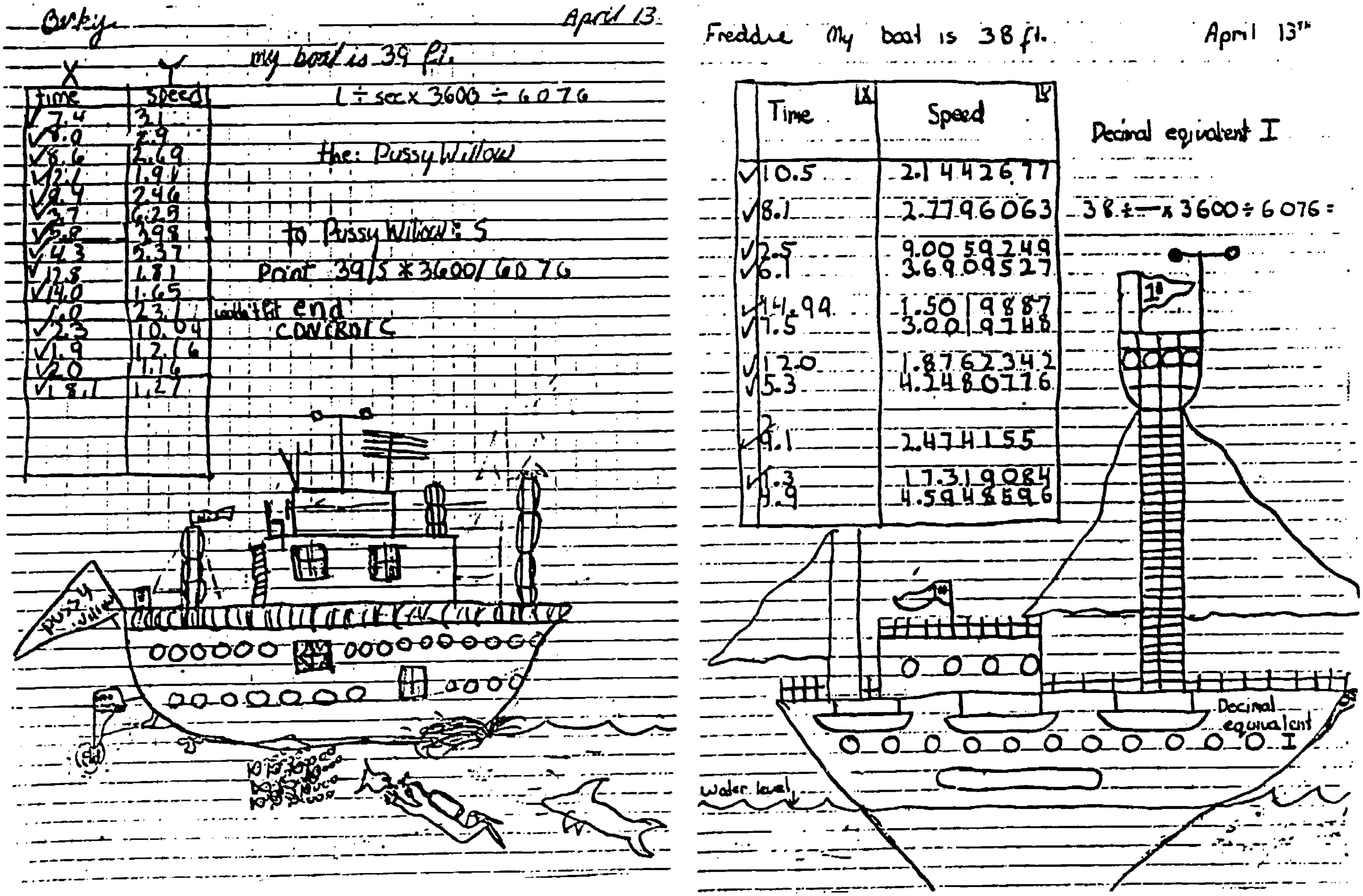
Figure 1

at the middle school level are familiar with functions that can be represented by straight lines on a graph. They may have learned to plot the points indicated by a set of ordered pairs, observe whether they fall on a line and then extend the line to find more points that fit the pattern. But the points they are beginning to see on this graph do not quite make a "straight" line; furthermore, extending the "line" in either direction to the x and y axes results in points that do not "make sense" to children: If the boat goes by the bread in 12 seconds, could its speed be 0? If the speed is 15 knots, would the boat go by the bread in 0 seconds? The real problem for which this graph is a symbolic representation gives a context in which children can think through a mathematical idea for themselves. They thus can move from the specifics of Mimi's time and speed (i.e., having observed that the more time it takes for the boat to go by the bread, the slower the boat is going) to understanding the concept of an inverse proportion.

When students begin to graph more points, they see that the line curves and does not cross either axis. They quickly become intrigued with finding out what the speed would be if the boat went by the bread in 1 second, or even half a second, doing complex computations to find a point on their graph--computations which would otherwise be nothing more than mundane and boring "practice." They are also motivated to do more accurate graphs by the nature of the problem rather than by the teacher's admonitions. The idea of approaching a limit but never reaching it, so central to higher mathematics, thereby emerges from the children's own curiosity. Seeing the representation of a regular relationship between quantities that turns out to be a curve on the graph rather than a straight line gives concrete substance to a new abstract idea. In middle school, one need do no more than expose children to the idea that such a graph is possible.

Direct and inverse proportions are a part of the math curriculum in the middle school years which always seem particularly puzzling to students. These concepts are at the same time profoundly simple and yet also frustrating in their complexity. Being able to imagine the speed as a function of the time it takes for the bread to go by the boat helps children to think about <u>why</u> the graph does not make a straight line or why the curve it does make never touches the x or y axis. They can talk about the very abstract idea of a "limit" in terms of the absurdity of imagining that the bread can get from the front of the boat to the back of it with no time having passed. They have familiar referents for understanding that one gets bigger while the other gets smaller, and that they do so in proportion to one another.

To solidify these ideas and assess children's understanding of them, a possible next step would be to have each member of the class invent his or her <u>own</u> boat of a reasonable length, longer or shorter than the "Mimi". If the length of the boat changes, do the strategies they developed for changing from feet per second to nautical miles per hour still apply? Do the graphs of time and speed look similar? How are they like the time/speed graph for the Mimi? How are they different? How is the graph for a 12-foot long sailfish different from the one for a 150-foot long "tall ship"? In more abstract terms, students are now being asked to think about what aspects of a situation <u>matter</u> in the analysis of a particular kind of problem. As they name their own boats and draw pictures of them (Fig. 2), they gain even more of a sense of "ownership" of the problem and its related mathematical ideas. Their creation of a new variable is connected with a new physical entity as they see their own and their classmates' boats being created on paper and contrasted with the Mimi. (See, for example, the graphs for "The Nasty Mooist" and "The Sea Otter" in Figure 3). The conversations in which they compare what happens to the graphs when <u>their own</u> boats go by the bread in one second or ten seconds or one hour are lively and focused. They push both their graphs and their own abilities to their limits as they use arithmetic, charting, and graphing as mathematical tools for understanding a problem that has meaning to them.



(a) (b)

Figure 2

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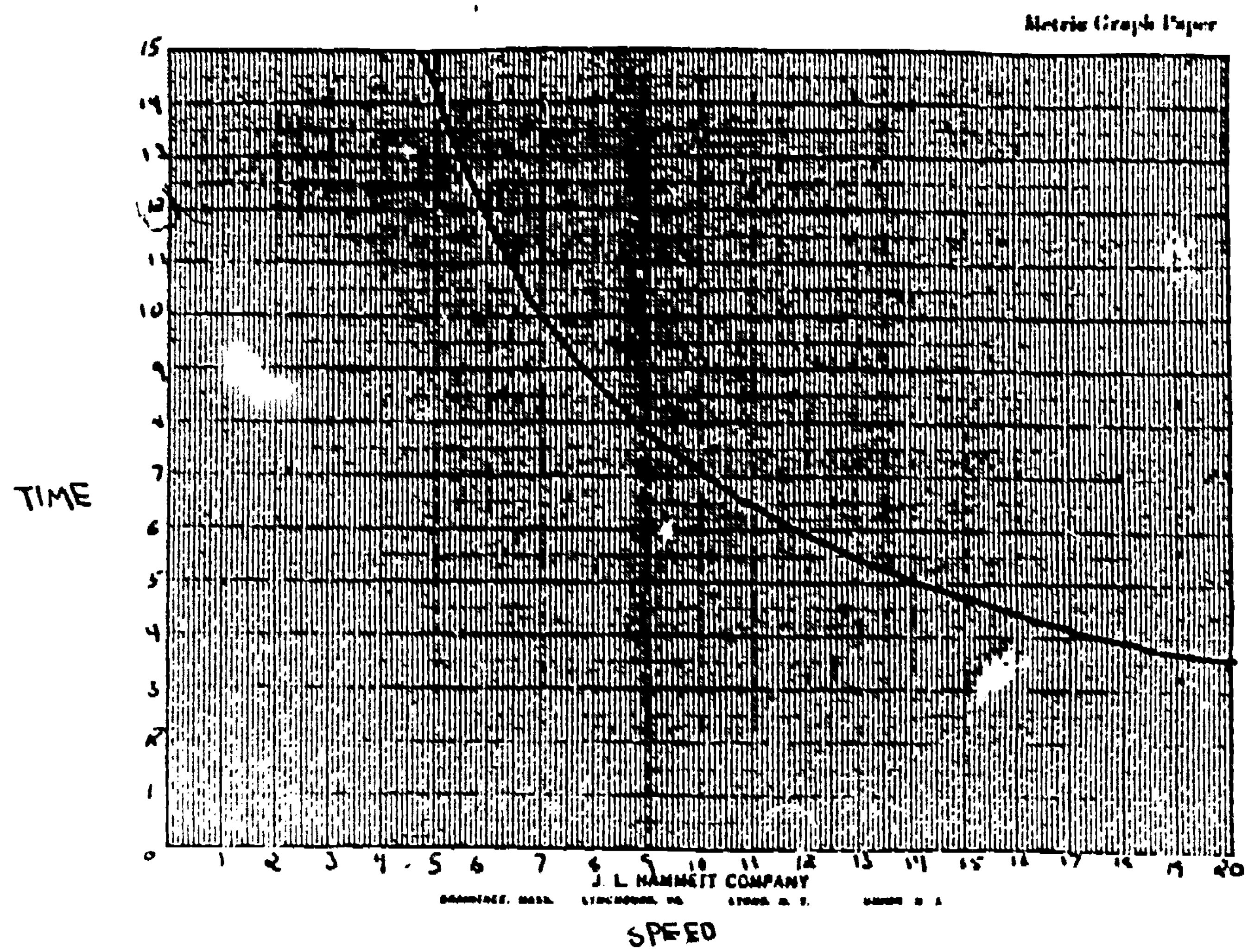
Jonathan "THE NASTY MOOIST" 120 Ft. Long



Charles Kalina

"The Sea Otter"

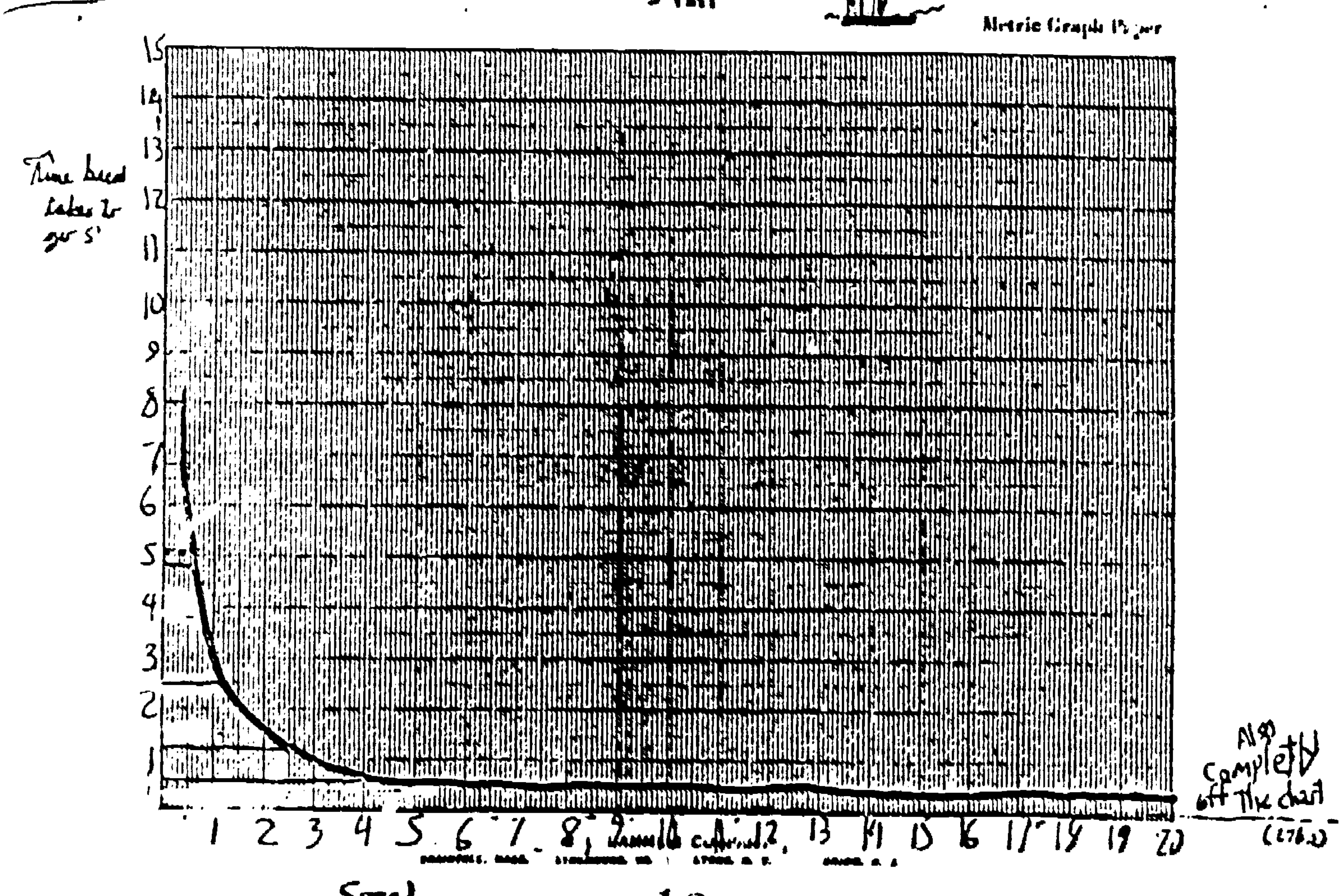


Figure 3

Other Sources of Real Problems

"The Voyage of the Mimi" is full of opportunities for such exploration, but there are also many other settings from which problems like this can be derived. Episodes of science-oriented TV series' like "3-2-1 Contact" and "Nova" include descriptions of real adults creating solutions to problems which can be analyzed using the kinds of mathematical tools described here. Several documentary episodes are also being prepared by the American Association for the Advancement of Science as videotapes that can be shown in the classrooms. They portray people like mountain climbers or architects or demolition experts confronting, analyzing, and solving mathematical problems. Newspaper stories about the space shuttle or pre-election polls or lottery winnings can be developed into stories in which a real mathematical problem can be examined at the middle school level. Current local events like the cleanup of a polluted river, the opening of a new shopping center, or the beginning of a balloon trip across the ocean offer opportunities for children to actually meet adult problem solvers who use mathematical relationships to understand what they are doing.

The more traditional mathematical "content" for grades 4 through 8, including operations on rational numbers, exponents and scientific notation, the geometry of similarity and congruence, concepts of ratio and proportion, and beginning work with functions, can all be related to settings in which people solve problems using mathematical tools that go beyond mere computation. And in addition, the mathematical ideas used in understanding these problems can be analysed by children when they have a context as a referrent to help them "make sense of what they are doing."

What do teachers need to know in order to develop and use a mathematical curriculum of this sort? Simply exposing children to situations in which mathematical ideas are being used is not enough. If classroom lessons are to bridge the gap from the concrete concepts and computational techniques associated with adding, subtracting, multiplying and dividing to the abstractions involved in recognizing number relationships, representing them symbolically, and manipulating those symbols to find an unknown, teachers need to know both where children are coming from and where they are going. Connecting the computations middle school students are able to do with the mathematics they will be learning in high school and beyond requires substantial knowledge of the subject beyond algorithms for arithmetical processes. Constructing serious mathematical explorations from real-life problems requires an appreciation of how mathematics can be used in a variety of situations to order, relate, and represent information in helpful ways. Organizing lessons that put the "answer" in context and help students to look more carefully at creating strategies for arriving at an answer requires confidence in one's own as well as students' abilities to see mathematics as a set of ideas that make sense. We would do well to think about how to better provide middle school teachers with these resources.

RESEARCH ON SIMILARITY AND PROPORTIONAL REASONING

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Introduction

Proportional reasoning is fundamental to the solution of problems frequently encountered in everyday life and in natural sciences. Many children, however, never reach the stage of cognitive development that allows them to consistently and correctly use proportionality.

Similarity is an instance of proportionality and has a wide range of applications in and of itself. Studying similar geometric shapes would seem to provide children with more concrete mental images for other types of proportional reasoning.

This paper will describe first some of the results that have been indicated by the extensive research on proportional reasoning.

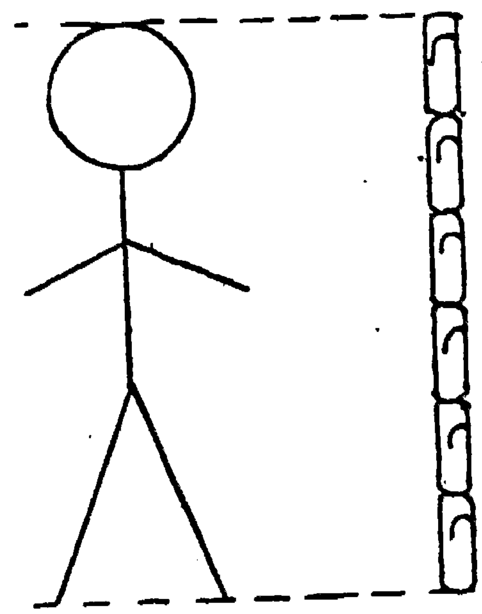
Two studies conducted by the Middle Grades Mathematics Project (MGMP) to determine the extent to which children in grades six, seven and eight exhibit an understanding of similarity both before and after a three week unit of instruction will be described next.

Research on Proportional Reasoning

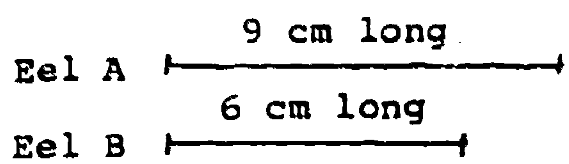
Proportional reasoning is frequently required in mathematics, natural science and everyday life. Inhelder and Piaget (1958) consider proportional reasoning as one of six abilities that characterize a formal-operational thinker.

In the 1970's, the ability of children and adults to reason proportionally was investigated extensively. Figure 1 presents a sample of questions that were employed in different studies on proportional reasoning. These studies seem to indicate that the majority of children and many adults lack the ability to use such reasoning effectively (Table 1).

a) You can see the height of Mr. Short measured with paper clips. Mr. Short has a friend Mr. Tall. When we measure their height with matchsticks, Mr. Short's height is 4 matchsticks and Mr. Tall's height is 6 matchsticks. How many paper clips are needed for Mr. Tall's height? (Adapted from the Karplus studies).



b) At the Zoo, eels A and B are fed minnows, and the number of minnows depends on the length of the eel. If eel B is fed [illegible] minnows, how many minnows should eel A be fed to match? (Adapted from Piaget's studies).



c) At a Lemonade contest, a team used this recipe to prepare lemonade concentrate.

Recipe: 2 spoons of sugar
3 spoons of lemon juice

How many spoons of lemon juice does the team have to use with 8 spoons of sugar? (Adapte` from the Karplus studies).

d) $\frac{2}{3} = \frac{\square}{9}$.

Figure 1. Sample of proportionality questions

TABLE 1

Results indicated by some proportionality studies

Grade Level	Study	Students using proportional reasoning
7-8	Wollman and Karplus (1974)	20%
	Abramovitz (1975)	25%
	Karplus, Karplus and Paulsen (1977)	25%
8-10	Karplus and Peterson (1970)	32%
11-12	Karplus and Peterson (1970)	80% (high SES)
College	Lawson and Wollman (1980)	50%
	Fuller and Thornton (1977)	50%
	Renner and Paske (1977)	50%
	Karplus, Pulos and Stage (1980)	74%

Besides these generalizations, it should be mentioned that level of performance in proportionality tasks differs according to task variables such as the numbers involved (Abramovitz, 1975), level of concreteness (Lunzer and Pumfrey, 1966) and according to subject variables such as socioeconomical status (Karplus and Peterson, 1970) and sex (Karplus et al. 1977).

The Concept of Similarity: A Primary Evaluation

The concept of similarity is important both as an essential stage in the development of children's geometrical understanding of their environment, and as a concrete aspect of proportional reasoning. Phenomena that require familiarity with enlargement, scale factor, projection, area growth, indirect measurement and other similarity-related concepts are frequently encountered by children in their immediate environment and in their studies of natural and social sciences. Because of its visual representation, mastery of similarity may very well be a first step towards an understanding of proportional reasoning.

The Similarity Unit developed by NSF-funded Middle Grades Mathematics Project (MGMP) presents to sixth, seventh, and eighth graders the basic similarity concepts in an informal, activity-oriented way.

A large-scale evaluation (Friedlander, 1984) of performance on similarity tasks at the middle grades level (N = 514) revealed a significant improvement as a result of a two to three week-long instruction with the Similarity Unit (Figure 2). In a comparison of grade levels, seventh graders seemed to gain the most when performance was measured by a 25-item Similarity Test. Both before and after

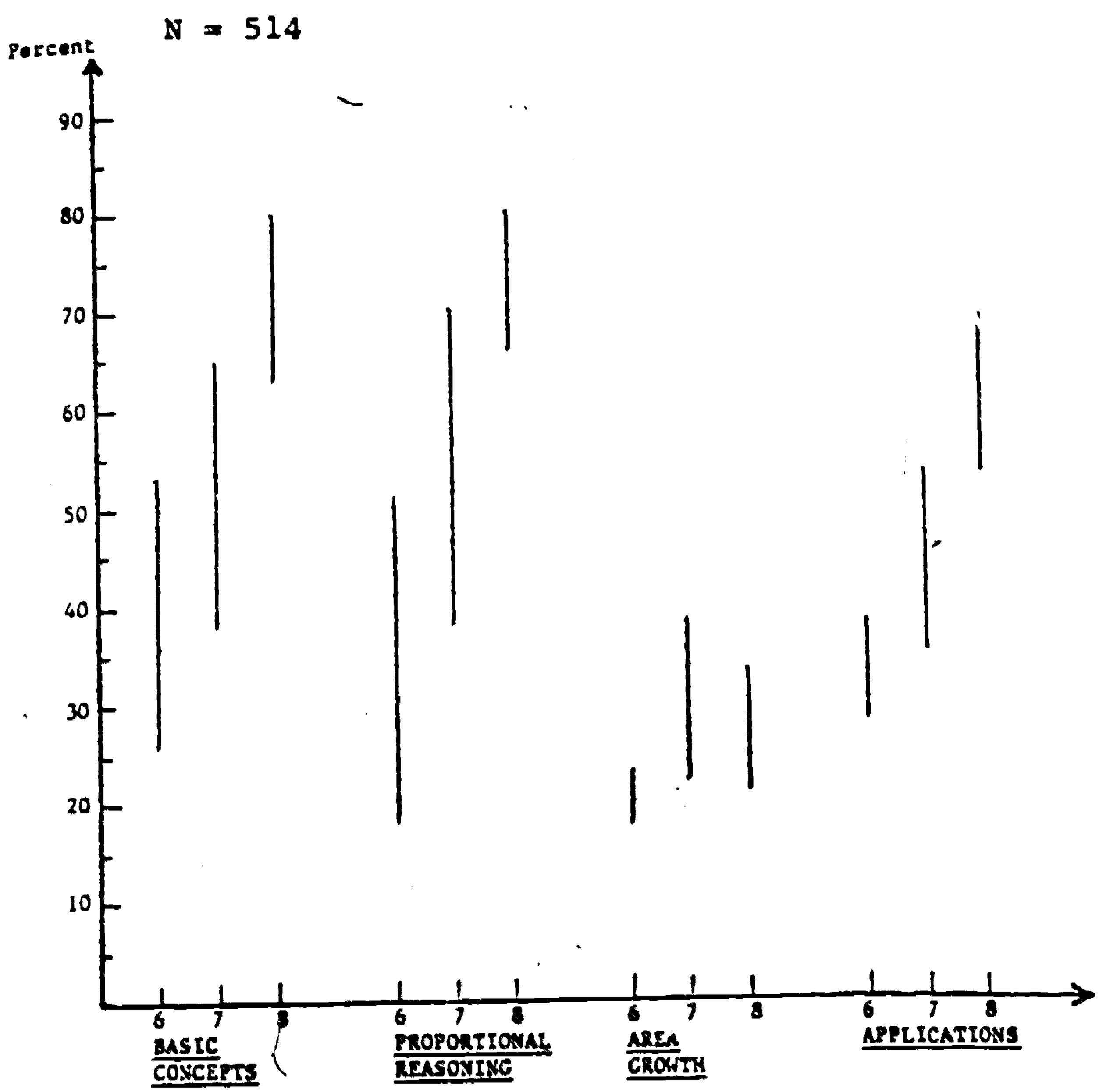


Figure 2. Gains in four similarity related topics

instruction, understanding of the area relationship of similar shapes proved to be very difficult for all grade levels in general, and for sixth graders in particular.

The Growth of Similarity Concepts: A Follow-up Study

As indicated by the vivid example of Benny in Erlwanger's case study, multiple choice, or other performance-oriented tests do not always indicate children's misconceptions on a topic. Therefore, a second study is currently conducted by the MGMP group. The purpose of this study is to find patterns, and to reveal effects of instructional intervention in children's perception of similarity.

Methodology

Two classes of sixth graders with the same mathematics teacher comprise the sample for the study. The classes were homogeneously grouped with the higher ability students in one and the average ability in the other. In-depth interviews were conducted with 17 students who scored in the middle range on two paper and pencil tests given pre- and post-instruction. These tests were the MGMP Similarity Test and a Ratio and Proportion Test which included the Karplus Mr. Tall/Mr. Short problem and selected items from the Concepts in Secondary Mathematics and Science (CSMP) Ratio and Proportion Test. Each audiotaped interview lasted from 30 to 60 minutes. The tasks to be performed were presented to each student in a uniform way. The interviewer asked students to explain their reasoning in detail. Students were presented tasks focusing on similarity of rectangles and those focusing on area growth. The two sets of results will be discussed separately.

The first part of the interview presented students with four different kinds of tasks related to determining similarity of rectangles. Each of these tasks was varied along a numerical scale designed to test the student's facility with handling proportions of increasing numerical difficulty. Table 2 shows the four tasks and the four ways in which the numbers in the proportions were varied.

Table 2. Interview Questions

TASK \ NUMERICAL TYPE (a by b, c by d)	1 (a|b, a|c)	2 (a∤b, a|c)	3 (a|b, a∤c)	4 (a∤b, a∤c)
1. Decide whether two drawn rectangles are similar or not.	3 by 6 and 9 by 18	2 by 3 and 8 by 12	3 by 9 and 4 by 12	6 by 8 and 9 by 12
2. Decide whether two cut-out rectangles are similar or not.	2 by 4 and 6 by 12	2 by 3 and 6 by 9	2 by 6 and 3 by 9	4 by 10 and 6 by 15
3. Given the lengths of three sides of two similar rectangles, find the fourth side. a b c ?	2 by 6 and 6 by ?	2 by 5 and 6 by ?	4 by 12 and 7 by ?	6 by 10 and 9 by ?
4. Cut a strip to make a rectangle similar to a given one.	2 by 4 and 6 by ?	2 by 3 and 8 by ?	2 by 6 and 5 by ?	4 by 6 and 6 by ?

The second part of the interview dealt with the concept of area growth in similar figures. Students were shown a rectangle representing a small room which cost $300 to carpet. The interviewer asked the student what would be the price of carpeting a larger room which is twice as long and twice as wide. The task was presented on three levels of concreteness: (1) no additional illustration, (2) student is given an assortment of shapes from which to choose a correct representation of the larger room and (3) student is given several copies of the small room and asked to represent the larger room. The question was repeated for a room enlarged by a scale factor of three.

Main Results

The performance on the four tasks that involved the concept of similarity between rectangles will be described first, and results related to the task involving the area relationship of two similar shapes will be described next. Due to the small number of subjects, the indicated trends should be considered with caution.

Student performance on the interview items related to rectangle similarity varied widely as a function of both task and the types of numbers involved. Figure 3 illustrates the level of success on each item expressed by the number of answers given that employ proportional reasoning. A general improvement in performance as a result of instruction is noticeable.

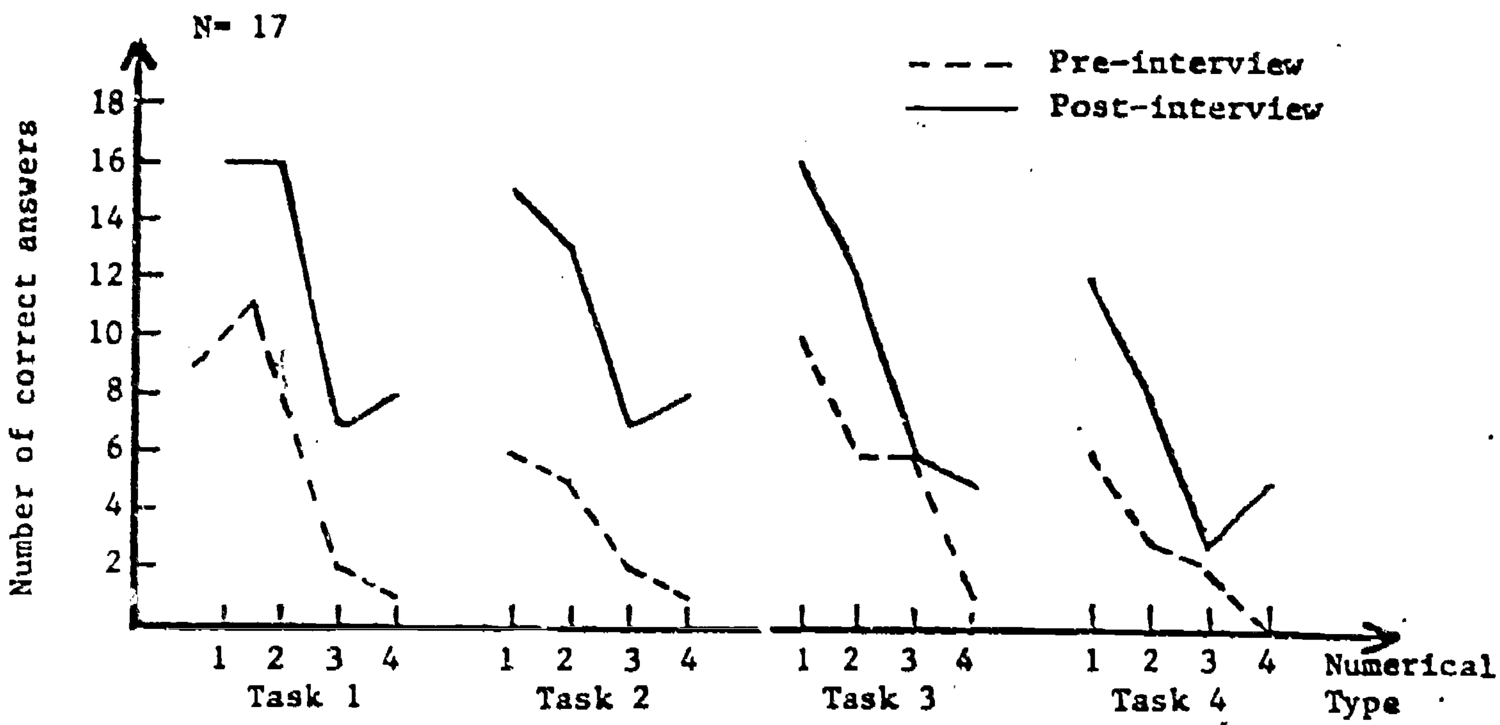


Figure 3. Level of success for rectangle similarity items

With some irregularities the level of success of the two groups tends to decrease as numerical difficulty increases. By collapsing results across tasks, a post-instructional level of success of 87, 72, 34 and 38 percent was observed on items involving proportions of numerical types 1,2,3, and 4 respectively. Thus, at sixth grade level, more than two-thirds of the students used proportional reasoning when the external ratios (i.e., width to width, and length to length) was an integer, but fewer were able to do so in other cases (even if internal ratios of length to width were an integer).

Similar differences in performance as a function of the numbers involved in other proportionality tasks were indicated by Abramovitz (1975), Karplus et al. (1980), and Quintero (1983).

Level of success in student performance varied as a function of task. Ordering of tasks according to level of success was difficult especially for pre-instructional performance. The results after instruction on similarity indicate a level of success of 69, 63, 57, and 41 percent on tasks 1, 2, 3 and 4 respectively.

These results seem to indicate that Task 1 (deciding similarity for pairs of drawn rectangles) was the least difficult, and Task 4 (creating a rectangle similar to a given one by drawing its fourth side) was the most difficult. Analysis of interviews allowed for the following categorization of student reactions to the presented similarity tasks:

* Proportional Reasoning -- setting up two ratios and a correct or incorrect consideration of their equivalence, or more frequently, considering the scale factor by which the small rectangle is enlarged.

* Whole Multiplication -- "fitting in" the sides of the small figure a whole (but not necessarily the same) number of times into the sides of the bigger figure. This kind of reasoning leads characteristically to the conclusion that if the scale factor is not an integer, the figures are not similar.

* Multiplication and Adjustment -- multiplying to enlarge, and "adjusting" by subtraction or addition (e.g., in the proportion 2:6::5:x, $x = 13$ because $5 = 2 \cdot 2 + 1$, and thus $2 \cdot 6 + 1 = 13$).

* Addition -- considering the difference rather than the ratio of the numbers.

* Visualization -- using intuition without considering the lengths of the sides.

With the exception of whole multiplication, these categories correspond to the classification of strategies used in other proportional reasoning tasks reported by Karplus and Karplus (1972).

The deterioration of performance according to increasing difficulty of numerical type of proportion indicates a remarkable lack of within-subject consistency in the employed strategies. Frequently, during four consecutive interview questions, students regressed within the same task from proportional reasoning to whole multiplication or multiplication/addition, and finally to using the additive strategy as the numerical type of the proportion became more difficult. Lack of consistency may be observed from the fact that no students used the same strategy for all sixteen items presented. During the pre-instructional interviews, one subject used four different strategies for the same task presented consecutively in four different numerical instances; eight subjects used three strategies for at least one task; seven subjects used two strategies for the same task; and only one student was consistent within each task. Instruction improved to a certain degree the within-task consistency: eight subjects used three different strategies for at least one task, seven subjects used two, and two subjects used proportional reasoning throughout the interview.

Tasks related to the area relationship of similar shapes had as a purpose the detection of student thinking strategies. Tasks were administered to the eight students in one of the two classes. Instruction did not improve performance on this task. The following thinking strategies could be detected:

* Area Scaling -- increasing the area by counting the number of times the small rectangle "fits into" the larger one. No student used the square of the linear scale as an argument.

* Scaling -- increasing the area by a factor which is different from the area, or the linear scale.

* Scaling and Adjustment -- the scaling is followed by an additive "adjustment".

* Periferal Counting -- considering the area of the original rectangles that "fit in across and down" the increased rectangle (characteristically counting the corner-rectangle twice) and ignoring the interior of the enlarged shape.

* No Reasoning -- not being able to explain, or not giving an answer.

Lack of consistency in employed strategies was noticeable in this case as well. Strategies tended to improve as the level of concreteness increased from an unillustrated presentation to the students' building of the enlarged rectangle with a number of cut-out original rectangles. It should be noted that at the post-instructional stage, fewer than 50 percent of the students were successful in this task even with this last presentation mode. The next phase of the study will be to expand the research to grades seven and eight, and to more schools and students.

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MOTIVATING ALGEBRA THROUGH PROBLEM SOLVING

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Algebra can be taught in a way which is both stimulating and interesting by continually posing interesting problems and applications which provide a motivation for the learning and understanding of algebra. By carefully selecting problems, a wide range of problem solving strategies and mathematical thinking can be developed. The attached list of problems serve to illustrate the strategies listed below.

1. Look for patterns
2. Make conjectures
3. Test conjectures
4. Ask questions backwards
5. Give examples and counter-examples
6. Try a simpler case
7. Guess and check

CUBE COLORING PROBLEM

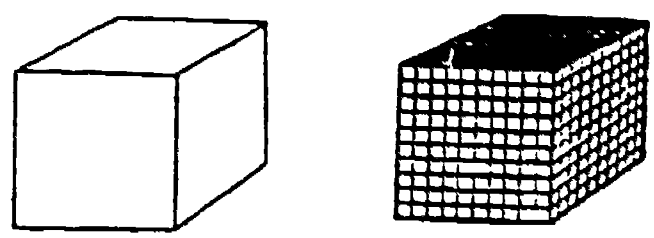
a) Suppose a large wooden cube, which has been painted black, is cup up into 1000 smaller cubes. Can you figure out how many cubes there would be with

3 faces (sides) painted black?

2 faces (sides) painted black?

1 face (side) painted black?

0 faces (sides) painted black?



b) What is the least number of cuts needed to cut the larger cube into 1000 smaller cubes?

c) If the volume of the original cube is 1 m^3, what is the volume of one of the smaller cubes?

d) Repeat parts a-c if the original cube is divided into n^3 smaller cubes.

CUBE COLORING PROBLEM

Age of Cube	Total No. of 1-cubes needed to build	3-faces	2-faces	1-face	0-faces	Total faces painted
2	8					
3						
4						
5						
6						
7						
⋮	⋮	⋮	⋮	⋮	⋮	⋮
100						
⋮	⋮	⋮	⋮	⋮	⋮	⋮
N						

PROBLEM SOLVING AND APPLICATIONS IN ALGEBRA

Part I Problem Solving

1-11 Looking for Patterns

1. a) How many diagonals does a convex polygon have?
   b) Use the pattern in 1 a) to find the sum of the first n positive integers.

2. A unit cube 1cm x 1cm x 1cm is cut into smaller cubes. If each edge is subdivided into n equal parts,
   a) how many cuts are needed to split the block?
   b) how many cubes result?
   c) what is the volume of each smaller cube?

3. A wooden cube is painted black and then cut up into smaller cubes (as in 2. above). If each edge is subdivided into n equal parts,
   a) how many cubes result?
   b) how many cubes have 3 faces painted black?
   c) how many cubes have 2 faces painted black?
   d) how many cubes have 1 face painted black?
   e) how many cubes have 0 faces painted black?

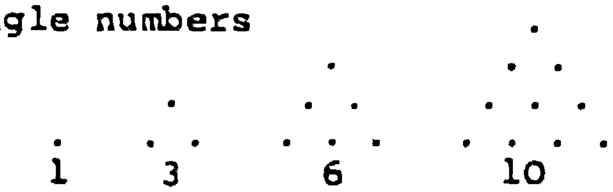
4. Write an algebraic statement to describe each pattern. Is your stat^ment true for all numbers? Why?

a) $(2 \cdot 2) + 3 = 2 + 5$
$(2 \cdot 4) + 5 = 4 + 9$
$(2 \cdot 5) + 6 = 5 + 11$

b) $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$
$\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$
$\frac{1}{9} - \frac{1}{10} = \frac{1}{90}$

c) $(\frac{1}{3})^2 + \frac{2}{3} = \frac{1}{3} + (\frac{2}{3})^2$
$(\frac{1}{5})^2 + \frac{4}{5} = \frac{1}{5} + (\frac{4}{5})^2$
$(\frac{2}{7})^2 + \frac{5}{7} = \frac{2}{7} + (\frac{5}{7})^2$

d) triangle numbers



5. A grasshopper jumps along a number line as follows: He starts at zero, moves forward one unit, then backward two units, forward three units, backward four units, forward five units and so on. Where will he be after

a) 15 moves?
b) 2,000 moves?
c) 2,001 moves?
d) n moves?
e) On what move will the grasshopper hit the point +100?
f) Will the grasshopper eventually hit every integer?

6. A grasshopper is on the number line again. He starts at zero and would like to hop to the point labeled 1. He has a slight problem: He can only hop half the distance from where he is to the point of destination. That is, after the first hop, he will be at the point 1/2. After the second hop, he will be at the point 3/4 (half of the distance from 1/2 to 1). After the third hop, he will be on 7/8 (half of the distance from 3/4 to 1), and so on.

a) Where will he be after 10 hops?
b) Where will he be after n hops?
c) Will he ever get to point 1?

7. Find a general rule for squaring a number ending in 5.

8. Do any positive integers exist (x and y) which satisfy the following equations? Why?

   a) $x^2 - y^2 = 48$ b) $x^2 - y^2 = 23$ c) $x^2 - y^2 = 45$

9. a) In the following sequence evaluate each term for $x = 1$ and $x = 2$.

   $$1 + \frac{1}{x} \qquad 1 + \frac{1}{1 + \frac{1}{x}} \qquad 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{x}}} \qquad 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{x}}}}$$

   b) Simplify the above expressions and evaluate.
   c) If this pattern continues, what is the value of the next three terms of $x = 1$, $x = 2$?

10. a) Use a calculator to find the sequence

   $$\sqrt{2} \qquad \sqrt{\sqrt{2}} \qquad \sqrt{\sqrt{\sqrt{2}}} \qquad \sqrt{\sqrt{\sqrt{\sqrt{2}}}} \qquad \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}}}}}$$

   b) Express the sequence using rational exponents?
   c) What is the nth term?

11-13 Puzzles

11. Box puzzle: Consider a 3 x 3 square.

   a) Pick four numbers, say 2, 8, 6, and enter them on the border as follows:

	6	3	Sum
2			
8			
Sum			

   b) Enter the product of the border numbers into the top four squares

	6	3 →	Sum □
2	12	6	
8 ↓	48	24	
Sum □			

   c) Add across the rows and columns

	6	3 →	Sum 9
2	12	6	18
8 ↓	48	24	72
Sum 10	60	30	?

↑ Grand Total

12. Graph Codes: Write a series of linear equation for $0 \leq y \leq 2$ whose graphs will spell out your first name or some other short word. For example, TIM can be spelled by graphing the following:

First Letter T	Second letter I	Third letter M
$y = 2$	$x = 0$	$x = 1$
$-3 \leq x \leq -1$		$y = -2x + 4$
$x = 2$		$y = 2x - 4$
		$x = 3$

13. Suppose two players play the following game concerning the equation $ax^2 + bx + c = 0$. Player A chooses a value for one of the coefficients. Player B chooses a value for one of the two remaining coefficients. Player A chooses a value for the third coefficient. Player A wins if both roots of the equation are real and Player B wins if the roots are complex. Is there a winning strategy for either players?

14-20 Mark each statement either AT for always true, ST for sometimes true, NT for never true. If a statement is ST give one example of when the statement is true and one example of when the statement is false.

14. The difference of two negative numbers is negative.

15. For any number x, the expression $-x$ is negative.

16. x^{-2} is a negative number.

17. $(-1)^n = -1$.

18. If $a \leq b$, then $\frac{1}{a} \leq \frac{1}{b}$.

19. The line $(y = a)$ is perpendicular to the y-axis at the point $(0,a)$.

20. If (a,b) is a solution to a linear equation, then (b,a) is also a solution.

Guess and check to find a general solution.

21. At noon the minute hand of a clock is directly over the hour hand. At what time between 1 o'clock and 2 o'clock will this phenomenon occur again? Guess 1:07 - check, then go back and try x minutes after 1 o'clock.

22-26 Give an example of:

22. Three equations which contain the point (0,1).

23. Three equations which intersect the x-axis at (-1,0).

24. Three equations with slope = 2.

25. Of six points which are four units from (0,0).

26. A 2 x 2 linear system that has the following property:
   a) Solution is (0,0)
   b) Linear system has no solutions
   c) Linear system has infinitely many solutions.

27-30 Reversing questions:

27. Which of the following integers are in the sequence: 4, 11, 18, 25 ...?
   a) 95
   b) 995
   c) 9,995

28. What positive integers can be written as the sum of
   a) Three consecutive integers?
   b) Four consecutive integers?

29. A lattice point is one in which both of its coordinates are integers. Find all the lattice points in the first quadrant which lie on each line.
   a) $4x + 5y = 77$
   b) $3x + 6y = 77$
   c) $x - y = 0$

30. For what values of c could you solve the equation $x^2 + 2x + c = 0$ by factoring?

Part II Applications

Problems which have the same mathematical model

Example 1: These problems appeared in an article by Zalman Usiskin in the April 1968 issue of the Mathematics Teacher. Although these problems are from different branches of mathematics (algebra, number theory, and geometry), their solutions all have something in common.

a) Express 1/2 as the sum of two different unit fractions. A unit fraction is of the form 1/n, where n is a positive integer.
b) Find all rectangles with integral sides whose area and perimeter are numerically equal.
c) What pairs of positive integers x and y have a harmonic mean equal to 4? The harmonic mean of x and y is given by $\frac{2xy}{x+y}$.

d) Find all pairs of integers whose product is positive and equal to twice their sum.
e) Given a point P, find all integers n such that the plane around P, can be covered by nonoverlapping congruent regular n-gons.
f) For which positive integers $n > 2$ is $(n-2)$ a factor of $2n$?

Example 2:

a) The Fence Problem. What is the maximum rectangular area that can be enclosed with a fence of perimeter 140 meters.
b) The Refrigerator Problem. The manager of an appliance store buys regrigerators at a wholesale price of $250 each. On the basis of past experience, the manager knows she can sell 20 refrigerators each month at $400 each and an additional one each month for each $3 reduction in selling price. What selling price will maximize the store's monthly profit?

"PROBLEMS"

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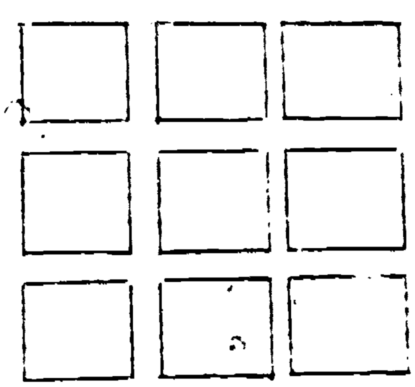
"Some Problems" and "More of the Same", although not all explicated in the presentation, represent a varied group of verbal problems that "springboard" a wide spectrum of mathematical responses. This random collection of "story problems" cover a wide gamut of skills and concepts underlying the middle school mathematics curriculum. In most cases the working mental tools involved go from simple arithmetic to algebra and geometry.

SOME PROBLEMS

1. List the ways 10 coins can be used to make change for 59¢.

2. Place 2,3,4,5,6,7,8,9, and 10 in the squares so the sum of three numbers in any straight line direction is always 18.

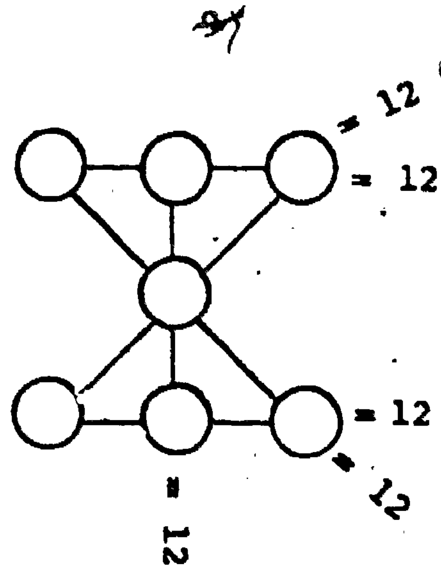


3. Lester had a code. In addition exercise, each letter stands for one of these numbers -- 2, 6, 7, 8

$$\begin{array}{r} PQ \\ PQ \\ +\ PQ \\ \hline RRS \end{array}$$

What number does each letter represent?

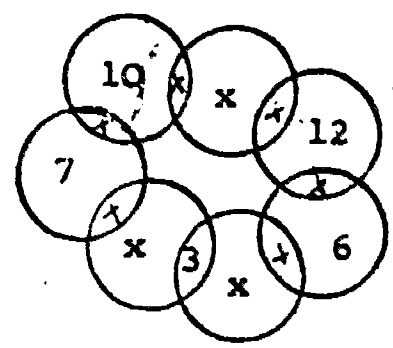
4. Place the numerals 1 through 7 in the circles so that the sum of each 3 circles in a straight line is 12.



5. Sue planned for a large Chinese party. What is the least number of guests she planned for if every 2 guests used a dish for rice between them, every 3 used a dish of gravy between them, and every 4 used a dish of meat between them?

6. How many 4 digit numbers can you make using number 1,2,3 4 only once? (Example: 4,321)

7. Replace the x's with the numbers 1 through 14 so the sum in each circle is 21. (3,6,7,10,12, have already been put in place.)



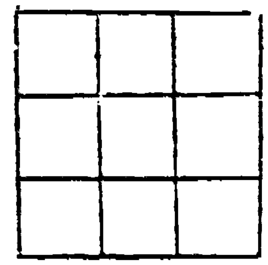
8. If Mr. Marra adds ten years to his present age and doubles it, he would be 82. How old is Mr. Marra now?

9. If Mr. Marra added 15 points to Shelly's test grade and multiplied the new number by four, it would equal 352. If 70 is passing, show by listing if Shelly passed the test.

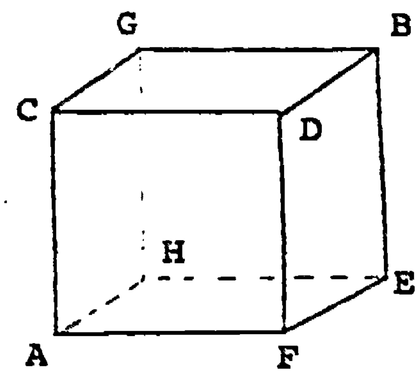
10. Mrs. Taylor tried to hide the amount of money she spent on clothes. She told Mr. Taylor that if he took the amount she spent, divided by 4, and then subtracted $20, he would get $5. How much did she spend?

11. Three foxes and three geese had to cross the river on a boat, two at a time. If there were ever more foxes than geese together, the geese would be eaten. How could all of them get across the river?

12. Place the number 1 through 9 in the cells so the sum in each direction is 15.



13. How many different 5-legged trips can an ant make in crawling from A to B?



14. Ralph, the computer, woke up one morning with a headache. He had been trying to figure out this problem. Can you help him? I am thinking of two numbers whose sum is 35 and their product is 250. What are the numbers?

15. For a school project Boy Scouts and Girl Scouts bought 3 Maple Trees and 3 Apple Trees to line the sidewalk from the parking lot to the school building. They wanted an equal number of trees on each side of the sidewalk. How many different ways can the trees be arranged along the sidewalk?

16. The license plate on Vicki's Vega has three digits. Their product is 216 and their sum is 19. What are the three digits?

17. You are counting railroad cars. 14 railroad cars go by in 1 minute. Railroad cars are about 70 feet long. Estimate the speed of the moving train in miles per hour.

18. The distance around a tennis court is 76 yards. The length of the court is 2 yards more than twice the width. What is the length and width of a tennis court?

19. Joe was reorganizing his stamp collection. He noticed that stamps that he had 3 of, were attached in several different ways. As he thumbed through his collection, he made a note of the different ways he found. How many ways did he discover?

20. In 24 hours how many times does the minute hand cross the hour hand?

MORE OF THE SAME

1. OBJECT: Think of a fourth word related to all three words listed below.

Cookies	Heart	Sixteen	________

The answer is "sweet". Cookies are sweet; sweet is part of the word "sweetheart" and part of the phrase "sweet sixteen".
Now try these words:

a.	surprise	line	birthday	________
b.	base	snow	dance	________
c.	rat	blue	cottage	________
d.	nap	bird	call	________
e.	golf	foot	country	________
f.	tiger	news	plate	________
g.	painting	bowl	nail	________
h.	maple	beet	loaf	________
i.	show	oak	plan	________
j.	light	village	golf	________

2. In a certain African village there live 800 women. Three percent of them are wearing one earring. Of the other 97 percent, half are wearing two earrings, half are wearing none. How many earrings all together are being worn by the women?

3. A logician with some time to kill in a small town decided to get a haircut. The town had only two barbers each with his own shop. The logician glanced into one shop and saw that it was extremely untidy. The barber needed a shave, his clothes were unkempt, his hair was badly cut. The other shop was extremely neat. The barber was freshly shaved and spotlessly dressed, his hair neatly trimmed. The logician returned to the first shop for his haircut. Why?

4. Smith gave a hotel clerk $15.00 for his cleaning bill. The clerk discovered he had overcharged and sent a bellboy to Smith's room with five $1.00 bills. The dishonest bellboy gave three to Smith, keeping two for himself. Smith has now paid $12.00. The bellboy has acquired $2.00. This accounts for $14.00. Where is the missing dollar?

5. A secretary types four letters to four people and addresses the four envelopes. If she inserts the letters at random, each in a different envelope, what is the probability that exactly three letters will go into the right envelopes?

6. If nine thousand nine hundred nine dollars is written as $9909, how should twelve thousand twelve hundred twelve dollars be written?

7. A customer in a restaurant found a dead fly in his coffee. He sent the waiter back for a fresh cup. After a sip he shouted, "This is the same cup of coffee I had before." How did he know?

8. "I guarantee," said the pet-shop salesman, "that this parrot will repeat every word it hears." A customer bought the parrot but found it would not speak a single word. Nevertheless, the salesman told the truth. Can you explain?

9. Give at least two ways a barometer can be used to determine the height of a tall building.

10. What number comes next in this series? 9, 16, 25, 36,...

11. In the box below, a rule of arithmetic applies across and down so that two of the numbers in a line produces the third. What is the missing number?

6	2	4
2	?	0
4	0	4

12. Find the number that logically computes the series: 2, 3, 5, 9, 17,...

13. See number 11 above.

6	2	12
4	5	20
24	10	?

14. If $A \times B = 24$, $C \times D = 32$, $B \times D = 48$ and $B \times C = 24$, what does $A \times B \times C \times D$ equal?

15. WANT TO BET?
    a. At a party you find 23 people present. What odds that no two were born on the same day of the same month?
    b. A gambling friend offers to bet that, of the license plates on the next 20 passing cars, at least two will match each other in their last two digits. Should you take the bet?
    c. Pick two Americans at random - Ms. A and Mr. B - and the chances are about 1 in 200,000 they'll know each other. But how likely is it that A will know someone who knows someone who knows B?
    d. If a family has three children, what's the likelihood the three will all be of the same sex?
    e. If a couple plan to have four children, which is more probable: a) two boys and two girls, or b) three of one sex and one of the other?
    f. You flip a coin and it comes up heads 10 times in a row. What odds, then, that it will come up tails on the next flip?

16. You wake up in a pitch-black room in a hunting lodge, and there's no light handy. In your duffel bag there are six black socks and six white ones, all mixed together. You want to pick out a matching pair. What is the smallest number of socks you can take out of the bag and be sure of getting a pair of the same color?

17. A bass plug and some tough-up paint cost a total of $2.50. The plug costs $2.00 more than the paint. What is the cost of each?

18. A deep-sea fishing boat is lying in the harbor. Over its side hangs a rope ladder, with its end just touching the water. Rungs of the ladder are one foot apart. The tide rises at the rate of eight inches an hour. At the end of six hours how many of the rings will be covered?

19. How much dirt is there in a hole 1 ft. by 1 ft. by 1 ft.?

20. A camp cook wanted to measure four ounces of syrup out of a jug but he had only a five-oz. and a three-oz. bottle. How did he manage it?

21. A logician vacationing in the South Seas finds himself on an island inhabited by the two proverbial tribes of liars and truth-tellers. Members of one tribe always tell the truth; members of the other always lie. He comes to a fork in the road and has to ask a native bystander which branch he should take to reach a village. He has no way of telling whether the native is a truth-teller or a liar. The logician thinks a moment, then asks one question only. From the reply he knows which road to take. What question does he ask?

21. Insert the missing numbers or letters:
    a. 3, 5, ___, 9, 11, ___, 15
    b. ZWY ___ W ___ U
    c. ACE ___ IK ___
    d. 100, ___, 400, 800
    e. YEL ___ OW
    f. 9, 7, 11, ___, 13, 11, ___, 13
    g. ___, ___, 9, 27, 81, ___
    h. 6, 8, 9, ___, 12, ___, 15
    i. ___, 24, 29, ___, 33, 34, 35
    j. A ___ ZYCDX ___ EF ___
    k. 2, ___, 2000, 2000, 200

22. Fill the blank squares with numbers which will make both the vertical columns and horizontal rows in each diagram add up to the sum shown at the right of that diagram. Use no number larger than 9. Do not use zero.

A

9			
		9	21
	3		

B

9				
		7		
	7			34

C

9				
		7		
			8	
	8			34

23. How would you cash a check for $63.00 with six bills but no one dollar bills?

24. Mr. Brown, Mr. Green and Mr. Black were lunching together. One wore a brown necktie, one a green tie, one a black. "Have you noticed," said the man with the green tie, "that although our ties have colors that match our names, not one of us has on a tie that matches his own name?" "By golly, you're right!" exclaimed Mr. Brown. What color tie was each man wearing?

25. My watch is ten minutes slow, though I'm under the impression its five minutes fast. Your watch is five minutes fast, though you think it's ten minutes slow. We both plan to catch a four o'clock train. Who gets there first?

A SUGGESTED OUTLINE FOR A COURSE IN TEACHING AND LEARNING PROBABILITY AND STATISTICS

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Goals

1. Examine the need for teaching probability and statistics
   a) Statistical literacy
   b) communication
   c) decision making
      i) mis-conceptions of probability
      ii) misuses of statistics
2. Become familiar with materials and supporting literature for teaching probability and statistics.
3. Experience small-group, activity-based, problem-solving approach to teaching probability and statistics.
4. Explore some statistical applications and probability "gems" problems.
5. Look at research on the teaching and learning of probability and statistics.

Expectations

1. Keep an updated notebook of data, analysis, and results of experiments, simulations, and problems done in class and at home.
2. Uncover at least 4 examples of misuses of statistics (ala Huff's book), write a short analysis and critique of each (about 1-2 pages), and be prepared to present them and share them with the class.
3. Choose one of the articles (listed below) from Statistics: A guide to the Unknown, write up a brief review and summary of the article (2-4 pages), and prepare a short (10 minute) summary of the article to share in class.
4. Put together a project in the teaching of probability and statistics. The project could be developing a unit on teaching some aspect of probability and statistics for your own classes, developing some new and/or extending some old problems and activities, writing and collecting computer programs which explore some aspects of probability and statistics, or anything else that turns you on and seems reasonable to the instructor (namely me.) You will conduct some short activity from your project, or make a short presentation on it, during the last week of the course. (Have your project direction chosen by Monday or Tuesday of the second week).

Titles for review articles from Statistics: A Guide to the Unknown. Pick one of these to review and present.

The biggest experiment (Salk Vaccine)
Deathday and Birthday
Setting Dosage Levels
Deciding Authorship
Drug Screening
Registration and voting
Measuring the Effect of Social Innovation by Time Series
Statistics, Sun, and Stars
Meaning of Words
How accountants save money by sampling
Probability of rain

References and Bibliography

1. There is an annotated list of materials referenced at the end of the handout "Stimulation with Simulation," which contains most of the major current best teaching sources for activity-based probability and statistics teaching.

2. The bibliography at the end of 1981 Yearbook on Teaching Probability is excellent and thorough.

3. There is a bibliography at the end of the paper "The psychology of inference and the teaching of probability and statistics: Two sides of the same coin" which references most of the recent research in the teaching and learning of probability and statistics, from the points of view of mathematics education and psychology.

4. In addition, we should also mention these references:

   Statistics by Freeman, Pisani, and Purves (Norton Publishers).

   The journal TEACHING STATISTICS, published at the University of Sheffield in England.

   The conference report of the FIRST INTERNATIONAL CONFERENCE ON THE TEACHING OF STATISTICS (ICOTS I) held in Sheffield, summer of 1982.

   Statistics: Concepts and Controversies by David Moore (Freeman publishers).

   EXPLORATORY DATA ANALYSIS by J. Tukey (Addison-Wesley) and another by Velleman and Hoaglin (ABC's of EDA).

Introduction to
Stimulation by Simulation

The questions in Stimulation by Simulation are somewhat open ended in nature. In many cases it is necessary to first define clearly what the problem is, and then to adopt a mode of simulation which will enable you to "model" an experiment which may be too cumbersome to actually carry out.

Before doing any of these problems, always write down your best guess first, and only then carry out a simulation of the problem.

A simulation of a probability experiment involves:

a) Modeling an experiment by using some apparatus with known probabilities (i.e., coins, dice, spinners, random numbers).
b) Performing the experiment many times with your apparatus; thus pooling results of several small groups may help to get a "large enough" sample.
c) Gathering, organizing, and analyzing the data (some statistical skills).
d) Calculating experimental probabilities, or other experimental outcomes (i.e., frequencies) from the data.
e) Making inferences or drawing conclusions from the experimental outcomes, that is, looking back.

If you have the means to construct a theoretical probability model for any of these problems, then do so and calculate the theoretical probabilities and compare them to the "experimental" probabilities from your simulation, and to your guesses.

HAVE FUN!

Stimulation by Simulation

1. A cereal company has put plastic models of 5 Star Wars characters in boxes of SPACY-O's. The company puts one character in each box. How many boxes would you expect to have to buy in order to get all five characters? (Write down a guess first.)

2. Five friends are sitting around a table at a party discussing the merits of astrology. What is the probability that at least two of them have the same astrological sign? (Assume the signs are equal in duration.)

3. Bill Walton made 60% of his foul shots last season. He received a tip from Lucas over the summer, tried a new style, and made 9 out of 10 foul shots. Did the new technique help or did the 9 for 10 just occur by chance based on Walton's 60% average?

4. Some data were obtained from the records of the Bureau of Meteorology in Perth, Australia. A sunny day was defined as a day on which the officially recorded number of hours of sunshine was 70% or more of the maximum possible hours of sunshine for that day of the year. The months April to September in 1973 seemed to be typical months in terms of average number of hours of sunshine. A classification of each day in this period as "sunny" or "dull" (S or D) and recording of the events led to the following:

   D followed by D: 79 occasions
   D followed by S: 33 occasions
   S followed by D: 33 occasions
   S followed by S: 37 occasions

   Thus the proportion of dull days followed by dull days was 79/112 = .70, and the proportion of sunny days followed by sunny days was 37/70 = .53.

   a) What is the "average" length of runs of consecutive dull days?
   b) What is the "average" length of runs of consecutive sunny days?

5. Two points (numbers) between 0 and 1 are chosen at random. The points divide the interval from 0 to 1 into three line segments. What is the probability that the three segments will form a triangle? (Hint: The "points" could be represented by... .)

6. About one-half the babies born are boys. Would there be more days during the calendar year when at least 60% of the babies born were boys:
   i) in a small hospital
   ii) in a large hospital
   iii) makes no difference

7. Suppose you are playing a game in which you have a 50% chance of winning. Bet a dollar. Each time you lose, double your bet. Each time you win, go back to betting a dollar. Is this a good betting system?

8. Fast Al suggests to Slow Sam that they play the following game. They will each throw either one or two fingers. If Sam throws one and Al throws two, Sam wins $30. If Sam throws two and Al throws one, Sam wins $10. If they throw the same number of fingers, Al wins $20.
   a) If each player plays a 50-50 strategy between their two choices, who wins in the long-run?
   b) What is the best strategy for Slow Sam, i.e., how should he divide his choices?

9. From where he stands, one step toward a cliff would send the drunken man over the cliff. He takes random steps, either towards the cliff or away from it. At any step, the probability that he takes a step toward the cliff is 1/3, and the probability that he steps away from it is 2/3. What is his chance of escaping a fall?

10. 16 teams are entered in a double elimination softball tournament. What is the expected number of games a team will play:
    i) if the team has a 50% chance of winning each game?
    ii) if the team has a 70% chance of winning each game?

11. What is the expected number of games played in a world series if the probability that, say, National League wins is .5, .6, .7, .8? (Try at least 100 series - do many trials in each group).

12. Several years ago a certain beer company claimed that they "tried their famous High Life taste test on many a beer expert, and no one had yet correctly identified which beer came from a can, which from a bottle, and which from a tap." How likely is it that the beer company was telling the truth? Or, perhaps another way to put it, how many beer experts do you suppose the company really tested?

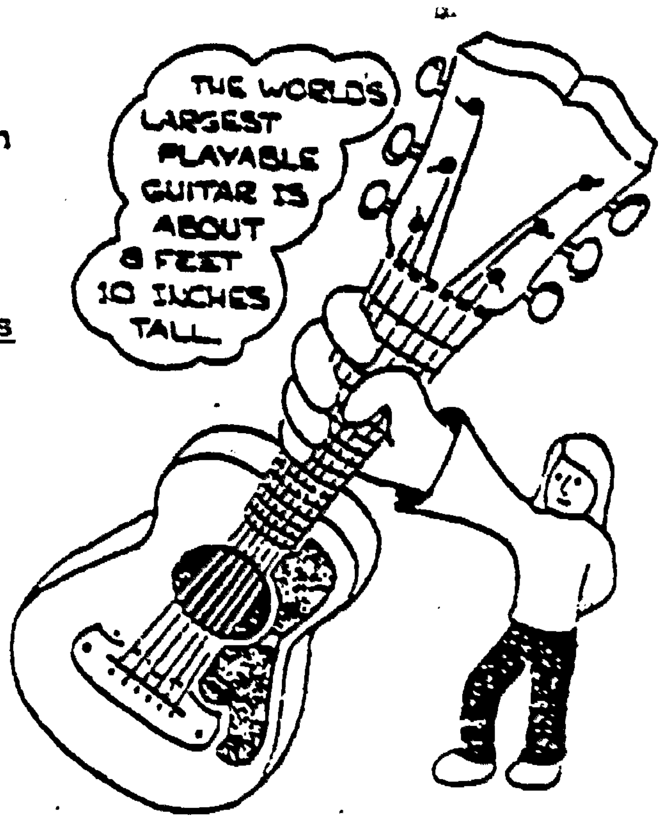
13. Suppose a man faces north and tosses a coin to decide whether to take one step north or one step south. The man does this for twenty-five tries.
    i) On the average, how far is he from the starting point?
    ii) On how many steps is he on the north side of his starting point? On the south side?
    iii) How often does he return to starting point?

14. Three towers are built of 8, 6, and 4 blocks. A tower is chosen at random and a block is removed.
    a) What is the expected number of blocks that will be removed before one of the towers has been "leveled". (i.e., no blocks left?)
    b) What is the probability the 4-block tower will be destroyed?
    c) Can you think of a "real world" situation for which this tower game is a model?

CAN YOU GUESSTIMATE?

I In the table write an estimate for each of the records.

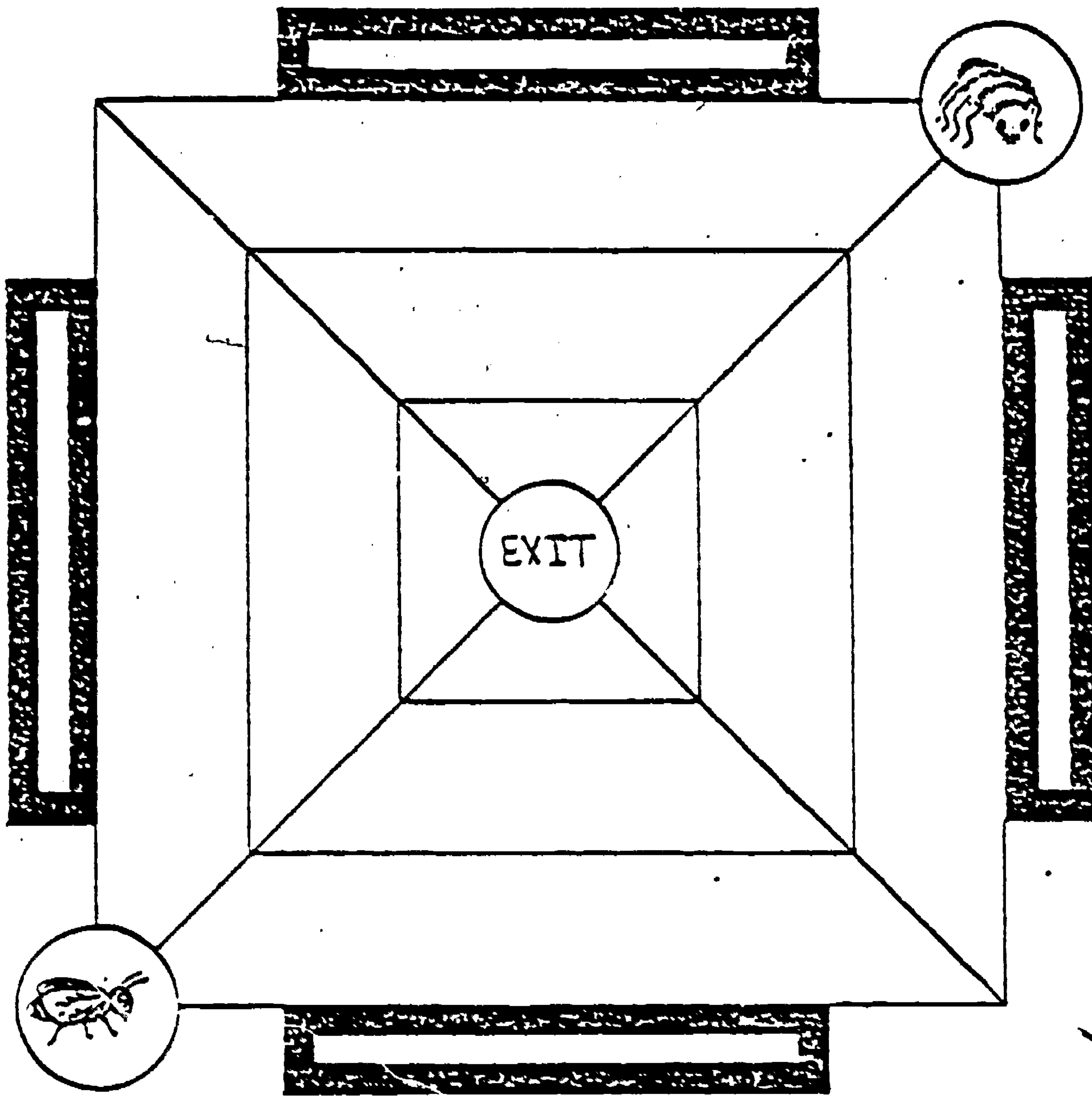
II Use the Guinness Book of World Records to check your estimates.



	Record	Estimate	Actual
1)	Longest time for continuous balancing on one foot	hr.	
2)	The length of the longest recorded fingernail	in.	
3)	The greatest number of years lived by a cat	yr.	
4)	The cost of the most expensive car	$	
5)	Height of the tallest monument	ft.	
6)	The greatest speed attained by a dog	mph	
7)	Most expensive pair of shoes	$	
8)	Diameter of the largest ball of string	ft.	
9)	Number of words in the world's longest poem		
10)	Wing span of the largest butterfly	in.	
11)	The greatest amount of snowfall in 24 hours	in.	
12)	Value in dollars of the most valuable stamp	$	
13)	The fastest speed attained by a race car	mph.	
14)	The height of the tallest woman	ft. in.	
15)	The length of the longest whale	ft.	

WILL THE SPIDER CATCH THE FLY?

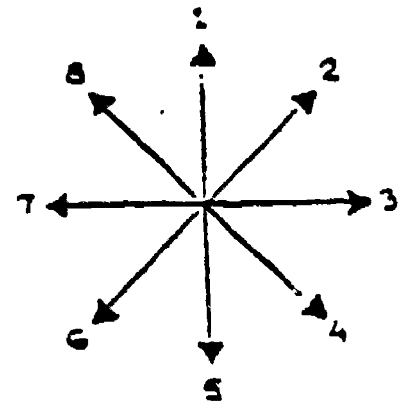
Materials needed: 2 markers
1 octahedral die



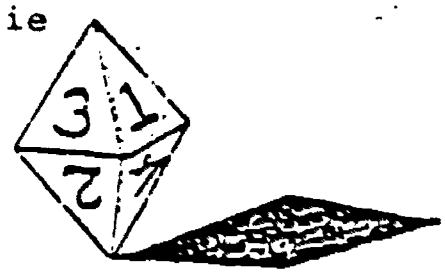
Idea from Al Shulte

Rules:

1) Fly rolls first.
2) The number face-up determines the direction of the move. See diagram to the right.
3) If a move is impossible, roll until a move can be made.
4) Play until
   a) the spider catches the fly or
   b) the fly escapes through the exit.
5) How many moves do you think it will take before the game is over? ________



Materials: 2 players each with an octahedral die



Rules:

1) Each player rolls the die.
2) Higher number is player A. Lower number is player B.
3) Player A rolls. Player B rolls. Divide A's number by B's number.
   a) If the 1st digit of the quotient is 1, 2, or 3, A wins.
   b) If the 1st digit of the quotiest is 4, 5, 6, 7, or 8, B wins.
   c) Example: A rolls 4; B rolls 2, 4 ÷ 2 = 2 A wins
      A rolls 3; B rolls 6, 3 ÷ 6 = .6 B wins
      A rolls 5; B rolls 8, 5 ÷ 8 = .625 B wins
      A rolls 5, B rolls 3, 5 ÷ 3 = 1.666... A wins.

4. Is this a "fair" game? Do both players have the same chance of winning? ________

5. Play the game 30 times. Record in the table below. You can do the division on a calculator to speed up the game.

A's roll															
B's roll															
Quotient															
Winner															

A's roll															
B's roll															
Quotient															
Winner															

RANDOM NUMBERS VIA A CALCULATOR

STEP 1

ENTER
A) DECIMAL POINT
B) THEN ANY SIX DIGITS
C) THEN ANY ODD DIGIT EXCEPT 5

STEP 2

MULTIPLY BY 147

STEP 3

SUBTRACT THE WHOLE NUMBER PART

STEP 4

FOR A 1-DIGIT NUMBER COPY THE FIRST DIGIT AFTER THE DECIMAL POINT.
FOR A 2-DIGIT NUMBER COPY THE FIRST TWO DIGITS AFTER THE DECIMAL POINT.
FOR A 3-DIGIT NUMBER ...

WITHOUT CLEARING THE CALCULATOR, REPEAT STEPS 2 THROUGH 4 TO GET AS MANY RANDOM NUMBERS AS YOU WISH.

EXAMPLE

STEP 1: .3186527

STEP 2: .3186527 X 147 = 46.841946

STEP 3: 46.841946 - 46 = .841946

STEP 4: FOR A 2 DIGIT NUMBER WRITE DOWN 84.

STEP 2: .841946 X 147 = 123.76606

STEP 3: 123.76606 - 123 = .76606

STEP 4: WRITE DOWN 76.

STEP 2: .76606 X 147 = 112.61082

Table 2. 2500 random digits

	01 02 03 04 05	06 07 08 09 10	11 12 13 14 15	16 17 18 19 20	21 22 23 24 25
01	61 44 34 03 09	03 64 20 54 24	65 69 66 39 80	13 97 76 73 34	41 17 26 81 06
02	85 19 76 44 59	08 60 20 66 68	42 99 28 71 47	73 73 97 24 18	38 25 89 37 20
03	41 17 95 60 40	12 77 51 80 36	46 07 70 39 27	56 65 18 08 70	10 98 45 88 44
04	05 75 79 35 85	60 89 12 66 30	41 25 62 64 54	32 35 64 42 96	16 66 34 64 80
05	36 28 97 63 85	39 85 19 76 44	90 21 61 91 82	77 24 27 32 37	47 98 96 59 75
06	73 56 63 38 40	16 66 09 76 20	08 26 63 16 38	14 16 20 79 87	74 54 96 70 02
07	94 78 81 49 27	50 50 73 46 27	27 04 75 38 83	97 69 20 30 53	19 54 35 62 28
08	26 15 00 92 41	16 37 97 99 37	90 76 87 87 44	34 76 45 94 83	96 07 46 21 97
09	38 37 11 05 75	43 72 07 36 66	43 43 91 05 71	31 59 50 95 94	37 64 64 61 12
10	49 83 04 05 41	49 73 07 47 93	81 28 16 10 73	87 08 07 10 25	01 38 98 14 19
11	85 40 88 30 95	50 02 66 99 44	23 38 99 04 73	69 67 52 23 61	84 89 18 92 79
12	07 55 13 99 88	48 59 19 75 51	96 24 56 41 26	43 14 85 11 47	07 46 03 44 18
13	82 67 78 13 04	83 85 53 97 36	11 63 50 67 81	97 08 14 96 86	82 89 66 78 07
14	83 92 56 64 46	18 94 30 33 99	77 71 38 98 19	31 52 23 35 38	38 55 63 29 61
15	19 60 98 77 85	31 71 71 73 23	08 39 54 26 54	28 61 22 55 59	49 08 28 22 12
16	90 48 51 41 65	32 99 72 09 94	28 43 36 12 21	34 20 08 96 60	96 22 72 84 34
17	90 92 84 99 45	45 79 20 51 76	67 05 60 68 84	10 02 27 22 49	55 93 13 41 34
18	61 28 77 99 36	11 00 00 83 58	83 67 15 23 69	59 60 02 12 94	56 25 84 14 93
19	41 25 77 73 71	53 30 05 60 05	69 30 20 59 30	54 79 23 59 75	78 30 07 29 36
20	58 05 97 70 59	90 92 27 57 38	39 87 40 42 64	44 30 94 33 52	61 53 27 17 60
21	99 60 50 50 60	12 48 08 01 88	60 51 73 38 54	47 41 67 43 84	21 35 12 90 43
22	88 61 29 18 05	31 29 56 94 33	58 88 68 42 10	08 11 15 96 06	15 51 43 39 96
23	19 28 81 63 24	30 96 40 11 59	36 16 01 02 60	36 16 26 93 53	41 09 50 85 01
24	53 61 62 34 47	04 37 74 97 09	77 36 92 80 45	99 26 28 24 24	54 63 43 63 54
25	40 03 44 30 11	42 25 70 19 79	90 12 36 16 80	28 70 24 86 07	76 17 01 50 80
26	65 15 18 13 54	05 13 69 91 51	84 57 52 89 88	12 52 03 39 71	19 48 20 94 16
27	95 79 58 84 86	00 04 73 69 94	89 12 93 84 29	72 62 79 66 98	65 17 54 69 56
28	75 26 86 16 42	65 03 22 43 68	87 68 70 09 18	92 85 94 60 32	97 44 95 82 72
29	92 45 48 29 84	56 60 50 64 07	71 46 35 31 52	21 80 61 25 30	31 99 58 07 04
30	43 00 97 26 90	99 85 55 75 16	09 55 34 16 16	94 32 12 12 07	32 90 97 62 47
31	14 99 59 97 84	18 40 71 98 04	89 24 19 23 56	06 01 68 65 28	23 90 28 10 90
32	79 93 51 89 07	25 25 29 18 02	50 48 21 47 74	61 37 03 51 60	87 97 63 86 43
33	61 37 36 14 84	94 14 96 55 57	05 34 47 88 62	57 73 75 02 34	49 18 83 92 43
34	71 00 51 72 62	59 18 87 82 84	74 04 46 24 66	39 82 30 87 75	41 10 53 02 29
35	76 21 40 24 19	56 19 89 13 48	27 53 41 07 14	28 62 58 67 84	53 16 26 16 97
36	26 97 03 03 30	88 39 46 67 21	17 83 46 74 11	35 54 29 36 86	30 32 06 47 37
37	58 54 81 74 22	32 45 26 40 88	30 91 66 86 52	71 42 99 54 75	12 94 11 09 83
38	45 33 94 97 70	96 27 03 89 63	37 57 46 16 18	78 55 78 07 98	03 46 57 47 39
39	27 26 48 62 10	83 63 45 30 92	48 32 96 67 26	95 90 65 50 46	09 95 58 67 29
40	57 74 50 98 61	50 30 38 41 58	86 28 79 50 71	48 30 58 93 23	70 76 72 42 06
41	68 78 34 95 35	91 63 55 60 22	19 10 77 88 59	11 36 40 56 55	56 29 76 58 93
42	97 63 57 69 62	65 00 51 67 52	21 53 52 16 86	73 67 24 16 68	09 05 74 93 63
43	20 32 35 52 41	47 17 53 83 72	45 20 28 25 04	21 94 00 18 55	26 80 19 50 20
44	53 90 99 23 17	76 44 15 99 65	91 04 22 64 00	39 80 65 21 47	68 75 28 48 16
45	36 42 17 95 78	02 29 66 50 33	65 61 43 77 29	93 34 62 39 42	36 07 61 92 07
46	40 09 18 94 06	62 89 97 10 02	58 63 02 91 44	79 03 35 47 69	14 11 42 33 99
47	33 19 98 40 42	13 73 63 72 59	26 06 08 92 65	63 08 82 45 85	14 45 81 65 21
48	69 49 02 58 44	45 45 19 69 33	51 68 97 99 05	77 54 22 70 97	59 06 64 21 68
49	17 49 43 65 45	04 95 82 76 31	85 53 15 21 70	59 17 27 54 67	07 76 13 95 00
50	43 13 78 30 55	30 80 88 19 13	13 89 11 00 60	41 86 23 07 60	22 77 93 30 83

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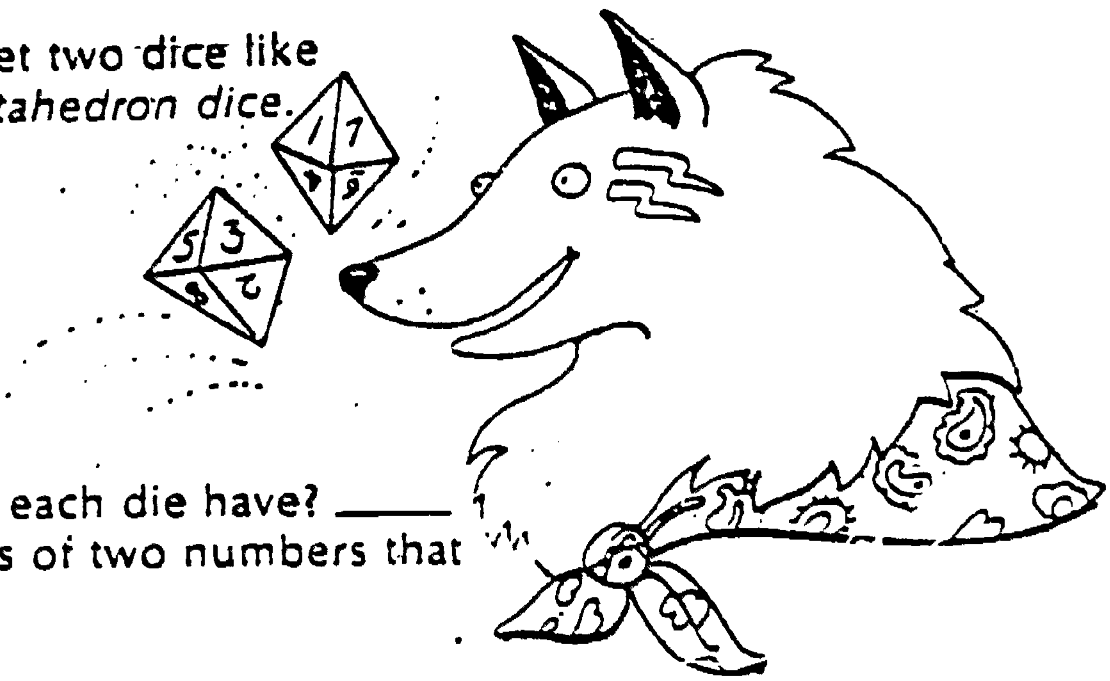
DIFFERENT DICE

Not all dice are cubes. Get two dice like these. They are called *octahedron dice.*



1. How many faces does each die have? _____
2. Make a list of the pairs of two numbers that can land up.

1,1 2,1 ___ ___ ___ ___ ___ ___

1,2 ___ ___ ___ ___ ___ ___ ___

1,3 ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

3. How many pairs are there altogether? _____
4. Write the sums for each pair in your list, below.
5. Which sum occurs most often? _____
6. a) How many times does a sum of 7 occur? _____
   b) Of 12? _____
   c) Of 1? _____
7. When you roll the dice, what is:
   a) Pr(sum of 7)? _____
   b) Pr(sum of 12)? _____
   c) Pr(sum of 5)? _____

Extension. Roll two octahedron dice 64 times. Keep a tally of the sums. Find you *experimental probabilities* of rolling sums of 7, 12, and 5. Compare these with the theoretical probabilities you found before.

MATERIALS: 2 octahedron dice, pencil.

GINNY'S GAME

Ginny has invented a game. She suggests to Tony, "Let's toss pennies. I'll toss mine first. If it comes up heads, I win. If it comes up tails, you toss. If your penny comes up heads, I win; if it's tails, you win."

Tony says, "That's not fair! You'll win *twice* as often."

Ginny says, "OK then! I get *one* point if I win, and you get *two* points if you win."

Tony says, "That's much better."

WHAT DO YOU THINK?

1. Is the game fair now? _____ Explain your answer. ______________________________

TRY IT OUT.

With a partner, try the game. One of you be Ginny and the other, Tony. Take 20 turns. Record your results in the table below.

2. Who won? __________
3. How many points did you each have? __________
4. Do you think the game is fair now? _____

Turn	Points Ginny	Points Tony	Turn	Points Ginny	Points Tony
1			11		
2			12		
3			13		
4			14		
5			15		
6			16		
7			17		
8			18		
9			19		
10			20		
Total			Total		

MATERIALS: penny, pencil, partner.

FINDING PROBABILITIES FROM EXPERIMENTS

Sometimes you can't figure out or even guess what a probability will be until you do an experiment. For example, suppose you toss a thumbtack. There are two ways it can land:

Point Down Point Up

There is no reason to suspect that the probability of "Point Up" is the same as the probability of "Point Down". In fact, for most thumbtacks, it isn't.

EXPERIMENT

1.. Put 10 identical thumbtacks into a paper cup.
Shake the cup and turn it face down on your desk.
Count the number of thumbtacks landing point up.
Do the experiment 20 times. Record your results on another paper.

What is the total number of tosses you have made? ____

2. What is your experimental probability of a thumbtack landing point up? ____
3. Express this figure as a two-place decimal. ____
4. Combine your results with those of other class members. Find the average probability of a tack landing point up for the whole class. ____

Extension. How do you think your results would be affected if the point of the tack were very long, for example, 1 meter long? If it were very short, for example, 1 millimeter long?

MATERIALS: paper cup, 10 identical thumbtacks, pencil, paper.

THE MARTINGALE SYSTEM

The Martingale system is an old and commonly used gambling system. It gives the gambler a good chance to win a small amount, balanced against a small chance of taking a large loss.

THE SYSTEM

Bet a certain amount. To make it easy, assume you bet $1.00.

Each time you lose, double your previous bet.

Each time you win, go back to the original bet, $1.00.

EXAMPLE

Toss a penny. Let a head(H)represent a win and a tail(T)represent a loss.

Toss	Amount Bet	Outcome	Amount Won on This Toss	Total Winnings to Date
1	$1.00	H	$1.00	$1.00
2	1.00	T	-1.00	0.00
3	2.00	H	2.00	2.00
4	1.00	T	-1.00	1.00

EXPERIMENT

Toss a penny 20 times to try out the system. Let a head (H) represent a win and a tail (T) represent a loss.

1. Record your results in a table like the one in the example.
2. Did you win or lose money after 20 tosses? _____ How much? _____
3. Do you think that this betting system is a good one? _____

Why or why not? ________________________________

Extension. Combine your results with those of a group or the whole class. Did the group win or lose as a whole? How much?

MATERIALS: penny, pencil, paper.

WHEN WERE THEY BORN?

1. Do you think that it is very likely that two people in your class have exactly the same birthday (month and day)? ____
2. Take a survey of your class to find out. Make a class graph that shows the 12 months and the students' names, like the one below.

			George Apr. 5								
Bob Jan. 2	Judy Feb. 15		Sue April 26		May June 3	Lyn July 13					
Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.

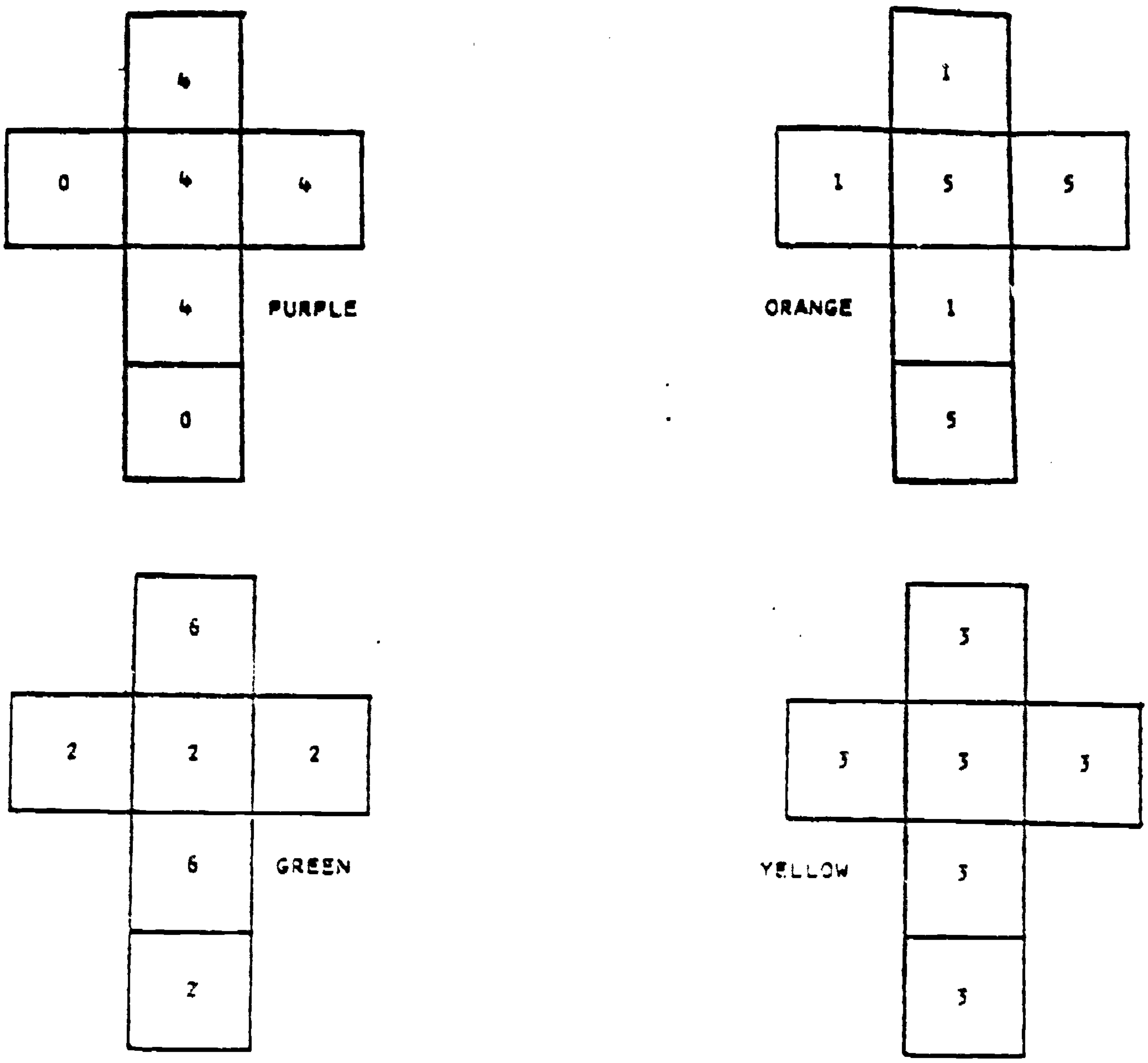
3. Were there any double birthdays? ____ How many? ____
4. Are you surprised by the results? ____
Why or why not? ____
5. If you took a survey of another class, do you think there would be any double birthdays in that class? ____
6. Do you think that someone in that class would have the same birthday as one of the students in your class? ____

Ask your teacher to arrange for one or two students to take a birthday survey in another class and share the results with your class.

7. Were there any double birthdays in the other class? ____
8. Combine the results from the two classes. How many double birthdays are there now? ____ Do these results surprise you? ____

MATERIALS: pencil, crayons or markers, graph paper.

NON-TRANSITIVE DICE



Expected value:

P(Purple) $E = \frac{2}{3}(4) + \frac{1}{2}(0) = 2\frac{2}{3}$

O(Orange) $E = \frac{1}{2}(5) + \frac{1}{2}(1) = 3$

G(Green) $E = \frac{1}{2}(6) + \frac{1}{2}(2) = 4$

Y(Yellow) $E = 3$

therefore: $G > O = Y > P$

However, in actual play, probability of winning reflect

$$P > O = Y > G.$$

Therefore, the expected value doesn't predict the actual outcomes.

SOME PROBABILITY BRAIN TEASERS

1. The game of craps, played with two dice, is one of the most popular gambling games. The rules are these. Only totals of the two dice count. The "thrower" tosses the dice and wins at once if the first throw is a 7 or 11. The player loses at once if the first throw is a 2, 3, or 12. Any other sum on the first throw is called a "point". If the first toss is a "point", the player throws the dice repeatedly until either he wins by throwing his point again, or loses by throwing a 7. What is the probability that the "thrower" will win? (Note: If a point is tossed on the first try, no sum on dice makes any difference after that except a 7 or that point!)

2. A three man jury has two members each of whom independently has probability "p" of making the correct decision, and a third member who flips a coin for each decision. Majority rules. A one-person jury has probability "p" of making the correct decision. Which jury has the better probability of making the correct decision? (The case of the Flippant Juror!)

3. (The big-bad-triangle, or three's a crowd.) Three duelers, the Good, the Bad, and the Ugly, are to fight a three cornered pistol duel. All know that the Ugly's chance of hitting his target is .30, the Bad's is .50, and the Good (of course!) never misses. They are to fire at their choice of target in succession in the order Ugly, Good, Bad, cyclically until only one man is left unhit. A hit man loses further turns, and is no longer shot at. Ugly has the first shot. What should he do to have the best chance of survival? (Running away doesn't count!)

SOME RESOURCES AND REFERENCES FOR PROBABILITY AND STATISTICS

A brief description of each reference follows. The labels S,J, and E indicate whether the materials are most appropriate for Senior High, Junior High, or Elementary (although these are good for all levels in some respects).

How To Lie With Statistics, By Darrell Huff, W.W. Norton Publishing (required for the workshop). Excellent and easy to read essay on ways statistics can be used to mislead or portray inaccurate information, either deliberately or accidentally. A must book for every consumer. (S,J, and E)

What Are My Chances Books A and B, Creative Publications. Ready to copy probability activities for introducing probability for the first time. (Mostly J and E).

Statistics and Information Organization, Math Resources Project, Creative Publications. Activities, problems, and concepts in statistics, with some probability. Excellent "content for teachers" and "didactics" sections. Activity sheets are ready to copy and use in the classroom. Excellent! (S,J, and E - mostly aimed at J when written).

Statistics by Example, Addison-Wesley Innovative Series. A series of essays that introduce statistical concepts through "neat" applications. In four modules. The first two books are probably the most appropriate. (S and J).

Fifty Challenging Problems in Probability with Solutions, Addison-Wesley. A list of challenging probability problems - for you, and for any gifted and talented kids you may wish to challenge. (S and J, mostly S).

Statistics: A Guide to the Unknown, Holden-Day. A book of essays on uses and importance of statistics in Health, Medicine, Government, Opinion poles, Communications, Consumerism, Weather, etc. Really a fun book. A good source for starting kids on a project of their interest which might involve statistics. (Mostly S 'cause of reading level).

Take A Chance With Your Calculator, By Lennart Rade, Dilithium Press. A book on how to use programmable calculators to carry out probability simulations - complete with neat problems and program solutions. Looks like a lot of fun. (Mostly S).

Probability With Statistical Applications, Addison-Wesley by Mosteller and others. A good "bible" of probability and statistics concepts for you to have as a reference book.

PART II - HISTORICAL PROBLEMS IN MATHEMATICS

The twenty items test on the history of important mathematical problems was developed several years ago by Prof. Shmuel Avital.

The initial purpose of this test was to investigate mathematics teachers' knowledge of some problems in the history of mathematics. In this workshop, the test provided leading questions in the history of mathematics that were assigned to the participating teachers in groups of two or three. The participants were in unanimous agreement that these assignments were useful and valuable to them.

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IMPORTANT MILE (KM) STONES IN
THE DEVELOPMENT OF MATHEMATICS

By

Shmuel Avital, Prof. Emeritus
The Technion, Haifa, Israel

1. Some people say that the introduction of a special symbol for the number zero is one of the greatest inventions of humanity. Explain:

2. (a) What is an "algorithm"?
   (b) What does one use the euclidean algorithm for?

3. (a) What are transcendental numbers?
   (b) List at least three such numbers?
   (c) How many transcendental numbers are there?

4. The set of numbers of the form $p+q\sqrt{2}$ where p and q are integers or fractions has a structure similar to the structure of the rational numbers. Explain on what this similarity depends.

5. (a) What mathematical structure is based on the axioms of Peano?
   (b) How many such axioms are there? List at least two of them.

6. What is the fundamental theorem of arithmetic?

7. (a) What are Diophantine equations? Explain and give at least one example.
   (b) What is the origin of the name of these equations?

8. Mathematicians of ancient Greece were concerned about the existence of incommensurable segments.

   (a) Explain what such segments are.
   (b) What is the algebraic implication of this problem?

9. Consider the equation: $\alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_1 x + \alpha_0 = 0$ where all α's are integer coefficients.

   (a) Do such equations always have a solution which is a real number? (Explain and give one example to substantiate your statement.)
   (b) What does the fundamental theorem of algebra say about such equations?
   (c) Is it always possible to find a solution in the form of an expression in which only operations of addition, subtraction, multiplication, division and extraction of roots are carried out on the coefficients? (Qualify your statement.)

10. Consider the series $1 + (-1) + 1 + (-1) + \ldots$ where we sum these numbers up to infinity. Writing this series in the form $[1 + (-1)] + [1 + (-1)] + [1 + (-1)] + \ldots$ we obtain zero as the sum to infinity. However, writing this series as $1 + [(-1) + 1] + [(-1) + 1] + [(-1) + 1] + \ldots$ we obtain 1 as the sum to infinity. What is the true sum to infinity of this series?

11. There are problems in mathematics which can easily be formulated and which nevertheless haven't been solved yet, even though mathematicians have tried very hard to solve them. List at least one such problem. (Give as many details as you know.)

12. There are three problems which were formulated in ancient Greece which mathematicians tried to solve for more than 1500 years and which were eventually solved in the 19th century. These problems are often referred to as "three famous problems of antiquity". What are they, and what is the solution?

    (i) Problem:
    Solution:
    (ii) Problem:
    Solution:
    (iii) Problem:
    Solution:

13. You have studied in school how to inscribe in a circle some regular polygons (such as a square, an equilateral triangle and others) with the help of a ruler and compass only. Can you inscribe every regular polygon in a circle with the help of these tools only? (Give as many details as you can.)

14. There are special requirements which a system of axioms has to satisfy. What are these requirements? Qualify, stating which of these requirements are more important than the others

15. (a) What is non-euclidean geometry?
    (b) When was this geometry developed and by whom?

16. (a) There exist in the plane regular polygons with any number of sides. Are there in space regular solids with any number of faces?
    (b) Such solids are connected with the name of a famous person. What is it?

17. (a) In what way has there been established a basic connection between algebra and geometry?
    (b) When was this connection originated and by whom?

18. Euclidean geometry may be described as dealing in isometric transformations (that is, transformations in which distances between points remain invariant). Explain.

19. The problem of drawing a tangent to a given curve at a given point is closely connected with the problem of finding areas.
    (a) What is this connection?
    (b) When was it discovered and by whom?

20. (a) There are sets of different infinities. What does this mean? Can you explain this idea using the set of rational numbers and the set of real numbers?
    (b) When was this discovered and by whom?

By the following scheme the participants were arbitrarily assigned to the problems of the Avital Test. In general, they felt it was an excellent way to get involved with each other.

Prob.	Teacher		Prob.	Teacher	
1	1,2,3	Dec. 3	11	2,15	Dec. 10
2	4,5,6		12	3,16	
3	7,8,9	Dec. 4	13	4,17	Dec. 11
4	10,11.12		14	5,18	
5	13,14,15	Dec. 5	15	6,19	Dec. 12
6	16,17,18		16	7,20	
7	19,20,21	Dec. 6	17	8,21	Dec. 13
8	22,23,25		18	9,22	
9	24,13	Dec. 7	19	10,23	Dec. 14
10	1,14		20	11,25	

Please prepare your written answer so it will be useful to a middle grade mathematics teacher. We will type and publish your answers after the workshop.

Answers should discuss the significance of the ideas in the mathematics curriculum including examples and references. Probably 1 to 6 pages is appropriate.

The following article from Mathematics Teacher was provided to assist the teachers with the proncunciation of famous mathematicians.

MATH'E MA TISH'ANS

By JAMES METZ
and JOSEPH CANELLA, C.S.V.

Griffin High School
Springfield, IL 62702

The following names and their pronunciations are offered as an aid to classroom teachers who may have only read the names and not heard them pronounced. Only a few names are listed and you may wish to add others. Most school libraries contain Webster's Biographical Dictionary (Springfield, Mass.: G. & C. Merriam Co., 1974), which was a primary source for the list. Two other helpful sources are Chamber's Biographical Dictionary, edited by J. O. Thorne (New York: St. Martin's Press, 1962), and the McGraw-Hill Encyclopedia of World Biography (New York: McGraw-Hill, 1973). The foreign language teachers in your school may also be helpful.

Abel (ay'bl), Niels Henrik (1802–29). Norwegian mathematician, concerned mainly with the theory of elliptical functions.

Bolyai (bo'lyay), János (1802–60). Hungarian mathematician, one of the founders of non-Euclidean geometry.

Cauchy (kō-shee), Augustin Louis, Baron (1789–1857). French mathematician who, along with Bolzano, developed the theory of functions.

Cavalieri (ka-val-yayr'ee), Francesco Bonaventura (1598–1647). Italian mathematician whose "method of indivisibles" began a new era in geometry and paved the way for the introduction of integral calculus.

DeMoivre (de-mwah'vr), Abraham (1667–1754). French mathematician who lived in England; helped to decide famous contest between Newton and Leibniz for the merit of the invention of fluxions, precursor of calculus.

Desargues (day-zarg), Gérard. (1593–1662). French mathematician, one of the founders of modern geometry.

Descartes (day-kart), René (1596–1650). French rationalist philosopher and mathematician whose greatest achievement was the discovery and formulation of coordinate geometry.

Diophantus (dī-ō-fan'tus) (3rd century A.D.). Greek mathematician, author of the earliest extant treatise on algebra.

Dirichlet (dee-re-klay'), Peter Gustav Lejeune (1805–1859). German mathematician, known especially for his work on theory of numbers, which he furthered by application of higher analysis, as well as for his work on definite integrals.

Eudoxus (yoo-dok'sus) (408-353 B.C.). Greek geometer and astronomer, studied in Egypt.

Euler (oy'ler), Leonard (1707–83). Swiss mathematician and physicist; one of the founders of the science of pure mathematics.

Fermat (fer-mah), Pierre de (1601–65). French mathematician who made many discoveries in the properties of numbers, probabilities, and geometry.

Fourier (foor-yay), Jean Baptiste Joseph, Baron de (1768–1830). French mathematician who discovered, in connection with work on heat flow, the theorem that bears his name, which states that any function of a variable can be expanded in a series of sines of multiples of the variable.

Galois (gah-lwah), Evariste (1811–32). French mathematician, noted for his group substitutions and theory of functions.

Harriot (har'i-ut), Thomas (1560–1621). English mathematician. His posthumously published work contains inventions that give algebra its modern form.

Lobachevski (lo-ba-chef 'ski), Nikolai (1793–1856). Russian mathematician, founder of non-Euclidean geometry.

Lebesque (le-bayg), Henri Léon (1875–1941). French mathematician famous for his researches in mathematical physics.

L'Hopital (lō-pee-tal), Guillaume François Antoine de (1661–1704). French geometer.

Leibniz (līb'nits), Gottfried Wilhelm (1646–1716). German philosopher and mathematician, pioneer of modern symbolic logic.

Poisson (pwa-sō), Siméon Denis (1781–1840). French mathematician famous for his researches in mathematical physics.

Riemann (ree'mahn), Georg Friedrich Bernhard (1826–66). German mathematician whose early work was an outstanding contribution to the theory of functions. He is best remembered for his development of the conceptions of Bolyai and Lobachevski, which resulted in a full-fledged non-Euclidean geometry.

Saccheri (sak-ayr'ee), Girolamo (1667–1733). Italian mathematician, the first to attempt a proof of Euclid's parallel postulate.

Thales (thay'leez) (640?–580 B.C.). Greek natural philosopher, said to have invented geometry by refining techniques of land surveying and astronomy by deductive reasoning.

Viete (vyayt), François (1540–1603). French mathematician, author of the earliest work on symbolic algebra; also wrote on trigonometry and geometry and obtained the value of π as an infinite product.

Question 1:

Some people say that the introduction of a special symbol for the number zero is one of the greatest inventions of humanity. Explain:

History of Zero

Definition of Zero: Identity element for addition $a + 0 = a$, e.g., $5 + 0 = a$. Multiplication property of zero $a \cdot 0 = 0$, e.g., $5 \cdot 0 = 0$. In subtraction, $a - 0 = a$ but in division, $a \div 0 =$ (undefined), e.g., $5 \div 0$ has no answer, so we say that division by zero is undefined.

Of the ten digits we use today, zero is the most recent discovery.

Earliest preserved samples of our present number system were found in India dating from about 250 B.C. but contained no zeros.

Zero must have been introduced into India sometime before 800 A.D. because it is found in Al-Khowarizmi about 825 A.D.

The word zero comes from the Arabic and Hindu words SIFR and SUNYA meaning void or empty.

Zero is important because it led to a place value system of writing numbers.

Example: To show the importance of place value put: 3 5 2 on board. Ask, what is the number? Answer: 352. Now, put zeros: 30,502.

The influence of the Roman empire dominated the Western cultur while Rome remained strong. Therefore, Roman numerals were used extensively in the west.

As the Roman empire declined, the Hindu-Arabic numerals (which we use today) began to be used.

Example: Have students attempt to add, subtract, multiply and divide with Roman numerals.

XIV	14	XIV	14	XIV	14
+ IX	+ 9	− IX	− 9	x IX	x 9

IX)XIV   9)14

Without the zero one must memorize a multitude of symbols to write numbers.

Example: Start out writing the numbers

1, 2, 3, 4, 5, 6, 7, 8, 9

what comes next? (No zero). Try to continue without using a zero.

When zero was introduced into the west, many debated whether it was a digit or not a digit. It only had meaning at certain times.

Example: 03 -> 0 has no meaning

30 -> gives the number 30

0.3 -> 0 has no meaning

.02 -> makes 2 smaller than .2

02.0 -> first zero has no meaning, second zero may be significant or may mean nothing.

No other digit behaves in this manner. Try, $3\overline{)12}$; after they are done some will have an answer: $\begin{array}{r}4\\3\overline{)12}\end{array}$. Others will have: $\begin{array}{r}04\\3\overline{)12}\end{array}$. This can be carried further using the examples:

1. $3\overline{)1.2}$ 3. $3\overline{).12}$ 3. $3\overline{)12.0}$ 4. $30\overline{)1.20}$

20th Century Quote: Just as the math teacher wanted to be an expert, the Professor a teacher, and the mouse an elephant, the zero puts on airs and pretended to be a digit.

Trivial Math Pursuit
1986 Ed.
Dr. Neppal Dlareqztif

Submitted by: Susan Apps
Dennis Cumpston
Maureen Denver

Euclid, Pythagoras, and all the rest
Will really put you to the test.
Match their wits and you will see
How you will fare in history!

Question 2:

(a) What is an "algorithm"?

(b) What does one use the Euclidean algorithm for?

(a) An algorithm is a systematic procedure for solving a problem, particularly a method that continually repeats some basic process. The development of an algorithm is an important aspect in problem solving in that it serves to create a future strategy for attacking problems of a nature similar to that of the original problem. In this age of technology, middle school students might benefit from the explanation of what an algorithm is and the demonstration of one to understand the methods and procedures a computer goes through in order to solve a problem.

The Euclidean algorithm is an example one might use to demonstrate to middle school students a process to find the greatest common factor of any two given positive integers. The flowchart given in Figure 1 may be helpful in the demonstration.

(b) The Euclidean algorithm is an understandable process to show students the meaning of algorithms. It is also a useful procedure in reducing fractions to lowest terms and/or finding a common denominator to add and subtract fractions. The algorithm is a simple approach. It is easily taught because it utilizes division, a process students probably already know and illustrates the need for accuracy in calculating remainders. Even students who have difficulty with the computational process of division can use the Euclidean algorithm with the aid of a calculator.

The Euclidean algorithm gives students a historical perspective of mathematics. Students may see the validity and the permanence of procedures developed hundreds of years ago. They may also observe the similarity between the Euclidean algorithm and more contemporary methods of factoring. (In algebra, the Euclidean algorithm is also an effective way of factoring polynomials.) Consider using the Euclidean algorithm the next time the concept of greatest common factor is taught.

Example:

GCF (18,24) = 2 x 3 LCM (18,24) = 2 x 3 x 3 x 4

```
      3    4
3 |  9   12
2 | 18   24
```

Pattern looks like an "L".

Student places both given numbers in a division box and divides by any common factor and continues to divide the quotients until the remaining quotients are relatively prime.

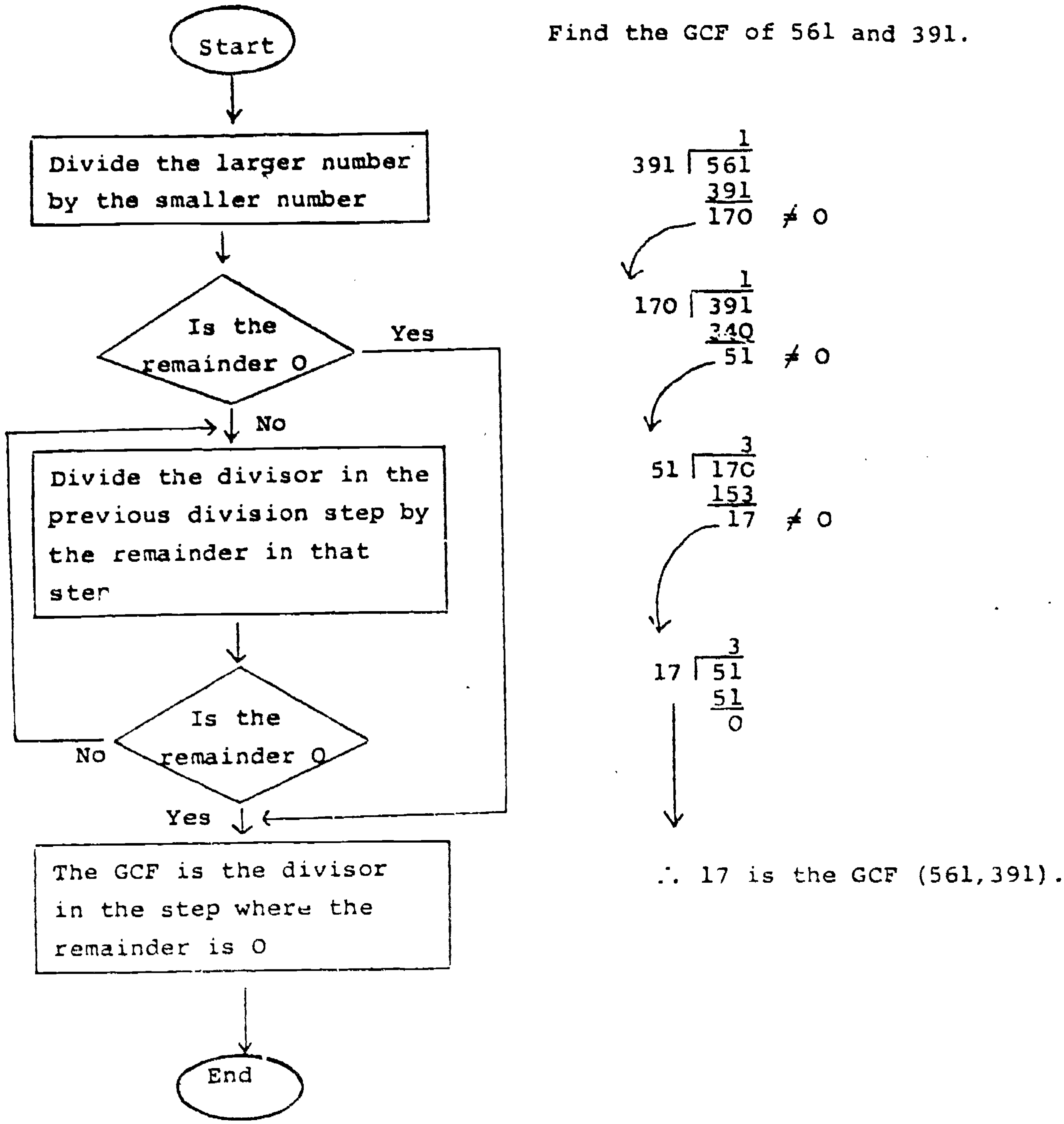


Figure 1. The Euclidean Algorithm

Reference:

Rogers, Hartly, Jr., "The Euclidean Algorithm As a Means of Simplifying Fractions", The Arithmetic Teacher, Volume 17, Number 8, Dec., 1970, pp.657-662.

Submitted by: Loretta Dixon
Gloria Fairchild
Robert Guzley

Question 3:

(a) What are transcendental numbers?
(b) List at least three such numbers.
(c) How many transcendental numbers are there?

Any number which is a root of an equation of the form

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

where the coefficients are rational numbers, $a_0 \neq 0$, is called an ALGEBRAIC number. For example, the irrational number $\sqrt{2}$ is algebraic because it is a root of the equation

$$x^2 - 2 = 0.$$

A number which is not algebraic is called TRANSCENDENTAL, for, as Euler said, "They transcend the power of algebraic methods." It was in 1844 that the existence of transcendental numbers was first proved, by Liouville. He showed that the sum

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^6} + \frac{1}{n^{24}} + \frac{1}{n^{120}} + \frac{1}{n^{6!}} + \quad (n > 1)$$

is transcendental. Although it was first thought that transcendental numbers were relatively scarce, it has since been proved that of all the numbers in mathematics, the transcendental ones are the most common.

Two famous transcendental numbers are e, the base of the natural logarithms (proved to be transcendental by Hermite in 1873) and π, the ratio of the circumference to the diameter of a circle (proved to be transcendental by Lindemann in 1882 using Hermite's method). In 1900, Hilbert proposed the problem to prove that $2^{\sqrt{2}}$ is transcendental, or even that it was irrational. In 1934, Gelfond proved that all numbers of the form a^b, where a is algebraic ($a \neq 0$ or $a \neq 1$) and b is irrational or imaginary are transcendental. "Transcendental number theory" is a highly active field of present day mathematical research. The number e^π is known to be transcendental, but the numbers e^e and π^π have not been definitely established.

The topic of transcendental numbers is not appropriate for most middle school classes. In an 8th grade algebra class an investigation of transcendental numbers seems appropriate as an enrichment or challenge activity. The following diagram may help students visualize the relationship of transcendental numbers to other real numbers.

REAL NUMBERS

← Rational →	← Irrational →	
← Algebraic →		← Transcendental →
$\frac{1}{2}$	$\sqrt{2}$	π

References:

1) Courant and Robbins, What is Mathematics?, Oxford University Press, 1941.

2) Eves, An Introduction to the History of Mathematics, Holt, Rinehart and Winston, 1964.

3) NCTM, Student Math Notes, Nov., 1983.

4) Newman, World of Mathematics, Simon and Schuster, 1956.

5) Rising and Wiesman, Mathematics in the Secondary Classroom: Selected Readings, Thomas Crowell, 1972, pp.100-103.

Submitted by: Dave Hallas
Joan Hall
Jack Halferty

Transcendental numbers can't be beat
π and e will ne'er repeat.
On and on and on they go
Where they stop we'll never know.

Question 4:

The set of numbers of the form $p + q\sqrt{2}$ where p and q are integers or fractions has a structure similar to the structure of the rational numbers. Explain on what this similarity depends.

Answer: The set of numbers in this form has the structure of a field. To be a field, a set of numbers must meet the following conditions: closure under addition and multiplication with additive and multiplicative inverses which also exist in the set. The following examples illustrate that these conditions are met in the set of numbers in the form $p + q\sqrt{2}$ where p and q are integers or fractions.

Example 1: If b, c, m and n are integers or fractions

$$b + c\sqrt{2} + m + n\sqrt{2} = b + m + c\sqrt{2} + n\sqrt{2}$$
$$= (b + m) + (c + n)\sqrt{2}$$

The expressions $(b + m)$ and $(c + n)$ must be fractions or integers since fractions or integers are in the set of rational numbers.

If $b = \frac{3}{8}$, $c = 5$, $m = \frac{1}{2}$ and $n = 6$

$$\frac{3}{8} + 5\sqrt{2} + \frac{1}{2} + 6\sqrt{2} = (\frac{3}{8} + \frac{1}{2}) + (5 + 6)\sqrt{2}$$
$$= \frac{7}{8} + 11\sqrt{2}.$$

Thus this result is in the specified form.

Example 2: If a, b, x and y are rational numbers

$$(a + b\sqrt{2})(x + y\sqrt{2}) = ax + ay\sqrt{2} + bx\sqrt{2} + 2by$$
$$= (ax + 2by) + (ay + bx)\sqrt{2}.$$

Since a, b, x and y are rational numbers, then the quantities $(ax + 2by)$ and $(ay + bx)$ must also be rational numbers.

If $a = -3$, $b = \frac{2}{5}$, $x = \frac{1}{2}$ and $y = 4$

$$(-3 + \frac{2}{5}\sqrt{2})(\frac{1}{2} + 4\sqrt{2}) = (-\frac{3}{2} + \frac{16}{5}) + (-12 + \frac{1}{5})\sqrt{2}$$
$$= \frac{17}{10} - \frac{59}{5}\sqrt{2}.$$

Therefore the result can also be written in the necessary form.

Example 3: If e and d are rational numbers, the additive inverse of $e + d\sqrt{2}$ would be $-(e + d\sqrt{2})$ and by simplification $-(e + d\sqrt{2}) = -e - d\sqrt{2}$. Since e and d are rational numbers, then $-e$ and $-d$ must be rational numbers. If $e = 7$ and $d = -\frac{1}{8}$, $-(7 - \frac{1}{8}\sqrt{2}) = -7 + \frac{1}{8}\sqrt{2}$. Thus the result can be written in the specified form.

Example 4: If r and s are rational numbers and r or $s \neq 0$, the multiplicative inverse of $r + s\sqrt{2}$ would be $\frac{1}{r + s\sqrt{2}}$. By simplification,

$$\frac{1}{r + s\sqrt{2}} = \frac{1}{r + s\sqrt{2}} \cdot \frac{r - s\sqrt{2}}{r - s\sqrt{2}}$$
$$= \frac{r - s\sqrt{2}}{r^2 - 2s^2}$$
$$= \frac{r}{r^2 - 2s^2} - \frac{s\sqrt{2}}{r^2 - 2s^2}$$
$$= \frac{r}{r^2 - 2s^2} + \left(\frac{-s}{r^2 - 2s^2}\right)\sqrt{2}$$

The quantities $\frac{r}{r^2 - 2s^2}$ and $\frac{-s}{r^2 - 2s^2}$ would be rational numbers since r and s are rational numbers.
If $r = 2$ and $s = -1$

$$\frac{1}{2 - 1\sqrt{2}} = \frac{2}{4 - 2} + \frac{1}{4 - 2}\sqrt{2} = 1 + \frac{1}{2}\sqrt{2}.$$

Therefore the result can be written in the necessary form.

Not only do the numbers of the form $p + q\sqrt{2}$ where p and q are rational numbers represent a field, but when $q = 0$, rational numbers are formed and when $p = 0$, irrational numbers are formed. In fact, it can be shown that numbers of the form $p + q\sqrt{k}$ where k is any positive rational number whose square root is irrational would have the structure of a field that would be an extension of the field of rational numbers. (Figure 1).

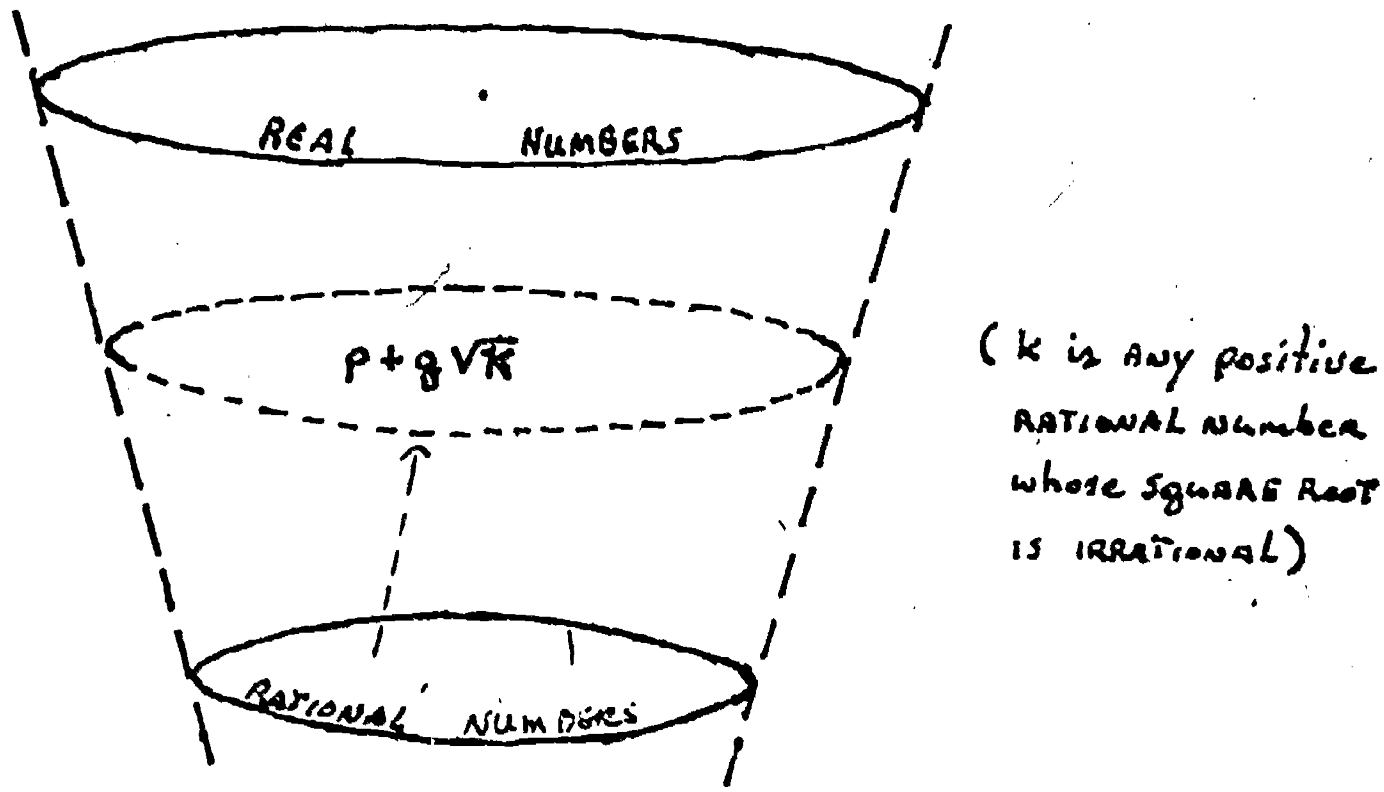


Figure 1. Extensions of the field of Rational Numbers

As for applications of this topic in the middle school area, this topic may be useful when irrational numbers and their properties are introduced. However, for most students this topic is inappropriate in the middle school and would best be left for Algebra II.

References

Courant, Richard and Herbert Robbins, What is Mathematics? London, Oxford University Press, 1941.

Stewart, San and David Tall, Algebraic Number Theory, London: Chapman and Hall, 1979.

Submitted by: Warren C. Hastings
Nancy Hemingway
Sally R. Hicks

Question 5:

(a) What mathematical structure is based on the axiom of Peano?

(b) How many such axioms are there? List at least two of them.

(a) It is possible to set up a 1-1 correspondence between the elements of the set of "numbers" that Peano's axioms describe and the set of natural numbers, such that the "1" of Peano's first axiom is paried with the natural number 1 and every successor of Peano's numbers are paired with the natural number n + 1.

This concept has been extended to finally include and set the structure for the set of real numbers.

Peano's axioms → natural numbers → positive rational numbers → positive real numbers → the real number system.

The real number system may be considered as a construction based on the Peano axioms.

(b) There are four axioms of Peano which deal with a set of elements, called $\mathcal{N}$ and a relation called s. "s" is defined as the relation "is the successor of". Therefore X s Y is read X "is the successor of" Y.

<u>Axiom 1</u> states that the set $\mathcal{N}$ contains an element, 1, such that there is no element for which 1 is a successor.

<u>Axiom 2</u> states that given an element X in $\mathcal{N}$ there is one unique element Y in the set which is it's successor, i.e., Y s X.

<u>Axiom 3</u> states that if Y s X (Y is the successor of X) and if Y is also the successor of Z, then X and Z must be the same number.

<u>Axiom 4</u> states: Consider a subset of $\mathcal{N}$ called G. If G contains the element, 1, and if every element of G has a successor which is also an element of G, then G = $\mathcal{N}$.

Discussion Questions

These questions are intended as suggestions of various areas to explore as an extension to this problem. The "answers" are not given as the "correct" or "only" responses but rather possible comments that might evolve from a discussion of these ideas.

1. How are the natural numbers a building block to all other numbers?

   All other number systems contain the natural numbers and are an extension of the natural numbers.

2. If we only had the natural numbers, what kinds of problems would we be unable to solve?

   a) Example: If you had $3 and gave your brother $5, how much money would have have left?

   b) If you had a candy bar and three friends, how much of the candy bar would each friend have? (No fractional solutions).

3. If you could add one more axiom to Peano's, what would you want to add?

   1 is a successor of α. (i.e., 1 s α).

4. Is the successor of a number always greater than that number? Can you think of any system of numbers where this is not the case?

   Clock arithmetic. Example: mod 7; 0 s 6.

5. How would you expect a number system on another planet to be similar to ours?

   Everyone needs a system of counting to be able to relate one amount to another either within their own system or in comparison to ours. A 1-1 correspondence would have to be established to relate this amount to a particular "number".

6. How would our money system be different if there were no fractional parts of dollars?

   In many ways it would be easier as you would be dealing only with whole-number values.

Reference

Introduction to the Foundations of Mathematics, Raymond L. Wilde, John Wiley and Sons, Inc., 1952.

Submitted by: Karen Higgins
D[illegible]na Jobson
Pam Hine

Question 6:
What is the fundamental theorem of arithmetic?

The fundamental theorem of arithmetic states that any integer other than one and zero can be factored as a unique product of prime numbers. (Of course, we allow for different arrangements of factors according to the commutative and associative properties). It is our opinion that this theorem is as important as any which middle school students will learn during their last years of arithmetic classes. This paper will present a brief history of the investigation of prime numbers, applications of the fundamental theorem to middle school curricula, and most importantly, a look at a numeration system which we suggest would be an extremely useful instructional aid in teaching about this most important theorem.

The Greeks were well aware of the existence of and special characteristics of the prime numbers. In book IX of The Elements, Euclid proved that the number of prime numbers is infinte. A brief summary of his proof is presented here. It is an indirect proof which assumes that the number of primes is finite. This finite set is expressed as $\{2,3,5,\ldots,p_n\}$. Given this set, it follows that there exists a number $z = 2 \times 3 \times 5 \times \ldots \times p_n$. Consider the number $z + 1$. This number is either prime or composite. If $z_n + 1$ is prime, the theorem that the number of primes is infinite (i.e., not finite) is proved. On the other hand, if $z_n + 1$ is composite, it must be divisible by a prime factor, q. If such a prime factor, q, is in the set $\{2,3,5,\ldots,p_n\}$, the number $z_n + 1$ will have a remainder of one, i.e., q is a prime number which is not in the finite set of primes. Again the theorem is proved. Two examples follow:

a) $(2 \cdot 3 \cdot 5 \cdot 7) + 1 = 211$, which is an additional prime not in the original finite set of primes.

b) $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) + 1 = 30{,}031$, which though not prime, is a product of 59 and 509, which are 2 primes not in the original finite set.

An interesting generalization of Euclid's theorem on the infinitude of primes was established by Lejeune-Dirichlet, who showed that every arithmetic sequence,

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

in which a and d are relatively prime contains an infinite number of primes. For example, $3n+1$ yields $\{1,4,7,10,13,\ldots\}$ and $6n+5$ yields $\{5,11,17,23,29,\ldots\}$. An interesting and instructive exercise for middle school school students might be to perform a few examples and then to identify the first 10 (20,30) primes. Such an exercise would also make clear to the students the definition of relatively prime numbers.

One way to find all the primes less than a given number is to use the Sieve of Eratosthenes, named after the Greek mathematician Eratosthenes (200 B.C.). If all natural numbers greater than 1 are considered, the numbers that are not prime are methodically crossed out. After crossing out all the multiples of 2,3,5 and 7 the remaining numbers in the sieve are primes. Students are quick to look for the short cuts in writing the sieve. Following are two other methods using this sieve:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102

The multiples are arranged in columns or diagonals, easy to cross out. Another method would be to arrange a table of the numbers congruent to 1,3,7,9, modulo 10. The numbers divisible by 3 are crossed with a single line, the survivors that are divisible by 7 are crossed out with 2 slant lines. The remaining numbers plus 2 and 5 are the 25 primes less than 100.

2	5	1	3	7	9
		11	13	17	19
		21	23	27	29
		31	33	37	39
		41	43	47	49
		51	53	57	59
		61	63	67	69
		71	73	77	79
		81	83	87	89
		91	93	97	99

The criterion for determining if a given number n is prime: If n is not divisible by any prime p such that $p^2 \leq n$, then n is prime.

Ex.: Is 397 composite or prime?

Solution: The possible primes p, such that $p^2 \leq 397$ are 2,3,5,7,11,13,17,19. Using divisibility rules we find each of these primes do not divide evenly into 397. Therefore 397 is prime.

As one can see, this theorem leads to many topics of Number Theory that appear in middle school curricula ; factors, divisors, multiples, common factors, common multiples, relatively prime numbers, composite numbers as well as finding the prime factorization by using divisibility tests. The divisibility tests most commonly used are:

test for 2 - if it is an even number

3 - if the sum of the digits are divisible by 3

4 - if the last two digits are divisible by 4

5 - if the units digit is a 5 or 0

6 - if the number is divisible by 3 and 2

7 - double the last digit, subtract this product from the remaining digits, if this is visibly divisible by 7 then the number is, or continue until you can check for divisible by 7.

8 - if the last 3 digits are divisible by 8

9 - if the sum of the digits is 0 (mod 9)

11 - find the difference between the sum of the odd power positions and the even power positions. If this difference is 0 or the number is divisible by 11 then the number is divisible by 11.

The Prime System

The prime system is discussed in an article in "The NCTM Enrichment Mathematics for the Grades" 27th Yearbook called "Numeration Systems" by Joseph N. Payne, pages 241-244.

This prime system is not a proof of the Fundamental Theorem of Arithmetic, but after presenting the system we feel students will have a better understanding that all numbers greater than one can be expressed uniquely as a product of primes.

The prime system may be introduced as a game of guess my rule and give pairs of numbers without giving away the rule. You look for patterns by pairing base 10 numbers with prime system numbers.

0	-	10	101	20		30	
1	0	11		21	1010	31	
2	1	12		22		32	5
3		13	100000	23		33	
4		14		24		⋮	
5	100	15		25			
6		16		26	'00001		
7		17		27			
8	3	18		28			
9		19		29			

Students will begin to see particular patterns such as the primes represented by powers of ten, the powers of 2 are the natural numbers, the powers of 3 are 10, 20, 30, etc.

PRIME SYSTEM

POWERS OF PRIME

	31	29	23	19	17	13	11	7	5	3	2
0											
1											0
2											1
3										1	0
4											2
5									1	0	0
6										1	1
7								1	0	0	0
8											3
9										2	0
10									1	0	1
11							1	0	0	0	0
12										1	2
13						1	0	0	0	0	0
14								1	0	0	1
15									1	1	0
16											4
17					1	0	0	0	0	0	0
18										2	1
19				1	0	0	0	0	0	0	0
20									1	0	2
21								1	0	1	0
22							1	0	0	0	1
23			1	0	0	0	0	0	0	0	0
24										1	3
25									2	0	0
26					1	0	0	0	0	0	1
27										3	0
28								1	0	0	2
29		1	0	0	0	0	0	0	0	0	0
30									1	1	1
31	1	0	0	0	0	0	0	0	0	0	0
32											5

After examining the prime system, it might help to investigate some interesting numbers. Here are a few:

Interesting Numbers in Prime System

Base Ten	Prime System	Base Ten	Prime System	Base Ten	Prime System
2	1	3	10	5	100
4	2	9	20	25	200
8	3	27	30	5^3	300
16	4	81	40	5^4	400
32	5	243	50	⋮	⋮
64	6	729	60		
128	7	3^7	70	7	1000
256	8	3^8	80	7^2	2000
512	9	3^9	90	7^3	3000
1024	?	3^{10}	[10]0	⋮	⋮
2048	?				

Base Ten	Prime System	Base Ten	Prime System
6	11	10	101
36	22	10^2	202
216	33	10^3	303
6^4	44	10^4	404
6^5	55	10^5	505
6^6	66	10^6	606
6^7	77	10^7	707
6^8	88	10^8	808
6^9	99	10^9	909
6^{10}	[10][10]	10^{10}	[10]0[10]

The prime system is an excellent opportunity to introduce some properties of exponents. The laws show that multiplying numbers with a like base we add the exponents of the number with the like base. Also division of numbers with a like base can be found by subtracting the exponents of the numbers with the like base.

To multiply numbers in the prime system we add the numbers digit by digit

For example: $12_p \cdot 10_p = 22_p$

$1010_p \cdot 101_p = 1111_p$

$6_p \cdot 4_p = [10]_p$ Not 10_p

To divide numbers in the prime system we subtract the numbers digit by digit

For example: $12_p \div 10_p = 2_p$

$101,000_p \div 1000_p = 100,000_p$

An extension of the prime system may be the introduction of negative exponents in division as well as by using them to write fractions.

For example: $3_p \div 20_p = ?$

$\therefore\ 03_p \div 20_p = [-2]3_p$

Base 10 $(2^3) \div (3)^2 = (\frac{1}{3^2})(2^3) = \frac{2^3}{3^2} = \frac{8}{9}$

Some examples of prime systems fractions are:

$\frac{1}{8} = [-3]_p$

$\frac{2}{8} = [-2]_p$

$\frac{3}{8} = 1[-3]_p$

$\frac{4}{8} = [-1]_p$

$\frac{5}{8} = 10^0[-3]_p$

$\frac{6}{8} = 1^0[-2]_p$

$\frac{7}{8} = 100^0[-3]_p$

$\frac{8}{8} = 0_p$

The introduction of the prime system is merely an extension of the idea of The Fundamental Theorem of Arithmetic. This theorem is a major part of the middle school math curriculum whether we name it as such or not.

Reference

1. Eves, Howard, An Introduction to the History of Mathematics, Holt, Rinehart, and Winston, 1969.

Submitted by: Steve Kirsner
Anita Koplyay
Ron Kovar

It's easy once you get the trick
A few ol'primes will make it click.
We'll never say that it is quick
The fundamental theorem of arithmetic.

Question 7:
- (a) What are Diophantine equations? Explain and give at least one example.
- (b) What is the origin of the name of these equations?

Reporter: Good morning! I am Wanda Crankcase, your roving CGM network reporter and this morning I'd like to take you back in time for a very rare interview with a most interesting personality. We will step into our miraculous time machine and travel back in time to 300 A.D. We will be talking with one of the last and most fertile mathematicians of the second Alexandrian school...Diophantus!, who lived about 400 years later than Euclid.

Mr. Diophantus, good morning. I have brought with me today some viewers who are very much interested in you and your work in solving algebraic equations. Would you tell us something about your life, and your work in general?

Mr. Diophantus: Well, most of what you will ever know of me personally is stated in this problem: I passed $\frac{1}{6}$ of my life in childhood, $\frac{1}{12}$ in youth, and $\frac{1}{7}$ more as a bachelor; five years after I married a son was born who died four years before me, at half my age.

$$\frac{AR}{6} + \frac{AR}{12} + \frac{AR}{7} + 5 + \frac{AR}{2} + 4 = AR$$

From this you will see I died at 84 which was about 330 A.D. Did I mention I'm Greek? Love that Greek salad in the State Room!

Anyway, I did write up two or three works in mathematics, but only part of them remain, most notably 7 of 13 books of Arithmetica. Well, there are a number of interesting little tid bits I could tell you about myself--

fractions as answers don't bother me,
since I don't like negatives,
$AR + 5 = 3$ can't be done.

What specifically would you like to know?

Reporter: We understand that in the solution of simultaneous equations, you managed with only one symbol for the unknown quantities, and arrived at answers, most commonly by the method of tentative assumptions. We called these Diophantine Equations, could you tell more about them?

Mr. Diophantus: Sure, during this era we don't have the system of symbols you have.

Consider this problem: John and James make a bet. If John loses, he must give James 30 marbles, and James will then have twice as many as John will. But if John wins, then James must give him 50 marbles, and then John will have three times as many as James will. How many marbles does each boy have to begin with?

How would you do this. Ah!. Use a variable!

Let AR = No. of John's marbles if he loses.

Let 2AR = No. of James marbles after winning.

But if John wins, he will be richer by 80 marbles than if he loses, and James will be poorer by 80, than if he wins. That is, John will have AR + 80 and John will have 2 AR − 80. But, John, we are told by the problem, will then have three times as many as James, which is 6 AR − 240.

$$\therefore\ \underset{\text{John}}{6\,\text{AR} - 240} = \underset{\text{John}}{\text{AR} + 80}$$

or 5 AR = 320

and AR = 64 So John starts with AR + 30 or 94 and James starts with 2 AR − 30 or 98 marbles.

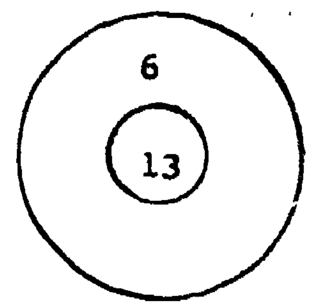
Pretty nifty eh! (Greek slang) Note, only one variable − none of that x and y stuff. And note what I set AR equal to. Want to know why they call this a Diophantine equation? I did it!

Another definition of a Diophantine Equation that you may find, states that it is an equation in which the number of solutions is greater than the number of equations...

Let me present an example using the simplest Diophantine equation: the linear Diophantine equation and your two variable system. This friend of mine, William "Fitzgerald" Tell, and his son were practicing for an archery demonstration. He wanted to know what possible combinations of arrows could land on his target to score 100.

Let x = No. of arrows in 6 area

Let y = No. of arrows in 13 area.



$6x + 13y = 100$

$$x = \frac{100 - 13y}{6}$$

								Neg.
y	1	2	3	4	5	6	7	8 →
x	$\frac{87}{6}$	$\frac{74}{6}$	$\frac{61}{6}$	8	$\frac{35}{6}$	$\frac{22}{6}$	$\frac{9}{6}$	$-\frac{4}{6}$

Which is the most sensible solution?
William...can you Tell?

Reporter: Thank you. This is Wanda Crankcase taking you back to the campus of Michigan State University, where we will visit with Dr. Jill Fitzmerle and perhaps get some insight as to the application of Diophantus' work with equations and how we might use his approach in 20th century middle school mathematics.

Dr. Fitzmerle: We will see if a solution can be found for an equation in the form of $ax + by = c$

$7x + 11y = 13$ GCF of (7, 11) = 1 and 13 is a multiple of 1
∴ there are solutions to the equation

$x = (13 - 11y)/7$

By the test and check method $x = 5$ and $y = -2$ and this is a solution.

To find more solutions:

If x_0 and y_0 is a solution of $ax + by = c$ so then is $x_0 + bt$ and $y_0 - at$ for any integer t.

Given $ax_0 + by_0 = c$ thus

$$a(x_0 + bt) + b(y_0 - at) = c$$
$$ax_0 + abt + by_0 - abt = c$$
$$ax_0 + by_0 = c$$

so let $x_1 = x_0 + bt$ and $y_1 = y_0 - at$.

If $t = 1$ then:

$x_1 = x_0 + bt$ $\quad$ $y_1 = y_0 - at$

$x_1 = 5 + 1 \cdot 11$ $\quad$ $y_1 = -2 - 7 \cdot 1$

$x_1 = 16$ $\quad$ $y_1 = -9$

$x = 16$ and $y = -9$ is another solution for $7x + 11y = 13$.

Now let us do a problem that might appear in an algebra or pre-algebra course.

A box contains beetles and spiders. There are 46 legs in the box, how many beetles are there?

x = beetles

y = spiders $\quad$ $6x + 8y = 46$ $\quad$ GCF of (6,8) is 2 and 46 is a multiple of 2 so therefore the equation has solutions.

$6x + 8y = 46$ can be

simplified to

$$3x + 4y = 23$$
$$x = (23 - 4y)/3$$

Test for x and y and one solution will be $x = 1$ and $y = 5$ but this solution doesn't satisfy the problem.

Let $t = 1$

$x_1 = x_0 + bt$ $\quad$ $y_1 = y_0 - at$

$x_1 = 1 + 4 \cdot 1$ $\quad$ $y_1 = 5 - 3 \cdot 1$

$x_1 = 5$ $\quad$ $y_1 = 2$

$x = 5$ and $y = 2$ is a solution for the original equation and it satisfies the problem.

So there can be 5 beetles.

Reporter: Thank you, Dr. Fitzmerle, and thank you all for tuning in. Until next time...this is Wanda Crankcase for CGM network. Good morning!

Submitted by: Cynthia Laurie
Marlene Montague
Gary Rhines

Diophantines have their name
Lots of letters are their game.
Solve us calmly, if you dare,
Then ask us if we will care!

Question 8:

Mathematicians of ancient Greece were concerned about the existence of incommensurable segments.

(a) Explain what such segments are.

(b) What is the algebraic implication of this problem?

(a) Two quantities, such as the diagonal and side of a square, are incommensurable when they do not have a ratio such as a (whole) number has to a (whole) number. Whole numbers and ratios are inadequate to compare the diagonal of a square, a cube, or a pentagon with its side. No matter how small the unit chosen, the segments remain incommensurable.

(b) It had been a fundamental tenet of Pythagoreanism that the essence of all things, in geometry as well as in the practical and theoretical affairs of man, is explainable in terms of <u>arithmos</u>, or intrinsic properties of whole numbers and their ratios.

In Plato's youth the discovery of the incommensurable had caused a veritable logical scandal, for it has raised havoc with theorems involving proportion. Students of Pythagoreanism were sworn to secrecy -- in one unfortunate case a student was drowned at sea for revealing the existence of the square root of 2.

Archytas was in a sense a transition figure in mathematics during Plato's time. Archytas was among the last of the Pythagoreans. He could still believe that number was all-important in life and in mathematics, but the wave of the future was to elevate geometry to the ascendency, largely because of the problem of incommensurability. It was through his influence that the quadrivium--arithmetic, geometry, music, and astronomy was established as the core of a liberal education.

The discovery of the incommensurable or irrational numbers brought about the following mathematical work. Proclus explained that it is only in arithmetic that all quantities bear "rational" ratios to one another, while in geometry there are "irrational" ones as well. Eudoxus, the founder of the theory of proportions, through meeting in the course of his investigations with proportions not expressible by whole number ratios, realized the necessity for a new theory of proportions which should be applicable to incommensurable as well as commensurable magnitudes.

References:

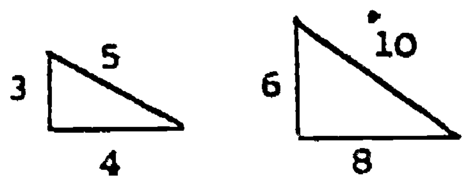
A History of Mathematics, by Carl B. Boyer, John Wiley and Sons, Inc., NY, London, Sidney.

Euclid's Elements, book ten.

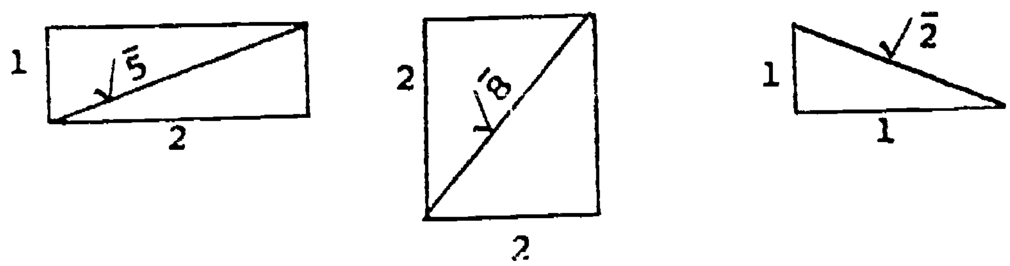
Implications for Middle School Mathematics:

Script taken from Square Root activity in Pre-Algebra with Pizzazz, book CC, by Steve and Janis Marcy to introduce irrational numbers and the impact of their discovery on the history of mathematics.

Uss of Pythagorean theorem to explore rational ratios, i.e.,



vs. irrational ratios, i.e.,



Challenge students to find other irrational ratios.

Super challenge for whiz kids: Using geoboard or dot paper, construct squares with areas 1,4,9,16,25. Construct a square with an area of 2 (HINT: what would be the measure of its side?) We have left out 3,5,6,7,8,10,11,12 13,14,15,17,18,19,20,21,22,23, and 24. Can you construct squares with any of those areas? Are they all possible?

Use to develop concept before beginning work with the Pythagorean theorem in Algebra I.

Commensurable vs Incommensurable

Characters: Aristotle, Zeno, Judge, Pythagoras, Eudoxus, Clerk

Hear ye! Hear Ye! The court of Commensurable Segments vs. Incommensurable Segments is now in session. All rise. And now - Here Come the Judge, Here Come the Judge...

Judge: Please be seated. Mr. Aristotle will you please state your case.

Mr. Aristotle: We have a case here, your Honor, in which the defendant has dared to dispute the fundamental tenet of Pythagorianism. - that the essence of all things are explainable in terms of whole numbers or their ratios. I intend to prove to this court not only that the defendant is guilty of almost demolishing the basis for the Pythagorean faith in whole numbers, but to show that he actually claims to have discovered irrational numbers AND disclosed his discovery without due regards to other mathematicians.

Judge: Call your first witness.

Mr. Aristotle: I call Mr. Pythagoras to the stand.

Clerk: Do you promise to tell the truth and nothing but the truth?

Mr. Pythagoras: I do.

Judge: You may be seated.

Mr. Aristotle: Mr. Pythagoras, will you tell this court what a perfect square is?

Mr. Pythagoras: A perfect square is a number whose square root is an integer. For example 81, is a perfect square, because 9 is the square root of 81.

Mr. Aristotle: Would you describe for this court what a repeating decimal is?

Mr. Pythagoras: A repeating decimal is a decimal in which the same sequence of digits repeats again and again and never ends.

Mr. Aristotle: Good. Now, Mr. Pythagoras, is a repeating decimal the only way to represent a rational number?

Mr. Pythagoras: No, Sir. Every rational number can be represented by either a repeating decimal or by a terminating decimal.

Mr. Aristotle: Thank you Mr. Pythagoras. The Plaintiff rests Your Honor.

Judge: Mr. Zeno, will you call your first witness?

Mr. Zeno: I call to the stand Mr. Eudoxus.

Clerk: Do you promise to tell the truth and nothing but the truth?

Mr. Eudoxus: I do.

Judge: You may be seated.

Mr. Zeno: To refresh our memory, would you name a few perfect squares?

Mr. Eudoxus: 1, 16, 49, 64, 144.

Mr. Zeno: Mr. Eudoxus, I don't recall your naming 2 in your list of numbers. Why not?

Mr. Eudoxus: Since 2 is not a perfect square, the square root of 2 is not an integer. The square root of 2 is a number which when squared equals 2 exactly.

Mr. Zeno: Would you explain to this court how you would try to find the square root of 2.

Mr. Eudoxus: The square root of 2 must be between 1 and 2 because $1^2 = 1$ and $2^2 = 4$. $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$. So the square root of 2 has to be between 1.4 and 1.5.

Mr. Zeno: Let me understand what you just said. The square root of 2 is between 1.4 and 1.5. How can that be?

Mr. Eudoxus: $(1.41)^2 = 1.9881$ and $(1.42)^2 = 2.0164$. Therefore the square root of 2 is between 1.41 and 1.42---and I'd like to take it to another decimal place. $(1.414)^2 = 1.999396$ and $(1.415)^2 = 2.002225$. Get the point? Therefore, the square root of 2 is between 1.414 and 1.415.

Mr. Zeno: Mr. Eudoxus, is there a terminating or repeating decimal for the square root of 2?

Mr. Eudoxus: No, Sir, there is not. It can be proved that there is no terminating decimal which, when squared, equals exactly 2. It can also be proved that there is no repeating decimal that, when squared, equals 2.

Mr. Zeno: What do you call a decimal that never terminates, and never repeats.

Mr. Eudoxus: The square root of every whole number is an irrational number unless the number is a perfect square.

Mr. Zeno: Thank you Mr. Eudoxus. In summation, your Honor, we are standing on the threshold of mathematical history. Mr. Eudoxus has courageously, and without regard for his own safety, revealed that the theory of proportions is inadequate to account for instances involving incommensurable segments, what we have here today termed irrational numbers. I beg you, your Honor, to consider the impact of what we decide here today upon mathematicians of the future. For instance, who can say when sometime, somewhere deep in the heart of Michigan some group of Honors Teachers may discover the relevance of Mr. Eudoxus' work to Middle School Mathematics? The possibilities are limitless. Your Honor, I rest my case.

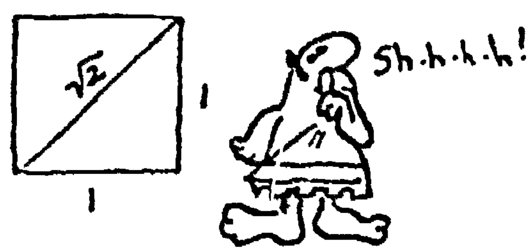
Judge: Thank you gentlemen. This court will recess for ten minutes. When I return, I will give you my decision in this case.

Milestone #8

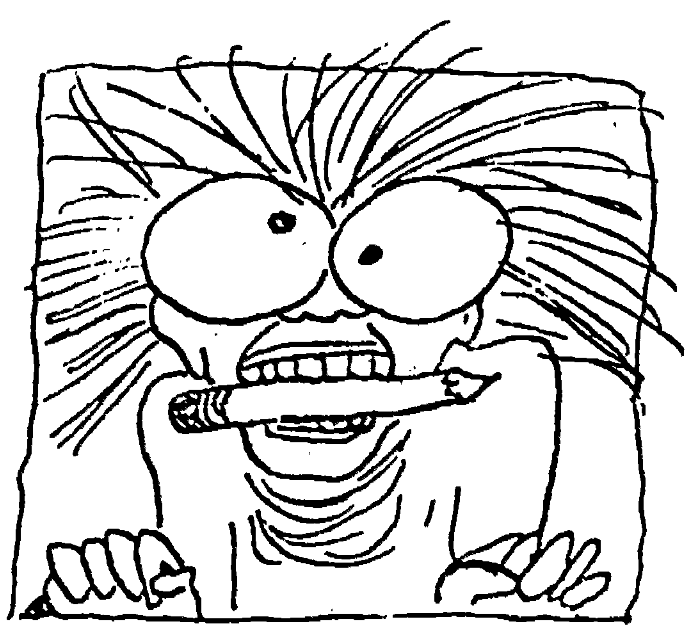
Mathematicians of ancient Greece were concerned about the existence of incommensurable segments.

a) Explain what such segments are.

Two quantities, such as the diagonal and side of a square, are incommensurable when they do not have a ratio such as a (whole) number has to a (whole) number.



b) What is the algebraic implication of this problem?



Problems like this make me IRRATIONAL!

with apologies to Marilyn Burns, Math for Smarty Pants for borrowing the art work.

Submitted by: Kenneth Servais
Marcia Swanson
Esther Williams

Root two, Root two, on the wall
Greatest grafitti of them all.
Will you ever come to ends?
Drop your point and join your friends.

Question 9:

Consider the equation: $\alpha_n x^n + \alpha_{n-1} x^{n-1} + \ldots + \alpha_1 x + \alpha_0 = 0$ where all α's are integer coefficients.

(a) Do such equations always have a solution which is a real number? (Explain and give one example to substantiate your statement.)

(b) What does the fundamental theorem of algebra say about such equations?

(c) Is it always possible to find a solution in the form of an expression in which only operations of addition, subtraction, multiplication, division and extraction of roots are carried out on the coefficients? (Qualify your statement.)

(a) $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ where all a's are integers does not always have a real solution.

Samples:

$x + 1 = 0$	$x^2 - 4 = 0$	$x^2 + 4 = 0$
$x = -1$	$x^2 = 4$	$x^2 = -4$
OK	$x = \pm 2$	?
	OK	$\square \cdot \square = -4$
		Nope!

Descarte in 1637 coined the terms Real and Imaginary, and created what is known as the "Great algebraic calamity".

Cardan - Find two numbers whose sum is 10 and whose product is 40.

$$5 + \sqrt{-15} \qquad 5 - \sqrt{-15}$$

He called these "sophisticated numbers" and said "to continue would involve arithmetic as subtle as it is useless".

The terms imaginary and real reflect the illusive qualities that numbers held for mathematicians of that day (or today!).

Of imaginary numbers, Leibniz said "the wonderful creature of an ideal world, almost an amphibian between things that are and things that are not".

(b) The "Fundamental Theorem of Algebra" states that in the field of complex numbers every polynomial has a solution.

(c) A solution of a polynomial equation in the form of an expression in which only operations of addition, subtraction, multiplication, division and extraction of roots are carried out on the coefficients is not always possible. In algebra the problem of solving equations of degree 5 or higher that led mathematicians to investigate the possibility of proving that certain problems could be solved.* During the 16th century, mathematicians learned that equations of the 3rd or 4th degree could be solved by processes similar to the method of solving quadratics. The methods have the properties that solutions or "roots" can be written as algebraic expressions obtained from the coefficients by a sequence of operations involving +, −, x, ÷ and extraction of $\sqrt{\ }$, $\sqrt[3]{\ }$, or $\sqrt[4]{\ }$. It was not until the 19th century that Ruffini (1765-1822) and Abel (1802-1829) made progress with the idea of the impossibility of solutions by means of radicals. This study led to the development of modern algebra and group theory.

> "Yes the solutions are really numbers, they are called imaginary numbers, not real numbers -- got it?"

Sources:

What is Mathematics, Courant and Robbins.

Historical Topics for the Classroom, 31st Yearbook of the National Council of Teachers of Mathematics.

Submitted by: Karen Higgins
Reg Waddoups

Question 10:

Consider the series $1+(-1)+1+(-1)+\ldots$ where we sum these numbers up to infinity. Writing this series in the form $[1+(-1)]+[1+(-1)]+[1+(-1)]+\ldots$ we obtain zero as the sum to infinity. However, writing this series as $1+[(-1)+1]+[(-1)+1]+[(-1)+1]+\ldots$ we obtain 1 as the sum to infinity. What is the true sum to infinity of this series?

For any geometric series G_n

$$G_n = a + aq + aq^2 + aq^3 + \ldots + aq^n = a\,\frac{1-q^{n+1}}{1-q}$$

where $q \neq 1$, therefore, in our series $a = 1$ and $q = -1$

so $1 \cdot \frac{1-(-1)^{n+1}}{1-(-1)} = \frac{1+1}{1-(-1)} = \frac{2}{2} = 1$ where n is even or

$1 \cdot \frac{1-(-1)^{n+1}}{1-(-1)} = \frac{1-1}{2} = \frac{0}{2} = 0$ where n is odd.

∴ this series is "divergent" meaning it does not have one limit since it can have two values depending upon whether n is an even or odd integer.

Application for Middle School Mathematics

Put the series $1+(-1)+1+(-1)+1+(-1)$ on the board. Ask students, "How long could you continue to do this?" Hopefully they will say "FOREVER" and a discussion of infinity will result.

Next, ask them, "How far is 'far enough'?" Now the series should look like this on the board:

$$1+(-1)+1+(-1)+1+(-1)\ldots$$

Have students copy the series on their paper and divide the class into two groups. Ask the first group to put their parentheses in the following manner:

$$[1+(-1)]+[1+(-1)]+[1+(-1)]\ldots$$

Have them determine the number of terms needed to find the sum of the series.

The other group is asked to put their parentheses in the following manner:

$$1 + [(-1) + 1] + [(-1) + 1] + [(-1) + 1]...$$

They are also asked to determine the number of terms needed to find the sum of the series.

Group one should reach the answer 0 and group two should reach the answer 1.

A discussion should follow concerning the use of parentheses and how the answer changes when placed in different positions.

Problems such as $(5 \times 2) + 3$ and $5 \times (2 + 3)$ could also be discussed here. Again discuss the question, "How far is far enough?" Because at one point, half the class has the correct answer, but given one more term, then the other half of the class has the correct answer.

What is a limit?

Consider the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

Ask students what comes next, continue writing on board. How far is far enough? Is there a way I can "summarize" the last term? Answer $\frac{1}{2^n}$. Have students add terms of the series.

Two terms: $\frac{1}{2} + \frac{1}{4} = .5 + .25 = .75$

Three terms: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = .5 + .25 + .125 = .875$

Four terms: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = .5 + .25 + .125 + .0625 = .9375$

Five terms: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} =$

$.5 + .25 + .125 + .0625 + .03125 = .96875$

Six terms: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} =$

$.5 + .25 + .125 + .0625 + .03125 + .015625 = .984375$

For an infinite geometric series $a + ar + ar^2 + \ldots$ the sum is $S = \frac{a}{1-r}$. In this example, $a = \frac{1}{2}$ and $r = \frac{1}{2}$ so $S = \frac{1/2}{1 - 1/2} = \frac{1/2}{1/2} = 1$.

Reference: What is Mathematics? Richard Courant and Herbert Robbins, Oxford University Press, 1953.

Submitted by: Susan Apps
Pam Hine

Question 11:

There are problems in mathematics which can easily be formulated and which nevertheless haven't been solved yet, even though mathematicians have tried very hard to solve them. List at least one such problem. (Give as many details as you know).

There are many interesting number relationships involving prime numbers which have yet to be proven. Teachers may use any of the following to explore the significance of prime numbers when studying the topics of factoring, GCF and LCM.

At this time it may be a good idea for students to discuss the idea of a proof. Students should understand that to prove something true, it must be true for EVERY example. However, to prove something FALSE you need only one counter-example. Students are sometimes too quick to say something is true without reviewing sufficient numbers and different types of examples.

PERFECT NUMBERS

No person is perfect...but some numbers are. Study the following table:

NUMBER	PROPER FACTORS	SUM OF THE PROPER FACTORS
2	1	1
3	1	1
4	1,2	3
5	1	1
6	1,2,3	6
7	1	1
8	1,2,4	7
9	1,3	4
10	1,2,5	8

Do you see any special relationship between the number and the sum of its proper factors? If the sum of the proper factors of a number is equal to that number, the number is said to be PERFECT. From the table above you can see that 6 is a PERFECT NUMBER.

Consider this method of looking at numbers generated by powers of 2:

1 + 2 = 3 (is the sum prime?)
(if yes, multiply sum X last power) 3 X 2 = 6 (PERFECT)

1 + 2 + 4 = 7 (prime → 7 X 4 = 28 (PERFECT)

[Let's look at the proper factors of 28: 1,2,4,7,14 → sum is 28]

1 + 2 + 4 + 8 = 15 (not prime)

1 + 2 + 4 + 8 + 16 = 31 (prime) → 31 X 16 = 496 (PERFECT)

1 + 2 + 4 + 8 + 16 + 32 = 63 (not prime)

Let's look at the table of PERFECT numbers we have found so far:

NUMBER	FACTORS ADDED	SUM OF PROPER FACTORS
6	1+2+3	= 6
28	1+2+4+7+14	= 28
496	?	= 496

Notice that the first perfect number is a single digit number, the second is a two digit number and the third is a three digit number. Do you suppose that the fourth perfect number will be a four digit number? It is !!!

8128	?	= ?

Observe from your table the values of the one's place digit in each perfect number. Do you see a pattern? Would you predict that the fifth perfect number would have 5 digits with a 6 in the one's place?

If you said yes, you're almost correct. The fifth perfect number is: 33,550,336.

Looking for patterns is helpful in making a guess, however each guess won't always be correct. Sometimes you win, but sometimes you haven't looked far enough to rely on a pattern yet.

At this point you should realize that the 6th perfect number will not have 6 digits, however, would you predict that the 6th perfect number would end in 8? Well take a look: The sixth perfect number is 8,589,869,056. But it is true that every KNOWN perfect number ends in either 6 or 28. To this date, 23 perfect numbers have been found (the largest of which requires more than 6000 digits to write.) It is still not known whether any odd perfect numbers exist. [Note: In some books the definition of a perfect number is one that equals twice the sum of all its factors. (6 → 1 + 2 + 3 + 6 = 12)].

GOLDBACH'S CONJECTURE (1742)

Any even number greater than 2 can be expressed as the sum of two primes.

For example:	4 = 2 + 2	18 = 11 + 7
	6 = 3 + 3	20 = 13 + 7
	8 = 5 + 3	48 = 19 + 29
	10 = 5 + 5	100 = 97 + 3 etc.

Student Exploration: Choose at least 5 different even numbers other than the examples shown and express them as the sum of two primes.

(Note: The difficulty with the proof is that prime numbers are defined in terms of multiplication while the problem involves addition.)

TWIN PRIMES

Prime numbers frequently occur in pairs in the form p and $p + 2$.

3,5   5,7   11,13   17,19   29,31   etc.

Are there infinitely many such pairs? Do these pairs occur less frequently as the range of numbers increases?

Accepting something as obvious in mathematics too often leads to incorrect conclusions therefore many things that appear or seem obvious are attempted to be proved as true. The problems above, although appear to be true have yet to actually been proven.

Submitted by: Donna Jobson
Dennis Cumpston

Question 12:
There are three problems which were formulated in ancient Greece which mathematicians tried to solve for more than 1500 years and which were eventually solved in the 19th century. These problems are often referred to as "three famous problems of antiquity". What are they, and what is the solution?

CLASSICAL PROBLEMS OF ANTIQUITY

Mathematicians often refer to the "three famous pıoblems of antiquity". The search for solutions to these problems was instrumental in leading to many important discoveries for centuries to come. This paper will discuss these classical problems and the way that they may be beneficial in the mathematics curriculum of middle schools.

One of the three famous problems is the "duplication of the cube", or the problem of constructing the edge of a cube having twice the volume of a given cube. This problem probably dates to the time of the Pythagoreans (ca. 450 B.C.) The P thagorean Theorem easily allows for the construction of a segment of length $\sqrt{2}$, and thus for a square with an rea of 2 units. The corresponding problem of finding a segment of length $\sqrt[3]{2}$ proved a much greater challenge to the Greeks, and indeed to mathematicians of many societies.

An interesting story explaining the origin of the problem says that the gods once sent a plague to the people of Athens. The Athenians sent a delegation to the oracle at Delos to ask how the gods would be appeased. They were told to double the size of the cubicle altar to Appolo. The Athenians proceeded to build a new altar with edges twice as long as the edges of the old altar. This, obviously, did not appease the gods. We have included this story as a possible introduction to the classical problems. We feel that middle school children often find mythology interesting. This story also suggests to children that the field of mathematics has relevance outside of the classroom.

An important restriction has been placed upon these problems - i.e., that only a straightedge and compass may be used to solve these three problems. The energetic search for solutions to these problems led the Greeks to many discoveries, such as that of the conic sections, of mahy cubic and quartic curves, and of some transcendental curves. Future investigation led to the development of important aspects of the theory of equations concerning domains of rationality, algebraic numbers, and group theory. It is important, we feel, to impress upon middle school children that persistent, thorough, and scholarly methods of inquiry are important ends in themselves, often at least as important as possible solutions to the original question being investigated.

Indeed, none of the three problems can be solved with straightedge and compass. The impossibility of the solutions was not established until the nineteenth century, more than 2000 years after the conception of the problems. The fact that these problems cannot be solved suggests to students that solutions to problems are not necessarily the most important result of problem solving activities - that, indeed, some problems worthy of inquiry don't even have solutions.

Just as the Pythagorean theorem lent itself to the question of duplicating the cube, the ease with which an angle can be bisected using only a compass and straightedge lent itself to the question of trisecting an angle. Eves suggests that the multisection of a line segment with Euclidean tools is simple and

> "it may be that the ancient Greeks were led to the trisection problem in an effort to solve the analogous problem of multisecting an angle. Or perhaps, more likely, the problem arose in efforts to construct a regular nine-sided polygon, where the trisection of a 60° angle is required." (p.35)

As is the case with each of the classical problems it was only after many centuries of investigation that the unsolvability of the trisection of an angle using only a straightedge and compass was formally proven. As was also the case with each problem, many significant discoveries were made in the attempts to come up with a solution to the trisection problem. Not only were mathematical advances made but Eves points out that "over the years many mechanical contrivances, linkage machines, and compound compasses, have been devised to solve the trisection problem" (p.87).

According to Eves, "probably no other problem has exercised a greater or a longer attraction than that of constructing a square equal in area to a given circle" (p.89). Indeed, this problem dates back to the Egyptians. As early as 414 B.C., Aristotle referred to the problem in his play Birds. The impossibility of performing the construction with compass and straightedge alone depends on the fact that π is a transcendental number.

Middle school students might think on first reflection that the construction might be impossible because π is an irrational number. It would then be helpful to construct a segment the length of $\sqrt{2}$ using the Pythagorean theorem as described above. The students would then understand that the property of irrationality alone does not disallow ipso facto, the construction of a segment of the given length. Although they might not be expected to comprehend the meaning of transcendental numbers, the introduction of the topic might be useful in stimulating interest.

Perhaps the most important aspect of these problems is the interrelationship between mathematics and other disciplines like history. All too often, because of our curricula, students fail to comprehend the interrelationship among disciplines. A look at the classical problems, we hope, will help students gain an appreciation of the development of modern societies.

Bibliography:

1. NCTM Yearbook 31st, Historical Topics for the classroom, 1978.

2. An Introduction to the History of Mathematics, Third Edition Howard Eves, Holt, Rinehart and Winston, 1964.

3. A History of Mathematics, Carl B. Boyer, John Wiley and Sons, Inc., New York, 1968.

Submitted by: Maureen Denver
Steve Kirsner

Trisect me, if you can!!
It's not been done by any man (or woman).
If you think you've got it down
You'll soon be made to look like a clown.

Qu..tion 13:

You have studied in school how to inscribe in a circle some regular polygons (such as a square, an equilateral triangle and others) with the help of a ruler and compass only. Can you inscribe every regular polygon in a circle with the help of these tools only? (Give as many details as you can.)

Determining which polygons can and cannot be inscribed in a circle using only a compass and a straight edge has intrigued mathematicians for centuries. A polygon is called an inscribed polygon when all the vertices of the polygon are points of a given circle. Polygons in which all the angles are congruent and all the sides are congruent are called regular polygons. The construction with a straight edge and compasses of regular polygons of three, four five six, and fifteen sides are discussed in Book IV of Euclid's Elements (first translation into English 1570). Not until almost the nineteenth century was it known that any other regular polygon could be constructed with these limited tools. In 1796 Gauss developed the theory that showed that a regular polygon having a prime number of sides can be constructed with these tools if and only if that number is of the form $f(n) = 2^{2^n} + 1$. For $n = 0,1,2,3,4$ we find $f(n) = 3,5,17,257,$ 65537, all are prime numbers. It is known that for no other value of n, $f(n)$ is a prime number. These findings of Gauss on the construction of regular polygons appear in his greatest single publication, Disquisitiones arithmeticae.

It has been said that it was Gauss' discovery at the age of 19, that a regular polygon of 17 sides can be constructed with a straight edge and a compass, that inspired his life to mathematics. He considered this 17-gon one of his master achievements, and he wanted a replica of his construction placed on his tombstone. Although this request was never fulfilled such a polygon is found on a monument erected to Gauss at his birth place in Brunswick.

Of the regular polygons having less than 20 sides, one can with Euclidean tools construct those having 3,4,5,6,8,10,12,15,16 and 17. The following constructions could be introduced in a middle school curricula:

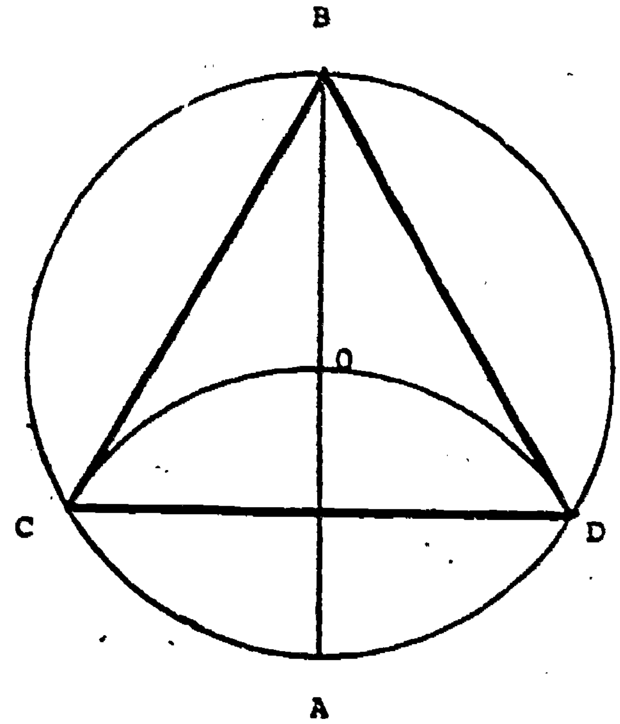
(See attached construction)

1. To inscribe an equilateral triangle:

   A. Draw a circle, center 8, diameter $\overline{AB}$

   B. Using A as center, radius $\overline{AO}$ draw an arc, endpoints C, D

   C. Connect B, C, D to form the triangle.



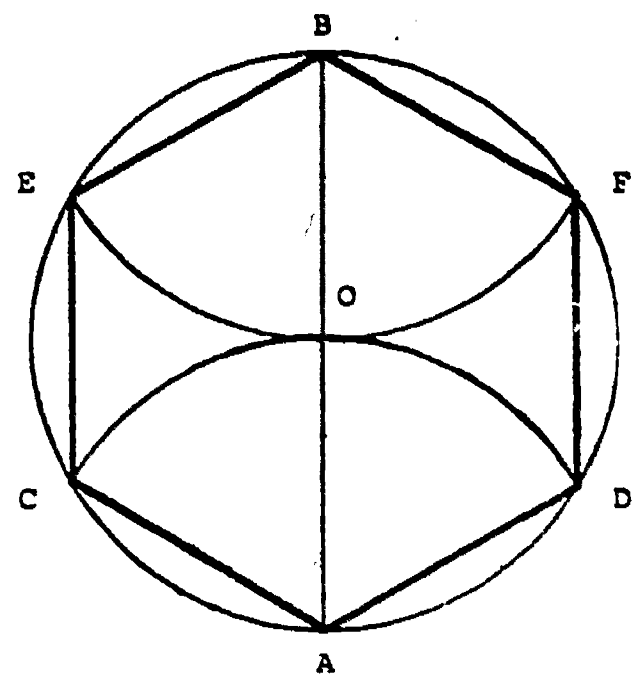
2. To inscribe a regular hexagon:

   A. Draw a circle, center O, diameter $\overline{AB}$

   B. Using A as center, radius $\overline{AO}$ draw an arc, endpoints C, D

   C. Using B as center, radius $\overline{BO}$ draw an arc, endpoints E, F

   C. Connect A, C, E, B, F, D, A to form the hexagon.

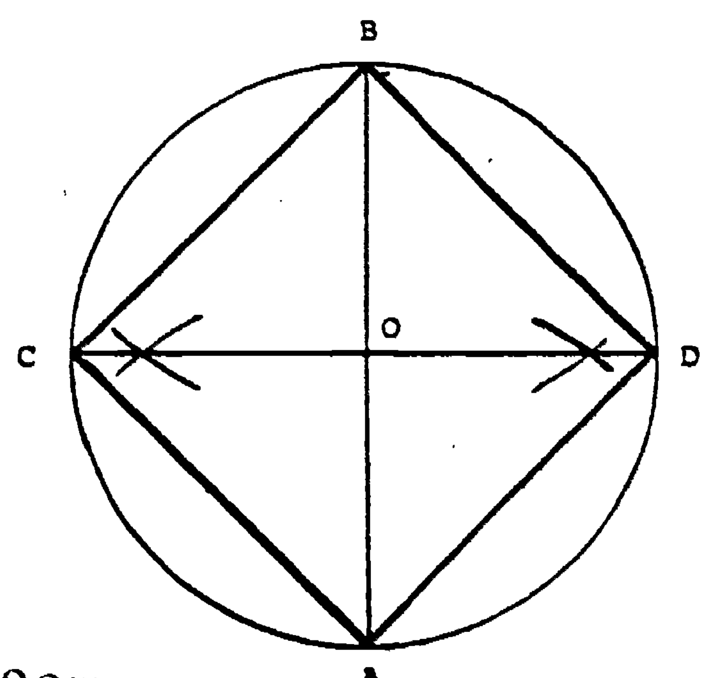


3. To inscribe a square:

   A. Draw a circle, center O, diameter $\overline{AB}$

   B. Construct perpendicular bisector of $\overline{AB}$, where it intersects the circle call the points C, D.

   C. Connect A-D-B-C-A to form the square.

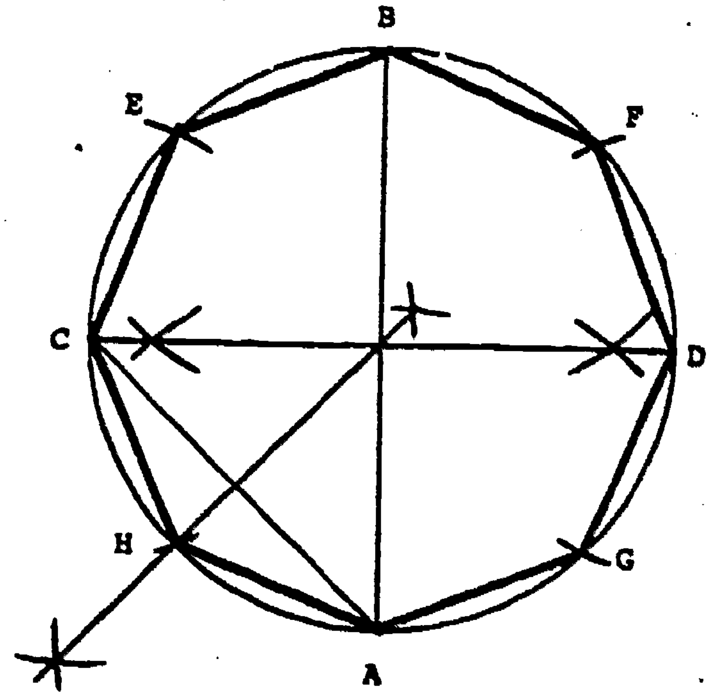


4. To inscribe a regular octagon:

   A. Locate A, B, C, D as in inscribing a square.

   B. Draw $\overline{AC}$ and bisect it. Call the point where the bisector intersects the circle H. Set compass with radius AH. From C mark arc, call the point on circle E. Mark from B call the point F. Mark from D call the point G.

   C. Connect A-H-C-E-B-F-D-G-A to form the octagon.



5. To inscribe a regular pentagon:

   A. Draw a circle, enter O, diameter $\overline{AB}$.

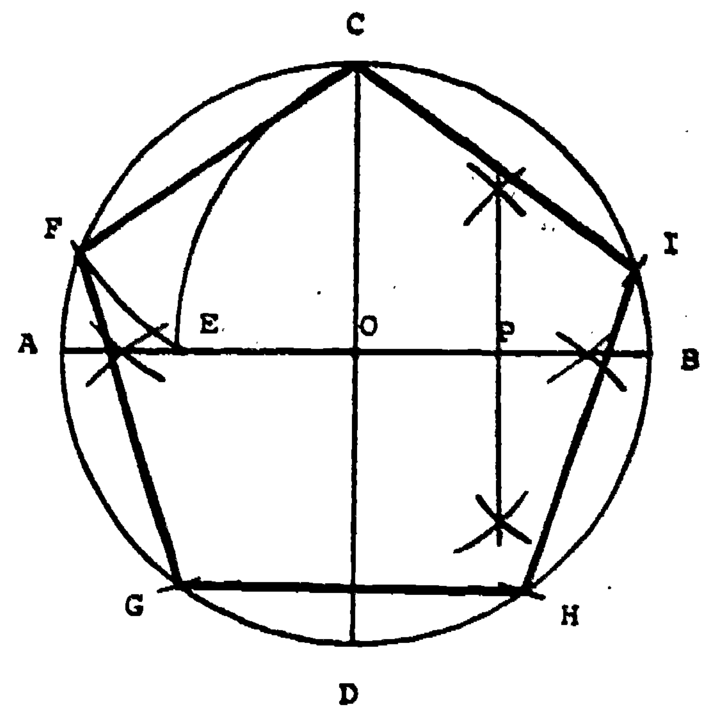
   B. Construct diameter $\overline{CD}$ perp. to $\overline{AB}$.

   C. Bisect OB. Label mid point P.

   D. Set radius at CP, draw an arc on OA, mark intersection as E.

   E. CE is the length of one side of the pentagon. Mark from C points F, G, H, I around the circle.

   F. Connect C-F-G-H-I-C to form the pentagon.



Reference:

1. Eves, Howard, An Introduction to the History of Mathematics, Holt, Rinehart and Winston, Inc., New York, 1964, p.116.

Submitted by: Anita Koplyay
Reg Waddoups

Question 14:
There are special requirements which a system of axioms has to satisfy. What are these requirements? Qualify, stating which of these requirements are more important than the others.

An axiomatic system is another name for a deductive system. The components of a deductive system are:

1) primitive terms (undefined terms)
2) defined terms (definitions)
3) axioms (nongeometric assumptions) or postulates (geometric assumptions)
4) laws of logic (rules for reasoning) and
5) theorems (proved statements).

The theorems of an axiomatic system follow from the definitions, axioms, and previous theorems in the system and are proven by deductive reasoning. With the addition of more axioms or definitions to a system more theorems can be derived. If a theorem cannot be proven directly by deductive reasoning, then an indirect proof may be used.

An axiomatic system must be consistent, complete, and independent. A system is consistent, when no two theorems deducible from the axioms are contradictory, complete, when every theorem in the system is deducible from the given axioms, and independent, when no axiom is a logical consequence of another given axiom. Of the three properties of an axiomatic system the least important is consistancy because the system is preserved if two or more axioms state the same information. However, if two axioms present counter statements or theorems cannot be logically deduced from the given axioms, then the system is no longer valuable as a system.

Knowledge of the axiomatic method is valuable to the middle school mathematics teacher not necessarily as a topic to be added to the curriculum, but rather as background information for the structure of mathematics. And yet an application of an axiomatic system that might be understandable to a middle school student is pool table math. In the February, 1977 issue of The Mathematics Teacher, the authors Hamel and Woodward, in an article entitled, "Developing Mathematics on a Pool Table", take a new look at pool tables and axiomatic systems combined into one exercise. With guidance this system might prove to be both an introduction to higher mathematics and some fun, too.

References:

1) Fitzgerald, Lindblom, Zetterberg, and Dalton, Geometry, Theory and Application, Laidlaw Brothers, River Forest, Illinois, 1971.

2) Hamel and Woodward, "Developing Mathematics On a Pool Table", The Mathematics Teacher, February, 1977, p.154-163.

Submitted by: Ron Kovar
Gloria Fairchild

Question 15:
(a) What is non-euclidean geometry?
(b) When was this geometry developed and by whom?

Many attempts were made to prove Euclid's 5th postulate. The earliest significant technique was used by Gerolamo Saccheri (1667-1733). He organized his attempt around a study of the properties of a special figure, a quadrilateral in which two equal sides are both perpendicular to a third side.

Then Johann Heinrich Lambert (1728-1777) followed by centering attention on another special figure, a quadrilateral, three of whose angles are right angles.

These attempts were followed by Adrien Marie Legendre (1752-1833) whose investigations of the sum of the angles of a triangle is either 1) equal to two right angles, or 2) greater than two right angles, or 3) less than two right angles.

The failure of all these attempts to prove the 5th postulate gave birth to a new conviction - that being that the 5th postulate was independent of the other axioms. Carl Friedrich Gauss (1777-1855) who was probably the greatest mathematician of all time turned his attention to this baffling problem. He confided some progress to his friend, Wolfgang Bolyai (1775-1856) and admitted that his work was not leading to the intended proof of the 5th postulate, but to something different and quite valid. He believed that Euclid's geometry was necessarily true, and believed that there must be another geometry. His assumption that the sum of 3 angles is less than 180° led him to a very curious geometry, but very consistent. He worked on this for over 30 years.

Gauss kept his discovery quite secret since he feared that his contemporaries biased by the Kantian philosophy would not understand his ideas and ridicule him.

He finally put his summary in writing in a letter in May, 1831, after 40 years.

But in January, 1832 Gauss received a letter from his friend Bolyai that informed him he needn't publish his findings since his own son, Johann had independently made the same discovery and already published it. He enclosed a copy of that paper.

Johann Bolyai (1802-1860) had acquired an interest in the theory of parallels from his father. In 1825, Johann sent his father the abstract of his discoveries. In 1829 he sent him the completed manuscript. Gauss received it from the elder Bolyai in 1832 and commented that the son had taken the same path that coincided with his own meditations.

Young Bolyai's joy of having created a new universe was dimmed when he heard he wasn't the first to publish it. Three years before his paper was published, Nicolai Ivanovitsch Lobatschewsky (1793-1856) had already published in a Russian paper on the same subject. This distressed him tremendously.

This new geometry discovered by Gauss, Bolyai, and Lobatschewsky came to be known as Non-Euclidean Geometry.

NON-EUCLIDEAN GEOMETRY

Non-Euclidean geometry developed as an attempted denial of Euclid's Fifth Postulate of Geometry. The Five Postulates are as follows:

1) One and only one straight line may be drawn joining two points.
2) A straight line may be produced continuously in either direction.
3) A circle may be drawn with any center and diameter.
4) All right angles are equal to each other.
5) Through a point outside a given straight line, there passes one and only one straight line that does not meet the given line, however far it is extended.

Several nineteenth century mathematicians attempted to prove that the "parallel postulate", as it came to be known is merely a result of some or all the preceding postulates. These attempts to make the Fifth Postulate were unsuccessful, and attempts to disprove it were fruitless as well.

The independence of the Fifth Postulate was established by construction of a geometry of points, lines, etc. where the parallel postulate did not hold. This became known as Non-Euclidean Geometry.

I. Hyperbolic

Hyperbolic geometry is present in a universe much different than a mere flat plane. It is similar to two adjacent trumpets with mouthpieces facing in opposite directions and is often referred to as a pseudosphere. Here a straight line is defined as a continuous trail along the trumpetlike surfaces. Though difficult to conceptualize, it can be seen that in this universe, given a straight line and a point not on that line, there are an <u>infinite</u> number of lines which pass through that point and are simultaneously parallel to the original line. Also, the sum of the angles of a triangle in this geometry is different from the sum of the angles of a Euclidean triangle in that it is less then 180°.

II. Elliptical

Elliptical geometry is much easier to visualize and understand because the universe is like an elongated sphere, much like our actual earth itself. Here a line is defined as a great circle on that sphere. It can be seen that with this definition, lines are not infinite, but they actually meet themselves and that there are no parallel lines (great circles) and, in fact, all lines intersect other lines in exactly two points.

The discovery of Non-Euclidean geometries ultimately led to the question of whether we actually live in a Euclidean or Non-Euclidean world. Carl Frederick Gauss reportedly attempted to solve this problem by measuring the angles formed by three distant mountain peaks. He was said to have found their sum to be 180°.

The original conclusion was that, because of this 180° sum, Euclidean geometry was more applicable to our real world. However it was later noted that in measuring relatively small lengths between mountains, the angle sum is hyperbolic geometry is so close to 180° that it could go undetected. This is similar to an analogy between Newton's and Einstein's physics where comparable answers are obtained for smaller values, but when larger magnitudes are involved, then Einstein's theories appear to be much more valid.

The effects of the discovery, of Non-Euclidean geometry are as follows:

1) It led to closer scrutiny of axiomatic structures, encouraging the questioning of the validity of postulates.
2) It led to the forming of new branches of mathematics in the twentieth century.
3) It showed that sometimes failure to disprove a widely held belief is not necessarily fruitless, but indeed can expand our view of mathematics.
4) It gave mathematicians a different way of looking at the world.

An elementary exposure to Non-Euclidean geometry may be enriching to the middle school student for several reasons. Primarily, it may teach the students that skepticism is a healthy attribute in learning if kept in perspective. Secondly, students may see that our systems of mathematics are just human inventions and that they are constantly changing as our knowledge of ourselves and our universe continues to grow. Finally, it may help students to understand that mathematics comprehension is certainly relative and is valid only within the context of which it is defined.

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Submitted by: Cynthia Laurie
Robert Guzley

Question 16:

(a) There exist in the plane regular polygons with any number of sides. Are there in space regular solids with any number of faces?

(b) Such solids are connected with the name of a famous person. What is it?

A polyhedra is a solid whose surfaces consists of a number of polygonal faces. A regular polyhedra has all faces and angles congruent.

The Greeks had known of regular polyhedra. Because of their association with Plato, these polyhedra were called the Platonic Solids.

The Platonic Solids are named on the following chart. "V" means vertices, "E" means edges and "F" means faces.

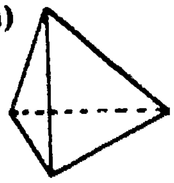
Platonic Solids	V	E	F
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

The Greeks proved that there were only five of these solids by metric methods, i.e., employing theorems about length, angle measurement and congruence.

The formula $V - E + F = 2$ was found in a manuscript of Descartes. It appears to have been written in the early 1660's. This formula was proven by Euler, and was published in 1752.

V = 4
E = 6
F = 4

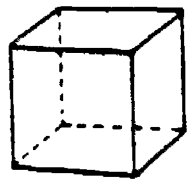
(a)



Tetrahedron

V = 8
E = 12
F = 6

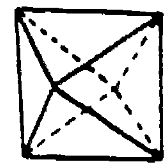
(b)



Cube

V = 6
E = 12
F = 8

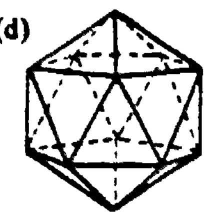
(c)



Octahedron

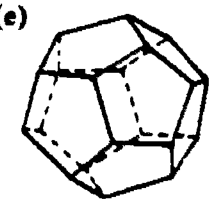
V = 12
E = 30
F = 20

(d)



Icosahedron

(e)



V = 20
E = 30
F = 12

Dodecahedron

The application of the Platonic Solids and Euler's Equation, in middle school mathematics, are as follows:

(1) area of faces
(2) surface area of a solid
(3) volume
(4) constructions
(5) pattern recognition
(6) verification of the formula by inspection
(7) problem solving as it relates to the above topics.

Some geometry construction books are: Geo-ring Polyhedra by Linda Silvey, Dual Discoveries Through Straw Polyhedra by Mary Laycock, Paper and Scissors Polygons by Linda Silvey and Loretta Taylor, Mathematical Models by Cundy and Rollett, Patterns in Space by Col. Robert S. Beard, and Puzzles in Space by David Stonerod.

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Submitted by: Jack Halferty
Marlene Montaque

Question 17:

(a) In what way has there been established a basic connection between algebra and geometry?

(b) When was this connection originated and by whom?

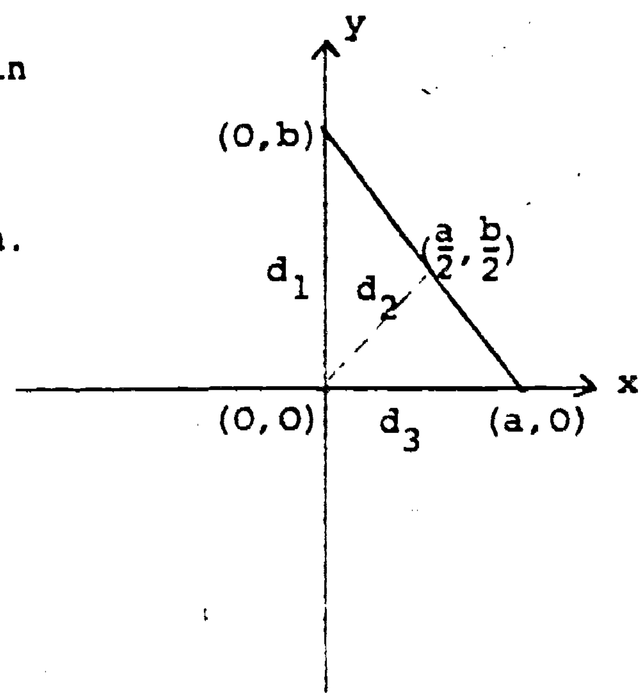
(b) This question is more easily approached by answering part (b) first. There seems to be some disagreement as to who originated the connection between algebra and geometry. Some authorities say that since Apollonius derived most of the geometry of the conic sections from the geometrical equivalents of Cartesian equations, analytic geometry was an invention of the Greeks. Others say the Egyptians and Romans originated the connection by using the idea of coordinates in surveying. However, most give credit to Rene Descartes for first devising the system of coordinates which made possible the development of analytic geometry in the early to mid 1600's.

(a) The method of analytic geometry in general consists of connecting algebra and geometry by means of systems of coordinates in which geometric curves may be represented by algebraic equations and vice versa. This method in many cases provides a much easier solution to geometric problems than is possible by the synthetic geometry of Euclid.

Most of us are familiar with the synthetic method of proof used in high school plane geometry courses. Here is an illustration of how a theorem of plane geometry may be proved by use of coordinates:

Prove analytically that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Solution: Since this problem is a little more sophisticated than beginning problems, there are a few analytic equations we can use that have already been derived.



I will locate the triangle so that its right angle is at the origin and the legs coincide with the axes. This is a general enough location. Let the legs be of length b and a. The coordinates of the vertices will then be (0,0), (0,b), and (a,0).

By using a derived formula for the coordinates of a midpoint

$$x = \frac{x_2 + x_1}{2}, \quad y = \frac{y_2 + y_1}{2},$$

the midpoint of the hypotenuse becomes:

$$x = \frac{a+0}{2} = \frac{a}{2} \qquad y = \frac{0+b}{2} = \frac{b}{2} \qquad (\frac{a}{2}, \frac{b}{2})$$

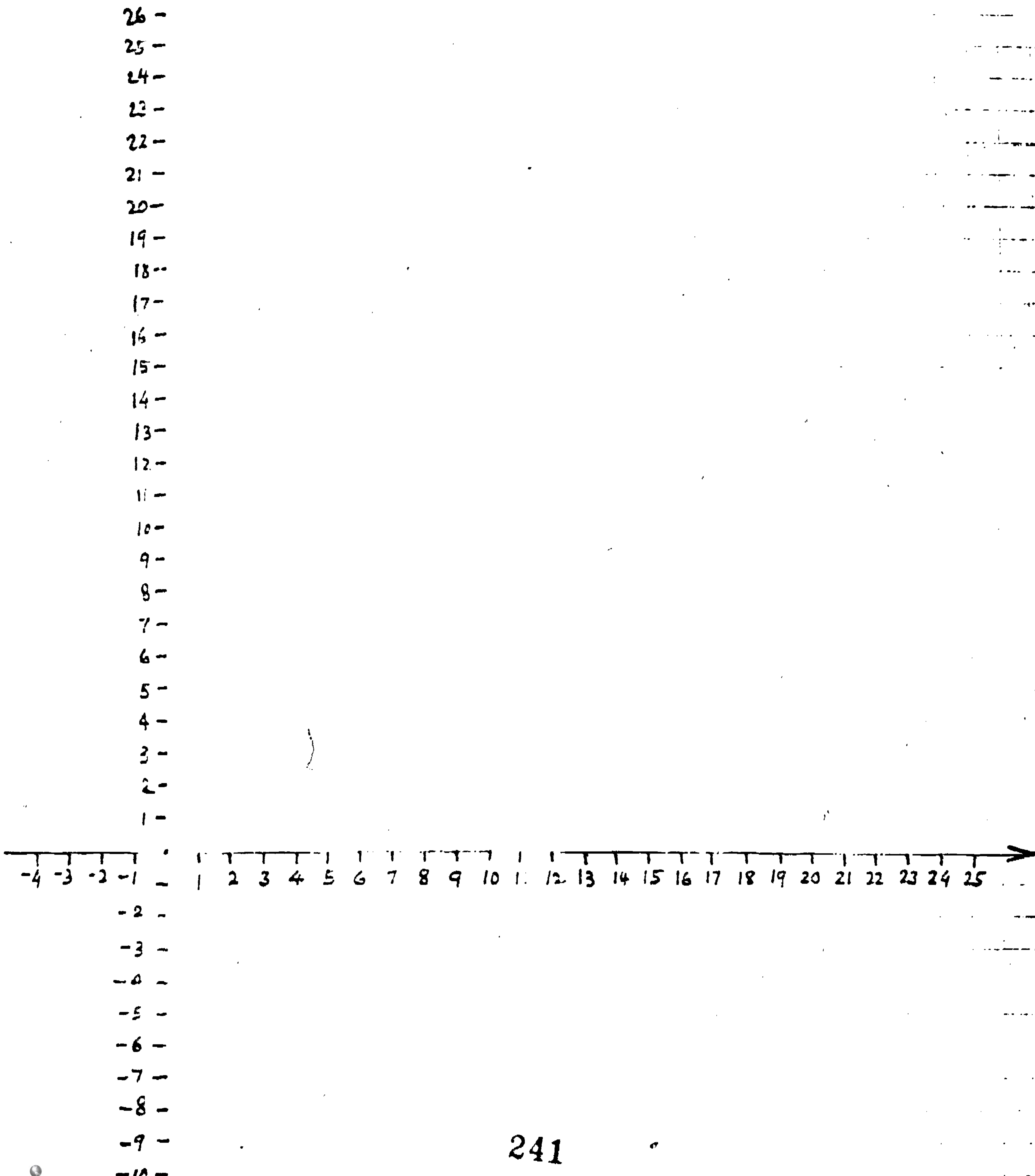
Then using a derived formula for the distance or length of a segment between two points $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, the distance between $(\frac{a}{2}, \frac{b}{2})$ and <u>each</u> of the three vertices can be shown to be $\frac{\sqrt{a^2 + b^2}}{\sqrt{4}}$;

$$d_1 = \sqrt{(\frac{a}{2} - 0)^2 + (\frac{b}{2} - b)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

$$d_2 = \sqrt{(\frac{a}{2} - 0)^2 + (\frac{b}{2} - 0)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

$$d_3 = \sqrt{(\frac{a}{2} - a)^2 + (\frac{b}{2} - 0)^2} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

HOW DO YOU MAKE AN ELEPHANT FLOAT?



LETTERS

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For  x = 4 to 6: set (x,26): set (x,25): set (x,23): Next x
For  x = 7 to 9: set (x,25): Next x
For  x = 11 to 13: set (x,26): set (x,24): Next x
For x = 17 to 19: set (x,26: Next x
For y = 23 to 26: set (7,y): set (g,y): set (10,y): set (11,y):
     set (16,y): set (18,y): Next y
For y = 25 to 26: set (4,y): Next y
For y = 23 to 25: set (6,y): Next y
For y = 24 to 26: set (13,y): Next y
For x = 13 to 15: set (x,-3): Next x
For x = 16 to 18: set (x,-2): Next x
For x = 15 to 17: set (x,-6): set (x,-7): Next x
For x = 18 to 20: set (x,-6): set (x,-7): set (x,-8): Next x
For x = 21 to 22.5: set (x,-7): Next x
For y = -2 to -5: set (13,y): set (15,y): set (17,y): Next y
For y = -6 to -8: set (14,y): set (17,y): set (18,y): set (20,y):
     set (22,y): Next y
For y = -6 to -7: set (15,y): set (21,y): Next y
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$y = -3x + 55 \quad 19 \leq x \leq 20$

$y = 3x - 55 \quad 20 \leq x \leq 21$

$y = -3x + 71 \quad 21 \leq x \leq 22$

$y = 3x - 71 \quad 22 \leq x \leq 23$

PICTURE (Continued)

Set H

(1) $y = 11 \quad 18.5 \leq x \leq 20$

(2) $x = 20 \quad 11 \leq y \leq 11.5$

PLOT (18.5,12.5)

Set I

(1) $y = -x + 31 \quad 17 \leq x \leq 18$

(2) $y = \frac{1}{2}x + \frac{11}{2} \quad 17 \leq x \leq 19$

Set J

(1) $y = -x + 27 \quad 18.5 \leq x \leq 19$

(2) $y = x - 11 \quad 19 \leq x \leq 20$

(3) $y = -3x + 69 \quad 19 \leq x \leq 20$

(4) $y = -\frac{1}{4}x + \frac{67}{4}$ $\quad 17 \leq x \leq 19$

(5) $y = \frac{1}{4}x + \frac{33}{4}$ $\quad 17 \leq x \leq 19$

(6) $y = -x + 32$ $\quad 18 \leq x \leq 19$

(7) $y = x + 4$ $\quad 18 \leq x \leq 19$

(8) $y = -x + 34$ $\quad 19 \leq x \leq 20$

(9) $x = 20$ $\quad 12 \leq y \leq 14$

(10) $y = -x + 32$ $\quad 20 \leq x \leq 23$

(11) $y = x - 14$ $\quad 21 \leq x \leq 23$

(12) $y = -x + 28$ $\quad 20 \leq x \leq 21$

Set K

(1) $y = 9$ $\quad 20 \leq x \leq 21$

(2) $y = x - 12$ $\quad 21 \leq x \leq 22$

PICTURE

Set A

(1) $y = -3x + 7$ $\quad 1 \leq x \leq 2$

(2) $y = 4$ $\quad 1 \leq x \leq 3$

(3) $y = -2x + 10$ $\quad 3 \leq x \leq 4$

(4) $y = \frac{1}{9}x + \frac{14}{9}$ $\quad 4 \leq x \leq 12$

Set B

(1) $y = -\frac{1}{3}x - \frac{1}{3}$ $\quad -4 \leq x \leq -1$

(2) $y = \frac{1}{3}x + \frac{1}{3}$ $\quad -1 \leq x \leq 2$

Set C

(1) $y = -\frac{1}{3}x + 9$ $\quad 18 \leq x \leq 21$

(2) $y = \frac{1}{3}x - 5$ $\quad 21 \leq x \leq 24$

(3) $y = -\frac{1}{3}x + 11$ $\quad 24 \leq x \leq 27$

Set D

(1) $x = 3$ $\quad 4 \leq y \leq 12$

(2) $y = 3x + 3$ $\quad 3 \leq x \leq 4$

(3) $y = \frac{2}{3}x + \frac{37}{3}$ $\quad 4 \leq x \leq 7$

(4) $y = -\frac{1}{3}x + \frac{58}{3}$ $\quad 7 \le x \le 10$

(5) $y = -2x + 42$ $\quad 10 \le x \le 11$

(6) $y = \frac{4}{4}x + \frac{26}{5}$ $\quad 11 \le x \le 16$

(7) $y = x - 6$ $\quad 16 \le x \le 17$

(8) $y = 11$ $\quad 17 \le x \le 18$

(9) $y = 2x - 25$ $\quad 17 \le x \le 18$

(10) $y = -\frac{1}{3}x + \frac{44}{3}$ $\quad 11 \le x \le 17$

(11) $y = -\frac{1}{4}x + \frac{55}{4}$ $\quad 11 \le x \le 12$

(12) $y = \frac{5}{12}x + \frac{21}{4}$ $\quad 9 \le x \le 12$

Set E

(1) $x = 4$ $\quad 11 \le y \le 15$

(2) $y = -\frac{1}{6}x + \frac{35}{3}$ $\quad 4 \le x \le 10$

(3) $y = -6x + 70$ $\quad 9 \le x \le 10$

Set F

(1) Plot (11,14)

Set F

(1) $y = 9$ $\quad 8.5 \le x \le 11$

(2) $x = 11$ $\quad 7 \le y \le 9$

(3) $y = \frac{1}{5}x + \frac{24}{5}$ $\quad 9.75 \le x \le 11$

Set G

(1) $y = -2x + 26$ $\quad 9 \le x \le 10$

(2) $y = x - 4$ $\quad 9 \le x \le 10$

(3) $y = 5$ $\quad 9 \le x \le 11$

(4) $x = 11$ $\quad 3 \le y \le 5$

Submitted by: Gary Rhines
Joan Hall

Question 13:

Euclidean geometry may be described as dealing in isometric transformations (that is, transformations in which distances between points remain invariant). Explain.

Euclidean geometry studies those properties of geometric figures that are not changed by moving the figures. There are two basic approaches to this subject:

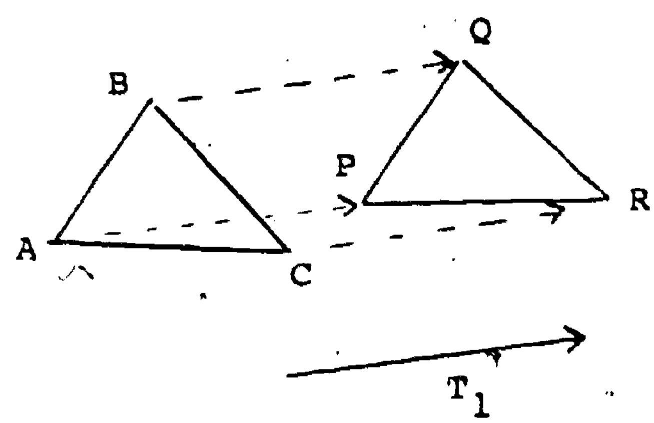
1) the deductive method
2) transformations

An <u>isometric transformation</u> is a "motion" in which the distance between any two points of a figure is not changed. There are three basic isometric transformations:

1) translation - a "slide"
2) reflection - a "flip"
3) rotation - a "turn"

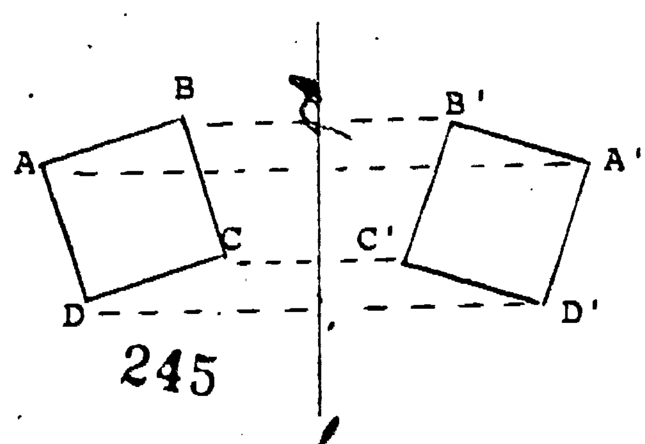
Let's look at each of these in turn.

A <u>translation</u> can be defined by a vector showing the distance and direction of the slide.



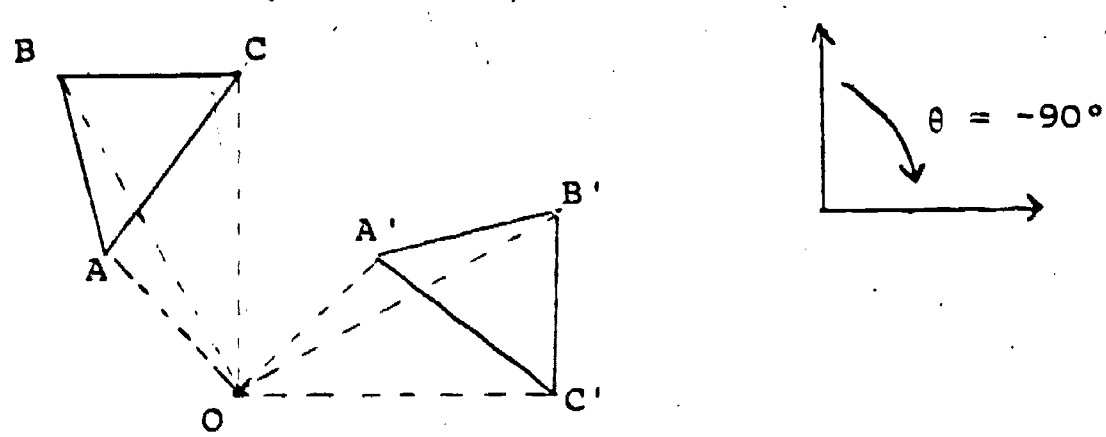
Triangle PQR is called the image of triangle ABC under translation T_1. P is the image of A, Q is the image of B, and R is the image of C.

A <u>reflection</u> can be defined as a "flip" in a given line of reflection, ℓ.



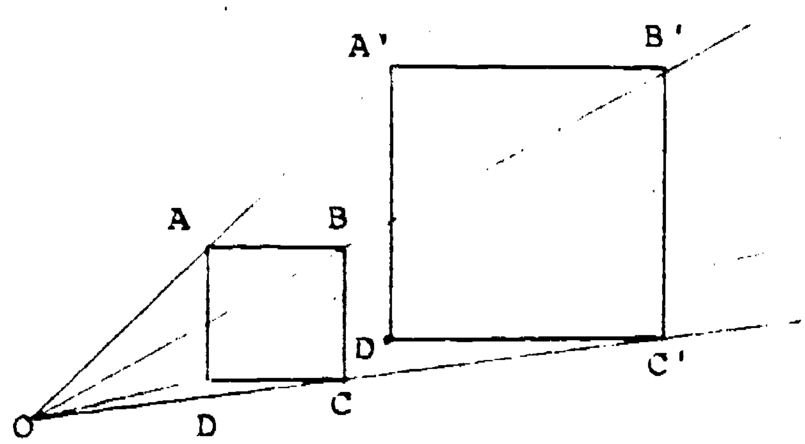
Note that the line of reflection, ℓ, is the perpendicular bisector of each line segment joining a point on the original figure to its image. Note also that reflection reverses orientation. To go from A to B in the original figure, one must traverse the figure in a clockwise direction, but to go from A' to B' one must traverse the figure in a counter-clockwise direction.

A <u>rotation</u> can be defined by a center of rotation, O, and a turn, θ (theta).



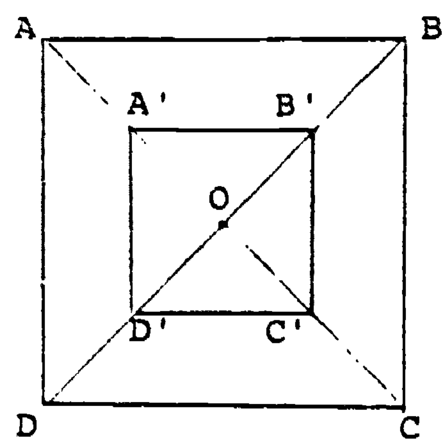
Note that $\angle AOA' \cong \angle BOB' \cong \angle COC' \cong \theta$ and that $\overline{OA} \cong \overline{OA'}$, $\overline{OB} \cong \overline{OB'}$, and $\overline{OC} \cong \overline{OC'}$.

Another transformation which is <u>not isometric</u> is a <u>dilation</u> or <u>enlargement</u>. Although figures transformed by enlargement are not congruent, they are similar. To define an enlargement requires a center of enlargement, O, and a scale factor, k.



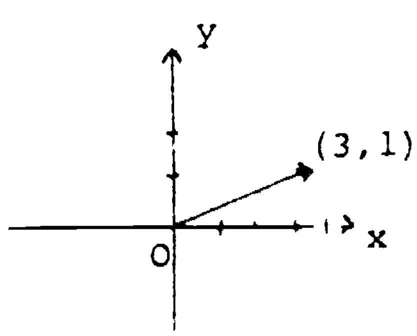
Note that the scale factor, $k = \frac{\overline{OA'}}{\overline{OA}} = \frac{\overline{OB'}}{\overline{OB}} = \frac{\overline{OC'}}{\overline{OC}} = \frac{\overline{OD'}}{\overline{OD}}$.

Here is another enlargement (dilation) with $k = \frac{1}{2}$ and O in the center of the figure.



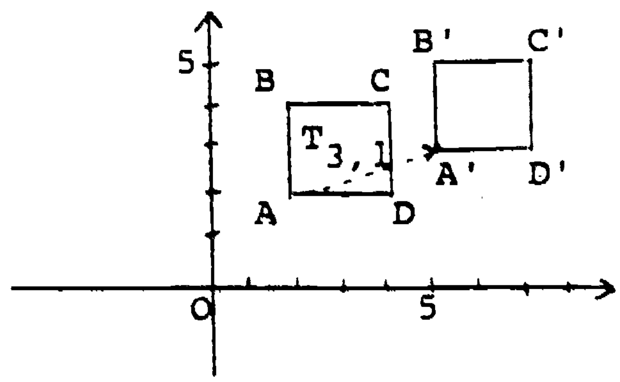
Note that $k = \frac{1}{2} = \frac{\overline{OA'}}{\overline{OA}} = \frac{\overline{OB'}}{\overline{OB}} = \frac{\overline{OC'}}{\overline{OC}} = \frac{\overline{OD'}}{\overline{OD}}$.

Transformations provide an additional way to link geometry and algebra. Initially transformations may be introduced completely in geometric terms. Later, transformations may be done on a coordinate plane, and the effect that various transformations have on the coordinates of a point may be systematically studied. The example below shows the effect of a translation.



$T_{3,1}$

A translation may be defined by the coordinates (a,b) of the endpoint of a vector beginning at the origin.



point	coordinates	image	coord.
A	(2,2)	A'	(5,3)
B	(2,4)	B'	(5,5)
C	(4,4)	C'	(7,5)
D	(4,2)	D'	(7,3)

$$(x,y) \xrightarrow[T_{3,1}]{} (x+3,y+1)$$

In general the image of a point with coordinates (x,y) under translation $T_{a,b}$ (as defined above) is the point with coordinates $(x+a,y+b)$. We say that $T_{a,b}$ maps the point with coordinates (x,y) onto the point with coordinates $(x+a,y+b)$ and write:

$$T_{a,b} : (x,y) \longrightarrow (x+a,y+b)$$

A systematic study of <u>rotations with center at the origin</u> will yield the following results:

$$R_{90°(ccw)} : (x,y) \longrightarrow (-y,x)$$

$$R_{90°(cw)} : (x,y) \longrightarrow (y,-x)$$

$$R_{180°} : (x,y) \longrightarrow (-x,-y).$$

A systematic study of reflections will yield the following results (the letter M is used to indicate a reflection and the subscript indicates the line of reflection):

$$M_{x\text{-axis}}: (x,y) \longrightarrow (x,-y)$$

$$M_{y\text{-axis}}: (x,y) \longrightarrow (-x,y)$$

$$M_{y=x} : (x,y) \longrightarrow (y,x)$$

$$M_{y=-x} : (x,y) \longrightarrow (-y,-x)$$

A dilation or enlargement with center at the origin and scale factor, k, has the following result:

$$D_k : (x,y) \longrightarrow (kx,ky)$$

We have attempted to give a brief introduction to the transformational approach to geometry. This approach seems very appropriate for middle school mathematics. There are several reasons for this:

1) Figures may be drawn on tracing paper and easily manipulated by students.
2) The link between geometry and algebra may be easily exploited.
3) "Escher-type" tessellations are an interesting extension.
4) Transformations may be explored on a computer at several levels:
   a) Using Logo from a strictly geometric point of view.
   b) Using BASIC or Logo from a coordinate-geometry perspective.
5) Transformations lay the foundation for more advanced work in computer graphics including 3-D transformations.

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Submitted by: David Hallas
Ken Servais

Question 19:
The problem of drawing a tangent to a given curve at a given point is closely connected with the problem of finding areas.
(a) What is this connection?
(b) When was it discovered and by whom?

(a) Calculus is defined by Encyclopedia Britanica as "the branch of mathematics dealing with the problem of defining precisely and calculating the slope of a curved line and the area inside a region bounded by a curved line." Finding slopes or more properly derivatives is the task of differential calculus, while finding areas more properly called integrals, is the task of integral calculus. The fundamental theorem of calculus relates these two concepts by stating that finding the area enclosed by the curve of a given function is equivalent to finding a function having the given function as its derivative. The two apparently unconnected limiting processes involved in the differentiation and integration of a function are intimately related. They are in fact, inverse to one another, as are addition and subtraction, or multiplication and division.

While differential calculus and integral calculus were once thought to be separate notions, the understanding that the two are intimately related led to the BIG idea in the evolution of mathematical analysis - there is but one calculus.

(b) It was as a student at Cambridge that Isaac Newton became acquainted with the work of Isaac Barrow who is generally credited with being the first to realize in full generality that differentiation and integration are inverse operations. Barrow resigned his position at Cambridge in favor of his student, Newton. It was in the year of his retirement, 1669 that his work was published.

Newton and Gottfried Wilhelm von Leibnitz are credited with the Fundamental Theorem of Calculus because, even though they worked independently of one another, they both succeeded in clearly recognizing and publicizing the notion. Newton tended to approach the notion abstracted from intuitive ideas of motion. English scholars tended to overlook the contributions of Leibnitz out of patriotic loyalty to Newton. Leibnitz, however, advocated mathematical reasoning or a calculus of deductive reasoning which laid the groundwork for the critical work of the 20th century on the foundations of analysis and mathematical thought.

It was left for future mathematicians such as Weierstrass, Euler, and Cauchy to refine the "new calculus", to include a more generally accepted, systematic approach to calculus.

References:

What is Mathematics, by Richard Courant and Herbert Robbins, Oxford University Press, London, 1941.

History of Mathematics, by Howard Eves, Holt, Rinehart and Winston, NY, 1964.

Development of Mathematics, by E.T. Bell, McGraw-Hill Book Co., Inc., NY, 1940.

History of Mathematics, by Florian Cajori, Macmillan and Co., NY, 1894.

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AREA OF CLOSED CURVES

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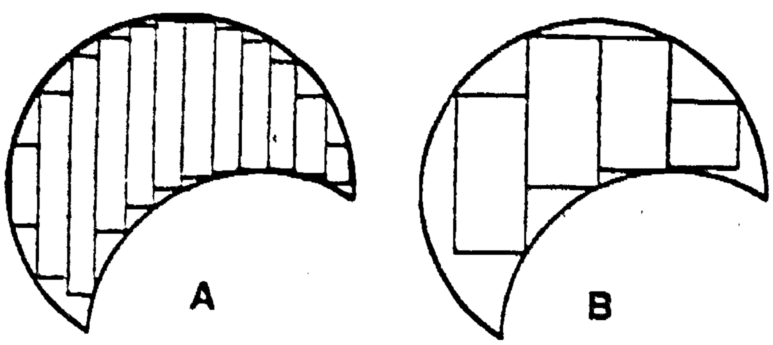
To find the approximate area of a closed curve, divide it into rectangles. Find the area of each rectangle. Then find the sum of those areas.

1. Which figure shows the sum of rectangles that most closely approximates the area of the curve?

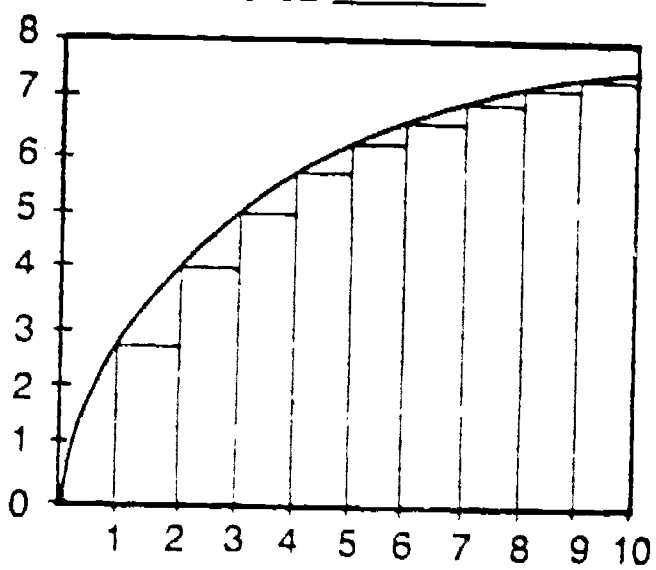
figure ______



A B

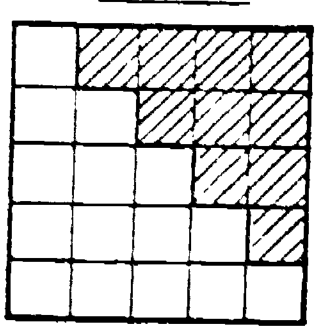
2. Estimate the area under the curve in square units.

area ______

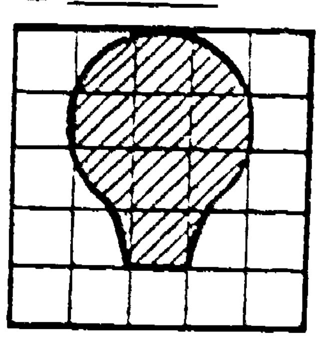


3. What fractional part of each figure is shaded?

A ______

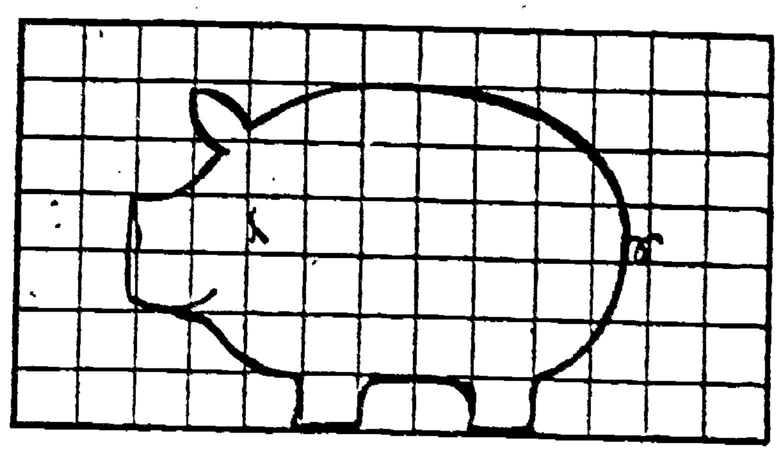


B ______



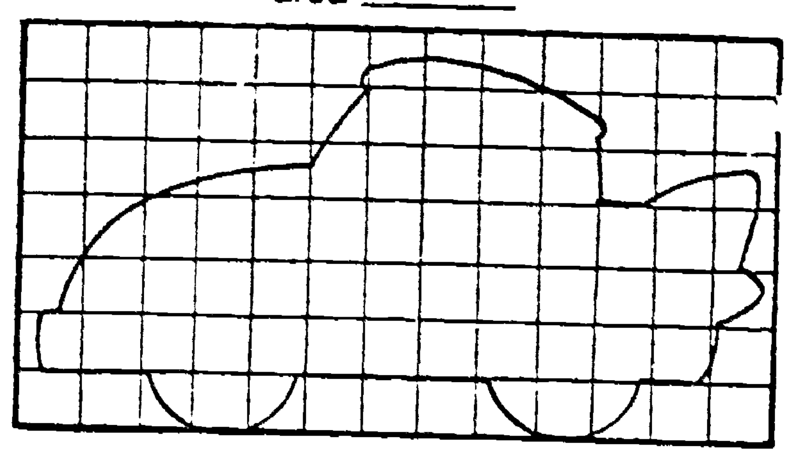
4. Estimate the area of the picture in square units.

area ______



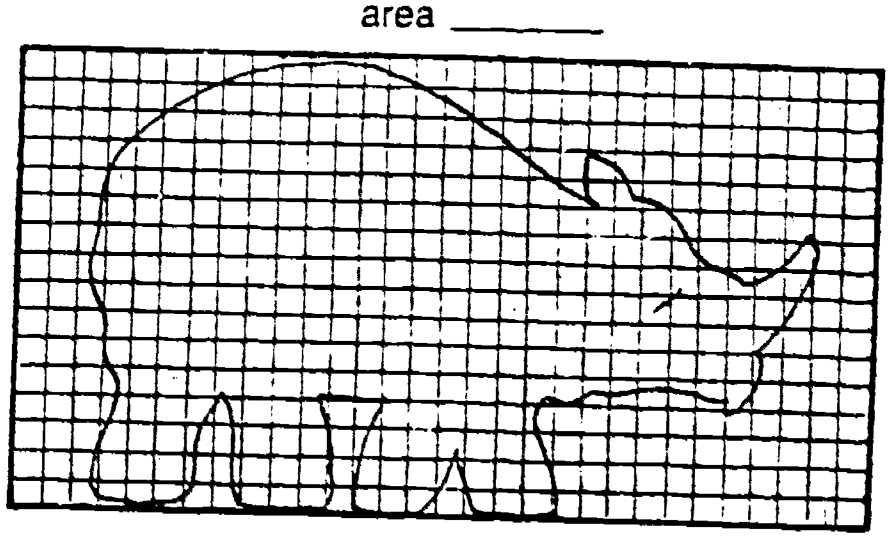
5. Estimate the area of the picture in square units.

area ______



6. Estimate the area of the picture in square units.

area ______



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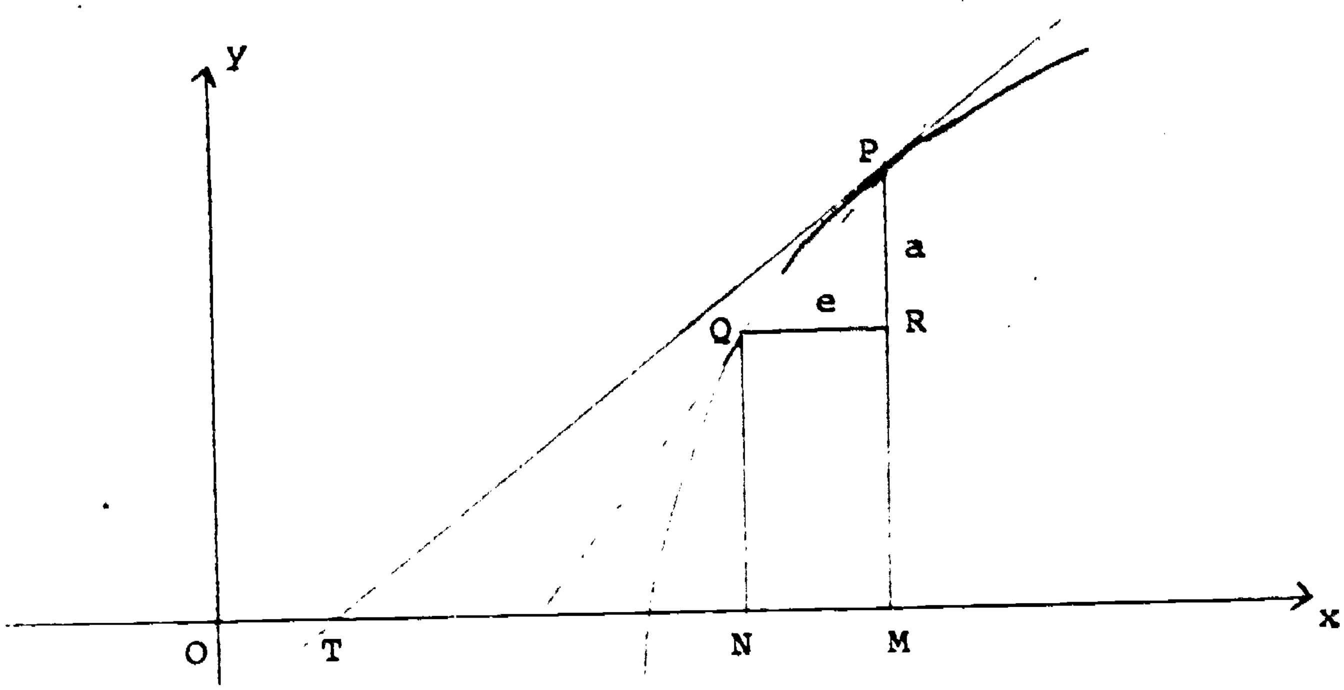
Lectiones opticae et geometricae

Isaac Barrow (1669)



Barrows approach - differential triangle

Required - to find the tangent at a point P on the given curve.

Method - Let Q be a neighboring point on the curve

- $\triangle PTM \sim \triangle PQR$
- as $\triangle PQR$ becomes infinitely small then $\frac{RP}{QR} = \frac{MP}{TM}$.
- Let QR = e and RP = a; P has coordinates (x,y) and Q has coordinates (x - e, y - a).
- Substitute the values into the equation of the curve. Neglecting squares and higher powers of e and a results in the ratio a/e which determines the slope of the tangent line.

Given the equation

$$x^3 + y^3 = r^3$$

let QR = e and RP = a

$$P = (x, y) \text{ and } Q = (x - e, y - a).$$

Then $(x - e)^3 + (y - a)^3 = r^3$

$$x^3 - 3x^2e + 3xe^2 - e^3 + y^3 - 3y^2a + 3ya^2 - a^3 = r^3.$$

Neglecting the square and higher powers of a and e, (as a and e become smaller their squares and higher powers become smaller exponentially and approach zero), we have:

$$x^3 - 3x^2e + y^3 - 3y^2a = r^3$$

Since $x^3 + y^3 = r^3$, by subtraction we have

$$-3x^2e - 3y^2a = 0$$

or

$$3x^2e + 3y^2a = 0.$$

Solving for a

$$a = -\left(\frac{x^2}{y^2}\right)e.$$

Then

$$\frac{a}{e} = -\frac{x^2}{y^2}$$

$\frac{a}{e}$ is the modern $\frac{dy}{dx}$.

If $\frac{dy}{dx} = -\frac{x^2}{y^2}$ then the integral of the expression should yield the original equation. In fact it does.

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$y^2dy = -x^2dx$$

$$\int y^2dy = \int -x^2dx$$

$$\frac{y^3}{3} = \frac{-x^3}{3} + c$$

$$\frac{x^3}{3} + \frac{y^3}{3} = c$$

$$x^3 + y^3 = 3c$$

which has the form $x^3 + y^3 = r^3$ and 3c approximates the value of r^3.

Submitted by: Warren Hastings
Marcia Swanson

Question 20:

(a) There are sets of different infinities. What does this mean? Can you explain this idea using the set of rational numbers and the set of real numbers?

(b) When was this discovered and by whom?

Infinity

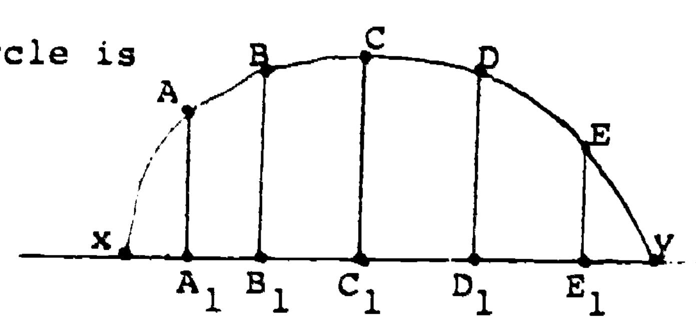
In 1870, the German mathematician Cantor (1845-1918) applied some of the ideas of symbolic logic to sets of numbers. He formed a theory that he called "the theory of sets", Cantor developed this theory because of his interest in infinite quantities. For example, he showed how the members of certain infinite sets could be matched one for one against each other such as

1	2	3	4	5
↕	↕	↕	↕	↕
2	4	6	8	10

George Cantor, the primary force in the development of the theory of the infinite introduced in 1874, was subjected to extreme ridicule by his colleagues, suffered mental breakdown, and died in a mental hospital.

Infinity is used in the potential sense to mean something approached, but never reached. In the example below, in theory, the end could never be reached.

Cantor defined the infinite number $\aleph_0$ called Aleph-null to represent the size of the infinite set. $\aleph$ -the natural numbers. Another infinite number $\aleph_1$ called Aleph-one represents the infinite number of points on a line or line segment. For example, for each point on the semicircle there is a point on the line segment, and for each point on the line segment there is a point on the semicircle. This one-to-one correspondence indicates that the cardinality of the semicircle is also $\aleph_0$



References

1. David, Philip J., The Lare of Large Numbers, The L.W. Singer Co., U.S.A., 1961.

2. NCTM Student Mathematic Notes, March, 1983.

3. Zulauf, Achim, The Logical and Set Theoretical Foundations of Mathematics, Oliver and Boyd, Edi~burg, 1969.

Submitted by: Nancy Hemingway
Esther Williams

PART III - EVALUATION

THE HONORS TEACHERS WORKSHOP: A FIRST EVALUATION

Alex Friedlander*

Purpose

This evaluation of the MSU Honors Teachers Workshop had two related purposes:

1. To detect the participant teachers' perception of their classroom teaching and of the workshop itself both before and after the workshop.

Table 1 describes in more detail the variables that were included in the investigation related to this purpose.

2. To detect relationships and patterns in the participant teachers' confidence in solving, confidence in teaching and level of performance in twenty tasks related to the mathematical content presented by the five MGMP instructional units.

The variables that were included in the investigation related to this purpose are presented in Table 2.

Instrumentation

Two questionnaires were given to the 25 participating teachers in one session of about 45 minutes both before and after the workshop: (1) a questionnaire on teacher perception of his/her own teaching and of the workshop, and (2) a questionnaire on twenty MGMP unit-related mathematical tasks. In order to assure respondent anonimity, but still allow for pre-post matching of answers, each teacher assumed a pseudoname that was systematically used on each questionnaire.

*The author would like to thank Allen Babugura for his patient and helpful assistance in the required computer work.

Table 1. Variables included in the evaluation of teachers' perception of their classroom teaching and of the workshop.

		Curriculum											Teaching style and strategies												
		Arithmetic	Number theory	Geometry	Spatial Visualization	Statistics	Probability	Measurement	Algebra	Calculators	Computers	History of Math	Drill and practice	Short problems	Complex projects	Whole class instruction	Group work	Seat work	Individual work	Open challenges	Originality of responses	Generalizations	Concrete manipulatives	Worksheets	Games
Classroom Teaching	Frequency of teaching before workshop																								
	Planned change in frequency of teaching																								
Workshop	Expectations from the workshop																								
	Extent of satisfaction from workshop																								

Table 2. Variables included in the evaluation of teachers' performance on mathematical tasks related to the MGMP instructional units.

	Number Theory (Factors and Multiples)	Area and Volume Growth (The Mouse and the Elephant)	Probability	Similarity	Spatial Visualization
Confidence in ability to solve the relevant tasks					
Confidence in ability to teach the relevant takss					
Level of performance in solving the tasks					

1. Questionnaire on teaching and workshop

In this questionnaire, the teacher was presented with a list of eleven mathematical topics and of teaching strategies. On each of these 45 items, the teacher was asked two basic questions related to: (a) their own frequency in teaching/using it, and (b) their perception of the workshop.

(a) The question related to pre-workshop frequency in teaching was: "How frequently do you teach the topic/use the strategy in your classes?" The answer options varied on a five-level scale from "Very frequently" to "Never" and were later graded accordingly from 5 to 1. After the workshop, the participants' change in teaching frequency was inquired through the question "Indicate the change in how frequently you plan to teach this topic/use this strategy in your classes as a result of the workshop". The answer options varied on a five-level scale from "Much more frequently than before" to "Much less than before", and were later graded accordingly from 5 to 1.

b) The participants' evaluation of the workshop was determined by their pre-workshop reactions to the question "Do you expect this workshop to deal with the topic/strategy?" (Scaled from "Definitely yes" -- 5, to "Definitely no" -- 1), and by their post-workshop reactions to the question "How effectively was the topic dealt with in the workshop"? (Scaled from "Much more than I expected" -- 5, to "Much less than I expected" -- 1).

A complete version of the pre-workshop questionnaire is presented in Appendix A.

2. Questionnaire on twenty MGMP unit-related tasks

This questionnaire was adapted from an evaluation study of a summer workshop for middle-grade mathematics teachers in Israel (the Weizmann Institute, Rehovot) and was originally designed by Fresko and Ben Haim (1984). The questionnaire (Appendix B) presents the teacher with twenty mathematical problems and requires the teacher (1) to grade the confidence in his/her ability to solve each of the items on a four-level scale varying from "I am positive that I can solve it" to "I am not familiar at all with the topic" (graded from 4 to 1); (2) to grade the confidence in his/her ability to teach the item on a three-level scale varying from "I am certain that I could teach it" to "I am not confident that I could teach it" (graded from 3 to 1); and then (3) to actually solve the items (graded 0/1 for wrong/right solution).

The teachers were presented with the same twenty problems before and after the workshop.

Main Findings and Discussion

Graphical representations of means are used in this section to describe patterns in the participant data gathered before the workshop, and to detect changes that can be attributed to the influence of the workshop. Complete tables of means and standard deviations may be found in Appendix C. Correlation, and t-test analyses were also occasionally employed to further clarify the data.

The findings are organized in three parts: (1) the participants' perception of their own teaching repertoire, (2) the participants' perception of the workshop's contribution, and (3) findings related to the twenty MGMP unit-related mathematical tasks.

1. Teaching repertoire: mathematical content and teaching strategies

(a) Content. The distribution of different mathematical topics in the teachers' pre-workshop class curriculum and the effect of the workshop on their planned curriculum is presented in Figure 1. The four regions in the graph are created by a vertical line that separates high and low frequency in teaching before the workshop, and a horizontal line that separates between a positive change ("more than before") and a negative change ("less than before") in teaching frequency.

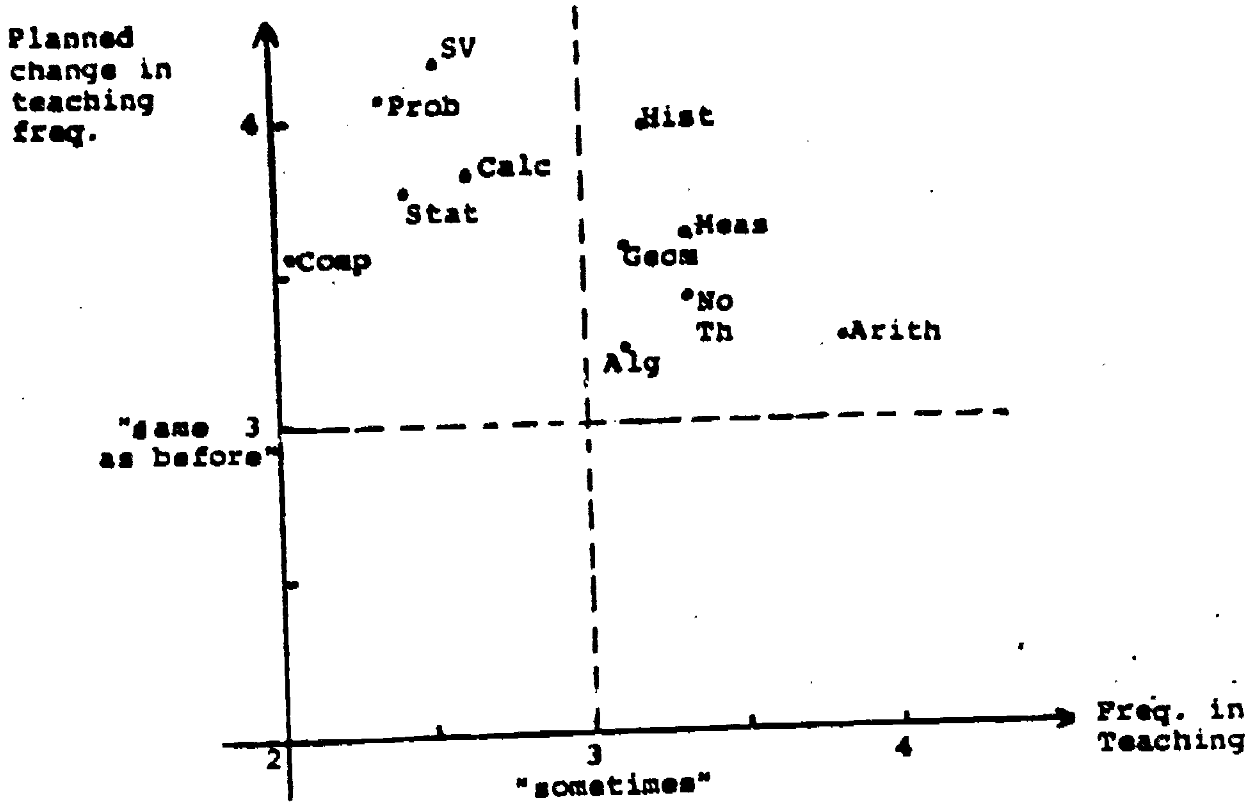


Figure 1. Frequency of teaching eleven mathematical topics before and after the workshop

The results indicate that probability, statistics, spatial visualization, use of computers and calculators were taught less frequently than the other topics. However as a result of the workshop, the teachers plan a rather large increase (above 3.5) in the frequency of teaching exactly these topics that were neglected before (particularly probability and spatial visualization) and also in the teaching of history of mathematics, geometry and measurement.

A strong influence of the workshop maybe detected here: the topics that were indicated by the teachers as subjects for future emphasis are the ones that were dealt with extensively by the workshop. Algebra and arithmetic were not emphasized by the workshop, and consequently did not show a similar growth in planned frequency of teaching.

Number theory that is the subject of one of the five MGMP units presented in the workshop did not have the same impact as the other units had on their related topics.

(b) Strategies. The four graphs presented in Figure 2 indicate the frequency of use of different teaching styles and strategies before and after the workshop. Again, a strong influence on teaching styles and strategies recommended in the workshop may be observed: The participants plan to increase the frequency of using more complex projects in teaching and mathematical topic, whereas the frequency of using drill and practice exercises is planned to stay at about the same level as before the workshop (Figure 2a). The frequency of group work is planned to increase, whereas the previously high frequencies in whole-class instruction and seatwork will probably be decreased (Figure 2b). Also planned for increased use in the classrooms are concrete manipulatives and games (Figure 2c), the posing of open challenges, gathering student responses and encouraging generalizations (Figure 2d). According to the group means, most of these strategies were only occasionally ("Sometimes") in use before the workshop.

2. Teacher perception of the workshop

Before the workshop, the teachers' assessment of their needs was not very discriminatory (Figure 3): they had relatively high expectations (above 3.5) with regard to all mathematical topics (except use of calculators) and with regard to most teaching strategies (except use of drill and practice exercises, seat work, and use of worksheets). After the workshop, however, the teachers rated spatial visualization, probability, and history of mathematics as being dealt with extensively (grade 4 out of a scale of 5) even as compared to their previously highly set expectations. A little lower but still "above expectation" ratings (between 3 and 3.5) were accorded to statistics, number theory, geometry and measurement. The ratings for the workshop's treatment of arithmetic, algebra, computers and calculators proved to be slightly below the participants' (originally highly set) expectations.

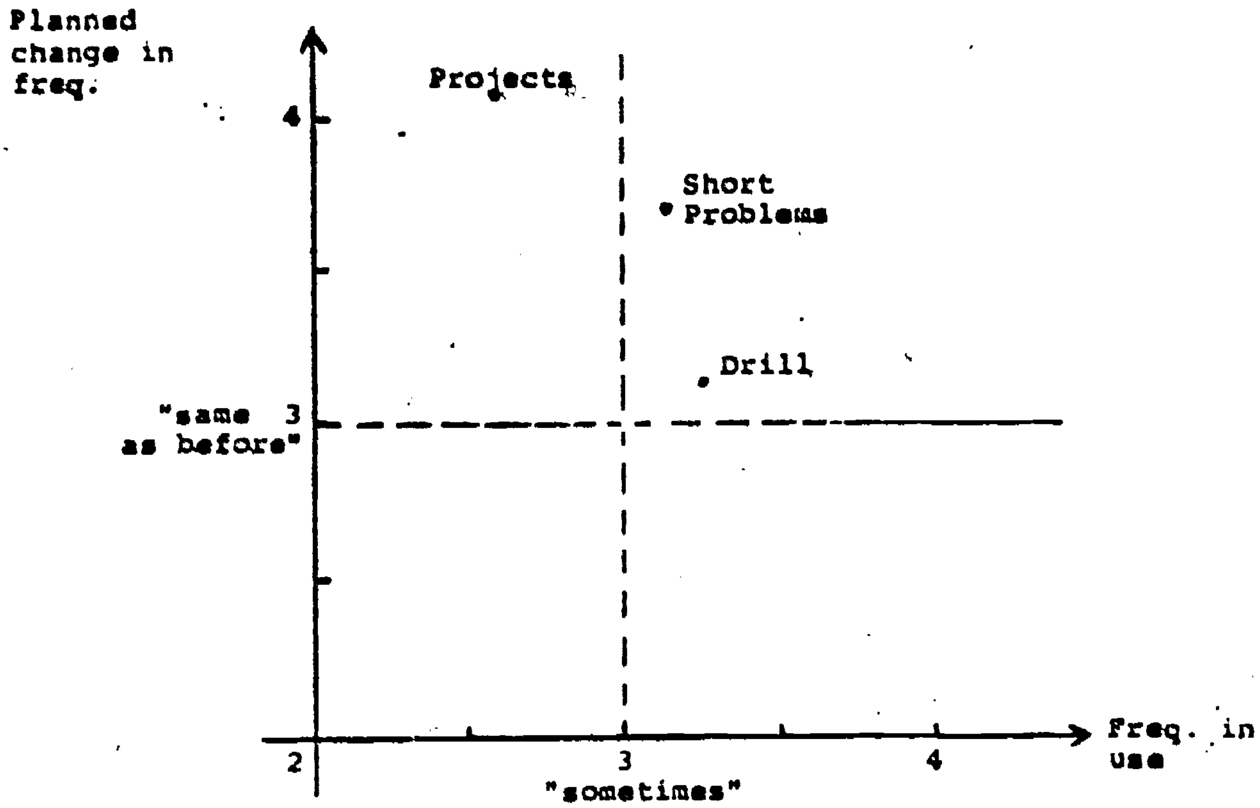


Figure 2a. Frequency of using drill and practice exercises, short word problems, and projects before and after the workshop

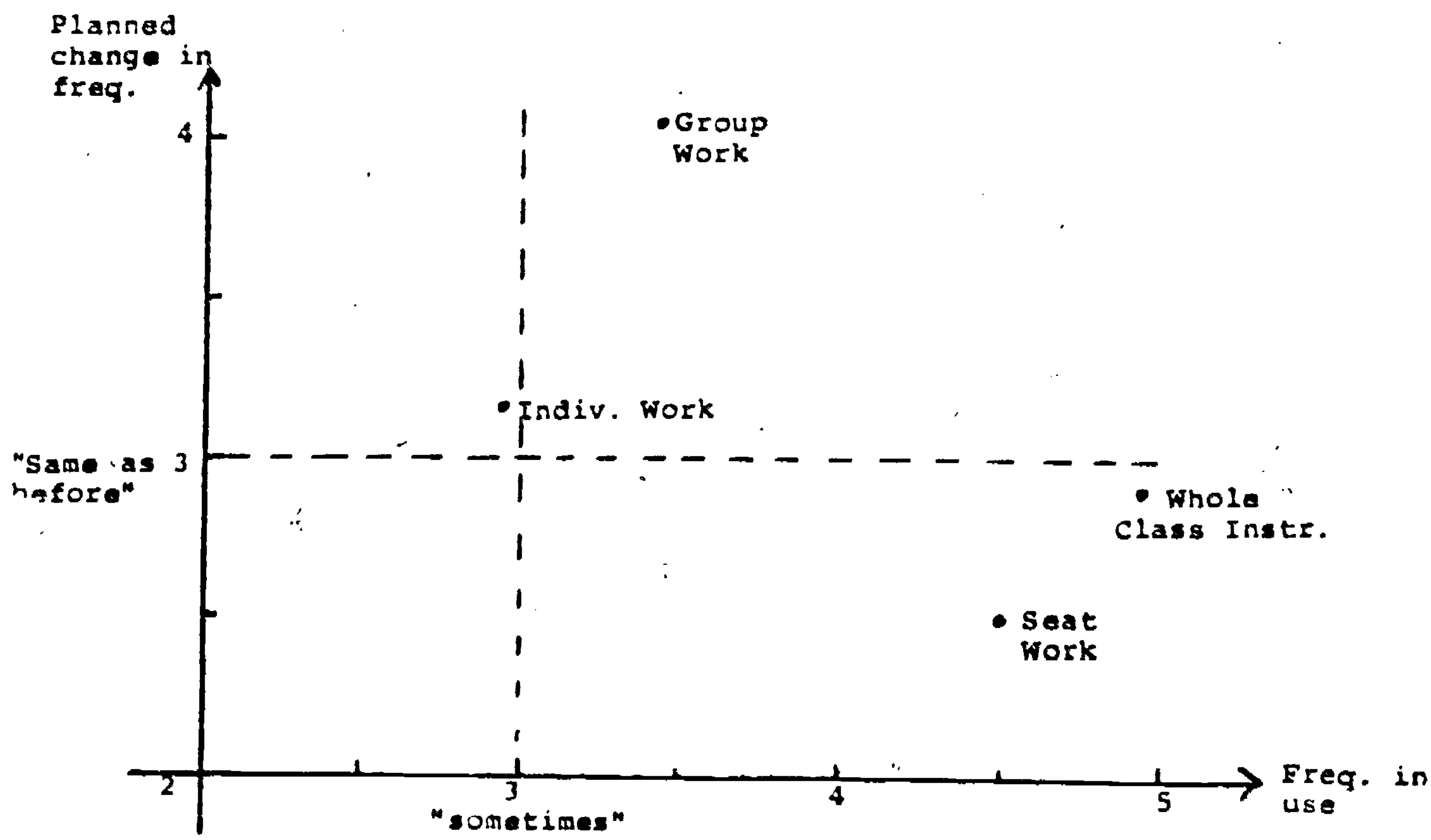


Figure 2b. Frequency of using group work, individualized work, seatwork and whole-class instruction before and after the workshop

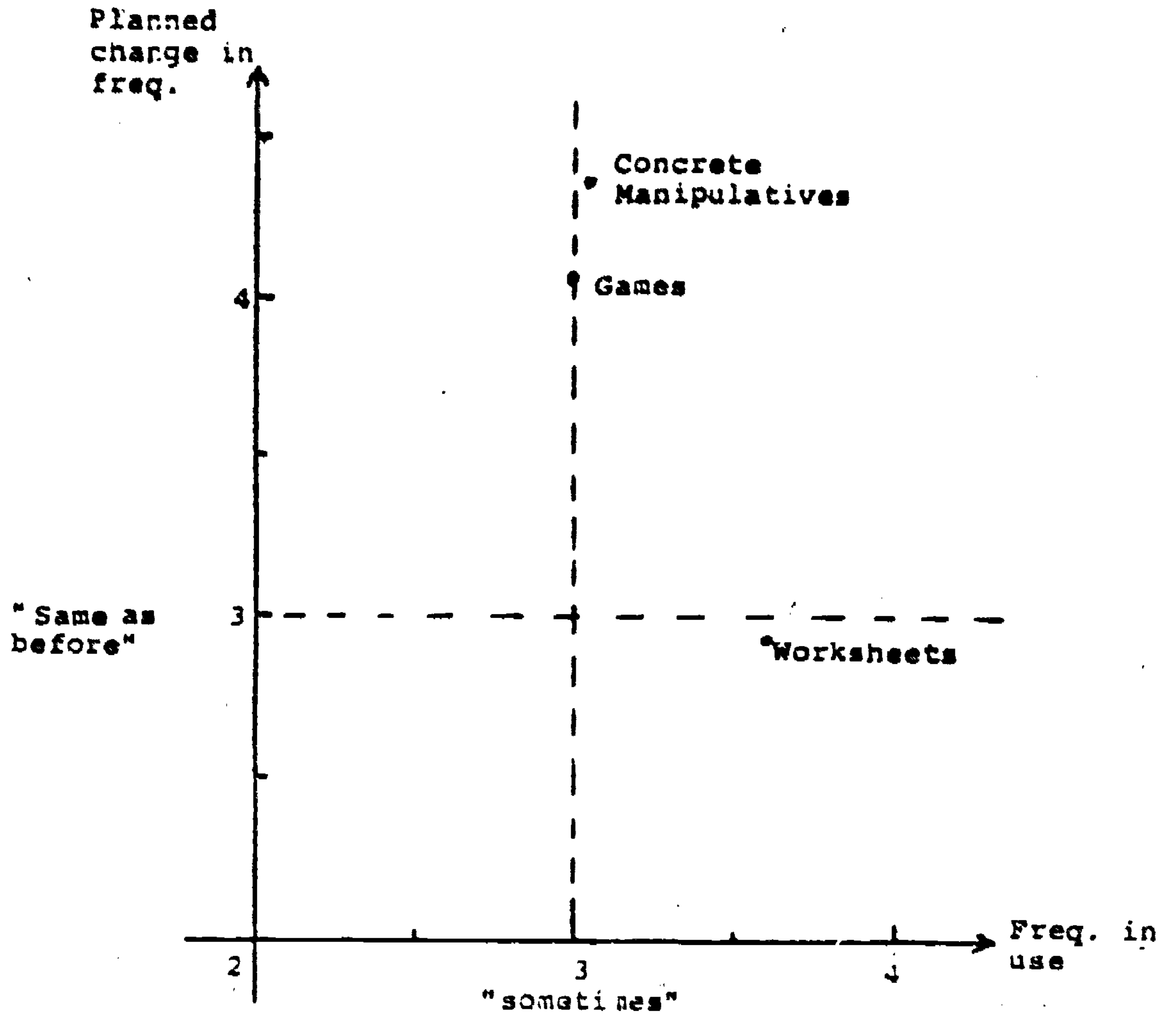


Figure 2c. Frequency of using concrete manipulatives, games, and worksheets before and after the workshop

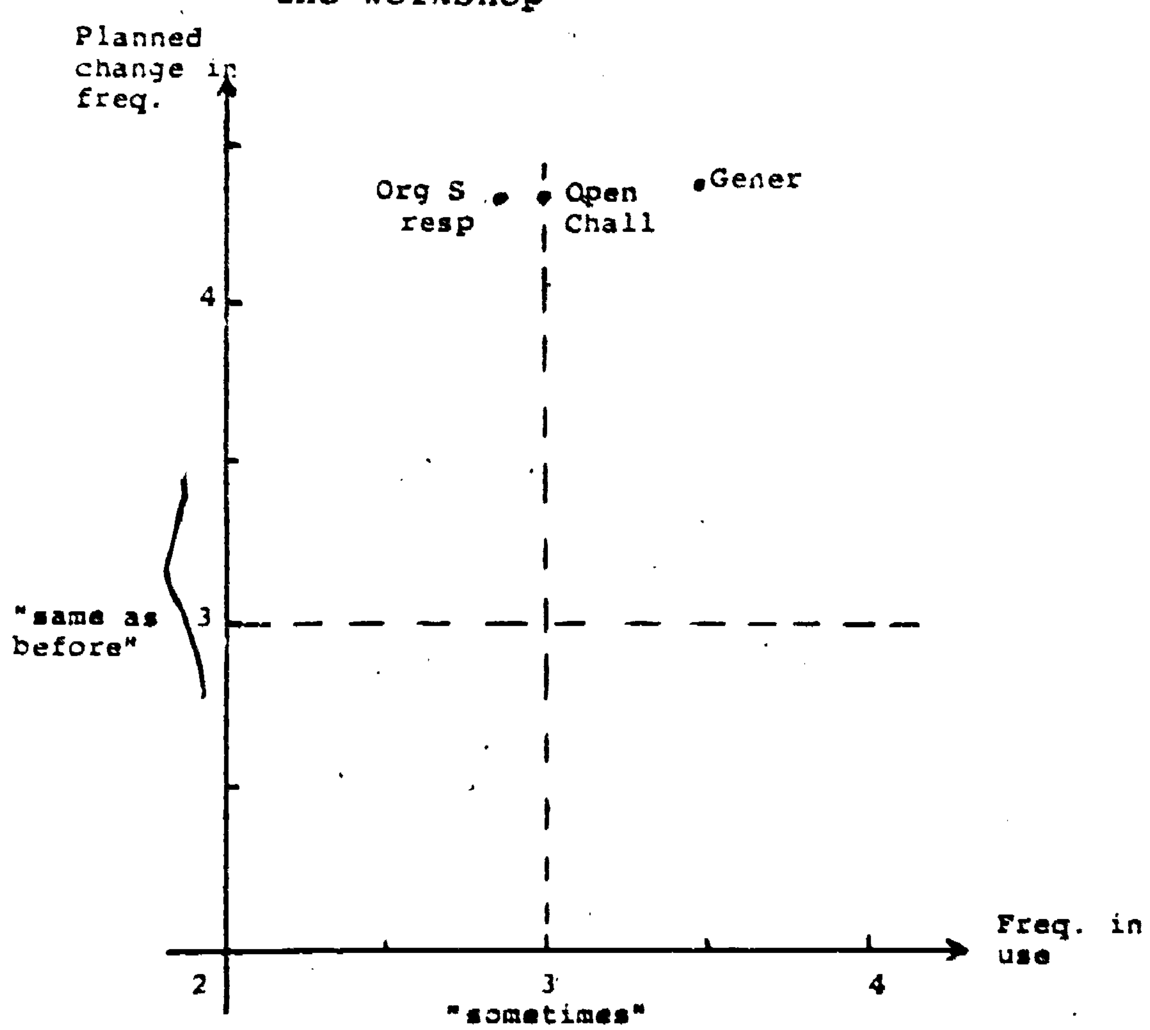


Figure 2d. Frequency of using open challenges, generalizations and organizing student responses before and after the workshop

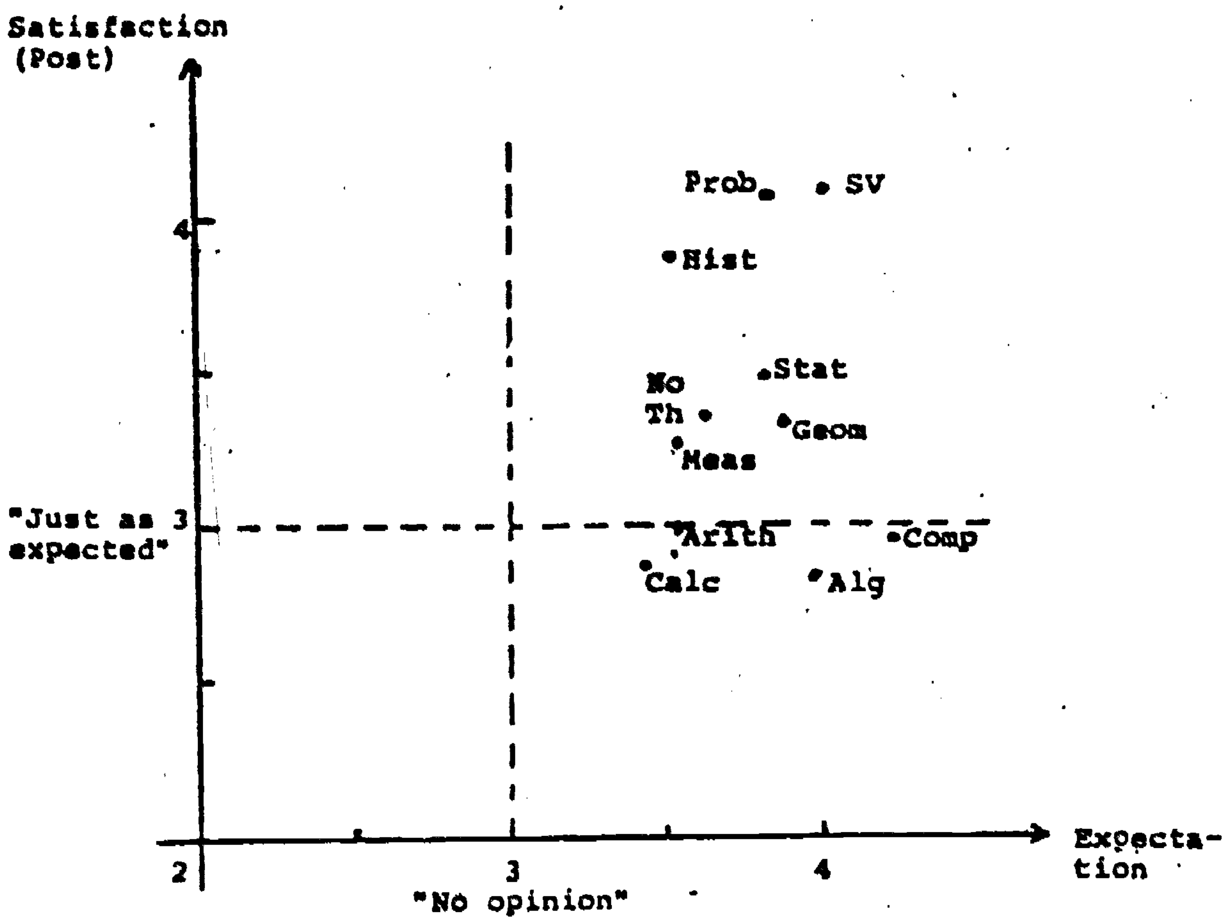


Figure 3. Teacher evaluation on the workshop's contribution with regard to eleven mathematical topics

The teachers also rated above expectation (more than 3.5 on a scale of 5) the workshop's treatment of the use of complex projects, concrete manipulatives, games, posing open-ended challenges, gathering student data, encouraging generalizations and group work (Table C.2).

All the trends indicated above should not be surprising to the workshop organizers: the teachers correctly identified the topics and the teaching strategies that were the main points of attention during the workshop, and rated them accordingly as being treated above their initial expectations. None of the workshop's major subjects was indicated as being dealt with in an unsatisfactory manner.

The fact that the workshop did not concentrate on algebra, use of calculators and computers was confirmed by the relatively low ratings in teacher satisfaction accorded to these topics. A greater emphasis on these topics should be considered in future workshops.

3. The twenty MGMP mathematical questions

In view of the fact that more than one third of the workshop was dedicated to the content and the teaching of the five MGMP units, the participants' level of performance on related mathematical problems is of particular interest. A complete list of participant scores and an item analysis may be found in Tables C.3 and C.4 (Appendix C). The group average increased from a level of 69 percent before the workshop to a post-workshop level of 79. Group averages on the other two dimensions investigated in this questionnaire (confidence in solving and confidence in teaching) were also computed and t-tested for pre-post workshop differences. These results are presented in Table 3.

Table 3. Participant mean scores, standard deviations, and results of t-tests for pre-post workshop differences

Variable	Mean Scores (S.D.)		t value	D.F.	p<
	Pre	Post			
Level of performance (1)	68.8 (16.5)	78.8 (18.0)	4.11	24	.001
Confidence in solving (2)	3.59 (.24)	3.71 (.30)	1.82	24	.09
Confidence in teaching (3)	2.41 (.36)	2.70 (.23)	4.88	24	.001

(1) in percent
(2) on a scale of 1 to 4
(3) on a scale of 1 to 3

At a significant level of .05, the results indicate an increase in level of performance and in ability to teach the twenty mathematical questions. No significant increase could be detected in the teachers' confidence in their ability to solve these questions.

An attempt to investigate any relationships among level of performance, confidence in teaching and confidence in solving led to the two graphs presented in Figure 4 (based on the mean scores from Table C.5) and to an analysis of correlation coefficients presented in Table 4.

The graph in Figure 4a describes the relationship between confidence in teaching and confidence in solving the twenty questions. Both before, and after instruction the confidence in solving was higher than the confidence in teaching -- a natural trend for good teachers who realize that not everything that one knows can also be easily taught. The graph clearly shows that as a result of the workhop, the confidence to teach increased more than confidence to solve. A possible explanation for this finding is the fact that the teachers' confidence in their ability to solve the questions was high already at the initial pre-workshop stage. A large growth in confidence in teaching spatial visualization and a slight regression in confidence in solving volume and area questions (on the "Mouse and Elephant" unit should also be mentioned. One can infer that (1) the Spatial Visualization unit was considered very useful in teaching this unconventional curriculum topic, and (2) the Mouse and the Elephant unit did not completely clarify all the complex notions involved in area and volume growth, but was successful in showing the teachers that the complexity of this topic may be disguised in deceptively simple questions.

The graph in Figure 4b compares the teachers' perception on their ability to solve certain mathematical tasks with their actual ability to solve these tasks. The graph indicates a general tendency to overestimate one's own ability to solve the questions. The line = x in the graph would indicate a correct estimation (i.e., a confidence in solving that is equal to the actual ability to solve). As a result of the

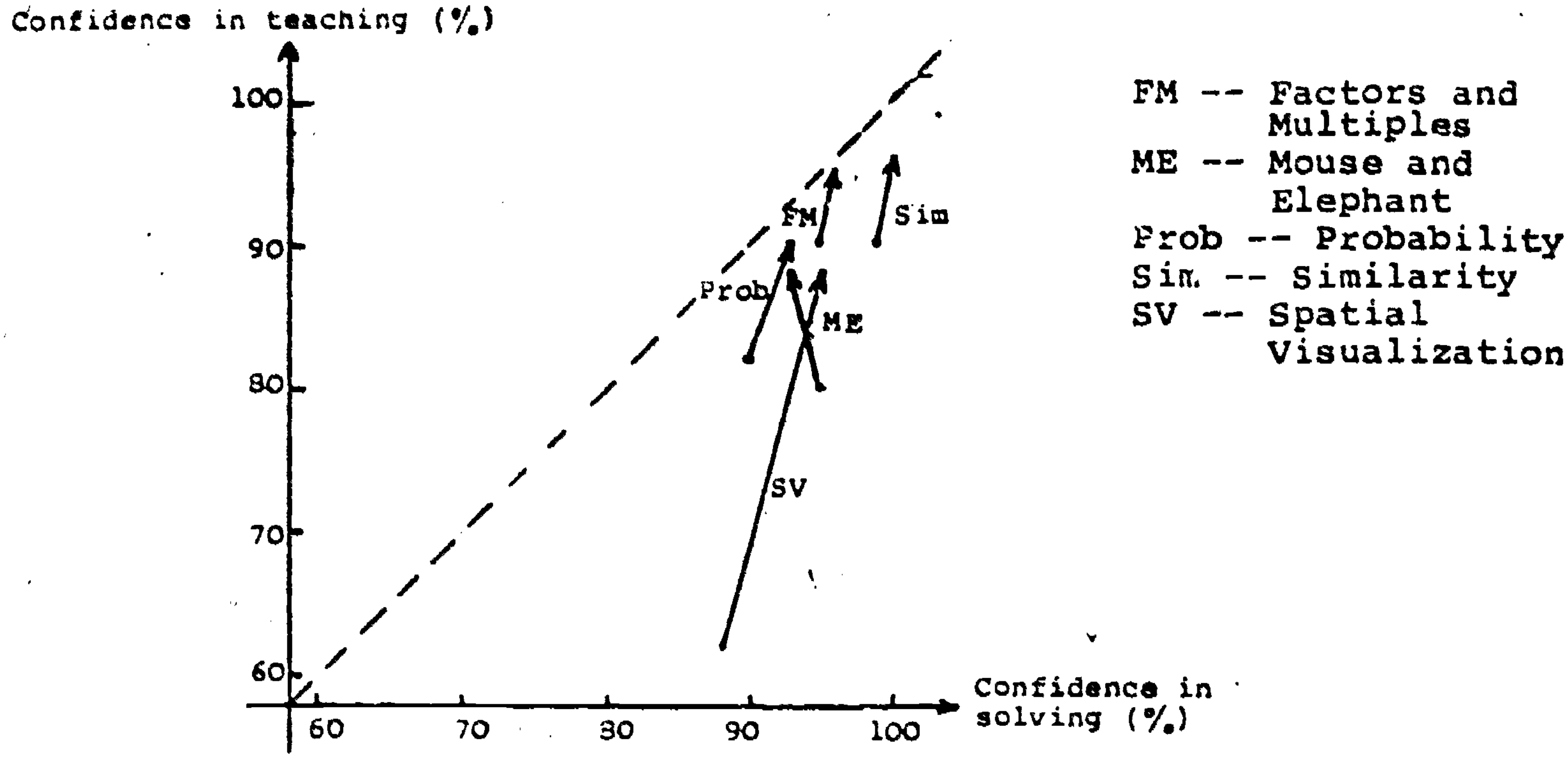


Figure 4a. Confidence in ability to solve versus confidence in ability to teach tasks related to the five MGMP units, before and after instruction

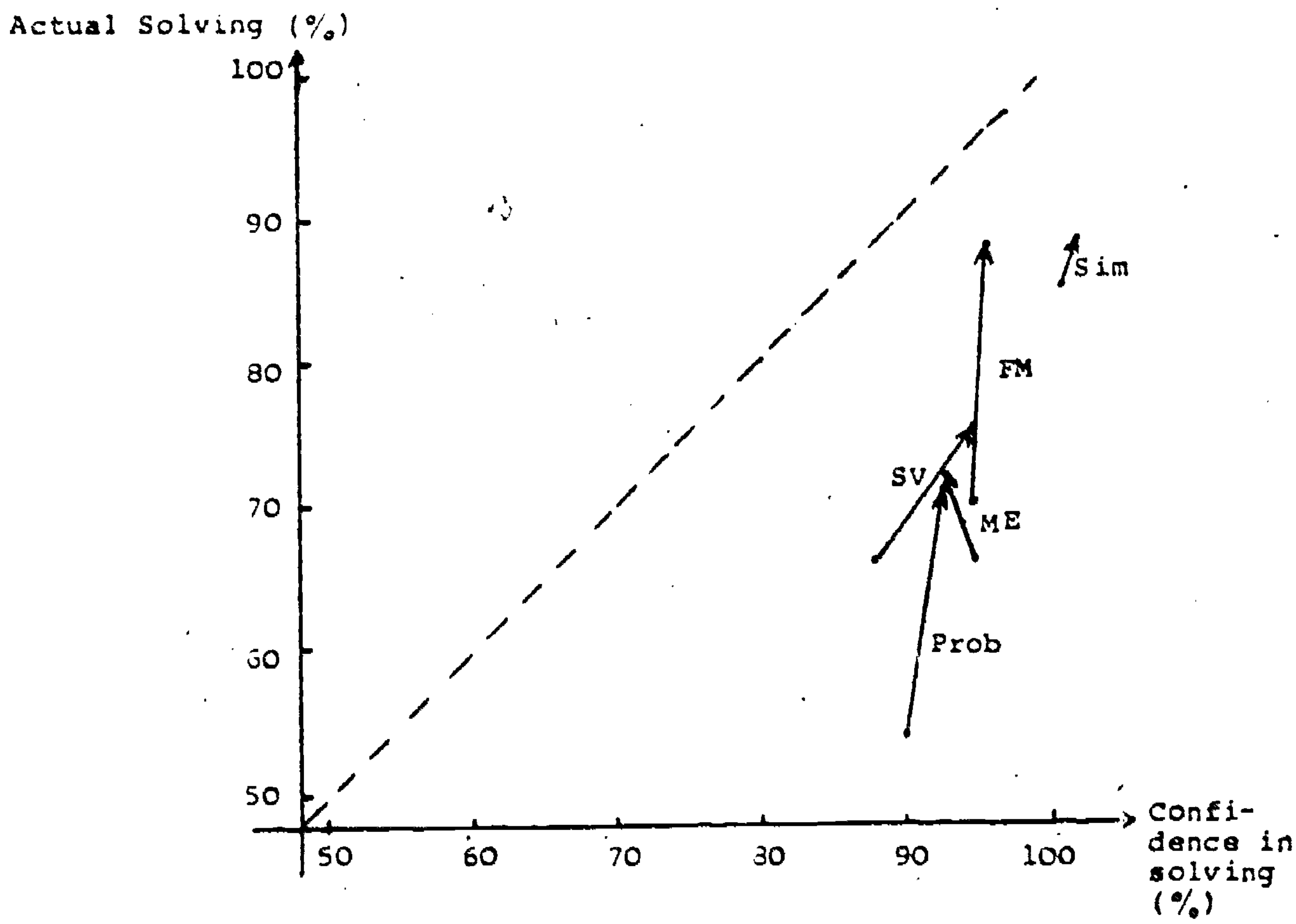


Figure 4b. Confidence in ability to solve versus actual level of performance on questions related to five MGMP units before and after workshop

workshop, the gap between the two variables decreased, due to a larger growth in ability to solve (especially in probability, number theory, and spatial visualization) than in confidence to solve.

The correlation coefficients in Table 4 indicate, at a .05 significance level the following relationship: (1) there is a significant relationship between confidence in solving and confidence in teaching both before and after the workshop (i.e., teachers that are more confident that they can solve a question are also more confident that they can teach it, and vice versa, (2) no significant relationship between confidence in solving and actual ability to solve could be detected before the workshop but such a relationship did show up after the workshop.

The latter result may be considered a positive influence due to the workshop: teachers became better estimators of their own ability to solve mathematical problems.

Table 4. Pearson correlation coefficients among level of performance, confidence in solving and confidence in teaching on the twenty MGMP questions

	Confidence in solving Pre (1)	Confidence in teaching Pre (2)	Level of performance Pre (3)	Confidence in Solving Post (4)	Confidence in teaching Post (5)	Level of performance Post (6)
(2)	.44*					
(3)	.22	.03				
(4)	.28	.34	.16			
(5)	.21	.56*	.05	.69*		
(6)	.10	.12	.76*	.43*	.20	

*significant at $\alpha = .05$

Conclusions

The data on the participating teachers' perception of their teaching repertoire and on their perception of the workshop seems to indicate that the workshop was successful in: (1) causing a considerable growth in the planned (and hopefully actual) frequency in teaching the mathematical topics, and using the teaching strategies that were emphasized throughout the workshop, and (2) highly satisfying many (though not all) of the participants' mathematical and pedagogical needs. No major areas of participant dissatisfaction could be detected.

The teachers had initially high expectations from the workshop with regard to most mathematical topics and teaching strategies. Therefore, to fully satisfy all these expectations would be difficult, if not impossible. The workshop seems to have had a particular high impact on the future plans to teach probability and spatial visualization -- topics that according to the teachers' own reports were neglected before. However, it should be mentioned that the participants were less satisfied with the workshop's treatment of algebra, arithmetic, calculators and computers as compared to other mathematical topics. Interestingly, the workshop seems to have raised the teachers' awareness to computers and calculators (as expressed by a planned growth in their use) in some indirect manner. The teaching of algebra seems to need a more intensive treatment, if any impact in this direction is considered desirable.

The workshop was also successful in increasing the teachers' planned frequency in future use of "high quality" teaching strategies that were the focus of most of its activities: presenting to students complex projects, gathering and organizing student responses, posing open challenges, encouraging generalizations, group work, use of mathematical games and of manipulatives. These changes are particularly important in view of the teachers' willingness to decrease the frequency of whole-class instruction, seatwork, and the use of worksheets related to the latter.

All these tendencies should be considered only an encouraging start, and their classroom implementation should be the subject of a follow-up evaluation.

The workshop dealt extensively with five mathematical topics: probability, geometrical similarity, spatial visualization, area and volume relationships, and number theory. The teachers' performance on these topics improved significantly and came close to a mastery level of 80 percent. The participants' tendency to overestimate their own ability to solve some of the related questions diminished after the workshop: a significant ($\alpha = .05$) improvement in level of performance was accompanied by only a slight, insignificant increase in confidence in ability to solve the problems. Therefore, a previously nonexistent correlation between ability to solve a given question and confidence in solving could be detected (at $\alpha = .05$) by the end of the workshop.

The teachers' self-reported growth in their ability to teach the five mathematical topics is a subjective measure. However, in view of the high quality of the teachers that were selected to participate in this workshop, actual changes at the classroom level may be considered certain.

Reference

Fresko, B. and Ben-Haim, D. (1984). An evaluation of two in-service courses for mathematics teachers, Unpublished technical report. Rehovot, Israel: Department of Science Teaching, The Weizmann Institute of Science.

PART IV - MISCELLANEOUS

CERTIFICATES OF ACHIEVEMENT

___________ is presented this award of distinction for outstanding performance and exceptional achievement in:

1. Susan - the phantom mission of rhyme composition.
2. Dennis - calculating the probability of anyone believing his 95‰ free throw average.
3. Maureen - understanding the principles of multiplication.
4. Loretta - "similarity" to Houdini and his famous vanishing act (but only in multiples of 5 days).
5. Gloria - successfully "combing" through voluminous workshop documents.
6. Bob - locating the "natural roots" of any problem.
7. Jack - finding the "perfect number" to insure his bets.
8. Joan - creative "Logo's" <u>on</u> and <u>off</u> the computer.
9. Dave - discovering lyric poetry through transcendental numeration.
10. Warren - Southern gentlemanly conduct while plowing through fields with toboggan.
11. Nancy - staying with us on overheads for AAA (Amiable Auto Accessibility).
12. Sally - calculating how many "mouse coats" she could pack in the extra suitcase she brought (and how to convince Warren to carry it).
13. Karen - the phantom character of iambic pentameter.
14. Pam - being "gifted" with escorts to basketball games.
15. Donna - bringing her own "Apple" for the teachers.
16. Steve - placing a bet with the North Pole and coming out "a-head".
17. Anita - being the Lone Star State's Rookie who works magic with squares.
18. Ron - making us use "deductive reasoning" to discover the identity of his dinner companions.
19. Cynthia - "reporting" on an unforgettable "time machine".
20. Marlene - walking in cadence on our trecks to Wells Hall.
21. Gary - epitomizes the slogan "speaks softly but carries a big fishing pole" (for Walleye!)
22. Ken - being the "Honors Doughboy" so we all could enjoy!
23. Marcia - "estimating" the length of the Oregon Trail (and coming with 2.5 cm.)
24. Reg - "proof" ing and "foop" ing his way through evening workshops.
25. Esther - continuing education without "getting her feet wet".

Sunday, December 9 was spent at a relaxing and pleasant brunch in the Lappan home. The menu follows:

MONASTERY LENTILS

1/4 (scant) cup olive oil (I use part mazola)

2 large onions chopped

1 carrot chopped

SAUTE 3 to 5 minutes

ADD 1/2 tsp each thyme and marjoram

1 cup seasoned stock (vegetable boullion cubes or beef or chicken stock)

1/4 cup chopped fresh parsley and add 1/4 cup dry sherry

2 cups canned tomatoes

dash of sugar

1 cup lentils

COOK until lentils are tender

TOP with grated swiss cheese (optional)

SERVE over brown rice

EGGS A DAY AHEAD (18-24 servings)

3 dozen large eggs beaten with

1/2 cup milk

1/2 cup butter or margerine

SAUCE (combine and heat)

2 cans mushroom soup

1/2 lbs. grated sharp cheddar cheese

1/2 cup dry sherry

Prepare 1 lb. fresh mushrooms (slice and saute in butter or margerine)

Prepare 1 lb. bulk sausage (saute and drain) (optional)

Soft scramble egg mixture in butter or margerine.

In 2 1/2 to 3 quart casserole put 1/3 eggs, 1/3 sauce - repeat twice. Cover and refrigerate 1 day.

Top with mushrooms, bake covered at 275 degrees for 50 minutes. Sausage is layered with eggs if used.

HUMMUS

1 1/2 cups Garbanzos (1 can), drained (reserve some liquid)
Juice of 1/2 lemon (about 1/6 cup)
2 cloves garlic, minced
1/4 cup olive oil
1/4 cup tahini
lots of fresh chopped parsley
Process first 4 ingredients in a blender until smooth, adding just enough liquid from the garbanzos to keep blender going. Blend in tahini till well mixed, stir in the parsley. Serve at room temperature with triangles of Middle Eastern pita bread and/or vegetable dippers.

HONORS TEACHERS WORKSHOP RICE

Oil as needed
1 cup raw BROWN RICE -- cooked as directed
1 small onion chopped
1/2 can apricot nectar (1 Tbsp. sherry optional)
1 cup mixed dried fruits chopped
2/3 cup mixed nuts chopped - sunflower seeds too
1/3 cup sesame seeds
1/4 to 1/2 tsp. cloves
1/2 tsp. salt
4 Tbsp. melted butter
Marinate fruit in apricot nectar and sherry several hours. Saute onion fruit, nuts, seeds in oil until golden. Add cloves and salt -- then mix with cooked rice. Put in a casserole. Pour butter over the mixture. Bake at 350 degrees for 15 to 20 minutes.

PEPERONATA

2 Tbsp. butter

1/4 cup olive oil

1 lb. onions sliced 1/8 in. thick about 4 cups

2 lb. green and red peppers, peeled by blanching first, seeded cut into 1 by 1/2 inch strips - about 6 cups

2 lb. tomatoes, peeled, seeded, coarsely chopped about cups

1 tsp. red wine vinegar

1 tsp. salt

fresh ground black pepper

In a heavy 12 inch skillet, melt 2 Tbsp. butter and 1/4 cup olive oil, moderate temp.

Add onions, turning, until soft and light brown about 10 min.

Add peppers, cover, reduce heat, 10 minutes.

Add tomatoes, vinegar, salt, a little pepper, cover, cook 5 minutes. Cook uncovered, high heat, stir gently, until almost all liquid has boiled away. Serve as a hot veggie dish or as an accompaniment.

APPLE BUNDT CAKE

2 cups granulated sugar

1/4 cup orange juice

1 cup vegetable oil

2 1/2 tsp. vanilla

4 eggs

3 cups flour

3 tsp. baking powder

1/2 tsp. salt

1 cup chopped nuts

powdered sugar topping

Beat first 5 ingredients at high speed. Sift dry ingredients then blend with first mixture. Fold in nuts.

Place 1/3 of batter in pan--then filling--batter--filling--batter. Be sure to end with batter.

FILLING: 2 cups apples - peeled and diced

1 tsp. cinnamon

1 tbsp. sugar

Cool, sprinkle with powdered sugar.

DANISH ALMOND PUFF

Part 1: 1 stick of butter/margarine
1 cup of flour
2 Tbsp. water

Mix, divide into 2 strips on a cookie sheet

Part 2: 1 cup water
1 stick butter/margarine

Bring to rolling boil; remove from heat; and add
1 1/2 tsp. almond extract
1 cup of flour

Mix and return to low heat 1 minute - stirring.

Remove from heat and whip in by hand
3 eggs until creamy.

Divide and put on the two strips of part 1.

Bake in preheated 350 degree oven for 1 hour.

Remove and cool.

FROST with a simple confectioner sugar frosting:
1 cup of confectioner sugar
1 Tbsp. softened butter/margarine
1 tsp. almond extract or vanilla
1-2 tsp. of water to desired consistency

Top with pecans or slivered almonds.

To our friends:
Math teachers, math teachers, math teachers, all
Read the ad and answered the call
The time has come for our good-byes
We all leave here a bit more wise
(and with tears in our eyes!)

With love and fond memories
The Phantom Prime Poets

APPENDIX A

QUESTIONNAIRE ON TEACHING REPERTOIRE AND ON PARTICIPANT SATISFACTION WITH THE WORKSHOP (PRE)

TOPIC	How frequently do you teach the topic in your classes (grades 6-8)? Very frequently	Frequently	Sometimes	Seldom	Never	Do you expect this workshop to deal with the topic? Definitely yes	I would like it	I don't have an opinion	I would not like it	Definitely no
Arithmetic (whole nos., fractions, decimals										
Drill										
Short word problems										
More complex problems or projects										
Number Theory (factors, primes, odd/even, etc.)										
Drill										
Short word problems										
More complex problems or projects										
Geometry										
Drill										
Short problems										
More complex problems or projects										
Spatial Visualization										
Drill										
Short problems										
More complex problems or projects										
Statistics										
Drill										
Short problems										
More complex problems or projects										
Probability										
Drill										
Short problems										
More complex problems or projects										

TOPIC	Teaching					Expectations from workshop				
	Very frequently	Frequently	Sometimes	Seldom	Never	Definitely yes	I would like it	I don't have an opinion	I would not like it	Definitely no
Measurement										
Drill										
Short problems										
More complex problems or projects										
Algebra										
Drill										
Short word problems										
More complex problems or projects										
Use of calculators										
For checking answers										
As a computational aid										
As a means for solving complex problems										
Use of computers										
Use of software for drill										
Use of software for simulations										
Use of software for solving problems										
Learning a programming language										
Programming for problem solving										
History of Mathematics										

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STRATEGY	How frequently do you use the strategy in your classes (grades 6-8)?					Do you expect the workshop to deal with the strategy?				
	Very frequently	Fre-quently	Sometimes	Seldom	Never	Definitely yes	I would like it	I don't have an opinion	I would not like it	Definitely no
Whole class instruction										
Group work										
Seat work (same assignment for all)										
Individualized work										
Posing open-ended challenges										
Gathering and organizing student responses										
Encouraging analysis and generalization										
Assigning homework										
Discussing homework										
Using concrete manipulatives										
Using worksheets										
Using games										

APPENDIX B

TWENTY MGMP UNIT-RELATED QUESTIONS

HONORS TEACHERS WORKSHOP EVALUATION

____
pseudoname

Indicate, by writing the appropriate number and letter in the boxes beside each question, to what extent you feel confident that you could solve the problem and teach the problem to middle school students. (You are not asked to solve the problems.)

Confidence in solving

1. I am positive that I can solve it.
2. I am somewhat certain that I can solve it.
3. I don't think I can solve it- the topic is not very familiar
4. I am not familiar at all with the topic.

Confidence in teaching

(A) I am certain that I could teach it.

(B) I am somewhat certain I could teach it.

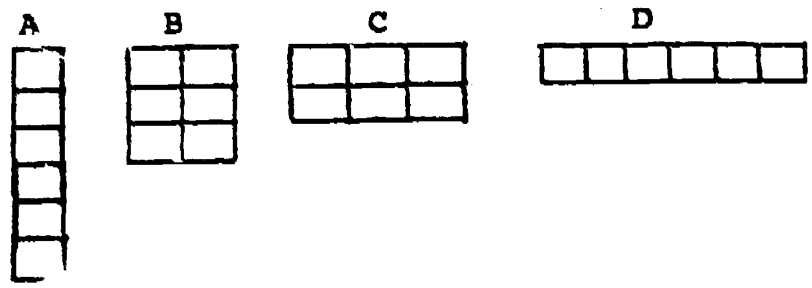
(C) I am not confident that I could teach it.

Question	Solving	Teaching
1.	☐	☐
2.	☐	☐
3.	☐	☐
4.	☐	☐
5.	☐	☐
6.	☐	☐
7.	☐	☐
8.	☐	☐
9.	☐	☐
10.	☐	☐
11.	☐	☐
12.	☐	☐
13.	☐	☐
14.	☐	☐
15.	☐	☐
16.	☐	☐
17.	☐	☐
18.	☐	☐
19.	☐	☐
20.	☐	☐

TWENTY MATHEMATICS PROBLEMS

1. A set of blocks can be separated into 6 equal piles. It can also be separated into 15 equal piles. What is the smallest number of blocks that could be in the set?

2. Here are the 4 different rectangles that cover 6 squares on a grid.

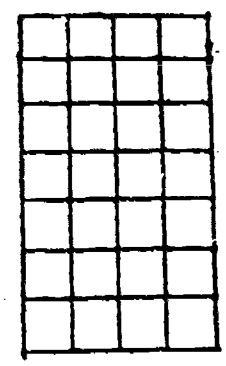


How many different rectangles can be made which would cover exactly 30 squares on a gird?

3. 3 is the greatest common divisor of 15 and another number. What is the other number?

4. What is the smallest prime number which is larger than 200?

5. What is the perimeter of this rectangle?



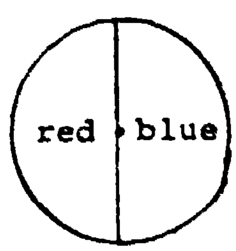
6. A rectangular field with an area of 240 square units has one edge of 20 units. What is the length of the other edge?

7. What is the surface area of a cube with an edge of 10?

8. You have a melon in your garden that is 3 inches across and weighs 3 ounces. If it grows to be 6 inches across, how much would it weigh?

9. What is the probability of getting a sum of 12 when two dice are thrown?

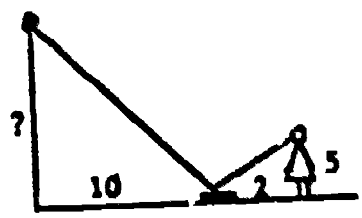
10. If the spinner shown is to be spun twice, what is the probability of getting red-red?



11. Two bills are drawn randomly from a bag containing a five dollar bill and 3 one dollar bills. If the experiment is repeated many times, what would you expect the average amount of money drawn per time to be?

12. What is the probability that a family of three children will have 2 girls and 1 boy?

13. Joan estimates the height of a flagpole by using a mirror.



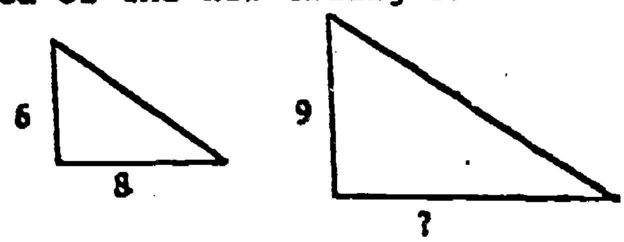
Distances

to eye level	5 ft.
Joan to mirror	2 ft.
Mirror to pole	10 ft.

How tall is the pole?

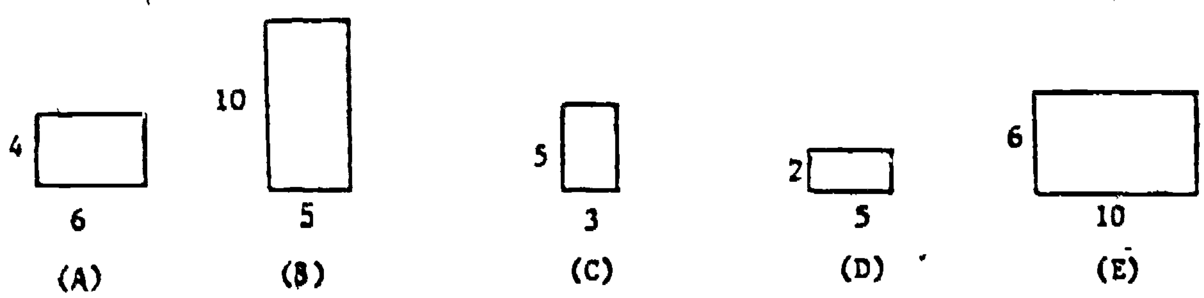
14. If the lengths of the sides of a triangle are each multiplied by 3, how much larger is the area of the new triangle?

15. These triangles are similar:

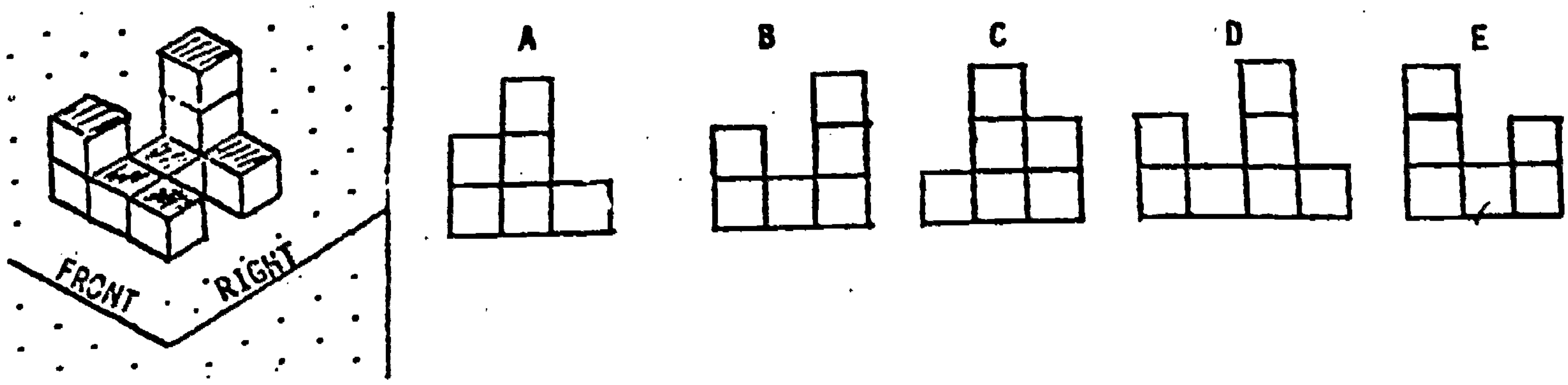


Find the missing length.

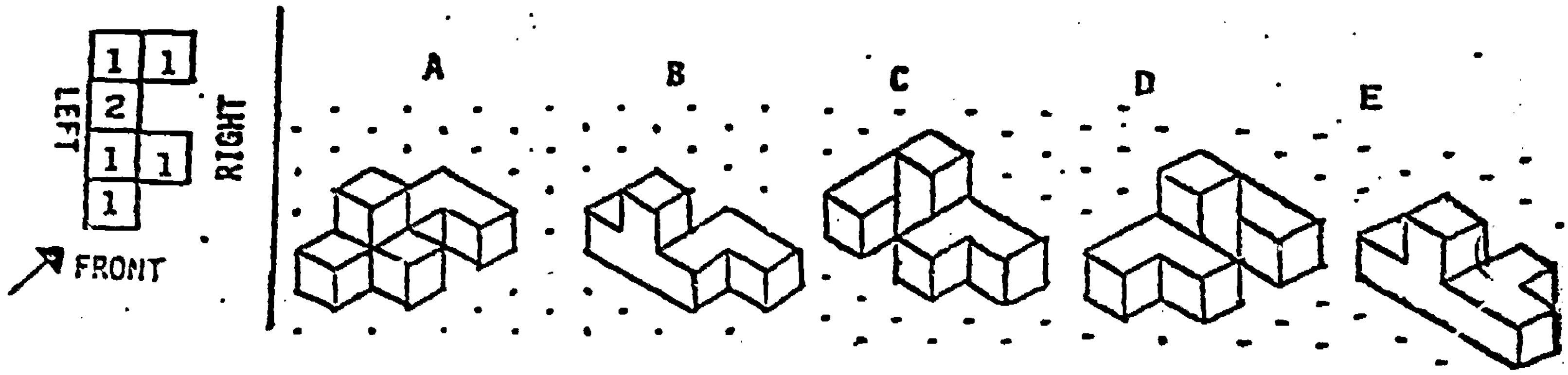
16. Mark which of the following rectangles is similar to a 10 x 15 rectangle?



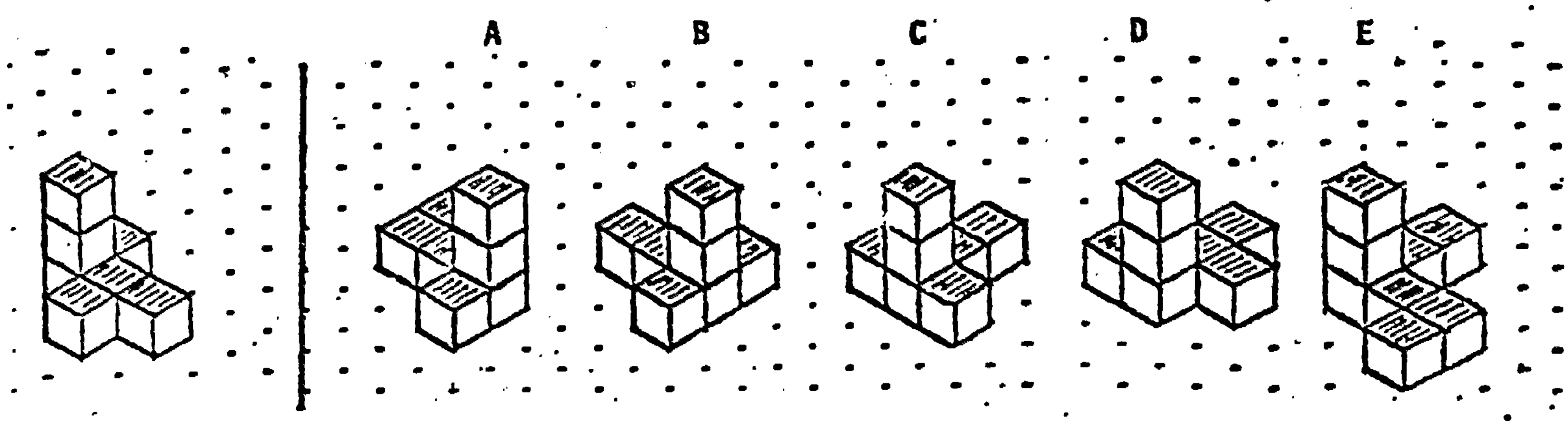
17. Mark the RIGHT VIEW. You are given a picture of a building drawn from the FRONT-RIGHT corner. Find the RIGHT VIEW.



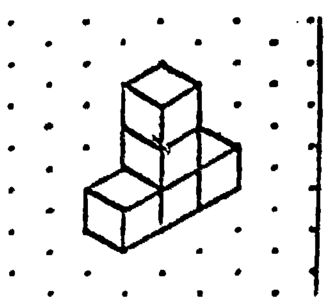
18. The number on each square indicates the number of cubes placed on that square. Mark the view from the FRONT-LEFT corner.



19. Mark another view of the first building.



20. What is the maximum number of cubes that could be used to build a building that has the given isometric drawing as a corner view?



APPENDIX C

MEAN SCORES AND STANDARD DEVIATIONS ON THE TWO EVALUATION QUESTIONNAIRES

Table C.1 Mean scores* in teachers' reaction to teaching frequency and workshop evaluation with regard to eleven mathematical topics

		Arithmetic	Number Theory	Geometry	Spatial Visualization	Statistics	Probability	Measurement	Algebra	Calculators	Computer	History of Mathematics
Teaching	Frequency in teaching	3.82	3.32	3.12	2.52	2.41	2.37	3.33	3.11	2.63	2.06	3.20
	Planned change in frequency in teaching	3.28	3.40	3.57	4.17	3.76	4.05	3.59	3.23	3.80	3.57	3.96
Workshop	Workshop expectations	3.55	3.63	3.92	4.03	3.83	4.08	3.57	3.99	3.43	4.27	3.52
	Workshop evaluations	2.99	3.32	3.31	4.07	3.49	3.84	3.25	2.81	2.88	2.95	3.84

*on a scale from 1 to 5

Table C.2 Mean scores* in teachers' reaction to teaching frequency and workshop evaluation with regard to teaching strategies

		Drill and Practice	Short Word Problems	Complex Projects	Whole-Class Instruction	Group Work	Seat Work	Individualized Work	Concrete Manipulatives	Worksheets	Games	Open-ended Challenges	Organizing Student Responses	Encouraging Generalizations
Teaching	Frequency in teaching	3.27 (.62)	3.13 (.47)	2.60 (.61)	4.96 (.20)	3.44 (.82)	4.52 (1.09)	2.96 (1.02)	3.04 (1.17)	3.60 (.91)	3.00 (.91)	3.00 (1.32)	2.88 (.83)	3.48 (.96)
	Planned change in frequency in teaching	3.11 (.83)	3.70 (.52)	4.08 (.37)	2.88 (.60)	4.12 (.60)	2.56 (.58)	3.16 (.55)	4.36 (.57)	2.92 (.64)	4.04 (.61)	4.32 (.48)	4.32 (.69)	4.36 (.50)
Workshop	Workshop expectations	3.35 (.72)	3.92 (.58)	4.18 (.42)	3.52 (1.00)	4.04 (.68)	3.16 (1.07)	4.12 (.60)	4.12 (1.01)	3.12 (1.01)	4.12 (1.01)	4.55 (.58)	3.92 (.49)	4.32 (.63)
	Workshop evaluations	3.13 (.76)	3.40 (.86)	3.62 (.89)	3.00 (.82)	3.72 (1.06)	2.68 (.95)	2.64 (.95)	3.96 (1.14)	2.96 (1.02)	3.60 (1.08)	3.88 (1.05)	3.96 (1.17)	4.00 (1.08)

*on a scale from 1 to 5

Table C.3 Individual Scores on the Twenty Mathematical Problems

Pseudoname	Pre	Post
Amber, Penny	18	20
Backwoods, Barney	15	17
Buffy	13	15
Berg, J.B.	13	20
Compute	13	16
Craig, Kelli	14	17
Cricket, Jimmy	13	13
Dean, Dizzy	8	8
Ebmeyer, Sally	13	18
Fern, Irma	16	19
Fenner, J.	16	19
Katheleen	3	9
Kendar, Molly	16	16
L.T.B.	16	18
Nixon, Richard	16	18
Perez, Tony	9	14
QED^2	17	19
Sandquist, Bob	16	18
Skara, Nama	13	15
Smith, Fred	11	10
Smyth, John W.	10	12
Spatzell, Vern	15	20
Sunshine, Sally	16	16
Wallace, Jim	13	18
Williamson, Julie	14	16

Table C.4 Distribution of Errors on Twenty Mathematical Problems

Number of errors

Problem	Pre	Post
1	0	1
2	14	6
3	2	0
4	15	5
5	2	5
6	12	10
7	5	2
8	16	10
9	6	3
10	2	2
11	19	17
12	18	6
13	4	2
14	6	2
15	2	1
16	2	1
17	4	1
18	5	1
19	2	3
20	23	19

Table C.5 Mean scores (and standard deviations) of teachers' reaction and performance on Twenty Mathematical Problems

	Confidence in solving(1)		Confidence in teaching(2)		Level of performance(3)	
	Pre	Post	Pre	Post	Pre	Post
Factors and Multiples	3.81 (.28)	3.85 (.19)	2.69 (.44)	2.85 (.20)	.70 (.22)	.88 (.19)
Mouse and Elephant	3.79 (.24)	3.72 (.39)	2.41 (.42)	2.63 (.33)	.66 (.22)	.72 (.21)
Probability	3.61 (.55)	3.71 (.32)	2.45 (.56)	2.69 (.34)	.54 (.26)	.71 (.25)
Similarity	3.94 (.16)	3.99 (.23)	2.70 (.47)	2.89 (.27)	.85 (.27)	.88 (.25)
Spatial Visualization	3.51 (.63)	3.80 (.30)	1.85 (.64)	2.64 (.42)	.66 (.23)	.75 (.18)

(1) on a scale from 1 to 4
(2) on a scale from 1 to 3
(3) 0 wrong/1 right

APPENDIX D

SELECTION CRITERIA AND PROCEDURES

Arrival Nov-27
Depart Dec 15

3 letters of nomination received before an application is sent

SELECTION CRITERIA AND PROCEDURES

This selection will be made by the staff of the Middle Grades Mathematics Project.

Qualifications for selection to the workshop:

1. Be certified to teach mathematics with a major, a minor, or an endorsement in mathematics.

We are expecting these teachers to serve as leaders in their community in mathematics education. Therefore, we want them to have the strongest possible background in mathematical content.

2. Be teaching a grades 6, 7, or 8.

3. Be selected and nominated by their principal and two peers.

To serve as a leader, the teacher must already be held in high regard in their community.

4. Be provided with a three-week paid leave of absence from their local school.

We want the school district to make a sufficient commitment to show that they are serious in their desire to improve their mathematics education program.

5. Receive assurance from local administrators that the applicant will be used as a local in-service resource person upon completion of the workshop.

Again, we want assurance that this potential leadership will be used.

The criteria which we will use for selection are:

Undergraduate record
Graduate record
Professional accomplishments
Previous recognition and honors
Evidence from the nominations

Reservations will be made by the department
Reasonable travel allowance, room and food. No stipend
All books will be furnished.

The participants in the Honors Teacher Workshop will enroll in two courses.

Mathematics 490 (9:00-12:00) Mathematical Problems (3 cr.)

This course will be coordinated by William Fitzgerald and will consist of a series of lectures and problem sessions conducted by various mathematicians and mathematics educators on recent new developments and ideas in mathematics and their applications.

Topics will include algebra, geometry, analysis, probability, statistics, finite methods, combinatorics, and graph theory. Particular attention will be paid to the generic origins of these ideas as they might be presented to middle school children. The use of micro-computers in teaching mathematics will be stressed where appropriate.

Speakers will also include users of mathematics. Contact has already been made with scientists and engineers across the campus from the physical and biological sciences who work in astronomy, chemistry, microbiology and biochemistry.

The group will have a guided tour of the MSU Heavy Ion Superconducting Cyclotron, the only one of its kind in the world.

Avital's test on the history of the important problems in mathematics will serve as a focus for the problem solving sessions.

Notes from the lectures and problem sessions presented by the mathematicians and scientists in Math 490 will be edited and published by the Michigan State University Mathematics Department to be available for other similar workshops around the United States.

Mathematics 405 (1:00-4:00) Mathematical Topics for Teachers (3 cr.)

In this course the students will become familiar with the five units of curriculum developed with NSF funding of the Middle Grades Mathematics Project at Michigan State University during 1980-82. Time and effort will also be devoted to planning for inservice activities in which the students will engage when they return to their home schools. This course will be coordinated by Glenda Lappan and assisted by the remainder of the MGMP staff.

During the last week of the workshop time will be devoted to discussions with and among the Honor Teachers about directions which NSF should pursue to be most effective in the mathematics curriculum improvement efforts.

A follow-up study by mail and phone will be conducted in May, 1985 of the participants to record and update their reactions to their in-service efforts.

APPLICATION

HONORS WORKSHOP FOR MATHEMATICS TEACHERS IN GRADES SIX, SEVEN, AND EIGHT

Michigan State University
November 26 - December 14, 1984

1. Name ______________________________

2. Home Address ______________ Home Phone ______________
____

3. School Address ______________ School Phone ______________
____

4. Educational History

Educational History	Institution	Degree	Year	Major	Minor
Secondary School					
Undergraduate					
Graduate					

5. Describe your present assignment in your school.

6. Describe your post-high school work experience.

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7. List the mathematics content and pedagogy courses by name that you have completed and the grades you earned to prepare you to be a middle grades mathematics teacher.

Content	Pedagogical

8. In what ways have you been involved in in-service activities?

9. In what ways have you influenced the mathematics program in your school?

10. What is the name and address of your local newspaper?

Please indicate in a statement below what is important to you in working with students in mathematics. What are the most important issues facing you in mathematics education? In what direction would you suggest mathematics education should be moving in the near future?

Describe in a preliminary way, how you will be used in your district as an instructional leader upon your return after the workshop. (Some of the workshop time will be spent planning these activities.) Obtain the appropriate signatures below.

These plans described above will be pursued upon the return of this applicant from the MSU Honors Teacher Workshop.

Principal	*District Official*

Include with your application transcripts of undergraduate and graduate courses and other documents which might strengthen your application.