DOCUMENT RESUME

ED 257 314 FL 015 063

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TITLE English for Students of Mathematics.
PUB DATE Sep 81
NOTE 92p.
PUB ?E Guides - Classroom Use - Materials (For Learner) (051)

EDRS PRICE MF01/PC04 Plus Postage.
DESCRIPTORS *College Mathematics; Diagrams; English for Special Purposes; Functions (Mathematics); Higher Education; *Instructional Materials; *Mathematical Concepts; Mathematics Materials; Matrices; Proof (Mathematics); *Reading Comprehension; Reading Skills; Semantics; *Sentence Structure; Syntax; Transformations (Mathematics; Verbs; Writing Skills
IDENTIFIERS *English for Science and Technology

ABSTRACT

A curriculum unit on English for mathematics students at the University of Haifa is presented. The text content concerns mathematics while integrating language principles. Exercises are included that involve comprehension of geometric diagrams, reading comprehension, word meanings, and restructuring sentences. The 12 sections cover: projective planes, infinite matrices and convergent sequences, relations, metric spaces, Peano's postulates for the natural numbers, monoids, matroids, graphs of functions, binomial coefficients, cross products of vectors, generalized inverses of linear transformations, and the Riemann integral. The lessons cover topics such as: when a word ending in "ing" is not a verb, modal verbs, simple sentences, active and passive verbs, verbs used as adjective/expanded forms, sentence function, phrases showing the writer's comments, general versus qualified statements, compound sentences, "since," clauses and phrases, parallel structures and enumeration, changing the subject without markers, language of proof, introduction and transition passages, intensifiers and qualifiers, statements of contrast, logical arguments, markers of emphasis, language of theorem and proof, sentence structure and information, following the argument, and sentence structure review. Appended is the Greek alphabet, including the name of each letter and the upper and lower case symbols for each letter. (SW)

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ENGLISH FOR STUDENTS OF MATHEMATICS

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September, 1981

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TEXT I: PROJECTIVE PLANES

A <u>projective plane</u> can be abstractly defined to be a certain system Π consisting of a set of objects called "points" and a set of objects called "lines", together with a relation called "incidence", which are all subject to the following axioms:

(0) If point P is incident with line L, then and only then is line L incident with point P.

(1) If P and Q are two distinct points of Π,* then there is one and only one line of Π incident with both P and Q.

(2) If L and M are two distinct lines of Π, then there is one and only one point of Π incident with both L and M.

(3) There exists at least one set $\{P_1,P_2,P_3,P_4\}$ of four distinct points of Π, no three of which are incident with the same line.

Note that axiom (3) implies that there exists at least one set $\{L_1,L_2,L_3,L_4\}$ of four distinct lines of Π, no three of which are incident with the same point. Indeed, if $\{P_1,P_2,P_3,P_4\}$ is the set of points the existence of which is postulated in axiom (3), then we can define the lines L_1, L_2, L_3, and L_4 as follows: L_1 is the unique line incident with P_1 and P_2; L_2 is the unique line incident with P_2 and P_3; L_3 is the unique line incident with P_3 and P_4; and L_4 is the unique line incident with P_1 and P_4.

Nothing in the above set of axioms implies that the number of points or

* Note: For names of the Greek letters, see Appendix, page 88.

the number of lines in an abstract projective plane is infinite. In fact, it is interesting to speculate what a projective plane with only finitely-many points and finitely-many lines would look like. Let n be a positive integer, and assume that Π is a projective plane in which there is a line which is incident with precisely $n+1$ distinct points. Then:

(A) Every line of Π is incident with precisely $n+1$ distinct points.

(B) Every point of Π is incident with precisely $n+1$ distinct lines.

(C) There are precisely n^2+n+1 distinct lines in Π.

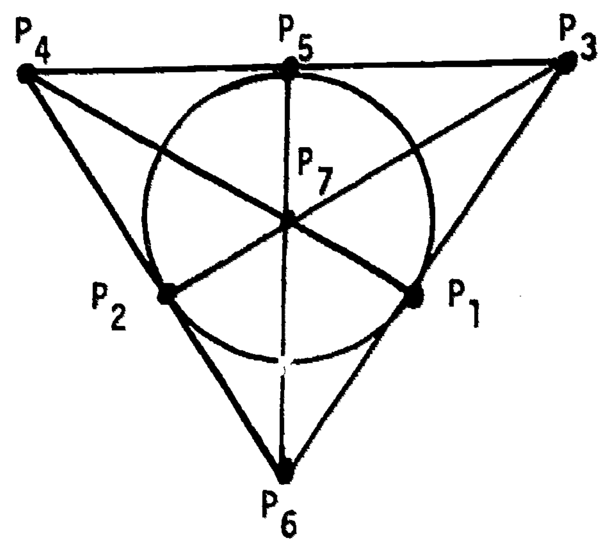
(D) There are precisely n^2+n+1 distinct points in Π.

If $n = 1$, there would only be three distinct points in Π, and this contradicts axiom (3). Therefore, we see that there must be at least three points of Π incident with each line. If there are precisely three points incident with each line in the plane, then the plane must consist of seven points and seven lines. What does such a plane look like? One way of representing such a plane is by the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

where the "points" of the plane are the rows of the matrix, the "lines" of the plane are the columns of the matrix, and a line is incident with a point if and only if the entry of the matrix in the intersection of the given row and given column equals 1.

Another way of representing the same plane is by the following diagram.



Check to make sure that all of the axioms of a finite projective plane are indeed satisfied by this example.

EXERCISES ON TEXT I:

I. Match the words in the first column with the words in the second column that have the same meaning.

1. row	______	*having*
2. distinct	______	*is assumed*
3. consisting of	______	*one and only one*
4. representing	______	*a certain number of*
5. precisely	______	*exactly*
6. unique	______	*not the same*
7. at least	______	*disagrees with*
8. contradicts	______	*no fewer than*
9. is postulated	______	*horizontal array of numbers*
10. intersection	______	*vertical array of numbers*
11. finitely-many	______	*meeting point of two lines*
12. satisfied	______	*condition fulfilled*
13. column	______	*showing*

II. To what do the following words in the text refer?

A. "which are all" (line 3)

B. "which" (line 16)

C. "this" (line 31)

D. "such a plane" (line 35)

III. Comprehension.

A. The following questions refer to the diagram on the preceding page. Answer each and give the number of the axiom to which it refers.

1. Is there a line incident with both the point P_3 and the point P_5?

   yes/no axiom no. _____

2. Is there more than one line incident with both the point P_1 and the point P_2?

   yes/no axiom no. _____

3. If L is the line incident with both the point P_1 and the point P_3 and if M is the line incident with both the point P_4 and the point P_5, which point(s) is (are) incident with both L and M?

   point(s) _____ axiom no. _____

B. According to lines 31-32, if $n = 1$ then axiom (3) is contradicted. Explain why.

C. What would happen if $n = 2$?

D. Are all points of a projective plane incident with the same number of lines?

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IV. *When a word ending in -ING is not a verb.*

We know that a word ending in -ING may often be a verb, as in the following example: "it is interesting to speculate" (line 23). The words "the following example" also contain a word ending in -ING. Here, however, "following" is not a verb; it is an adjective describing what kind of example, the example which follows. But by changing the sentence, we can make "follows" a verb. (See below, line 4)

Change the following phrases, where the word ending in -ING is not a verb, in the same way. The first one has been done for you.

line	*-ING word*	*verb*
2	*consisting of*	*which consists of*
4	the *following axioms*	the axioms which follow
35	*One way of representing*	
41	*the following diagram*	

V. *Modal verbs.*

Mark the sentence that means the same as the sentence from the text.

A. *(line 1) "A projective plane can be abstractly defined ..."*

1. *It is possible to abstractly define a projective plane*
2. *One may possibly abstractly define a projective plane*
3. *A projective plane could be abstractly defined*

B. *(line 16) "... we can define the lines ..."*

1. *It is possible to define the lines*
2. *One may possibly define the lines*
3. *The lines could be defined*

C. *(line 23) "It is interesting to speculate what a projective plane ... would look like."*

1. *Such a plane might have been interesting to imagine*
2. *Such a plane was interesting to imagine*
3. *Such a plane will be interesting to imagine*

D. (line 31) "... there would only be three points ..."

1. There were only three points
2. There will only be three points
3. There would only have been three points

E. (line 32) "... there must be at least three points ..."

1. There had to be at least three points
2. There have to be at least three points
3. There would have been at least three points

F. (line 34) "... the plane must consist of seven points ..."

1. The plane has to consist of seven points
2. The plane might consist of seven points
3. The plane could consist of seven points

VI. Simple sentences.

A simple sentence contains one clause consisting of a verb and a subject. We find the subject of the sentence by asking "who" or "what" did the action. Look at the following example:

(line 27) Every line of Π is incident with precisely $n+1$ distinct points.

verb: is incident
subject: every line of Π

Find the verb and subject in each of the following sentences.

A. Every po..nt of Π is incident with precisely $n+1$ distinct lines.

verb:

subject:

B. There are precisely n^2+n+1 distinct lines in Π.

verb:

subject:

C. There are precisely n^2+n+1 distinct points in Π.

verb:

subject:

D. *What does such a plane look like?*

verb:

subject:

E. *Another way of representing the same plane is by the following diagram.*

verb:

subject:

VII. *<u>Writer's comments</u>.*

Sometimes the writer explains or comments on the mathematical ideas he presents. One such sentence is the following (lines 21-22): "Nothing in the above set of axioms implies that the number of points or the number of lines in an abstract projective plane is infinite." The writer points out, in case the reader hasn't noticed, that the axioms do not imply that the plane is infinite.

Explain what the writer is trying to say in each of the following sentences:

A. *(lines 22-24): In fact, it is interesting to speculate what a projective plane with only finitely-many points and finitely-many lines would look like.*

B. *(line 35): What does such a plane look line?*

TEXT II: INFINITE MATRICES AND CONVERGENT SEQUENCES

Let $\mathbb{N}$ be the set of all natural numbers. An <u>infinite real matrix</u> T is a function from the Cartesian product $\mathbb{N} \times \mathbb{N}$ to the set of real numbers. We denote T by $[t_{ij}]$, where $t_{ij} = T(i,j)$ for each pair (i,j) of natural numbers. If $\{a_j\}$ is a sequence of real numbers having the property that the infinite series $\Sigma_{j=1}^{\infty} t_{ij}a_j$ converges for each natural number i, then we can define a sequence of real numbers $\{b_i\}$ by setting $b_i = \Sigma_{j=1}^{\infty} t_{ij}a_j$ for each natural number i. Such a sequence is called a <u>transform</u> of $\{a_j\}$ by the infinite matrix T.

An infinite real matrix $T = [t_{ij}]$ is called a <u>Toeplitz matrix</u> if and only if the following conditions are satisfied:

(1) For every convergent sequence $\{a_j\}$, the transform $\{b_i\}$ by the matrix T of $\{a_j\}$ is well-defined; and

(2) $\lim_{i\to\infty} b_i = \lim_{j\to\infty} a_j$.

For example, the infinite real matrix $T = [t_{ij}]$ defined by

$$t_{ij} = \begin{cases} 1/i & \text{if } 1 \leq j \leq i \\ 0 & \text{otherwise} \end{cases}$$

is a Toeplitz matrix.

A divergent sequence may be transformed by a Toeplitz matrix into a convergent sequence. For example, if $a_j = (-1)^{j+1}$ for each natural number j, then the sequence $\{a_j\}$ diverges, but its transform by the Toeplitz matrix T, defined in the above example, is the sequence $\{b_i\}$

with $b_i = [1 + (-1)^{i+1}]/2i$, which converges to 0.

More generally, if $\{a_j\}$ is *any* divergent sequence each of the terms of which is equal to 1 or to -1, then there exists a Toeplitz matrix T which transforms it into a convergent sequence. Such a matrix $T = [t_{ij}]$ can be defined in the following manner: let $\{n_i\}$ be a strictly increasing sequence of natural numbers having the property that $a_{n_i} = -a_{n_i+1}$ for each natural number i. Define t_{ij} to equal 1/2 if $j = n_i$ or if $j = n_i+1$. Otherwise define t_{ij} to equal 0. Then $T = [t_{ij}]$ is a Toeplitz matrix which transforms $\{a_j\}$ into the 0-sequence.

On the other hand, for any given Toeplitz matrix T there exists a sequence $\{a_j\}$, each of the terms of which is equal to 1 or to -1, which is transformed by T into a divergent sequence. In other words, we see that some Toeplitz matrices transform some sequences the terms of which are equal to 1 or to -1, into convergent sequences, but no Toeplitz matrices transform all such sequences into convergent sequences. The proof of this statement is based on a characterization of Toeplitz matrices first proven by Toeplitz in 1911:

THEOREM: *An infinite real matrix* $T = [t_{ij}]$ *is a Toeplitz matrix if and only if the following conditions are satisfied:*

(1) *There exists a real number* r *such that* $\Sigma_{j=1}^{\infty} |t_{ij}| \leq r$ *for all natural numbers* i;

(2) $\lim_{i\to\infty} \Sigma_{j=1}^{\infty} t_{ij} = 1$;

(3) $\lim_{i\to\infty} t_{ij} = 0$ *for every natural number* j.

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EXERCISES ON TEXT II:

I. *Match the words in the first column with the words in the second column having the opposite meaning.*

1.	*following*	_____	*equivalent*
2.	*infinite*	_____	*meaning*
3.	*converge*	_____	*decreasing*
4.	*increasing*	_____	*diverge*
5.	*equal*	_____	*previous*
		_____	*verge*
		_____	*indefinite*
		_____	*unequal*
		_____	*finite*

II. *To what do the following words in the text refer? Choose a or b.*

A.	"Such a sequence (line 7)	a. $\{a_j\}$	b. $\{b_i\}$
B.	"its" (line 19)	a. natural number j	b. sequence $\{a_j\}$
C.	"which" (line 21)	a. Toeplitz matrix T	b. sequence $\{b_i\}$
D.	"which" (line 23)	a. terms	b. any divergent sequence $\{a_j\}$
E.	"which" (line 24)	a. Toeplitz matrix T	b. $\{a_j\}$
F.	"it" (line 24)	a. Toeplitz matrix T	b. any divergent sequence $\{a_j\}$
G.	"which" (line 31)	a. Toeplitz matrix T	b. sequence $\{a_j\}$
H.	"which" (line 33)	a. Toeplitz matrices	b. some sequences
I.	"such sequences" (line 35)	a. convergent sequences	b. some sequences

III. Verbs: active and passive.

The following sentence contains the verb "denote" in the active voice: (line 3) "We denote T by $[t_{ij}]$, where ..." . This sentence can be rewritten using the passive voice as follows: "The matrix T is denoted by $[t_{ij}]$, where ...".

Rewrite the following using the passive voice.

A. (line 6) " ... we can define a sequence of real numbers ..."

B. (line 33) "... some Toeplitz matrices transform some sequences ..."

C. (lines 34-35) "... but no Toeplitz matrices transform all such sequences into convergent sequences."

Rewrite the following using the active voice.

D. (line 17) "A divergent sequence may be transformed by a Toeplitz matrix ..."

E. (lines 24-25) "Such a matrix $T = [t_{ij}]$ can be defined in the following manner:"

F. (lines 30-32) "... there exists a sequence $\{a_j\}$, ... which is transformed by T into a divergent sequence."

G. (lines 35-36) "The proof of this statement is based on a characterization of Toeplitz matrices ..."

IV. Verb used as adjective/expanded forms

In the following sentence the word "defined" is the past participle of the verb "to define": (lines 14-15) "For example, the infinite matrix $T = [t_{ij}]$ defined by ... is a Toeplitz matrix." Since the word "defined" is not accompanied by the auxiliary verb "is" or "was", however, it is not used as a verb in this sentence. Instead, it functions as an adjective to describe the noun phrase "the infinite matrix $T = [t_{ij}]$". The verb in this sentence is "is" (line 16).

This sentence can be expanded by using the relative pronoun "which" in the following way: "For example, the infinite matrix $T = [t_{ij}]$, which is defined by ..., is a Toeplitz matrix."

Expand the following phrases and clauses from the text. The first one is done for you.

A. (lines 19-20) " ... but its transform by the Toeplitz matrix T, defined in the above example, is ... "

expanded form: "... matrix T, which is/was defined in ..."

B. (lines 25-26) "a strictly increasing sequence"

C. (line 26) "natural numbers having the property ... "

D. (line 30) "for any given Toeplitz matrix T"

E. (lines 36-37) "a characterization ... first proven by Toeplitz in 1911"

F. (line 39) "the following conditions"

Rewrite the following clauses without the relative pronoun "which".

G. (lines 23-24) "... then there exists a Toeplitz matrix T which transforms it into a convergent sequence.

H. (line 29) "... a Toeplitz matrix which transforms $\{a_j\}$ into the 0-sequence"

Fill in the following table by changing these sentences in the text. The first one has been done for you.

lines	active	passive	adjective	expanded
4-5	---	---	$\{a_j\}$ is a sequence of real numbers having the property ...	$\{a_j\}$ is a sequence of real numbers which have the property ...
6	we can define a sequence of real numbers by setting $b_i = \Sigma_{j=1}^{\infty} t_{ij}a_j$			
14			the infinite real matrix $T = [t_{ij}]$ defined by ...	
24-25		Such a matrix can be defined in the following manner:		

V. Sentence function

A. Definitions: A definition gives the meaning of a word or phrase. For example, the sentence in lines 1-3 is a definition: "An infinite real matrix T is a function from the Cartesian product $\mathbb{N} \times \mathbb{N}$ to the set of real

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numbers." The skeleton of this sentence is:

An ... is a ... .

Definitions can also be recognized from the following key words and structures:

A ... is called a ...

We define ... by ... where ...

We can define ... if ...

and so forth. In the table below, list the sentences in the text which contain a definition, indicating the words which mark the skeleton of the sentence. The first one has been done for you.

lines	*key words of definition*
1-3	*An infinite real matrix T is a function from ... to ...*

B. General statements: Some statements assert that something is always true. Such statements are marked by words such as "every", "always", "any", and "all". In the following table, list the sentences containing general statements, and give the key word(s) which show this. The first one has already been done for you.

lines	*key words of general statement*
1	*Let ... be ... all*

C. <u>Conditional statements</u>: Often a statement asserts that something is true under certain conditions. Such statements are signaled by words such as "some", "if and only if", "such", "otherwise", "where ... for ...", and so forth. In the following table, list the sentences containing conditional statements, and indicate the key word(s) which show this. The first one has already been done for you.

lines	*key words of conditional statement*
3-4	*We denote ... by ... where ... for each pair ...*

Note that, in a mathematical context, "some" implies existence, whereas "every" does not. Look at the following two sentences:

Every pink elephant has ten feet.

Some pink elephants have ten feet.

The first sentence can be true; it states that if there were such things as pink elephants then they would all have ten feet. The second sentence, however, states that there are existing pink elephants and that some of them have ten feet while perhaps others have some other number of feet. The second sentence is false.

VI. <u>Simple sentences</u>.

In Exercise VI on Text I we characterized simple sentences. In the following table, list the simple sentences appearing in Text II. For each, give the subject and the verb. The first one has been done for you.

line(s)	*subject*	*verb*
1	*you (understood)*	*Let*

VII. *Comprehension*

A. *What is the relationship between* $\{a_j\}$ *and* $\{b_i\}$?

B. *In lines 14-16, what will happen if we omit the condition "0 otherwise"?*

C. *Can any Toeplitz matrix transform all divergent sequences, each of whose terms equals 1 or -1, into convergent sequences? (Give line numbers in the text to support your answer.)*

D. *Can every divergent sequence, each of whose terms equals 1 or -1, be transformed by a Toeplitz matrix into a convergent sequence? (Give line numbers in the text to support your answer.)*

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TEXT III: RELATIONS

If S and T are nonempty sets, then we define the Cartesian product of S and T to be the set of all ordered pairs of the form (s,t), where $s \in S$ and $t \in T$. We denote the Cartesian product of S and T by $S \times T$. Any nonempty subset of $S \times T$ is called a relation between S and T. We will denote the relations between S and T by capital Greek letters.

EXAMPLE: If $S = T =$ the set of all real numbers then

$$\Phi = \{(s,t) \mid s = t + 7\}$$

and

$$\Psi = \{(s,t) \mid s^2 + t^2 = 1\}$$

are relations between S and T.

A relation on a nonempty set S is defined to be a relation between S and itself, that is to say, a nonempty subset of $S \times S$. The two relations defined in the above example are relations on the set of real numbers. Among the various types of relations one can define on a set S, we single out for special emphasis a class of relations known as equivalence relations. An equivalence relation on a set S is a relation Φ satisfying the following three conditions:

(1) (Reflexivity) If $s \in S$ then $(s,s) \in \Phi$.

(2) (Symmetry) If $(s,s') \in \Phi$ then $(s',s) \in \Phi$.

(3) (Transitivity) If $(s,s') \in \Phi$ and if $(s',s'') \in \Phi$ then $(s,s'') \in \Phi$.

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EXAMPLE: Let $S = \{1,2,3\}$. Then $S \times S$ has nine elements, and so there are 511 possible relations defined on S. Of these, only five are equivalence relations, namely

$\Phi_1 = S \times S$;

$\Phi_2 = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$;

$\Phi_3 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$;

$\Phi_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$;

$\Phi_5 = \{(1,1),(2,2),(3,3)\}$.

If Φ is a relation on a set S, it is sometimes convenient to write $s \, \Phi \, s'$ instead of $(s,s') \in \Phi$. This is particularly true in the case of equivalence relations. Such relations are often denoted by one of the following symbols: $\sim$, $\simeq$, $\approx$, $\equiv$. Thus we write $s \sim s'$ instead of $(s,s') \in \sim$.

Equivalence relations defined on a nonempty set S give rise to partitions of the set, in the following sense: a collection $\{A_i \mid i \in \Omega\}$ of nonempty subsets of S is said to be a <u>partition</u> of S if and only if the following two conditions are satisfied:

(1) $S = \cup_{i \in \Omega} A_i$;

(2) $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Any partition $\{A_i\}$ of S defines an equivalence relation $\sim$ on S as follows: if s and s' are elements of S, then $s \sim s'$ if and only if s and s' both belong to the same set A_k of the partition.

Conversely, suppose that $\sim$ is an equivalence relation defined on a set S. For each element s of S, let $B(s) = \{s' \in S \mid s \sim s'\}$. This set is called the <u>equivalence class of</u> s with respect to the relation ~. Let T be a subset of S consisting of precisely one representative from each equivalence class of elements of S. Then $\{B(s) \mid s \in T\}$ is a partition of the set S. To see why this is true, we note that for any two elements s and s' of S, we have $s \sim s'$ if and only if $B(s) = B(s')$.

EXAMPLE: Let $\mathbb{Z}$ be the set of integers. We define an equivalence relation ~ on $\mathbb{Z}$ by saying that $n \sim k$ if and only if $n-k$ is an even number. Then this relation defines two distinct equivalence classes of integers: B(1), which is the set of all odd numbers; and B(2), which is the set of all even numbers. Clearly $\{B(1), B(2)\}$ is a partition of $\mathbb{Z}$.

EXERCISES ON TEXT III:

I. Match the symbol in the first column with its name in the second column by writing the correct number in the blank space.

1.	$\in$	____	*is an element of*
2.	$\times$	____	*is equivalent to*
3.	'	____	*double prime*
4.	~	____	*intersection*
5.	$\cup$	____	*Cartesian product*
6.	$\cap$	____	*prime*
7.	''	____	*union*
8.	s^2	____	*s squared*

II. Match the mathematical expression in the first column with its translation into words in the second column by writing the correct number in the blank space.

1.	(line 3)	$s \in S$	______	the set of A_i, as i ranges over Ω
2.	(line 4)	$S \times T$	______	s and s' are equivalent with respect to Φ
3.	(line 22)	$S = \{1,2,3\}$	______	s is an element of S
4.	(line 31)	$(s,s') \in \Phi$	______	the set S having elements 1,2,3
5.	(line 33)	$s \sim s'$	______	S equals the union of the A_i, as i ranges over Ω
6.	(line 35)	$\{A_i \mid i \in \Omega\}$	______	the Cartesian product of S and T
7.	(line 38)	$S = \cup_{i \in \Omega} A_i$	______	s is related to s' by $\sim$

III. To what words in the text do the following words refer?

A. (line 11) "itself"

B. (line 13) "the two relations"

C. (line 23) "Of these"

D. (line 48) "This is true"

E. (line 52) "This relation"

IV. Find the verb and the subject of each of the following sentences:

A. (lines 4-5) "Any nonempty subset of $S \times T$ is called a relation between S and T."

verb:

subject:

B. (lines 5-6) "We will denote the relations between S and T by capital Greek letters."

verb:

subject:

C. (lines 17-18) "An equivalence relation on a set S is a relation satisfying the following three conditions:"

verb:

subject:

D. (lines 13-14) "The two relations defined in the above example are relations on the set of real numbers."

verb:

subject:

E. (lines 31-32) "This is particularly true in the case of equivalence relations."

verb:

subject:

F. (lines 32-33) "Such relations are often denoted by one of the following symbols: ..."

verb:

subject:

G. (line 33) "Thus we write ..."

verb:

subject:

H. (lines 44-45) "This set is called the equivalence class of S with respect to the relation $\sim$."

verb:

subject:

I. (lines 47-48) "Then $\{B(s) \mid s \in T\}$ is a partition of the set S."

verb:

subject:

J. (line 50) "Let $\mathbb{Z}$ be the set of integers."

verb:

subject:

V. Sentence function

A. Definitions: In Exercise V on Text II we discussed definitions and how to recognize them. In the table below, list the sentences in the text which contain a definition, indicating the words which mark the skeleton of the sentence.

line(s)	key words of definition

B. Denotation: The writer often explains the signs by which he will express a quantity or a relationship by saying that he will "write" or "denote" it in a particular way. Denotation differs from definition in that the writer is not defining what a concept is, but merely assigning it a symbol. In the table below, list the sentences in the text which contain statements of this kind. The first one has been done for you.

line(s)	key words of denotation
3-4	We denote ... by $S \times T$

C. Classification: A classification is a statement that places a given item within a certain class or group. One example of a classification is this statement: "An apple is a kind of fruit." In the table below, list the sentences containing classification statements. The first one has been done for you.

line(s)	key words of classification
13-14	The two relations ... are relations on the set of real numbers

VI. Verbs: active and passive

The active and passive voices were discussed in Exercise III on Text II and expanded forms were discussed in Exercise IV.

A. In the table below, list the sentences containing verbs in the active voice. Give line numbers and the verb of each sentence. The first one has been done for you.

line(s)	*verb*
1	*define*

B. *In the table below, list the sentences containing verbs in the passive voice. Give line numbers and the verb of each sentence. The first one has been done for you.*

line (s)	*verb*
4	*is called*

C. *It is possible to expand a sentence containing a past participle (-ED, -T) or a present participle (-ING) of a verb by adding the word "which". To the -ED form, add the auxiliary verb IS, ARE, WAS, or WERE to form the passive form of the verb. The -ING form becomes active when it is expanded by using "which".*

Expand the following phrases and clauses from the text.

1. *(lines 13-15) "The two relations defined in the above example ..."*

2. *(lines 17-18) "An equivalence relation on a set S is a relation ... satisfying the following three conditions."*

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3. (lines 22-23) "Then $S \times S$ has nine elements, and so there are 511 possible relations defined on S."

4. (line 34) "Equivalence relations defined on a nonempty set S ..."

5. (lines 45-47) "Let T be a subset of S consisting of precisely one representative ..."

VII Comprehension

A. Which special type of relation is discussed in the text?

B. According to lines 17-29, which of the following conditions does the relation Φ_2 satisfy?
   1. reflexivity
   2. symmetry
   3. transitivity
   4. all of the above

C. Look at the following sentences:
   1. (lines 40-41) "Any partition $\{A_i\}$ of S defines an equivalence relation $\sim$ on S ..."
   2. (lines 43-44) "Conversely, suppose that $\sim$ is an equivalence relation defined on a set S."

   Which one of the following sentences would you expect to follow sentence 2(l.3-4,
   a. "Equivalence relations defined on a nonempty set S give rise to partitions of the set."
   b. "A collection of nonempty subsets of S is said to be a partition of S...
   c. "This set is called an equivalence class of elements of S."

   How does the verb "define" help you to answer?

D. According to lines 50-52, what would happen if $n-k$ were an odd number?

TEXT IV: METRIC SPACES

A distance function defined on a nonempty set S is a function d from $S \times S$ to S such that for each s_1, s_2, and s_3 in S the following conditions are satisfied:

(1) $d(s_1,s_2) = 0$ if and only if $s_1 = s_2$;

(2) $d(s_1,s_2) = d(s_2,s_1)$;

(3) $d(s_1,s_3) \leq d(s_1,s_2) + d(s_2,s_3)$.

As a rule, several distance functions can be defined on the same set. For example, if E is the usual Euclidean plane, then we can define three distance functions on E as follows: if $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are points in S, set

$$d_1(P_1,P_2) = [(x_2 - x_1)^2 + (y_2 - y_1)^2]^{1/2};$$

$$d_2(P_1,P_2) = \max\{|x_2 - x_1|, |y_2 - y_1|\};$$

$$d_3(P_1,P_2) = |x_2 - x_1| + |y_2 - y_1|.$$

A nonempty set S, together with a fixed distance function d defined on it, is called a metric space. If s is a point in a metric space (S,d) and if r is a positive real number, then we define the ball of radius r around the point s to be $B_s(r) = \{s' \in S \mid d(s,s') < r\}$. To see that the metric spaces (E,d_1), (E,d_2), and (E,d_3) are not the same, we note that

(1) In (E,d_1) the ball $B_{(0,0)}(1)$ consists of the interior of the disc of radius 1 around the origin $(0,0)$.

(2) In (E,d_2) the ball $B_{(0,0)}(1)$ consists of the interior of the square with vertices $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$.

(3) In (E,d_3) the ball $B_{(0,0)}(1)$ consists of the interior of the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$.

If (S,d) is a metric space, we say that a sequence $\langle s_i \rangle$ of points in S <u>converges</u> to a given point s^* in S if and only if, for any $\varepsilon > 0$, there exists a natural number n such that $d(s_i,s^*) < \varepsilon$ for all $i > n$. As far as convergence is concerned, all three of the above distance functions are equivalent. That is to say, a sequence of points in E converges to a given point in E with respect to one of these distance functions if and only if it converges to that point with respect to the other two.

To see why this is so, we must consider the notion of a topology defined on a set S. If S is a nonempty set, then a <u>topology</u> defined on S is a set U of subsets of S, containing both $\emptyset$ and S, which is closed under taking finite intersections and arbitrary unions of its members. A subset V of a topology U on S is called a <u>basis</u> for U if and only if U is just the set of all possible unions of sets of members of V.

If d is a distance function defined on a nonempty set S, let $\{B_s(r) \mid s \in S; r > 0\}$ be the basis for a topology on S. We will say that two distance functions defined on S are <u>equivalent</u> if and only if they give rise to the same topology on S in the above manner. Another way of saying this is the following: two distance functions d and d' defined on a set S are equivalent if and only if for every point S in S, and for every positive real number r, there exist positive real numbers r' and r'' such that

(1) $d(s,s') < r \rightarrow d'(s,s') < r'$; and

(2) $d'(s,s') < r \Rightarrow d(s,s') < r''$.

With this definition in mind, the equivalence of the distance functions d_1, d_2, and d_3 defined on E is immediate.

Needless to say, the above argument does not imply that we cannot define two nonequivalent distance functions on the same set. For example, we can define another distance function d_4 on E as follows:

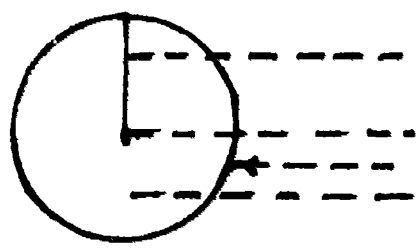
$$d_4(P_1,P_2) = \begin{cases} 1 & \text{if } P_1 \neq P_2 \\ 0 & \text{if } P_1 = P_2 \end{cases} .$$

This function is clearly not equivalent to any of the other three.

EXERCISES ON TEXT IV:

I. Fill in the drawing by writing the number of the appropriate word at the end of the broken line.

1. *center*
2. *radius*
3. *disc*
4. *interior*
5. *vertex*
6. *distance function*



II. To what in the text do the following words refer?

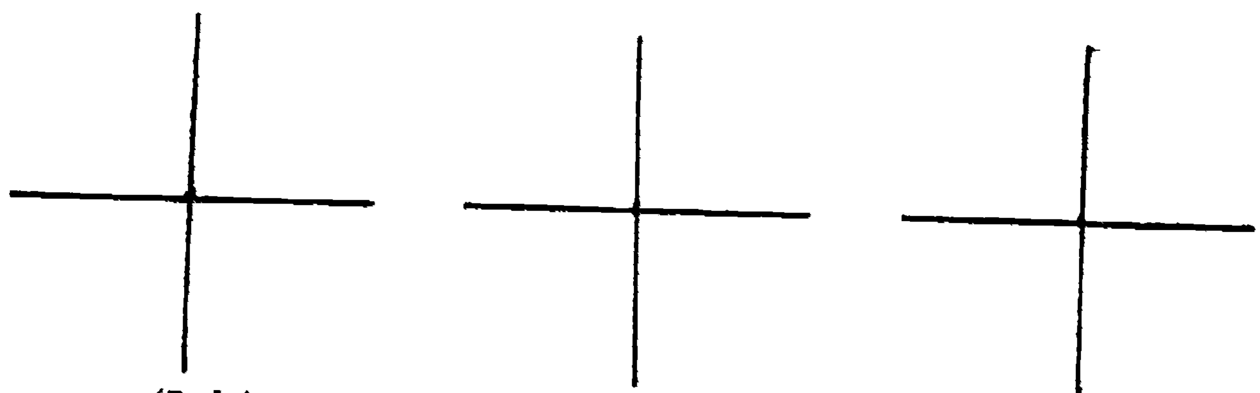
1. *(line 15) "it"*
2. *(line 30) "it"*
3. *(line 31) "that point"*
4. *(line 31) "the other two"*
5. *(line 32) "this"*
6. *(line 34) "which"*
7. *(line 35) "its"*
8. *(line 40) "they"*
9. *(line 54) "the other three"*

III. *Comprehension*

A. *What is the connection between the metric spaces* (E,d_1) *and* (E,d_2)?

B. *In terms of convergence, is there any difference between* (E,d_1) *and* (E,d_3)?

C. *Draw three graphs of the ball* $B_{(0,0)}(1)$ *in* (E,d_1), (E,d_2), *and* (E,d_3).



(E,d_1) (E,d_2) (E,d_3)

IV. *Phrases showing the writer's comments.*

To explain the importance of a certain idea or sentence, a writer may begin a phrase with a comment. Although such phrases are not essential to the meaning of the sentence, they add a framework which clarifies the writer's meaning. The phrase "for example" shows that the writer is about to give an example. Here is another kind of marker:

(line 7) "As a rule, several distance functions can be defined on the same set."

The sentence would not lose its meaning if the phrase "As a rule" were left out. However the phrase "as a rule", which means "in general", emphasizes the fact that the idea expressed in this sentence is usually true.

Explain the purpose or function of each of the following phrases:

A. *(lines 17-18) "To see that the metric spaces ... are not the same,"*

B. *(lines 27-28) "As far as convergence is concerned,"*

C. *(lines 28-29) "That is to say,"*

D. (line 32) "To see why this is so,"

E. (line 41) "Another way of saying this is the following:"

F. (line 47) "With this definition in mind,"

G. (line 49) "Needless to say,"

V. General vs. qualified statements

Some words signal statements that are always true. Other words signal statements that are true only under certain conditions. Such words include "if", "if and only if", "such that", "which", "where", "when", etc. These words qualify the general statement, making it more specific.

Indicate whether the following sentences are general or qualified, and underline the key word(s). The first one has been done for you.

Line(s)	sentence	general	qualified
1-7	A distance function ... is a ... <u>such that</u> for each ... the following conditions are satisfied:		x
7-13	For example, if E is the usual ..., then we define three distance functions ...		
25-27	If (S,d) is a metric space, we say that ... if and only if, for any, ..., there exists ... such that ... for all ...		
35-37	A subset V of a ... is called ... if end only if ... the set of all possible unions of ...		
54	This function is clearly not equivalent to any of the other three.		
7	As a rule, several distance functions can be defined on the same set.		

VI. Definition, explanation/expansion, example

The following sequence of sentences is very common in mathematical writing: definition, explanation/expansion, example.

Look at the following table, which contains lines 1-13 of the text:

line(s)	*key words*	*sentence function(s)*
1-7	*A distance function ... is a function ... such that for each ... the following conditions are satisfied:*	*definition* (qualified)
7	*As a rule, several ... can be defined ... .*	*general statement, expansion*
8-13	*For example, if E ..., then we define three distance functions ... as follows:*	*example, definition*

Fill in the following table in the same way.

line(s)	*key words*	*sentence function(s)*
25-27		*qualified definition*
27-28		*expansion*
28-31		*explanation*
33-35	*If ..., then a topology defined on S is ..., containing ..., which is closed under ...*	
35-37	*A subset V of a topology ... is called a basis for U if and only if ...*	
38-39	*If d is a distance function defined on ..., then ... is the basis for a topology on S.*	
39-41	*We will say that two ... are equivalent if and only if they ... in the above manner*	
41-46		*rephrasing of definition*

(continuation)

line(s)	key words	sentence function(s)
49-50	..., the above argument does not imply that we cannot define two nonequivalent...	
51-53		definition/example
54	This ... clearly not equivalent to any....	

VII. Compound sentences.

Two simple sentences may be combined into a single, long compound sentence by placing a semi-colon (;) or comma and connecting word (, and , but , or) in place of the period. For example, look at the following two sentences:
1. We can define another distance function d_4 on E.

2. This function is not equivalent to the others.
These sentences may be connected in a number of ways. For example:
A. We can define another distance function d_4 on E; this function ... .
B. We can ... on E, and this function ... .
C. We can ... on E, but this function ... .
In example B, the second half of the sentence, beginning with "and", follows from the first half. In example C, the second half of the sentence, beginning with "but", contrasts with the first half. Example A is ambiguous; it may suggest either a logical sequence or a contrast.

For each of the following sentences, find the verb and the subject. Then combine them into a compound sentence.

A. 1. Several distance functions can be defined on the same set.

verb:

subject:

2. We can define three distance functions on E.

verb:

subject:

Compound sentence combining 1 and 2:

B. 1. A nonempty set S ... is called a metric space.

verb:

subject:

2. *The metric spaces* (E,d_1), (E,d_2), *and* (E,d_3) *are not the same.*

verb:

subject:

Compound sentence combining 1 and 2:

C. 1. *The equivalence of the distance functions* d_1, d_2, *and* d_3 *defined on* E *is immediate.*

verb:

subject:

2. *The above argument does not imply the converse.*

verb:

subject:

Compund sentence combining 1 and 2:

TEXT V: PEANO'S POSTULATES FOR THE NATURAL NUMBERS

In 1889 the Italian mathematician G. Peano formulated a set of five postulates which state in precise mathematical language those properties of the natural numbers which we feel to be intuitively obvious. These postulates are the following:

(I) 1 is a natural number.

(II) To every natural number n there is assigned a natural number $S(n)$, called the <u>successor</u> of n.

(III) If n and m are different natural numbers, then $S(n) \neq S(m)$.

(IV) There is no natural number n satisfying $S(n) = 1$.

(V) If U is a set of natural numbers containing 1 and having the property that if n belongs to U then $S(n)$ belongs to U, then U equals the set of all natural numbers.

We normally denote the successor of a natural number n by $n+1$ instead of $S(n)$.

Peano's fifth postulate is often called the <u>Principle of Mathematical Induction</u>, and there are many equivalent ways of formulating it. One way in which this prinicple is often stated is the following: if P is a property of the natu [illegible] that

(i) 1 has property P;

(ii) If n has property P, then so does $n+1$;

then all natural numbers must have property P. A proof which makes use of the Principle of Mathematical Induction must therefore have two parts: the basis, which shows that 1 has property P, and the step, which shows that if n has property P, then so does $n+1$. To see how this works, let us consider a simple proof by mathematical induction.

THEOREM: If n is a natural number, then n^2-n is always an even number.

PROOF: If $n = 1$ then $1^2 - 1 = 0$, and this is an even number. Thus we have proven the basis of the induction. Now assume that n is a natural number having the property that n^2-n is even. Then $(n+1)^2 - (n+1) = (n^2+2n+1)-(n+1) = n^2+n = (n^2- n)+2n$. But n^2-n is an even number, and $2n$ is also an even number. Since the sum of two even numbers is even, this proves that $(n+1)^2-(n+1)$ is also an even number. Thus we have proven the step of the induction. □

We can replace postulates (II), (III), and (IV) by a single postulate:

(*) There exists a one-to-one function S from the set of natural numbers to itself, the image of which does not contain 1.

Since there can exist no one-to-one correspondence between a finite set and one of its proper subsets, postulate (*) implies that the set of all natural numbers is infinite. It is therefore sometimes called the Postulate of Infinity. Is there a natural number other than 1 which does not belong to the image of the function S? The answer to this question provides another good example of the application of the Principle of Mathematical Induction.

THEOREM: The only natural number which is not the successor of a natural number is 1.

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PROOF: Let W be the image of the function S; that is, W is the set of all natural numbers of the form S(n), where n is a natural number. Let U = {1} ∪ W. Then surely 1 ∈ U, and if n belongs to U, then S(n) certainly belongs to U (since it belongs to W by the definition of W). Therefore, by the Principle of Mathematical Induction, U equals the set of all natural numbers. This means that every natural number other than 1 is a member of W and so is the successor of some other natural number. □

EXERCISES ON TEXT V:

I. Match the word in the first column with the word in the second column having the same meaning.

1. containing	______	*following*
2. obvious	______	*characteristic*
3. surely	______	*clear*
4. proper	______	*fulfilling*
5. succeeding	______	*including*
6. provide	______	*state*
7. formulate	______	*substitute*
8. property	______	*not equal to the whole*
9. replace	______	*give*
10. satisfying	______	*certainly*
11. precise	______	*exact*

II. To which words in the text do the following words refer? Choose a or b.

1. (line 3)	*"which"*	*a. numbers*	*b. properties*
2. (line 24)	*"so does"*	*a. has property P*	*b. have two parts*
3. (line 36)	*"itself"*	*a. function S*	*b. set of natural numbers*
4. (line 38)	*"its"*	*a. postulate (*)*	*b. a finite set*
5. (line 39)	*"It*	*a. postulate (*)*	*b. a finite set*
6. (line 48)	*"it"*	*a. S(n)*	*b. U*

III. <u>Since</u>

The word "since" has two meanings, depending how it is used in the sentence. One meaning is "from a certain time": He's been here <u>since</u> yesterday. The *other meaning occurs when "since" is used as a connecting word; it is then used to mean "because": <u>Since</u> it was raining, I decided to take an umbrella.*

In the following sentences, which of the two meanings does "since" have: (a) "from a certain time", or (b) "because"?

A. *(lines 31-32) "Since the sum of two even numbers is even, this proves that $(n+1)^2-(n+1)$ is also an even number."*

B. *(lines 37-30) "Since there can exist no one-to-one correspondence ..., postulate (*) implies that the set of all natural numbers is infinite."*

C. *(lines 47-48) "Then surely ... (since it belongs to W by the definition of W)."*

IV. *<u>Phrases showing the writer's comments</u>.*

Phrases showing the writer's comments were discussed in Exercise IV on Text IV. For each of the following sentences from the text, indicate whether it is (1) in the formal language of mathematics, or (2) the writer's comments about mathematics. The first one has been done for you.

A. *(lines 27-28) "Thus we have proven the base of the induction."* 2

B. *(lines 31-32) "Since the sum of two even numbers is even, this proves that $(n+1)^2-(n+1)$ is also an even number."* ____

C. *(lines 35-36) "There exists a one-to-one function S from the set of natural numbers to itself, the image of which does not contain 1."* ____

D. *(lines 2-') "... those properties of the natural numbers which we feel to be intuitively obvious."* ____

E. *(line 5) "1 is a natural number."* ____

F. *(lines 24-25) "To see how this works, let us consider a simple proof by mathematical induction."* ____

G. *(line 27) "If $n = 1$ then $1^2 - 1 = 0$, and this is an even number."* ____

V. *Clauses and phrases.*

A clause contains a verb and a subject. For example, in

(lines 27-28): "Thus we have proven the base of the induction."

the underlined part is a clause, the verb of which is "have proven" and the subject of which (found by asking "who?" or "what?") is "we". Note that the past participle "proven" cannot be considered a verb without the auxiliary "have". In general, the present participle (-ING) and the past participle (-ED, -EN) of a verb cannot be considered as verbs without an auxiliary. Examples of auxiliary verbs are "has," "have," "is," and "are."

In the following sentences, indicate which verb forms function as verbs.

A. *(lines 6-7) "To every natural number n there is assigned a natural number S(n), called the successor of n."*

1. *is assigned*
2. *called*

B. *(line 9) "There is no natural number n satisfying S(n) = 1."*

1. *is*
2. *satisfying*

C. *To every natural number n there is assigned a natural number S(n) which is called the successor of n.*

1. *is assigned*
2. *is called*

D. *There is no natural number n which satisfies S(n) = 1.*

1. *is*
2. *satisfies*

A group of words which does not have either a subject or *a verb is called a phrase. The following are phrases: (line 7) "called the successor of n", and (line 9) "satisfying S(n) = 1".*

VI. *Compound sentences.*

Compound sentences were discussed in Exercise VII on Text IV. When two simple sentences are joined to make up a compound sentence, each becomes a clause. Since each of these clauses has its own verb and subject, it is called independent.

For example, here are two simple sentences:

Let A be an even integer. Let B be an odd integer.

They can be joined in either of the following ways:

1. Let A be an even integer; let B be an odd integer.
2. Let A be an even integer, and let B be an odd integer.

Each of these compound sentences has two independent clauses, as follows:

	connector	subject	verb
first independent clause	---	you (understood)	Let
second independent clause	; / , and	you (understood)	let

Note that there is no connector at the beginning of the first clause, whereas there is a connector to join the second cause to the first.

In the following compound sentences, give the verbs, subjects, and connector.

	connector	subject	verb
A. (lines 15-16) "Peano's fifth postulate is often called the <u>Principle of Mathematical Induction</u>, and there are many equivalent ways of formulating it."			
B. (lines 30-31) "But n^2-n is an even number, and $2n$ is also an even number."			

VII. Comprehension

Each of the following statements is either true or false. For each statement, give the line number, and if it is false, explain why.

	lines	T/F - why
A. Peano's postulates state properties of the natural numbers that are difficult to understand.		
B. We usually denote the successor of a natural number n by $n+1$.		
C. There is only one way of formulating Peano's fifth postulate.		

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	lines	*T/F - why*
D. *A proof which makes use of the Principle of Mathematical Induction need have only one part: either the base or the step.*		
E. *The sum of two even numbers can never be an odd number.*		
F. *We can replace postulate (II) by postulate (*).*		
G. *Between a finite set and one of its proper subsets it is sometimes possible to construct a one-to-one correspondence.*		
H. *The question in lines 40-41 is answered in lines 41-42.*		

TEXT VI: MONOIDS

If S is a nonempty set then a function from $S \times S$ to S is called a binary operation on S. A nonempty set together with at least one binary operation satisfying certain given conditions is called an algebraic structure. There are several types of algebraic structures of varying degrees of complexity which have proven to be of use in mathematics and which have been extensively studied.

Let us look at a very simple type of algebraic structure. A set S with a single binary operation $*$ defined on it is called a semigroup if and only if this operation is associative, i.e., if and only if $a*(b*c) = (a*b)*c$ for any three elements a b, and c in S. The set $\mathbb{Z}$ of integers together with the operation of addition forms a semigroup. The set $\mathbb{Z}$ of integers together with the operation of subtraction does not form a semigroup. The set of positive integers together with the operation of taking greatest common divisor forms a semigroup. So does the set of negative integers together with the operation of taking maximum.

A semigroup $(S,*)$ is called a monoid if and only if there exists an element e of S satisfying the condition that $s*e = s = e*s$ for all elements s of S. Such an element is called an identity of the monoid. Any monoid has at most one identity. Indeed, if f and e are both identities of a monoid $(S,*)$, then $e = e*f = f$. Therefore, any monoid has precisely one identity.

Since monoids are more "complicated" algebraic structures--they satisfy more conditions--we expect them to be more "interesting" to the mathematician and at the same time expect to find less of them. The set of positive integers together with the operation of addition is a semigroup which is not a monoid. The set of integers, together with the operation of taking maximum, is another example of a semigroup which is not a monoid.

A function α from a monoid $(S,*)$ to itself is said to be an <u>endomorphism</u> of the monoid if and only if the following two conditions are satisfied:

(1) $\alpha(s)*\alpha(s') = \alpha(s*s')$ for all $s, s' \in S$;

(2) $\alpha(e) = e$.

Let us denote the set of all endomorphisms of a monoid $(S,*)$ by End(S). This set is nonempty since it surely contains the function δ defined by $\delta(s) = s$ for all $s \in S$.

Composition of functions is a binary operation on End(S). That is to say, if $\alpha, \beta \in \mathrm{End}(S)$ then the function $\alpha^{o}\beta$ defined by $\alpha^{o}\beta(s) = \alpha(\beta(s))$ for all $s \in S$ is an endomorphism of S. To see this, note that for all $s, s' \in S$ we have $\alpha^{o}\beta(s*s') = \alpha(\beta(s*s')) = \alpha(\beta(s)*\beta(s')) = \alpha(\beta(s))*\alpha(\beta(s')) = \alpha^{o}\beta(s)*\alpha^{o}\beta(s')$. Moreover, $\alpha^{o}\beta(e) = \alpha(\beta(e)) = \alpha(e) = e$. In fact, End(S) together with the operation of composition is a semigroup, since composition of functions is easily seen to be associative. For any $\alpha \in \mathrm{End}(S)$, we have $\alpha^{o}\delta = \alpha = \delta^{o}\alpha$, and so we see that δ is the identity element of End(S). Therefore End(S) is in fact a monoid.

Thus we have seen how to build a monoid End(S) from a given monoid S.

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We can, of course, repeat this process and define another monoid $End_2(S)$ to be $End(End(S))$. In general, we can define the monoid $End_n(S)$ to be $End(End_{n-1}(S))$ for every positive integer n greater than 2.

If α is an endomorphism of a monoid S, then the <u>image</u> of α is a submonoid of S. That is to say, it is a subset of S which is a monoid under the same operation used in S. We will denote this submonoid of S by $im(\alpha)$. If we apply α to every element in $im(\alpha)$, we obtain a submonoid of $im(\alpha)$, which we denote by $im(\alpha^2)$. More generally, we can define $im(\alpha^n)$ to be the submonoid of $im(\alpha^{n-1})$ obtained by applying α to every element of $im(\alpha^{n-1})$, where n is any positive integer greater than 2.

EXERCISES ON TEXT VI:

I. *Match the words in the first column with the words in the second column having the same meaning.*

1. *satisfying*	______	*that is to say*
2. *conditions*	______	*fulfilling*
3. *several*	______	*exactly*
4. *varying*	______	*widely*
5. *extensively*	______	*requirements*
6. *i. e.*	______	*many*
7. *precisely*	______	*get*
8. *moreover*	______	*different*
9. *since*	______	*because*
10. *therefore*	______	*furthermore*
11. *obtain*	______	*thus*

II. *To what do the following words in the text refer?*

A. *"which" (line 6)*

B. *"it" (line 8)*

C. "do" (line 14)

D. "them" (line 23)

E. "it" (line 50)

III. Parallel structures and enumeration.

A writer often presents and develops a group of ideas together as a unit. The definition of one term may be necessary to understand and define another term. On the other hand, ideals may be contrasted with their negatives or with related ideas.

A writer may wish to show how ideas are related by using sentences of parallel structure. Look at lines 10-15 of the text:

lines	key words
10-11	The set ... together with the operation of ... forms a semigroup.
12-13	The set ... together with the operation of ... does not form a semigroup.
13-14	The set ... together with the operation of ... forms a semigroup.
14-15	So does the set ... together with the operation of ... .

By repeating key words, the writer shows the similarity among the four ideas. In the last sentence, the words "So does" show that the data in this sentence behaves like the data in the previous sentences. There is, however, one sentence of contrast: the data in lines 12-13 does not behave in the same way.

Parallel structures appear again in lines 24-27. Give line numbers and key words.

lines	key words

Here, in addition to parallel structures, the word "another" marks the fact that this example is the second in a series.

Enumeration is the numbering of a series of items. Often writers of mathematics mark a series very clearly. Look at lines 28-32 in the text:

lines	*key words*
28-30	*A ... is said to be ... if and only if the following two conditions are satisfied:*
31	*(1)*
32	*(2)*

In the first sentence, the writer states that two conditions must be satisfied. Then he enumerates the two conditions, using the markers (1) and (2) before each one.

IV. Changing the subject without markers.

Markers of enumeration such as "one", "another", "first", "second", "third", etc. can be very helpful in finding each of a series of important points and examples. Sometimes, however, writers do not give clear markers; the reader must infer them from the context. Let us look at lines 1-7 in the text:

line	*key words*
1-2	*If ... then ... is called a binary operation ... .*
2-4	*A ... with at least one binary operation ... is called an algebraic structure.*
4-6	*There are several types of algebraic structures of varying degrees of complexity ... .*
7	*Let us look at a very simple type of algebraic structure.*

First the writer defines a binary operation because he needs to use this term in the definition of an algebraic structure. The third sentence contains a general statement about algebraic structures. In the fourth sentence the writer expresses his intention of beginning his disucssion with a simple type of algebraic structure.

Look at lines 22-24: "Since monoids are more 'complicated' algebraic structures ..." . Here again we have the markers of enumeration. Among the algebraic structures there are two types: "simple" and "more complicated". These markers are not as clear as words such as "first" and "second"; they have to be recognized by the reader from the context.

Sometimes the writer may change the topic of discussion without a marker. Consider line 36: "Composition of functions is a binary operation on End(S)." From here on, the writer discusses End(S) and not S. Finally, from line 49 the author changes the subject again to a discussion of the image of an endomorphism.

V. *Language of proof,*

A mathematical proof consists of four elements: definitions of the relevant terms, a statement of the hypotheses assumed and of the assertion to be proven, the writer's argument or reasoning, and the conclusion in which the writer repeats what has been proven. Often, however, som of these elements may be missing if the author feels that they are obvious to the reader. Let us look at lines 16-21 of the text:

lines	*key words*	*sentence function*
16-18	*A ... is called ... if and only if ...*	*definition*
18	*Such a ... is called an ...*	*definition*
19	*Any monoid has at most one identity*	*statement to be proven*
19-20	*Indeed, if ..., then ...*	*writer's argument*
20-21	*Therefore, any monoid has precisely one identity*	*another statement to be proven*

No argument is given for the last statement, since the writer assumes that the reader sees that it is an obvious conclusion drawn from the definitions on line 16-18 (which state that any monoid has at least one identity) and the statement on line 19 (which states that any monoid has at most one identity).

Another proof appears in lines 36-44. Fill in the following table by using any of *the following possible sentence function(s): introduction, expansion, writer's argument, definition, satement to be proven, conclusion.*

lines	*key words*	*sentence function*
36	*Composition ... is a ...*	
36-38	*That is to say, if ... then ...*	
38-39	*To see this, note that for all ...*	
40	*Moreover, ...*	
41-42	*In fact, ..., since ...*	
42-43	*For any ... we have ...* and so we see that . . .	
43-44	*Therefore End(S) is in fact a monoid*	

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VI. Comprehension

A. Explain why the set $\mathbb{Z}$ *together with the operation of subtraction does not form a semigroup. (line 12)*

B. Explain why the set of positive integers together with the operation of addition is a semigroup which is not a monoid. (lines 24-26)

TEXT VII: MATROIDS

It is often the case in mathematics that a concept can be defined in several equivalent ways, each of which presents new insights into the concept or suggests new applications of it. We illustrate this by considering the notion of a matroid.

A matroid $M = (E,S)$ consists of a nonempty finite set E, together with a nonempty collection S of subsets of E (called independent sets), which satisfies the following conditions:

(1.1) Any subset of an element of S belongs to S.

(1.2) If I and J are elements of S with J having more elements than I, then there exists an element x of J not in I with the property that $I \cup \{x\}$ is an element of S.

For example, let E be a set of vectors which span a finitely-generated vector space over a field F, and let S be the set of all linearly-independent subsets of E.

It is easy to show that any independent set in M is contained in a maximal independent set, called a base, and that any two bases have the same number of elements. This suggests another definition of a matroid, which can be shown to be equivalent to the first: a matroid $M = (E,B)$ consists of a nonempty finite set E, together with a nonempty collection B of subsets of E (called bases), which satisfies the following conditions:

(2.1) No element of B properly contains any other element of B.

(2.2) If I and J are elements of B and if $x \in I$, then there exists an element y of J such that $(I \setminus \{x\}) \cup \{y\}$ belongs to B.

Let us call the number of elements in a base the <u>rank</u> of the matroid. (This is well-defined since all bases have the same number of elements.) Using the notion of rank, we can give yet another definition of a matroid: a matroid $M = (E,\rho)$ consists of a nonempty finite set E, together with an integer-valued function ρ (called its <u>rank function</u>), which is defined on the set of all subsets of E and which satisfies the following conditions:

(3.1) For any subset A of E, $\rho(A)$ is a nonnegative integer no greater than the number of elements in A.

(3.2) If $A \subseteq B \subseteq E$ then $\rho(A) \leq \rho(B)$.

(3.3) If A and B are subsets of E then $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$.

If A is a subset of E then the <u>closure</u> of A in E consists of the set of all those elements x of E satisfying the condition that $\rho(A \cup \{x\}) = \rho(A)$. Clearly A is contained in its closure. This suggests another definition of matroids, which can be shown to be equivalent to all of the previous ones: a <u>matroid</u> $M = (E,c)$ consists of a non-empty finite set E, together with a function c from the family of all subsets of E to itself, which satisfies the following conditions:

(4.1) For any subset A of E, $A \subseteq c(A) = c(c(A))$.

(4.2) If $A \subseteq B \subseteq E$ then $c(A) \subseteq c(B)$.

(4.3) If A is a subset of E and if x, y are elements of E satisfying $x \in c(A \cup \{y\})$ but $x \notin c(A)$, then $y \in c(A \cup \{x\})$.

Let us conclude by considering some additional examples of matroids:

(A) If E is a set having at least k elements, then we can define a matroid structure on E by taking as independent subsets of E all those subsets having precisely k elements.

(B) If E is the set of edges of a finite graph Γ then we define a matroid structure on E as follows: if A is a subset of E then define ρ(A) to be the number of vertices in the subgraph of Γ determined by A minus the number of connected components in that subgraph.

(C) Let E be any finite set of real numbers. Then we can define a matroid structure on E by taking as independent subsets of E all sets of elements of E which are roots of some polynomial with rational coefficients.

EXERCISES ON TEXT VII:

I. Match the words in the first column with the words in the second column having the same meaning.

1.	*concept*	______	*use*
2.	*insight*	______	*understanding*
3.	*illustrate*	______	*maximal independent set*
4.	*considering*	______	*notion*
5.	*span*	______	*examining*
6.	*previous*	______	*member*
7.	*precisely*	______	*show*
8.	*base*	______	*exactly*
9.	*element*	______	*before*
10.	*application*	______	*use*

II. To which words in the text do the following words refer?

A. "each of which" (line 2)

B. "it" (line 3)

C. "this" (line 3)

D. "This" (line 24)

E. "ones" (line 37)

III. Comprehension

A. The text gives four equivalent definitions of the matroid structure on a nonempty finite set E. Using these definitions, fill in the following table:

definition	notation	defining structures on E	special term for it
1	$M = (E,S)$	nonempty collection S of subsets of E	independent sets
2			
3			
4			

B. For each of the examples A, B, and C, which definition is suitable?

example	definition
A	
B	
C	

IV. Introductions and transition passages.

In an introduction the writer explains the purpose or importance of the information which will follow. The first paragraph of the text (lines 1-14) contains two such sentences. The first (lines 1-3) presents a general statement, and the second (lines 3-4) explains that the rest of the text is an illustration of this general statement. The introduction gives the text a frame without which it would not be meaningful.

A transition passage appears between ideas; it concludes one and/or introduces the next. One transition passage appears in lines 15-17. The sentence shows how one definition of a matroid leads to the next definition, which is introduced in the following sentence.

Find two other examples of transition passages in the text and give key words.

lines	*key words*

V. *Parallel constructions.*

(*In Exercise III on Text VI we talked about parallel constructions.*) *Look at the last paragraph of the text. It contains three examples, each of which begins with a given assumption and follows with a definition. The key words, however, vary in each example. Fill in the following table:*

example	*given/key words*	*definition/key words*
A	*If E is a set having ...*	*then we can define a matroid structure on E by ...*
B		
C		

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TEXT VIII: GRAPHS OF FUNCTIONS

If f is a real-valued function of a real variable, then it is often useful to represent f pictorially by considering the subset $\{(x,y) \mid y = f(x)\}$ of the plane. This subset of the plane is called the <u>graph</u> of the real-valued function f.

While the graph of any real-valued function of a real variable is well-defined in theory, there are some functions whose graphs are impossible, or essentially impossible, to draw in practice. For example, one cannot draw an accurate representation of the graph of the function f defined by:

$$f(x) = \begin{cases} x^2 & \text{when } x \text{ is a rational number} \\ \sin(x) & \text{when } x \text{ is an irrational number} \end{cases}$$

Even if the function is differentiable, its graph may not be accurately drawable. For example, consider the function

$$g(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

which is everywhere differentiable but whose graph cannot be drawn accurately in the vicinity of the origin.

Nonetheless, if the function is sufficiently "nice", its graph can be drawn and provides valuable information about the behavior of the function. For example, the graph of a constant function $f(x) = c$ is a line parallel to the X-axis. If $f(x)$ is an increasing function (namely, if $f(x_1) > f(x_2)$ when $x_1 > x_2$), then the graph of $f(x)$ reflects this by rising as one goes to the right and falling as one goes to the left. The opposite is true for the graph of a decreasing function. The graph of the function $h(x)$ defined by

$$h(x) = n \quad \text{if } n \text{ is an integer and } n-1 < x \leq n$$

looks like a series of steps, and so functions such as this are called <u>step functions</u>.

Not every curve in the plane is the graph of a function. Indeed, from the very definition of a graph, we see that for any real number x there can be only one real number y such that (x,y) belongs to the graph. Pictorially, this means that every line in the plane parallel to the Y-axis intersects the graph at only one point. Thus, for example, a circle cannot be the graph of a real-valued function of a real variable. This condition is also sufficient. That is to say, if Γ is a curve in the plane having the property that every line in the plane parallel to the Y-axis intersects Γ once and only once, then Γ is the graph of some real-valued function of a real variable.

It is perfectly possible that a line in the plane parallel to the X-axis may intersect the graph of a real-valued function of a real variable several times. Indeed, we have already noted that if $f(x)$ is a constant function then its graph is itself such a line. However, a necessary and sufficient condition for a function f to be invertible is that this cannot happen; i. e., f is invertible if and only if lines parallel to the X-axis intersect the graph of f at most once. If f and g are real-valued functions of a real variable, then these functions are inverses of each other if and only if the graph of f is the reflection of the graph of g across the diagonal x = y. (This condition means that a point (a,b) of the plane is in the graph of f if and only if the point (b,a) is in the graph of g.)

In short, one can tell, by merely looking at the graph of a function, whether the function is increasing, decreasing, or neither. One can also tell whether the function is even (i.e., $f(x) = f(-x)$ for all x) or odd (i.e., $f(x) = -f(-x)$ for all x). One can also see if the function has local minima or maxima and whether the function rises or falls at a faster or slower rate. Thus the graphical representation of functions is a compact way of presenting information about them in a form from which human observers can easily extract relevant data.

EXERCISES ON TEXT VIII:

I. Match the words in the first column with the words in the second column having the same meaning.

1.	*consider*	______	*show*
2.	*while*	______	*basically*
3.	*essentially*	______	*give*
4.	*representation*	______	*increase*
5.	*accurate*	______	*enough*
6.	*in the vicinity of*	______	*near*
7.	*sufficient*	______	*drawing*
8.	*provide*	______	*think about*
9.	*reflection*	______	*image*
0.	*nonetheless*	______	*although*
11.	*rise*	______	*exact*
12.	*fall*	______	*however*
13.	*reflect*	______	*obtain*
14.	*extract*	______	*decrease*
15.	*information*	______	*data*

II. *A word may take on different forms, depending on its part of speech. Fill in the table below, giving the adjective, verb, and noun forms of some words. The first line has been done for you.*

adjective	*verb*	*noun*
pictorial	*picture*	*picture*
		variable
		practice
essential	-----	
		representation
drawable		
differentiable		
		inverse

III. *Intensifiers and qualifiers.*

Some words emphasize or strengthen (intensify) the meaning of a sentence. These words include "itself", "very", "indeed", and others. Other words weaken or qualify the meaning of a sentence; these may include "sometimes", "may", and "would", among others.

For each of the following phrases, put a plus (+) if the underlined word(s) strengthen(s) the meaning and a minus (-) if the underlined word(s) qualify(-ies) the meaning. The first two have been done for you.

A. (lines 1-2) it is <u>often</u> useful —

B. (line 15) <u>everywhere</u> differentiable +

C. (lines 6-7) <u>essentially</u> impossible

D. (line 29) the <u>very</u> definition

E. (line 30) <u>only</u> one

F. (line 38) it is <u>perfectly</u> possible

G. (line 39) may intersect

H. (line 40) Indeed

I. (line 41) graph is itself

J. (line 43) if and only if

K. (line 44) at most once

L. (line 49) merely looking

IV. To what words in the text do the following words refer?

A. (line 11) "its"

B. (line 21) "this"

C. (line 31) "this"

D. (line 41) "itself"

E. (line 42) "this"

V. Statements of contrast.

Statements of contrast are marked by words such as "but", "although", "while", "nevertheless". Consider the following sentence:

(lines 5-7) "While the graph of any real-valued function of a real variable is well-defined in theory, there are some functions whose graphs are impossible, or essentially impossible, to draw in practice."

This sentence contrasts the possibility of drawing the graph of any real-valued function of a real variable in theory, on one hand, with the impossibility, in practice, of drawing certain graphs, on the other hand. One word marking the contrast in this sentence is "While".

In the following sentences, underline the words marking contrast.

A. (lines 17-18) "Nonetheless, if the function is sufficiently "nice", its graph can be drawn ... ."

B. (line 28) "Not every curve in the plane is the graph of a function."

C. (lines 41-44) "*However, a necessary and sufficient condition for a function f to be invertible is that this cannot happen; i.e., f is invertible if and only if lines parallel to the X-axis intersect the graph of f at most once.*"

VI. Logical arguments.

A logical argument is usually composed of three elements:
A. Beginning or introduction: definition, general statement;
B. Middle or development: expansion, example(s), comparison/contrast;
C. End: result, conclusion, summary.
A mathematical text, even one without formal proofs, is usually composed as a series of logical arguments where one idea is stated, developed, and followed by another idea. The new idea may either agree with (i.e., add to) or disagree with (i.e., contrast with) the previous idea.

In Text VIII, lines 1-4 contain the introduction to the entire text. Lines 5-48 contain four main ideas, each one of which is developed, and lines 49-56 contain the conclusion of the text.

Let us analyze lines 5-48 further.

A. The impossibility of drawing certain graphs.

lines	key words	sentence function
5-7	While the graph of any real-valued function is well-defined in theory, there are some functions whose graphs are impossible to draw in practice.	contrast
7-10	For example, one cannot draw	example #1
11-12	Even if ..., its graph may not be accurately drawable.	another case
12-14	For example, consider	example #2

B. Functions whose graphs can be drawn.

The next idea, appearing in lines 17-27, contrasts with the previous idea but agrees with the introduction. This paragraph contains four examples.

Fill in the table below:

lines	key words	sentence function
17-18	Nonetheless, if ..., its graph can be drawn ...	contrast
19-20	For example,	example #1
		example #2
		example #3
		example #4, definition

C. Not all curves are graphs of functions.

The next idea, appearing in lines 28-37, is a qualification explaining that some curves may be drawn that are not functions.

1. How many examples are given? Name it (them).

2. What is it (are they) an example (s) of?

3. What are the key words that help you to find the answers?

D. The graphs of invertible functions.

Lines 38-48 return to the idea that it is possible to draw graphs of functions. The sentence in lines 34-37, in the previous paragraph, discusses a line parallel to the Y-axis. The first sentence in this paragraph (lines 38-40), by contrast, discusses a line parallel to the X-axis.

1. (line 40) "Indeed, we have already noted ..." In which lines has the writer noted this idea previously in the text?

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2. *The idea about invertible functions is developed in lines 41-48. Fill in the following table, giving key words and sentence functions.*

lines	*key words*	*sentence function*
41-44		
44-47		
47-48		

3. *The conclusion of the text (lines 49-56), begins with three statements of parallel form. Given the sentence function of each of them.*

lines	*key words*	*sentence function*
49-50	*In short, one can tell ... whether the function is ..., or ... .*	
50-52	*One can also tell whether the function is ... or ... .*	
52-54	*One can also see if the function has ... or ... and where the function ...*	

4. *The main theme of the text is stated four times. Find these statements and give the line numbers.*

1.

2.

3.

4.

VII. Comprehension

A. Why is it not possible to draw a graph of these functions:

1. The function f, defined in lines 9-10?

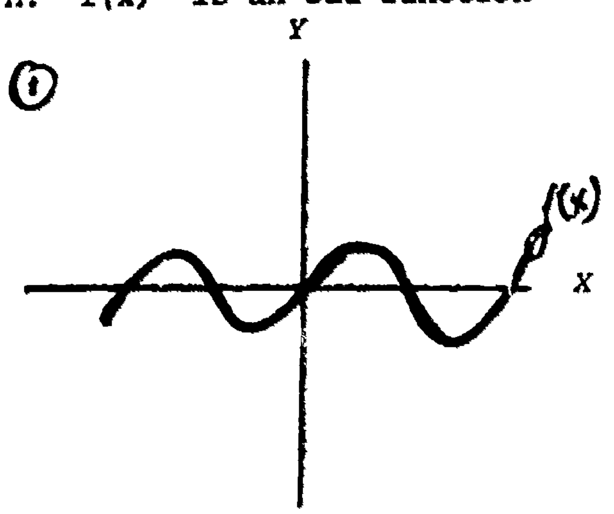
2. The function g, defined in lines 13-14?

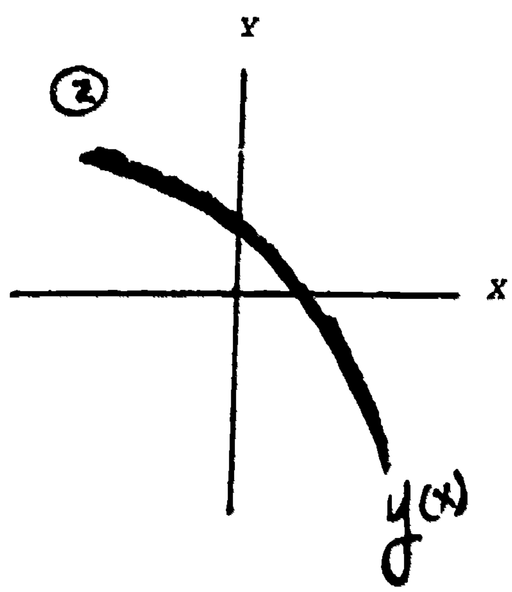
B. *According to lines 17-18, under what conditions is it possible to draw the graph of a function?*

C. *Why is it not possible for a circle to be the graph of a real-valued function of a real variable?*

D. *Match these descriptions with the graphs that follow them.*

a. $f(x) = c$ is a line parallel to the X-axis; f is a constant function
b. $f(x)$ is an increasing function; $f(x)$ increases as x increases and decreases as x decreases
c. $f(x)$ is a decreasing function; $f(x)$ decreases as x increases
d. $f(x)$ is a step function
e. the graph of f is a curve such that every line in the plane parallel to the Y-axis intersects the graph of f only once
f. the graph of f is the reflection of the graph of g across the diagonal $x = y$
g. $f(x)$ is an even function
h. $f(x)$ is an odd function





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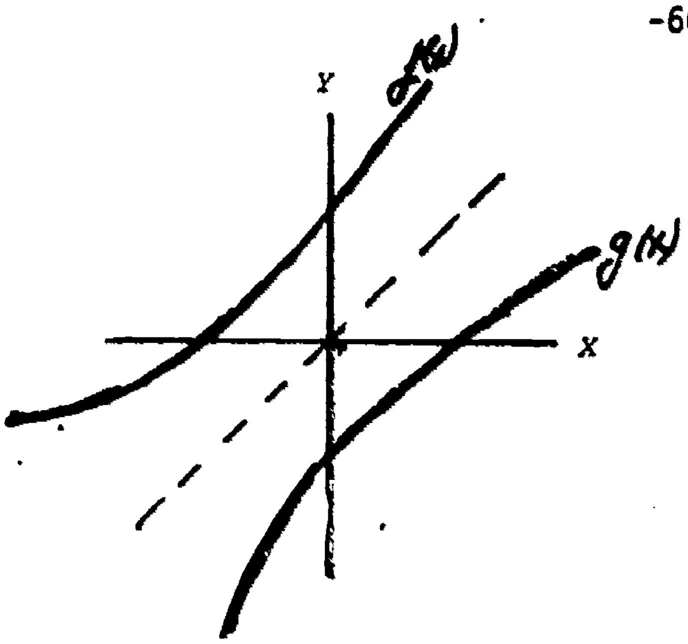

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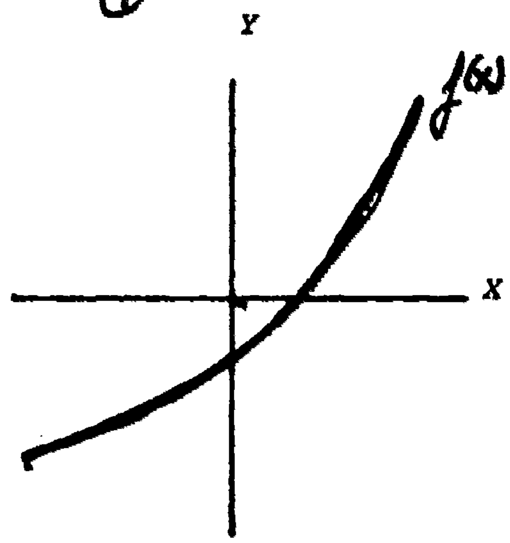

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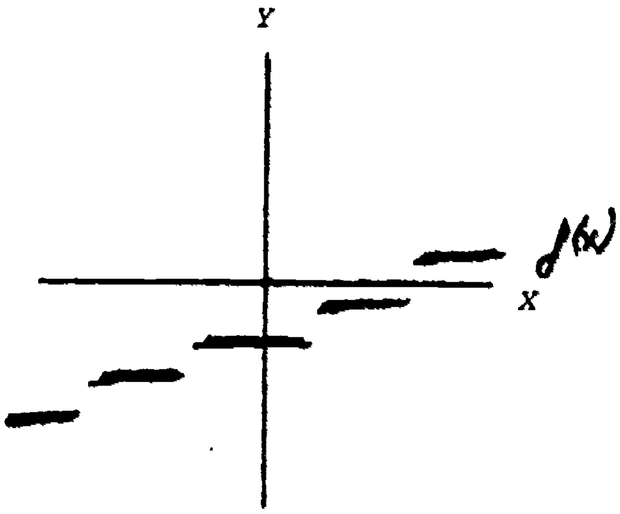

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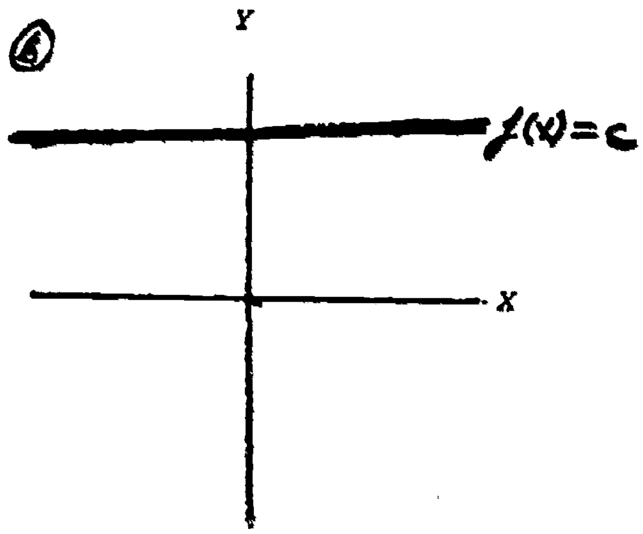

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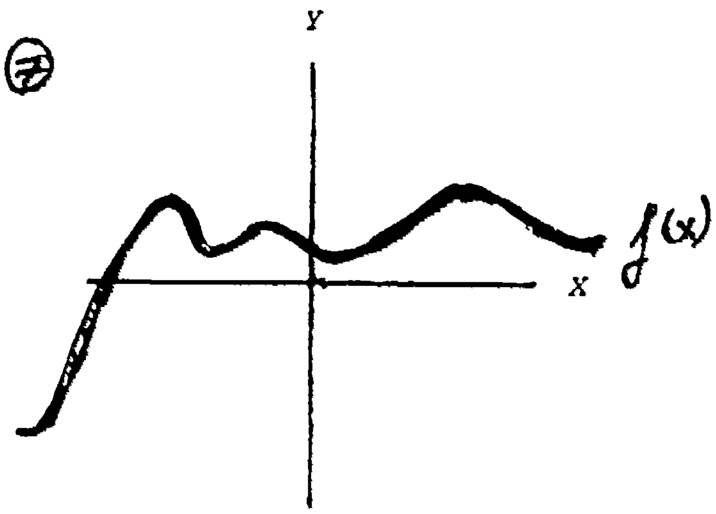

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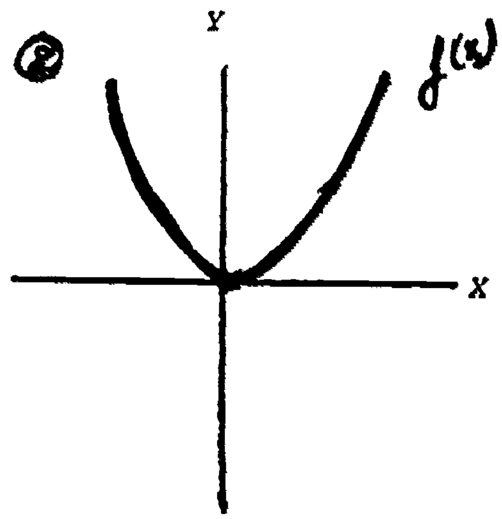

TEXT IX: BINOMIAL COEFFICIENTS

If n is a positive integer, we define the number $n!$ (read as "n factorial") to be the product of all of the positive integers from 1 through n. By convention, we also set $0!$ equal to 1. If $k \leq n$ are nonnegative integers, we define the number $\binom{n}{k}$ to equal $\frac{n!}{k!(n-k)!}$. The first theorem will show that this number is always a positive integer and, indeed, it will provide a recursive method of computing it.

THEOREM 1. <u>If</u> n <u>is a positive integer then</u>

(1) $\binom{0}{0} = \binom{n}{0} = \binom{n}{n} = 1.$

(2) <u>For</u> $1 \leq k < n$, <u>we have</u> $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

PROOF: Part (1) of the theorem follows directly from the definition of $\binom{n}{k}$. To prove part (2), we note that

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left[\frac{n}{k(n-k)}\right] \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!}\left[\frac{1}{n-k} + \frac{1}{k}\right] \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \binom{n-1}{k-1} + \binom{n-1}{k}. \qquad \square
\end{aligned}$$

The following theorem is known as the <u>Binomial Theorem</u>. Because of it, the integers $\binom{n}{k}$ are known as the <u>binomial coefficients</u>.

THEOREM 2. <u>If</u> n <u>is any positive integer then</u>

$$(x + y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

PROOF: We will prove this theorem by induction on n. For $n = 1$, we have $(x + y) = \binom{1}{0}x + \binom{1}{1}y$, as desired. Therefore assume that $n > 1$ and that we have already established the formula

$$(x + y)^{n-1} = \Sigma_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k}y^k.$$

We can then compute $(x + y)^n$ as follows:

$$\begin{aligned}(x + y)^n &= (x + y)(x + y)^{n-1} \\ &= x(x + y)^{n-1} + y(x + y)^{n-1} \\ &= x[\Sigma_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k}y^k] + y[\Sigma_{k=0}^{n-1} \binom{n-1}{k} x^{n-1-k}y^k] \\ &= \binom{n-1}{0}x^n + \Sigma_{k=1}^{n-1}[\binom{n-1}{k-1} + \binom{n-1}{k}]x^{n-k}y^k + \binom{n-1}{n-1}y^n.\end{aligned}$$

By Theorem 1 we have $\binom{n-1}{0} = \binom{n}{0} = \binom{n-1}{n-1} = \binom{n}{n} = 1$ and $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ for all $0 < k < n$, which proves that

$$(x + y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^{n-k}y^k. \quad \square$$

The binomial coefficients $\binom{n}{k}$ have another, very important, interpretation in combinatorics, as we see from the following theorem.

THEOREM 3. *If* $0 \leq k \leq n$ *are nonnegative integers then* $\binom{n}{k}$ *is precisely the number of* k-*element subsets of a set containing* n *elements*.

PROOF: We proceed by induction on n, the case $n = 1$ being obvious. Let A be a set of n elements, where n is assumed now to be greater than 1. Select an element a_0 from A, and let $B = A \setminus \{a_0\}$. Any k-element subset of A either contains a_0 or it does not. Subsets of the first type are precisely those obtained by taking a $(k-1)$-element subset of B and adding a_0 to it. By the induction hypothesis, there are $\binom{n-1}{k-1}$ of these. Subsets of the second type are precisely those obtained by taking k-element subsets of B. By the induction hypothesis, there are $\binom{n-1}{k}$ of these. Therefore the total number of k-element subsets of A is $\binom{n-1}{k-1} + \binom{n-1}{k}$. By Theorem 1, this equals $\binom{n}{k}$. $\square$

EXERCISES ON TEXT IX:

I. A word may take on different forms, depending on its part of speech. Fill in the following table. The first line has been done for you.

	verb	*noun*
1.	*define*	*definition*
2.		*product*
3.	*equal*	
4.	*provide*	
5.	*compute*	
6.		*proof*
7.		*induction*
8.	*assume*	
9.		*formula*
10.		*interpretation*
11.	*select*	
12.	*add*	
13.		*hypothesis*
14.		*combinatorics*

II. *To what words in the text do the following words refer?*

A. (line 5) "this number"

B. (line 5) "it"

C. (line 6) "it"

D. (line 17) "it"

E. (line 40) "it"

F. (line 40) "the first type"

G. (line 41) "it"

H. (line 42) "these"

I. (line 42) "the second type"

J. (line 44) "these"

K. (line 45) "this"

III. Markers of emphasis

Some words are used to emphasize a point. For each word below, give the idea the writer is emphasizing. The first one has been done for you.

A. (line 5) "indeed" the first theorem will show how to compute n recursively

B. (line 22) "as desired"

C. (line 23) "already"

D. (line 33) "very important"

E. (line 38) "now"

F. (line 40) "precisely"

IV. The language of theorem and proof.

The text contains three theorems and their proofs. We will examine each theorem and proof separately, looking for the following sentence functions:

given data, definition, denotation, computation, logical argument, conclusion, writer's statement of intention, writer's comment.

A. Let us examine lines 1-16, which contain Theorem 1 and its proof, as well as the remarks preceding it.

line(s)	*key words*	*sentence function*
1-2	*If n is a positive integer we define ... from 1 through n.*	*definition, denotation*
3	*By convention, we also set 0! equal to 1.*	*definition, denotation*
4	*If ... we define ...*	*definition, denotation*
4-6	*The first theorem will show ...*	*writer's intent*
7-9	*If n is a positive integer then ...*	*statement of given data and desired conclusion*
10-16	*PROOF: ... □*	*logical argument and computation*

B. For lines 17-32, which contain Theorem 2 and its proof, as well as the remarks preceding it, fill in the table below

line(s)	*key words*	*sentence function*
17		
17-18	*... are known as ...*	
19-20		
21	*We will prove ... by ...*	
21-22	*For n = 1, we have ...*	
22-24		*given data*
25-29		*writer's intent, computation*
30-32	*By ... we have ... and ... for all ... which proves ... .*	*computation, conclusion*

C. *Fill in the following table for lines 33-45.*

line(s)	key words	sentence function
33-34	The ... have another, very important, interpretation ..., as we see ...	
35-36		statement of given data and desired conclusion
37		writer's intent
38		given data
39		
39-40	Any ... either ... or it does not	logical argument
40-41		
41-42		
42=43		
43-44		
44-45		conclusion
45		

V. *Complex sentences.*

A. *A complex sentence has at least two clauses: one independent clause and one dependent clause. An independent clause makes sense as a complete thought; it can be a separate statement. The following is an independent clause:*

(lines 1-2) " ... we define the number $n!$ ... to be the product of all positive integers from 1 through n.

The verb in it is "define", and its subject is "we". The next statement is also an independent clause:

(line 1) "... n is a positive integer"

Here the verb is "is" and the subject is "n".

By adding the connecting word "If" before the clause, the above statement becomes a dependent clause:

(line 1) "If n is a positive integer, ... "

This clause depends on another, independent, clause to follow it in order to finish the statement and complete the thought. Thus, we have:

(lines 1-2) "If n is a positive integer, we define ... to be the product ..."

clause	*connecting word*	*subject*	*verb*
dependent	*If*	*n*	*is*
independent	-----	*we*	*define*

The dependent clause is not complete without the independent clause. Together, they form a complex sentence.

B. There are other connecting words which introduce dependent clauses Some of these are: "that", "as", "where", "when", "since", and "although". In the following complex sentences, underline the connecting word at the beginning of the dependent clause.

1. *(lines 3-4) "If $k \leq n$ are nonnegative integers, we define ... to equal ..."*

2. *(lines 33-34) "The binomial coefficients $\binom{n}{k}$ have another ... interpretation, as we see from the following theorem."*

3. *(line 38) "Let A be ..., where n is assumed not to be greater than 1."*

C. The complex sentence should not be confused with the compound sentence, where each clause is independent. Connecting words which permit the clause to remain independent are "and", "or", "but". In the following compound sentences, underline the connecting word at the beginning of the independent clause.

1. *(line 39) "Select an element a_0 from A and let $B = A \setminus \{a_0\}$."*

2. *(lines 39-40) "Any k-element subset of A either contains a_0 or it does not."*

D. Note that the following sentence is made up of three clauses:

(lines 4-6) "The first theorem will show <u>that</u> this number is always ... <u>and</u> ... it will provide ... ."

The first half of the sentence contains two clauses, an independent and a dependent clause (beginning with the connecting word "that"). The third clause begins with the word "and" which signals the beginning of an independent clause. This sentence, containing a dependent and two independent clauses, is called a compound-complex sentence.

clause	*connecting word*	*subject*	*verb*
independent	-----	*The first theorem*	*will show*
dependent	*that*	*this number*	*is*
independent	*and*	*it*	*will provide*

VI. Comprehension

A. How does the answer in line 16 prove part (2) of Theorem 1?

B. In line 37, the writer says that the case $n = 1$ is obvious. Why?

TEXT X: CROSS PRODUCTS OF VECTORS

In general, it is not possible to define an algebraically-interesting product of vectors in a vector space of arbitrary dimension over the field $\mathbb{R}$ of real numbers. On the line $\mathbb{R}^1$ we have, of course, the ordinary product of real numbers; in the place $\mathbb{R}^2$ we have the product obtained from the usual identification of $\mathbb{R}^2$ with the field of complex numbers. In the space $\mathbb{R}^3$ one can define a product, called the *cross product*, which is far less interesting from the algebraic point of view but which is of use in advanced calculus.

Let $u = (a_1,a_2,a_3)$ and $v = (b_1,b_2,b_3)$ be vectors in $\mathbb{R}^3$. We define the *cross product* of u and v, denoted by $u \times v$, to be the vector $(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)$. Thus, for example, we note that

$$(1,0,0) \times (0,1,0) = (0,0,1),$$
$$(0,1,0) \times (0,0,1) = (1,0,0),$$
$$(0,0,1) \times (1,0,0) = (0,1,0).$$

By straightforward computation, we can now prove the following elementary properties of the cross product.

THEOREM 1: *If* $a \in \mathbb{R}$ *and if* $u, v, w \in \mathbb{R}^3$ *then*

(1) $u \times v = -(v \times u)$;

(2) $(u + w) \times v = (u \times v) + (w \times v)$;

(3) $a(u \times v) = (au) \times v = u \times (av)$;

(4) $(u \times v) \times w + (v \times w) \times u + (w \times u) \times v = 0$.

Note that if $v \in \mathbb{R}^3$ then by (1) we have $v \times v = -(v \times v)$. Since the

0-vector is the only vector equal to its own negative, we immediately obtain the following corollary to Theorem 1.

COROLLARY: *If* $v \in \mathbb{R}^3$ *then* $v \times v = (0,0,0)$.

THEOREM 2: *If* $u, v \in \mathbb{R}^3$ *are nonzero vectors then* $u \times v = (0,0,0)$ *if and only if there exists a nonzero scalar* $a \in R$ *satisfying* $u = av$.

PROOF: If $u = av$, then $u \times v = (0,0,0)$ by Theorem 1(3) and by the Corollary to Theorem 1. Conversely, assume that $u \times v = (0,0,0)$, where $u = (a_1,a_2,a_3)$ and $v = (b_1,b_2,b_3)$. Then

(*) $\quad a_2b_3 - a_3b_2 = a_3b_1 - a_1b_3 = a_1b_2 - a_2b_1 = 0.$

Since $v \neq (0,0,0)$ then one of the b_i is nonzero. Say $b_1 \neq 0$. By (*) we see that

(**) $\quad a_1 = a_1b_1/b_1; \quad a_2 = a_1b_2/b_1; \quad a_3 = a_1b_3/b_1$

so $u = (a_1/b_1)v$. A similar result is obtained if we assume $b_2 \neq 0$ or $b_3 \neq 0$. $\square$

The cross product has a clear geometric interpretation. Indeed, $u \times v$ is the vector in $\mathbb{R}^3$ which is perpendicular to both u and v and the length of which equals $|u||v|\sin\theta$, where θ is the angle between the vectors u and v in the plane generated by them. Note that the length of $u \times v$ is precisely equal to the area of the parallelogram with sides u and v. The direction of the vector $u \times v$ is chosen according to the "right-hand rule".

Using this geometric interpretation we see that Theorem 2 can be rephrased as follows:

COROLLARY: If u and v are nonzero vectors in $\mathbb{R}^3$ then $u \times v = (0,0,0)$ if and only if u and v are parallel.

Compare this result with the theorem stating that if u and v are nonzero vectors in $\mathbb{R}^3$ then $u \cdot v = 0$ if and only if u and v are perpendicular.

EXERCISES ON TEXT X:

I. Match the words in the first column with the words in the second column having the same meaning.

1.	*ordinary*	______	*repeated in different words*
2.	*straightforward*	______	*basic*
3.	*elementary*	______	*usual*
4.	*properties*	______	*get*
5.	*immediately*	______	*right away*
6.	*assume*	______	*direct*
7.	*precisely*	______	*exactly*
8.	*generated by*	______	*defined by*
9.	*rephrased*	______	*fulfilling (the condition)*
10.	*satisfying*	______	*characteristics*
11.	*obtain*	______	*suppose*

II. Results

In mathematics, a line of reasoning often ends in a result. Spotting the result can help us understand the writer's chain of thought. Look at lines 8-20 in the text:

lines	*key words*	*sentence function*
8	*Let ... and ... be ... .*	*given data*
8-10	*We define ... .*	*definition*
11-13	*Thus, for example, ...*	*example*
14-15	*By ..., we can now prove the following ... .*	*intent of writer*
16-20	*Theorem 1*	*theorem*

Theorem 1 is <u>a result of</u> the definition and explanations in lines 8-15. (They do not constitute a <u>proof</u> of Theorem 1 of course, but form the chain of thought which leads the writer to propose Theorem 1.)

Sometimes a result and its reasoning appear in the same sentence. Consider the following lines in the text:

(lines 21-23) "Since the 0-vector is the only vector equal to its own negative, we immediately obtain the following corollary to Theorem 1."

Here the reason is "Since the 0-vector is ..." and the result is "we ... obtain ..." . Often the connection between them is very blatant:

(line 24) "If $v \in \mathbb{R}^3$ then $v \times v = (0,0,0)$."

Here the reason is "if $v \in \mathbb{R}^3$" and the result is "then ...".

In the following sentences, give the key words showing results:

lines	*key words*
25-26	
27	
29-30	
31-34	
34-35	

III. Sentence structure and information

When reading a very long, complicated sentence, it is sometimes useful to break it down into smaller clauses in order to understand it more easily. To do this, it is helpful to look for the verb(s), subject(s), and connectors in the sentence.

Look at lines 1-3 of the text. This long sentence has only one (independent) clause: one verb and one subject. This is a simple sentence (see Exercise VI on Text I) with one bit of information: it is not possible to define.

Now look at lines 3-5 in the text:

lines	*clause*	*connector*	*subject*	*verb*
3-4	independent	-----	we	have
4-5	independent	;	we	have

This is a compound sentence (see Exercise VII on Text IV); it has two independent clauses connected by a semi-colon. It gives us two equivale bits of information: one about $\mathbb{R}^1$ *and one about* $\mathbb{R}^2$.

Finally, look at lines 5-7 in the text:

lines	*clause*	*connector*	*subject*	*verb*
5-6	independent	-----	one	can define
6-7	dependent	which	which	is
7	dependent	but which	which	is

This is a complex sentence (see Exercise V on Text IX) with three clauses: one independent and two dependent. The main bit of information is that one can define the cross product in $\mathbb{R}^3$. *The two other bits of information are less important: that this product is less interesting than the products in* $\mathbb{R}^1$ *and* $\mathbb{R}^2$, *but that it is useful nevertheless.*

Analyse the following sentences in the same way as above, and give the main idea of each sentence.

A.

lines	*clause*	*connector*	*subject*	*verb*
21-23 ("Since...")				

main idea:

B.

lines	*clause*	*connector*	*subject*	*verb*
36-39 ("Indeed...")				

main idea:

C.	*lines*	*clause*	*connector*	*subject*	*verb*
	46-47				

main idea:

IV. *Comprehension*

A. "Say $b_1 \neq 0$." (line 31) This sentence means the same as:

a. b_1 is always unequal to 0.

b. let us decide that the value of i for which $b_i \neq 0$ is 1.

B. "A similar result is obtained if we assume that $b_2 \neq 0$ or $b_3 \neq 0$." (lines 34-35) This sentence means the same as:

a. It doesn't matter whether $b_1 \neq 0$, $b_2 \neq 0$, or $b_3 \neq 0$, we always end up with $u = (a_i/b_i)v$ for some i.

b. the result will always be the result in line 33.

TEXT XI: GENERALIZED INVERSES OF LINEAR TRANSFORMATIONS

Let V and W be finite-dimensional vector spaces over the same field F, and let α be a linear transformation from V to W which assigns to each vector v in V the vector $v\alpha$ in W. If V and W have the same dimension, and if α is nonsingular (i.e. if $v\alpha \neq 0$ whenever $v \neq 0$), then α has an *inverse* β, which is a linear transformation from W to V having the property that $\alpha\beta$ and $\beta\alpha$ equal the identity transformations on V and on W, respectively, In particular, β satisfies the equality

(1) $\alpha\beta\alpha = \alpha.$

If the dimension of V does not equal the dimension of W, or if these dimensions are equal but α is singular, then α does not have an inverse; nonetheless, there may still exist a linear transformation β from W to V satisfying equality (1). Such a transformation is called a *generalized inverse* of α. Generalized inverses tend to be very useful tools in linear algebra. Their existence is guaranteed by the following theorem.

THEOREM: *Let* V *and* W *be finite-dimensional vector spaces over the same field* F, *and let* α *be a linear transformation from* V *to* W *with kernel* K *and image* Y. *Let* K' *be a subspace of* V *satisfying* $V = K \oplus K'$, *and let* Y' *be a subspace of* W *satisfying* $W = Y \oplus Y'$. *Then there exists a unique linear transformation* $\beta: W \to V$ *which is a generalized inverse of* α *and which satisfies the following conditions*:

(1) K' *is the image of* β;

(2) Y' *is the kernel of* β.

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PROOF: We first prove the existence of at least one such generalized inverse β. To do this, we begin by noting that the restriction of α to K' is one-to-one. Indeed, if x and x' are elements of K' satisfying $x\alpha = x'\alpha$, then $x-x'$ belongs both to K' and to K, the kernel of α. But $K \cap K' = \{0\}$, and so we must have $x = x'$. Moreover, if y is a vector in Y, the image of α in W, then there exists a vector x in K' satisfying $x\alpha = y$. Indeed, if $v\alpha = y$ for some vector v in V, then we can write $v = k + x$, where k is in K and x is in K'. Then we note that $y = v\alpha = k\alpha + x\alpha = 0 + x\alpha = x\alpha$.

We have thus shown that the restriction of α to K', which we will denote by $\bar{\alpha}$, is an isomorphism between K' and Y, and so has an inverse which we will denote by $\bar{\alpha}^{-1}$. If $\pi: W \rightarrow Y$ is the restriction transformation, we now define the linear transformation $\beta: W \rightarrow V$ to be $\pi\bar{\alpha}^{-1}$. Note that the kernel of β equals Y', the kernel of π, and that the image of β equals K', the image of $\bar{\alpha}^{-1}$. Moreover, if v is a vector in V, and if $v = k + x$ is the unique decomposition of v into the sum of an element of K and an element of K', then

$$v\alpha\beta\alpha = (k + x)\alpha\beta\alpha = k\alpha\beta\alpha + x\alpha\beta\alpha = 0\beta\alpha + x\bar{\alpha}\pi\bar{\alpha}^{-1}\bar{\alpha} = 0 + x\bar{\alpha} = v\alpha.$$

(Here we used the fact that $y\pi = y$ for any vector y in Y.) Therefore $\alpha\beta\alpha = \alpha$, and so β is a generalized inverse of α satisfying conditions (1) and (2).

We now prove that there can be no more than one generalized inverse of α satisfying conditions (1) and (2). Indeed, assume that β_1 and β_2 are two such linear transformations. If w is a vector in W, then we can write $w = y + z$, where y is a vector in Y and z is a vector in Y'. Since Y' equals the

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kernel of β_1 and the kernel of β_2, this implies that $w\beta_1 = y\beta_1$ and $w\beta_2 = y\beta_2$. As we have already seen, there is a unique element x of K' satisfying $x\alpha = y$, and so $y\beta_2\bar{\alpha} = y\beta_2\alpha = x\alpha\beta_2\alpha = x\alpha = x\alpha\beta_1\alpha = y\beta_1\alpha = y\beta_1\bar{\alpha}$. But $\bar{\alpha}$ is one-to-one, and so this implies that $w\beta_1 = y\beta_1 = y\beta_2 = w\beta_2$. This is true for every vector w in W, and so we have $\beta_1 = \beta_2$. □

EXERCISES ON TEXT XI:

I. Match the words in the first column with the symbols in the second column used to denote them in the text.

1.	*finite-dimensional vector space*	______	α
2.	*field*	______	$W \to Y$
3.	*linear transformation*	______	β
4.	*vector in V*	______	v
5.	*inverse of α*	______	k
6.	*composition of β and α*	______	$\beta\alpha$
7.	*function from W to Y*	______	V
8.	*direct sum*	______	$\bar{\alpha}$
9.	*restriction of α to K'*	______	z
10.	*inverse of restriction of α to K'*	______	F
11.	*kernel of π*	______	$\bar{\alpha}^{-1}$
12.	*vector in Y*	______	Y'
13.	*vector in K*	______	$\oplus$

II. To what words in the text do the following words refer?

A. "these dimensions" (lines 9-10)

B. "such a transformation" (line 12)

C. "their existence" (line 14)

D. "one such generalized inverse" (lines 23-24)

E. "do this" (line 24)

F. "two such linear transformations" (lines 44-45)

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G. *"this implies" (line 47)*

H. *"this implies" (line 50)*

I. *"This is true" (line 50)*

III. *Following the argument.*

A. *Ideas in sequence. When explaining a point, the writer often uses key words such as "In particular", "Moreover", and "Indeed" to emphasize that he is continuing a point made in the sentence before. In the table below, list sentences in the text in which the writer continues a point he has begun in the sentence before. The first one has been done for you.*

lines	*first words in the sentence*
7	*In particular, β satisfies the equality*

B. *Conditionals. Conditional statements state that if certain conditions hold then something is true. In the table below, list the sentences in the text containing conditionals. The first one has been done for you.*

lines	*if*	*and if / or if*	*then*	*which/where/satisfying*
3-7	*V and W have the same dimension*	*α is non-singular*	*α has an inverse β*	*is a linear transformation on V and W respectively*

C. *Result*. Some sentences contain a statement and a result, which is often marked by the words "and so." Fill in the following table with such sentences. The first sentence has been done for you.

lines	*statement*	*result*
32-35	We have thus shown that the restriction ... is an isomorphism between Y and Y,	and so has an inverse which we will denote by α^{-1}.

D. *Writer's comments*. Sometimes the writer points out some useful information outside of the formal disucssion or argument. Such comments are often marked by "Note that". Fill in the table below using sentences from the text. The first one has been done for you.

lines	*writer's comments*
13-14	Generalized inverses tend to be very useful tools in linear algebra.

E. *Writer's intent*. Sometimes the writer may explain the procedure by which he intends to organize the material in the proof or parts of the text. Some key [illegible] are "We must first prove", "To do this we begin by", and "We will thus [illegible]". Fill in the table below using sentences from the text. The first one [illegible] been done for you.

lines	*writer's intent*
23-24	*We first prove the existence of at least one such generalized inverse* β.

IV. *Comprehension*

A. *Under which two conditions is it possible that* α *not have an inverse?*

B. *Which phrases in the proof of the theorem show both the existence and the uniqueness of* β*?*

C. *According to the last paragraph, how many generalized inverses of* α *satisfy both conditions (1) and (2)?*

D. *According to the last paragraph, how many elements* x *of* K' *satisfy* $x\alpha = y$*?*

ERIC

V. *Sentence structure review.*

Look at the following table.

Sentence	*clause*	*joined by*	
Simple *Simple*	*1 independent* *1 independent*	--- ---	*A is positive.* *B is negative.*
Compound	*2 independent*	*;*	*A is positive; B is negative.*
Compound	*2 independent*	*;*	*A is positive; nonetheless, B is negative.*
Compound	*2 independent*	*connecting word "and"*	*A is positive, and B is negative.* *A is positive, and so B is negative.*
Complex	*1 dependent,* *1 independent*	*connecting word "if"*	*If A is positive, then B is negative.*
Complex	*1 dependent,* *1 independent*	*connecting word "when"*	*When A is positive, B is negative.*

Answer the following questions. NOte that more than one choice may be correct.

A. *In this text, conditionals have the form of a*
   1. *sirple sentence*
   2. *compound sentence*
   3. *complex sentence*

B. *In this text, result has the form of a*
   1. *simple sentence*
   2. *compound sentence*
   3. *complex sentence*
   4. *compound-complex sentence*

TEXT XII: THE RIEMANN INTEGRAL

Let us consider the set A of all ordered pairs (I,f), where $I = [a,b]$ is a closed interval on the real line, and where f is a continuous function from I to the nonnegative real numbers. Then the Riemann integral can be thought of as a function from A to the nonnegative real numbers which assigns to each pair (I,f) the area of the region in the plane which is bounded by the graph of f and by the lines $X = 0$, $Y = a$, and $Y = b$. We emphasize this outlook by writing $R(I,f)$ instead of $\int_I f(t)\,dt$ to denote the Riemann integral.

Two properties of the function R are evident:

(1) If $I = [a,b]$ is a closed interval on the real line, and if c is an interior point of I, then c divides I into two closed subintervals, $I_1 = [a,c]$ and $I_2 = [c,b]$, the intersection of which is a single point. If f is a continuous function from I to the nonnegative real numbers, then f is continuous on each of I_1 and I_2, and we have

$$R(I,f) = R(I_1,f) + R(I_2,f).$$

This property of R is called additivity.

(2) If $I = [a,b]$ is a closed interval on the real line, and if f is a continuous function from I to the nonnegative real numbers, then there exist points c and d in I such that $f(c) = \min_{t\in I} f(t)$ and $d(d) = \max_{t\in I} f(t)$. Moreover, we see that the region in the plane bounded by the graph of f and the lines $X = 0$, $Y = a$, and $Y = b$ is clearly contained in the rectangle bounded by the lines $X = 0$, $X = f(d)$, $Y = a$, and $Y = b$; and it clearly

contains the rectangle bounded by the lines $X = 0$, $X = f(c)$, $Y = a$, and $Y = b$. Therefore we have

$$(b - a)f(c) \le R(I,f) \le (b - a)f(d).$$

This property of R is called *betweenness*.

What is amazing is that these two properties--additivity and betweenness--together fully characterize the Riemann integral.

THEOREM: *Let* F *be a function which assigns a nonnegative real number* $F(I,f)$ *to each pair* (I,f) *in* A *and which satisfies the following conditions*:

(1) *If* $(I,f) \in A$ *and if* $I = I_1 \cup I_2$, *where* I_1 *and* I_2 *are closed subintervals of* I *which intersect at a single point, then* $F(I,f) = F(I_1,f) + F(I_2,f)$.

(2) *If* $(I,f) \in A$, *where* $I = [a,b]$, *and if* c *and* d *are points in* I *satisfying* $f(c) = \min_{t \in I} f(t)$ *and* $f(d) = \max_{t \in I} f(t)$, *then* $(b - a)f(c) \le F(I,f) \le (b - a)f(d)$.

Then $F(I,f) = \int_I f(t)\,dt$ *for any* $(I,f) \in A$.

PROOF: Suppose that $(I,f) \in A$, where $I = [a,b]$. Let us recall the definition of the Riemann integral $\int_I f(t)\,dt$. For any finite set of points

$$a = a_0 < a_1 < \dots < a_n = b$$

in I, and for each $0 \le i <$, we select points c_i and d_i in $[a_i,a_{i+1}]$ satisfying $f(c_i) = \min_{t \in [a_i,a_{i+1}]} f(t)$ and $f(d_i) = \max_{t \in [a_i,a_{i+1}]} f(t)$. The sum $\Sigma_{i=0}^{n-1}\ (a_{i+1}-a_i)f(c_i)$ is called a *lower sum* of f on I, and the sum $\Sigma_{i=0}^{n-1} (a_{i+1}-a_i)f(d_i)$ is called an upper sum of f on I. The lower integral of f on I is defined to be the least upper bound of all possible lower sums of f on I, and the upper integral of f on I is defined to be the greatest lower bound of all upper sums of f on I. For (I,f) in A, these values are equal to the same number, called the Riemann integral of f on I, and denoted by $\int_I f(t)\,dt$.

Consequently, to prove the theorem, it suffices to show that $F(I,f)$ is greater than or equal to any lower sum of f on I, and that it is less than or equal to any upper sum of f on I. Indeed, let

$$a = a_0 < a_1 < \ldots < a_n = b$$

be a finite set of points on I. For each $0 \leq i < n$, choose points c_i and d_i in $[a_i, a_{i+1}]$ as defined at the beginning of the proof. Then (2) implies that

$$(a_{i+1} - a_i)f(c_i) \leq F([a_i,a_{i+1}],f) \leq (a_{i+1} - a_i)f(d_i)$$

for each $0 \leq i < n$, and thus

$$\Sigma_{i=0}^{n-1} (a_{i+1} - a_i)f(c_i) \leq \Sigma_{i=0}^{n-1} F([a_i,a_{i+1}],f) \leq \Sigma_{i=0}^{n-1} (a_{i+1} - a_i)f(d_i).$$

But (1) implies that $\Sigma_{i=0}^{n-1} F([a_i,a_{i+1}],f) = F(I,f)$, thereby proving the theorem.

EXERCISES ON TEXT XII:

I. Match the words in the first column with the symbols used to denote them in the text from the second column.

1. integral	_____	$\geq$
2. set of ordered pairs	_____	(I,f)
3. ordered pair	_____	f
4. continuous function	_____	c
5. interior point	_____	A
6. interval	_____	$[c,b]$
7. is an element of	_____	$\int$
8. greater than or equal to	_____	$\in$

II. Match the words in the first column with the words in the second column having the same meaning.

1. examine	_____	assumed
2. designate	_____	consider
3. subset of a plane	_____	suffices
4. characteristics	_____	select
5. clear	_____	assign
6. supposed	_____	properties
7. remember	_____	denoted by
8. choose	_____	recall
9. indicated by	_____	evident
10. is enough	_____	region

III. In the text, find one example of each of the following:

sentence line numbers

A. Simple sentence

B. Compound sentence

C. Complex sentence

IV. To what words in the text do the following words refer?

A. "this outlook" (line 7)

B. "it" (line 22)

C. "these two properties" (line 27)

D. "(2)" (line 56)

V. *Below is a list of sentence functions. Match this list to each of the sentences in the text. (Some sentence functions appear more than once, whereas some may not appear at all.)*

Sentence functions | *sentence line numbers*

A. *definition*

B. *given data*

C. *example*

D. *writer's comment*

E. *clarification*

F. *introduction*

G. *explanation*

H. *condition*

I. *result*

J. *writer's intention*

K. *expansion (addition)*

VI. *In the statement of the theorem, decide which sentences belong to the introduction, which to the argument, and which to the conclusion. Underline the key words which helped you decide.*

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VII. In the proof of the theorem, decide which sentences belong to the introduction, which to the argument, and which to the conclusion. Underline the key words which helped you decide.

VIII. Comprehension

A. Why do the expressions $R(I,f)$ and $\int_I f(t)\,dt$ denote the same thing?

How do they emphasize different things?

B. According to the property of additivity, which two values are added to obtain the value of the Riemann integral on a given interval?

C. According to the property of betweenness, between which two values do we find the value of the Riemann integral?

D. How are the points c_i chosen to obtain the lower sum?

E. How are the points d_i chosen to obtain the upper sum?

F. Find the key sentence that shows the writer's procedure in concluding the proof.

APPENDIX: THE GREEK ALPHABET

lower case	upper case	name
α	Α	alpha
β	Β	beta
γ	Γ	gamma
δ	Δ	delta
ε	Ε	epsilon
ζ	Ζ	zeta
η	Η	eta
θ	Θ	theta
ι	Ι	iota
κ	Κ	kappa
λ	Λ	lambda
μ	Μ	mu
ν	Ν	nu
ξ	Ξ	xi
ο	Ο	omicron
π	Π	pi
ρ	Ρ	rho
σ	Σ	sigma
τ	Τ	tau
υ	Υ	upsilon
φ	Φ	phi
χ	Χ	chi
ψ	Ψ	psi
ω	Ω	omega