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ABSTRACT

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Characteristics of Problem Representation Indicative of Understanding in Mathematics Problem Solving

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Running head: PROBLEM REPRESENTATION CHARACTERISTICS

Abstract

This study investigated the problem representations formed by college students while solving mathematics problems. Problem representation characteristics indicative of understanding were identified by analyzing audio-tapes and written work of sixteen subjects, ages 16 to 24, who solved mathematics problems using the think-aloud technique. These characteristics fall into three broad categories 1) content, 2) external code, and 3) processes involved in establishing the representation. This characterization is summarized in a problem representation instrument which can be used to assess the degree of understanding exhibited during problem solving. Significant positive correlations of the characteristics with follow-up tasks assumed to be indicative of understanding were obtained.

Characteristics Of Problem Representation
Indicative Of Understanding In Mathematics Problem Solving

The study of problem solving is receiving much attention currently in several different areas of research (Chi, Glaser, & Rees, 1981; Greeno, 1980; Lester, 1980; Suydam, 1980). Information processing researchers have focused on the development of computer programs to model human problem solving, mathematics education researchers have focused on the processes used by students as they solve mathematics problems, and some psychologists have focused on differences between expert and novice performance on physics problems.

Research in mathematics problem solving makes frequent reference to "understanding" a problem without using a consistent and explicit definition of "understanding". Even a cursory glance at the literature shows that "understanding" is used in a wide variety of ways. For example, Polya (1957) used understanding to refer to the first of four stages of problem solving. The understanding stage involves identifying the unknown, the data and the conditions. Schoenfeld (1980) broadened Polya's view of understanding by including problem analysis, design and exploration as part of the understanding of a problem. Kilpatrick (1968) and Days (1978) each defined a category of processes as understanding processes. Kilpatrick included such processes as identifies the unknown and draws a diagram. Days had a narrower view of understanding processes. He introduced a category of processes called representational which is distinct from the understanding category. Days included "draws a diagram" in the representational category rather than the understanding category. In other instances researchers have used the term "understanding" in an intuitive sense. Lucas (1972) awarded points to problem solvers on the basis of whether or not they displayed an "understanding" of the problem. Webb (1979b) awarded points to solvers on the basis of whether or not they "understood what the problem was asking".

While the term "understanding" has been used by many mathematics education researchers few have had the study of understanding in mathematics problem solving as a primary focus. It seems that such a study is both timely and important. In most teaching situations the teacher aims to have students develop an understanding of concepts and procedures. And in problem solving situations it is a truism that it is more desirable for a student to solve a problem with understanding than without. Further, most teachers would claim to be able to identify problem solutions which exhibit greater understanding and those which exhibit less understanding. In order to study understanding in mathematics problem solving it is necessary to first clarify what is meant by "understanding" in problem solving and to establish a means for assessing the degree to which it is present in particular solutions.

Information processing research on problem solving is helpful at this point. As researchers in information processing have attempted to model human problem solving performance, differing aspects of problem solving have been identified. In particular, the process of understanding a problem has been distinguished from the process of solving the problem. "Understanding" has been defined in terms of problem representation and it is generally agreed that the degree of understanding exhibited is reflected in the nature of the problem representation (Greeno, 1977). Consequently, it seems as though the study of problem representation may provide greater clarity to the term. Furthermore, constructivists assume that problem solvers build idiosyncratic knowledge structures and thus view understanding in terms of the mental representations constructed by the learner (von Glasersfeld. Steffe, Richards, & Thompson, 1983). For the constructivist, understanding is a function of a problem solver's prior experience which influences the nature of the problem representation developed.

Definitions

This section presents the definitions of problem, understanding, and problem representation as they are used in this study. Each is defined and discussed.

The definition of problem which is used in this study is: "A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution" (Lester, 1978, p.54). This definition is consistent with the definition used by researchers in both information processing (Newell & Simon, 1972) and in mathematics education (Lester, 1980).

The definition of understanding a problem that is used in this study is: Understanding a problem is the building of a problem representation. According to this definition, understanding is a process. This definition is used by Simon (Simon & Hayes, 1976) and by Greeno (1977) and is consistent with the concept of understanding used in the literature on language comprehension. In the study of language the concept of understanding is used to refer to the construction of a representation of some information such as a sentence, paragraph or story (Greeno, 1978). The representation constructed is based on the actual input, that is, on the specific materials presented to the subject and on the subject's conceptual knowledge (diSibio, 1982). Conceptual knowledge includes grammatical knowledge, factual knowledge, and specific content-domain knowledge (Mayer, 1982; Minsky, 1975). Judgments about the degree of a solver's understanding are made in terms of the features of the representation.

The definition of problem representation that is used in this study is: A problem representation is a cognitive structure which is constructed by a solver when interpreting a problem on the basis of his domain-related knowledge and its organization. This definition is similar to that used by Chi, Feltovich, & Glaser (1981) in studying problem solving in physics and is consistent with the use of the term representation in the cognitive psychology literature where the term is widely used, particularly in the literature on the study of language comprehension. A discussion of the concept of representation follows.

To study how the mind functions, cognitive psychologists present a subject with tasks which the subject interprets in terms of his or her conceptual knowledge. The result of this processing is called the mental or internal representation of the task (Mayer, 1978; Greeno, 1977). The representation may or may not resemble the presentation iconically. For example, research has shown that the mental representation of the word "four" is likely to not be verbal, rather it is more likely to be "visual" (Shephard & Podgorny, 1978). Similarly, the symbol "4" is not likely to be represented mentally in terms of the shape of the symbol, rather in terms of an associated concept such as a pattern. On the other hand, a human face is likely to be encoded visually, so the mental representation will iconically resemble the perceived stimulus (Shephard & Podgorny, 1978).

Sentences are typically processed, not as sequences of . . or even as sequences of words, but rather in terms of their meaning (Bransford, 1979). A string of nonsense syllables, on the other hand, may be encoded exactly in that form, i.e. as a string of syllables or even as a sequence of letters, because it does not signify anything to the subject. Words, hence sentences, have meaning in as much as they point to and elicit associations with knowledge previously constructed by the subject. Persons studying language comprehension do not view the mental or internal representation of a sentence as consisting of the words in the sentence. Rather, the representation includes the meaning associated with the words and is necessarily linked with conceptual knowledge. For example, the sentence "Ida borrowed the tablecloth from Jan" is linked mentally to the concept of "borrowed". Thus, included in the mental representation is the notion that the tablecloth was at one time in Jan's possession, possession then changed to Ida and possession has or will return to Jan.

Since a mental representation of a sentence includes links with prior conceptual knowledge, the representation is subject dependent. Consider the sentence "Bill bought the red car." An individual who does not know who Bill is or which car is being referred to as "the red car" will form a mental representation with relatively few associations. Imagine a friend of Bill's who was interested in buying the same red car. The mental representation that the friend has of the sentence will have relatively more associations than the representation in the previous case. For example, for the friend, the sentence will be linked with the knowledge that he too wanted to buy the car and possibly to information about the cost of the car. Previously constructed knowledge influences the mental representation evoked by a stimulus.

The view adopted here is that in mathematics problem solving, when an individual is presented with a problem, he or she uses the information to form a mental representation of the problem. As with sentence representation, problem representation includes more than is directly provided in the problem statement. Associations are established with conceptual knowledge. Different individuals have different conceptual knowledge and will make different associations with their knowledge. Relationships between elements also may be established depending on the person's existing schemas. In some cases an individual may fail to mentally use some of the information provided by the problem statement, may establish some relationships contrary to the problem statement, or may fail to establish certain relationships. Consequently, problem representation is quite subject dependent. Further, as a subject attempts to solve a problem, new relationships between problem elements or between problem elements and other knowledge the subject already has, may be formed. Thus the formation of a problem representation is a dynamic constructive process and depends on the the individual forming the representation.

Purpose

The purpose of this study was to develop a list of characteristics of problem representation that are indicative of understanding. Greeno's view that the degree of understanding of a problem is indicated by the nature of the problem representation was adopted.

The study was carried out in two phases. Phase one had as its goal the development of a list of characteristics of problem representation indicative of understanding. Greeno's criteria of understanding in problem solving, based on the theory of language comprehension, (Greeno, 1977) and the qualities of schema outlined by Skemp (1979) were useful guides in the development.

Phase two consisted of determining the relationships between performance on a set of follow-up tasks and the ratings of characteristics of problem representation outlined in phase one. The question to be answered in phase two was:

"Do problem solving protocols which have higher ratings on the problem representation characteristics identified in phase one correspond to greater success on the follow-up tasks than problem solving protocols which have lower ratings on the problem representation characteristics?"

Each of the follow-up tasks was selected because of its relevance to understanding. It was assumed that successful performance on the follow-up tasks indicates that the original problem was solved with understanding.

Phase One

The purpose of the first phase was to develop a list of problem representation characteristics that are indicators of understanding.

Method

The subjects for phase one were sixteen students, ages 16 to 24, including nine undergraduate and five graduate students at a large midwestern university and two high school juniors. Subjects were selected who were able to convey, verbally or through their written work, their mental processing while solving mathematics problems. Subjects were scheduled for two-hour problem solving interivews with the experimenter, in pairs when scheduling permitted. Four of the subjects were interviewed individually. The idea of using pairs of subjects in collecting data on problem solving is due to Schoenfeld (1981). His rationale is that when two subjects work together to solve a problem they must reveal what they are thinking to their partner. Consequently, more verbalization occurs without prompting or interference from the interviewer. During the interview subjects worked together to solve a variety of mathematics problems while thi. ing aloud, One example of the problems used is the following:

> The surface of Clear Lake is 35 feet above the surface of Blue Lake. Clear Lake is twice as deep as Blue Lake. The bottom of Clear Lake is 12 feet above the bottom of Blue Lake. How deep are the two lakes?

The problems were presented one at a time. The experimenter gave no feedback about the correctness of the solution but asked questions when necessary to encourage the subjects to reveal their thought processes. The interviews were audio-tape recorded. The experimenter kept a detailed written record of observations during the problem solving interview. This written record aided in the coordination of a subject's verbal and written records and also served as a means of recording significant and interesting behaviors that were not apparent by later review of the subject's written work or the audio tape. The subject's written work, the audio-tape, and the experimenter's written record formed the protocol.

Subjects were asked to try to complete each problem before proceeding to the next problem. However, subjects were notified after they had worked on a problem for twenty minutes without completing it so that they would have ample opportunity to attempt a large number of the problems. The number of problems completed differed for different subject pairs. Since the purpose of these interviews was to gain information useful to the development of a list of characteristics of problem representation it was not essential that all subjects complete exactly the same set of problems.

The protocol analysis in phase one of this study proceeded as follows. An initial list of anticipated observable events was made. This list included such things as attention to types of notation used, diagrams used, amount of rereading of portions of the problem statement and which portions are reread, evidence of planning, identification of problem components, and evidence of use of information not explicitly stated in the problem. The Ericsson and Simon model of verbal reporting suggests that periods of silence may be indicative of a reorganization of a representation or strategy (Ericsson & Simon, 1980). Consequently, observations of the above and similar events were made throughout the problem solving process with careful attention to changes after periods of silence. Schoenfeld (1981) had suggested that it is possible to identify decision points in a problem solving protocol where a solver may be redirecting his problem solving. According to Schoenfeld, problem solving protocols can be divided into "macroscopic chunks of consistent behavior", e.g. reading, analysis, exploration, transition, which he labels episodes. The points between episodes are called decision points. If decision points between episodes are potential places where a solver redirects his problem solving, they may be points where the solver revises his problem representation. Although the protocols were not parsed into episodes in the study reported here, attention was given to evidence of changes in the characteristics of the representation at apparent decision points.

Protocols were analyzed using the initial list. Modifications were made in the list to account for important observations not accounted for by the initial list. The analysis and modification process continued until clarification emerged, i.e. until the list adequately accounted for observed behaviors. The characteristics of problem representation that resulted are necessarily the experimenter's own interpretation of the evidence but are founded on the experimenter's understanding and knowledge of related research. The analysis was a continual process and was concluded when a consistent account emerged. Because it offered additional helpful information, data from a pilot study, conducted in a manner similar to phase one but using individual interviews rather than pairs, were also used in the development of the list of characteristics of problem representation.

Results

Phase one resulted in a list of characteristics of problem representation that can be used to evaluate a problem solving episode. Greeno (1977) identified correspondence, coherence, and connectedness as qualities of problem representation indicat[illegible] of understanding. This study extends Greer[illegible]'s work by providing a means of assessing these qualities and by elaborating additional representation characteristics. The characteristics developed here fall into three broad categories: content, external code, and processes. The content category is used to evaluate "what" is represented, the external code category to evaluate "how" the content is represented, and the processes category to evaluate specific features of how the problem solver proceeded in developing the attained problem representation. Each category is divided into subcategories for purposes of providing a detailed characterization of problem representation. The complete list of characteristics developed is shown in the problem representation instrument in Figure 1.

Insert Figure 1 about here

The distinction between the content and the code of the representation parallels similar distinctions made in studying language representation (Glass, Holyoak, & Santa, 1979). The inclusion of the processes category characterizes the way the problem representation was formed.

In this research the position was taken that there is no one ideal representation for a problem. Consequently, each problem representation is evaluated on its own merits rather than by comparison with some predetermined representation. The method of evaluating the characteristics identified reflects this position. For example, within the content category, the subcategories accuracy and completeness are evaluated by identifying inaccuracies and incompleteness in the solver's representation. The default is accurate and complete.

The content category contains the subcategories accuracy, completeness, and generalizability. Accuracy is a measure of the extent to which the solver's representation is consistent with the statement of the problem. A problem representation is considered to be accurate unless inaccuracies are found. Inaccuracies, or errors, in problem representation may be due to a variety of factors. Phase one of the study resulted in the identification of the following factors as causes of inaccuracy: encoding error, unjustified assumption, incorrect inference, lack of knowledge, computational error, and inaccurate goal. Definitions of these factors and their ratings are given in the appendix.

Completeness of a problem representation refers to the extent to which the information extracted from the problem statement and the relationships established while building and elaborating the representation are sufficient for the solution of the problem. In the process of solving a problem the solver must form relationships by encoding the problem statement in terms of his or her conceptual knowledge. To be complete a problem representation must contain needed explicit and implicit relationships and a representation of the goal. A relationship is called explicit if it is based only on information provided explicitly in the problem statement. A relationship is called implicit if it is inferred from aspects of the problem statement. Incompleteness in problem representation can occur in several different ways. Phase one of this study identified the following factors as causes of incomplete problem representation: absence of needed explicit relationship, absence of needed implicit relationship, lack of knowledge, and absence of goal. The definitions of these factors and their ratings are given in the appendix.

Another characteristic of the content of representation considered in this research is generalizability. The generalizability of a problem representation refers to the extent to which the representation is useful for solving problems similar in structure to the given problem. A precise definition and the rating method are given in the appendix.

The code of a representation refers to "how" the content is represented. In this study, there was no attempt to assess the internal code, only the external code. The solver's written work and verbalizations are taken as external code. Characteristics of external code identified by this study as important for describing the nature of the representation are level of abstraction, analogical versus analytical features, and specific types of code, such as a diagram, equations, chart or list. A code is analogical if it iconically resembles what is being represented. A code is called analytical if it consists of arbitrary relationships between the representation and what is being represented. Language, verbal, written, and mathematical, are all examples of analytical code. Definitions of the external code categories and their ratings are given in the appendix.

The third broad category of problem representation characteristics identified in phase one describes processes by which the representation is established. The factors selected for this category are: identify versus build, immediacy of relationships, types of connections, and strength of connections. Identify versus build refers to the extent to which the problem solver approaches the problem by treating it as a type for which he has available a schema or general representation which indicates the solution process. Immediacy of relationships indicates the extent to which the solution process is dominated by the establishment of relationships or by carrying out needed mathematical procedures. Types of connections refers to the extent to which the connections established are based on rote memorization or syntactic processing versus conceptual or semantic processing. Strength of connections refers to the solver's confidence in an established representation as evidenced, in part, by persistence with the representation. Precise definitions of these factors and their ratings are given in the appendix.

The evaluation of a problem representation on the characteristics identified in this study involves subjectivity. Specifically, knowledge about a subject's mathematical knowledge and background influences the ratings. Further, to use the problem representation instrument effectively the evaluator should be present during the problem solving interview. This limitation is not viewed as a weakness since typically intuitive evaluations of "understanding" require that the evaluator have considerable knowledge about the solver and his solution process. Further, the goal of this study was not to produce an instrument for use by independent evaluators, rather it was to explicate the nature of problem representation by identifying characteristics that are indicative of understanding. The characterization given here is best clarified through examples which illustrate the characteristics and their ratings, and which clarify the distinctions between the various characteristics. A detailed discussion is given in Yackel (1984).

Phase Two

The purpose of phase two was to verify that the characteristics of problem representation identified in phase one are in fact indicative of the understanding attained by the solver.

Method

Phase two consisted of presenting subjects with problem solving tasks, assessing the problem representations developed during the problem solving in terms of the characteristics outlined in phase one, and then presenting subjects with follow-up tasks. The follow-up tasks were selected so that success on the tasks could be reasonably assumed to indicate understanding of the original problem task. Performance on each follow-up task was then compared to the ratings given on the problem representation characteristics.

The subjects for this phase of the study were 36 students enrolled in an introductory level statistics course, taught by the experimenter, at a large midwestern university, who volunteered for the study. All of the four classes, freshman, sophomore, junior, and senior, were represented. There were an equal number of males and females and the students represented a variety of ability levels.

The follow-up tasks used were recollection of the problem immediately upon its completion, solution of a similar problem, creation of a problem with a similar solution method, and recollection of the problem at the end of the interview. Use of the problem recollection tasks is based on work of Silver (1979) and Krutetskii (1976). Use of the similar problems task to assess understanding of a problem is based on work of Gagne (1966) and Greeno (1977). Use of the problem creation task as a means of assessing understanding is based on work of Krutetskii (1976).

Subjects were interviewed individually, by the experimenter in two-hour sessions. Subjects were asked to think aloud during the interviews which were audio-taped. In the interview each subject was presented with four problems to solve and four accompanying follow-up tasks. The problems used for the problem solving tasks were problems which require no mathematical knowledge beyond arithmetic for successful completion. The problems were presented one at a time, typed on individual cards. A subject was allowed 45 minutes to solve a problem. Upon completion of the problem the experimenter removed the problem card and the solver's written work. The first three follow-up tasks were then presented.

The first follow-up task was the immediate recollection of the problem. Subjects were asked to "Repeat the problem statement." If a subject did not understand the task he was asked, "What did the problem on the card say?" The subject responded verbally to this task.

Upon completion of the first follow-up task the subject was presented with a similar problem, typed on a card. The subject was not told that this was a follow-up task or that the problem was similar to the original problem. Upon completion of the problem the experimenter removed the problem card and the solver's written work. Follow-up task 3 was then presented.

For follow-up task 3 subjects were instructed to "Make up a problem having a solution method like the solution method of ---, (the name of the problem solving task)." It should be noted that the problem solving task was not the problem the subject had just completed since the similar problem had been solved in the intervening time. Subjects did not have access to either the statement of the original problem, the statement of the similar problem, or their written work for either problem during this task.

As a fourth follow-up task, all subjects, at the end of the interview, were again asked to restate each of the four problems previously presented. Since the subject had also completed the similar problems and had not been told that some tasks were follow-up tasks it was necessary for the experimenter to identify for the subject which problems were to be recalled. This was done by saying something such as, "You did a problem about a football league and the draft. Tell me what the problem said." If the subject proceeded to explain his problem solution the experimenter said, "Tell me what the problem statement on the card was."

Throughout the interview the experimenter kept extensive notes of the subject's activity. These notes were used to coordinate the subject's written work with the audio-tape as well as to record information that would not be apparent from later review of the solver's written work or the audio-tape.

For each of the four initial problems presented in phase two, the subject's problem representation was characterized using the instrument developed in phase one. The solver's written work, the audio-tape and the experimenter's written record were used in this evaluation process. The follow-up tasks were evaluated as follows. The immediate and final problem recollections were rated as correct in details only, correct in structure only, correct in both details and structure or correct in neither. The solution of the similar problem was evaluated on appropriateness of solution method and correctness of answer. The problem created by the subject was rated as similar in details to the given problem, same in structure as the given problem, both or neither.

The data obtained from phase two of the study were analyzed as follows. Each of the items listed on the problem representation instrument was treated as a separate random variable. For example, within the category accuracy there were six variables, encoding error, unjustified assumption, incorrect inference, lack of knowledge, and computational error, and inaccurate goal. Within the category external code, analogical vs analytical was taken as one variable. The portion of the external code category labeled "types" was coded so that each type of code formed a separate variable. Presence of that type of code was rated 1 and absence rated 0. For example, the variable diagram was rated 1 if there was a diagram present and was rated 0 if no diagram was present. The variables defined by the problem representation instrument are referred to as the representation variables. There are 23 variables in all including answer on original problem. Even though it is not viewed as a characteristic of problem representation in this study, answer on original problem was included since it records the product of the problem solving process and as such the result of the solver's use of his or her problem representation. All of these random variables except analogical vs analytical are ordinal level. An additional variable recorded for each of the problem solving tasks was solution time in minutes. The follow-up task variables were immediate recollection, final recollection, problem creation, solution time on similar problem, method on similar problem, and answer on similar problem. Five of the six follow-up variables, the exception being solution time on similar problem, are ordinal level.

Three different analyses of the data were conducted. The first was the consideration of frequencies of the representation variables and the follow-up task variables except for solution time on similar problem. Those variables which were heavily concentrated on a single variable value were omitted from the second statistical analysis of the data.

The second analysis conducted was measures of association between the representation variables and the follow-up task variables. Since almost all of these variables were ordinal level, the appropriate measure of association was the Kendall's tau correlation coefficient. When the pair of variables to be correlated had equal number of possible values Kendall's tau b was used, otherwise tau c was used, (Kendall, 1970). Kendall's tau b and tau c are appropriate when the data have a large number of ties as was the case here (Agresti & Agresti, 1979; Kendall, 1970).

The final analysis was the computation of Kendall's tau correlation coefficients for the representation variables with each other. This analysis determines whether or not the variables are related to each other.

Results

The frequency data shows that each of the representation variables achieved each of its values except for certain variables in the accuracy and completeness categories. The variables encoding error, unjustified assumption, lack of knowledge and inaccurate goal, within the accuracy category, and the variables lack of knowledge and absence of goal, within the completeness category, were concentrated on the highest possible value. This means that for each of these variables all or most of the instances indicated no inaccuracy or incompleteness due to these factors.

The follow-up task responses for immediate and final problem recollection were concentrated on correct in both details and structure, with final recollection responses slightly less concentrated than the immediate recollection responses. The problem recollection follow-up tasks did not provide as much information as anticipated. The variables used to assess performance on the similar problem task and the problem creation task proved more useful, especially method and answer on the similar problem task. For a discussion of the frequencies results see Yackel (1984).

The major result of phase two of the study was that the characteristics of problem representation identified in phase one of the study are indicative of understanding when measured by answer or method on the similar problem task and by the problem creation task. Table 1 shows the correlations of the representation variables and the follow-up task variables. Strong positive correlations indicate that high ratings on the representation variables are associated with high ratings on the follow-up tasks and hence are indicative of understanding on the original task.

Insert Table 1 about here

Correlations of follow-up task variables with the immediate and final recollection tasks were relatively small in magnitude when they were significant. The concentration of the recollection responses on a single value limited the utility of these tasks in differentiating between qualitatively different problem representations. There are several reasons for this. These are discussed in detail in Yackel (1984).

Kendall's correlation coefficients were computed to determine the extent of relationship between the representation variables. Table 1 shows that the categories of variables on the problem representation instrument are not independent of each other but that within the category accuracy the variables are, for the most part, unrelated and within the category completeness some of the variables are not related. Even when a significant correlation exists between variables it is not appropriate to conclude that the variables are not measuring distinct characteristics. For example, absence of needed explicit relationship and absence of needed implicit relationship certainly measure two distinct characteristics of problem representation, yet they are positively correlated. For the problems and subjects used in this study high ratings on one variable occurred simultaneously with high ratings on the other variable and low low combinations also occurred. These occurrences outweighed any high low combinations which occurred.

Discussion

The overall plan of the study was to identify characteristics of problem representation potentially indicative of understanding, in phase one, and to verify that the characteristics are in fact indicative of understanding, in phase two, through the use of follow-up tasks assumed to be indicators of understanding. The study has shown that characteristics of problem representation which are indicative of understanding are accuracy, completeness, generalizability, and certain process variables. Specific causes of inaccuracy and incompleteness have been identified. Those which were not concentrated on a single variable value were shown to be indicative of understanding, as assessed by most of the follow-up tasks. Consequently the meaning of "understanding" in mathematics problem solving is clarified through the characterization of problem representation developed here.

The degree of understanding exhibited by a solver in a mathematics problem solving task can be assessed directly by considering characteristics of the problem representation formed by the solver. It is not necessary to use subsequent tasks to assess the degree of understanding of the original task.

In this study incorrect inference and inaccurate goal were the most frequently occurring causes of an inaccurate problem representation. Absence of needed explicit relationship and absence of needed implicit relationship were the most frequently occurring causes of an incomplete problem representation. Several of the factors listed as causes of inaccuracy and incompeleteness were observed only infrequently in this study. The infrequent occurrence of some of these is explained by the very specific nature of these factors as causes of inaccuracy or incompleteness and by the criteria for problem selection used in this study. Some of the variables, such as encoding error, are very specific but are necessary to provide a complete description of sources of inaccuracy and incompleteness. A variable such as incorrect inference is less specific and hence encompasses more errors. Consequently its frequency as a cause of error is much higher. The criteria for problem selection used in this study limited the likelihood of occurrence of some of the factors as causes or error or inaccuracy. For example, problems were selected which require no mathematics knowledge beyond that of the typical college student, thus reducing the frequency of lack of knowledge as a cause of error or of incompleteness.

Analysis of the problem solving protocols and performance on the follow-up tasks showed that a diagram plays at least two significant, but distinct roles in problem solving. It serves as a means of expressing information in the solver's current mental representation, that is, a solver uses a diagram to record spatial information given in the problem statement or information he or she has derived from the problem statement. Once drawn it also serves as a means of aiding the solver in further developing the representation, especially in establishing additional relationships that have spatial features. This second function is especially important since in problem solving a major task is to establish relationships between problem components.

Also of interest is the role diagrams serve in recalling problems. In this study some subjects, when given the problem creation task which required creation of a problem with a similar solution method to the original problem, recreated a diagram drawn for the original problem as an aid in its recollection, thus providing evidence that their internal code was spatial in nature. Further investigation of the role of diagrams in problem solving and of the potential of the problem creation task in assessing spatial features of the internal code of a problem representation is indicated.

Implications

This study has several implications for research. First it has shown that a problem solving task can be meaningfully assessed for degree of understanding. This can be done by considering the characteristics of problem representation identified in this study. It is not necessary to use a subsequent task to assess the degree of understanding exhibited during the problem solving episode. The construct of mental representation is useful in studying problem solving. The term problem representation then assumes a much broader meaning than it has been given in most previous mathematics education contexts. Terminology used by cognitive psychologists, such as "code", and "analog" and "analytic" code, is useful in describing problem representations.

Second, this study has shown that it is useful to study problem solving from a global approach. Much has been learned from problem solving process research which studies a problem solver's activity by checking processes used and recording the sequence of their use. Such research looks at problem solving from a microscopic view. Schoenfeld (1981) has called for research which takes a macroscopic view of problem solving. Use of the construct problem representation permits a macroscopic view; investigating the quality of a problem representation using the problem instrument requires analysis of the problem solving protocol as a whole. Further research can take advantage of the development and clarification provided for problem representation and understanding in problem solving presented in this study.

The research reported in this study has important implications for the teaching of problem solving. Current emphasis in problem solving is on the teaching and use of heuristics. Such emphasis is well founded. Familiarity with and ability to use heuristics in problem solving is essential for the successful solver. This research shows that attention to the formation of a problem representation is another important aspect. Specific aspects of representation such as accuracy and completeness need to be emphasized by teachers. Students can be specifically directed to attend to the qualities of accuracy and completeness. The intent is not that a student will be able to judge his representation for accuracy and completeness. But, that he will be explicitly aware that failure to solve a problem may be due to interpreting the problem differently than intended or failing to use additional information or establish relationships. The distinction between explicit and implicit information can be made. Once the distinctions are clarified students can become cognizant of information used in problem solving which is explicit and information which is implicit. General awareness on the part of a student that implicit information is often needed to solve a problem should encourage him to consider potentially related information.

The various factors causing inaccuracy and incompleteness that have been outlined can be useful to teachers in identifying inadequacies in a student's problem representation. These factors can be used by the teacher to help provide direction to a student without telling the student specifically what is in error or lacking or exactly how to proceed to successfully solve the problem. Successful use of specific aspects of representation, such as accuracy and completeness, by teachers presumes that they have extensive experiences solving problems themselves and observing others solve problems.

Use of the problem creation task has shown its potential as a teaching tool. Students should be asked to create problems with similar solution methods to problems they have solved. In the process of problem creation, students are forced to consider the relationship between various mathematical relationships and the verbal statements which express those relationships. Problem creation is a difficult task for students. Success on this task may require a higher degree of understanding than successful solving of a similar problem.

Two additional implications for teaching problem solving relate to the identify vs build and the immediacy of relationships variables. Students must learn that in problem solving (as defined herein) an appropriate problem representation will not be available within the solver's existing knowledge. Problem representations are built by establishing relationships between problem components and possible additional knowledge the solver identifies as relevant to the solution. Establishing these relationships typically dominates the problem solving process. Carrying out necessary mathematical procedures and computations is secondary. Students should not expect to know exactly what to do immediately. To use Polya's terminology, "carrying out the plan" is secondary to "understanding the problem" and "devising a plan".

There are several possible ways to help students learn the distinction between establishing relationships and carrying out mathematical procedures. One way is to have students outline possible solution plans rather than to try to solve the problem. Schoenfeld (1982) has used this approach although for a different purpose. A simple way to emphasize carrying out mathematical procedures is to provide students with aids that minimize the amount of time and effort required to complete the procedures, such as calculators, integration tables, and lists of formulas. Finally, students are heavily influenced by the way they are evaluated. Tests that emphasize problem formulation through the types of questions asked and the way they are scored have potential for helping students make the distinctions suggested here.

A final implication of this research for teaching problem solving is that students need to have the capability to express their mental representation. In most cases students are unable to completely solve problems mentally without using an external expression of their thoughts, such as paper and pencil work. Students need to be able to use diagrams and mathematical language, such as conventional symbolism and notation to effectively express their mental representations. Ability to use mathematical language and diagrams facilitates students' development of a problem representation. In this study the experimenter observed that most subjects read a problem and either during the first reading or immediately thereafter recorded, on paper, information they obtained from the verbal problem statement. Repeatedly it was observed that when a subject did not know how to express on paper what he had just read, he would stop as if no progress could be made with the problem solving until that information could be expressed. In many of these cases it was difficult for the subject to come to the decision to proceed without having expressed on paper the information provided by a sentence or a phrase. Without directly verbalizing it, most of these subjects seemed to be implying that unless they could express the meaning of the phrase or sentence on paper, they had not established it mentally. Apparently, ability to use mathematical language and notation facilitates the development of a problem representation. A confounding factor in this study was that most of the subjects had little previous experience solving problems. It is quite likely that many of them intended to solve the problems by manipulating symbols and were stymied if they could not find an appropriate symbolic expression for the problem. This further emphasizes the need for the distinction between the content of a problem representation and the external code used to express the representation.

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Appendix

Definitions of Characteristics of Problem Representation and Their Ratings

Accuracy of representation

The following factors in this category will be rated 0,1,2.

0 More than one error due to this cause.

1 One error due to this cause.

2 No errors due to this cause.

Encoding error: An error in encoding information occurs when the solver(s) misreads or misinterprets a word or phrase. There must be evidence from the protocol that had the encoding error been noticed a correct representation would have resulted. This error is characterized by quick processing and failure to carefully analyze the problem information.

Unjustified assumption: An error due to the solver(s) making an assumption that is not justified on the basis of the information provided in the problem statement.

Incorrect inference: An error made when the solver(s) makes an erroneous inference from the situation described in the problem statement.

Lack of knowledge: An error in the representation which results from the solver(s) lacking some knowledge, eg. in interpreting the meaning of the words in the problem statement, or in translating the information into mathematical notation.

Computational error: An error in the representation which results from an error in computation made by the solver(s).

The following factor will be rated 0,1,2.

0 Inaccurate goal.

1 The solver shows evidence of having identified the correct goal early in the problem solving but stops working after attaining a major subgoal.

2 Accurate goal.

Inaccurate goal: The solver(s) has identified a goal different from the intended goal of the problem.

Completener of Representation

The following factors in this category will be rated 0,1,2.

0 More than one omission due to this cause.

1 One omission due to this cause.

2 No incompleteness due to this cause.

Absence of needed explicit relationship: A relationship explicitly given in the problem information is omitted.

Absence of needed implicit relationship: An implicit relationship necessary for the solution is omitted.

Lack of Knowledge: Failure to encode, to assign meaning to, one or more portions of the problem statement.

The following factor will be rated 0,1.

0 Omission of goal.

1 Goal present.

Absence of goal: The solver is unable to determine a goal for the problem, i.e. has no specific goal.

Generalizability of representation (0,1,2)

The generalizability of a problem representation refers to the extent to which the representation is useful for solving similar problems.

0 (isolated): The representation is useful only for the problem given. Nonsystematic trial and error falls at this level.

1 (some evidence of integration): Some aspects of the representation will be useful for solving similar problems. Systematic trial and error as well as general statements made verbally but not explicitly written down will be taken as evidence of some integration.

2 (integrated): The problem representation is descriptive of or uses notation or a solution method which clearly indicates that it is one of a class of similar problems. A problem involving similar relationships could be solved using the same representation making needed modifications to account for the new data.

External Code

Analogical vs analytical. The following types will be checked if they are observed to occur.

none: Essentially no external code is used. Several numbers may be written on the solver's paper but nothing else.

analogical only: The code used in some ways resembles what is being represented, i.e. it has some properties similar to actual perception or has some features of what is being represented.

analytical only: The code is based on an arbitrary relationship between the representation and what is being represented, eg. mathematical symbolism or notation.

both analogical and analytical: Both types of code are present in the solver's work.

Level of abstraction (0,1,2)

Level of abstraction refers to the extent to which what is expressed by the code is more general than the input which is actually presented. When more than one code is present the most abstract code will be rated.

0 (low): The external code used describes only what is actually presented. There is no generalized notation.

1 (moderate): Limited use is made of symbolic notation, equations or diagrams which abstract the mathematical features of the problem.

2 (high): Extensive use is made of symbolic notation, equations, diagrams which abstract the mathematical features of the problem. The code expresses a general mathematical model.

Types of code:

To facilitate the above rating of level of abstraction the following types of code will be checked when they are observed to occur. The category "other" provides for indicating a form of code not among those listed that is observed to occur and is viewed as relevant to the determination of the level of abstraction.

diagram

symbolic notation

equations

chart or list

other

Process Of Establishing The Problem Representation

The variables in this category characterize the process of establishing the problem representation.

Identify vs build (0,1,2)

0 (identify): The solver treats the problem as one of a type for which he already knows a solution procedure and identifies a representation that can be used to solve the problem.

1 (mainly identify, some evidence of building): The solver identifies the problem as one of a type he knows how to solve but must establish a number of relationships and processes semantically for that purpose.

2 (build): The solver establishes relationships on the basis of the information in the problem statement and not on the basis of treating it as a problem type.

Immediacy of relationships (0,1,2)

0 (low): The solver(s) is slow to sense the relationships. An appropriate representation is not established until near the end of the solution process or not at all.

1 (moderate): Establishing the necessary relationships dominates the solution process. Some time may be spent initially on exploration. Relationships may be modified during the solution. Some time is spent carrying out the necessary mathematical procedures but this is secondary.

2 (high): The needed relationships are established almost immediately by the solvers. Carrying out the necessary mathematical procedures dominates the solution process. Establishing the needed relationships is secondary.

Types of connections (0,1,2)

0 (associative): Connections are established on the basis of rote memory or statement syntax.

1 (mainly associative, some evidence of conceptual): Most of the

connections are associative. There is evidence that some are conceptual.

2 (conceptual): There is evidence that the connections are conceptual or that most of the processing is semantic.

Strength of connections (0,1,2)

Strength of connections refers to confidence in the problem representation.

0 (weak): Solvers have not attained a solution or have no confidence in their representation. Relationships established are readily abandoned at the suggestion of a solution partner or an observer.

1 (moderate): Connections are used in a very tentative way. The solver expresses some uncertainty about some of the relationships that have been established. Obtaining an answer to the problem which the solver has reason to believe is correct may be the only way the solver is certain that the relationships are correct.

2 (strong): The solver exhibits strong confidence in the problem representation.

Answer on Original Problem (0,1)

0 (incorrect): No answer is given or the answer given is not correct.

1 (correct): The answer given to the problem is correct.

Table 1
Kendall's Correlation Coefficients of Representation Variables with Follow-Up Task Variables

	Follow-up Task Variable					
	recollection		similar problem			problem
Representation variable	immediate	final	method	answer	time	creation
Accuracy						
incorrect inference	.12**	.13*	.40**	.41**	.21**	.27**
inaccurate goal	.03	.12**	.07	.08*	-.08	.11**
accuracy total	.13**	.20**	.43**	.44**	.13*	.30**
Completeness						
absence of needed explicit relationship	.10**	.12**	.17**	.15**	-.11	.21**
absence of needed implicit relationship	.04	.05	.62**	.42**	.22**	.25**
completeness total	.18*	.11*	.61**	.43**	.10	.29**
Generalizability	.04	.13*	.57**	.45**	.19**	.16*
Level of abstraction of external code	-.06	.07	.34**	.14*	.12	-.02
Processes						
identify vs build	.02	-.05	.23**	.24**	.41**	.16**
immediacy of relationships	.03	.08	.61**	.36**	.18**	.24**
types of connections	.05	.07	.32**	.34**	.22**	.26**
strength of connections	.07	.20**	.36**	.20*	-.19**	.21**
process total	.07	.11*	.61**	.46**	.16**	.34**
Answer on Original Problem	.07	.16*	.44**	.39**	.21**	.27**

* p<.05
** p<.01

PROBLEM REPRESENTATION INSTRUMENT

Subject_____
Problem_____
Date_____

CHARACTERISTICS OF PROBLEM REPRESENTATION

A. Accuracy of representation

factors determining inaccuracy of representation	more than one error	one error	no errors
encoding error	0	1	2
unjustified assumption	0	1	2
incorrect inference	0	1	2
lack of knowledge	0	1	2
computational error	0	1	2
inaccurate goal	0 yes	1	2 no

B. Completeness of representation

factors determining incompleteness of representation	more than one omission	one omission	no omissions
absence of needed explicit relationship	0	1	2
absence of needed implicit relationship	0	1	2
lack of knowledge	0	1	2
absence of goal	0 yes	1 no	

C. Generalizability of representation

0	1	2
isolated	some evidence of integration	integrated

External code

Analogical vs analytical

none_____
analogical only_____
analytical only_____
both analogical and analytical_____

Level of abstraction

0	1	2
low	moderate	high

types:

diagram_____
symbolic notation_____
equations_____
chart or list_____
other (describe)_____

Process of establishing the problem representation

identify vs build

0 identify	1 mainly identify, some evidence of building	2 build

immediacy of relationships

0 low	1 moderate	2 high

types of connections

0 associative	1 mainly associative some evidence of conceptual	2 conceptual

strength of connections

0 weak	1 moderate	2 strong

Correctness of answer

0 incorrect	1 correct

Figure 1

Problem Representation Instrument