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ABSTRACT

This chapter from a longer work concerns the ability of children to solve arithmetic word problems. The studies reviewed suggest that, with age, children's improved ability to solve word problems primarily involves an increase in the complexity of conceptual knowledge required to understand the situations described in those problems. Considered in the various sections are conceptual and procedural knowledge in problem solving, approaches to analyzing knowledge in problem solving, a review of research on children's word problem solving, a theory of the knowledge required to solve word problems, the locus of improvement in problem-solving skill, stages of conceptual knowledge, related analysis of conceptual understanding in problem solving, and a summary discussion. (MNS)

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DEVELOPMENT OF CHILDREN'S PROBLEM-SOLVING ABILITY IN ARITHMETIC

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CHAPTER 4

Development of Children's Problem-Solving Ability in Arithmetic[1]

This cnapter is concerned with the development of an important aspect of children's problem-solving skill in arithmetic—the ability to solve arithmetic word problems. There are several factors that might enable older children to perform better in problem-solving tasks than younger children, including the complexity of conceptual knowledge about the problem domain and the sophistication of problem-solving procedures. The studies reviewed here suggest that, with age, children's improved ability to solve word problems primarily involves an increase in the complexity of conceptual knowledge required to understand the situations described in those problems. We will describe these findings in this chapter and consider some general issues about the development of problem-solving skill.

CONCEPTUAL AND PROCEDURAL KNOWLEDGE IN PROBLEM SOLVING

One major issue we will address concerns the relationship between conceptual and procedural knowledge in performance and development. In the past few years much has been learned about how improvements in either conceptual

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knowledge (e.g., Chi, 1978; Gentner, 1975; Stein & Trabasso, 1981) or problem-solving procedures and strategies (e.g., Baylor & Gascon, 1974; Brown, 1978; Groen & Resnick, 1977; Klahr & Robinson, 1981; Young & O'Shea, 1981) separately contribute to improvements in performance as children get older. However, most tasks that children perform involve an interaction between both knowledge and procedures, and the nature of this interaction is a significant theoretical question for models of cognitive development. Some progress is being made. Recent studies by Siegler (1976) and Siegler and Klahr (1981) have identified the importance of understanding the relevant features and relations of a particular task in the acquisition of more advanced problem-solving procedures. Chi (1982) discusses the interactive role of domain-specific knowledge and various kinds of strategies in affecting children's memory performance. Also, Resnick (1981; Chapter 3, this volume) has related increased knowledge of relevant goals and constraints in subtraction to improvements in children's computational procedures using blocks. This knowledge can then be used to learn similar procedures in another context where these goals and constraints are less salient, as in the formal syntax of arithmetic. The theoretical and empirical studies of word problem solving that we will describe provide further analyses of interactions between conceptual and procedural knowledge.

A second and related issue concerns the knowledge (either conceptual or procedural) that we attribute to children on the basis of their problem-solving performance. Frequently children are said to understand a concept if their performance on some task is consistent with that concept. Children whose performance is inconsistent are said to lack understanding. We will argue that such an all-or-none view of children's understanding is too limiting. Our argument will be based on two lines of evidence. First, children who appear to lack understanding of a concept on one task often show performance that is consistent with that concept on other tasks (e.g., Gelman & Gallistel, 1978; Trabasso, Isen, Dolecki, McLanahan, Riley, & Tucker, 1978), thus implying some understanding of the concept. Second, even when several children perform successfully on the same problem-solving task, this does not necessarily imply that they share the same underlying knowledge (e.g., Dean, Chabaud, & Bridges, 1981). For example, children may differ in their representations of the problem, and this can affect the kinds of procedures required for solution, as well as the ability to solve related problems. We will discuss some recent theoretical analyses of children's understanding that provide explicit descriptions of the knowledge underlying different stages of problem-solving skill.

Our discussion of these issues will be based on the development of a specific hypothesis about the nature of children's skill in solving arithmetic word problems. According to this hypothesis, improvement in performance results mainly from improved understanding of certain conceptual relationships. This is not to say that knowledge of formal arithmetic lacks importance for children.

Indeed, an important possibility is that acquisition of certain conceptual structures depends upon the knowledge of formal arithmetic that children acquire through school instruction. However, we are unable to conceptualize knowledge of formal arithmetic in a way that makes it sufficient for solving word problems. Furthermore, problems with the same arithmetic structure but different conceptual structures differ substantially in their difficulty for children. We take this as evidence that conceptual understanding is required if the texts of word problems are to be mapped onto arithmetic relationships and operations. Our goal is to emphasize the importance of informal concepts in problem solving and to provide a detailed analysis of informal concepts that are needed for a class of arithmetic word problems.

APPROACHES TO ANALYZING KNOWLEDGE IN PROBLEM SOLVING

The analyses we will describe have been influenced to a large extent by recent cognitive theories of problem solving and language understanding. These theories have provided increasingly rigorous concepts and methods for understanding the knowledge underlying problem-solving performance. Early analyses of problem solving around 1950 focused on fairly general connections between actions performed during problem solving. Behaviorists (e.g., Maltzman, 1955) and associationists (e.g., Underwood & Richardson, 1956) analyzed solutions of problems using concepts such as strength of associations and competition between responses. More recent information-processing analyses (e.g., Newell & Simon, 1972) have provided more specific concepts and more rigorous methods for analyzing performance in problem-solving situations. We now can analyze the cognitive processes required for solving problems in considerable detail, providing hypotheses about specific cognitive procedures as well as the general strategies that are involved in successful performance.

Recent developments in cognitive theory have also made possible a more detailed and rigorous analysis of the role of conceptual knowledge in problem solving. The importance of conceptual knowledge for understanding and representing problems has long been recognized. Gestalt theorists such as Duncker (1945), Kohler (1927), and Wertheimer (1945/1959) conceptualized solution of a problem as achievement of understanding the problem as a whole and as the relations of problem elements and solution procedures to the whole. A primary contribution of recent cognitive theories has been the development of concepts and methods that allow more specific hypotheses about the conceptual knowledge required to solve complex problems in domains such as physics (McDermott & Larkin, 1978; Novak, 1976) and high school geometry (Anderson, Greeno, Kline, & Neves, 1981; Greeno, 1977, 1978). An important theoretical resource in this development has come from cognitive theories of language understanding (e.g., Anderson, 1976; Norman & Rumelhart, 1975; Schank & Abelson, 1977). In these theories, understanding a sentence or a story is viewed as the construction of a coherent representation of the various elements in the message, with individual elements interconnected in a network of relationships. Understanding in problem solving is characterized similarly as a process of representing problem information or solution components in coherent relational networks constructed on the basis of general conceptual knowledge.

We now sketch the contents of the remaining sections of this chapter. The next section characterizes a class of addition and subtraction word problems. We review the major factors that have been used to characterize aspects of word problems, including their relative difficulty. The next three sections concern the knowledge underlying children's performance on these problems. The section entitled "A Theory of the Knowledge Required to Solve Word Problems" presents a theoretical analysis of the knowledge and strategies that we hypothesize to be involved in successful performance. The section on the locus of improvement in problem-solving skill presents evidence that even very young children have available a range of strategies for solving word problems, but they differ from older children in the conceptual knowledge required to apply those strategies. The section on the stages of conceptual knowledge specifies some of these differences in conceptual knowledge as they relate to differences in the success, efficiency, and generality of problem-solving performance. In the section entitled "Related Analyses of Conceptual Understanding in Problem-Solving we present a brief summary of analyses of word problem solving in domains other than elementary arithmetic. Finally, the discussion section relates children's performance on word problems to more general issues in developmental theory and methodology.

REVIEW OF RESEARCH ON CHILDREN'S WORD PROBLEM SOLVING

Our review of research on addition and subtraction word problems and factors that influence their difficulty has two main sections. First, we summarize findings of studies concerned with global factors. These include studies of general structural features, such as the grammatical complexity of the problem statement, and studies concerned with the effect of having materials such as blocks available as aids for problem representation. Second, we review literature on more detailed semantic factors. We present a survey of analyses that have categorized problems based on semantic relationships among quantities in the problem situation. Then we summarize empirical studies that have compared the difficulty of problems differing in their semantic characteristics.

Global Factors

STRUCTURAL FEATURES OF PROBLEM STATEMENTS

Several studies have considered general surface characteristics of problem statements as factors influencing problem difficulty. Variables such as problem length, grammatical complexity, and order of problem statements have been shown to have significant effects on ease of solution (Jerman, 1971, 1973–1974; Jerman & Rees, 1972; Loftus, 1970). Regression analyses indicate that a large proportion of variance in problem difficulty can be accounted for by these factors (Loftus & Suppes, 1972).

The type of number sentence represented by the relations among quantities in the problem has also been related to problem difficulty. (Grouws, 1972; Lindvall & Ibarra, 1980a: Rosenthal & Resnick, 1974). Problems represented by sentences where the unknown is either the first ($? + a = b$) or second ($a + ? = b$) number are more difficult than problems represented by equations where the result is the unknown ($a + b = ?$). Rosenthal and Resnick provided an explicit model to account for these differences in difficulty. Their model focused on the process of translating the problem text into an equation, and difficulty was predicted as a function of the number and kinds of transformations required to translate the equation into its canonical form for solution (e.g., either $a + b = ?$ or $a - b = ?$).

CONCRETE AIDS

Another factor that has been examined is the availability of concrete materials such as blocks as aids in solving word problems. Several studies have found that the availability of blocks (e.g., Bolduc, 1970; Hebbeler, 1977; LeBlanc, 1968; Steffe, 1968, 1970; Steffe & Johnson, 1971) and/or reference dolls or pictures (e.g., Harvey, 1976; Ibarra & Lindvall, 1979; Marshall, 1976; Shores & Underhill, 1976) facilitate solution of problems, particularly for young children. As an illustration, Tables 4.1 and 4.2 show data from children's performance with and without concrete objects. The exact nature of the "change" and "combine" problem types in these tables will be discussed later in this section. The data in Table 4.1 are from a study by Riley (Greeno & Riley, 1981; Riley, 1981) in which kindergarten children solved problems that described a change in some quantity. Table 4.2 shows data from a study in which Steffe and Johnson (1971) asked first-graders to solve problems like those in Table 4.1, as well as problems involving combinations of quantities. The point to be made here is that in both studies there was a general improvement in children's performance when they used objects to solve problems. The fact that concrete aids did not facilitate

Table 4.1
Proportions of Kindergartners Who Performed Correctly on Arithmetic Tasks[a]

Problem type	Without objects	With objects
Change (1)	.70	.87
Change (2)	.61	1.00
Change (3)	.22	.61
Change (4)	.30	.91
Change (5)	.09	.09
Change (6)	.17	.22

[a]From Riley, 1981.

kindergartners' performance on change (5) and change (6) problems relates to some interesting theoretical issues that we will discuss later.

Additional support for the facilitation effect of objects was obtained by Carpenter, Hiebert, and Moser (1981) who showed that, given a choice between solving word problems with or without blocks, first-graders preferred to use blocks. Some data indicate that merely observing (not manipulating) concrete representations of problem solutions improves word problem performance (e.g., Buckingham & MacLatchy, 1930; Gibb, 1956; Ibarra & Lindvall, 1979).

OTHER GLOBAL FACTORS

In addition to the factors we have discussed here, the ability of individual children to solve word problems has been studied in relation to their general reading ability and various instructional methods. Extensive reviews of this research have been provided by Aiken (1971, 1972) and Barnett, Vos, and Sowder (1979).

Table 4.2
Proportions of First-Graders Who Performed Correctly on Arithmetic Tasks[a]

Problem type	Without objects	With objects
Change (1)	.67	.85
Change (2)	.43	.61
Change (3)	.41	.67
Change (5)	.41	.58
Combine (1)	.67	.80
Combine (2)	.35	.55

[a]From Steffe & Johnson, 1971.

Specific Factors

Although the analyses of global characteristics of word problems has provided a basis for reasonably accurate predictions of problem difficulty, significant differences have been found between problems for which these factors are held constant (e.g., Gibb, 1956; LeBlanc, 1968; Schell & Burns, 1962). Furthermore, the effect of certain words, such as *altogether* and *less*, has been shown to depend upon whether the operation suggested by the word matches the operation required for problem solution (e.g., Dahmus, 1970; Jerman, 1971; Linville, 1976; Nesher & Teubal, 1974). For these reasons and others, many recent analyses of word problems have focused on specific problem characteristics involving the relationships among quantities described in the problem. The main finding from these analyses is that the understanding of quantitative relationships in problems involves factors other than the arithmetic equations that express the relationships; the conceptual structure of the problem must also be taken into account. In the remainder of this section, we review the kinds of conceptual relations that describe simple addition and subtraction problems; we also review empirical studies of differences in problem difficulty that are associated with these semantic variables.

PROBLEM TYPES

A word problem identifies some quantities and describes a relationship among them. Table 4.3 shows examples of several kinds of word problems that have been included in various research studies. Each of these problems describes a simple situation involving either addition or subtraction. The categories in Table 4.3 include the change, combine, and compare categories used in an analysis by Heller and Greeno (1978). These categories are representative of categorical schemes that have been used by several investigators (e.g., Carpenter & Moser, 1981; Fuson, 1979; Nesher, 1981; Vergnaud, 1981) in analyses of simple addition and subtraction problems, although the names used to refer to the categories have varied. The equalizing category in Table 4.3 is from the work of Carpenter and Moser (1981).

Semantic Structure. One of the ways the problems in Table 4.3 differ is in the semantic relations used to describe the problem situation. By *semantic relations* we refer to conceptual knowledge about increases, decreases, combinations, and comparisons involving sets of objects.

The first two problem categories shown in Table 4.3—change and equalizing—describe addition and subtraction as actions that cause increases or decreases in some quantity. For example, in change (1) the initial quantity or *start set* of Joe's three marbles is increased by the action of Tom giving Joe five more marbles (the *change set*). The resulting quantity or *result set* is eight. Equalizing

Table 4.3
Types of Word Problems[a]

Action	Static
CHANGE	COMBINE
Result unknown	Combine value unknown
1. Joe had 3 marbles. Then Tom gave him 5 more marbles. How many marbles does Joe have now?	1. Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have altogether?
2. Joe had 8 marbles. Then he gave 5 marbles to Tom. How many marbles does Joe have now?	Subset unknown
	2. Joe and Tom have 8 marbles altogether. Joe has 3 marbles. How many marbles does Tom have?
Change unknown	COMPARE
	Difference unknown
3. Joe had 3 marbles. Then Tom gave him some more marbles. Now Joe has 8 marbles. How many marbles did Tom give him?	1. Joe has 8 marbles. Tom has 5 marbles. How many marbles does Joe have more than Tom?
4. Joe had 8 marbles. Then he gave some marbles to Tom. Now Joe has 3 marbles. How many marbles did he give to Tom?	2. Joe has 8 marbles. Tom has 5 marbles. How many marbles does Tom have less than Joe?
Start unknown	Compared quality unknown
5. Joe had some marbles. Then Tom gave him 5 more marbles. Now Joe has 8 marbles. How many marbles did Joe have in the beginning?	3. Joe has 3 marbles. Tom has 5 more marbles than Joe. How many marbles does Tom have?
6. Joe had some marbles. Then he gave 5 marbles to Tom. Now Joe has 3 marbles. How many marbles did Joe have in the beginning?	4. Joe has 8 marbles. Tom has 5 marbles less than Joe. How many marbles does Tom have?
EQUALIZING	Referent unknown
1. Joe has 3 marbles. Tom has 8 marbles. What could Joe do to have as many marbles as Tom? (How many marbles does Joe need to have as many as Tom?)	5. Joe has 8 marbles. He has 5 more marbles than Tom. How many marbles does Tom have?
2. Joe has 8 marbles. Tom has 3 marbles. What could Joe do to have as many marbles as Tom?	6. Joe has 3 marbles. He has 5 marbles less than Tom. How many marbles does Tom have?

[a]From Riley, 1981.

problems involve two separate quantities, one of which is changed to be the same as the other quantity. In equalizing (1) the problem solver is asked to change the amount of Joe's set to be the same as the amount of Tom's set.

The remaining categories—combine and compare—involve static relations between quantities. In combine (1) there are two distinct quantities that do not change—Joe's three marbles and Tom's five marbles—and the problem solver is asked to consider them in combination: *How many marbles do Joe and Tom have altogether?* Compare (1) also describes two quantities that do not change, but this time the problem solver is asked to determine the difference between them: *How many marbles does Joe have more than Tom?* Since in this case Joe's marbles are being compared to Tom's, Joe's marbles are called the *compared set* and Tom's marbles are called the *referent set*. If the question had been *How many marbles does Tom have less than Joe?* then Tom's marbles would have been the compared set and Joe's would have been the referent set.

Identity of the Unknown Quantity. In addition to the various semantic relations, there are other ways in which the problems in Table 4.3 differ. In each kind of problem—change, equalizing, combine, and compare—there are three items of information. Different problems can be formed by varying the items of information given and those to be found by the problem solver. In change problems, the three items of information are the start, change, and result sets. Any of these can be found if the other two are given, yielding three different cases: The unknown may be the start, the change, or the result. Furthermore, the direction of change can either be an increase or a decrease, so there are a total of six kinds of change problems. Change problems involving increases are referred to collectively as change/join problems; change problems involving subtraction are referred to as change/separate problems.

A similar set of variations exists for compare problems, where the direction of difference may be more or less and the unknown quantity may be the amount of difference between the referent set and the compared set, or either of the two sets themselves. Equalizing problems usually restrict the unknown to the difference between the given quantity and the desired quantity, although a total of six variations are possible. In combine problems there are fewer possible variations: The unknown is either the combined set or one of the subsets.

RELATIVE DIFFICULTY

There have been many empirical studies concerned with the relative difficulty of problems similar to those in Table 4.3. The basic procedure usually involves having children individually solve selected problems that are read to them by the experimenter. Memorial and computational difficulties are kept at a minimum by reading the problems slowly, repeating them if necessary, and by restricting the size of the numbers in the problems such that sums are less than

10. In addition, concrete objects are often provided for children to use in solving the problems.

The main findings from these studies are summarized next. In general, older children perform better than younger children, which is not surprising. Both semantic structure and identity of the unknown consistently influence relative problem difficulty.

Semantic Structure. Evidence for the influence of semantic structure in problem solution comes from studies showing that problems described by different semantic structures are not equally difficult, even when they require the same operation for solution. This suggests that solving a word problem requires more than just knowing the operations and having some general skill in applying them.

Tables 4.4, 4.5 and 4.6 show the results from three separate studies in which children solved sets of word problems using blocks. All studies followed a procedure like the one just described, with the exception that in the Carpenter *et al.* (1981) study, sums of the given numbers were between 11 and 15. The following is a summary of the main findings from these three studies.

Compare problems (3) and (6) are more difficult than either change (1) or combine (1) problems, although all four problem solutions involve a simple addition. Similarly, problems involving subtraction can also vary in difficulty across semantic structures. Combine (2) problems and virtually all compare problems involving subtraction are, in general, more difficult than change problems (2) and (4). These findings agree with those from other studies. Compare (1) problems have been consistently shown to be more difficult than change (2) problems for first-graders (e.g., Gibb, 1956; Schell & Burns, 1962; Shores & Underhill, 1976). Combine (2) problems are, in general, more difficult than change (2) for kindergartners and first-graders (e.g., Gibb, 1956; Ibarra &

Table 4.4
Proportions of First-Graders Who Performed Correctly[a]

Problem type	With objects
Change (1)	.95
Change (2)	.91
Change (3)	.72
Equalizing (1)	.91
Equalizing (2)	.91
Combine (1)	.88
Combine (2)	.77
Compare (1)	.81
Compare (3)	.28

[a]From Carpenter, Hiebert, & Moser, 1981.

Table 4.5
Proportions of Children Who Performed Correctly Using Objects[a]

	Grade			
Problem type	K	1	2	3
Change (1)	.87	1.00	1.00	1.00
Change (2)	1.00	1.00	1.00	1.00
Change (3)	.61	.56	1.00	1.00
Change (4)	.91	.78	1.00	1.00
Change (5)	.09	.28	.80	.95
Change (6)	.22	.39	.70	.80
Combine (1)	1.00	1.00	1.00	1.00
Combine (2)	.22	.39	.70	1.00
Compare (1)	.17	.28	.85	1.00
Compare (2)	.04	.22	.75	1.00
Compare (3)	.13	.17	.80	1.00
Compare (4)	.17	.28	.90	.95
Compare (5)	.17	.11	.65	.75
Compare (6)	.00	.06	.35	.75

[a]From Riley, 1981.

Lindvall, 1979; LeBlanc, 1968; Nesher & Katriel, 1978; Vergnaud, 1981), but are slightly easier than compare (1) problems (Schell & Burns, 1962). Interestingly, children in the Carpenter *et al.* study performed relatively well on combine (2) and compare (1) problems for reasons we will discuss later.

Another source of evidence for the influence of semantic structure on problem difficulty comes from children's solution procedures with blocks. Carpenter *et al.* (1981) report that the dominant factor in determining the children's solu-

Table 4.6
Proportions of Kindergartners Who Performed Correctly[a]

Problem type	With objects
Change (1)	.89
Change (2)	.91
Change (3)	.08
Change (4)	.64
Change (5)	.32
Change (6)	.12
Combine (1)	.83
Combine (2)	18

[a]From Tamburino, 1980.

tion strategy was the structure of the problem. For example, change (2), change (3), and compare (1) all require the child to find the difference between the two numbers given in the problem; however, the strategies children used to solve each of these problems were quite different. Almost all children used a subtraction strategy (separating or counting down) to solve change (2). For change (3) almost all children used an addition strategy (adding on or counting up). For compare (1) a matching strategy was frequently used.

The final source of evidence we present for the influence of semantic structure comes from studies showing children's lack of reliance on formal arithmetic in solving word problems. There are data showing that very young children can solve some word problems before they have received any formal introduction to the syntax of arithmetic (e.g., Buckingham & MacLatchy, 1930; Carpenter *et al.*, 1981; Carpenter & Moser, 1981; Ibarra & Lindvall, 1979). There also is evidence that translating simple word problems into equations is not a necessary, or even usual, step in the solution processes of most children who have studied the formal notation of arithmetic. Second-grade children sometimes find it difficult or impossible to write equations for problems they have already solved (Lindvall & Ibarra, 1980a; Riley, 1981). Carpenter (1980) found that about one-fourth of first-graders solved the problem before they wrote a number sentence, in spite of instructions to the contrary. Together these studies suggest that children base solutions on an understanding of the semantic relations in the problem situation and do not need to employ standard, written procedures.

We have presented three main kinds of evidence for the influence of a problem's semantic structure on children's solutions to word problems. As mentioned earlier, we believe the various semantic structures correspond to specific concepts—concepts of quantitative change, equalization, combination, and comparison. On the basis of the aforementioned findings, it might be tempting to speculate that these four concepts emerge at different times in cognitive development. For example, at a certain age, a specific child might have the concepts of change and combination, but not the concept of comparison. The findings we present next suggest that is too simplistic.

Identity of the Unknown Quantity. The main source of evidence that these concepts are not acquired in a sequential, all-or-none fashion comes from studies showing that problems having the same semantic structure also vary in difficulty.

Referring again to the change problems in Tables 4.4, 4.5, and 4.6, children had no difficulty solving change problems when the start and change amounts were given and they were asked for the result. Even preschool children can solve these problems (e.g., Buckingham & MacLatchy, 1930; Hebbeler, 1977). However, many kindergartners and first-graders had difficulty if the start and the result were given and they were asked to find the amount of change. Problems like (5) and (6), where the result and change were given with the start set unknown, were difficult at all grade levels (see also, Hiebert, 1981; Lindvall &

Ibarra, 1980a; Vergnaud, 1981)—even more difficult than the combine (2) and compare (1) problems previously discussed.

As with change problems, the difficulty of combine and compare problems also varies depending on which value in the problem is unknown. Combine (2) problems in which one of the subsets is unknown are significantly more difficult than combine (1) problems in which the two subsets are known and the problem solver is asked to determine their combined value. Compare problems (5) and (6) in which the referent is unknown are more difficult than any of the other compare problems.

Clearly we must consider more than just a problem's semantic structure in our effort to understand the problem-solving skills of children at different ages. Specific features within each semantic structure, like the identity of the unknown quantity, must also be taken into account.

In summary, word problems differ both in the semantic relations used to describe a particular problem situation and in the identity of the quantity that is left unknown. The resulting problem types have been related to fairly systematic differences in children's performance at various grade levels. Some problems are relatively easy for preschoolers, whereas other problems remain difficult for many third-graders, even when concrete aids are made available. However, simply identifying which problems are more difficult than others tells us little about why they are difficult. In the following sections we present a theoretical analysis that has attempted to relate differences in performance on word problems to the knowledge children have available at different ages.

A THEORY OF THE KNOWLEDGE REQUIRED TO SOLVE WORD PROBLEMS

In this section we describe the current version of a theoretical analysis of word problem solving. The analysis is in the form of computer simulation models that solve word problems like the ones in Table 4.3. The conceptual knowledge and procedures represented in these models represent specific hypotheses about the knowledge required to solve word problems. The categories of knowledge we propose are similar to those found in other analyses (e.g., Fuson, 1979; Nesher, 1981; Vergnaud, 1981) that distinguish between semantics of problems and the semantics of addition and subtraction operations. In our analysis, we distinguish three main kinds of knowledge during problem solving: (*a*) problem schemata for understanding the various semantic relations discussed earlier, (*b*) action schemata for representing the model's knowledge about actions involved in problem solutions; and (*c*) strategic knowledge for planning solutions to problems. When a model is given a word problem to solve, it uses its knowledge of problem schemata to represent the particular problem situation being described.

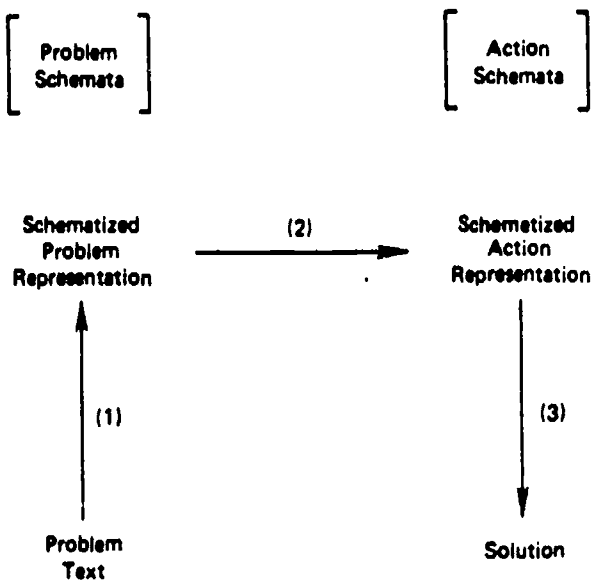

Figure 4.1. Framework of a model of problem understanding and solution. Arrows represent processes of (1) comprehension, (2) mapping from conceptual relations to quantitative procedures. and (3) execution of procedures.

The model's planning procedures then use action schemata to generate a solution to the problem. The general framework for this solution process is shown in Figure 4.1.

Problem Schemata

Our use of schemata is similar to the uses of that term in recent theories of language understanding (e.g., Anderson, 1976; Norman & Rumelhart, 1975; Schank & Abelson, 1977). In these theories, schemata have been used to organize the information in a sentence or story and to expand the representation of the message to include components that were not explicitly mentioned, but are nevertheless required to make the representation coherent and complete. Similarly, we view the process of understanding a word problem as fitting the components of the problem into a coherent structure.

The analysis that we have developed proposes three main types of problem schemata for understanding simple change, combine, and compare word problems. The representations have the form of semantic network structures consisting of elements and relations between those elements. For example, Figure 4.2 shows a representation of change (2). The representation has three main components. First, there is an initial quantity that represents the start set (1) of Joe's eight marbles. Second, there is some event that causes a change, in this case a

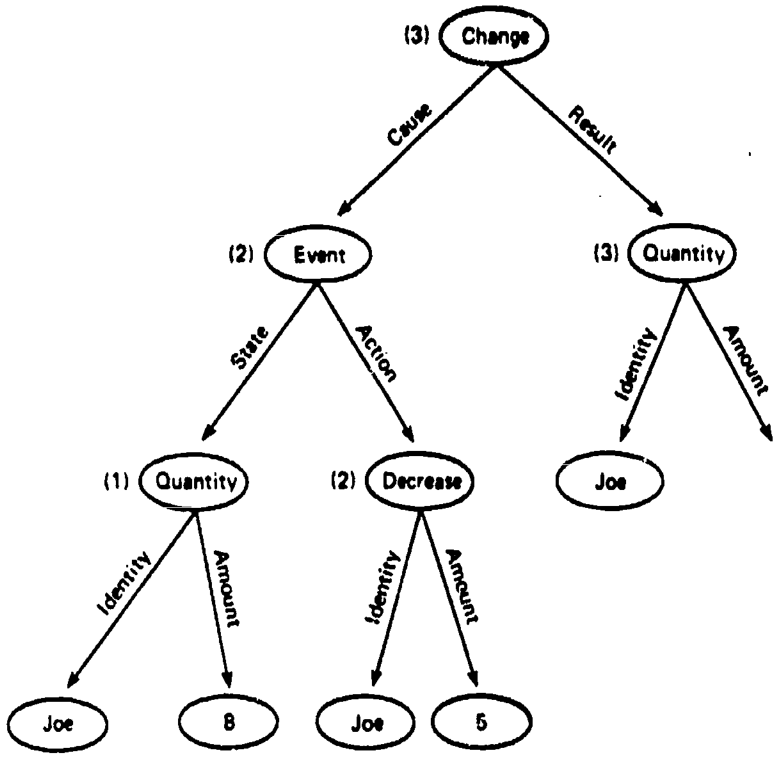

Figure 4.2. Schematized representation of change (2).

decrease, in the start set; the amount of this change is called the change set (2). The result of this change is represented as an unknown final quantity of marbles, or the result set (3).

The model that correctly solves all the change problems builds the first component (1) when it receives *Joe had 8 marbles.* When the sentence *Then he gave 5 marbles to Tom* is received, the model infers that the problem is about a change and constructs the rest of the structure in Figure 4.2, explicitly indicating that it expects to hear about a result. Finally, when the question *How many marbles does Joe have now?* is received, the model understands this as a request to determine the amount of the result set. At this point the model refers to its knowledge of action schemata and planning procedures.

Action Schemata

Once a model has represented a problem situation, it must have some way of relating this representation to its problem-solving procedures. We believe this requires a second type of schema that represents knowledge about actions used in planning solutions to problems. These action schemata are associated with the problem representation during problem solving and mediate the choice of an action or operation to solve the problem. Examples of these action schemata are shown in Figure 4.3.

The organization of action schemata into prerequisites and consequences is patterned after Sacerdoti's (1977) model of planning in problem solving called NOAH. Prerequisites are conditions in the problem situation that must be present for an action to be performed. Consequences are those conditions that will exist in the problem situation once the action is carried out. We should point out that the format we have used to illustrate these action schemata is simply a matter of convenience; they just as easily could have been drawn as network structures similar to the change schema in Figure 4.2.

Referring to Figure 4.3, make-set is the schema associated with the model's action for building a set X with an amount *N*. The prerequisite of this action is that the model begin with an empty set: The availability of objects and locations on which to put them is assumed. The consequence is a set X containing *N* objects. Thus, the action make-set (Joe, 8) would result in a set of eight objects belonging to Joe. Other action schemata include put-in, take-out, and count-all. Put-in adds *N* objects to an existing set X. Its prerequisite is that X exists with an amount *M*, and its consequence is that X now contains *N* more objects. Take-out removes *N* objects from set X. Count-all is an action that determines the number of members of set X by counting all the members of X (as opposed to counting on from a subset of X). In the diagram, all the action schemata have labels—"make-set," "take-out," etc.—although a child could have schemata without labels.

MAKE-SET: X, N
  Prerequisite: X,
  Consequence: X, N

PUT-IN: X, N
  Prerequisite: X
  Consequence: X, N More

TAKE-OUT: X, N
  Prerequisite: X
  Consequence: X, N Less

COUNT-ALL: X, ?
  Prerequisite: X
  Consequence: X, N

Figure 4.3. Action schemata.

COMPARE X, More, Y
(X has N1, Y has N2)

1. MAKE-SET: X, N1

2. MAKE-SET: Y, N2

3. MATCH: Z in X match Y
   a. Form subset of X equal to Y
   b. Z = matching subset

4. GET-REMAIN: W in X from Z
   a. SEPARATE X: Z and remainder
   b. W = remainder

5. COUNT-ALL: W

Figure 4.4. "Compare" action schema.

Action schemata are organized into different levels to enable efficient planning. This is because some schemata are actually composites of several other schemata and are therefore more global. For example, Figure 4.4 shows a global schema called *compare* that determines the amount of difference between the numbers of members of two sets. This particular procedure depends on counting, rather than subtraction of numbers. However, an alternative assumption might be that compare's subschemata involve subtracting numbers. This change would not alter compare's role in the knowledge structure; it would merely modify the way in which compare would be executed. More will be said about how compare is executed in the next section: the focus here is on its general structure.

Compare is composed of four specific actions—make-set, match, get-remain, and count-all—and is therefore more global than they are; get-remain is in turn more global than its component action, separate. Although not shown explicitly, each of these action schemata also contains information about its corresponding prerequisites and consequences. As will be described, the organization of action schemata into these levels of generality allows the model to consider global solution methods before taking into account all the details of implementing any particular method.

Strategic Knowledge

In addition to problem schemata and action schemata, the models also have strategic knowledge for planning solutions to problems. Strategic knowledge is represented by production rules organized in a way that permits top-down planning of the kind studied by Sacerdoti (1977) and implemented in many current models of human problem solving (e.g., Chi, Feltovich, & Glaser, 1981; Greeno, Magone, & Chaiklin, 1979; Polson, Atwood, Jeffries, & Turner, 1981). Knowledge for planning involves the action schemata just discussed as well as simple associations between goals and procedures relevant to attaining those goals. When a model is given a problem to solve, it sets a goal either to make the external situation correspond to some given information or to obtain some requested information. The model then uses its knowledge about actions to plan how to achieve that goal in the current problem situation.

Planning involves working out a solution from the top down, that is, choosing a general approach (e.g., match) to a problem, then deciding about actions that are somewhat more specific, and only then working out the details. After a plan is selected, the model tries to carry out the actions associated with that plan in an attempt to achieve the current goal. If the action prerequisites are satisfied in the current problem situation, then the plan can be carried out immediately. If not, some further work must be done, and this requires setting one or more subgoals. The model's planning knowledge includes knowledge of subgoals that are useful in achieving a plan. Once generated, the new subgoal replaces the earlier goal, but the previous goal is stored in memory to be retrieved when the new subgoal is either achieved or is determined to be impossible. This process of setting goals and subgoals and planning how to achieve them continues until the problem is solved.

THE LOCUS OF IMPROVEMENT IN PROBLEM-SOLVING SKILL

We have identified three main components of knowledge needed for successful performance in the domain of word problems. Children's difficulties in solving some problems may be caused by the absence of one or more of these components of knowledge. In the next section we will present an analysis of different levels of children's problem-solving skill in which the major factor is assumed to be acquisition of an improved ability to represent problem information. In this section, we present some evidence for this hypothesis, comparing it with an alternative hypothesis that a main source of children's difficulty is their lack of knowledge about the actions required to solve certain word problems.

We discuss evidence from studies of compare problems and combine problems. The gist of the findings is that problems that are difficult in their usual wording are made much easier by changing the wording in appropriate ways. These findings are similar to those obtained in recent studies of class inclusion (Dean *et al.*, 1981; Markman, 1973; Trabasso *et al.*, 1978) and, as in those studies, argue against the hypothesis that children lack the action schemata required to solve the problem. In our analysis the main locus of children's improvement in problem-solving skill is in the acquisition of schemata for understanding the problem in a way that relates it to already available action schemata.

Compare Problems

We present a brief summary of results obtained by Hudson (1980) in a study of young children's performance on compare problems of the kind called compare (1) in Table 4.3. Recall that problems involving comparison are difficult, at least for young children. One possibility is that children lack the action schemata required to plan a solution to the problem. Indeed, the compare procedure is fairly complex, as was shown in Figure 4.4. It involves first using make-set to create two sets to be compared. Then match finds a subset Z of the larger set X whose elements are in one-to-one correspondence with the elements of the smaller set Y. The procedure get-remain then uses separate to identify the difference between the two sets. Separate removes all the elements in X that are not part of Z, and identifies these elements as the remainder W. Finally, count-all determines the number of members of W. It is tempting to conclude that younger children have not yet acquired this relatively complex procedure and that this lack of a problem-solving method is responsible for their poor performance on problems involving comparisons.

This interpretation is contradicted by data collected by Hudson (1980), who presented problems of the kind shown in Figure 4.5 to 12 nursery-school, 24 kindergarten, and 28 first-grade children. Two different questions were asked. One was the usual comparative question, in this case, *How many more birds than worms are there?* The other question was an alternative that Hudson devised: *Suppose the birds all race over and each one tries to get a worm! Will every bird get a worm? . . . How many birds won't get a worm?*

The results were striking, as shown in Table 4.7. Hudson gave eight questions of each type, and the proportions here are for children who gave six or more correct responses. A correct response was the difference between the sets—for example, *One more bird than worms*, or *One bird won't get a worm*. The most

Table 4.7
Proportions of Children with Consistent Correct Responses

Grade	How many more?	How many won't get?
Nursery school	.17	.83
Kindergarten	.25	.96
First	.64	1.00

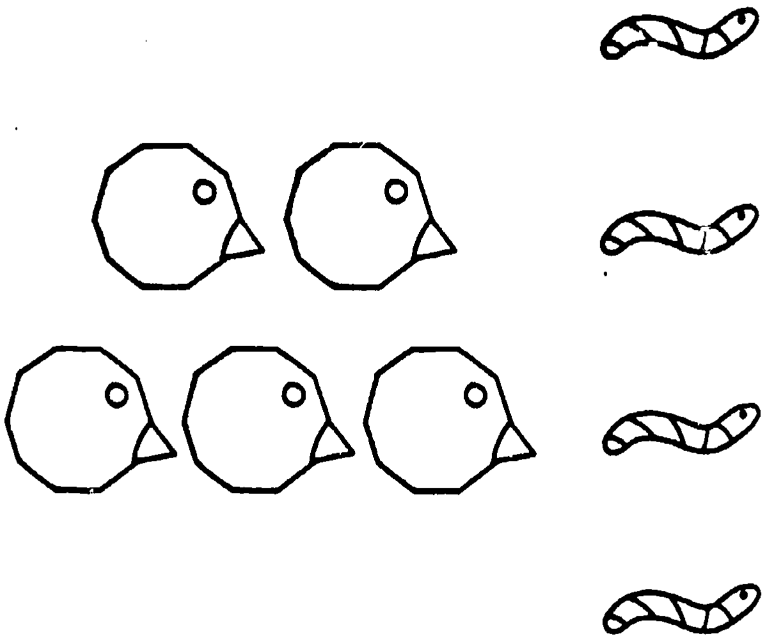

Figure 4.5. Example of problems used in Hudson's study.

frequent incorr . response was the number in the larger set—for example, *Five*, or *Five bird* . Another frequent error was to give both set sizes—for example, *Five birds and four worms*. Very few of the nursery-school or kindergarten children answered the *How many more?* questions by giving the difference between the sets. However, nearly all the children of all three ages answered the *How many won't get?* questions correctly.

Thus, Hudson's data do not support the hypothesis that children lack a procedure for finding a set difference; to answer the *Won't get* questions, children used a match procedure to form a correspondence between the two sets and to count the remaining subset of the larger set.

Combine Problems

Similar effects of rewordings have been obtained for combine (2) problems like the one in Table 4.1:

Joe and Tom have 8 marbles altogether.
Joe has 5 marbles.
How many marbles does Tom have?

As noted earlier, these problems are quite difficult for young children even though the solution procedure involves three relatively simple actions: make-set, take-out, and count-all. Here make-set counts out a set X of eight blocks to represent the marbles that Joe and Tom have altogether; take-out removes five blocks from X to represent Joe's marbles; and count-all counts the number of blocks remaining in X to determine how many marbles Tom has. We can assume that most children have this solution procedure readily available since they use it to solve change (2) problems, which also require subtraction. Furthermore, as in the Hudson study, it appears that slight rewordings of the combine (2) problem enable many children to solve it correctly using this procedure. Carpenter *et al.* (1981) report that 33/43 first-graders did in fact solve combine (2) problems correctly when asked,

> *There are 6 children on the playground.*
> *4 are boys and **the rest** are girls.*
> *How many girls are on the playground?*

Lindvall and Ibarra (1980b) found that combine (2) problems like,

> *Together, Tom and Joe have 8 apples.*
> *Three **of these** apples belong to Tom.*
> *How many **of them** belong to Joe?*

are significantly easier for kindergarten children than the combine (2) problems like the one in Table 4.1. We will return to the specific nature of the facilitation effect of such rewordings later. The point to be made here is that once again we cannot attribute children's problem-solving difficulties to a deficiency involving problem-solving actions. Instead, we hypothesize that acquisition of skill is primarily an improvement in children's ability to understand problems—that is, in their ability to represent the relationships among quantities described in problem situations in a way that relates to available solution procedures.

STAGES OF CONCEPTUAL KNOWLEDGE

In this section we present specific hypotheses about the nature of conceptual development that results in improved skill in solving arithmetic word problems. The hypotheses are based on data obtained by Riley (1981) in a developmental study of performance on word problems like the change, combine, and compare types in Table 4.3. Riley designed computational models of word problem solving that include processes of representing the problem information. The models were intended to simulate children's performance at different levels of skill. Levels of skill corresponded to different patterns of performance typical of children at different ages. Within each of the three semantic

Table 4.8
Patterns of Performance on Change Problems

Example of problems	Levels of performance 1	2	3
Result unknown			
1. Joe had 3 marbles. Then Tom gave him 5 more marbles. How many marbles does Joe have now?	+	+	+
2. Joe had 8 marbles. Then he gave 5 marbles to Tom. How many marbles does Joe have now?	+	+	+
Change unknown			
3. Joe had 3 marbles. Then Tom gave him some more marbles. Now Joe has 8 marbles. How many marbles did Tom give him?	"8"	+	+
4. Joe had 8 marbles. Then he gave some marbles to Tom. Now Joe has 3 marbles. How many marbles did he give to Tom?	+	+	+
Start unknown			
5. Joe had some marbles. Then Tom gave him 5 more marbles. Now Joe has 8 marbles. How many marbles did Joe have in the beginning?	"5"	"5"	+
6. Joe had some marbles. Then he gave 5 marbles to Tom. Now Joe has 3 marbles. How many marbles did Joe have in the beginning?	NA	NA	+

Table 4.9
Proportions of Patterns Consistent with Models

	Level	Grade K	1	2	3
Change problems	1a	.04	.22	0	0
	1b	.30	.17	0	0
	2	.39	.17	.10	.05
	3a	.09	.17	.30	.15
	3b	09	.22	.60	.80
	Residual	.09	.05	0	0

categories included in the study, three levels of skill were identified, each associated with a distinctive pattern of correct responses and errors on the six problems of the type.

The patterns of performance for the change problems are shown in Table 4.8. A "+" means the child answered correctly, "NA" indicates no answer, and numbers indicate the characteristic error for that problem; for example, on change (3) the specific number that was given was the result set, whatever that number was. (Different children solved these problems with different numbers involved.) Thus, reading vertically, a child at Level 1 would answer change problems 1, 2, and 4 correctly, respond with the result set for change (3) problems, give the change set for change (5), and no reply for change (6).

Table 4.9 shows the proportions of children in each of four grades whose performance was consistent with the identified patterns. These data are from performance when blocks were available. Level 1a children responded correctly on change problems 1 and 2 only. Level 1b children responded correctly on change problems 1, 2, and 4, as specified by the pattern for Level 1 in Table 4.8. Children who responded correctly on all problems except change (5) *or* change (6) were classified as being in Level 3a in Table 4.9. A child classified at Level 3b was correct on all problems. The proportions of children in the residual columns of Table 4.9 are those whose performance was not consistent with any of the patterns.

We should point out that some of the children identified in the residual column actually did respond consistently, but in ways not accounted for by our models. For example, there were a few children who consistently put out an arbitrary number of blocks for problem statements involving the word *some*. This can lead to predictable confusions when the arbitrary set does not correspond to the actual answer, but at the same time occasionally allows for fortuitous correct responses to difficult problems like change (5). A more detailed discussion of this behavior is provided by Tamburino (1980) and Lindvall and Tamburino (1981) in their account of why change (5) problems were easier than change (3) problems for some children in Tamburino's study (see Table 4.6), although the reverse is usually true in the literature.

Riley designed models to simulate each of the performance patterns in Table 4.8. The knowledge structures and procedures represented in these models represent hypotheses about the kinds of information-processing components needed to explain the different patterns of performance on the various problems. The models that simulate the three levels of change problem performance have been implemented, and we will describe their characteristics in some detail. The models for combine and compare performance have been designed, but not implemented. We will summarize their main features.

Development of Schemata for Change Problems

Riley's analysis of processes for change problems consists of three models that simulate the different levels of children's performance on these problems. That is, each model solves the six change problems in a way that leads to one of the different patterns of performance that Riley identified. All of the models employ the same general approach to problem solving, as was shown in Figure 4.1. That is, problem schemata are used to represent the current problem situation, and knowledge of action schemata and planning procedures are used to determine a solution. The main differences between the models relate to the ways in which information is represented and the ways in which quantitative information is manipulated. Models with more detailed representational schemata and more sophisticated action schemata represent the more advanced levels of problem-solving skill. Model (1) understands quantitative relations by means of a simple schema that limits its representations of change problems to the external displays of blocks. Model (2) has a change schema for maintaining an internal representation of increases and decreases in the sets of blocks it manipulates; the process of building this representation is still relatively "bottom-up" in the sense it depends upon the external display of objects. Model (3) also has a change schema for representing features internally, but can use its change schema in a more "top-down" way than model (2) to direct understanding independent of the external display of blocks. Models (2) and (3) also have a richer set of action schemata for producing and manipulating quantitative information and a richer understanding of certain relations between numbers; for example, model (3) has an understanding of part–whole relations. We will discuss the relationship between these different kinds of knowledge during performance and development in the next section.

MODEL (1)

The lowest level of performance on the change problems is represented by model (1). The knowledge that model (1) has available for problem solving includes the action schemata in Figures 4.3 and 4.4, procedures for planning in the way described on pp. 169–170, and the simple schema for representing quantitative information shown in Figure 4.6. This knowledge is sufficient to solve change problems (1), (2), and (4), but leads to predictable errors on change problems (3), (5), and (6). The first three problems share two main characteristics: The actions required to solve the problem can be selected on the basis of local problem features, and the solution set is available for direct inspection at the time the question is asked. For example, solving change (4) involves reducing Joe's initial set of eight blocks to three blocks in response to *Now Joe has 3 marbles*, with the effect that the change and result sets are now physically

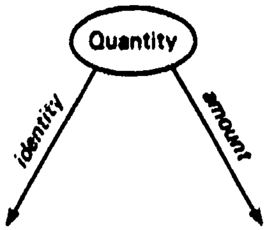

Figure 4.6. Model (1)'s schema for representing quantitative information.

separate. Thus, the model can easily identify the change set when asked *How many marbles did Joe give Tom?* and responds correctly even though it did not keep a memory record of the structural relationships in the problem.

Now consider how model (1) solves change (3) in which the solution set is not available for direct inspection. The model has no difficulty carrying out the correct procedures to solve the problem; failure is due to problem representation. The model counts out three blocks in response to *Joe has 3 marbles*, and uses the simple schema in Figure 4.6 to represent these blocks as a quantity whose identity is Joe and amount is three.

Next the model attempts to put in more blocks in response to *Then Tom gave him some more marbles*. But, since it does not yet know exactly how many to put in, it does nothing and therefore does not change its representation of the problem situation. The next sentence, *Now Joe has 8 marbles*, results in a goal to create a set of eight blocks. The model counts the three blocks, then continues to add in additional blocks until there are eight blocks total. The resulting set is represented as a quantity whose identity is Joe and amount is eight.

The difficulty arises when model (1) is asked to determine the number of marbles that were added in to change the initial set. Since the start set and change set are not distinguished in model (1)'s final representation, the question is simply interpreted as a request to determine the total number of marbles in the set. It therefore counts all the marbles and incorrectly answers *Eight*. To solve change (3) problems correctly, the child would have to represent, in [illegible], additional information about the sets in the problem. This probably accounts for why change (3) problems are generally more difficult for young children than change (4) problems, even though both problems involve an unkonwn change set (see also Hiebert, 1981; Tamburino, 1980).

The idea that children's failure on change (3) is due to a failure to represent the separate start and change sets is consistent with several findings. Many studies have shown that even when children have little difficulty selecting and carrying out the appropriate actions to solve change (3) problems using blocks, many of them give the value of the result set as their answer (e.g., Riley, 1981; Tamburino, 1980). Another kind of evidence comes from a study by Harvey (1976). He successfully trained first-graders to solve similar problems using external partitions to distinguish the two sets. Children initially solved the problem using a single paper plate with a partition: The start set was placed on one side of the partition and the change set was placed on the other side. The next step involved using two paper plates. Finally, children solved the problem correctly using a single plate with no partition. The success of Harvey's training procedure suggests that, prior to training, a main part of children's difficulty in solving these problems was due to a failure to distinguish the start and change sets.

MODEL (2)

The main difference between model (1) and model (2) is that model (2) represents internally additional information about the problem situation. This involves a schema for change problems (Figure 4.2) in which there is a mental record kept of the structural role of each item of information. This additional structural information enables model (2) to give the correct answer to change (3) problems where model (1) failed.

Model (2)'s behavior in response to the first two sentences of change (3) is identical to model (1)'s. It simply counts out three blocks in response to *Joe has three marbles.* and represents this as a set belonging to Joe with an amount three. The model attempts to put in additional blocks in response to *Then Tom gave him some more marbles.* but since no amount is mentioned, it does nothing. At this point model (1)'s and model(2)'s understanding of the problem are identical. The difference between the two models becomes evident from the way model (2) responds to the next input: *Now Joe has 8 marbles.* In addition to simply increasing the existing set until there are a total of eight blocks, model (2) also identifies the set of eight as the result set that was produced by increasing the start set of three blocks by some unknown amount. The resulting problem representation is shown in Figure 4.7. Thus when model (2) is asked, *How many marbles did Tom give Joe,* it can identify the separate change set in its problem representation and determine the set's numerosity by counting all but three of the blocks.

Although model (2) has a more complete internal representation than model (1), it still lacks an important ability for top-down processing in its representation of problem information. This is seen in model (2)'s performance on change (5) problems. Recall that children at this level cannot solve this problem and give *Five* (the value of the change set) as their most frequent incorrect response. The model receives the first sentence, *Joe had some marbles,* and attempts to create a set of blocks to represent these marbles, but realizes it does yet know exactly how many Joe has. Therefore the model does nothing but simply remember the fact that it heard about Joe. The second sentence results in the model putting out five blocks for Joe. However, because the model failed to represent explicitly the

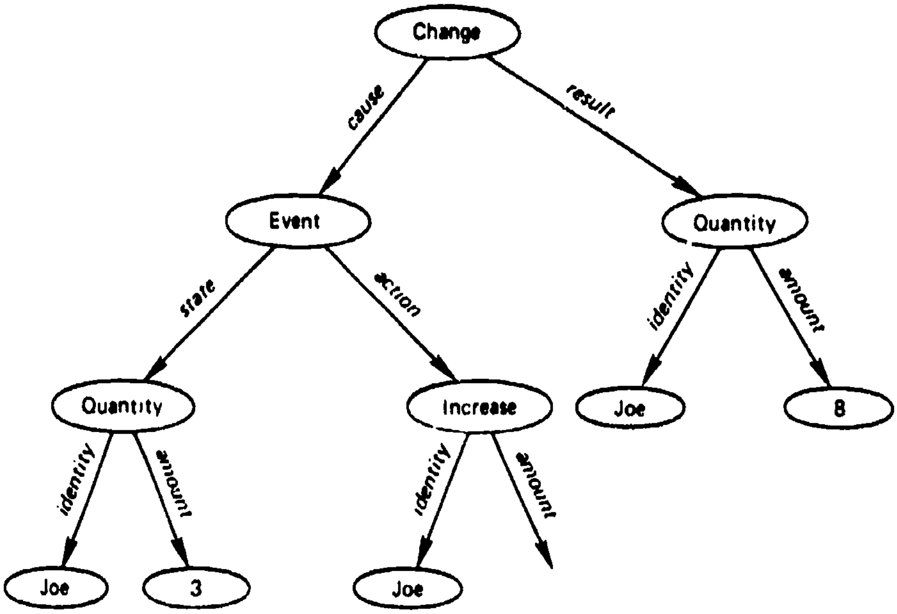

Figure 4.7. Model (2)'s representation of change (3) before determining the amount of the change set.

unknown start set, the additional five blocks are not represented as a change in the initial set, but simply as a set of five belonging to Joe. This means that when the model receives *Now Joe has 8 marbles*, the problem situation is the same as that in change (3). The model increases the set of five to eight, resulting in a representation identical to the one that was shown in Figure 4.7. Notice that the set of five blocks is identified as the start set, although it is actually the change set in the original problem. This accounts for why model (2) answers *Five* when asked *How many marbles did Joe have in the beginning?*

MODEL (3)

Model (3), like model (2), has a change schema for maintaining a structural representation of the problem situation. However, unlike model (2), model (3) can use its change schema in a top-down fashion to build a representation of the entire problem before actually solving it. This permits model (3) to operate on a quantity whose value is unknown, as required in change problems (5) and (6).

Model (3)'s ability to solve change problems (5) and (6) also involves the schema for representing part–whole relations shown in Figure 4.8. The reason for this will become clearer as we continue, but basically when the start set is unknown, the action required to solve the problem is not immediately available from the initial problem representation. We hypothesize that these problems are best understood in terms of the part–whole relations between the quantities.

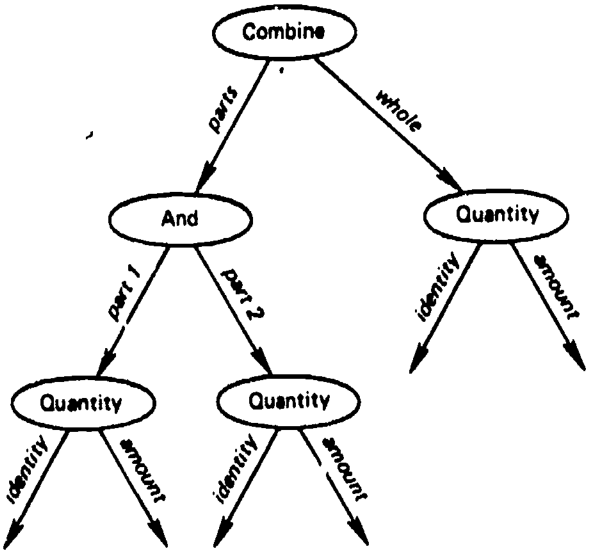

Figure 4.8. Model (3)'s schema for representing part–whole relations between quantities.

The implications of model (3)'s additional conceptual knowledge become apparent from the way it solves change (5). The model's understanding of the sentence, *Joe had some marbles*, is represented as a quantity whose identity is Joe and amount is unknown. *Then Tom gave him 5 more marbles* is understood as an increase in the initial quantity, causing the rest of the change schema to be instantiated (Figure 4.9), and the model puts out five blocks for Joe. Thus, model (3)'s representation of change (5) maintains a record of Joe's five marbles as the amount of change in the as-yet-unknown start set. This is in contrast to model (2), where no record was kept of the unknown start set, with the eventual consequence that the change set of five blocks was represented incorrectly as the start set.

Model (3) represents the third sentence of change (5), *Now Joe has 8 marbles*, as the amount of the result set and increases the existing set until it contains eight blocks. When model (3) is asked, *How many marbles did Joe have in the beginning?* it sets the goal of determining the value of the start set, but has not direct referent for this set in its blocks representation as was the case in the change (3) example (see the section on model (2)). We therefore hypothesize that identifying the appropriate action requires additional inferences about the part–whole relations in the problem, as shown in Figure 4.10.

Since the direction of change is an increase, the model infers that both the start and change sets are parts of the result set. On the basis of this inference, the model determines that the start set must consist of the additional blocks that were added to the change set to make a total of eight blocks. The model then counts these additional blocks and anwers *Three*.

Actually, the same basic solution to change (5) could just as easily have been represented by an alternative blocks procedure—one that children also frequently use (Carpenter, 1980; Riley, 1981). That is, model (3) could have delayed putting out any blocks until it inferred the part–whole relations between the quantities in the problem. Then the model could have put out the eight blocks first, used separate to remove five of the blocks, and finally identified the remaining three blocks as the answer.

Considering either blocks procedure, model (3)'s solution to change (5) suggests an alternative explanation for model (2)'s failure on change problems in which the start set is unknown. It is possible that some children did in fact use their change schema to represent correctly the problem situation with the start set unknown, but lacked the part–whole schema required to infer the appropriate operation.

The proposal that children require an understanding of the part–whole relation to solve change problems (5) and (6) is supported in Riley's study in which few children in any age group correctly solved these two problem types without first being able to solve combine problems with one of the subsets unknown. (We assume subset unknown problems also require an understanding of part–whole relations.) Furthermore, there is evidence suggesting that once children understand part–whole relations, they can use this knowledge to understand all change problems, even though our analyses have shown that knowledge of these relations is not required to solve change problems (1), (2), (3), and (4). Carpenter

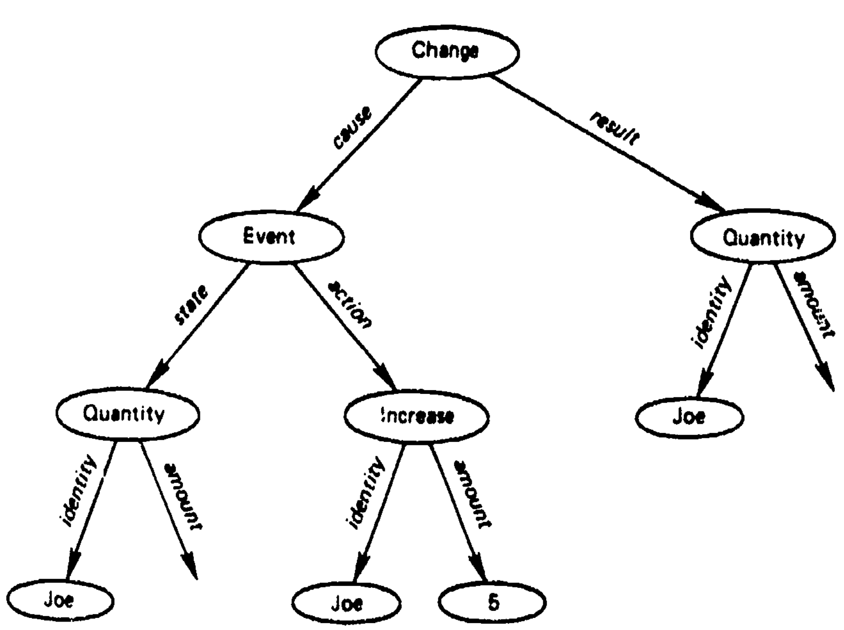

Figure 4.9. Model (3)'s representation of change (5) before determining the amount of the start and result sets.

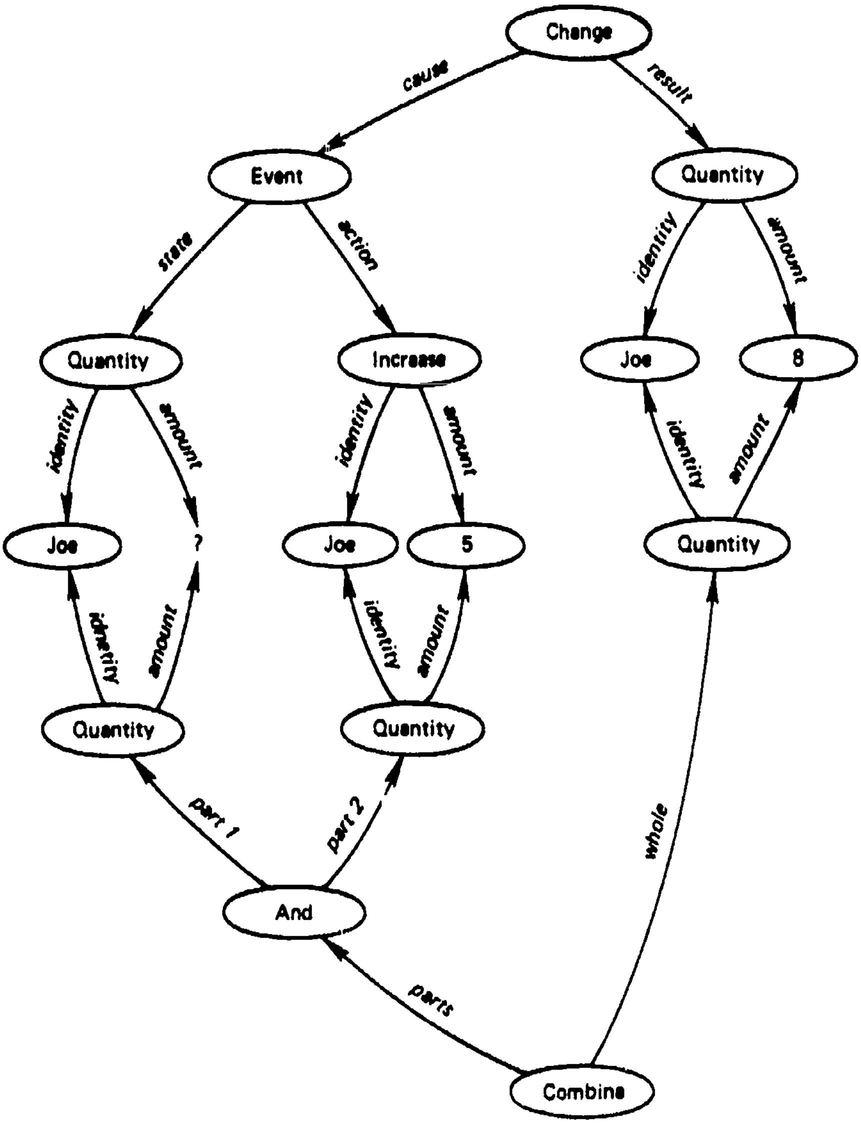

Figure 4.10. Model (3)'s inference about the part–whole relations in change (5).

(1980) reports a study of first-graders' strategies in solving word problems before and after receiving instruction in analyzing the part–whole relations of these problems. Prior to instruction, children's solution processes modeled the actions or relationships described in the problem. That is, children used separate, count-on, and match to solve change (2), change (3), and compare (1) problems, respectively, even though all of these problems involve finding the difference between two quantities. After instruction, children generally used separate for all subtraction problems. Apparently, these children were basing their solutions on the part–whole relations in the problems. A similar trend is indicated by results from a study reported by Zweng, Gerarghty, and Turner (1979) who found that the majority of third-graders and almost all the fourth-graders used subtraction to solve problems like change (3).

Models for Combine and Compare Problems

Riley's simulation models to explain children's performance on combine and compare problems are similar in their general features to those we have just described. Riley assumes that, at the lowest level, the child's representations of problems are limited to the external displays of blocks; at an intermediate level there are schemata for representing, internally, additional information about the relationships between quantities; and at the most advanced level, schemata are available that direct problem representations and solutions in a more top-down manner. The results for these models were similar to those obtained for the change problems, although the proportions of children not consistent with any of the models was somewhat greater for the compare problem set, as shown in Table 4.10. Overall, the models seem to provide a reasonable first approximation to the nature of the increased skill that children showed in solving these problems. As we will briefly describe, the models also provide a theoretical framework for integrating the various findings related to children's performance on combine and compare problems.

Recall that compare (1) problems are usually quite difficult for kindergarten and first-grade children. In Riley's compare models, failure is associated with the lack of a schema for understanding the problem situation in a way that makes

Table 4.10
Proportions of Patterns Consistent with Models

		Grade			
	Level	K	1	2	3
Combine problems	1	.52	.28	.05	0
	2	.22	.34	.10	0
	3	.13	.23	.85	.95
	Residual	.13	.17	0	.05
Compare problems	1	.52	.61	.05	0
	2	.04	.06	.40	.45
	3	0	.06	.20	.50
	Residual	.[illegible]4	.27	.35	.05

contact with the model's available action schemata—in this case the match action schema. However, with instruction, first-graders apparently have no difficulty learning to apply match to solve compare (1) problems (Carpenter, 1980; Marshall, 1976). Carpenter (1980) reports a study in which children who had received such instruction were tested on a variety of problems once in February and again in May. Compare (1) problems were relatively easy for these children at both testing times; correct proportions were .67 and .74, respectively. At the same time, compare (3) problems remained difficult at both interview (.28 and .46). In February, errors on compare (3) consisted of responding with one of the givens. (Similar errors have been reported by Gibb, 1956; Hudson, 1980; Marshall, 1976; Riley, 1981; and Shores and Underhill, 1976.) In May, errors were almost equally divided between responding with one of the givens and choosing the wrong operation. This suggests that the children had learned to associate their match procedure with the *how many more than* question in the same way that children associated *giving* and *taking* with the actions put-in and take-out. But as with put-in and take-out, the children had not yet learned to represent the important relationships between the sets involved in the procedure and therefore failed to generalize the instruction to other compare problems (e.g., compare [3]). Furthermore, children did not acquire the entire compare schema at once but, as Carpenter *et al.* suggested, first focused on the difference relationships, as indicated by the increase in "wrong operation" errors for compare (3).

Riley's models propose a similar sequence of understanding to account for performance on combine problems. Even preschool children have little difficulty solving combine (1) problems. This does not, however, mean that solving these combine (1) problems involves any understanding of set inclusion or part–whole relations. Our lowest level combine model solves this problem by a simple association between the *how many altogether* question and its count-all action schema. In fact, it is this model's lack of understanding of part–whole relations that accounts for its failure on combine (2) problems. The more advanced models have a combine schema that allows them to infer the part—whole relation between, for example, the eight marbles that Joe and Tom have altogether in combine (2) and the five marbles that Joe has. These relations are not mentioned explicitly in the problem, and without this schema, children simply interpret each line of the problem separately as in change model (1) and have no way to infer the relation between the two sets. They therefore put out a set of eight blocks to represent the marbles that Joe and Tom have altogether and a separate set of five blocks to represent Tom's marbles. This leads to the incorrect response of *Eight* when asked how many marbles Tom has. The facilitation effect of rewordings like *of them* and *the rest* can be attributed to circumventing the need for a combine schema by making the source of Joe's marbles more explicit, allowing the child to remove the five from the set of eight.

Acquisition of Problem-Solving Procedures

Children's conceptual knowledge of the elements and relations in word problems also seems to be related to the acquisition of more sophisticated counting procedures. For example, model (2)'s change schema represents the model's understanding that changing a given quantity into a desired quantity involves either increasing or decreasing the given quantity by a specific amount. We therefore attribute a more sophisticated counting procedure to model (2) called count-on. Count-on allows the model to extend or count a set by beginning with the value of an existing set if it is known. Thus, if model (2) already has three blocks, but needs eight blocks, it can begin the count with three and simply keep adding in blocks until it gets to eight, instead of recounting the three as model (1) did with its count-all procedure.

Evidence for a relationship between the availability of the change schema and count-on comes from Steffe and Thompson (1981) who report a positive correlation between the ability to count-on and the ability to solve change (3) problems. Hiebert (1981) found that count-on (referred to in his study as add-on) was the procedure most frequently observed in first-graders' correct solutions to change (3) problems using blocks. The developmental relationship between the availability of count-on and the availability of the change schema will be considered in the discussion section.

Children's conceptual knowledge may also be related to the acquisition of the more efficient Min counting procedure (Groen & Resnick, 1977) in which the number counted-on changes from being the second addend given to being the smaller addend given. The mathematical property that allows this more efficient procedure is commutativity, which we believe corresponds to an implicit understanding of the part–whole relations between the addends a and b and their sum, c. That is, a and b are both parts of c, and therefore $a + b$ and $b + a$ are equavalent operations. It seems likely, therefore, that acquisition of the Min counting procedure would be related to children's understanding of combine (2) problems, as these problems require an understanding of part–whole relations. Fuson (1979) proposes the same basic idea by pointing out that the commutativity relation between a and b may vary with the addition problem type: Commutativity would seem to be less obvious when the roles played by the two numbers differ (as in change problems) than when the roles coincide (as in combine problems). Thus, although the sequential property of change addition problems may facilitate the transition from count-all to count-on, it may make commutativity less apparent. Fuson suggests that the transition to the more efficient Min procedure might be facilitated in the context of combine, rather than change, addition problems. It is clear that more empirical and theoretical work is required to clarify the relationship between conceptual and procedural knowledge in the development of problem-solving skill.

Conclusions

The models described in this section provide a detailed hypothesis about changes that occur in children's ability to understand relationships among quantities and to use their representations of these relationships to solve problems. We cannot claim uniqueness for the models that we have described here. Indeed, there are some redundant features in the models so that somewhat simpler accounts could be given to explain the observed improvements in skill. For example, model (2)'s ability to solve change (3) problems is related to the ability to construct a representation of the separate start and change sets and to the use of the count-on procedure. Even so, we are confident that the children's improvement in skill in these problems involves something along the lines of these models. The more skillful models are more accurate because they understand problems better. That is, their representations of problems include the relevant features of the problems more completely and in ways that lead to the choice of appropriate problem-solving actions.

RELATED ANALYSES OF CONCEPTUAL UNDERSTANDING IN PROBLEM SOLVING

We have argued that successful problem-solving performance by children depends on their understanding of certain concepts, which we have characterized as schemata used in representing information in problem situations. We want to avoid the impression that this point applies only to young children; it is equally true of adult problem solvers in domains much more complex than primary-grade arithmetic. In this section we briefly review findings from studies in two domains of intermediate complexity—algebra and physics—in which the central findings serve to emphasize the importance of conceptual knowledge in problem solving.

Word Problems in Algebra

High-school instruction in algebra usually includes solution of word problems, where the solution method presented involves translating the text into equations. The program, Student, developed by Bobrow (1968), solves algebra word problems using a method of translation into equations like the one presented in most instruction. Student uses little conceptual knowledge, focusing instead primarily on syntactic information to translate the English problem statement directly into a corresponding set of equations. It then solves the set of equations for the requested unknown. Table 4.11 presents an example of the kind of word problem Student was able to solve along with the simplified trace of

Table 4.11
Student's Direct Translation Strategy[a]

Problem

If the number of customers Tom gets is twice the number of advertisements he runs, and the number of advertisements is 45, what is the number of customers Tom gets?

Solution strategy

1. Partition the problem into phrases:
   If / the number of customers Tom gets / is twice /
   the number of advertisements he runs /, and /
   the number of advertisements he runs / is 45 /
   , what is / the number of customers Tom gets / ? /
2. Translate phrases into algebraic terms:

The number of customers Tom gets:	x
is:	=
twice:	2*
the number of advertisements he runs:	y
is	=
45:	45
what:	?

3. Organize algebraic terms into equations:
   $x = 2 * y$
   $y = 45$
   $? = x$
4. Simplify equations into single equation:
   $x = 2 \times 45$

Note. Asterisks stand for times.
[a]Adapted from Roman & Laudato, 1974.

Student's direct translation strategy. Notice how Student's solution relies almost entirely on syntactic information in the problem to guide the solution process with little understanding of the problem structure.

Although Student successfully solves many problems, a comparison of Student's performance with that of human students revealed limitations caused by its lack of conceptual knowledge. Paige and Simon (1966) and Hinsley, Hayes, and Simon (1977) found that whereas human performance was similar to Student's in important ways, humans also used a number of processes that Student did not have, but that corresponded to an understanding of the relations in the problem.

Paige and Simon (1966) noted that some human subjects proceeded, not by simply translating the verbal statements into algebraic equations (which was what they were requested to do), but by constructing a physical representation of the problem and then drawing information from the representation. Consider the following problem:

A board was sawed into two pieces. One piece was two-thirds as long as the whole board and was exceeded in length by the second piece by 4 feet. How long was the board before it was cut?

If one solves the problem by the direct translation strategy outlined in Table 4.11, an equation is obtained that yields a negative number for the length of the original board, a physically impossible result. Some students noticed this before setting up any equations, indicating that they had not proceeded by direct translation, but had instead constructed a physical representation of the situation described.

Word Problems in Physics

Important contributions regarding the influence of conceptual knowledge in problem solving have been made by comparing experts' and novices' solution procedures. Larkin, McDermott, Simon, and Simon (1979) and Simon and Simon (1978) found that skilled physics problem solvers work from elaborated representations of the problem, rather than directly from the problem description. These representations often include diagrams that make certain relationships and constraints highly salient. In effect, experts have more conceptual knowledge about problem situations than novices, and it is this conceptual knowledge that guides their more effective and efficient solutions. Although experts and novices may be observed to break problems into subparts or set subgoals to deal with difficulties, these procedures are apparently executed by experts with an emphasis on the problem "Gestalt," whereas novices tend to solve problems on the basis of more local problem features. Support for this idea comes from a study by Chi, Feltovitch, and Glaser (981) in which expert and novice physics subjects were asked to sort problems into categories. The groups formed by novices contained problems with similar objects—for example, rotating objects. In contrast, the groups that experts formed contained problems related to general principles of physics, such as conservation of energy. Together these findings emphasize the importance of conceptual knowledge for constructing and transforming problem representations throughout solution, as well as the role of these representations in determining the nature and amount of procedural knowledge required to achieve a solution. Similar findings have also been obtained for the learning of early geometry proof exercises (Anderson, Greeno, Kline, & Neves, 1981; Greeno, 1980).

DISCUSSION

We have presented a theoretical analysis of both the conceptual knowledge and the cognitive procedures underlying children's performance at different stages of skill in solving word problems. In this section we summarize how these two forms of knowledge interact during problem solving and consider the role of this interaction as skill develops. Finally, we consider the general implications of this interaction for interpreting children's performance on other tasks.

Relationship between Knowledge and Procedures in Performance and Development

PERFORMANCE

We have identified three main ways conceptual knowledge and procedures interact during problem solving. One way involves the role of schemata in the selection of actions. In the models that we described on pp. 176–184, both problem schemata and action schemata are required to relate the problem statement to the actions required to solve the problem. Problem schemata are involved in interpreting the problem text. They range from model (1)'s simple schema for representing quantitative relations, to model (3)'s more complex change schema, to the schemata required for representing the complex relationships in combine (2) and complex forms of compare problems. For all the models, these schemata are associated with goals either to change the current problem situation or to obtain some information from the problem situation. Planning procedures then identify an action whose consequence matches the current goal. Sometimes there is a direct match between this goal and one of a model's action schemata—for example, the goal to increase the amount of a given set and model (1)'s put-in schema. In other cases, additional schemata are required to infer important relations in the problem situation before the appropriate action can be selected—for example, model (3)'s inferences about the part–whole relations between the quantities in change (5). In either case, the application of even simple actions in problem solving requires some mediating conceptual knowledge in the form of schemata.

The second way that conceptual and procedural knowledge interact involves the use of schemata to monitor the effects of selected actions on a problem situation. For example, whenever model (2) performs the action count-on, it uses its change schema to maintain a record of the effects of that action. This record includes information about the values of the separate start and change sets and is important for correctly answering problems like change (3). Failure to monitor the effects of actions can result in predictable errors on some problems (e.g., model [1]'s incorrect answer to change [3]).

Finally, conceptual knowledge can influence which actions get selected. For example, model (2) and model (3) solve change (3) problems correctly; however, differences in their conceptual understanding of the relationships between quantities in a change situation lead to differences in the actions chosen for solution. Model (2) understands the problem as an increase of some unknown amount in the start set and uses count-on to solve the problem. Model (3) also represents the problem as a change problem with an unknown change set, but then it infers the part–whole relations between the quantities in the problem, identifies one of the parts as the unknown, and solves the problem with reference to its understanding of part–whole relations between numbers.

We also suggested that a child's conceptual knowledge of the relations between quantities in a word problem is related to the acquisition of more efficient counting procedures. Thus, there are at least two motivations for acquiring more advanced schemata—necessity and efficiency. The question remains how more advanced schemata and procedures develop and how they interact during development.

DEVELOPMENT

Recent theories of learning and development suggest some interesting possibilities for how the acquisition of sophisticated problem-solving procedures may be related to the acquisition of conceptual knowledge. For example, Klahr and Wallace (1976) and Neches (1981) postulate some principles to constrain development that emphasize the avoidance of redundant or unnecessary processing in the developing cognitive system. They propose that once a procedure is acquired, its operation is monitored by the child by means of what is called a procedural trace—that is, a record of the procedure's functioning in some situation. This procedural trace allows "detection of consistent sequences" and eventual "elimination of redundant processing." It is feasible that procedural traces not only result in more efficient procedures, but are also the basis for the development of more advanced problem schemata. For example, consider the following mechanism to account for the transition from model (1) to model (2). If a set of a known amount—say three blocks—is already present and the child is asked to increase it to make it eight, model (1) children typically do not begin counting and adding in from the known set value, but rather use the procedure start-count-set to begin the count all over again, starting with the existing set. According to both Neches's theory and Klahr and Wallace's theory, the transition from this procedure to model (2)'s count-on would involve (*a*) "tracing" the operation of start-count set, (*b*) thereby noticing the redundancy of counting the three over again, and (*c*) finally eliminating this redundancy by beginning with three and counting on to the desired result set. Thus, it is not that model (1) does not form any representation of its solution procedure: It has to have some procedural trace to advance to model (2). However, the units of model (1)'s procedural traces are different from those of models (2) and (3) and do not correspond to the structural information required by some of the change problems. Anyway, at some point in transitioning between model (1) and model (2), the model's procedural trace and what we want to claim is the model's developing change schema are probably indistinguishable.

Children's available schemata may also influence what procedures get acquired. That is, levels of conceptual understanding function as intermediate steps in acquiring new procedures. Thus, it is unlikely that a child would acquire count-on until that child at least had the simple schema for representing quantitative information shown in Figure 4.6. As Kamii (1980) points out, it is impossible to put two numbers into a relationship unless the numbers themselves are solidly present in the child's mind.

In summary, we have made some general suggestions that children's procedural knowledge leads to the acquisition of schemata, and these schemata in turn are involved as intermediate steps in acquiring more advanced procedures. More work is required to explicate further the nature of this interaction.

Knowledge Underlying Problem-Solving Performance

The analyses just discussed also have some important general implications concerning the knowledge we attribute to children on the basis of their problem-solving performance. Piaget (Piaget & Szeminska, 1952) pointed out that children lack understanding of some very important concepts: conservation of number, class inclusion, seriation, and so on. Evidence for these failures of understanding came from performance that was inconsistent with the general concepts; for example, when a child sees two sets with the same number of objects and says one has more, that performance is inconsistent with the concept of number conservation. Recently, numerous investigators have shown that in other circumstances children will show performance that is consistent with those concepts. For example, Gelman and Gallistel (1978) provided considerable evidence for preschool children's understanding of number concepts involving small sets, and Trabasso *et al.* (1978) summarized a substantial body of evidence that under appropriate circumstances, children show that they understand the concept of class inclusion. Results of studies of word problem solving by Hudson (1980), Lindvall and Ibarra (1980a), and Carpenter *et al.* (1981) also fall in this category; they show that with appropriate rewordings, children are quite capable of showing that they understand concepts of quantitative comparison and set inclusion.

Implications of this are clear. Children's failure to show understanding of a concept on one kind of task should not be taken as firm evidence that they lack understanding of the concept; there may be other tasks in which their performance shows that they understand the concept quite well. At the same time, we cannot attribute the same understanding to all children who pass a simplified version of a task when these children may differ considerably in their performance on more standard versions of the task. We need to account for the differences in knowledge between children who demonstrate understanding of a concept on a single task only and children whose conceptual understanding generalizes across a range of tasks that apparently involve the same concept. The analyses discussed in this chapter have made some progress in identifying some of these differences.

Children who are more skilled have acquired schemata that act as principles for organizing the information in a problem The schema appears to be used in a top-down fashion so that it overrides distracting features of the problem situation. Children who do not have these schemata cannot make these inferences and are dependent on modified problem situations where the relations are made explicit through rewordings or perceptual changes. Therefore we are inclined to view as very important the development of a schema to the point where it can be used to organize a problem situation and thereby override distracting, irrelevant factors. Piaget may have been wrong to assert that children lacked understanding of a schema if they failed his tests for that understanding, but it is equally misguided to assert that a schema is understood if we can find evidence for that understanding in some limited task domain. What we need is an analysis of the process of understanding in various problems situations, as well as an account of the features that are required for children at different states of development to produce an appropriate understanding of the situation and the task.

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REFERENCES

Aiken, L. R., Jr. Verbal factors and mathematics learning: A review of research. *Journal for Research in Mathematics Education*, 1971, *2*, 304–313.

Aiken, L. R., Jr. Language factors in learning mathematics. *Review of Educational Research*, 1972, *42*, 359–385.

Anderson, J. R. *Language, memory, and thought*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1976.

Anderson, J. R., Greeno, J. G., Kline, P. J., & Neves, D. M. Acquisition of problem-solving skill. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Barnett, J. C., Vos, K., & Sowder, L. A review of selected literature in applied problem solving. In R. Lesh, D. Mierkiewicz, & M. Kantowski (Eds.), *Applied mathematical problem solving*. Columbus, Ohio: ERIC, November 1979.

Baylor, G. W., & Gascon, J. An information-processing theory of aspects of the development of weight seriation in children. *Cognitive Psychology*, 1974, *6*, 1–40.

Bobrow, D. G. Natural language input for a computer problem-solving system. In M. Minsky (Ed.), *Semantic information processing*. Cambridge, Mass.: MIT Press, 1968.

Bolduc, E. J., Jr. A factorial study of the effects of three variables on the ability of first-grade children to solve arithmetic addition problems (Doctoral dissertation, University of Tennessee, 1969). *Dissertation Abstracts International*, 1970, *30*, 3358A.

Brown, A. L. Knowing when, where, and how to remember: A problem of metacognition. In R. Glaser (Ed.), *Advances in instructional psychology*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978.

Buckingham, B. R., & MacLatchy, J. The number abilities of children when they enter grade one. In *29th Yearbook of the National Society for the Study of Education*. Bloomington, Ind.: Public School Publishing, 1930.

Carpenter, T. P. *The effect of instruction on first-grade children's initial solution processes for basic addition and subtraction problems*. Paper presented at the annual meeting of the American Educational Research Association, Boston, April 1980.

Carpenter, T. P., Hiebert, J., & Moser, J. The effect of problem structure on first-grader's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, 1981, *12*(1), 27–39.

Carpenter, T. P., & Moser, J. M. The development of addition and subtraction problem-solving skills. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: Developmental perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Chi, M. T. H. Knowledge structures and memory development. In R. S. Siegler (Ed.), *Children's thinking: What develops?*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978.

Chi, M. T. H. Interactive roles of knowledge and strategies in development. In S. Chipman, J. Siegel, & R. Glaser (Eds.), *Thinking and learning skills: Current research and open questions* (Vol. 2). Hillsdale, N.J.: Lawrence Erlbaum Associates, 1982.

Chi, M. T. H., Feltovich, P. J., & Glaser, R. Categorization and representation of physics problems by experts and novices. *Cognitive Science*, 1981, *5*, 121–152.

Dahmus, R. M. How to teach verbal problems. *School Science and Mathematics*, 1970, *70*(2), 121–138.

Dean, A. L., Chabaud, S., & Bridges, E. Classes, collections, and distinctive features: Alternative strategies for solving inclusion problems. *Cognitive Psychology*, 1981, *13*, 84–112.

Duncker, K. On problem-solving. *Psychological Monographs*, 145, *58*(270), 1–112.

Fuson, K. C. *Counting solution procedures in addition and subtraction*. Paper presented at the Wingspread Conference on the Initial Learning of Addition and Subtraction Skills, Racine, Wisconsin, November 1979.

Gelman, R., & Gallistel, C. R. *The child's understanding of number*. Cambridge, Mass.: Harvard University Press, 1978.

Gentner, D. Evidence for the psychological reality of semantic components: The verbs of possession. In D. A. Norman & D. E. Rumelhart (Eds.), *Explorations in cognition*. San Francisco: Freeman, 1975. Pp. 241–246.

Gibb, E. G. Children's thinking in the process of subtraction. *Journal of Experimental Education*, 1956, *25*, 71–80.

Greeno, J. G. Process of understanding in problem solving. In N. J. Catellan, D. B. Pisoni, & G.R. Potts (Eds.), *Cognitive theory* (Vol. 2). Hillsdale, N.J.: Lawrence Erlbaum Associates, 1977.

Greeno, J. G. A study of problem solving. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 1). Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978.

Greeno, J. G. *Instruction for skill and unaerstanding in mathematical problem solving*. Paper presented at 22nd International Congress of Psychology, Leipzig, July 6–12, 1980.

Greeno, J. G., Magone, M. E., & Chaiklin, S. Theory of constructions and set in problem solving. *Memory and Cognition*, 1979, *7*, 445–461.

Greeno, J. G., & Riley, M. S. *Processes and development of understanding* (LRDC Publication, 1981). Pittsburgh, Pa.: Learning Research and Development Center, University of Pittsburgh, 1981.

Groen, G. J., & Resnick, L. B. Can preschool children invent addition algorithms? *Journal of Educational Psychology*, 1977, *69*, 645–652.

Grouws, D. A. Differential performance of third-grade children in solving open sentences of four types. (Doctoral dissertation, University of Wisconsin, 1971). *Dissertation Abstracts International*, 1972, *32*, 3860A.

Harvey, C. O. *A study of the achievement and transfer effects of additive subtraction and class inclusion training*. Unpublished doctoral dissertation, University of Houston, 1976.

Hebbeler, K. Young children's addition. *The Journal of Children's Mathematical Behavior*, 1977, *1*, 108–121.

Heller, J. I., & Greeno, J. G. *Semantic processing in arithmetic word problem solving*. Paper presented at the Midwestern Psychological Association Convention, Chicago, May 1978.

Hiebert, J. *Young children's solution processes for verbal addition and subtraction problems: The effect of the position of the unknown set*. Paper presented at the 59th annual meeting of the National Council of Teachers of Mathematics, St. Louis, April 1981.

Hinsley, D., Hayes, J. R., & Simon, H. From words to equations: Meaning and representation in algebra word problems. In M. A. Just & P. A. Carpenter (Eds.), *Cognitive processes in comprehension*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1977.

Hudson, T. Young children's difficulty with "How many more ______ than ______ are there?" questions. (Doctoral dissertation, Indiana University, 1980). *Dissertation Abstracts International*, July 1980, *41*,(01).

Ibarra, C. G., & Lindvall, C. M. *An investigation of factors associated with children's comprehension of simple story problems involving addition and subtraction prior to formal instruction on these operations*. Paper presented at the annual meeting of the National Council of Teachers of Mathematics, Boston, April 1979.

Jerman, M. *Instruction in problem solving and an analysis of structural variables that contribute to problem-solving difficulty* (Tech. Rep. No. 180). Stanford: Institute for Mathematical Studies in the Social Sciences, 1971.

Jerman, M. Problem length as a structural variable in verbal arithmetic problems. *Educational Studies in Mathematics*, 1973–1974, *5*, 109–123.

Jerman, M., & Rees, R. Predicting the relative difficulty of verbal arithmetic problems. *Educational Studies in Mathematics*, 1972, *4*(3), 306–323.

Kamii, C. *Equations in first-grade arithmetic: A problem for the "disadvantaged" or for first graders in general?* Paper presented at the annual meeting of the American Educational Research Association, Boston, April 1980.

Klahr, D., & Robinson, M. Formal assessment of problem-solving and planning processes in preschool children. *Cognitive Psychology*, 1981, *13*(1), 113–148.

Klahr, D., & Wallace, J. G. *Cognitive development: An information-processing view*. New York: Halstead Press, 1976.

Kohler, W. *The mentality of apes*. New York: Harcourt Brace, 1927.

Larkin, J. H., McDermott, J., Simon, D. P., & Simon, H. A. *Expert and novice performance in solving physics problems* (CIP Working Paper 410). Pittsburgh, Pa.: Department of Psychology, Carnegie-Mellon University, December 1979.

LeBlanc, J. *The performance of first-grade children in four levels of conservation of numerousness and three IQ groups when solving arithmetic subtraction problems*. Unpublished doctoral dissertation, University of Wisconsin, 1968.

Lindvall, C. M., & Ibarra, C. G. Incorrect procedures used by primary grade pupils in solving open addition and subtraction sentences. *Journal for Research in Mathematics Education*, 1980, *11*(1), 50–62. (a)

Lindvall, C. M., & Ibarra, C. G. *A clinical investigation of the difficulties evidenced by kindergarten children in developing "models" in the solution of arithmetic story problems*. Paper presented at the annual meeting of the American Educational Research Association, Boston, April 1980. (b)

Lindvall, C. M., & Tamburino, J. L. *Information processing capabilities used by kindergarten children when solving simple arithmetic story problems*. Paper presented at the annual meeting of the American Educational Research Association, Los Angeles, April 1981.

Linville, W. J. Syntax, vocabulary, and the verbal arithmetic problem. *School Science and Mathematics*, 1976, *76*(2), 152–158.

Loftus, E. J. F. *An analysis of the structural variables that determine problem solving difficulty on a computer-based teletype* (Tech. Rep. No. 162). Stanford: Institute for Mathematical Studies in the Social Sciences, 1970.

Loftus, E. J. F., & Suppes, P. Structural variables that determine problem solving difficulty in computer-assisted instruction. *Journal of Educational Psychology*, 1972, *63*(6), 531–542.

McDermott, J., & Larkin, J. H. Re-representing textbook physics problems. In *Proceedings of the Second National Conference, Canadian Society for Computational Studies of Intelligence*. Toronto, Canada: University of Toronto, 1978.

Maltzman, I. Thinking: From a behavioristic point of view. *Psychological Review*, 1955, *62*, 275–286.

Markman, E. The facilitation of part–whole comparisons by use of the collective noun "family." *Child Development*, 1973, *44*, 837–840.

Marshall, G. *A study of the achievement and transfer effects of comparison subtraction and one-to-one correspondence training*. Unpublished doctoral dissertation, University of Houston, 1976.

Neches, R. *Models of heuristic procedure modification*. Unpublished doctoral dissertation, Carnegie-Mellon University, 1981.

Nesher, P. Levels of description in the analysis of addition and subtraction. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: Developmental perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Nesher, P. S., & Katriel, T. *Two cognitive modes in arithmetic word problem solving*. Paper presented at the second annual meeting of the International Group for the Psychology of Mathematics Education, Onabruck, West Germany, September 1978.

Nesher, P., & Teubal, E. Verbal cues as an interfering factor in verbal problem solving. *Educational Studies in Mathematics*, 1974, *6*, 41–51.

Newell, A., & Simon, H. A. *Human problem solving*. Englewood Cliffs, N.J.: Prentice-Hall, 1972.

Norman, D. A., & Rumelhart, D. E. *Explorations in Cognition*. San Francisco: Freeman, 1975.

Novak, G. S. Computer understanding of physics problems stated in natural language. *American Journal of Computational Linguistics*, 1976. (Microfiche 53)

Paige, J. M., & Simon, H. A. Cognitive processes in solving algebra word problems. In B. Kleinmuntz (Ed.), *Problem solving: Research, method, and theory*. New York: Wiley, 1966. Pp. 51–119.

Piaget, J., & Szeminska, A. *The child's conception of number*. New York: Norton, 1952. (Original French edition, 1941.)

Polson, P., Atwood, M. E., Jeffries, R., & Turner, A. The processes involved in designing software. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Resnick, L. B. Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: Developmental perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Riley, M. S. *Conceptual and procedural knowledge in development.* Unpublished Master's thesis, University of Pittsburgh, 1981.

Roman, R. A., & Laudato, N.C. *Computer-assisted instruction in word problems: Rationale and design* (LRDC Publication 1974/19). Pittsburgh, Pa.: Learning Research and development Center, University of Pittsburgh, 1974.

Rosenthal, D. J. A., & Resnick, L. B. Children's solution processes in arithmetic word problems. *Journal of Educational Psychology*, 1974, *66*, 817–825.

Sacerdoti, E. D. *A structure for plans and behavior.* New York: Elsevier-North Holland Publishing, 1977.

Schank, R. C., & Abelson, R. P. *Scripts, plans, goals, and understanding: An inquiry into human knowledge structures.* Hillsdale, N.J.: Lawrence Erlbaum Associates, 1977.

Schell, L. M., & Burns, P. C. Pupil performance with three types of subtraction situations. *School Science and Mathematics*, 1962, *62*, (3, Whole No. 545), 208–214.

Shores, J., & Underhill, R. G. *An analysis of kindergarten and first-grade children's addition and subtraction problem-solving modeling and accuracy.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, April 1976.

Siegler, R. Three aspects of cognitive development. *Cognitive Psychology*, 1976, *8*, 481–520.

Siegler, R., & Klahr, D. When do children learn? The relationship between existing knowledge and the acquisition of new knowledge. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 2). Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Simon, D. P., & Simon, H. A. Individual differences in solving physics problems. In R. Siegler (Ed.), *Children's thinking: What develops?* Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978.

Steffe, L. P. The relationship of conservation of numerousness to problem-solving abilities of first-grade children. *Arithmetic Teacher*, 1968, *15*, 47–52.

Steffe, L. P. Differential performance of first-grade children when solving arithmetic addition problems. *Journal for Research in Mathematics Education*, 1970, *1*, 144–161.

Steffe, L. P., & Johnson, D. C. Problem-solving performances of first-grade children. *Journal for Research in Mathematics Education*, 1971, 2, 50–64.

Steffe, L. P., & Thompson, P. W. Children's counting in arithmetical problem solving. In T. P. Carpenter, J. M. Moser, & T. Rombert (Eds.), *Addition and subtraction: Developmental perspective.* Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Stein, N. L., & Trabasso, T. Waht's in a story: Critical issues in comprehension and instruction. In R. Glaser (Ed.), *Advances in the psychology of instruction* (Vol. 2). Norwood, N.J.: Lawrence Erlbaum Associates, 1981.

Tamburino, J. L. *An analysis of the modelling processes used by kindergarten children in solving simple addition and subtraction story problems.* Unpublished Master's thesis, University of Pittsburgh, 1980.

Trabasso, T., Isen, A. M., Dolecki, P., McLanahan, A. G., Riley, C. A., & Tucker, T. How do children solve class-inclusion problems? In R. S. Siegler (Ed.), *Children's thinking: What develops?* Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978.

Underwood, B. J., & Richardson, J. Some verbal materials for the study of concept formation. *Psychological Bulletin*, 1956, *53*, 84–95.

Vergnaud, G. A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. Romberg (Eds.), *Addition and subtraction: Developmental perspective.* Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Wertheimer, M. *Productive thinking.* New York: Harper & Row, 1945. (Enlarged edition, 1959)

Young, R. M., & O'Shea, T. Errors in children's subtraction. *Cognitive Science*, 1981, *5*, 87–119.

Zweng, M. J., Gerarghty, J., & Turner, J. *Children's strategies of solving verbal problems* (Final rep. to NIE, grant no. NIE-G-78-0094). Iowa City, Iowa: University of Iowa, August 1979