DOCUMENT RESUME

ED 247 996 JC 840 510

AUTHOR Page, Warren
TITLE Knowledge Transmission and Acquisition: Cognitive and Affective Considerations.
PUB DATE Jul 84
NOTE 25p.; Paper presented at the Sloan Foundation Conference on New Directions in Two-Year College Mathematics (Atherton, CA, July 11-14, 1984).
PUB TYPE Speeches/Conference Papers (150) -- Reports - Descriptive (141)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *College Mathematics; *Instructional Improvement; *Learning Strategies; *Mathematics Instruction; Two Year Colleges

ABSTRACT

Arguing that college mathematics education must be made more effective, especially for technology, engineering, mathematical sciences, and physical sciences students, this paper presents nine general principles to enhance math instruction for all students. Introductory material argues that changes in perception, attitudes, and role models are needed to realize the goals of integrating knowledge acquisition and knowledge utilization and exploring metacognitive instructional considerations. Next, a historical and futuristic overview is provided of important mathematical issues of the 20th century. Then, the general principles for mathematics instruction are presented, discussed, and illustrated with examples: (1) "what" one communicates in mathematics instruction includes the intrinsic nature and value of the discipline; (2) "how" one communicates goes beyond the exchange of ideas and information to long-lasting psychosocial values; (3) math teachers must appreciate individual differences and their impact on learning styles; (4) multimodal representation of concepts has the potential for synergistic learning; (5) math principles should be presented as the basis for solving classes of problems; (6) students need to learn to reformulate and restructure problem representations; (7) teachers must anticipate and preempt students' misinterpretations; (8) control knowledge must be appreciated as part of knowledge acquisition and accumulation; and (9) students must be responsible partners in an interactive and collaborative learning environment. (AYC)

KNOWLEDGE TRANSMISSION AND ACQUISITION:
COGNITIVE AND AFFECTIVE CONSIDERATIONS

Warren Page
New York City Technical College
Brooklyn, New York

Presented at the Sloan Foundation Conference on New Directions in Two-Year College Mathematics, Atherton, CA, July 11-14, 1984

Knowledge Transmission and Acquisition: Cognitive and Affective Considerations

Warren Page
New York City Technical College
Brooklyn, New York 11201

Introduction

College mathematics education will be further challenged in the 1980's by demographic changes, the enrollment of nontraditional students (older people, for example), and by society's inevitable demands for increased mathematical knowledge and competence. Thus, new initiatives for teaching widely differing student populations must be found and explored. The mathematical knowledge bases required to meet the scientific, technical, vocational, cultural, and functional needs of such varied student populations must also be closely examined. But without proper focus NOW, our approaches to these issues will be inadequate and the benefits will be ephemeral.

What is needed now is a revolution in intellectual, philosophical, and social perspectives - perspectives which reflect the very dramatic changing nature of the mathematical enterprise. Indeed, it is my belief that:

(I) College mathematics education, and in particular the mathematics training of TEMP* - career students, must be made more effective. Knowledge acquisition must be embedded into, and integrated with, knowledge utilization in order that learning be functional and relevant.

*TEMP = Technology, Engineering, Mathematical sciences (including computer science), Physical science.

(II) New, metacognitive educational considerations must be explored and given greater prominence in order that students be able to parlay their current mathematics education and beginning career status into productive future learning and professional growth.

(III) Changes in perception, attitudes, and role models are needed in order to realize (I) and (II).

These beliefs reflect and interface with important aspects of classroom instruction, artificial intelligence research, and cognitive (including neurobiological) research. Unfortunately these nodes of mathematical endeavor are not as well interrelated as they could be. Thus, it is my hope that this conference, and this paper in particular, will help to stimulate further interest in strengthening these connections.

In this paper, I attempt to (loosely!) depict mathematical knowledge as the resultant vector whose components are interactive processes such as the acquisition, representation, utilization, organization, and management of information. For each person, the coordinates of these component vectors are individual-matrix dependent. Accordingly, mathematical knowledge should be thought of as a dynamic vector that grows and changes orientation in one's intellectual space.

The instructional strategies advocated in this paper are intimately intertwined with behavioral objectives, information-processing, and styles of learning. They are offered as general principles that can enhance mathematics instruction for all students. For TEMPs, these approaches should be viewed as first-order guiding principles that constitute the logical prerequisities and pragmatic basis for higher order considerations - including, for example, metacognition and learning how to learn - that will be the focus of a forthcoming paper in progress.

New Awareness

The present crisis in college mathematics instruction is not so much one of "what specific course content and to whom it should be taught" as it is a reflection of continued failure by the mathematical community to properly communicate what mathematics is and how it can be of value to different, changing student populations.[1] This is sine qua non! Without such understanding and guidance, students will find easier or more rewarding academic disciplines beckoning; why bother with mathematics and its demands?

Most high school students and college freshmen are curriculum captives insofar as they must usually complete certain mathematical course requirements. But given their first opportunity to make choices, college sophomores and juniors increasingly vote to abandon mathematics by enrolling in other courses of study [32]. As adults, they'll also vote with their political and financial influence. These votes have ominous implications for the future concerns and allocation of resources for college mathematics education.

The first two years of college mathematics is particularly crucial for influencing and partially reversing these voting patterns. However, new perspectives and attitudes are required to bring about such changes. Indeed, it is my belief that the

---

1. "...the number of Native Americans, Hispanics, Orientals, Mormons, and Seventh Day Adventists are all increasing rapidly." "...any surge of new enrollments during the next two decades in higher education will be led by minorities, particularly blacks and Hispanics." "In most community colleges today, the average age of students is thirty-six and climbing." For further details and a demographic portrait of students in the 1990s, see [8].

two most critical factors in teaching mathematics concern "what" one conveys and "how" communication takes place. Both factors are intimately intertwined with information-processing and learning; each has affective as well as cognitive dimensions.

> "What" one communicates in mathematics instruction transcends the elucidation of mathematical concepts; the teacher of mathematics also conveys (consciously and unconsciously) a great deal to students about the intrinsic nature and value of the discipline itself. Students' impressions and attitudes about mathematics play an important role in their motivation (therefore, commitment and perseverence) and ultimate success or failure in mathematics courses. *(GP)*[1]

Thus, effective mathematics instruction must begin by making students want to study mathematics.

> "How" one communicates in mathematics instruction goes beyond the exchange of ideas and information. Classroom learning experiences and attitudes give rise to long-lasting psycholosocial values on what it means to do mathematics and who should do it. *(GP)*[2]

Implicit in each of these factors is the realization that effective learning is rarely possible if teachers of mathematics cannot introduce and develop concepts in a manner commensurate with their students' information- processing abilities and levels of understanding. This realization subsumes an awareness of the fact that a large constellation of behavioral patterns may be at work in predisposing students to success or failure in their mathematics courses, particularly so at the basic skills level.[2]

---

2. In "Mindscape and Science Theories" [11], Maruyama uses the term <u>mindscape</u> to mean "a structure of reasoning, cognition, perception, conceptualization, design, planning, and decision making that may vary from one individual, profession, culture, or social group to another." He distinguishes four pure mindscapes and their combinations, and illustrates their aspects at the overt, covert, and abstract levels.

O. J. Harvey administered psychological tests to university students over a number of years. In [7], he identified four epistemological types and their distribution among first-year university students.

Thus:

> Teachers of mathematics must appreciate individual differences and understand how psycho-physio-social factors impact on styles of learning. $(GP)_3$

In this vein, it is singularly important for instructors to realize that they too have their own cognitive preferences. The types of exams they prefer and develop, for instance, reflect their own cognitive styles and not necessarily those of their students. Thus, students' success (or failure) may not depend only on course content, but may also be related to the information-presenting strategies and instructional demands of their teacher. People do learn to learn differently! Instructional procedures which may be beneficial to some students can disadvantage and be counterproductive to other groups of students. Behavioral differences <u>must</u> be taken into consideration, if people having different styles of learning are to interact fruitfully.[3] A few simple examples suffice to illustrate this point.

The quality and quantity of interaction in the classroom are important ingredients for learning. While some students prefer, and do better, working alone, others learn best through some form of give-and-take. The nature of interaction conducive to learning will vary according to the student's background and psychological profile. Since setting, ambience, and interaction are interrelated, it is not immediately clear if students'classroom inactivity result from culturally-related

---

3. An appropriate modification E. T. Hall's statement [6] is: most instructors are only dimly aware of the elaborate and varied behavioral patterns which prescribe our handling of time, spatial relations, and our attitudes toward work, play, and learning. Accordingly, we insist that everyone else do things our way ... and those who do not are often regarded as "underachievers."

reasons, because they are consciously (or unconsciously) separating and disinvesting themselves from classroom instruction, or because they feel anxious and uncomfortable in the educational environment.[4] Instructors alert to these nuances can enhance learning through classroom-teaching strategies that are appropriate to their students' behavioral needs. (For examples of student-student and teacher-student interactive strategies, see [13], [14], [28].)

Unfortunately, most instructors require that all students take the class exam at the same time, despite the fact that individuals learn and grow at different intellectual rates. This requirement clearly stacks the odds against the slower learners as well as those who (appreciating time other than as a preciously dwindling commodity)[5] have not yet learned to plan sufficient time for study. Exam grades for these "out of phase" students do not reflect their actual subject mastery once such students have caught up. Accordingly, their final grade - based on grades which reflect their states of unpreparedness - may not be commensurate with their knowledge at the end of the course. This disadvantage can be diminished, if not eliminated, by

---

4. Styles of participation conducive to learning also vary with culture. North American Indians learn best through observation. Oriental students seem to do well without heavy emphasis on classroom participation. Americans generally require more interaction than students from Anglo-French cultures, but not as much as Hispanic students.

5. Variations in the perception and utilization of time become evident as one moves westward and southward from the northeastern part of the United States. As a rule, however, Americans think of time as being linear, sequential, and quantifiable. "It should take x time to cover this material; we'll plan an exam for y date." Other cultures share neither our sense of urgency nor our immutable compartmentalization of time: it makes more sense to disregard time constraints and work at the job until it is completed than it does to abandon one unfinished task in order to begin a new one.

broadening the constraint of "fixed day" for an exam to "fixed period" for that exam. For example, students can be given the opportunity to take one of three variants of the exam (test T_i on day D_i for i=1,2,3) during a fixed exam week. In this humanistic context, exams can do more than attempt to subjectively quantify levels of understanding; they can (and should!) be used as pedagogical tools for motivating and rewarding further learning. For instance, students who did poorly on an exam will be highly motivated to clear up specific areas of weakness if they are allowed to take another variant of this exam (during its fixed test period, for example)-in which case, their overall grade for that class's exam is the average of the two exams taken. For another variation on this theme [15], students can be made aware of the fact that each class test will contain one arbitrarily chosen problem from each of the preceeding class tests.

Knowledge Transmission and Aquisition

The notion of learning has a wide range of interpretations among people - both in terms of what "knowledge" means and what is required to reach that state of knowing. Unfortunately, far too many students view mathematics as a lifeless body of facts and formulas to be memorized or stored for short-term, cued recall; doing mathematics is too widely interpreted as concept-identification, formula substitution, symbol manipulation, and problem solving in a very narrow, artificial doman. Why is this so? Why have so many students been lulled into these misconceptions, and how can we help them to better appreciate what mathematical knowledge means and what is required to reach that state of knowing?

Each of the above questions must have a multiplicity of answers. But surely what instructors expect and demand of students is pivotal. Thus, we must accept the responsibility for this imprinting and we must take the initiative for bringing about some very fundamental changes in our students' perceptions. A necessary first step is to make it convincingly clear that:

<u>Knowledge acquisition does not imply knowledge utilization</u>.

Just being able to identify a geometric figure (say, a rhombus) reveals nothing about the intrinsic properties of that figure. And symbolic manipulation without understanding is only slightly more meaningless than solving a trivial variant of the same problem for the twentieth time. That such superficial forms of knowledge are minimally functional can easily be demonstrated, and must be driven home, by instructors. It is also very important to alert students to the impact of a powerful anxiety-reducing drug, commonly called 'pocket-calculator.' It alleviates students' motivation to learn by making them feel that they can use it to solve all their mathematics problems. This myth is also easily dispelled. For example:

<u>Problem 1</u>. Enter any number $x > 0$ on your calculator and repeatedly use the $\sqrt{\ }$ key. What do you get? Why?

<u>Problem 2</u>. On your calculator, enter 2 and take $\sqrt{\ }$. Continue to repeat this pattern of adding 2 followed by taking $\sqrt{\ }$. What do you get? Why?

Problems which can be posed, but not solved, by a calculator are effective for demonstrating to students that their head-held calculator is much more powerful than their hand-held calculator and that although calculators can be helpful for computing, they should not be antidotes for the headache of having to think.

Dispelling students' myths is not enough. There still remains the question of how to help them appreciate mathematics as a dynamic and multilayered activity - a richly rewarding and evolving synergism of process and product. This we now consider in greater detail.

In most instances, mathematics instruction is considerably more effective when several modes of perception are used - as may be the case, for example, when (left-hemispherically oriented) technology students and, say, (more right-hemispheric) humanities majors are in the same course.[6,7] Thus, both the symbolic-analytic approach and the visospatial-relational approach may be used to prove (Figure 1) that "the geometric series $1+\frac{1}{2}+\frac{1}{4}+\cdots$ converges to 2."

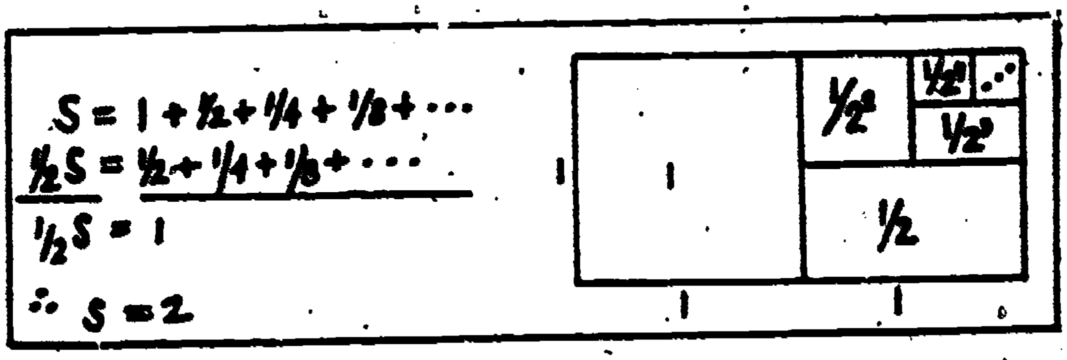


Figure 1.

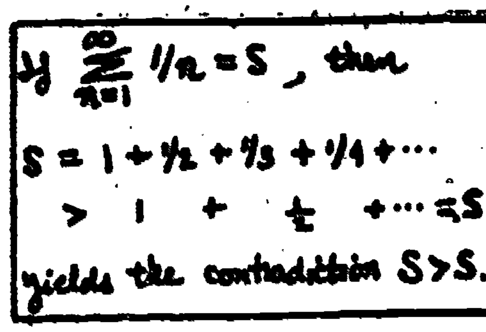


Figure 2.

In the same spirit, analytic proofs (viz, converging-series tests) that the harmonic series diverges may be supplemented with, or made more plausible by, following a (right-hemispheric) analogical tact as in Figure 2.

---

6. Today, it is well known that there exists major differentiations of functions between the brain's left and right hemispheres. In the most simplistic terms, left-hemispheric thinking resembles the discrete, sequential processing of a digital calculator; right hemispheric thinking simulates the concurrent, relational activity of an analog computer.

7. Cohen [2], [3] found that white middle-class children tend to be analytical in orientation, whereas Chicano and black children tend to be rational. She also found difference in orientation among professions [4].

We all know that $\sum_{k=1}^{n} k = n(n+1)/2$ can be (and usually is) proved by induction. But, as is often the case, students feel cheated: "here, induction is an accessory after the fact. How did one know the formula to be verified in the first place?" Instructors, of course, can invoke Gauss' (more right-hemispheric) relational approach to obtain the aforementioned conjectured formula for verification (Figure 3).

$n=5$: $1+2+3+4+5$ equals "$2\frac{1}{2}$ pairs of 6," or $\frac{5}{2}(5+1)$

$n=6$: $1+2+3+4+5+6$ equals "3 pairs of 7," or $\frac{6}{2}(6+1)$

↓

$$\sum_{k=1}^{n} k \stackrel{?}{=} \frac{n}{2}(n+1)$$

Figure 3.

One can also obtain $\sum_{k=1}^{n} k = n(n+1)/2$ by counting the dots in the right triangle of Figure 4. (The right triangle of dots, when reflected with respect to its hypotenuse, produces a square of dots plus an extra superimposed diagonal. Thus, $2\sum_{k=1}^{n} k = n^2 + n$.) Figure 4 also illustrates some of the author's visually-induced proofs of other known results [21].

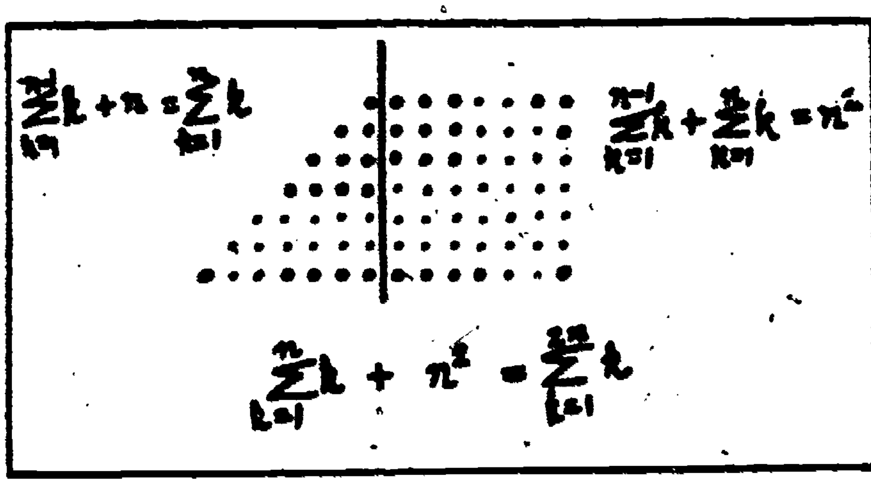


Figure 4.

Since pictures are usually more easily recalled than procedures, visual portrayals of algebraic processes can enhance the retrieval of information. Synthetic division and synthetic multiplication [10] offer two good illustrations. The point being emphasized is:

> Multimodal representation of concepts can do more than convince students of concepts' veracity; they have the potential for synergistic learning - as, for example, when concepts are introduced by one modality and students are asked to finds representaions/proofs in other modes of thought. (GP)

But let's not stop here! This leads in a natural manner to a whole new dimension of thinking. An example or two suffices to make this clear.

1. Having symbolically demonstrated that S = 2 (Figure 1), the author was surprised to see that some students felt tricked and less than convinced of this result. "How did you know to multiply by 1/2 and then subtract?" "Where did the whole series disappear to?" Interestingly, the visual proof- stumbled upon during class session - was perfectly acceptable to everyone. A few additional remarks, between pauses, began to lead students to a new awareness. It soon became clear that our visual proof was also "an accessory after the fact." How, after all, did I know to begin with a 1 x 2 - sized rectangle in the first place? The symbolic proof was also challenged as being bogus since it too was based on the aprioi knowledge that S was a finite number. Next, we also discovered that the same algorithmic process can produce meaningful as well as meaningless results (replacing the ratio "1/2" by "r" for r>1, we still obtain $S = 1/(1-r)$), and that algorithmic, existential, and constructivistic thinking are intimately interrelated. Finally, it was intuitively clear that the analytic proof generalizes much more efficiently than a

geometric one to arbitrary converging geometric series. (Students may enjoy attempting a visospatial proof, or they can refer to the author's discovered generalization [17].)

2. Given the motivation and opportunity to experiment, even the weakest students will quickly discover that the distributive multiplication depicted below

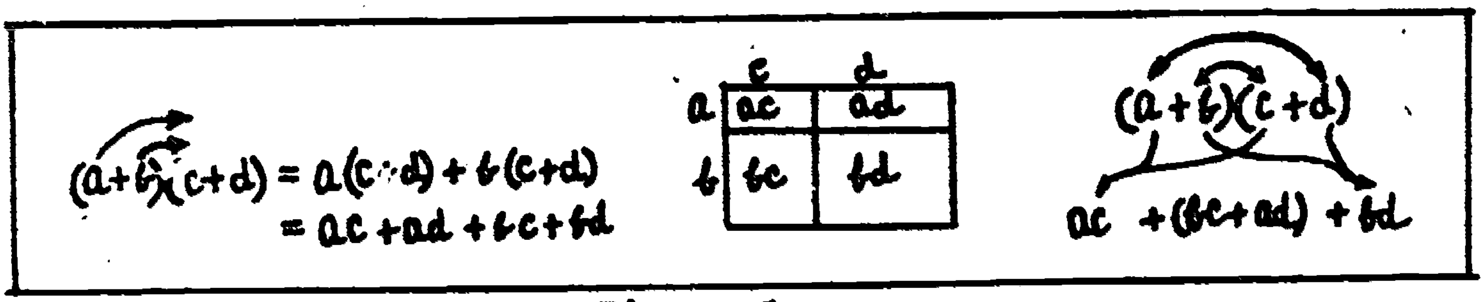


Figure 5.

is the representation that can best be extended to the multiplication of multinomials.

3. A few well-selected examples made it clear that Gauss' combinatoric approach (of using pairs of numbers) had greater potential for adaptation to other contexts than do dot proofs, but it was not as pervasive as mathematical induction. (Here was the beginning of a new appreciation and respect for induction.)

Comparing and analyzing the efficiency, extendability, and generalizability of representations is an important first step toward developing the types of awareness students will need in their algorithmic and computer-related mathematics learning. (See, for example, [9].)

Experimenting with alternate modes of representation can also be stimulating and informative to instructors. Figure 4, for example, yielded a newly discovered visual proof by the author of the fact that <u>the sum of the first n terms of an arithmetic progression plus n^2 times the difference equals the sum of the next n terms of the progression.</u>[8] By examining the various representations students use, we can better judge how well they understood the concept in question.[9]

Embedding concepts in processes can help students appreciate mathematics as a dynamic and multilayered activity - an evolving synergism of process and product. These perceptions must begin, so to speak, at the molecular level. Numbers, variables, shapes, formulas and equations, as well as, other such basic entities, must not be perceived as passive, static notions, but rather as interactive processes and actions. This impacts on how information itself is presented.[10] To use Herb Simon's analogy [29].

A physician's knowledge of how to treat diseases is useless if the physician can't tell when the patient has the disease. Thus, a large part of medical knowledge consists of condition-action pairs; the condition being the disease symptoms, and the action being the appropriate treatment.

---

8. If the dots on the hypotenuse of the right triangle are labeled "a" and all other dots on the trapezoid are labeled "d" then (since the dots on the triangle plus the dots on the square comprise the dots on the trapezoid) $\sum_{k=0}^{n-1} (a+kd) + n^2 d = \sum_{k=n}^{2n-1} (a+kd)$.

9. Greeno [5] offers three general criteria for judging the degree of understanding of a represented concept: internal coherence of the representation, its connectedness to other relevant knowledge, and how accurately it captures the concept's essential features.

10. Too many students think of a variable x as being a fixed unknown (rather than as an actively roaming entity - an operator whose character changes depending on where it is encountered in its domain); formulas are perceived as receptors passively waiting for substituted numbers (rather than as the algebraic or visually portrayed embodiments of how variables relate to each other); and equations are considered as fixed states of equilibria (rather than as reversable processes, where each side eyeballs the other and can get there by an appropriate sequence of transformations).

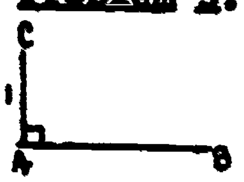
This is not the format of mathematics/science knowledge, in general. We are much more explicit in enunciating principles than in describing when and how they can be applied. Formulas and theorems, for example, do not always carry internal information about contexts or situations that should evoke their use. Greeno [5], is probably correct in his impression that "most teaching of algorithmic processes often focus almost entirely on the actions to be performed, with little attention to the issue of when to perform them." Mathematics texts, on the other hand, seem to assume that once students are shown a few worked out problems, they'll be able to generate their own situation-action responses for solving problems. This is not always the case, and even less so for students in their earlier college mathematics courses.[11]

The point being stressed here is that every important mathematical result should be presented as the action component of condition-action pairs. For such a "production," the conditions needed for the result to apply are built into the presentation. In broader terms:

---

11. For instance, knowing that $x^0 = 1$ is useless in Problem 1 if students don't realize that $x^{1/n} \to x^0$. Knowing how to solve quadratic equations is useless in Problem 2 if students don't realize that $y = \sqrt{2 + \cdots \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$ can be expressed as $y^2 = 2 + y$. Finally, students' knowledge of the Pythagorean theorem is useless they cannot use it in appropriate situations (Problem 3) and they attempt to apply it to inappropriate contexts (Problem 4).

<u>Problem 3</u>. Using only a compass, measure off length $\sqrt{3}$ along AB.



[Solution: Mark off D on AB such that AD = 1; then CD = $\sqrt{2}$. Mark off E on AB such that AE = $\sqrt{2}$; then CE = $\sqrt{3}$.]

Every key notion and every important principle should not only be considered in terms of its intrinsic properties, but also as the basis for solving a primitive class of problems.

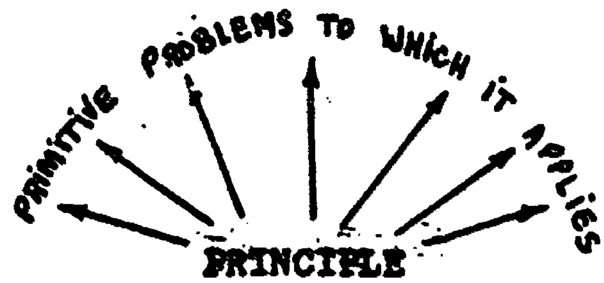


Figure 6.

A nice illustration of this principle is "An Approach to Problem-Solving Using Equivalence Classes Modulo n" by J.E. Schultz and W.F. Burger [29].

There is another important pedagogical facet to $(GP)_4$ - namely, the manner in which this type of thinking and awareness can be broadened to solve problems. Indeed, it is well known that the manner in which a problem is described is of critical importance in determining how easily the problem can be solved or whether it can be solved at all.

Problem 4. Find the length of hypotenuse c.

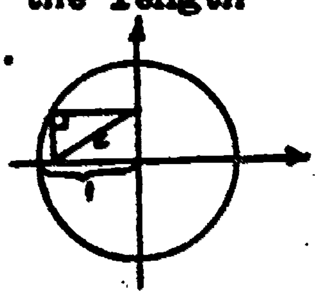


Figure 7.

Problem 5. Find the area of the parallelogram plus the area of the square.

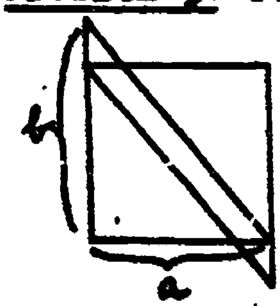


Figure 8.

Attempts to solve Problems 4, 5 corroborate the findings of research experiments: subjects don't ordinarily search for the most efficient representation of the problem; they tend to adopt the representation of the problem from the language of its statement.[12] Thus, as Simon points out [30], it should be clear that:

> Instructors need to help students improve their skills in reformulating and restructuring problem representations. It is most important to make students understand that the value of their mathematically-related career skills will, in large part, depend on their ability to recognize and construct contexts that evoke appropriate mathematical principles and processes. $(GP)_6$

As the instructional dual to $(GP)_5$, where principles served as "seekers" of conditions and contexts where they apply, problems can serve as "attractors" for as many distinctly different solutions as possible.

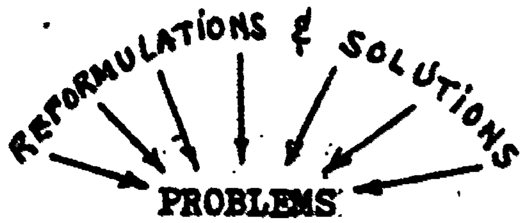


Figure 9.

An especially nice illustration of this is J. Staib's "Answer Finding Versus Problem Solving" [32], where the class discovered nine different ways to find the distance from a point to a line. Also see "Convexity in Elementary Calculus: Some Geometric Equivalences" [1], and Pedersen and Pólya's "On Problems with Solutions Attainable in More Than One Way" [25].

---

12. In Problem 4, the "hypotenuse" c equals 1 since the other diagonal of the rectangle is the radius of the circle. If the "parallelogram" and the "square" (in Werthheimer's Problem 5) was restructured as overlapping right triangles of base a and height b, the desired area is immediately seen to be ½ab.

The perception of objects, systems, and processes vary considerably amongst people, and this has tremendous bearing on how mathematical notions are perceived and utilized. This is especially true in the classroom. As teachers are expounding on mathematical notions and principles, students are busily concocting their own idiosynchratic versions based on their own consistent private logic. Such cognitive misinterpretations, however, are not confined only to developmental mathematics students or to those whose backgrounds do not reward clear and precise thinking. In the margins of Bourbaki's advanced level texts, for instance, the roadway danger signal ☡ (caution!) is followed by elaborative comments designed to help prevent readers from making wrong interpretations that are consistent with the antecedent exposition. The point being stressed here is that:

> In presenting information, it is vitally important for teachers to anticipate and preempt students' misinterpretations. $(GP)_7$

The instructional strategies summarized in $(GP)_1$ – $(GP)_7$ can help teachers monitor, and become more attuned to, the nature of these misinterpretations. To the extent that we examine, analyze, and modify our instructional strategies, we gain a higher form of instructional knowledge and an increased capacity for becoming better <u>imparters</u> of knowledge.

There are also important information management considerations for the <u>acquirerers</u> of knowledge. Consider, for instance, students who do well on homework assignments or quizzes covering each specific aspect of a problem situation but still do poorly on exams where they are no cues as to which solution strategies to apply to the problems as a whole. In short, they lack certain aspects of control knowledge (that is, information management). Other manifestions of deficiences or weaknesses in control knowledge include: incorrect or incomplete categorization of problem protoypes, lack of coherent knowledge structure and organization, inability to recall or retrieve information, nonassessment of concept attainment, and disregard for solution verification.

There are many strategies for helping students to overcome these deficiences. A contextually-representative sample might be the following:

- Classroom exams. " Without actually solving problem P, carefully describe and/or set up in as many different ways as possible how to obtain the answer to P."

- Homework assignments. "Compare and contrast the types of problems (and how they are solved!) in this chapter with those in . . ."

- Interactive discussions. "How do you know that your method is correct? Your answer is reasonable? ..." [13]

- Term papers or course-related projects. "Summarize the chapter's (course's) key concepts and principles, and be sure to discuss or depict their interrelationships." (See, for example, [26], [27].)

- Realistic role models "...Okay, I'll try to solve and analyze this mathematical problem you've encountered in the physics lab (on the job, for contest X, ...). I'm not really sure where to begin...Suppose we first attempt...because ... "

---

13. In "Questions in the Round - An Effective Barometer of Understanding" [14], the instructor proceeds around the room requiring that each student either ask a question (to be answered by the instructor) or else be asked a question by the instructor. This strategy provides an excellent opportunity for instructors to ask questions of control knowledge.

The most effective strategy, however, is to make students realize that while it is natural to form misconceptions and make errors, specific actions for their detection and analysis are also important mathematical activities. We must demand, and students must be made to appreciate, that verification and analysis are necessary in doing mathematics. Thus:

> Control knowledge must be appreciated as being an integral part of knowledge acquisition and accumulation. (GP)

It seems clear that both teachers and students can receive and impart important types of knowledge from each other. Accordingly:

> Teachers must invite and encourage students to be responsible partners in an interactive collaborative learning environment. $(GP)_9$

Interactive and collaborative aspects of $(GP)_9$ have already been considered earlier. The invitation I urge is not explicit in nature, but rather implicit in the way we teach and do mathematics in the classroom - manifestations,soto speak, of being "great teachers" in the sense of J. Epstein's edited volume of essays Masters: Portraits of Great Teachers [12]:

> "What all the great teachers appear to have in common is a love of their subject, an obvious satisfaction in arousing this love in their students, and an ability to convince them that what they are being taught is deadly serious."

The most natural embodiment of $(GP)_9$ is for teachers to guide, assist, and/or collaborate with students in actually doing mathematics that has meaning to them. There are many ways to proceed, depending on the students' capabilities and levels of mathematical sophistication:

Mathematical problems, puzzles, and games have been popular since antiquity, and their solutions have contributed much to the development of modern mathematics. Thus, Leibnitz appears to have been correct when he said, "Men are never so ingenius as when they are inventing games." Recreational mathematics and examples from everyday life always stimulate students' curiosity and whet their intellectual appetites for more.

Weaving mathematical tapestries can be fascinating. Combining and interlacing novel ideas from diverse areas of mathematics (as distinct from applying mathematics to other disciplines) is a beautiful way to impress students with the fact that mathematics is indeed a coherent, harmonious whole.

Doing mathematical research cannot fail to convey the challenge and excitement of attempted discovery. Fruitful research exists at all levels.[14] The rewards of successful research - giving an invited (classroom) lecture, seeing one's results(s) in publication, and other forms of peer acknowledgement - can be the biggest payoffs and reinforcers for students to stay invested in the study of mathematics.

Concluding Remarks

Finally, as we began, let us pause to reflect on where college mathematics could be heading. To the extent that we succeed in going beyond changing our students' votes and actually imbue our more capable students with positive perceptions of (and feelings toward) mathematics, we increase the likelihood that the focus of mathematics instruction will not only be as a "seeker" of contexts and domains of application, but will also

14. For examples of mathematical research that can be undertaken by, or shared with, students in their earlier years of college mathematics, see ([18], geometry), ([21], precalculus), ([19], calculus), ([20], number theory), ([23], statistics), ([22] and [24], general).

become an "attractor" for significant contributions from many of these serviced disciplines. Mathematically competent and well predisposed students entering careers in computer science, the social and biological sciences, and the humanities will most likely be more motivated and better equipped to bring their expertise to bear on improving and enhancing mathematics instruction.

By giving careful attention to the what and how factors of mathematics education, college mathematics instructors can play an important role in the evolving vitality and future growth of mathematics instruction at all levels. It is not an opportunity that should be cavalierly disregarded.

References

[1] Victor A. Belfi, "Convexity in Elementary Calculus: Some Geometrical Equivalences." The College Mathematics Journal 15:1 (January 1984) 37-41.

[2] R. A. Cohen, "Conceptual Styles, Cultural Conflict, and Nonverbal Texts of Intelligence." American Anthropologist 71 (1969) 825-856.

[3] ______________, "The Influence of Conceptual Rule-sets on Measures of Learning Ability." In Race and Intelligence: Anthropological Studies 8 (1971) 41-57.

[4] ______________, "Relational and Analytic Intelligence Among Designers and Researchers." In Environmental Design Research Vol. 1 Selected Papers, edited by Wolfgang Preiser; Dowden, Hutchinson and Ross Publ. Stroudsburg, PA. 1973.

[5] James Greeno, "Understanding and Procedural Knowledge in Mathematics Instruction." Educational Psychologist 12:3 (1978) 262-283.

[6] E. T. Hall, The Silent Language. Anchor, Garden City, NY, 1973.

[7] O. J. Harvey, Experience, Structure and Adaptability. Springer, New York, 1966.

[8] Harold L. Hodgkinson, "Guess Who's Coming to College." AAUP Academe (March-April 1983) 13-20.

[9] Dan Kalman and Warren Page, "Efficient Exponentiation Algorithms for Calculators." The College Mathematics Journal 16:1 (January 1985) 58-60.

[10] Kenneth R. Kundert, "Why Not Teach Synthetic Multiplication?" The Two-Year College Mathematics Journal 11:2 (1980) 121-122.

[11] Magoroh Maruyama, "Mindscapes and Science Theories." Current Anthropology 21:5 (October 1980) 589-600.

[12] Masters: Portraits of Great Teachers, ed. Joseph Epstein, Basic Books, New York, 1981.

[13] Warren Page, "Small Group Strategy for Enhanced Learning." American Mathematical Monthly 86:10 (1979) 856-858.

[14] ______________, "Questions in the Round: An Effective Barometer of Understanding." The Two-Year College Mathematics Journal 10:4 (1979) 278-279.

[15] ______________, "Exams Can Leverage Learning." The Two-Year College Mathematics Journal 10:1 (1979) 38.

[16] ______________, "Count the Dots." Mathematics Magazine 55:5 (March 1981) 97.

[17] ______________, "Con' ergent Geometric Series," Mathematics Magazine 54:4 (September 1981) 201.

[18] ______________, "Compass Construction $\sqrt{n}$." The Mathematics Association of Two-Year Colleges Journal 10:13 (1976) 180-181.

[19] ______________, "The Formula for Arc Length Does Measure Arc Length." In Two-Year College Mathematics Readings. ed. Warren Page, The Mathematical Association of America, 1981, pp. 111-114.

[20] ______________, "A General Approach to p• q r-Cycles." In Two-Year College Mathematics Readings, ed. Warren Page, The Mathematical Association of America, 1981, pp. 263-274.

[21] Warren Page and Leo Chosid, "Synthetic Division Shortened." The Two-Year College Mathematics Journal 12:5 (November 1981) 334-336.

[22] Warren Page and Harold Dorwart, "Numerical Patterns and Geometrical Configurations." Mathematics Magazine 57 (March 1984) 82-92.

[23] Warren Page and Vedula N. Murty, "Nearness Relations Among Measures of Central Tendency and Dispersion." The Two-Year College Mathematics Journal: Part I in 13:5 (November 1982) 315-327, and Part II in 14:1 (January 1983) 8-17.

[24] Warren Page and K.R.S. Sastry, "Polygonal Numbers on a Square Lattice." To appear.

[25] Jean Pederson and George Pólya, "On Problems with Solutions Attainable in More than One Way." The College Mathematics Journal 15:3 (June 1984) 1218-228.

[26] Frederick Reif, "Teaching Problem Solving-A Scientific Approach." The Physics Teacher (May 1981) 310-316.

[27] Frederick Reif and Joan I. Heller, "Knowledge Structure and Problem Solving in Physics." Educational Psychologist 17:2 (1982) 102-107.

[28] Alban J. Roques, "Homework - A Problem With a solution." The Two-Year College Mathematics Journal 10:2 (1979) 116.

[29] James E. Schultz and William F. Burger, "An Approach to Problem Solving Using Equivalence Classes Modulo n." The College Mathematics Journal. 15 (November 1984) 401-405.

[30] Herbert A. Simon, "Problem Solving and Education." In Problem Solving and Education: Issues in Teaching and Research, eds. D. T. Tuma and F. Reif. Lawrence Earlbaum Associates, Hillsdale, NJ, 1980, pp. 81-96.

[31] John Staib, "Problem Solving Versus Answer Finding." In Two-Year College Readings, ed. Warren Page, The Mathematical Association of America, 1981, pp. 221 227.

[32] Gail S. Young, "Who Takes Elementary Mathematics Courses? Why? A Guess and Some Problems for Change." In The Future of College Mathematics, ed. A. Ralston and G. S. Young, Springer-Verlag, New York, 1983, pp. 13-26.