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ABSTRACT

The objectives of this study were to: (1) develop a working model based on Pierre van Hiele and Dina van Hiele-Geldof's five levels of thought development in geometry; (2) characterize the thinking in geometry of sixth (N=16) and ninth (N=16) graders in terms of these levels (examining at what level they are at, if they show potential for progress within a level or to a higher level, and what difficulties they encounter); and (3) to analyze the grades K-8 geometry strand of three commercial textbook series. A wide range in levels of thinking among the subjects was found; some were consistent in identifying, naming, comparing and operating on geometry figures; others were able to give informal deductive explanations (level 2). It was also found that the inability to advance in level of thinking may be related to their deficiencies in language, both in knowledge of geometry vocabulary and ability to use it precisely and consistently. In addition, textbook material on geometry provides students with little opportunity to make progress to higher levels of thinking and may actually impede such progress by concentrating on level 0 thinking and reducing the level of thinking for topics which can be treated at levels 1-2. (JN)

AN INVESTIGATION OF VAN HIELE LEVELS OF THINKING IN GEOMETRY AMONG SIXTH AND NINTH GRADERS: RESEARCH FINDINGS AND IMPLICATIONS

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Background

The purpose of this research was to investigate geometric thinking of adolescents in inner city schools. The conceptual framework was built on a model consisting of five levels of thought development in geometry, presented by the Dutch educators P. M. van Hiele and his late wife, Dina van Hiele-Geldof in 1957. This model has motivated considerable research and resultant changes in geometry curriculum by Soviet educators. In recent years, interest has been growing in the United States.

As experienced teachers in Montessori schools, the van Hieles were greatly concerned about the difficulties their students encountered with secondary school geometry. They believed that secondary school geometry involves thinking at a relatively high "level" and students have not had sufficient experiences in thinking at prerequisite lower "levels." Their research work focused on levels of thinking in geometry and the role of instruction in helping students move from one level to the next. The van Hieles completed companion dissertations on levels of thinking and the role of insight in learning geometry at the University of Utrecht in 1957. Dina van Hiele-Geldof's work focused on a didactic experiment aimed at raising a student's thought level while Pierre van Hiele formulated the structure of thought levels and principles designed to help students gain insight into geometry.

The van Hiele Model

According to the van Hieles, the learner, assisted by appropriate instructional experiences, passes through the following five levels, where the learner cannot achieve one level of thinking without having passed through the previous levels.

Level 0: The student identifies, names, compares and operates on geometric figures (e.g. triangles, angles, intersecting or parallel lines) according to their appearance.

Level 1: The student analyzes figures in terms of their components and relationship among components and discovers properties/rules of a class of shapes empirically (e.g. by folding, measuring, using a grid or diagram).

Level 2: The student logically interrelates previously ...vered properties/rules ... ving or following ...mal arguments.

Level 3: The student proves theorems deductively and establishes interrelationships among networks of theorems.

Level 4: The student establishes theorems in different postulational systems and analyzes/compares these systems.

The van Hieles (1958) noted that learning is a discontinuous process and that there are jumps in the learning curve which reveal the presence of "levels". They observed that at certain points of instruction:

> The learning process has stopped. Later on it will continue itself ... . In the meantime, the student seems to have "matured." The teacher does not succeed in explaining the subject. He seems to speak a language which cannot be understood by pupils who have not yet reached the new level. They might accept the explanations of the teacher, but the subject taught will not sink into their minds. The pupil himself feels helpless, perhaps he can imitate certain actions, but he has no view of his own activity until he has reached the new level. (1958, p. 75)

Overall, the van Hieles made certain observations about the general nature of these levels of thinking and their relationship to teaching. P.M. van Hiele (1959a) notes:

> At each level there appears in an extrinsic way that which was intrinsic at the preceding level. At level 0, figures were in fact determined by their properties, but someone thinking at level 0 is not aware of these properties. (p.202)

Van Hiele (1959b) states that the levels are "characterized by differences in objects of thought." (p. 14) For example, at level 0, the objects of thought are geometric figures. At level 1, the student operates on certain objects, namely, classes of figures, which were products of level 0 activities and discovers properties for these classes. At level 2, these properties become the objects that the student acts upon, yielding logical orderings of these properties. At level 3, the ordering relations become the objects on which the student operates and at level 4, the objects of thought are the foundation of these ordering relations. Van Hiele (1959a) also points out that:

> Each level has its own linguistic symbols and its own system of relations connecting these symbols. A relation which is 'correct' at one level can reveal itself to be incorrect at another. Think, for example, of a relation between a square and a rectangle. Two people who reason at different levels cannot understand each other... Neither can manage to follow the thought processes of the other ... (p.202)

Language structure is a critical factor in the movement through the van Hiele levels - from global (concrete) structures (level 0) to visual geometric structures (level 1-2) to abstract structures (level 3-4). In stressing the importance of language, van Hiele notes that many failures in teaching geometry result from a language barrier - the teacher using the language of a higher level than is understood by the student.

The van Hieles propose a sequence of five "phases" of learning to move students from one thought level to the next. Basically these five phases constitute an outline for organizing instruction. The phases within the levels are described as follows with examples given for transition from level 0 to level 1:

Information: Through working with material presented to them, students become acquainted with the structure of the material (e.g. examine examples and non-examples).

Guided orientation: Students' investigation of the material is now guided by certain questions or directions provided by the teacher (e.g. folding, measuring, looking for symmetry).

Explicitation: Students learn to express what they have learned about the material in correct language (e.g. express ideas about properties of figures).

Free orientation: Students now apply their new languasge in further investigations of the material, possibly by doing tasks which can be completed in different ways (e.g. knowing properties of one kind of shape investigates these properties for a new shape, such as kites).

Integration: Students acquire an overview of the material they have learned (e.g. properties of a figures are summarized).

Progress from one level to the next, asserts van Hiele (1959a), is more dependent upon instruction than on age or biological maturation, and types of instructional experiences can affect progress (or lack of it).

> It is possible however that certain methods of teaching do not permit the attainment of the higher levels, so that methods of thought used at these levels remain inaccessible to the student. (p. 202)

The van Hieles (1958) point out that it is possible to te[illegible] material to students above their actual level - for exa[illegible]:

> Arithmetic of fractions without telling what fractions mean ..differentiation without knowing differential quo[illegible]ents ... . (p.76)

This then results in a reduction of the subject matter to a lower level.

In summary, the major characteristics of the van Hiele "levels" are that (1) the levels are sequential, (2) each level has its own language, set of symbols, and network of relations, (3) what is implicit at one level becomes explicit at the next level, (4) material taught to students above their level is subject to reduction of level, (5) progress from one level to the next is more dependent on instructional experience than on age or maturation, and (6) one goes through various "phases" in proceeding from one level to the next.

Research Project: Objectives, Methods, Design and Analyses

The general question that this research addressed is whether the van Hiele model describes how students learn geometry. Three major objectives were: (1) to develop a working model of the van Hiele levels, based on several sources which we had translated from Dutch into English; (2) to characterize the thinking in geometry of sixth and ninth graders in terms of levels - in particular, at what levels are students?, do they show potential for progress within a level or to a higher level?, and what difficulties do they encounter?; (3) to analyze current geometry curriculum as evidenced by American text series (grades K - 8) in light of the van Hiele model.

The first objective was achieved after an analysis of van Hiele source material, in particular, Dina van Hiele-Geldof's doctoral dissertation and Pierre van Hiele's article, "La pensee de l'enfant et la geometrie," which were unavailable in English until the Project translated them. (See Project's publication, English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre van Hiele.) Based on specific quotations from the van Hiele sources, the Project formulated a detailed model of the levels (see Appendix, pp.17-20 for level descriptors). Pierre van Hiele and two other van Hiele researchers, Alan Hoffer (1981) and William Burger (1982) examined the level descriptors and validated them for each level.

The second objective was implemented through a clinical study that was carried out in several phases. The first involved the development and validation of three modules based on the model and designed for use as a research tool in clinical interviews. Modules dealt with Properties of Quadrilaterals, Angle Sum of Polygons, and Area of Quadrilaterals. The module on Angle Sums was based on the approaches and materials used by Dina van Hiele-Geldof in her doctoral research which involved a geometry teaching experiment for twelve-year-olds. The modules included instructional activities along with key assessment tasks that were correlated with specific level descriptors. Modules were pilot tested and revised along with scripts for the interviewers. See Appendix (pp.21-30) for description of content of modules and for sample activities.

To facilitate analysis of student responses to tasks in the clinical interviews, the Project developed protocol forms for each module. These forms, to be completed by reviewers of the videotapes, contained not only check lists and questions to assess student's use of vocabulary/language, responses to different tasks, responses to key questions, van Hiele level of response, use of materials, and types of difficulties but also spaces for reviewers' descriptive comments about student's attitude, style of learning, non-verbal communication, and preference of materials. For each task, the time required and tape location were noted. The modules together with the protocol forms were validated by the researchers cited above for the level descriptors validation.

In the second phase, clinical interviews were conducted with 16 sixth graders and 16 ninth graders. In six to eight 45-minute sessions, these subjects worked with an interviewer through the modules. Sessions were videotaped.

The final phase dealt with the analysis of the videotapes and synthesis of results for the sixth and ninth graders. This was done in three stages. First, videotapes for individual subjects were reviewed by one member of the Project staff who completed the protocol forms for that student. Based on the information in the lengthy protocol forms, the reviewer then prepared a summary statement (1-2 pages for each module) on the student's performance and summary index cards noting briefly the student's level of thinking (initial and progress), difficulties, language, learning style, and miscellaneous. The next stage involved a review and validation of the initial analysis of each student's performance by one or more other members of the Project staff. This review included discussing information recorded on the protocol forms and viewing again key portions of the student's videotapes. In the final stage of the data analysis, one Project member reviewed and synthesized results for the sixth graders and another did the ninth graders. These overall results were then discussed and refined by the Project staff.

Concurrently, the third research objective, an analysis of the geometry strand of three commercial textbook series (grades K - 8), was initiated in order to determine: (1) what geometry topics are taught by grade level in order to measure the richness and continuity of instruction; (2) at what van Hiele level are the materials at each grade level; (3) if the van Hiele level of material is sequenced by grade level; (4) if there are jumps across van Hiele levels; (5) if the text presentation of geometry topics is consistent with didactic principles of the van Hieles. Criteria for selection of the text series were frequency of use both in the United States (as reported in the Science Education Databook, Directorate for Science Education, National Science Foundation, 1980), and in local Brooklyn school districts (as reported by mathematics coordinators) from which students were to be drawn for the clinical study. In general, geometry materials intended for the average student were reviewed, although activities for an enriched program were also examined. Data forms were used to collect and record the text's page by page introduction and use of vocabulary at each grade level, aim of each lesson, and the van Hiele level of the expository material, of the exercises, and of the test questions for each geometry lesson in the three terxt series, grades K - 8. The levels of exposition, exercises and test questions of a text lesson were determined by using the Project-developed level descriptors. Completed data forms were analyzed and summarized with comparisons being made among the three text series.

Findings from Clinical Study

The Project-developed modules described above were used as the basis for assessment and instruction in the clinical one-on-one interviews with sixth and ninth grade students. All interviews began with Module 1 which dealt with Properties of Quadrilaterals and involved mainly level 0 and 1 thinking about types of shapes (squares, rectangles, parallelograms) and their properties (e.g. equal sides, parallelism of sides, equal angles). Module 1 also included instructional branches that reviewed prerequisite concepts such as angle, parallel lines. After Module 1, subjects went on to Module 2 (Angles and Angle Sums for Triangles and Quadrilaterals) and/or Module 3 (Area of Rectangles, Right Triangles, Parallelograms, Triangles, Trapezoids) and Extensions (see Appendix, pp.29-30). While Modules 2 and 3 activities involved level 0 and 1 thinking, they also included tasks which called for level 2 thinking, e.g. explaining why, giving arguments or simple proofs.

The Project assessed the "entry level" of thinking of students relative to geometry topics that are commonly studied in grades 4 - 6. This was done mainly through key questions or tasks throughout Module 1 and at the beginning of Modules 2 and 3. These tasks, to which students could respond at levels 0, 1, or 2, were presented with little or no prompting from the interviewer who accepted whatever response the student gave. Since, according to the van Hieles, level of thinking is determined in part by prior learning experiences, such "static assessments" may not accurately assess the student's ability to think in geometry if the student has had little or no learning experiences on the topic involved. Therefore, the Project also assessed what might be termed the student's "potential level" by examining the student's responses as the student moved through the instuction in the interviews. This more dynamic form of assessment during a learning experience, as Dina van Hiele-Geldof did in her teaching experiment, enabled the Project to examine changes in a student's thinking, within a level or to a higher level, and also difficulties which impeded progress. Results of the sixth grade phase of this Project are reported below.

Sixth Grade Subjects

Sixteen sixth graders (9 boys and 7 girls of whom 12 were minority students - 9 Black, 3 Hispanic) from inner-city schools in New York City were involved in the study. Subjects were drawn from three achievement levels as determined by grade equivalency scores on mathematics and reading tests. There were 3 below average subjects (1 to 2 years below grade level), 5 average, and 8 above average (at least 1 year above grade level). All sixteen subjects completed Module 1, some in 3 sessions and the slowest in 7. Six subjects completed Modules 1, 2, and most of 3.

Analyses of the videotaped interviews indicated that these sixth graders fall roughly into three groups: three level 0 thinkers; five level 0 thinkers who made progress into level 1; and eight students whose entry level was 0-1 and who made progress within level 1 and even towards level 2.

Three of the 16 were strictly level 0 thinkers. They began at level 0 and for the most part, remained at level 0, even after instruction. Their thinking showed a lack of analysis of shapes in terms of their parts, lack of familiarity with basic geometric concepts and terminology, and poor language (vocabulary and grammar) both generally and in mathematics, especially expressive language. These students frequently forgot terms and concepts even shortly after they had been introduced by the interviewer. All had a weak background in school geometry and also difficulty with arithmetic concepts and skills. In fact, these three seemed to be "geometry deprived." They showed little knowledge of basic geometric concepts and language, and they reported having seldom studied geometry in grade school.

Five of the 16 began in level 0, much like the three students above, but made progress with level 0 (learning basic concepts and terms) and into level 1 (using these concepts to describe shapes and to formulate properties for some classes of shapes, in particular, familiar ones such as squares, rectangles). However, they had difficulty characterizing less familiar shapes (e.g. parallelograms) in terms of properties. Their progress was marked by instability between level 0 and level 1. Careful instruction and frequent review of concepts and terms was needed to sustain their progress. While they began to think about shapes in terms of properties (level 1), they did not try to relate properties in a logical ordering (level 2). One of these five was below grade level, the other four were on grade level. They tended to be more verbal than the three "level 0 thinkers," but they had difficulty expressing themselves using standard geometric terms and often used manipulatives in checking properties (e.g. placing D-stix on sides of a shape to show parallelism) or in explaining. These students also had a weak background in geometry. They tended to respond more easily to the interviewer than the three students above and also were less dependent on the interviewer for feedback and reinforcement.

Eight of the 16 students showed thinking at levels 0 and 1 at the start of Module 1, although most had to fill in or review some concepts (right angle, opposite sides and angles) at level 0. They also needed to become more fluent with level 1 language for describing shapes in terms of properties (e.g. "opposite sides are parallel"). These students progressed toward level 2 by following and then summarizing arguments, for example, why the opposite angles of a parallelogram are equal via saws and ladders. A few progressed farther and began to give explanations (or simple proofs) more independently and with more details and rigor. Initially, however, most students equated "proof" with generalization by examples (i.e. inductive reasoning) and only gradually after experiencing some deductive explanations in Modules 2 and 3 did they seem to acquire a sensitivity to an informal deductive approach. Some students, however, did not yet seem sure of the power of their deductive arguments even though they could follow an argument or give one on their own. They did not yet see the need for such deductive arguments. These students were all above grade level in achievement. They were quite verbal and tended to express themselves confidently. They also seemed more reflective about the questions and problems in the modules and about their own thinking.

Ninth Grade Subjects

Sixteen ninth graders (5 boys and 11 girls of whom 13 were minority students - 10 Black, 1 Hispanic, 2 Oriental) from inner-city schools in Brooklyn were involved in the study. Subjects were drawn from three achievement levels as determined by grade equivalency scores on mathematics and reading tests. There were 2 below average subjects (1 to 2 years below grade level), 8 average, and 6 above average (at least 1 year above grade level). All subjects completed Module 1, ten completed the three instructional modules with eight of these completing some Extensions (see Appendix pp. 29-30), and five subjects completed only two of the modules.

Since the seventh and eighth grade mathematics curriculum has large units on informal geometry, it was expected that most ninth graders would have a stronger background in basic geometric concepts than sixth graders. Analyses of the videotaped interviews indicated that the ninth grade subjects fell into three groups: two level 0 thinkers; seven students entering at level 0-1 made progress to be classified as level 1 thinkers with some progress towards level 2; and seven students entering at level 1 (with occasional lapses to level 0) made progress to be classified as level 1-2 or in some instances as level 2.

The ninth grade level 0 thinkers had the same characteristics as those described above for the sixth grade level 0 thinkers. Their decision-making about shapes and properties was always on a "looks like" basis. Particularly noticeable was their poor language, i.e. their inability to express an idea clearly in a complete sentence.

The entering level of seven ninth graders was assessed as level 0-1. Most had to fill in or review some basic grometric concepts. They thought of certain shapes (triangles, rectangles and squares) in terms of their properties but they had less or no knowledge of parallelograms and trapezoids. They used or learned level 1 language for describing shapes and their properties, only occasionally reverting to level 0 type explanations. In order to justify conclusions, they frequently resorted to an inductive approach (i.e. measuring a number of specific cases). In some instances prior learning and/or misconceptions interfered with progress (e.g. "a right angle points to the right;" "one ray of an angle must be horizontal;" "a square cannot be a rectangle"). As with the comparable group of sixth graders, some of these students progressed toward level 2 by following and/or summarizing arguments and by trying to relate some properties by a logical ordering.

The remaining group of seven ninth graders (whose entry level was assessed as level 1) needed very little review of basic concepts and used appropriate but sometimes non-standard language to describe properties of figures. They readily explained subclass relations and learned to relate properties in a logical ordering. They not only followed arguments but learned to provide simple deductive explanations, thereby showing characteristics of level 2 thinking. In addition, some were able to formulate definitions and justify necessary and sufficient conditions in given tasks. All of this group were able to do at least some of the Extensions beyond the three basic instructional modules (see Appendix pp. 29-30).

Discussion of Results

Results indicate that there was a wide range in levels of thinking among the subjects - from some who were consistent level 0 thinkers to some who were able to give informal deductive explanations (level 2). The entry level of the sixth graders was mainly level 0 or at times level 1 and for ninth graders, it was mainly level 0-1 or level 1. Students tended to identify and sort shapes on an "it looks like" basis rather than by using properties. Initial descriptions of shapes (e.g. a rectangle) were generally imprecise and level 0 in nature, even for students who later exhibited level 1 thinking. It was only after some instruction that students began to express themselves more precisely in terms of properties of shapes. Reduction of level was also observed for topics that could have been treated at levels 1 or 2 - for example, the angle sum of a triangle or area rules for rectangles and triangles which students knew only by rote rather than by inductive or deductive explanations.

There are several possible explanations for this level 0 thinking. One is the lack of experience in doing geometry in school. Several students, even some of the higher ability ones, reported doing little geometry in grades 4-8. A second explanation, as will be seen later, is that even when geometry was studied, the text material probably did little to encourage higher levels of thinking. A third explanation is that the entry assessment tasks which involved cut-out figures or diagrams and can be responded to at levels 0, 1 or 2, are done most naturally at level 0 which matches the format in which they are presented. Also, since the interviewer accepted level 0 responses as correct and reinforced them with comments such as "okay....good..", the student may have been led to believe that this was the expected kind of response. Students who made progress profited from well-defined directives from the interviewer about the kind of reponse that was acceptable.

While the entry level of the sixth graders was mainly level 0 or at times level 1, potential level for all but three of the 16 sixth graders was at level 1 or even 2. The lack of progress into level 1 for the three sixth graders and the two level 0 ninth graders can be attributed to several factors. First, they seemed to lack general ability, in particular in the use of language. They did not exhibit language needed for level 1 such as the use of quantifiers (e.g."all these rectangles have ...) and conditional expressions e.g. "if the shape has..., then it is a ...). They also lacked most prerequisite geometric and measurement concepts. The instructional materials in Module 1 were designed to review topics normally covered in grades 4-8, not develop them for such weak students. Also, the interview schedule did not permit time needed to carefully develop topics with these students. Additional research is needed to determine whether other materials and extended instruction would have enabled low ability students to make progress into level 1.

Progress of the other subjects was also influenced by instruction and ability, in particular, language ability. Students who were in transition to level 1 needed considerable help, guidance, and encouragement. They had difficulty remembering geometric terms, although they did not have trouble with related concepts. They needed directives that focused their attention on parts of shapes and on sets of shapes (level 1) to prevent them from lapsing back into level 0 thinking about individual shapes on an "it looks like" basis. Students who were solid level 1 thinkers or progressed toward level 2 picked up ideas quickly, remembered terminology and use it appropriately. They became more fluent in talking geometry as they moved through the modules. They also displayed the ability to reason well, both inductively and deductively, as was reflected by their use of language associated with levels 1 and 2, for example "all...any...because...I can explain...it follows that...it is true because...I can prove that."

At times students' progress was impeded by their lack of expectation to explain. At the beginning, many of the ninth graders used a rather algorithmic or procedure-oriented approach to tasks. Their usual reply, when asked for an explanation, was "that's a rule." This was also true for some sixth graders. Gradually those students who made progress toward level 2 realized that explanations were expected and began spontaneously offering reasons or giving arguments to justify their statements.

The interviewer and instructional materials played a special role in helping students to progress within a level or to a higher level. The interviewer provided instruction designed to move students to a higher level. Also the interviewer guided student responses through questioning and directives about the quality of responses, helping students to learn the rules of the game as it were - for example, to observe relationships between parts of a figure and to make generalizations (level 1) or to give deductive explanations (level 2). Students on a given level realize this, but students in transition need guidance about expectations, and the interviewer-teacher can use a meta-language about thinking to communicate such expectations to the student. For example, the two sixth grade students who gave careful proofs of area rules did so to "be technical" or "to clinch it" (their words for the kind of response they thought the interviewer wanted).

Two additional comments about the levels of thinking found for students in this study can be made. The first concerns the differences in performance for most subjects on tasks designed to assess entry level of thinking and on tasks designed to measure potential level of thinking as a result of instruction. Several of this Project's tasks for measuring entry level were similar to those used by Burger (1982) who conducted two 45-minute interviews with students in grades K - 12. Burger found mainly level 0 thinking for subjects in grades K - 8, just as this Project did for most assessments of entry level. However, in this study, most students performed at a higher level on "potential" assessment tasks than on "entry" tasks, especially students who progressed toward level 2. This suggests that conventional tests or assessments of level of thinking may not adequately characterize the student's ability to think at certain levels, especially when there has been little or no opportunity to experience topics and logical reasoning processes in school. Second, these results indicate that some sixth graders and almost all ninth graders are quite capable of engaging in geometry activities that call for level 1 and even level 2 thinking which is probably a higher level than they experienced in school.

Text Analysis

It is not the purpose of this paper to give a detailed report on the Project's text analysis. However, some aspects of this analysis are reported below because they are particularly relevant to the student's progress or lack of progress in the clinical part of the study.

Before presenting the Project's findings from the text analysis, a brief description of a similar text analysis done by Soviet educators will be given. Using the van Hiele model, Soviet researchers Pyshkalo (1968) and Stolyar (1965) investigated the levels of thinking embodied in their Soviet school mathematics program. A detailed analysis of the standard textbooks for Soviet grades 1-5 (our grades 3-7) revealed the absence of any systematic choice of geometric material, large gaps in its study, and a markedly late and one-sided acquaintance with many of the most important geometric concepts. Also, Pyshkalo found jumps in levels (primarily from level 0 to 2) with respect to a majority of the concepts studied. In addition, the study of geometric concepts in grades 1-5 continued on level 0, and then only from the quantitative aspect, such as measuring length or area.

Results of the Project's text analysis are remarkably similar to those of the Soviet study. Results confirm the existence of some gaps in van Hiele terms. For example, while level 0 experience is very rich in some series, the lack of extensive level 1 experience in grades K - 8 indicates that many students enter high school geometry courses (which require level 2 thought) with a level 0 background. There are also frequent gaps in level in individual text pages, where the exposition is at a higher level than the exercises required of the student. Tests are usually at level 0. Also level 0 development is frequently misleading, since shapes are shown in standard orientations. Non-examples are not presented along with examples of the shapes or concepts thus making it difficult for students to formulate a working description of the shape or concept. Students are seldom asked to formulate properties, and the text itself frequently generalizes from one instance.

There are relatively few examples of materials in the text series surveyed where the next level of thought is implicitly used - for example, the angle sum of a triangle is often developed through an experiment, measuring the three angles and adding the measures, a method which lacks the potential for later understanding provided by other approaches (e.g. as used by Dina van Hiele-Geldof in her teaching experiment).

Many of the lessons devoted to "geometry" or "measurement" have as aims the practice and development of vocabulary, direct measurement, or application of a formula. Students are seldom asked to explain formulas for measurement, and hence lessons involving formulas are susceptible to "reduction of level." When explanations are not asked for, students tend to focus on memorizing the rule as the important thing rather than explaining why it works.

The clinical study conducted by the Project indicated that students' inability to advance in level of thinking may be related to their deficiencies in language - both in knowledge of geometry vocabulary, and ability to use it precisely and consistently. The text analysis indicates that students do not receive much help in developing language ability from their texts. A suggestion for text writers might be to include more questions that require spontaneous recall of geometry vocabulary (e.g. "describe the sides of this figure" rather than "identify which sides are parallel"), and also questions that require formulation of thoughts into sentences (e.g. "explain why...").

As noted above, unfortunately USA textbook material on geometry provides students with little opportunity to make progress to higher levels of thinking and may actually impede such progress by concentrating on level 0 thinking and reducing the level of thinking for topics which can be treated at levels 1-2. This supports van Hiele's original contention that difficulties in tenth year geometry are related to a gap in level of thinking created by insufficient experiences at levels 0-2. Closing this gap by improving the geometry curriculum and instruction, particularly in grades 4-8, is a challenge that the Soviets educators have already acknowledged and one that USA mathematics educators are beginning to accept, in particular "levelists" who find the van Hiele model a useful tool for formulating and addressing issues about learning and thinking in geometry.

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10. van Hiele, P.M. Development and Learning Process. Acta Paedagogica Ultrajectina. Groningen: J. B. Wolters, 1959b, pp. 1-31.

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*12. van Hiele, P.M. Begrip und Inzicht. Muusses Purmerend, 1973.

13. van Hiele, P.M. Wie Kann Man Im Mathematikunterricht Den Denkstufen Rechnung Tragen? Educational Studies in Mathematics, 7 (1976) pp. 157-169.

14. van Hiele, P.M. van a tot z (Text series for secondary school students of different abilities). Muusses Purmerend, 1976, 1977, 1978, 1979.

15. van Hiele, P.M. Levels of Thinking, How to Meet Them, How to Avoid Them. Paper presented at Special Interest Group for Research in Mathematics Education, Presession of National Council of Teachers of Mathematics Annual Meeting, April 1980.

16. van Hiele, P.M. and van Hiele-Geldof, D. A Method of Initiation into Geometry at Secondary Schools. In H. Freudenthal (Ed.), Report on Methods of Initiation into Geometry. Groningen: J. B. Wolters, 1958, pp. 67-80.

*17. van Hiele-Geldof, Dina. De didaktick van de Meetkunde in de eerste klass van het V.H.M.O.; includes English summary. Doctoral dissertation, University of Utrecht, 1957, pp. 1-183.

*18. van Hiele-Geldof, Dina. De didaktick van de Meetkunde als leerproces voor volwassenen, 1958.

19. Wirszup, Izaak. Breakthrough in the psychology of learning and teaching geometry. In J. L. Martin (Ed.), Space and Geometry: Papers from a Research Workshop. Columbus, Ohio: ERIC/SMEAC, 1976.

*Project Translations into English of Selected van Hiele Works:

1. Complete doctoral dissertation of D. van Hiele-Geldof: De didaktick van de Meetkunde in de eerste klas het V.H.M.O., Utrecht, 1957, pp. 1-183.

2. Selected portions and summaries of chapters of P.M. van Hiele's Begrip en inzicht.

3. Complete article by D. van Hiele-Geldof: De didaktiek van de meetkunde als leerproces voor volwassenen, 18 pages, 1958. (Last article by D. van Hiele-Geldof)

4. Complete article of P.M. van Hiele: La pensée de l'enfant et la geometrie. Bulletin de L'Association des Professeures de Mathématiques de l'Enseignment Public, 1959a, pp. 199-205.

APPENDIX

Descriptors of van Hiele Levels

LEVEL 0: Student judges and operates on shapes (e.g. squares, triangles) and other geometric figures (e.g. lines, angles, intersecting and non-intersecting lines) according to their appearance.

The student:

1. identifies instances of a figure by its appearance as a whole:

   (a) in a simple drawing, diagram, or set of cutouts (e.g. squares, right angles);

   (b) in a photograph or physical object;

   (c) in a shape or other more complex configuration (e.g. angles in a quadrilateral or in two intersecting lines; shapes in a pattern of a triangular grid; edges, faces, vertices of a cube).

2. recognizes shapes and other geometric figures in different positions/orientations.

3. copies a shape (on a geoboard, on dot/graph/grid paper).

4. names or labels shapes and other geometric fig[illegible] names and labels appropriately.

5. compares and sorts shapes on the basis of their appearance as a whole (e.g. on an "it looks like basis").

6. verbally describes shapes by their appearance as a whole (e.g. a rectangle "looks like a window", a parallelogram "looks like a slanty rectangle", and angle "looks like hands on a clock").

7. operates on shapes by folding, measuring, coloring, constructing, manipulating (e.g. making patterns with pattern blocks or by coloring a triangular grid; solving a geometric puzzle) to experience how shapes are made of parts, or how different shapes are related.

8. applies other geometric notions to figures as a whole (e.g. finds area of shape by covering it with tiles or counting squares on a grid overlay; symmetry; coordinates and graphing).

9. solves routine problems by measuring, counting, etc. rather than by using properties which apply in general.

10. identifies parts of figure but does <u>NOT</u> analyze a figure in terms of its components, does <u>NOT</u> think of properties as characterizing a class of figures.

LEVEL 1: Student analyzes figures in terms of their components and relationships between components, and establishes properties of a class of figures empirically (e.g. by folding, measuring, making a model, using a diagram on a grid or cutout shapes).

The student:

1. identifies and tests relationships among components of a figure (e.g. congruence of opposite sides of a parallelogram; congruence of angles in a tiling pattern).

2. recalls and uses appropriate vocabulary for components and relationships (e.g. opposite sides, corresponding angles are congruent, diagonals bisect each other).

3. (a) compares two shapes according to relationships among their components (e.g. notes how a square and rectangle are alike and different in terms of sides and angles);

   (b) sorts shapes in different ways according to certain properties, including a sort of all instances of a class from non-instances.

4. (a) interprets and uses a verbal description of a figure in terms of its properties and uses this description to draw/construct the figure;

   (b) interprets verbal or symbolic (e.g. $A= bxh$) statements of rules and applies them.

5. discovers properties of specific figures empirically and generalizes properties for that class of figures (e.g. angle sum of a triangle is 180° by observing several examples).

6. (a) describes a class of figures (e.g. parallelograms) in terms of its properties, and

   (b) tells what shape it is, given certain properties.

7. identifies which properties used to characterize one class of figures also apply to another class of figures and compares classes of figures according to their properties (e.g. both rectangles and parallelograms have opposite sides equal and parallel).

8. discovers properties of an unfamiliar class of figures (e.g. kites).

9. solves geometric problems by using known properties of figures or by insightful approaches.

10. searches for properties/relationships (guided by teacher/materials, or spontaneously on own) but:

    (a) can<u>NOT</u> explain how certain properties of a figure are interrelated (e.g. how "opposite angles are equal" can follow from "opposite sides are parallel" for quadrilaterals);
    (b) can<u>NOT</u> explain subclass relationships (e.g. all rectangles are parallelograms);
    (c) does <u>NOT</u> see a need for proof of generalizations discovered empirically (e.g. why angle sum of any triangle must be 180°).

LEVEL 2: Student formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows deductive arguments.

The student:

1. (a) identifies different sets of properties that characterize a class of figures;

   (b) identifies minimum sets of properties that can characterize a figure.

2. gives informal arguments (using diagrams, cutout shapes that are folded, or other materials):

   (a) having drawn a correct conclusion from given information, justifies the conclusion (e.g. explains why the third angles of two triangles are equal if the other two angles in each triangle are equal);

   (b) orders classes of figures (e.g. explains why all rectangles are parallelograms);

   (c) orders two properties (e.g. sum of the angles of a triangle is "the ancestor" of sum of the angles of a quadrilateral; area rule for parallelogram is used to derive the area rule for triangle);

   (d) discovers new properties by deduction (e.g. angle sum of a pentagon must be $3 \times 180^{\circ}$, or 540°);

   (e) interrelates several properties (e.g. makes a genealogical tree showing how "saws/ladders" are used to explain why the sum of the angles of a triangle equal 180°).

3. (a) follows a deductive argument and can supply parts of the argument (e.g. why the diagonal of a parallelogram divides it into two congruent triangles, and establishes the rule bh/2 for the area of a triangle);

   (b) can give a summary of a deductive argument.

4. (a) can sometimes give more than one correct explanation/argument to prove something, (e.g. compares two explanations for the area of a trapezoid or sum of the angles of a quadrilateral), and

   (b) relates different explanations to a family tree.

5. recognizes differences between a statement and its converse and can test their truth or falsity.

6. identifies and uses strategies or insightful reasoning to solve problems.

7. recognizes the role of "proof" but:

   (a) does <u>NOT</u> grasp its meaning in an axiomatic sense (e.g. does <u>NOT</u> see the need for definitions and basic assumptions);

   (b) can<u>NOT</u> yet establish interrelationships between networks of theorems.

Content of Instructional Modules

Module 1: Properties of Polygons

Activity 1 Introductory game
Activity 2 Shapes in Pictures
Instructional Branches - Basic Concepts (as needed on parallel, angle and right angle, opposite sides and angles, congruence)
Activity 3 Sorting and Properties of Groups of Shapes
- a. Guess My Rule (by sorting number of sides)
- b. Sorting quadrilaterals
- c. Names and Properties of quadrilaterals
- d. Inclusion Relations
- e. Guess My Rule (sorting by parallelism of sides)
- f. Guessing shapes particlly uncovered
- g. Guessing shapes from partial list of properties
- h. Minimal properties for defining a shape

Activity 4 Kites - Properties and Inclusion Relation

Module 2: Angle Measurement and Angle Sum for Polygons

Activity 1 Angle Measurement - Assessment
Instructional Branch (angle measurement)
Activity 2 Making Tilings and Grids
Activity 3 Saws and Ladders
Activity 4 Coloring angles (establishing congruence of angles)
Activity 5 Developing Properties from Grids (Angle sum for triangle; quadrilaterals)
Activity 6 Family Trees (to interrelate properties and develop a logical hierarchy)

Module 3: Area

Activity 1 Introductory activity - Tangrams
Activity 2 Area concepts - Assessment
Activity 3 Area of rectangle
Activity 4 Area of right triangle
Activity 5 Area of parallelogram
Activity 6 Area of triangle
Activity 7 Area of trapezoid

Extensions

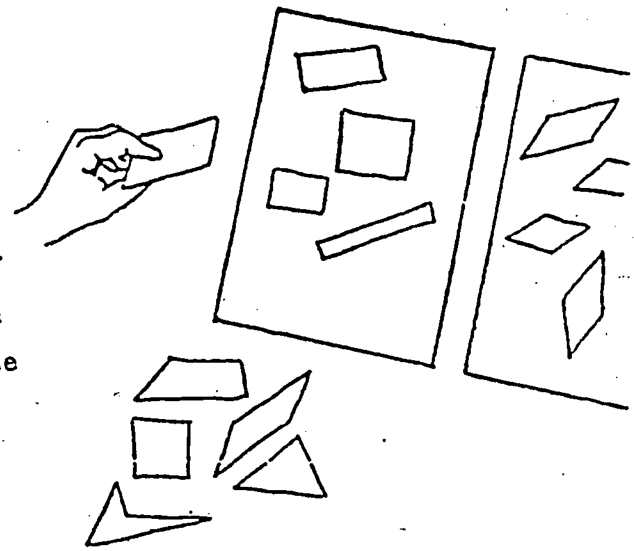
Activity 1 Areas of figures having all vertices on two parellel lines
Activity 2 Proving opposite angles/sides of a parallelogram congruent
Activity 3 Exterior angle of a triangle
Activity 4 Interrelationships of family trees

SAMPLE ACTIVITIES FROM INSTRUCTIONAL MODULES

Below are given selected sample activities from the instructional modules. Some activities are included primarily to diagnose van Hiele levels of thought, while others play an instructional role and are intended to help the subject move from one level to the next. Given the age group of subjects with which this investigation is concerned (grades 6 and 9), the Project designed the instructional modules to attain only Level 2 thinking.

MODULE 1: INTRODUCTORY ACTIVITIES – PROPERTIES OF POLYGONS

LEVEL 0

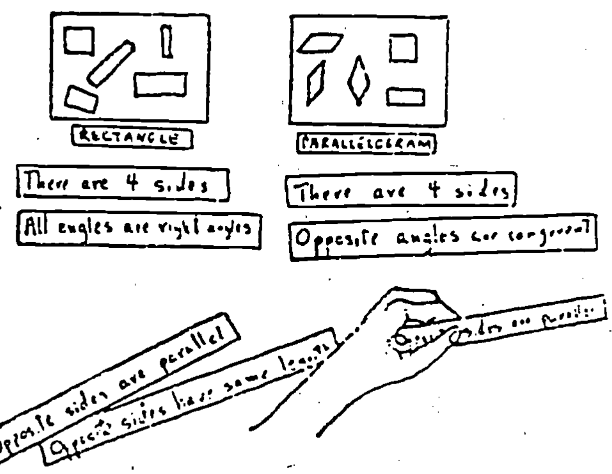


Tasks: Subject is asked to identify figures in photographs of houses, city landscape.

Subject is asked to sort a collection of cut-out shapes and to verbalize why each piece is placed where it is. (A response of "Because it looks like those" indicates Level 0 thought.)

Instructional branches are provided for concepts essential for later activities.

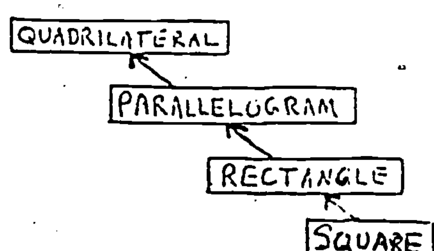
LEVEL 1



Tasks: With shapes available for sorting, subject is asked to arrange picture cards, name cards and property cards for a selection of shapes.

As a polygon is slowly uncovered by a piece of paper, subject is asked what the polygon could be, based on what is visible.

LEVEL 2



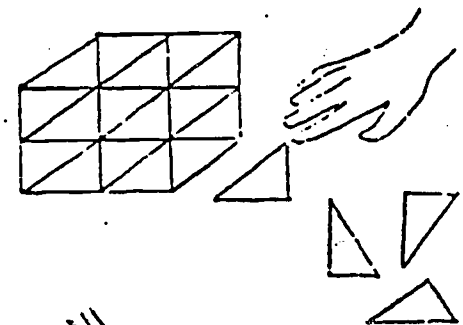
Tasks: Subject is asked to show the inclusion relations among classes of shapes.

MODULE 2: TILING AND ANGLE MEASUREMENT

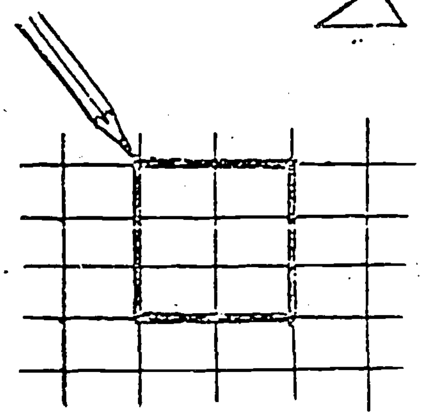
Below are given sample activities from Module 2. (Note: Many of these ideas are adapted directly from the work of Dina van Hiele-Geldof.)

LEVEL 0

Tasks: Subject is asked to use cut-out congruent shapes to make tilings.

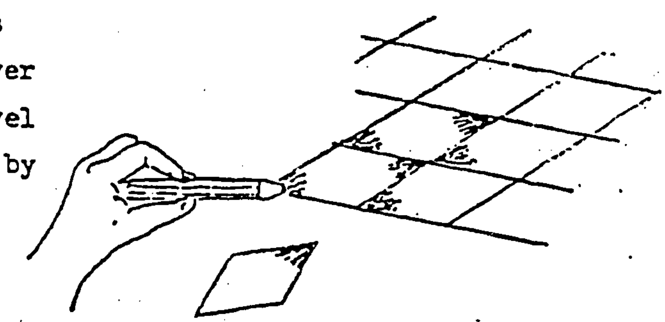


Subject is asked to make a sketch of tiling having available only one cut-out to trace. (At Level 0, subject tends to copy pattern by tracing shape by shape, rather than to look for and use families of parallel lines systematically, using a ruler.)



Subject is asked to find examples of figures in tiling grids. (At Level 0, subject will justify figures found "by eye", on a basis of "because it looks like one", and does not spontaneously refer to properties.)

Subject is asked to color congruent angles in tiling grids the same color, perhaps fitting the corner of a cut-out tile over the angle to check congruence. (At Level 0, subject will tend to color in angle by angle, rather than to use a systematic procedure such as 'saws' and 'ladders', described on the next page.)

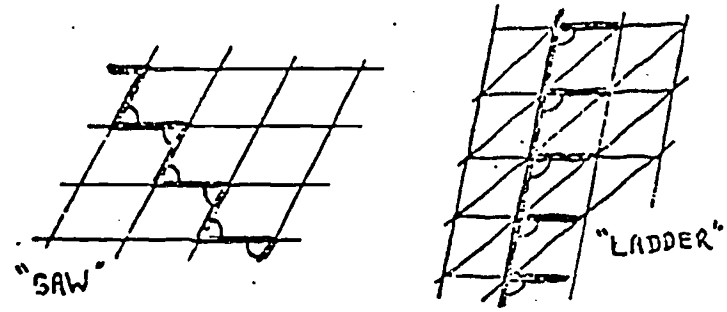


While doing these activities, Level 0 subjects will be gaining experience which can lead to Level 1 thought, for example by discovering properties of shapes in the course of tiling, or noting patterns when coloring angles.

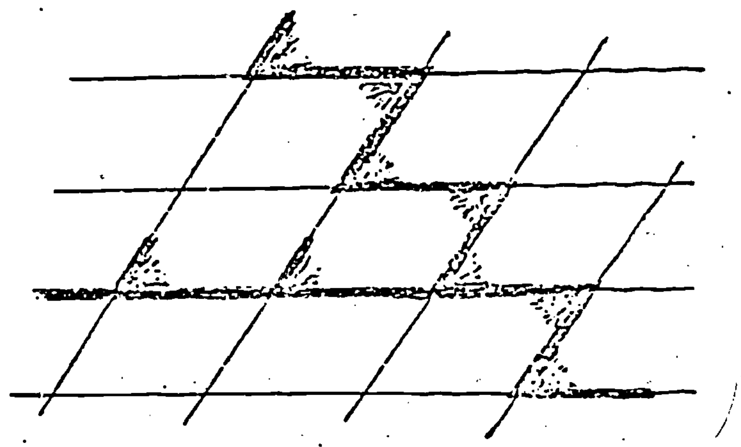
For subjects who are not skillful in angle measurement, appropriate instruction is provided; beginning with direct comparison of angle (using a movable model), then using wedges as non-standard units and introducing the degree as a standard unit, and finally using an acetate overlay "angle tester" to measure angles.

LEVEL 1

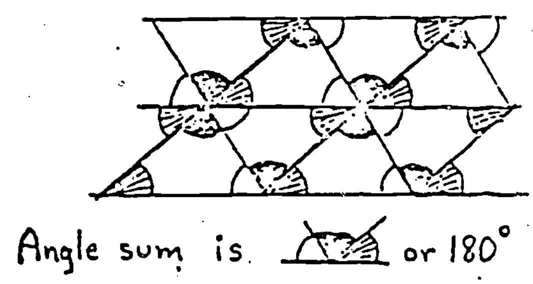
Tasks: Subject is asked to find "saws" and "ladders" in tiling patterns, to describe what these are in terms of parallel lines, and to note congruent angles.

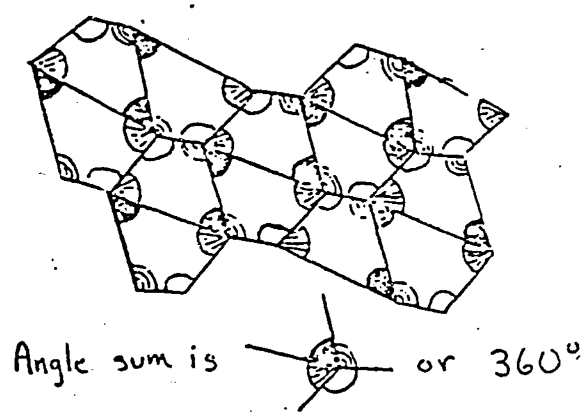


Subject is asked to color in angles on tiling grids using a cut-out shape as a model, and to note how "saw" and "ladder" can be used to color congruent angles in a grid systematically.



Subject is asked to note relationships such as congruence of opposite angles of a parallelogram, and angle sum of a triangle and of a quadrilateral.

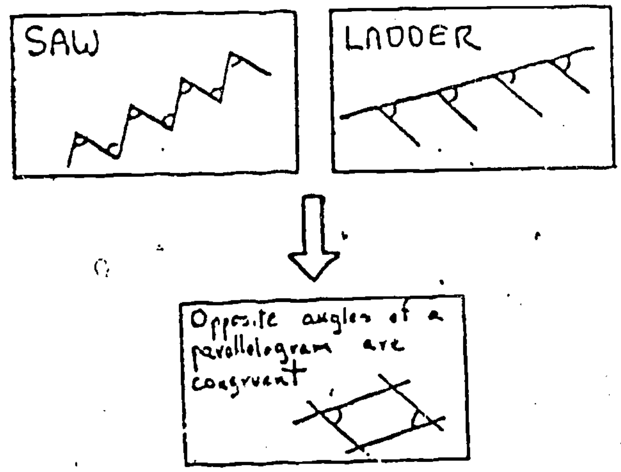


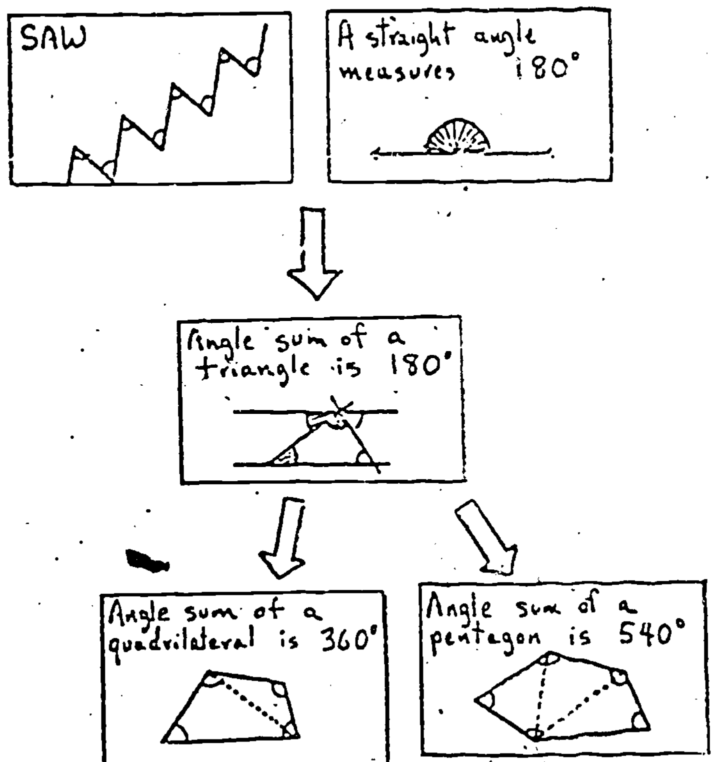


While doing these activities, Level 1 subjects will begin implicitly using relationships among properties, thus laying the groundwork for explicit consideration of these relationships at Level 2.

LEVEL 2

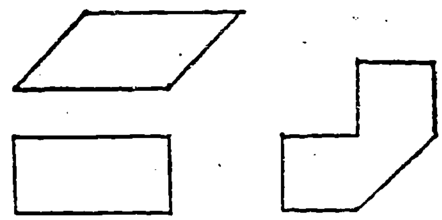
Tasks: Subject is asked to explain the logical ordering among relationships discovered in Level 1 activities. Using the image of a "family tree" with cut-out arrows to suggest "ancestry", subjects are asked to arrange property cards (which were developed in earlier activities) to show logical inter-relationships.



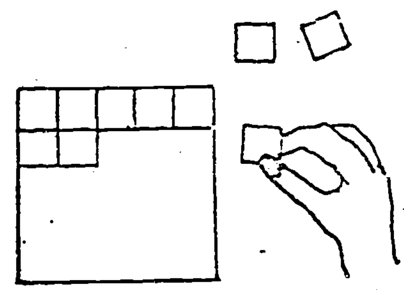


Below are given sample tasks from the instructional module on area. The modules begin with an informal introductory activity that involves Tangram puzzles. The second activity is disgnostic in order to assess the subject's initial level of thinking relative to area topics and to determine branching to subsequent instructional activities on area rules for rectangles, parallelograms, triangles, and trapezoids. Each of these activities involves informal arguments for deducing the area rule.

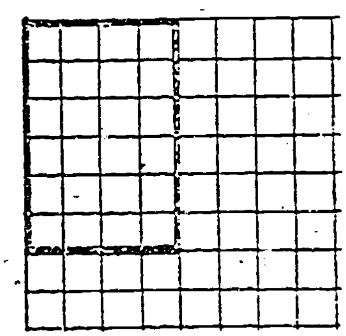
LEVEL 0



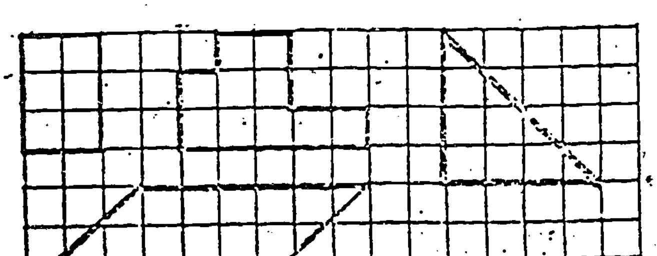
Tasks: Subject is asked to compare the area of shapes. The subject may place Tangram pieces on the shapes to check.



Subject is asked to determine which of two cutout rectangles (5x5 and 6x4) is larger. The question is posed informally as "which of these needs more gold foil to cover it?" Subjects at Level 0 may use square inch tiles and count, or use a transparent square inch grid. Later, the subject interprets "area" as how many square units cover a shape.



Student is asked to find the area of various shapes. A Level 0 response is to count squares and half squares.



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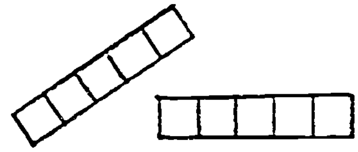
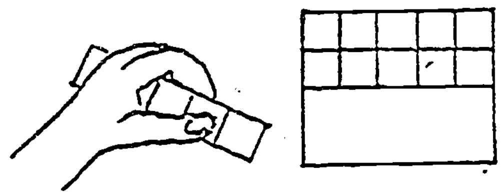
LEVEL 1

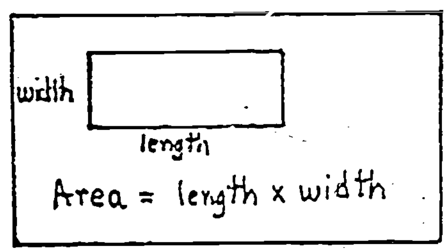
Here the subject is guided to discover ways to find the area of different shapes. Rules are then formulated and applied.

<u>Tasks</u>: The subject is asked to find the areas of several rectangles, using strips of squares to cover each rectangle. This leads to a repeated addition or multiplication notion for area of a rectangle. The subject is asked if this procedure (which the subject may have expressed as "multiply how many rows times how many squares in a row") works for all rectangles. The procedure is then refined to yield: to find the area of a rectangle we multiply the length x width. An area rule card summarizes this.

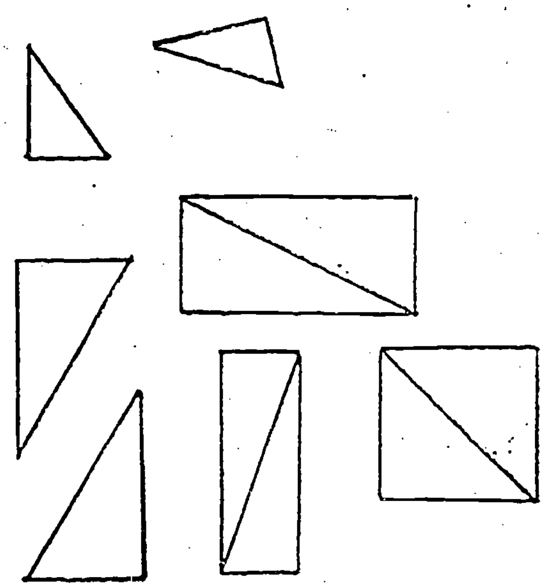




Four strips of 5, or 20

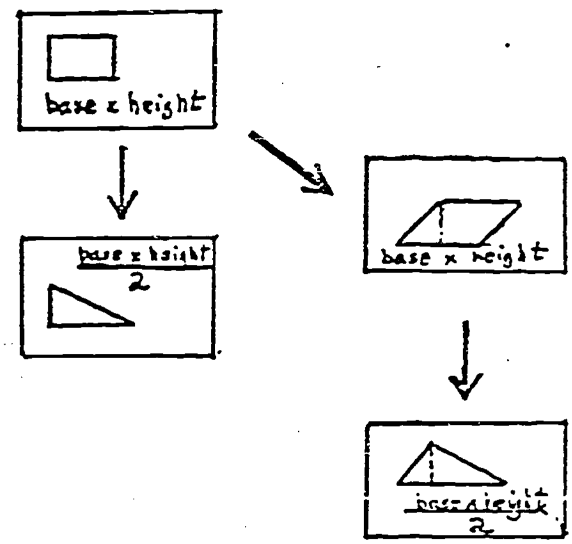


The subject is presented with pairs of congruent right triangles and asked to make shapes with each pair. The subject is guided to discover that the area of any right triangle is one-half the area of the related rectangle. Later this is expressed more formally as area = (base x height)/2 The subject is asked to apply both the procedure and the rule.



LEVEL 2

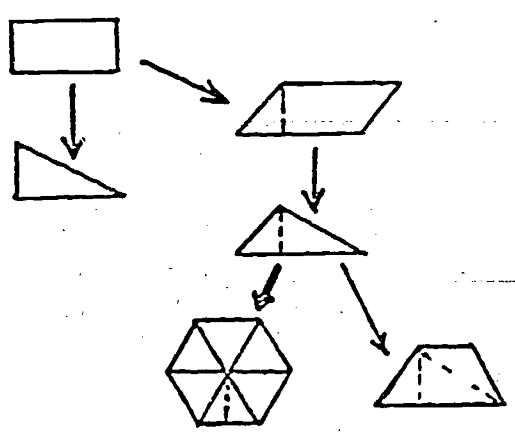
The student is asked to explain how certain area procedures and rules can be derived from others. The area rules are logically interrelated using a "family tree" schema.



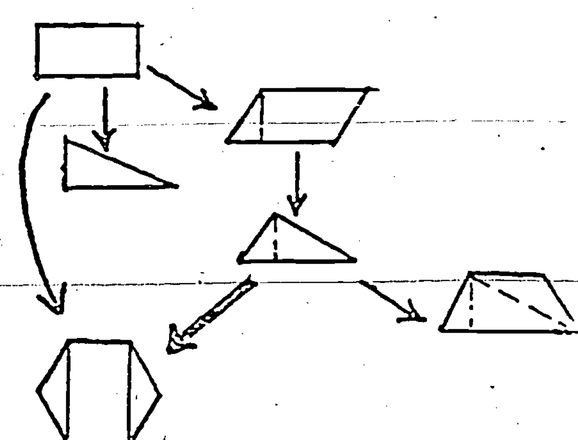
Tasks: Having discovered a procedure for finding the area of any triangle (one-half the related parallelogram), the student is asked to relate this procedure to those already established. The student places the "follows from" arrow and the triangle area rule card in the previously established "family tree" and explains why they were placed in that way.

Having discovered different procedures for finding the area of a regular hexagon, the student is asked to relate each way to a previously established "family tree" of area rules.

One Procedure



Another Procedure

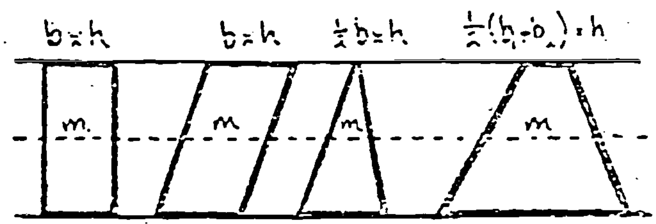


REVIEW AND EXTENSIONS

This material is designed only for those subjects who have indicated potential for Level 2 thought by their performance in Modules 1-3. It allows a distinction to be drawn between subjects who can follow an argument with considerable guidance (often the case in Modules 1-3) and those who can also retain and recreate arguments at Level 2. It also assesses subjects' ability to provide a new logical justification (an "informal proof") and to recognize inter-relationships among previously developed principles. Four activities are sketched below.

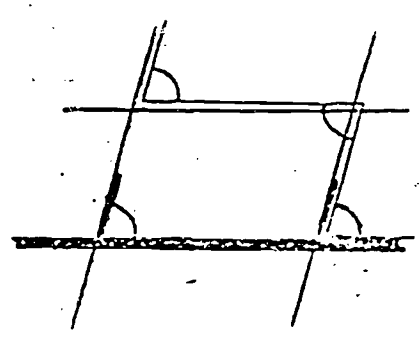
LEVEL 2

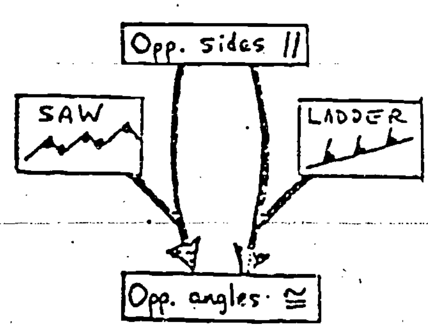
Tasks: Activity 1: Subjects are led to discover and justify the general principle for finding area of figures whose vertices lie on two parallel lines (area = midline x height) using previously developed principles.



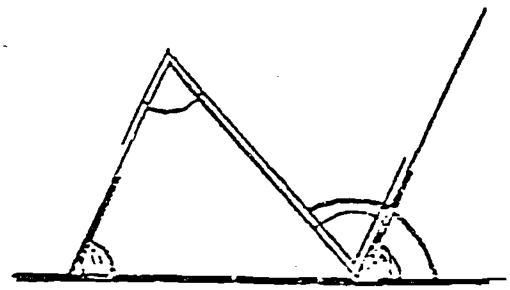
$A = m \times h$

Activity 2: Subjects are asked to determine a minimal set of properties required to define a parallelogram, and to apply "saw" and "ladder" properties to prove opposite angles are congruent.





Activity 3: Subjects are led to discover that an exterior angle of a triangle is the sum of the opposite interior angles, and they justify this (i.e. create a proof, not just follow one) using angle sum of a triangle, or "saws" and "ladders".



Activity 4: Subjects are asked to recognize and point out relationships among previously developed "family trees", and to consider questions such as whether "saws" and "ladders" have ancestors. Subjects are asked whether principles in other areas of math (e.g. arithmetic and algebra) might have "family trees", and if there are common ancestors.