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ABSTRACT

An inquiry attempted to build an agenda for research that would result in a cognitive theory of instruction capable of informing educational practice and extending the limits of knowledge about how people learn and develop. What would such a theory look like, how close are we to having one, and what directions must be followed to further its development are among the questions explored. A brief history is presented of psychological theories of instruction from the viewpoints of E. L. Thorndike, B. F. Skinner, Gestalt psychology, and Piaget. It is pointed out that a cognitive theory of instruction must be both descriptive (explaining why instruction works and why it does not) and prescriptive (suggesting what to do next time for better results). Within this framework, three components of such a theory of instruction are described and analyzed: (1) specification of capabilities to be acquired; (2) description of the acquisition processes; and (3) principles of intervention. (JD)

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# LEARNING RESEARCH AND DEVELOPMENT CENTER

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## TOWARD A COGNITIVE THEORY OF INSTRUCTION

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# 1 Toward a Cognitive Theory of Instruction

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We are now well accustomed to noting the cognitive "revolution" that has characterized the last decade or two of psychology. The human mind has been rediscovered, or at least reaffirmed; reasoning and thought are central objects of scientific study; and the nature of human cognitive abilities is being examined in fresh ways. It seems evident that the new conceptions of human competence that are emerging ought to affect the practice of education—that a cognitive theory of instruction ought to be emerging alongside our increasingly elaborated theories of cognitive performance and development. What would such a theory look like, how close are we to having one, and what directions must be followed to further its development? These are the questions explored in this chapter. The goal of this inquiry is to build an agenda for research that will result in a cognitive theory of instruction capable of informing educational practice and at the same time extending the limits of our knowledge about how people learn and develop.

Let us begin with some definitions that will serve to set the boundaries of the inquiry. First, I define as instruction *anything that is done in order to help someone else acquire a new capability*. This is an intentionally broad definition. It means that instruction is not limited to traditional "teacher's tasks," such as lecturing or conducting recitations or setting homework assignments—although these are certainly activities that may qualify as instruction. Rather, any act that intentionally arranges the world so that somebody will learn something more easily qualifies as instruction. I think it will become clear as the chapter proceeds why this broad definition of instruction is essential—indeed is dictated by—the view of human learning that is being elaborated by current cognitive psychology.

With this view of instruction as a point of departure, we can now consider the elements of a theory of instruction. Such a theory must be both descriptive, explaining why instruction works and why it does not, and prescriptive: suggesting what to do the next time for better results. For these purposes three requirements must be met. First, a theory of instruction must specify the new capabilities that we are trying to help somebody acquire—that is, the goal of the instructional effort. Second, it must provide a theoretical account of how people acquire these desired capabilities. Finally, an instructional theory must specify how something done by an instructor interacts with the individual's processes of acquisition so that something new is acquired. There are, then, three components to a theory of instruction: (1) specification of capabilities to be acquired; (2) description of acquisition processes; and (3) principles of intervention.

## A BRIEF HISTORY OF PSYCHOLOGICAL THEORIES OF INSTRUCTION

The effort to build a theory of instruction is rooted in today's cognitive psychology and poses a new challenge, but this is by no means the first time that psychologists have addressed this task. A brief review of some past efforts at drawing instructional implications from psychological theory will help us to appreciate both the goals and the potential pitfalls of our new venture.

### E. L. Thorndike and the Theory of Bonds

Our account begins with Edward L. Thorndike, the prominent American associationist. Thorndike had a well-developed instructional theory that grew directly out of his general associationist theory of how the human mind works. For Thorndike, new capabilities to be acquired could be described as collections of "bonds"—that is, associations between stimuli or between stimuli and responses. Thorndike took so seriously the notion of defining instructional goals in these terms that he actually undertook an analysis of school subject matter. In 1922 he published a book entitled *The Psychology of Arithmetic*, which contains many lists of the bonds he thought made up the subject matter of arithmetic. The book thus essentially offered what we might now call a *task analysis* of arithmetic, in terms consonant with associationist learning theory. In keeping with associationist principles, there was minimal organization imposed on the lists of bonds. Thorndike implicitly recognized some deeper structure than that reflected in a simple collection of bonds; he proposed that bonds that "go together" should be taught together. Thus, he clustered addition bonds in one list and subtraction bonds in another, and so forth, largely following common sense views of arithmetic content. But his book offered little guidance as to what made things go together.

Despite this limitation, Thorndike's task analysis proved very powerful. This was in large part because it was accompanied by a strongly articulated theory of acquisition. This theory specified that one acquires new bonds through a trial-and-error process in which associations that are rewarded become stronger, whereas those that are punished or ignored gradually die out. This is the "law of effect." The law of effect pointed in turn to a very clear theory of instructional intervention. An instructor should organize practice in a way that would strengthen correct [illegible] by reward, and weaken incorrect ones. This theory led to several [illegible] of research in mathematics education in which investigators tried to determine empirically which bonds were easiest to form and which were hardest, so that practice could be organized from easiest to hardest. Such practice would give maximum opportunities for rewarding correct answers and thus strengthening correct bonds.

This approach to mathematics teaching still continues. For example, much computer-assisted drill-and-practice instruction can be viewed as a sophisticated manifestation of Thorndike's theory. The Stanford CAI (computer-assisted instruction) programs for math (Suppes & Morningstar, 1972), for example, fit that theory very well even though there is no mention in any of the program descriptions of association theory. Thus the Thorndikian theory of instruction has had a real influence on educational practice.

## Skinner and Operant Conditioning

Another psychologist who has had a profound impact on the theory and practice of instruction is B. F. Skinner (Glaser, 1978; Skinner, 1958). His effect was to lead instruction even further away from a central concern with the structure of knowledge and its interrelatedness. Skinner and other radical behaviorists denied that a science of mental life was possible because mental events were not open to public observation. With respect to instruction, the radical behaviorist position dictated a definition of the capabilities to be taught entirely in terms of observable performances. This has led to an entire technology of behavioral objectives (cf. Mager, 1961), still one of the more powerful influences on curriculum design and teaching practice.

Although the Skinnerian formulation was explicit about the terms in which capabilities to be induced through instruction should be stated, Skinner himself never did the kind of detailed work on the analysis of instructional subject matter that Thorndike did. Thus, there were no guidelines in Skinner's own writing explaining how to arrive at the content of objectives or how to order them. Robert Gagné's theory of cumulative learning (Gagné, 1962, 1968) and the methods of task analysis and learning hierarchy specification based on it (cf. Resnick, 1973) filled this gap, providing a method of task analysis that is still very influential.

As was the case with associationism, there was a strong acquisition theory associated with the Skinnerian view of learning. Much was shared with Thorndike, since learning was seen to be the result of patterns of reinforcement, or reward. But Skinner went beyond Thorndike: He proposed that wrong responses produce such negative side effects in learning that it would be best to avoid them completely. He and his associates (e.g., Terrace, 1963) showed that "errorless learning" was possible through shaping of behavior by small successive approximations. This led naturally to an interest in a technology of teaching by organizing practice into carefully arranged sequences through which an individual gradually acquires the elements of a new and complex performance without making wrong responses en route. This was translated for school use into "programmed instruction"—a form of instruction characterized by very small steps, heavy prompting, and careful sequencing so that children would be led step by step toward ability to perform the specified behavioral objectives. Meanwhile, the same general principles were applied to methods of organizing and maintaining desired social behavior in the classroom and keeping children's attention on the assigned work. This line of application became known as "behavior modification" (Kazdin, 1981).

Both associationism and behaviorism, then, provided a coherent theory of instruction that included methods of specifying the capabilities to be taught, a general theory of acquisition, and principles for intervention. Neither, however, offered a thorough analysis of thinking or knowledge, and so both were often judged inadequate by educators and psychologists interested in promoting reasoning and understanding. These groups found the theories of Piaget and other psychologists, such as those of the Gestalt school, more compatible with their concerns. We turn next to these early cognitive psychologists.

## Gestalt Psychology and the Structures of Thinking

Although they do not come to mind immediately as instructional theorists, Gestalt psychologists—especially Max Wertheimer (1945/1959)—were in fact very interested in education. Wertheimer spent time in schools and tried to develop a theory of education that would promote "productive thinking" and "meaningful" learning. Compared with the formulations offered by associationists and behaviorists, the instructional theory that can be induced from Wertheimer's writing is very sketchy. Nevertheless, it represents an early cognitive theory of instruction and thus is of considerable interest to our present inquiry.

For Wertheimer, the important capabilities to be promoted through instruction were principles and structured knowledge rather than unordered collections of bonds or behaviors specified without reference to the thoughts behind them. The essential character of Gestalt thought on education is well illustrated by reference to Wertheimer's famous parallelogram problem. Wertheimer reports going into a classroom of children who had been taught to find the area of a parallelogram by dropping a perpendicular line and then multiplying the perpendicular by the base of the parallelogram. Performance on this task was excellent as long as the parallelogram was presented in the standard way, as shown in the top of Fig. 1.1. But when Wertheimer asked the class to find the area of a parallelogram in a

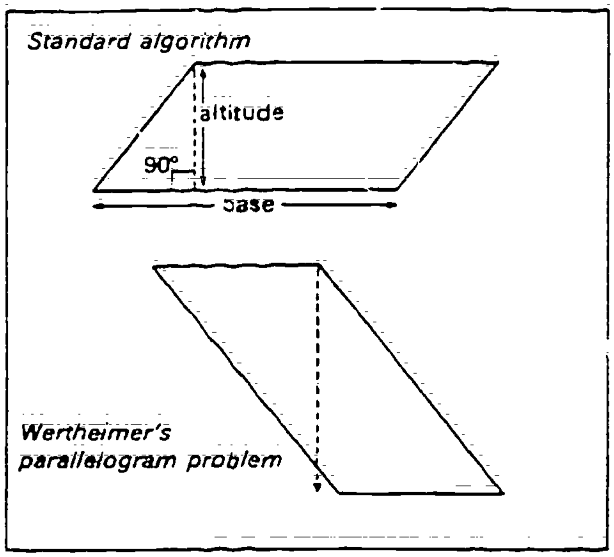


FIG. 1.1. Finding the area of a parallelogram using the standard algorithm. Children were confused when applying it to Wertheimer's problem figure. (From Resnick and Ford, 1981. Reprinted with permission.)

different position (as in the bottom of the figure) the typical response was "That's not fair" or "We haven't had that yet"—from teacher as well as children! The difficulty was that the standard formula did not seem to apply with the "up-ended" figure when a perpendicular was dropped from a top angle.

Wertheimer used this negative example to point to what one ought *not* to seek as an educational outcome—rote learning of procedures and answers—and to an alternative goal. He was interested in instruction that would lead children to recognize the principles that lay behind procedures so that they could solve problems that were not identical to those they had already encountered. For the parallelogram, this would mean recognizing: (1) that "area" refers to the number of unit squares that can be superimposed on a figure, but that this requires a figure that has right angles; (2) that nonrectangular figures can be converted into rectangular ones by cutting and repiecing figures; and (3) that the added perpendicular in the standard formula for the parallelogram is simply a convenient way of simulating the effects of this cutting and repiecing. Recognition of these three principles is what Wertheimer would have viewed as essential to a "structural" solution to the parallelogram problem. It was that kind of structural knowledge that he proposed as the appropriate objective of instruction.

Wertheimer thus proposed the terms in which capabilities for instruction should be analyzed. Unfortunately, however, the other portions of a theory of instruction—a theory of acquisition and a theory of intervention—are largely missing in the Gestaltist formulation. For the Gestaltists, structural knowledge was essentially either present or absent. Little attention was paid to how it developed.

With respect to intervention, the Gestalt emphasis on underlying structures of knowledge led to an interest in "discovery" methods of teaching. The notion was that if one discovered something rather than being told or shown it, then the underlying principles rather than just a performance pattern would be acquired. This theme was directly pursued in work by Katona (1940/1967), who tried in a number of experiments to show that learning by memorizing actually interfered with the recognition of principles and organized structures The theme of discovery learning was also picked up by a number of educational psychologists (see Shulman & Keislar, 1966). However, a close analysis of a number of the discovery learning experiments suggests that it was not the discovery *methods* of teaching so much as the different *content* made available to students that accounts for different learning outcomes (see Resnick & Ford, 1981, pp. 144–146).

A more robust principle of intervention that can be drawn from Gestalt theory is the importance of providing instructional *representations* that highlight the relations and structural features that we want students to acquire. This principle—well illustrated by the variety of "structural materials" for teaching mathematics that were developed during the 1950s and 1960s—is also in accord with developmental theories of instruction offered by Bruner (1960, 1966) and by Piaget.

## Piaget

One can hardly consider the possibilities for a cognitive instructional theory without attending to Piaget. Piaget himself had little to say about instruction; yet despite this, there have been numerous efforts to draw educational implications from his work, and a variety of different educational programs have been labeled "Piagetian" (Collis, 1975; Furth & Wachs, 1975; Kamii & DeVries, 1977). Is there a coherent instructional theory to be found beneath the label? The answer requires a look at the work of a number of psychologists and educators who consider themselves to be applying Piagetian theory.

Consider first the question of the capabilities to be fostered through instruction. A central core of Piaget's work has been concerned with characterizing the emergence in children of the general logical deductive capacities that are the structural bases of thinking. This led some educators, especially in the first flush of excitement over Piaget, to propose the teaching of operational thinking—assessed by the various "Piagetian tasks" such as conservation—as the goal of instruction (e.g., Kamii, 1972; Lavatelli, 1970; Weikart, Rogers, Adcock, & McClelland, 1971). Improvement of performance on various Piagetian scales, which are themselves based on the tasks used in Piaget's studies, has sometimes been proposed as a criterion of effective education even when the actual instruction focuses on traditional school subject matter (Almy, 1970). Others have proposed Piagetian reinterpretations of school subject matter, especially in mathematics (e.g., Lovell, 1971) and science (Lawson, this volume); or early education programs aimed at general forms of operational thinking (e.g., Furth & Wachs, 1975; Kamii & DeVries, 1977). Often, however, it is not very clear in what sense the goals are specifically Piagetian, since the concepts to be taught have not been studied by Piaget. In any case, this kind of interpretive analysis of subject matter is quite different from the very detailed instructional task analyses of Thorndike or Gagné.

The difficulty in making clear connections between Piagetian theory and the tasks of school instruction persists when we consider the question of acquisition. Piaget does, of course, offer a broad theory of development and hence of the acquisition of capabilities. (See Gallagher & Reid, 1981, for an introduction to Piaget's theory of learning; see Inhelder, Sinclair, & Bovet, 1974, for some Genevan instructional studies.) The key elements in this theory are *interaction* and *equilibration*. Broadly, the interactionist position specifies that biological endowment interacts with the environment so that a child growing up in its appropriate socio-ecological niche will develop in certain directions. Equilibration refers to the complementary processes of assimilation and accommodation by which the child constructs successively more complex and powerful schemes that are used to interpret the stimuli encountered in the environment. The process of equilibration is sparked by cognitive conflict, or the noting of contradictions. Clearly, there is a good deal more of an acquisition theory here than in the Gestalt work. Nevertheless, the Piagetian acquisition theory remains quite general. The actual *processes* involved in assimilation and accommodation are not specified; so the *nature* of the organism–environment interaction is not made clear.

As a result, Piagetian theory provides a very weak guide for instructional efforts. This becomes apparent as one attempts to derive an intervention theory from Piaget's writing. Only some very general principles are forthcoming: Set up an environment in which the child can interact and be actively engaged with things, with other children, and with adults. Promote the natural activities of children in interaction with their environments. Do not do too much drill and practice as this leaves little room for the construction of ideas and relationships by the child. Leave room for invention and discovery. Point out contradictions and let the child work on resolving them.

It is, for the moment, very hard to derive anything more than these general principles from the Piagetian view. Although the principles have been inspiring to many educators, they have proved difficult to translate into specific practices for the classroom. Again, we see a contrast with the explicit prescriptions of behavioral and associationist theories of instruction.

These, then, are some of the predecessors we have to build upon as we approach the task of developing a cognitive theory of instruction. As we have seen the theories that are strong on prescribing interventions are theories that do not have much to say about thought processes. Even worse, they are theories that almost entirely ignore structure, organization, and meaning as central aspects of learning. On the other hand, the more cognitive theories—those that treat mental life as real and important and that are concerned with the structure of knowledge—have been very weakly developed as instructional theories. Despite some elegant examples of the *kinds* of instructional goals that might be promoted, neither Gestaltist nor Piagetian analyses have proceeded very far in specifying these outcomes. Further, we can draw only very general theories of acquisition or intervention from Piaget, and virtually none from Wertheimer and the Gestaltists.

Is this choice between cognition and a vigorous instructional theory necessary? Or can we envisage a strong theory of acquisition and intervention based on cognitive analyses of instructionally relevant tasks? That is the question addressed in the remainder of this chapter. I will consider the characteristics of current cognitive science research as they bear on these instructionally relevant tasks in order to suggest directions for a cognitive science of instruction. My account begins by characterizing the current state of cognitive task analysis and then turns to the implications of information-processing theories for cognitive theories of acquisition and intervention.

## COGNITIVE TASK ANALYSIS: NEW DEFINITIONS OF CAPABILITIES TO BE DEVELOPED THROUGH INSTRUCTION

A central concern of cognitive science today is the analysis of complex task performances. As basic principles of cognitive processing have become established, largely through work on simple, laboratory-like tasks, cognitive scientists (including psychologists, linguists, and computer scientists) have increasingly turned their attention to the more complex tasks that occur in the real world. Among the kinds of tasks now under study are comprehension of extended written and spoken messages, solving physics problems, solving mathematics problems ranging from simple arithmetic to geometry and algebra, programming computers, repairing electrical equipment, reading X-ray films, and performing medical diagnoses. All these tasks are the kinds that form part of school, university, or technical curricula. Because cognitive analyses of performance on instructionally relevant tasks automatically afford descriptions of capabilities to be fostered by instruction, it is possible to characterize a large part of current mainstream cognitive psychology as directly contributing to a theory of instruction.

The flavor of the research on cognitive task analysis, and the kinds of capabilities that are being identified, can best be conveyed by considering examples of this research. I focus here on three domains from the school curriculum—reading comprehension, science, and mathematics.

## Reading Comprehension

There is no instructionally relevant domain of investigation that has experienced as spectacular a growth in the past several years as reading comprehension. Furthermore, work on the understanding and comprehension of natural language points to some of the major themes in cognitive theory. Thus, this is a good place in which to begin a consideration of cognitive task analysis. Current work on text comprehension represents a blending of questions and methods from at least three disciplines: psychology, linguistics, and artificial intelligence. My examples come largely from psychology, but it is a psychology that is by now heavily influenced by and influential in the other two fields; the sources of influence and points of collaboration are mentioned as I proceed.

Three major themes can be detected in the current line of research on understanding and learning from text. These are: (1) the importance of prior knowledge in understanding a text passage, (2) the central role of inference in reading, and (3) the flexibility of the reading process—its adaptation to local conditions and demands. Each of these is considered in turn.

*Prior Knowledge.* In 1972, Bransford and Johnson offered a dramatic example of the extent to which texts become comprehensible by virtue of being assimilated into existing cognitive structures. To make the demonstration, they showed that certain especially ambiguous passages could not be understood at all until some hint of what they were "about" was provided. For example, the passage in Fig. 1.2 is incomprehensible until a picture related to the story is seen (Fig. 1.3).

Following Bransford and Johnson, there came a number of other demonstrations that ambiguous (but perhaps not so totally obscure) passages took on meaning either according to hints provided in advance by the experimenter or according to the subjects' predilections. Anderson, Reynolds, Schallert, and Goetz (1977), for example, showed that music students interpreted a passage as describing an evening of playing music, whereas physical education students interpreted the same passage as an evening of playing cards. Others showed that the context in which information in a passage was conveyed could influence what was remembered from the passage and how the memory was organized. For example, in research by Anderson, Spiro, and Anderson (1978) a "restaurant" story produced a different pattern of recall for an identical set of food items than did a "supermarket" story.

These various demonstrations of the role of organizing schemata on understanding and recall echoed an earlier line of work, by Ausubel (1968), on "advance organizers." Ausubel had published a series of studies that showed

If the balloons popped the sound wouldn't be able to carry since everything would be too far away from the correct floor. A closed window would also prevent the sound from carrying, since most buildings tend to be well insulated. Since the whole operation depends upon a steady flow of electricity, a break in the middle of the wire would also cause problems. Of course, the fellow could shout, but the human voice is not loud enough to carry that far. An additional problem is that a string could break on the instrument. Then there could be no accompaniment to the message. It is clear that the best situation would involve less distance. Then there would be fewer potential problems. With face to face contact, the least number of things could go wrong.

FIG. 1.2 Ambiguous textual passage. (From Bransford & Johnson, 1972. Reprinted with permission.)

that learning of information from text could be improved if, in advance of reading the text, students were provided with an organization or structure within which to interpret it. Recent reviews and experiments (e.g., Mayer, 1979) have made it clear that advance organizers are an advantage when certain conditions hold: (1) the learner does not already know enough about the topic to provide the organizer for himself or herself; (2) the organizer is at least somewhat more "abstract" than the text itself, so that it provides a general structure, rather than just a preview of what is to come; (3) the material to be learned does not itself contain the necessary organizing and anchoring ideas; (4) the test comes long enough after the exposure to organizer and text that the learner cannot recall the material directly but must create an interpreted version.

Work on "story grammars" is a particularly well-developed example of how prior knowledge works in the process of understanding. Dealing with a particular kind of text, the narrative story, a number of investigators (Mandler, 1978; Stein & Glenn, 1979; Thorndyke, 1977) have shown that stories have a prototypical structure in which several categories of information must occur in order: a setting, an initiating event, an internal response, an attempt (to obtain a goal), an outcome or consequence, and a reaction. Some of the categories in this structure are more central than others. This is shown by substantial regularities in which portions of stories people omit in their retellings or in the stories they make up (Goldman, 1979). Story grammars represent an attempt to develop a domain-specific theory of text understanding. That is, it is proposed that in order to understand stories people must already have an idea of what a story is supposed to be like. This idealized story—a "schema" of a story—organizes and directs their interaction with the particular story they are now reading or hearing.

The idea that a prototypic version of a situation is used to interpret specific instances is shared by much of the recent artificial intelligence (AI) work on natural language understanding (Schank & Abelson, 1977). A number of computer programs have been constructed that are capable of "understanding" and answering questions about texts on a number of topics. Much of the ability of these AI programs to understand derives from their domain specificity, which allows them to use previously stored knowledge to interpret the new text. But the specificity is also a shortcoming, for it makes it difficult to develop a general program for text comprehension.

*Inference.* The most comprehensive attempt to build a generalized model of text understanding is that of Kintsch and van Dijk (1978). Their efforts are, in a sense, the necessary complements to models of processing that depend on representing an individual's knowledge of a particular domain. Their theory focuses on the extent to which a text is internally "coherent" (i.e., explicitly interconnected) and on the processes required for a reader to infer connections that are not explicit in the text. A text is coherent in the Kintsch and van Dijk analysis to the extent that each new proposition (i.e., actor–action–object sequence) makes explicit reference to prior propositions. In Fig. 1.4, for example, propositions 1–4 and 5–11 are fully coherent because the actor (subject) in each proposition has already been named in a close prior proposition. Line 5 is not explicitly coherent with its predecessors, however. To understand the text—that is, to construct a fully coherent representation of it—the reader must infer a linking proposition: for example, "The Swazi tribe had warriors." The number of missing propositions and the distance in the text that must be traversed to find explicit links affect not only the processing time for a text but also the short-term memory demands it makes. Too much local incoherence can render a text incomprehensible, but a completely explicit text would be uninteresting—like a primer. Optimal texts thus require "just the right amount" of inferential work by the reader—as if there were an implicit contract between the writer and the reader.

The Kintsch and van Dijk theory highlights the central role of *inference* in understanding a text. This is a feature of comprehension that emerges in virtually every cognitive analysis. Texts and oral messages are rarely, if ever, complete in specifying everything that has to be known to make sense of the situation described or the argument being made. It is the task of the reader to fill in the gaps. To do this, when one reads a text one builds up in the mind a knowledge structure that fits the situation in the text. This knowledge structure is not a direct match to the text; it leaves out some things that the text mentions and adds some things

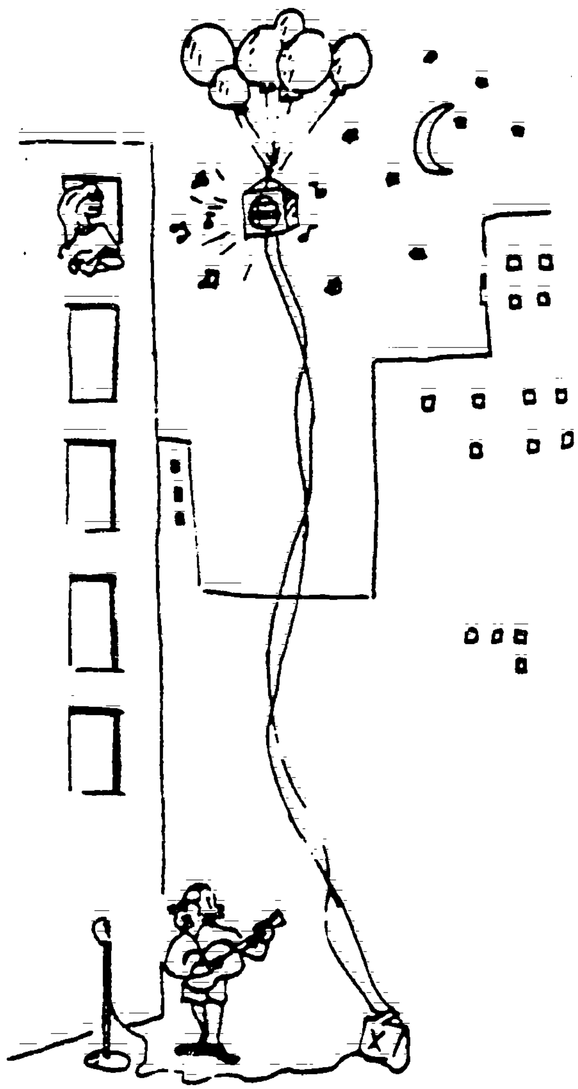

FIG. 1.3. Appropriate context for ambiguous passage (From Bransford & Johnson, 1972. Copyright 1972 by Academic Press. Reprinted with permission.)

Text:
The Swazi tribe was at war with a neighboring tribe because of a dispute over some cattle. Among the warriors were two unmarried men named Kakra and his younger brother Gum. Kakra was killed in a battle. . .

Propositional analysis of text:

1. The Swazi tribe was at war.
2. The war was with a neighboring tribe.
3. The war had a cause.
4. The cause was a dispute over some cattle.

5. There were warriors.
6. The warriors were two men.
7. The men were unmarried.
8. The men were named Kakra and Gum.
9. Gum was the younger brother of Kakra.
10. Kakra was killed.
11. The killing was in a battle.

FIG. 1.4. A text and its propositions. (Adapted from Kintsch, 1979.)

that the text does not mention. What it leaves out is what the reader construes as not being important or central to the main argument of the text. What it adds are things that the text has failed to specify but which "have to" be true if the situation being represented is to make sense.

The ability to detect and use centrality to a theme is an important part of the inference process. For a given story line or line of argument, readers tend to agree fairly well on what statements are important or central and which ones are subordinate, perhaps functioning only as elaborations. Meyer (1975) has used this regularity to develop a method for coding the statements in a passage for their relative centrality; other less systematic ways of judging centrality have also been developed. Using measures of this kind, it has been possible to show that the material most likely to be forgotten or left out of a summary is the material lowest in the hierarchy of importance. Conversely, if material high in the hierarchy is not specified in the text, people will have trouble interpreting the text at all, will tend to insert missing high-level propositions in their summaries, and will spend a long time "studying" the portion of the passage where the high-level organizing material is expected to be (Kieras, 1977). Also, when asked whether a given statement was or was not present in the text, people are likely to assert with great confidence that highly central material that is consonant with the gist of the text was there—even when it was not. In a series of studies that further link the Kintsch general coherence-plus-inference model with schema theories. Voss and his colleagues (Chiesi, Spilich, & Voss, 1979; Spilich, Vesonder, Chiesi, & Voss, 1979) have shown that readers' ability to make inferences depends on what they already know about the topic of the text.

*Flexibility.* Various studies have shown that skilled readers adapt their reading to local features of the text as well as to their own purpose for reading. For example, reading rates are slower and there is more checking back at points in a text where ambiguous, inconsistent, or incoherent information is encountered (Kieras, 1977). Eye-movement data demonstrate additional processing activity at points where information from important clauses must be integrated and inferences made (Just & Carpenter, 1981), and on parts of the text that are relevant to a particular kind of information the reader is trying to get from a passage (Rothkopf & Billington, 1979). Skilled readers also adjust their reading rates to the general readability of the text (Bassin & Martin, 1976; Coke, 1976) and to the kinds of information they seek to acquire (McConkie, Rayner, & Wilson, 1973; Samuels & Dahl, 1975). Finally, the number and types of inferences made depend on the purpose for which a text is read.

Although this flexibility on the part of skilled readers has been frequently documented, the processes underlying it have only recently begun to be studied. This recent research has made it clear that the likelihood of using varied processes under normal reading conditions must depend on individuals' abilities to monitor their own comprehension (Brown, 1980). These "metacomprehension" abilities, which are at least partly a function of age, seem to depend critically on sensitivity to important relationships among the propositions in a text. Thus, flexibility, like inference, depends on prior knowledge of the topic of the text.

## Science and Mathematics

A growing body of work in the learning and performance of science and mathematics tasks is pointing to some general characteristics of performance in these domains that accords well with the emphasis on prior knowledge, inferences, and flexible strategies in reading comprehension. In particular, research on science and problem solving suggests that the knowledge structures of individuals who are highly skilled in a domain are different in kind from the knowledge structures of novices. As a result, experts solve problems in different ways from novices, and—we may conjecture—they go about learning new information in their domain of expertise differently. At the same time, it is becoming clear that even very young children invent theories that allow them to construct procedural and predictive rules in simple mathematics and science domains.

*Novice–Expert Differences in Physics.* Chi, Feltovich, and Glaser (1981) have shown that the initial representation of a problem in mechanics is different for expert physicists (advanced graduate students) than for novices (undergraduate students). When asked to sort and classify problems, the novices did so on the basis of the kind of apparatus involved (lever, inclined plane, balance beam, etc.), the actual terms used in the problem statement, or the surface characteristics of the diagram presented. Experts, however, classified problems according to the underlying physics principle needed to solve them (e.g., Energy Laws, Newton's Second Law). Some typical novice classifications are shown in Fig. 1.5; the contrasting expert classifications are shown in Fig. 1.6.

Larkin, McDermott, Simon, and Simon (1980), in a complementary set of studies, have shown that the process of solution is also different for novices and experts. The novices seem to directly translate the given information into for-

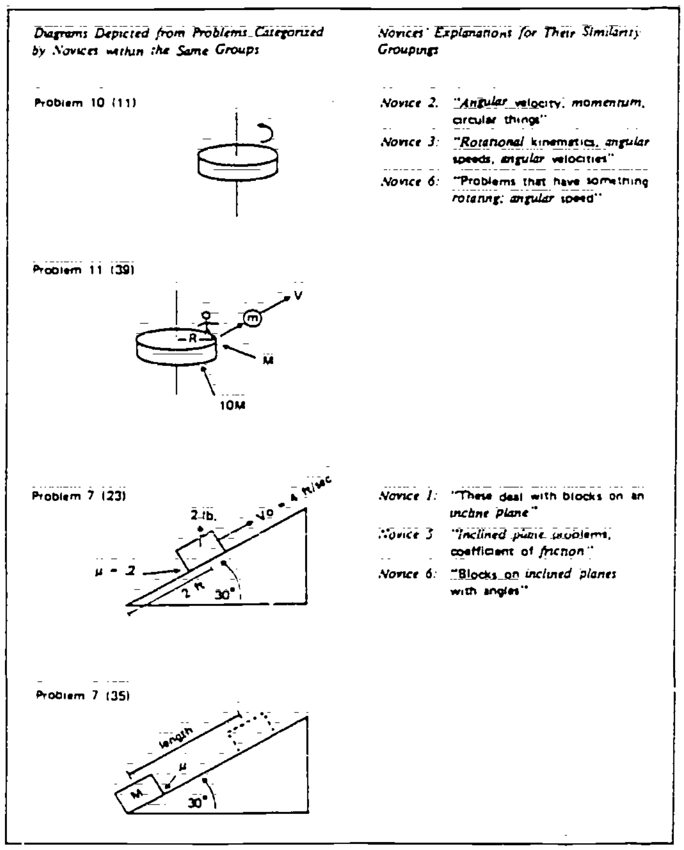


FIG. 1.5. Diagrams depicted from two pairs of problems categorized by novices as similar and samples of three novices' explanations for their similarity. Problem numbers given represent chapter, followed by problem number from Halliday and Resnick, 1974. (From Chi et al., 1981. Copyright 1981 by Ablex Publishing Corp. Reprinted with permission.)

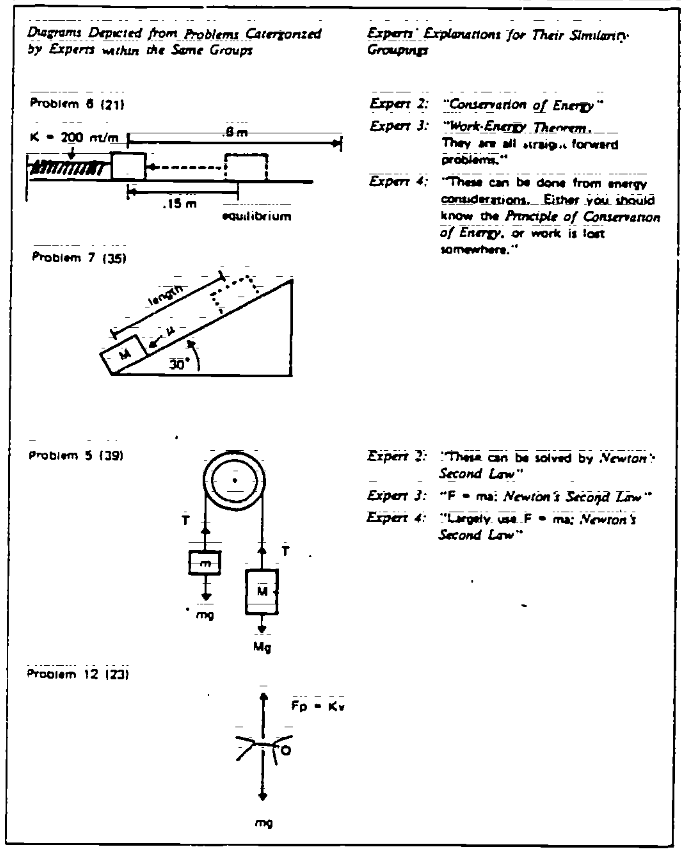


FIG. 1.6. Diagrams depicted from pairs of problems categorized by experts as similar and samples of three experts' explanations for their similarity. Problem numbers given represent chapter, followed by problem number from Halliday and Resnick, 1974 (From Chi et al., 1981; Copyright 1981 by Ablex Publishing Corp. Reprinted with permission.)

mulas and then work more or less algebraically on the formulas, looking for substitutions and transformations that will yield the required answer to the problem. Experts, by contrast, do a fair amount of interpretation that allows them to represent the problems as instances of certain general laws of physics and to restructure the relationships between elements of the problem. As a result, they usually have only one or two equations to write and the problem is virtually solved by the time they figure out what it is about. A general characterization of Larkin's finding is that the novices behave as if they are doing puzzles in which the terms to be manipulated have little external reference—their protocols look very much like those of people solving cryptarithmetic and other puzzlelike problems (Newell & Simon, 1972). They are characterized by what has come to be called "means–end analysis," in which they work backward from a goal. Experts, by contrast, seem to be working forward from the information given in the problem.

*Invented Routines and Rule-Driven Behavior.* Examination of school textbooks does not reveal in any direct way what children actually do when they perform arithmetic tasks. This is revealed in a striking way by consideration of simple, single-digit addition and subtraction. Textbooks typically teach addition as a process of counting out the two named subsets and then recounting the combined set, and everyone expects children rather quickly to give up any kind of counting in favor of memorizing the addition "facts." However, experiments have now revealed that there is an intermediate period during which children continue to solve addition problems by counting—but not by the method initially taught in school. Instead, they use a procedure that seems to imply an understanding of commutativity and that is elegantly simpler than the procedure taught. This procedure, typical of 6-year-olds and up, is known as the *min* model, because the smaller (minimum) of the two addends is added to the other in a counting-on procedure. For example, to add 3 + 5, the child starts at 5 (even though it is named second) and counts on: "5 . . . 6, 7, 8." This procedure has been documented in reaction-time and interview studies of a number of children in different countries and of different measured mental abilities (Groen & Parkman, 1972; Svenson, 1975; Svenson & Broquist, 1975). A study by Groen and Resnick (1977) shows that the *min* procedure can be invented by 4- and 5-year-old children.

A similar story can be told for subtraction. The textbooks demonstrate either a counting-out procedure in which the minuend set is established, a specified number of objects is removed, and the remainder counted; or a matching procedure in which sets to represent the minuend and the subtrahend are established, objects from these sets are paired one-for-one, and the remaining unmatched objects are counted. However, after practice, children do something rather different from either of these: they *either* count down from the minuend *or* count up from the subtrahend, *whichever will take the fewest counts.* Thus for 9 − 2 they say, "9 . . . 8, 7," but for 9 − 7, they say, "7 . . . 8, 9" (Svenson & Hedenborg, 1979; Woods, Resnick, & Groen, 1975). Children who invent these procedures behave as if they understand the commutativity principle of addition and the complementarity of addition and subtraction. However, it is not yet clear how explicit such understanding actually is (cf. Resnick, in press).

These inventions by children are no doubt heartening for those who would apply a constructivist theory of learning and development to education by leaving children free to explore and discover within only loosely defined boundaries. But the presence of inventions tells only part of the story, for not all inventions are mathematically successful. Several investigators (Brown & Burton, 1978;

 

Ginsburg, 1977; Lankford, 1972) have shown that children's errors on slightly more complex arithmetic tasks (e.g., multidigit written subtraction) are actually systematic applications of the wrong algorithmic procedure, rather than random failures to recall the appropriate "facts." These wrong procedures are variants of the correct ones; they are analogous to computer algorithms with "bugs" in them and have thus been christened "buggy algorithms." A finite number of bugs, which in various combinations make up several dozen buggy algorithms, have been identified for subtraction—which is the most often studied arithmetic domain so far. The children who display these buggy algorithms are systematically applying rules that no one could have taught them (for no one would deliberately teach a wrong rule). Buggy algorithms are thus clear examples of inventions that are unsuccessful.

Despite their failure as rules of calculation, buggy algorithms demonstrate an important characteristic of human learning and performance. Close analysis of the various incorrect algorithms that have been observed among children makes it clear that most of them are rather sensible and often quite small departures from the correct algorithm. As the examples in Fig. 1.7 reveal, they tend to "look right" and to obey a large number of the important rules for written calculation: The digit structure is respected, there is only a single digit per column, all the columns are filled, and so forth. In the sense of being an orderly and reasonable response to a new situation, the buggy algorithm looks quite sensible.

To illustrate this point, consider the third bug shown in Fig. 1.7. The written response looks more or less correct and follows most of the rules for written subtraction, such as operating on each column only once, having only a single digit in each column, and showing a borrow digit with a crossed-out and decremented digit. The syntax of the procedure is more or less correct. Yet the algorithm violates some fundamental mathematical constraints. In particular, it behaves as if its inventor does not understand that the borrow digit adds units via an exchange with another column, and that the exchange must maintain equivalence of the overall quantity. This knowledge, if applied, would not permit what this particular buggy algorithm does: It "borrows" 100 but "returns" only 10. Other bugs, too, have the character of respecting much of the syntax of written arithmetic but violating the "semantics"—or underlying meaning (Resnick, 1982).

A similar emphasis on the sensible, rule-driven character of "wrong" responses appears in Siegler's (1978) work on the balance beam and other similar tasks. In the balance task various numbers of weights can be hung at various distances from the fulcrum; the child must predict which, if either, side will go down. Children do not completely solve this problem successfully until adolescence. Siegler showed, however, that at each of several preceding stages children actually follow algorithmic rules for deciding which side of the balance will go down. The rules that are followed become successively richer, not only in the amount of information that children call on but in the extent to which they are

1. Smaller-From-Larger. The student subtracts the smaller digit in a column from the larger digit regardless of which one is on top.

| | |
|---|---|
| 326 | 542 |
| − 117 | − 389 |
| 211 | 247 |

2. Borrow-From-Zero. When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.

| | |
|---|---|
| 602 | 802 |
| − 437 | − 396 |
| 265 | 506 |

3. Borrow-Across-Zero. When the student needs to borrow from a column whose top digit is 0, he skips that column and borrows from the next one. (Note: This bug must be combined with either bug 5 or bug 6.)

| | |
|---|---|
| 802 | 804 |
| − 327 | − 456 |
| 225 | 308 |

4. Stops-Borrow-At-Zero. The student fails to decrement 0, although he adds 10 correctly to the top digit of the active column. (Note: This bug must be combined with either bug 5 or bug 6.)

| | |
|---|---|
| 703 | 604 |
| − 678 | − 387 |
| 175 | 307 |

5. 0 − N = N. Whenever there is 0 on top, the digit on the bottom is written as the answer.

| | |
|---|---|
| 709 | 6008 |
| − 352 | − 327 |
| 457 | 6321 |

6. 0 − N = 0. Whenever there is 0 on top, 0 is written as the answer.

| | |
|---|---|
| 804 | 3050 |
| − 462 | − 621 |
| 402 | 3030 |

7. N − 0 = 0. Whenever there is 0 on the bottom, 0 is written as the answer.

| | |
|---|---|
| 976 | 856 |
| − 302 | − 409 |
| 604 | 407 |

8. Don't-Decrement-Zero. When borrowing from a column in which the top digit is 0, the student rewrites the 0 as 10, but does not change the 10 to 9 when incrementing the active column.

| | |
|---|---|
| 702 | 205 |
| − 368 | − 9 |
| 344 | 1106 |

9. Zero-Instead-Of-Borrow. The student writes 0 as the answer in any column in which the bottom digit is larger than the top.

| | |
|---|---|
| 326 | 542 |
| − 117 | − 389 |
| 210 | 200 |

10. Borrow-From-Bottom-Instead-Of-Zero. If the top digit in the column being borrowed from is 0, the student borrows from the bottom digit instead. (Note: This bug must be combined with either bug 5 or bug 6.)

| | |
|---|---|
| 702 | 508 |
| − 368 | − 489 |
| 454 | 109 |

FIG. 1.7. Samples of Brown and Burton's (1978) buggy subtraction algorithms invented by children. (Adapted from Resnick, 1982. Copyright 1982 by Lawrence Erlbaum Associates. Reprinted by permission.)

able to coordinate that information. The early rules consider either weight or distance alone; the child seems to be unable to consider both at once. Subsequently, weight and distance are combined; but, at first, not in an accurate computation of torque on the balance beam. Nevertheless, the rules are systematic and produce predictable patterns of responses. Here, too, is evidence for the principled character even of errors.

The preceding sketch of the status of cognitive task analysis provides convincing evidence that one part of the agenda of building a cognitive theory of instruction is well under way. Analyses of the kind that have been described are capable of providing rigorous and formal statements of instructional objectives concerned with meaning and understanding and their relation to performance skills. We no longer have to accept the choice between the rigor of behavioral objectives and our desire for explicit recognition of cognitive structures and thought. We are moving closer to being able to make useful specifications of *cognitive* objectives for instruction (cf. Greeno, 1976).

## COGNITIVE THEORIES OF ACQUISITION

When we move to a consideration of the other parts of the instructional theory agenda, however, we are on less well-developed ground. The reemergence of cognition in American psychology was accompanied by a loss of interest in learning and acquisition processes. Until very recently, cognitive psychologists have been focusing almost exclusively on the issue of cognitive performance while ignoring the issue of how these performances are acquired. In contrast, the older learning theories—those represented by Thorndike and Skinner, for example—were deeply interested in transitions. Their theories were largely intended to account for changes in performance as a result of certain kinds of experiences in the environment.

Although work toward cognitive theories of acquisition is relatively recent, such work is now recognized by many cognitive scientists as a major agenda for the field (see, e.g., Anderson, 1981). Further, the research on cognitive task analysis—some of which I have reviewed here—points both to the standards of rigor to be expected in eventual cognitive learning theories and to some of the broad characteristics that these theories are likely to have. I consider some of these characteristics in the following section.

### Constructivism

Even a cursory consideration of the emerging findings from cognitive task analyses makes it clear that our new theories of acquisition will have to take account of the important role of mental constructions and interpretations by the learner. The central role of inference in text understanding, the evidence for inventions of mathematical procedures, and the characterization of expert problem solvers as individuals who reformulate problems before beginning to work on them all point toward the active role that the learner himself plays in acquiring new knowledge. For those who have been committed to cognitive interpretations of development, this emphasis on active construction of knowledge by the learner is not new. It has been a central theme in Piaget's work and theory for decades and has been much emphasized in many of the most sensitive and influential explorations of the implications of Piaget's work for education (Duckworth, 1979:

Furth, 1970; Ginsburg & Opper, 1969; Kamii & DeVries, 1977). Yet, curiousl this central theme has been little elaborated even within cognitive development theory. As noted earlier, Piaget's theory of construction of new schemes v assimilation and accommodation provides only a skeleton of a theory of acquis tion. It has not been fleshed out, nor has it been developed for any particul domain of knowledge to a degree sufficient to provide a useful example of ho the processes might actually work in the course of cognitive construction.

*Sensible Constructions on Limited Data.* A central feature of the construc tions that characterize acquisition is that they operate without complete informa tion. Rather than waiting until all the evidence is in, people seem to work t make sense of the world on the basis of the information they have. The researc on buggy algorithms, on rule-based developmental sequences, and on inferenc in text comprehension all point to the fact that people seek sensible solutions an explanations within the limits of their knowledge.

A close consideration of buggy arithmetic algorithms and their origin high lights this point. Brown and VanLehn (1982) have developed a formal theory, i the form of a computer simulation, of the origin of bugs in arithmetic. Accordin to their "repair" theory, buggy algorithms arise when an arithmetic problem i encountered for which the child's current algorithms are incomplete or inap propriate. The child, trying to respond, eventually reaches an impasse, a situa tion for which no action is available. At this point, the child calls on a list o "repairs"—actions to try when the standard action cannot be used. The repai list includes strategies such as performing the action in a different column skipping the action, swapping top and bottom numbers in a column, and sub stituting an operation (such as incrementing for decrementing). The outcome generated through this repair process is then checked by a set of "critics" tha inspect the resulting solution for conformity to some basic criteria, such as nc empty columns, only one digit per column in the answer, only one decrement pe column, and the like.

Together, the repair and critic lists constitute the key elements in a "generate and test" problem-solving routine. This is the same kind of "intelligent" problem solving that characterizes many successful performances in other domains (cf. Simon, 1976, pp. 65–98). With buggy algorithms, the trouble seems to lie not in the reasoning processes but in the inadequate data base applied. Inspection of the repair and critic lists makes it clear that the generation and the test rules in this particular system can all be viewed as "syntactic." That is, they all concern the surface structure of the procedure and do not necessarily reflect what we can call the "semantics" of subtraction (Resnick, 1982).

Repair theory is, in fact, a detailed theory of acquisition for a small domain of arithmetic. Its broader implications for cognitive theories of acquisition is that these theories must recognize people's tendency to organize and structure whatever information they have—even though the information may be grossly incomplete or downright inaccurate. They do not simply acquire information passively until there is enough of it for "correct" rules and explanations to emerge. This tendency to construct ordered explanations and routines even in the absence of adequate information can account at least partly for another phenomenon, observed thus far mainly in science learning: robust beliefs that are resistant to change even when instruction (and thus better information) *does* come along. We will consider this phenomenon again, in the context of coherence and integration in learning.

## Coherence in Learning: The Integration of Old and New Knowledge

As we have seen, research on reading comprehension makes it clear that the job of the reader is to connect new information in a text to the old, and it shows how comprehension falters when these "given-new" (cf. Clark & Haviland, 1977) links are difficult to establish. An extension of these notions of coherence-building suggests that we view the acquisition of new capabilities as a process of building appropriate links between knowledge already held and new knowledge. Stated this way, the role of coherence in learning sounds deceptively simple. The established behaviorist notion of building new performances out of the components of old ones, stated most elegantly in Gagné's (1962) theory of prerequisites and cumulative learning, appears to describe its role almost completely. It might seem that all that is needed to extend the coherence principle to a cognitive theory of acquisition are the detailed task analyses that would allow us to specify the mental structures that are to be extended at the next stage of learning. Even this would be no small task, but there is evidence that the problem of coherence will prove even more complicated than cumulative learning theory would suggest.

A growing body of evidence, mostly collected in studies of science learning, is now showing how prior knowledge can actually interfere with new learning. A recurrent finding in studies of physics instruction is that people bring with them to the study of physics a quite powerful set of beliefs about how the physical world works. These beliefs are robust and resistant to the new data and theoretical principles that are taught in physics courses (Champagne, Gunstone, & Klopfer, in press; Selman, Krupa, Stone, & Jaquette, in press; Viennot, 1979). Their "naive" beliefs allow people to construct explanations of various phenomena that accord quite well with their perceived experiences. The difficulty is that these beliefs do not match well with the Newtonian principles taught in physics courses, yet they are not always abandoned as the result of instruction in Newtonian physics. Some students can perform adequately on the textbook problems in a high school or college physics course, yet when given practical problems that are not easily recognized as applications of the textbook formulas—problems that force them to construct their own representation of the situation—they will revert to the conceptions they had before the course began. These students show evidence of having well-integrated knowledge structures that will have to be given up or altered radically before they can acquire Newtonian conceptions at a level beyond mechanical equation solving.

These findings force us to broaden our thinking about the relations between established and new knowledge in the course of acquisition. We must think not only about the cumulation and linking of knowledge structures, but also about what kind of *confrontation* between old conceptions and new ones may be needed for the new to take hold. If schema theories of discourse comprehension and the performances of expert physicists point toward the positive role that prior knowledge can play in performance and learning, these studies of difficulties in physics teaching reveal its negative role. This is a far more subtle role for "entering capabilities" than was allowed in the Gagné analyses of prerequisites. Now that the phenomenon has been demonstrated, the task ahead is to analyze it. How, precisely, do already held schemata drive attention *away* from competing interpretations? Is the process simply preemptive, or is there a more complex relationship between the two schema systems to be uncovered? And what happens to old schemata as new ones take over? Are they simply "left behind" or are they incorporated into a new framework? As answers to questions like these begin to take shape, we will have a theory of acquisition capable of more directly guiding instruction in these complex subject matters.

### The Nature of Theory Change: Insight Versus Incremental Change

The preceding discussion suggests that it may be useful to view cognitive acquisition as a process of knowledge restructuring, a definition that immediately brings to mind both Gestalt and Piagetian theory. These structuralist theories both stressed the role of invention and insight in the process of such restructuring. The notion that insight, presumably resulting in an immediate restructuring of knowledge, is characteristic of learning contrasts sharply with traditional theories of learning in which acquisition is described as a gradual and incremental process—a function of extended practice. Debates among learning theorists concerning incremental versus "all-or-none" learning of simple discriminations can, in fact, be viewed as one of the important predecessors of the cognitive revolution in experimental psychology. Given this history, it is not surprising that there has been a general tendency to equate cognitive theories of human behavior with insight-oriented theories of acquisition. In fact, a cognitive view of education has up to now almost always carried the implication that learning does not proceed in smooth steps, but instead is marked by occasional moments of insight resulting in immediate qualitative differences in the nature of thinking.

It is no longer so likely, however, that cognitive theories of acquisition will emphasize momentary insightful restructurings rather than the gradual acquisition of new understanding. A number of cognitive scientists are now turning their attention to the question of acquisition. A recent edited volume (Anderson, 1981) provides an excellent overview of the direction that one major class of these theories—the ones committed to formalization through computer simulation—is taking. Work of this kind by Neches (1981) is directly relevant to the examples of task analysis presented earlier in this chapter and illustrates well the power of incremental theories. Neches has constructed a computer program that invents the *min* model of mental addition. That is, it begins with a procedure for addition in which both addends are counted in the order presented; but after a period of practice on addition problems, it performs by "naming" the larger addend and then counting up by the amount of the smaller. The conversion is accomplished by a set of mechanisms that seek performance efficiency by eliminating redundant steps and by strengthening responses that are made while weakening potentially competing responses. The program is self-modifying, all transformations of knowledge are its own. It makes a major transformation in its procedure, yet there is no single moment of insight. Indeed, it is not clear that the system needs to "understand" commutativity in order to behave in accord with it.

The Neches program, like other self-modifying systems, provides a plausible account of how acquisition might proceed. But the account is only that *plausible*, but by no means proven. One difficulty with this line of work is that up to now it has proceeded with only a limited amount of data on the actual human processes that are being modeled. Typically, as with Neches, there exists strong evidence for a particular initial performance and a final one. However, there is usually no observation of the acquisition process itself. As a result, the theorist is free to build a self-modifying system without the requirement that any particular features of the acquisition sequence itself be matched. Intriguing as the current self-modifying programs are, then, we must withhold acceptance of them as actual descriptions of cognitive activity until they can be more fully constrained by observations of human learning.

We must not, however, assume that as a more substantiated theory of acquisition is developed it will necessarily support insight as opposed to incremental accounts of acquisition. Over the past 3 years my colleagues and I have spent several hundred hours studying protocols taken on a small number of children as they were taught subtraction using a method intended to induce understanding of the meaning of the various scratch marks and manipulations involved in subtraction with borrowing. This instruction emphasizes the analogy between two different representations of subtraction—one, the symbols used in standard written notation and the other, a concrete representation using blocks designed to highlight the quantitative meaning of place value. The child is required to make alternating moves in the two representations, in order to build a mental mapping between operations in each. This procedure helps the child to apply his or her understanding of the blocks action to the writing. The procedure is summarized in Fig. 1.8.

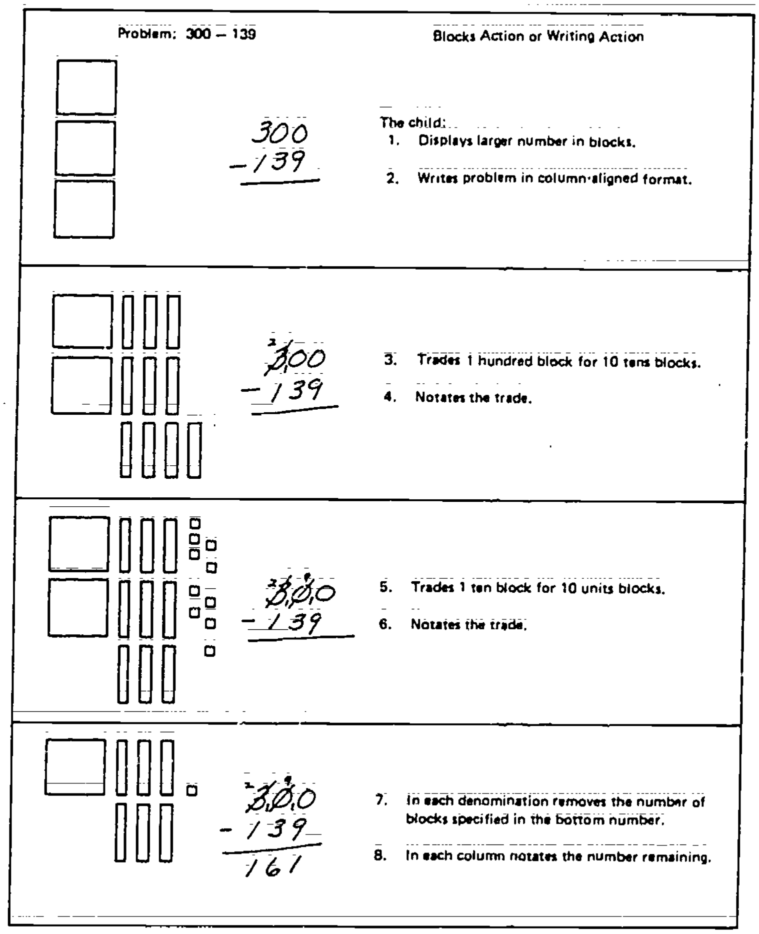


FIG. 1.8. Instructional procedure for mapping blocks action to writing in teaching subtraction with borrowing. (From Resnick, 1982. Copyright 1982 by Lawrence Erlbaum Associates. Reprinted by permission.)

We have paid special attention to the case of one child, Molly, who quite clearly experienced a moment of insight in the course of this instruction. She told us so by her exclamation, "Oh, neat! Now I get it," at Step 2 (Fig. 1.8), and an accompanying change in the pace and rhythm of her working through the remaining steps. Recently, impelled by the need to begin serious work on a theory of acquisition for this domain of mathematics understanding, we have begun to reexamine this child's protocol. There is no doubt that Molly *felt* that she had understood something new at the moment of her exclamation. But on closer examination it is not at all clear what it was that she really "got." She certainly did not suddenly know the correct algorithm to use, nor did she automatically and quickly generate correct descriptions of the operations she was doing. In fact, it took a delay of several weeks and another hour or so of probing for Molly to generate an explanation of written subtraction that convinced us that she understood *why* the written subtraction algorithm "worked." There was, in other words, no immediate restructuring of knowledge at Molly's moment of felt insight.

Yet something clearly happened at that moment. What? Our best current interpretation is that at the moment of insight, Molly learned *where* to look for explanations. That is, she learned that the blocks and the writing are analogous and thus, perhaps, the schemata she had for blocks might also apply to writing. This insight allowed her to inspect the two routines and to apply her understanding of one to the other—a process, however, that was to take considerable time beyond the actual moment of insight.

The story of Molly's insight is, for the moment, more a conjecture than a theory. I offer it now, despite its very tentative status, to make two points. First, the feeling of having an insight does not necessarily mean that a complete restructuring of knowledge has taken place; it may only mark a moment in which some clue as to how to go about the process of restructuring becomes apparent. Second, our study of Molly highlights the kind of detailed observations—freed of assumptions about insight and immediate mental restructuring as the foundation of acquisition—that we are going to need as we build a cognitive theory of acquisition.

## A COGNITIVE THEORY OF INTERVENTION

We come finally to the question of intervention. If the route has been long, I think it has not been unnecessarily circuitous. For to propose principles of intervention in the absence of a strong theory of acquisition is to operate in the kind of theoretical vacuum that can only produce an endless series of empirical experiments on different instructional methods, with no basis for interpretation.

It should not be surprising that with a cognitive theory of acquisition only beginning to emerge, we can hardly point to a vigorous cognitive theory of intervention at this time. Nevertheless, the developments outlined in the course of this chapter surely suggest the kind of instructional theory that we can expect to build. Specifically, the accumulated body of cognitive task analysis and the emerging work on cognitive theories of acquisition clearly signal the need for a *constructivist theory of instruction*. It now seems absolutely certain that our task is to develop a theory of intervention that places the learner's active mental construction at the very heart of the instructional exchange. Instruction cannot simply put knowledge and skill into people's heads. Instead, effective instruction must aim to place learners in situations where the constructions that they naturally and inevitably make as they try to make sense of their worlds are *correct* as well as sensible ones.

A constructivist theory of instruction need not avoid prescriptions for intervention; nor need it specify only a very general environmental arrangement, as in some of the Piagetian proposals for designing schools and classrooms. Instead, a constructivist intervention theory must address all the traditional concerns of instructional design: how to present new information to students, what kinds of responses to demand from students, how to sequence and schedule learning episodes, and what kinds of feedback to provide at what points in the learning sequence. These traditional instructional concerns take on new substance and direction when approached from a constructivist perspective. Although the enterprise of building a constructivist instructional theory has barely begun, it is possible to sketch some of the questions that will need to be addressed as work proceeds. I do this in the concluding pages of this chapter.

## Instructional Representations

One of the central questions for any theory of instructional intervention concerns the form in which information is best conveyed to the learner. In traditional instructional design it was tacitly assumed that a task analysis that specified the performance or knowledge of skilled people in a domain would automatically yield not only "objectives" for instruction in that domain but an outline of the form in which information should be presented to learners. Implicit in this assumption was the notion that instruction should communicate as directly as possible the "mature" or "expert's" form of a concept or skill. Research of the kind discussed in this chapter, however, makes it clear that this assumption does not adequately recognize the work of the learner in *constructing* the mature form of knowledge. Novice–expert contrastive studies have shown that the mental representations of beginners differ qualitatively from those of people more experienced in a domain of knowledge. Furthermore, there are hints that novices may not be able to assimilate or use the categories and representations of experts when these are directly presented; yet we know that extensive practice allows people who begin as novices to discover the representations and skillful performances of experts. If this is so, then the task of the instructor is not to search for ways of presenting information that directly match the thought or performance patterns of experts. Rather, it is to find instructional representations that allow learners to gradually construct those expert representations for themselves.

Until quite recently, the question of representations for instruction has been the almost exclusive concern of curriculum developers and pedagogical subject-matter specialists. Mathematics educators, for example, have developed an extensive repertoire of concrete and pictorial representations of mathematical concepts (Resnick & Ford, 1981, chap. 5). Only recently have psychologists begun to analyze these materials and their function in the learning process. In an earlier paper (Resnick, 1976), I suggested that instructional representations must: (1) represent the concept or idea to be acquired in a veridical, if simplified way; (2) be "transparent" to the learner (i.e., represent relationships in an easily apprehended form or decompose procedures into manageable units); (3) map well onto expert modes of understanding and skill. As an example, Dienes blocks (of the kind displayed in Fig. 1.8) seem to meet these requirements nicely for the principles of place value. They represent the relations among numbers that are embedded in place value notation, and they are in a form that is easy for children to recognize and manipulate. The ten-for-one relationship between adjacent denominations is visible and physically verifiable by superimposition or alignment of the blocks, and exchanges between denominations can be made in a way that parallels the steps in calculation algorithms. In another domain of learning, Gentner (1980) has been analyzing the role of analogies in learning and teaching scientific concepts. Research on instructional representations in a variety of subject matters should eventually lead to a generalizable theory of representations for teaching that is capable of guiding instructional design efforts. Such a theory will need to take explicit account of how learners interpret representations and how they build mental representations and procedures from the instructional materials that are presented.

## Interventions That Take Account of Previous Knowledge

As I have noted earlier, the theme of building on past learning is an old one in instructional psychology. In behavioral theories, it has generally been assumed that previous knowledge, when present, facilitated subsequent learning. That is, new capabilities were to be built of the components of older ones, and knowledge and skill would thus cumulate hierarchically. As we have seen, however, prior knowledge can also interfere with acquiring new concepts. This means that instructional methods are needed that explicitly take this interference into account. A theory of intervention quite different from the one derived from cumulative learning theory will be needed to guide instruction of this kind. As a point of departure, the Piagetian notion that cognitive growth occurs as the result of conflict between competing schemes might be elaborated in the context of instructional subject matters and perhaps linked more explicitly to schema-driven theories of comprehension and acquisition. This could provide one basis for intervention studies that explore different approaches to relating new learning to old. What is best: ignoring prior incompatible conceptions and helping students construct strong new ones, or directly confronting the conflicts between the old and the new conceptions? These kinds of questions have rarely been raised in the context of instruction, except by people exploring the educational implications of Piaget (e.g., Duckworth, 1979). They will surely have to be addressed in the constructivist instructional theory of the future.

## A New Theory of Practice in Learning

The time is also ripe for a new look at an old instructional question: the role of practice in learning. It is interesting to contrast our current views of practice with those of Thorndike. For Thorndike. the role of practice was straightforward and self-evident. It "stamped in" correct associations through reward. The assumption, maintained in subsequent associationist and behaviorist instructional theories, was that competencies did not change in any fundamental qualitative way as a result of practice: they merely became stronger, faster and more reliable. The evidence from recent cognitive research is that practice also provides the occasion for productive cognitive inventions and that skill and understanding may undergo qualitative changes in the course of practice. This, I believe, forces a reconsideration of the role of practice in learning. Why does practice sometimes lead to productive inventions, and sometimes to the kind of rigid, "distorted" thought deplored by Wertheimer? A recent paper by Anderson, Greeno, Kline, and Neves (1981) suggests that the same mechanisms that account for the acquisition of skill in constructing high school geometry proofs may also explain phenomena such as set and functional fixedness. We do not know how general such a "trade-off" between the benefits of smooth and skillful performance and the disadvantages of rigid performance sequences may be. Anderson's theory accounts nicely for data on the developing abilities of a group of high school students in a traditional geometry course in which daily practice in proving theorems of a fairly standard type is provided. However, this does not mean that forms of instruction and practice might not be devised that would foster skill acquisition without promoting set or functional fixedness.

If practice is to provide the occasion—and perhaps the motivation—for the invention and testing of new procedures, then the traditional distinction between skill acquisition and understanding may need to be substantially modified. Practice, leading to skillful performance, may turn out to be necessary to the development of deep understanding, at least in certain domains of learning. Piaget's insistence on reflexive abstraction and his demonstration that successful performance often precedes understanding of certain phenomena (Piaget, 1974/1978) suggest such a possibility, as do our own demonstrations of procedural inventions by children (Groen & Resnick, 1977). Surely the *kinds* of practice afforded by instruction and the ways in which procedural practice is interspersed with invitations to reflect and construct explanations will influence the development of understanding. Here, then, is another set of questions about practice that a constructivist theory of intervention will have to address.

## Assessing Understanding: The Links Between Knowledge and Performance

Assessment of students' entering capabilities and of the results of their learning efforts is an integral part of an instructional intervention. A cognitive theory of intervention must therefore include principles for assessment that are capable of revealing a learner's state of knowledge. Although the need for diagnostic assessment has long been recognized by instructional and test theorists (Glaser. 1981; Greeno, this volume) existing tests fall far short of permitting strong inferences about the cognitive state of learners. Traditional achievement tests are designed more to facilitate comparisons between people than to permit strong descriptions of individual competence. Meanwhile. content-referenced tests (Glaser. 1963). which *are* intended to reveal directly the learner's capabilities, lack any principled basis for linking observations of performance to inferences about cognitive competence. This is hardly surprising when one considers the behavioral roots of the content-referenced testing movement. According to strict behavioral theories, a person's competence is nothing more than the sum of that person's performances. There was thus no reason to try to develop a method for inferring underlying knowledge from the observed performances.

A cognitive theory of instruction, by contrast, requires exactly such a method. By focusing on the knowledge to be inferred, rather than on the performances per se. the objectives of instruction are likely to change—in directions that promote transfer and further learning (cf. Greeno. 1976. this volume). Principled bases for inferring that knowledge from performance must therefore be developed. Cognitive psychologists regularly make such inferences in their laboratories. But the effort is usually intensely individual and ad hoc: Those inferences survive that, after inspection of a large amount of data and research findings of other scientists, remain plausible. For instructional assessment, tasks and allowable inferences about underlying competence will need to be specified more systematically. The translation of methods of cognitive analysis into forms usable for instructional assessment constitutes another major agenda for a cognitive theory of intervention.

## CONCLUSION

I have tried in this chapter to sketch an emerging revolution in the psychology of instruction. The cognitive perspective that now permeates psychology and its related disciplines has profound implications for the ways in which we ought, in the future, to think about instruction. As I have suggested here, the new view of the learner as an active constructor of knowledge forces a deep reconsideration of many of the assumptions of the instructional theories we have been living with. Some of the directions in which the new constructivist assumptions may lead instructional theory have been suggested in the final section of the chapter. But the particular questions addressed there are only early examples of the ways in which the traditional concerns of instructional theory can be expected to take on new substance as work toward a cognitive theory of instruction gathers momentum.

One of the more important influences on the direction of research in cognition is the cumulating evidence for the central and complicated role of prior knowledge in performance and learning. As this phenomenon has been recognized it has had the effect of directing the efforts of cognitive scientists toward intensive study of human performance in particular domains of skill or knowledge. Instead of searching for general laws of learning or development many cognitive scientists are now devoting attention to the analysis of specific task domains—including many that are of direct interest to the educator. Although this has the effect of making large segments of basic cognitive science immediately relevant to the task of developing an instructional theory, it also renders the task more complex than it might have seemed in the past. If we are to find general laws of cognitive learning that can be applied to instruction, it will be only through the detailed study of particular domains of knowledge. It is only in this way that we will be able to understand how knowledge accumulates and influences new cognitive constructions.

## REFERENCES


Almy, M. *Logical thinking in second grade*. New York: Teachers College Press, 1970

Anderson, J. R. *Cognitive skills and their acquisition*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Anderson, R. C., Reynolds, R. E., Schallert, D. L., & Goetz, T. E. Frameworks for comprehending discourse. *American Educational Research Journal*, 1977, *14*, 367–381.

Anderson, R. C., Spiro, R. J., & Anderson, M. C. Schemata as scaffolding for the representation of information in connected discourse. *American Educational Research Journal* 1978, *15*(3), 433–440.

Ausubel, D. *Educational psychology: A cognitive view*. New York: Holt, Rinehart, & Winston, 1968.

Bassin, C. B., & Martin, C. J. Effects of three types of redundancy reduction on comprehension, reading rate, and reading time of English prose. *Journal of Educational Psychology*, 1976, *68*(5), 649–652.

Bransford, J. D., & Johnson, M. K. Contextual prerequisites for understanding: Some investigations of comprehension and recall. *Journal of Verbal Learning and Verbal Behavior*, 1972, *11*, 717–26.

Brown, A. L. Metacognitive development and reading. In R. J. Spiro, B. C. Bruce, & W. F. Brewer (Eds.), *Theoretical issues in reading comprehension*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1980.

Brown, J. S., & Burton, R. R. Diagnostic models for procedural bugs in basic mathematical skills. *Cognitive Science*, 1978, *2*, 155–192.

Brown, J. S., & VanLehn, K. Toward a generative theory of bugs in procedural skills. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1982.

Bruner, J. S. *The process of education*. Cambridge, Mass.: Harvard University Press, 1960.

Bruner, J. S. *Toward a theory of instruction*. Cambridge, Mass.: Harvard University Press, 1966.

Champagne, A. B., Gunstone, R. F., & Klopfer, L. E. Cognitive research and the design of science instruction. *Educational Psychologist*, 1982, *17*(1), 31–53.

Chi, M. T. H., Feltovich, P., & Glaser, R. Categorization and representation of physics problems by experts and novices. *Cognitive Science*, 1981, *5*, 121–152.

Chiesi, H. L., Spilich, G. J., & Voss, J. F. Acquisition of domain-related information in relation to high and low domain knowledge. *Journal of Verbal Learning and Verbal Behavior*, 1979, *18*, 257–274.

Clark, H. H., & Haviland, S. E. Comprehension and the given-new contract. In R. O. Freedle (Ed.), *Discourse production and comprehension* (Vol. 1). Norwood, N.J.: Ablex, 1977.

Coke, E. V. Reading rate, readability in variations in task-induced processing. *Journal of Educational Psychology*, 1976, *68*, 167–173.

Collis, K. F. *The development of formal reasoning* (Project No. HR2434/1 Report). Social Research Council of Great Britain, May 1975.

Duckworth, E. Either we're too early and they can't learn it or we're too late and they know it already: The dilemma of "applying Piaget." *Harvard Educational Review*, 1979, *49*(3), 297–312.

Furth, H. G. *Piaget for teachers*. Englewood Cliffs, N.J.: Prentice-Hall, 1970.

Furth, H. G., & Wachs, H. *Thinking goes to school: Piaget's theory in practice*. New York: Oxford University Press, 1975.

Gagne, R. M. The acquisition of knowledge. *Psychological Review*, 1962, *69*(4), 355–365.

Gagne, R. M. Learning hierarchies. *Educational Psychologist*, 1968, *6*(1), 1–9.

Gallagher, J., & Reid, D. K. *The learning of Piaget and Inhelder*. Belmont, Calif.: Brooks-Cole, 1981.

Gentner, D. *The structure of analogical models in science*. (BBN Report No. 4451) Cambridge, Mass.: Bolt Beranek & Newman, Inc., 1980.

Ginsburg, H. *Children's arithmetic: The learning process*. New York: Van Nostrand, 1977.

Ginsburg, H., & Opper, S. *Piaget's theory of intellectual development*. Englewood Cliffs, N.J.: Prentice-Hall, 1969.

Glaser, R. Instructional technology and the measurement of learning outcomes: Some questions. *American Psychologist*, 1963, *18*, 519–521.

Glaser, R. The future of testing: A research agenda for cognitive psychology and psychometrics. *American Psychologist*, 1981, *36*, 923–936.

Glaser, R. The contributions of B. F. Skinner to education and some counterinfluences. In P. Suppes (Ed.), *Impact of research on education*. Washington, D.C.: National Academy of Education, 1978.

Goldman, S. R. *Knowledge children use in producing stories about problem solving*. Paper presented at the annual meeting of the American Psychological Association, New York, 1979.

Greeno, J. G. Cognitive objectives of instruction: Theory of knowledge for solving problems and answering questions. In D. Klahr (Ed.), *Cognition and instruction*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1976.

Groen, G. J., & Parkman, J. M. A chronometric analysis of simple addition. *Psychological Review*, 1972, *79*(4), 329–343.

Groen, G. J., & Resnick, L. B. Can preschool children invent addition algorithms? *Journal of Educational Psychology*, 1977, *69*, 645–652.

Halliday, D., & Resnick, R. *Fundamentals of physics*. New York: Wiley, 1974.

Inhelder, B., Sinclair, H., & Bovet, M. *Learning and the development of cognition*. Cambridge, Mass.: Harvard University Press, 1974. (Originally published, 1974.)

Just, M. A., & Carpenter, P. A. Inference processes during reading: Reflections from eye fixation. In J. W. Senders, D. F. Fisher, & R. A. Monty, (Eds.), *Eye movements and higher psychological functions*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Kamii, C. An application of Piaget's theory to the conceptualization of a preschool curriculum. In R. K. Parker (Ed.), *The preschool in action: Exploring early childhood programs*. Boston, Mass.: Allyn & Bacon, 1972.

Kamii, C., & DeVries, R. Piaget for early education. In M. C. Day & R. K. Parker (Eds.), *The preschool in action*. Boston, Mass.: Allyn & Bacon, 1977.

Katona, G. *Organizing and memorizing: Studies in the psychology of learning and teaching*. New York: Hafner, 1967. (Originally published, 1940.)

Kazdin, A. E. Behavior modification in education: Contributions and limitations. *Developmental Review*, 1981, *1*, 34–57.

Kieras, D. Problems of reference in text comprehension. In M. A. Just & P. A. Carpenter (Eds.), *Cognitive processes in comprehension*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1977.

Kintsch, W. On modeling comprehension. *Educational Psychologist*, 1979, *14*, 3–14.

Kintsch, W., & van Dijk, T. A. Toward a model of text comprehension and production. *Psychological Review*, 1978, *85*, 363–394.

Lankford, F. G. *Some computational strategies of seventh grade pupils* (Final report, Project No. 2–C–013). HEW/OE National Center for Educational Research and Development and the Center for Advanced Studies, University of Virginia, October 1972.

Larkin, J. H., McDermott, J., Simon, D. P., & Simon, H. A. Expert and novice performance in solving physics problems. *Science*, 1980, *80*(4450), 1335–1342.

Lavatelli, C. S. *Piaget's theory applied to an early education curriculum: A Piaget program*. New York: Learning Research Associates, 1970.

Lovell, K. *The growth of understanding in mathematics: Kindergarten through grade three*. New York: Holt, Rinehart, & Winston, 1971.

Mager, R. F. *Preparing objectives for programmed instruction*. San Francisco: Fearon, 1961.

Mandler, J. M. A code in the node: The use of a story schema in retrieval. *Discourse Processes*, 1978, *1*, 14–35.

Mayer, R. E. Can advance organizers influence meaningful learning? *Review of Educational Research*, 1979, *49*, 371–383.

McConkie, G. W., Rayner, K., & Wilson, S. J. Experimental manipulation of reading strategies. *Journal of Educational Psychology*, 1973, *65*(1), 1–8.

Meyer, B. J. F. *The organization of prose and its effect on recall*. Amsterdam: Elsevier, 1975.

Neches, R. *Models of heuristic procedure modification*. Unpublished doctoral dissertation, Carnegie-Mellon University, 1981.

Newell, A., & Simon, H. A. *Human problem solving*. Englewood Cliffs, N.J.: Prentice-Hall, 1972.

Piaget, J. *Success and understanding*. Cambridge, Mass.: Harvard University Press, 1978. (Originally published, 1974.)

Resnick, L. B. (Ed.). Hierarchies in children's learning: A symposium. *Instructional Science*, 1973, *2*, 311–362.

Resnick, L. B. Task analysis in instruction design: Some cases from mathematics. In D. Klahr (Ed.), *Cognition and instruction*. Hillsdale, N.J., Lawrence Erlbaum Associates, 1976.

Resnick, L. B. Syntax and semantics in learning to subtract. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1982.

Resnick, L. B. A developmental theory of number understanding. In H. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic Press, 1983.

Resnick, L. B., & Ford, W. W. *The psychology of mathematics for instruction*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1981.

Rothkopf, E. Z., & Billington, M. J. Goal-guided learning from text: Inferring a descriptive processing model from inspection times and eye movements. *Journal of Educational Psychology*, 1979, *71*(3), 310–327.

Samuels, S. J., & Dahl, P. R. Establishing appropriate purpose for reading and its effect on flexibility of reading rate. *Journal of Educational Psychology*, 1975, *67*(1), 38–43.

Schank, R., & Abelson, R. *Scripts, plans, goals, and understanding. An inquiry into human knowledge structures*. Hillsdale, N.J., Lawrence Erlbaum Associates, 1977.

Selman, R. L., Krupa, M. P., Stone, C. R., & Jaquette, D. S. Concrete operational thought and the emergence of the concept of unseen force in children's theories of electromagnetism and gravity. *Science Education*, 1982, *66*, 181–184.

Shulman, L. S., & Keislar, E. R. (Eds.). Learning by discovery: A critical appraisal. Chicago Rand McNally, 1966.

Siegler, R. The origins of scientific reasoning. In R. Siegler (Ed.), *Children's thinking: What develops*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978.

Simon, H. A. Identifying basic abilities underlying intelligent performance of complex tasks. In L. B. Resnick (Ed.), *The nature of intelligence*. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1976.

Skinner, B. F. Teaching machines. *Science*, 1958, *128*, 969–977.

Spilich, G. J., Vesonder, G. T., Chiesi, H. L., & Voss, J. F. Text processing of domain-related information for individuals with high and low domain knowledge. *Journal of Verbal Learning and Verbal Behavior*, 1979, *18*, 275–290.

Stein, N. L., & Glenn, C. G. An analysis of story comprehension in elementary school children. In R. O. Freedle (Ed.), *New directions in discourse processing* (Vol. 2). Norwood, N.J.: Ablex, 1979.

Suppes, P., & Morningstar, M. *Computer-assisted instruction at Stanford, 1966–1968: Data, models and evaluation of the arithmetic programs*. New York: Academic Press, 1972.

Svenson, O. Analysis of time required by children for simple additions. *Acta Psychologica*, 1975, *39*, 289–302.

Svenson, O., & Broquist, S. Strategies for solving simple addition problems: A comparison of normal and subnormal children. *Scandinavian Journal of Psychology*, 1975, *16*, 143–151.

Svenson, O., & Hedenborg, M. L. Strategies used by children when solving simple subtractions. *Acta Psychologica*, 1979, *43*, 1–13.

Terrace, H. S. Discrimination learning with and without errors. *Journal of Experimental Analysis of Behavior*, 1963, *6*, 1–27.

Thorndike, E. L. *The Psychology of Arithmetic*. New York: Macmillan, 1922.

Thorndyke, P. W. Cognitive structures in comprehension and memory of narrative discourse. *Cognitive Psychology*, 1977, *9*(1), 77–110.

Viennot, L. Spontaneous reasoning in elementary dynamics. *European Journal of Science Education*, 1979, *1*(2), 205–221.

Weikart, D., Rogers, L., Adcock, C., & McClelland, D. *The cognitively oriented curriculum: A framework for preschool teachers*. Urbana: University of Illinois, 1971.

Wertheimer, M. *Productive thinking*. New York: Harper & Row, 1959. (Originally published, 1945.)

Woods, S. S., Resnick, L. B., & Groen, G. J. An experimental test of five process models for subtraction. *Journal of Educational Psychology*, 1975, *67*(1), 17–21.