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AUTHOR	Greeno, James G.; And Others
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ABSTRACT

A framework is presented for characterizing competence for cognitive tasks, with a detailed hypothesis about competence for counting by typical 5-year-old children. It is proposed that competence has three main components that are called conceptual, procedural, and utilizational competence. Conceptual competence, which is discussed in greatest detail, is the implicit understanding of general principles of the domain. Procedural competence is understanding of general principles of action and takes the form of planning heuristics. Utilizational competence is understanding of relations between features of a task setting and requirements of performance. A characterization of conceptual competence for counting is presented, in the form of action schemata that constitute understanding of counting principles such as cardinality, one-to-one correspondence, and order. This hypothesis about competence is connected explicitly to a detailed analysis of performance in counting tasks. The connection is provided by derivations of planning nets for procedures that are included in process models that simulate children's performance. It is concluded that the nature of young children's understanding reflects competence that supports the understanding of counting as well as later development, such as explicit understanding of the role of one-to-one correspondence in definitions of equivalence. (Author/JN)

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CONCEPTUAL COMPETENCE AND CHILDREN'S COUNTING

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JAMES G. GREENO, MARY S. RILEY, AND ROCHEL GELMAN

University of Pittsburgh

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James G. Greeno
University of Pittsburgh

Mary S. Riley
University of California, San Diego

Rochel Gelman
University of Pennsylvania

Learning Research and Development Center
University of Pittsburgh

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Conceptual Competence and Children's Counting

JAMES G. GREENO AND MARY S. RILEY

University of Pittsburgh

AND

ROCHEL GELMAN

University of Pennsylvania

A framework is presented for characterizing competence for cognitive tasks, with a detailed hypothesis about competence for counting by typical 5-year-old children. It is proposed that competence has three main components that are called conceptual, procedural, and utilizational competence. Conceptual competence, which is discussed in greatest detail in this article, is the implicit understanding of general principles of the domain. Procedural competence is understanding of general principles of action and takes the form of planning heuristics. Utilizational competence is understanding of relations between features of a task setting and requirements of performance. A characterization of conceptual competence for counting is presented, in the form of action schemata that constitute understanding of counting principles such as cardinality, one-to-one correspondence, and order. This hypothesis about competence is connected explicitly to a detailed analysis of performance in counting tasks. The connection is provided by derivations of planning nets for procedures that are included in process models that simulate children's performance.

I. INTRODUCTION

We distinguish between hypotheses about performance and hypotheses about competence. Hypotheses about performance postulate cognitive processes and structures that are used in performing tasks, and often are

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formulated in programming languages as computational models that simulate performance in the tasks.

Hypotheses about competence postulate general concepts and principles that we assume are used in constructing or acquiring procedures for use in a conceptual domain, for example, the domain of number. The principles, which often cannot be articulated by the subjects, account for the fact that the diverse performance procedures that appear in diversely structured tasks all have a set of properties that are required by the principles.

I.A. Competence in Counting

Consider an example of generative performance, which illustrates the kind of phenomenon that we believe requires an analysis of competence.

Gelman and Gallistel (1978) asked children to count a set of five objects, arranged in a straight line. Children typically do this by starting at one end and counting along the line. After a chid counted the objects, the experimenter pointed to the second object in the line and said, "now count them again, but make this the 'one'." The task was repeated, asking for the second object to be "the 'two'," "the 'three'," "the 'four'," and "the 'five'." The series was repeated using the fourth object in the line. Many children performed successfully on the modified counting task: a majority of 5-year-old children gave correct performance on either 9 or all 10 of the constrained counting trials.

To count correctly in Gelman and Gallistel's task, children were required to modify the procedure for counting that they used normally. They almost surely had not been taught to count with the constraint of associating a specific numeral with a specific object. A procedure for counting can be modified in many ways, only some of which are consistent with principles of counting and number. It is reasonable to infer that many children who generated correct procedures understood that every object should be tagged once, no object should be tagged more than once, the numerals should be used in their standard order, and the order of tagging objects can be changed.

The domain in which we have worked out an analysis of conceptual competence is simple counting of sets of objects.[1] The question of implicit understanding of principles in this domain was raised by Piaget

[1] The term "counting" is ambiguous. In our use, we refer to performance where there is a set of objects to count. We do not discuss the task of just reciting the string of numerals. In formal mathematical terms, counting is a procedure for determining the cardinality of an assigned set; that is, finding a standard set for which there is a one-to-one mapping to the assigned set. The standard set is an initial segment of an ordered set of symbols. We only consider procedures in which the one-to-one mapping between the standard set and the assigned set is established explicitly; it is not, for example, in "counting by fives."

(1941/1952), whose observations of children's performance in conservation, class-inclusion, and seriation tasks led to the conclusion that children lack understanding of the principles of number that underly counting. According to this quite standard view, preschool children's apparent counting behaviors reflect nothing more than rote performance of a procedure involving recitation of a string of words and coordinated tagging of objects. Gelman and Gallistel (1978), however, disagreed. Based on children's performance in a variety of tasks involving counting, they maintained that preschoolers' counting reflects implicit understanding of counting principles, which guides performance in counting as well as the acquisition of skill in applying the counting procedure. The principles involved include cardinality, one-to-one correspondence, and the relation of ordering, as well as principles pertaining to the conditions under which these three can be applied.

The analysis that we present is the result of our effort to become clearer about the understanding of these principles and their relation to counting performance. The hypothesis that we developed has two components: a process model that simulates salient aspects of children's performance, and a hypothesis about competence that relates relevant components of the process model to the principles of counting.

We began by developing the model of performance in counting tasks; we call this model SC, for Simulation of Counting. SC is a hypothesis about children's cognitive structures and processes that account for their performance in counting.

It is possible to interpret SC as a hypothesis about children's understanding of counting principles. A disadvantage of this interpretation is that in SC the principles remain implicit; that is, the principles are not represented directly in SC, although they were in our minds and influenced our decisions as we designed the model. This led us to develop a different sort of model, a model in which the counting principles are specified explicitly and give rise to suitable procedures through a derivational system that constructs procedures that are consistent with the principles. We propose this formulation of counting principles as a hypothesis about children's implicit understanding in this domain, the understanding that underlies what they do when they count, not what they say about what they do.

I.B. Processes Reflecting Cardinality

We will describe SC in detail in Section IV. We describe one aspect of the model now, to illustrate the implicit nature of counting principles in that model. The illustration deals with the principle of cardinality.

Understanding of cardinality involves knowledge that the number of objects in a set is a property of the set, and that the number of a set corresponds to the last numeral used when the set is counted. This knowledge is simulated in SC by four components of the counting procedure. First, when a request to count some objects is presented, SC constructs a representation that includes a symbol for the set; the objects are associated with the set as members during the counting process. Second, SC sets a goal of finding the number of the set. Third, when all the objects have been counted, SC retrieves the goal of finding the number. Finally, an association is stored in memory connecting the symbol for the set of objects with the numeral used last in the counting procedure, indicating that the concept named by the numeral is the number of the set.

We included these features in SC to simulate children's performance that is taken as evidence for their understanding of cardinality. One form of such evidence involves identifying situational factors that influence the frequency of errors in performance. SC is primarily a model of correct performance, and as such, it includes components that depend on reliable execution of subprocedures. Errors can be interpreted by hypothesizing that subprocedures of the model are less likely to be performed reliably in task situations where errors are more frequent.[2] To illustrate, consider cases where young children vary in their tendency to indicate the cardinal value of a collection.

Preschool children have a tendency to recount when they are asked "how many?" after they have previously been asked to count a set of objects (e.g., Markman, 1979; Schaeffer, Eggleston, & Scott, 1974). This tendency has been interpreted as indicating that young children do not understand the cardinal principle; for if they do, they should repeat only the last numeral said during the count trial (e.g., Schaeffer et al., 1974). Recounting is not a universal feature of children's performance by any means. In Gelman and Gallistel's (1978) experiments where the same set of objects was presented repeatedly in different spatial arrangements, many children did not count de novo after each arrangement; they simply repeated the last tag used on a previous counting trial. Even so, the frequency of recounting is sufficient to require an explanation. Markman's findings provide a clue.

Markman (1979) observed children's frequency of recounting in two conditions: a condition where the objects were referred to with a class

[2] In many discussions, "competence" refers to an ability to perform correctly, and "performance" includes factors that can produce errors. Our distinction is quite different. As we use the term, SC is a model of performance; it simulates correct performance and provides interpretations of incorrect performance. By "competence" we refer to something else, namely, implicit understanding of general principles that is not represented in SC. If the general principles are understood and applied correctly, then performance should be correct; therefore, interpretations of errors as results of situational factors help support the claim that children understand the principles.

noun (e.g., "Here are some soldiers; count the soldiers"; and "How many soldiers are there?") and a condition where the set was referred to explicitly, using a collection noun (e.g., "Here is an army with some soldiers; count the soldiers in the army"; and "How many soldiers are there in the army?"). In the class condition children recounted on about one-half of the trials; in the collection condition, recounts were observed on only 13% of the trials. Markman and we conclude that use of a collection noun makes it more likely that children will represent the objects to be counted as a set, and that this representation is needed to store the number of the set in memory. In our model of counting performance, this is simulated by the inclusion of a symbol for the set of objects in the representation that the model constructs. Storage of the number of the set at the completion of counting provides SC with information needed to answer the "How many?" question. Failures could be simulated trivially by omitting the representational process at the beginning of counting, and Markman's finding would be simulated by a psycholinguistic process in which the representation would depend on the form of the question, with a higher probability of including a referent to the set when a collection noun occurs.

A second kind of evidence involves performance that relates to retention of the goal to store the number of the set in memory while counting is carried out. Gelman and Gallistel (1978) noted that preschoolers' repetitions of the last numeral used in counting, which probably indicates that children appreciate its significance, occurred less frequently after counting large sets than after counting smaller sets. Similarly, Gelman and Meck (1983) found that 3- and 4-year-olds were better able to indicate the cardinal value of a display when a puppet did the counting than when they counted the objects themselves. In SC, the numeral–set association is formed to satisfy the goal of finding the number of objects in the set. This goal has to be retained in memory during counting and retrieved when counting has been completed. Both of the empirical findings are consistent with a hypothesis that failures to associate the numeral and the counted set can result from failure to retain the cardinality goal in memory because of interference from performance of the counting process, rather than failures to understand the cardinality principle.

While the cognitive processes and structures postulated in SC seem appropriate as explanations of children's performance, they do not provide a satisfactory representation of understanding the principle of cardinality. This is because the principle is nearly as implicit in the model as it is in children's performance. Processes of representing a set and forming an association between the set and a numeral are related to understanding of cardinality, but they are not what we mean by understanding of cardinality. To characterize understanding of cardinality, and the other counting principles as well, we need to postulate a knowledge structure in which the principles appear in more explicit form.

Explicit formulations of the principles of counting and number are well known, in the form of definitions and axioms of arithmetic (e.g., Halmos, 1960/1974; Russell, 1964). We have developed a different formulation of the principles to enable our hypotheses about competence to be connected explicitly with models of performance.

I.C. Theoretical Framework

In our analysis, competence and performance are related through a logic of planning. Procedures for performing tasks are generated by a planning system containing three components of competence that we call conceptual, procedural, and utilizational competence. Conceptual competence includes understanding of general principles of the task domain that constrain and justify correct performance. Procedural and utilizational competence are required for these principles to become manifest in performance. Procedural competence includes understanding of general principles of action, relating actions with goals and with conditions of performance. Utilizational competence includes understanding of relationships between features of task settings and goals that can be achieved by using those features. These three components are shown in Fig. 1, in a diagram that reflects their use in derivations of performance structures.

Conceptual competence represents understanding of principles in a form that enables their use in planning. The principles are represented as schematic action units. For example, representation of the principle of one-to-one correspondence includes a schematic action for increasing sets by adding corresponding elements to the sets, and representation of the principle of ordering includes schematic actions for retrieving members of ordered sets in their correct sequence.

By "procedural competence" we refer to knowledge of general principles involving relations of goals, actions, and requisite conditions for actions. Procedural competence includes heuristic rules for planning: the procedures that recognize goals of different types during planning, that search for action schemata with consequences that match goals that have been recognized, and that determine when planning is successfully completed. Procedural competence also includes theorem-proving methods that search for features of the task setting that can be used to prove that conditions will be satisfied, and additional planning heuristics that use these theorem-proving methods when they are needed.

The third component of competence, utilizational competence, includes knowledge used by the theorem prover in its efforts to relate features of the task setting to goals of planning. An example of knowledge

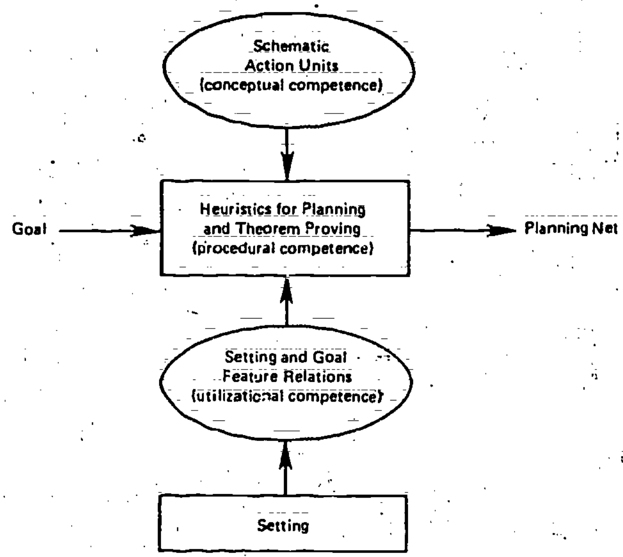

FIG. 1. Components of competence.

included in utilizational competence is a proposition involving objects that are arranged in a straight line. The planner is able to prove a theorem that the straight-line feature of a set of objects can be used to form an ordering of the objects, a feature that satisfies a planning goal of forming partitions of the set of objects to be counted.

We want to formulate hypotheses about competence that are connected explicitly with models of performance. We obtain these connections using a formalism of planning nets, introduced by VanLehn and Brown (1980). Hypotheses about competence are premises in a derivation in which components of a performance model are derived.

The general idea is illustrated in Figs. 2 and 3. Figure 2 shows a simplified model of performance for counting sets of objects, and Fig. 3 shows a simplified planning net for that model of performance. Our hypotheses about competence for counting are cognitive structures that are used in deriving nets like the one in Fig. 3. Thus, planning nets provide definite and explicit connections between competence hypotheses and models of children's performance.

The model shown in Fig. 2 simulates performance of counting when objects are arranged in a straight line, so that the operations of retrieving the "first object" and the "next object after *bound*" can be applied. We call the set of objects L (because they are in a line) and N denotes the set of numerals. SL and SN refer to initial segments of the sets L and N, respectively.

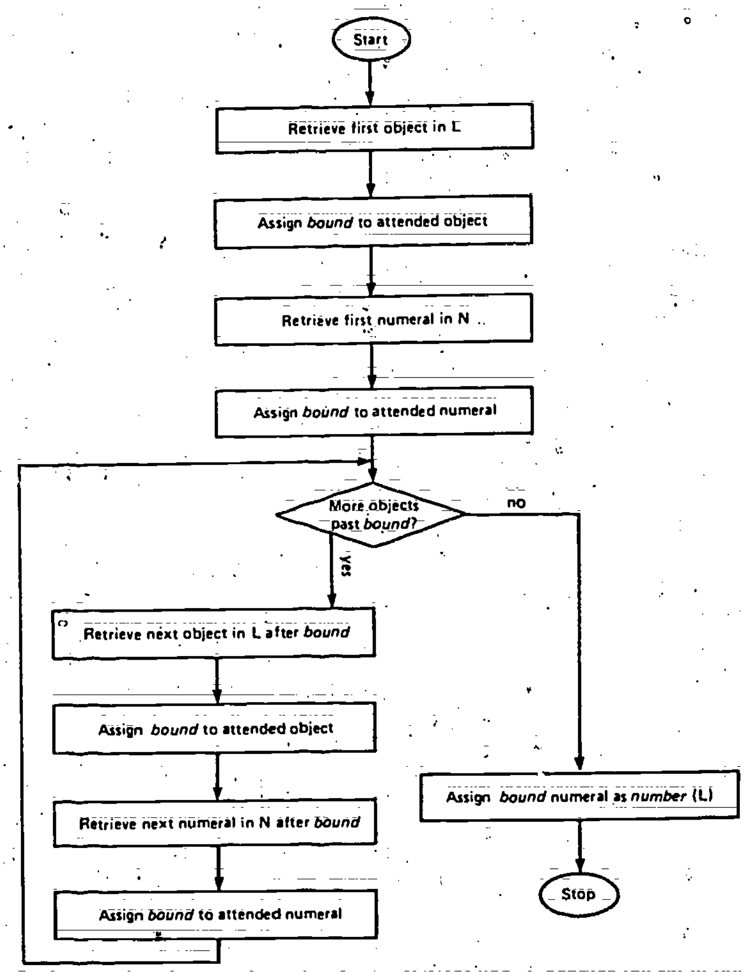

FIG. 2. A procedure for counting objects in a straight line. *L* denotes the set of objects to be counted; *N* denotes the set of numerals.

To begin counting, the model locates the object at one end of the line (called "Retrieve first object in *L*") and remembers that object (called "Assign *bound* to attended object").[3] The model then recalls the numeral "one" (called "Retrieve first numeral in *N*") and remembers it (called

[3] The components of Fig. 2 are summaries of quite complex activities; for example, retrieving and assigning *bound* to objects involve processes of perceptual scanning and memory of the direction of scanning, as well as memory of the current *bound* object. These processes are simulated in more detail in our performance model, SC. We discuss the grain size of a competence hypothesis in Section IV. The choice of grain size provides one way of distinguishing between competence and performance.

"Assign *bound* to attended numeral"). From then on, the counting process is a sequence of cycles. In each cycle, there is test whether there are more objects. If there are, the model finds the next object in the line ("Retrieve next object") and points to it ("Assign *bound*"), retrieves the next numeral, and remembers it. When there are no more objects, the model stops and remembers that the last numeral used is the number of objects in the counted set ("Assign *bound* numeral as *number*(*L*)").

Figure 3 shows a simplified planning net with components of Fig. 2 as its terminal nodes. A planning net includes components of a procedure along with more abstract components, linked together by a set of planning relations. The procedural components are terminal nodes in the net, and correspond to parts of a model of task performance. The components of the planning net include action units (shown as rectangles), goals (shown as hexagons), and tests (shown as diamonds).

The importance of a planning net is that it provides an explicit connection between hypotheses about competence and a model of performance. A planning net is derived from the components of competence discussed earlier and shown in Fig. 1. The action units in a planning net are instantiated versions of action schemata that are included in conceptual competence. The planning heuristics that are included in procedural competence provide rules for selecting action schemata on the basis of their consequences, and setting new goals on the basis of requisite conditions of schemata that have ben selected. Connections between goals and actions in the network correspond to relations that are stored in the action schemata. These relations include consequences of actions; for example, COUNT(*L*) is linked to *number*(*L*) becuse a consequence of the action COUNT is that the number of the set is determined. Relations also include requisite conditions for actions; for example, COUNT(*L*) is connected to *equal*(*L*,*SN*) because forming equal sets is one of COUNT's requisite conditions.

In addition to providing an explicit logical connection between hypotheses about competence and a model of performance, the planning net also shows a structural analysis of the procedure. Action units at one level are related to action units at higher levels as constituents, and these relations are mediated by goals. For example, [RETRIEVE-FIRST object in *L*] and [ASSIGN *bound* to object] are constituents of the abstract action INITIALIZE(*L*,*SL*). INITIALIZE(*L*,*SL*) and INCREMENT(*L*,*SL*) are connected to the goal *one-more*(*L*,*SL*,*a*), and so on. (Goals between the terminal action units and their parents are omitted from Fig. 3 for simplicity. Some of these are shown in Fig. 5.)

I.D. Discussion

In our analyses, we use concepts and methods from three major lines of cognitive theory. We use Piaget's (e.g., 1941/1952) fundamental insight

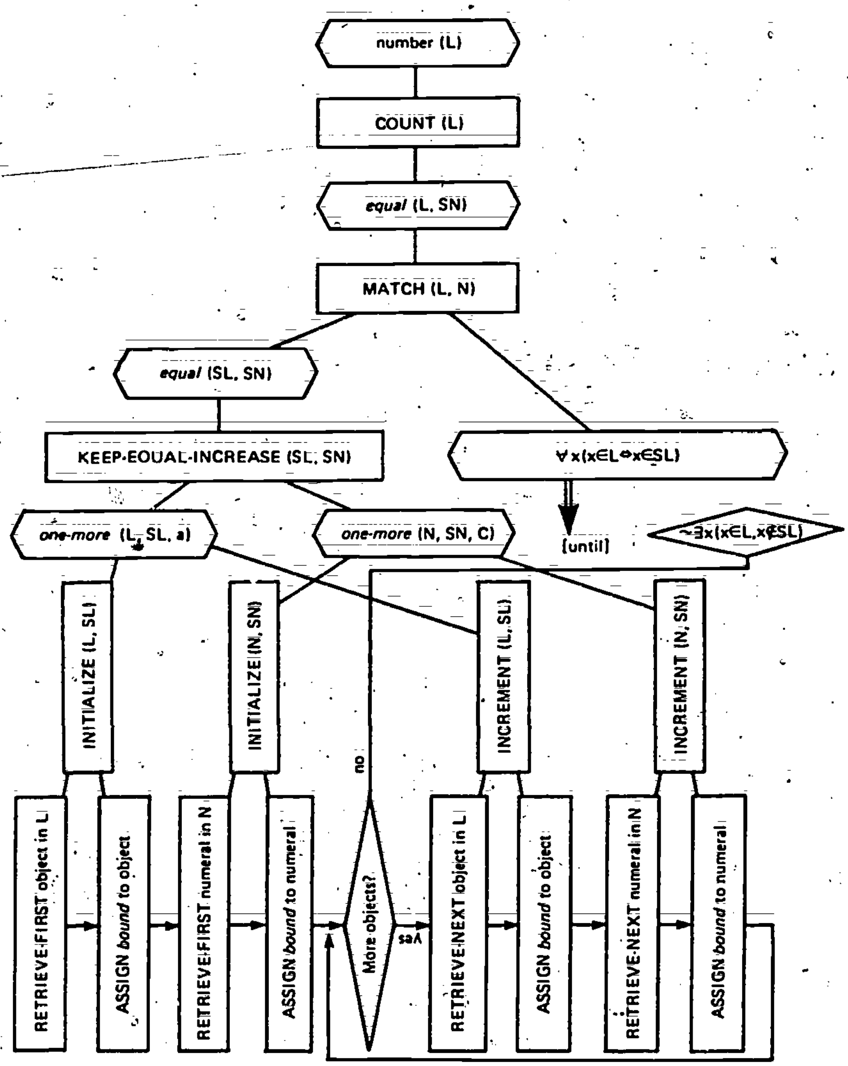

FIG. 3. Planning net for standard counting. *SN* denotes an initial segment of numerals; *SL* denotes an initial segment of objects.

that children's performance reflects understanding (not necessarily articulate) of general principles. However, Piaget did not attempt to derive relationships between the principles and properties of performance, as we do in our analysis. (We also differ with Piaget in our substantive conclusions about the competence that we attribute to preschool children;

we discuss this in Section V.) We use Chomsky's (e.g., 1965) theoretical method of analyzing competence with formal derivations that connect postulated competence with properties of performance. Our analysis differs from Chomsky's and other linguistic analyses in the objects that are derived. In linguistic analyses, the derived objects are sentences, corresponding to sequences of behavior. We derive cognitive procedures, which are capable of producing sequences of behavior. Another difference between our analysis and those standard in linguistics is that we use observations of performance, rather than linguistic intuition, as the main source of evidence about the competence that we attribute to children (cf. Pylyshyn, 1973). Finally, we use concepts and methods introduced by Newell and Simon (1972) that have become standard in cognitive psychology to analyze and represent structures and processes in models that simulate performance in cognitive tasks.

We note that we do not necessarily identify the *process of derivation* of planning nets as a plausible psychological hypothesis. As with other hypotheses about competence, we restrict our claim of psychological reality to the *content* of the knowledge that is attributed to individuals and to the *structures* that are implied by that knowledge. In our analysis, the relation between competence and peformance structures has the form of derivations in which the performance structures are consequences of competence structures, derived by a planning system. However, we have not tried to determine the form of the dependene between competence and performance structures in human cognition. (We discuss this matter further in Section V.)

II. FORMULATION OF CONCEPTUAL COMPETENCE

In this section we present a characterization of conceptual competence in the domain of counting and number. Results of empirical studies of children's counting performance make it reasonable to attribute this degree of competence to typical 5-year-olds in industrialized societies.

We represent conceptual competence in the form of action schemata similar to those used by Sacerdoti (1977) in his analysis of knowledge structures for planning. Each of the schemata corresponds to an action, represented in general form. Each schema specifies one or more consequences of performing the action and requisite conditions that must be satisfied for the action to be performed. Knowledge in this form is appropriate for a planner, which can search for schemata with consequences that match the goals that arise during planning, and can set goals for further planning corresponding to requisite conditions of schemata that it has selected.

To illustrate our notation, we present two simple schemata (which we do use in our analyses): PICK-UP and PUT-DOWN.

(1) PICK-UP(*a*)
Prerequisites: *movable*(*a*);
empty(*Hand*).
Consequence: $a \in$ *Hand*.

(2) PUT-DOWN(*a*,*location*)
Prerequisite: $a \in$ *Hand*.
Consequence: *location*(*a*).
Effect: *empty*(*Hand*).

Think of a schema as knowledge of a kind of action that has specified results and specified requirements for its performance, so that a planner can include the action in order to achieve a desired condition if the required conditions are satisfied. PICK-UP and PUT-DOWN are simplified versions of knowledge for moving an object. Suppose there is a goal to have some object in a specified location, for example, this journal on your desk. PUT-DOWN has the appropriate consequence, with *a* given the value "the journal," and *location* given the value "the desk." Prerequisite conditions must be true before an action can be performed; thus, there must be a *Hand* with the journal in it, and this can be set as a goal. "$a \in$ *Hand*" is a consequence of PICK-UP, and since *a* is the journal, PICK-UP may be put into the plan. PICK-UP requires *empty*(*Hand*), which may be true in the situation; if this can be proved, the plan can be confirmed. Note that PUT-DOWN has both a consequence, *location*(*a*), and an effect, *empty*(*Hand*). Both consequences and effects are results of performing actions, and the distinction between them is somewhat arbitrary. Consequences are generally the desired outcomes of actions in the task setting, and effects are other outcomes. (We also use effects to get around technical difficulties in planning that we do not attempt to analyze fully, cf. Footnote 5, below.)

A significant requirement for the formulation is that the schemata should provide a sufficient basis for performance of counting. Schemata related to the various principles are motivated further by evidence that supports attribution of understanding the principles, and we mention some characteristics of that relevant evidence in this discussion.

The case for understanding of principles is strongest if a child is required to generate a new procedure or a modification of a known procedure, and the procedure that is formed is consistent with the principles. In such cases, the child's performance is the outcome of a problem-solving process for which the particular task circumstances, together with the principles, define the problem. This kind of evidence is frequently used in developmental psycholinguistics, where knowledge of the rules of language is attributed to children when they systematically produce sentences of a given complexity (e.g., Brown, 1973). Evidence based on novel performance is especially compelling if the performance correctly and systematically follows the principle but is incorrect or unconventional in some other way. In such cases, we can rule out the hypothesis that the observed performance was acquired by direct tuition or observation of adults. In psycholinguistics, special attention has been given to children's systematic production of errors like "mouses," "footses," "wented," and "haded," that reflect overgeneralizations of rules that govern the production of regular cases (e.g., Berko, 1958; Brown, 1973; Clark & Clark, 1977).

Evidence for understanding also is obtained if a child can evaluate performance as correct or incorrect with respect to a principle, as when children are shown examples of counting and can identify errors that are made, or when they spontaneously correct their own errors. Evaluation of examples is often used to test whether someone has acquired a categorical concept (e.g., Bruner, Goodnow, & Austin, 1956), and is the standard test in computational linguistics for a system that is alleged to know the grammar of a language (e.g., Hopcroft & Ullman, 1969). A third form of evidence is provided by performance that is systematically consistent with a principle, especially when it occurs in a wide variety of contexts so that consistency would be unlikely in the absence of knowledge of the principle.

Evidence for understanding of principles always is problematic to some degree. Any single piece of evidence can be explained without recourse to a hypothesis of understanding: performance consistent with the principle could be learned by rote, evaluation could involve simple comparison of the example performance and covert performance of a rote procedure, and novel correct procedures could be generated by trial and error. Even so, a combination of evidence of these various kinds can constitute a compelling argument that principles are understood significantly, i.e., are understood as constraints on performance, even if the understanding is implicit, i.e., they cannot be stated by the child.

We present the schemata that we postulate as competence for counting in four groups, related to principles of counting identified by Gelman and Gallistel (1978). Gelman and Gallistel's principles were (1) cardinality: the last numeral reached in a count is the symbol for the number of items in the counted set; (2) one-to-one correspondence: each object in the counted set must be tagged with a unique numeral and every used numeral must be applied to an object; (3) indifference of object order: the objects in the set may be counted in any order; (4) stable order of numerals: the numerals must be used in their standard sequence; (5) abstraction: objects in a set need not be all of one kind to be counted. We begin with schemata involving knowledge of set relations in abstraction, because these are simpler than the others, and we proceed through schemata involving understanding of order, one-to-one correspondence, and finally we discuss schemata of cardinality.[4]

II.A. Sets, Subsets, and Abstraction

One basic requirement of counting is to keep track of which objects have been counted. The task therefore requires cognizance of a subset, identified in some way, which is empty initially, and which gains members as counting proceeds until all of the objects have been counted. A schema representing this understanding is ADD-TO:

(3) ADD-TO(X,A)
Prerequisites: $A = \{x: property(x)\}$;
$a \in X$;
$\sim property(a)$.
Postrequisite: $property(a)$
Consequence: $one\text{-}more(X,A,a)$
Effects: $a \in A$;
$\forall x\ ((x \neq a) = \rangle\ x \in A$ before $\langle = \rangle\ x \in A$ after).

ADD-TO represents knowledge for increasing a subset by adding a single new member to it from a specified set of objects. The first prerequisite of ADD-TO represents knowledge that subsets can be identified by a property. For ADD-TO to be performed, members of the subset have to be identified by some property, and there has to be an object that does not yet have the property. When the object has been given the property, ADD-TO has been completed, and its consequence and effects will be achieved. A postrequisite is a condition for completion of the action. ADD-TO is completed when the object a has been given the *property*. Performance of ADD-TO has three outcomes: A has the property *one-more*, which means that it has a member that was not in A before; a becomes a member of the subset A; and except for adding a, the membership of A has not changed.[5]

ADD-TO is a schema for a global action that cannot be executed directly. More elementary actions are required for global actions to be performed. One possibility that we use involves identifying a subset by a location—that is, A is the subset of objects located in a specified place.

[4] We do not intend to suggest that understanding of the principles discussed here necessarily develops in the sequence in which we present the principles.

[5] This last effect reflects a technical issue in planning that we have not tried to solve. The lack of change in membership of A other than adding a involves the general problem of knowing which features of a situation remain *unchanged* by an action, as well as knowing the consequences and side effects of an action. This problem, the so-called "frame problem," probably cannot be solved in a general way in the context of planning. A full analysis would require identifying conditions that must be monitored during execution.

In that case, the *property* in ADD-TO is the specified place, and when ADD-TO's postrequisite is set as a goal, the planner can choose PUT-DOWN and then PICK-UP to achieve it.

In another case that we have analyzed, a subset is designated by placing physical marks on the objects. This uses the schema ADD-MARK:

(4) ADD-MARK(*a,marker*)
Prerequisite: ~ *on*(*a,marker*).
Postrequisite: *on*(*a,marker*).
Consequence: *marked*(*a*).

ADD-MARK can be implemented using PICK-UP and PUT-DOWN to place a marker on the object *a*.

The schemata that we have discussed provide competence that is required for correct counting performance. Gelman and Gallistel (1978) also discussed another aspect of competence that they called the principle of abstraction, responding to discussions by Gast (1957), by Klahr and Wallace (1973), and by Werner (1957), suggesting that children might have to form intensional representations of a set in order to count it. For example, to count a set containing apples and oranges, the child would have to know and apply the concept "fruit," and to count a set containing persons and pieces of furniture an abstract concept of "things" would be required.

Empirical observations do not support the hypothesis that children initially restrict their counting to homogeneous sets. Gelman and Gallistel (1978) presented heterogeneous arrays of object for children to count, and observed no resistance to counting. Gelman (1980) observed that 3- to 5-year-olds comfortably counted sets as heterogeneous as "all the things in the room," including people. Strauss and Curtis (1981) showed that infants abstract cardinality of small sets differing in the kinds of objects they contain, and Starkey, Spelke, and Gelman (1983) have found that infants abstract the cardinality of heterogeneous sets, including matching the number of objects in a heterogeneous diplay with the number of drum beats they hear.

The principle of abstraction is a permissive principle, rather than a constraint, and it is represented in our competence hypothesis by the absence of a restriction, rather than by any definite assertion. In the schema ADD-TO, and in other schemata, there is no requirement that the set to be counted contains only objects of a single recognizable kind. Indeed, if we were to analyze a task such as "Count the horses" in a display containing different kinds of animals, we would have to provide a conceptual basis for introducing a test of category membership into the counting procedure.

II.B. Ordering and Initial Segments

Correct counting requires that an ordered set of numerals is available to the counter, and requires that the members of that list be used in the counting process in the order that they are given in the set. We include two simple schemata for retrieval of the items from ordered sets.

(5) RETRIEVE-FIRST(*X*)
Prerequisites: *order*(*X*);
first(*X*,*a*).
Consequence: *attend*(*a*).

(6) RETRIEVE-NEXT(*X*,*x*)
Prerequisites: *order*(*X*);
next(*X*,*x*,*a*).
Consequence: *attend*(*a*).

The consequence of these actions is that during execution, the performer will be attending to the retrieved object, which enables other actions involving the object (cf. Schema 9, below). Prerequisites include that the set of items, denoted *X*, is ordered; that is, there is a unique first member of *X* and each member of *X* has a unique successor (except the last member, if that exists). Prerequisites also include designation of the first member of *X* or the successor of a given member of *X*.

RETRIEVE-FIRST and RETRIEVE-NEXT would be sufficient for reciting the numerals in order, but they are not sufficient for use of the numerals in counting. For counting, the ordering is used to partition a set into the subset that has already been included in the count and the subset that has not yet been included. The set already included is an initial segment of the ordered set, containing the items up through a designated item, the upper bound. We denote an initial segment of a set *X* as *SX*, for "segment of *X*." Understanding of this set-theoretic concept is represented by two schemata: INITIALIZE, which places the first member of *X* in *SX*, and INCREMENT, which adds a new member to *SX* by moving the bound to the successor of the current upper bound.

(7) INITIALIZE(*X*,*SX*)
Prerequisites: *order*(*X*);
empty(*SX*);
first(*X*,*a*).
Postrequisite: *bound*(*SX*,*a*).
Consequence: *one-more*(*X*,*SX*,*a*)
Effects: $\sim$ *empty*(*SX*);
$a \in SX$;
$\forall x \, (x \neq a \langle = \rangle x \notin SX)$.

(8) INCREMENT(X,SX)
Prerequisites: *order*(X);
$\sim$ *empty*(SX);
bound(SX,b);
next(X,b,a).
Postrequisite: *bound*(SX,a).
Consequence: *one-more*(X,SX,a).
Effects: $a \in SX$;
$\forall x\ ((x \neq a) \Rightarrow x \in SX$ before $\Leftrightarrow x \in SX$ after).

One more simple schema is needed for the action of assigning the property *bound* to an item.

(9) ASSIGN (*property*,x)
Prerequisite: *attend*(x).
Consequence: *property*(x).

The schemata for ordering that we have given provide a basis for the use of an ordered set in coun' ng. In addition, there is evidence that children appreciate that use of a stably ordered set of numerals is an essential feature of counting.

One kind of evidence involves systematic performance that follows the rule of stable order with a sequence that is idiosyncratic and thus would not have been learned. Gelman and Gallistel (1978) and Fuson, Richards, & Briars (1982) have reported the tendency of 2½-year-old children to count with unconventional sequences that are used systematically. Sequences such as "1, 2, 3, 4, 8, 10, 11" are used by very young children even when they are asked to count small sets, and by somewhat older children when they count larger sets. Despite the use of nonstandard lists, the lists are used systematically. Thus, for example, a 30-month-old child might say, "Two, six," when counting a two-item set, and "Two, six, ten," when counting a three-item array. This child used his own list over and over again, even though he was corrected repeatedly by his parent, R. Gelman.

Performance in Gelman and Gallistel's (1978) modified counting task, described earlier, also provides evidence of understanding the stable-order principle that involves generation of new procedures. When a constraint of using a specified numeral for a specified object is imposed, there is a conflict between using the numerals in their standard order and tagging the objects in their spatial order. Correct performance involves using the order of numerals and therefore changing the order in which objects are tagged. The correct performance given by a preponderance of children, especially by 5-year-olds, provides evidence that children appreciate the stable-order principle as a requirement of counting.

Evidence also has been obtained that children can detect errors in application of the stable-order principle. Briars and Siegler (in press) and Gelman and Meck (1983) gave children the task of observing as a puppet counted and telling the experimenter whether the puppet made an error. Gelman and Meck's puppet made three kinds of errors involving the order of numerals: reversing a pair of numerals, skipping a numeral, or using a random string. Briars and Siegler's puppet made two kinds of errors: omitting a numeral or using an extra numeral. In both studies, 5-year-olds performed at a very high level: Gelman and Meck's subjects detected 96% of the order errors and Briars and Siegler's subjects detected 92% of the order errors. Three- and four-year-olds also showed substantial ability to detect order errors: 76% and 96% of errors detected to Gelman and Meck's study, and 54% and 78% of errors detected in Briars and Siegler's study. (Simple correct counts, with the numerals used in correct order and objects counted from end to end, were called correct over 95% of the time by children of all ages in both studies.)

In our formulation of competence, the schemata for retrieving items from ordered sets and using initial segments (Schemata 5–9) provide the capability of using the order relation in counting, but do not represent understanding that it is a required feature. Cognizance of the stable-order principle as a requirement is represented in the schema called COUNT (Schema 12), in which use of an ordered set of numerals is a prerequisite condition.

II.C. One-to-One Correspondence

In correct counting, for each object counted exactly one numeral must be used. This prohibits tagging any object more than once or omitting any object; it also prohibits using a numeral more than once or skipping any numerals. If these constraints are satisfied, and if all the objects are tagged, then there is a correspondence between numerals and objects. Furthermore, the objects can be tagged in any sequence, providing that each object is tagged exactly once.

Evidence for generative knowledge of the one-to-one requirement was obtained in Gelman and Gallistel's (1978) modified counting task. Recall that by 5 years of age, most children gave near-perfect solutions; that is, they honored the object–numeral pairing constraint and correctly counted all the items in the array. As the constraint was varied on different trials, different pairings of objects and numerals were required. Thus, these children were successful in generating performance that preserved one-to-one correspondence with differing sequences of tagging the objects.

Children's evaluations of puppets' counting also provide evidence for understanding of the one-to-one requirement. Errors of skipping an object or counting an object twice were detected on 95% of the trials by 5-year-olds (Briars & Siegler, in press), on 82% (Gelman & Meck, 1983) and 89% (Briars & Siegler, in press) of the trials by 4-year-olds, and on 67% (Gelman & Meck) and 60% (Briars & Siegler) of the trials by 3-year-olds. Children also saw puppets count correctly with unusual sequences of tagging objects, either starting in the middle of an array, working to one end, then returning and counting the remaining objects, or counting alternate objects (which were of one color), then reversing direction and counting the remaining objects (of a different color). Gelman and Meck's subjects called nearly all of these counts correct: 96% of the trials by both 3- and 4-year-olds. Briars and Siegler's subjects often called these counts incorrect: 35% by 3-year-olds, 65% by 4-year-olds, and 53% by 5-year-olds, but the 5-year-olds rejected these nonstandard counts significantly less frequently than they rejected counts that were incorrect.

(Gelman & Meck (1983) suggested that the different findings were due, in part, to young children's tendency to respond before the puppet finishes a trial. Gelman & Meck pretrained their subjects to wait until a counting sequence was completed. Additionally, these unconventional count trials pose a problem of ambiguity in instructions. Insofar as a procedure is unconventional it may be judged as wrong because it is different and not because the child thinks a counting principle is violated. Indeed, in a current study by Gelman, Meck, & Greeno, a 5-year-old told us that standard count trials were "right and right," that error trials that were unconventional and violated a principle were "wrong and wrong," and that uncoventional trials that did not violate a principle were "wrong but right.")

In Gelman and Gallistel's (1978) data, children's counting performance honored the one-to-one requirement in the preponderance of cases. Almost all one-to-one errors involved counting an object twice or skipping over an object as a child systematically moved his or her finger from object to object. Such errors would be expected if the children's counting procedures were appropriately constrained by the one-to-one requirement but were subject to occasional failures of keeping track of just what objects had already been counted. It is noteworthy that children who use idiosyncratic lists of number words honor the one-to-one requirement with those lists, as do children who use the standard list. Gelman and Gallistel also observed children counting the same set of objects in varied spatial arrangements, and children who recounted the arrays showed no tendency to keep assigning the same numerals to the same items. These children were apparently indifferent to the order in which the objects were counted.

We represent conceptual competence regarding one-to-one correspondence with two schemata:

(10) MATCH(X, Y)
Prerequisites: *empty*(A);
empty(B).
Corequisites: *subset*(A,X), where $A = \{x: tagged(x)\}$;
subset(B,Y), where $B = \{y: used(y)\}$;
equal(A,B).
Postrequisite: $\forall x\ (x \in X \Rightarrow x \in A)$.
Consequence: *equal*(X,B)

(11) KEEP-EQUAL-INCREASE(X,A,Y,B)
Prerequisite: *equal*(A,B)
Corequisites: $\forall x\ ((x \neq a) \Rightarrow x \in A$ before $\Leftrightarrow x \in A$ after);
$\forall y\ ((y \neq b) \Rightarrow y \in B$ before $\Leftrightarrow y \in B$ after).
Postrequisites: *one-more* (X,A,a);
one-more(Y,B,b).
Consequence: *equal*(A,B).

The arguments of MATCH are two sets, denoted X and Y. Its consequence is a subset of Y that is *equal* to X. Prerequisites of MATCH include designation of subsets A (of X) and B (of Y) that are initially empty. Corequisites of actions are conditions that must be maintained throughout performance of the action. Corequisites of MATCH include maintaining the partitions of X and Y during counting, so that A contains the members of X that have been tagged and B contains the members of Y that have been used. Another corequisite of MATCH is that the subsets A and B are to be kept equal. MATCH is complete (i.e., its postrequisite is satisfied) when all the members of X have been included in A.

KEEP-EQUAL-INCREASE provides a way to increase two sets while keeping them equal. The prerequisite and consequence of KEEP-EQUAL-INCREASE is the equality of two sets, A and B. The postrequisite is that A and B should each receive a new member; A receives a member from X and B receives one from Y. The condition to be maintained (i.e., the corequisite) is that no members should be lost from A or B, and no members other than the designated objects a and b should be added.

MATCH and KEEP-EQUAL-INCREASE provide a procedural definition of the predicate *equal* in our analysis. The definition can be stated less formally as follows: two sets are *equal* if they are initially empty, then each is increased by a single member, however many times the joint increase occurs.

The predicate *equal* is logically prior to the concept of *number* in our analysis, which may seem counterintuitive, but is consistent with some evidence about young children. Miller (in press) showed two toy turtles, each with a pile of candy, and a third pile of candy to be shared equally between the two turtles. The most common method used by children 3 years old and older was to progressively distribute one candy at a time to each turtle, repeating this procedure until all the candies were distributed. This concern for equality of number extended to other tasks not strictly numerical in nature. In dividing equal lengths and areas, preschoolers often showed a similar strategy, cutting many pieces of apparently arbitrary size, but taking care that the same *number* of pieces was given to each recipient.

II.D. Cardinality

The principal of cardinality is the significance of the counting procedure. Counting determines the number of objects in a set, which is represented by the last numeral used when the count is finished.

Some observations were discussed in Section I that support attribution to understanding cardinality to young children. We mentioned Markman's (1979) finding that recounting occurs less frequently when collection nouns are used to refer to sets than when only class nouns are used. We also mentioned evidence given by Gelman and Gallistel (1978) and by Gelman and Meck (1983) that failures to store the cardinality of a counted set in memory can be interpreted as results of interference from task demands of performing the counting process.

Important evidence of generative knowledge of cardinality comes from observations that children invent novel procedures in arithmetic. A striking example was provided by Groen and Resnick's (1977) observations of preschool children's procedures for solving addition problems. Children 4½ years old were taught to solve simple addition problems by counting out two groups of objects equal in value to the two addends, combining the objects into one group, and then counting the combined group. After several sessions of practice, one-half of the children spontaneously employed a more efficient algorithm that they had not been taught. This was to count on from the cardinal value of the larger addend. Neches (1981) has developed a plausible analysis of the process of generating new procedures, based on noticing invariance of results of components of the procedure already in place. The invariances needed for development of the count-on addition algorithm are results of component counting procedures, that is, the cardinalities of subsets and the total set in the situation. Children's modifications of procedures with cardinality preserved as an invariant support the conclusion that they understand the meaning and significance of the counting procedure.

Evidence of understanding cardinality also is provided in children's performance on an evaluation task. Gelman and Meck's (1983) puppet counted sets of 5, 7, 12, or 20 objects, and sometimes made mistakes in answering "how many?" questions with a value less than the last numeral stated, or one greater than the last numeral stated. These errors were almost always detected by children: the 3-year-olds chose to correct the puppet on 70% of the error trials; the 4-year-olds on 90% of such trials. Having corrected the puppet, the 3- and 4-year-old children gave the correct answer, respectively, 94 and 95% of the time.

Further evidence includes Gelman and Gallistel's (1978) observation that children frequently repeat the last numeral used in counting, often with emphasis. Repetition of the final numeral suggests that children appreciate that it signifies something special, and evidence from other experiments (Gelman & Tucker, 1975) indicates its significance as the cardinality of the set. Gelman has also observed that a child who uses an idiosyncratic list of numeral terms in counting repeats the last term used when asked how many objects are present.

Finally, another supporting observation is that children count objects spontaneously, that is, without instructions to count, when the number of objects is relevant in some way for a task. This indicates that the children understand that counting is the appropriate procedure for determining cardinality. Evidence of this kind was obtained in experiments by Gelman (1977), who showed children displays with two set of objects and taught the children that the "winner" was always the set with a greater number of objects. Then on a sequence of trials, a display was shown, then hidden briefly, and then shown again with some change made in the meantime. Children's appreciation that number was relevant was indicated by their reactions of surprise and what they said when they encountered this change: e.g., "Took one! Was three—one, two three. Now two." (Children did not react as strongly if the change involved the positions of objects in the displays and maintained they still won because the number was as expected. Even when an item of a different type or color was substituted and children were surprised, they insisted the display was still the winner because it had the expected number.) Although there was no explicit instruction to count the objects, children were often observed to count them aloud, indicating an association between the goal of finding numbers of sets and the counting procedure.

We represent implicit understanding of cardinality with the schema COUNT:

(12) COUNT(X)
Prerequisites: Set of numerals, N;
order(N).
Postrequisites: *equal*(X,SN);
bound(SN,n).
Consequence: *number*(X) = n.

COUNT gives a procedural definition of the number of a set. In it, the concepts of order and equality are synthesized.[6] The use of an ordered set of numerals is a prerequisite of counting. Counting is complete when there is an initial segment of the numerals that is equal to the set to be counted. The upper bound of that initial segment corresponds to the number of objects in the counted set.

III. PLANNING NETS

In Section II we have presented a characterization of conceptual competence for counting. Now we present derivations of planning nets[7] that relate that competence to performance in counting tasks. To derive planning nets we require assumptions about planning heuristics and knowledge about the task setting, which we refer to as procedural and utilizational competence.[8] We discuss these components of competence briefly, and then present derivations of planning nets for counting procedures.

III.A. Procedural Competence

In the derivation of planning nets, the schemata described in Section II function as premises, and planning nets are the theorems that are derived. A set of inference rules is needed, and these are provided by a set of planning heuristics.

The structure of action schemata was patterned after Sacerdoti (1977). Consequences and requisite conditions are included in each action schema, which permits planning to occur essentially through means–ends

[6] Our treatment of cardinality and order here is similar to Gelman and Gallistel's (1978, Chap. 11) characterization of young children's understanding of equivalence and ordering relations.

[7] The idea of a planning net used here is generally similar to that developed by VanLehn and Brown (1980), but differs in some significant details. One of these is our use of action schemata as the premises of the derivation; VanLehn and Brown used constraints expressed as logical forms. Another is that VanLehn and Brown included heuristics for deriving sequential properties of procedures; that is omitted in our analysis, where we allow planning to stop when a sufficient set of procedural components has been derived.

[8] The heuristic rules that we have used in deriving planning nets for counting are standard in the literature on planning (e.g., Fikes, 1977). A planning system was not implemented in the work that we report here, so there is some uncertainty about the adequacy of planning rules and the exact formulation of the other knowledge that was postulated. However, the extensions of standard planning methodology that would be required to plan procedures for counting do not seem to raise conceptual difficulties that would change the main results of the analysis. In subsequent work, a planning system is being implemented, and although this has not been completed at the time of this writing, preliminary results have been obtained and reported in Smith and Greeno (1983). The results thus far bear out the expectation that the main conclusions reported here are valid.

analysis (Newell & Simon, 1972). Planning begins with the presentation of a main goal, to find the number of a set of objects. The planner searches in the set of action schemata for a schema that has a consequence that matches the goal. When one is found, it is tentatively included in the plan, and its requisite conditions are examined. Prerequisites have to be satisfied before an action can be performed; corequisites have to be satisfied throughout performance of the action; and postrequisites have to be satisfied in order to complete the action.

For each requisite condition of a schema that the planner has included, the planner first tests whether the condition is satisfied in the setting. The planner has some theorem-proving procedures that use information about features of the setting along with general principles that link settings to conditions required for schemata. (Understanding of those general linking principles is called utilizational competence in our analysis, and we discuss it in the next subsection.) Requisite conditions can also be satisfied by effects of other actions in the plan. If the requisite condition is satisfied, the planner asserts the specific features in the setting or the side effects that are to be used in satisfying the condition.

Requisite conditions that are not satisfied are set as goals for planning. Planning proceeds by considering each goal that has been set, searching for an action schema whose consequence matches the goal. If more than one schema is available, the planner keeps a record of alternatives, enabling return to the choice point if the alternative chosen first cannot be developed successfully. If there are alternatives that require different prerequisite conditions, both (or all) the schemata can be included in the plan, along with an explicit test that will determine which of the actions should be performed during execution. Actions that require multiple steps can be included, along with tests for their completion.

Planning is complete when all the goals that have been set are satisfied by consequences of actions that have been included in the plan.

III.B. Utilizational Competence

The understanding represented by conceptual schemata and planning heuristics must be combined with knowledge about the setting in which counting will occur to derive a procedure for performing the task. The system includes general principles that can be used to prove theorems about the satisfaction of requisite conditions by features of the task setting. We refer to this as utilizational competence, since it is knowledge that enables features of the setting to be utilized for the application of conceptual competence. Utilizational competence enables the planner to determine that features of the task setting can be used in developing its plan. In making these determinations, the planner uses a simple theorem prover that contains rules for making inferences based on features of the setting.

An illustration of utilizational competence involves the way in which objects to be counted are arranged, and restrictions that may be placed on moving the objects. In one situation that we have analyzed, the objects are arranged in a straight line. This is relevant to the requirement of maintaining a partition between the set of objects that have been tagged and the objects that remain untagged during counting. (This requirement is in the conceptual schemata as a corequisite of MATCH.) Utilizational competence includes a proposition that objects in a straight line can be ordered, starting at one end and proceeding to the other. Then the partition that is required can be maintained by using the spatial sequence in tagging the objects.

We also will discuss counting in a setting where objects are not arranged in a straight line, but can be moved from one location to another. Using a proposition in utilizational competence, the planner determines that the partition of tagged and untagged objects required by MATCH can be achieved by designating a spatial region for locating the tagged objects.

III.C. A Planning Net for "Standard Counting"

In this section we derive a planning net for counting in one situation, where the objects to be counted are arranged in a straight line. In sections that follow, we discuss generalizations of the analysis involving variations in the setting and with constraints imposed on counting objects that are in a straight line.

To fix the target of the analysis, recall Fig. 2, a procedure for counting objects in a straight line. This is a simplified version of the procedure that was implemented in the process model SC, which we discuss in Section IV. Our goal is to provide a structural analysis of this procedure that shows how it is generated from the principles of counting, in the form of the action schemata described in Section II.

Figure 4 shows a portion of the planning net that is generated from a goal of finding the number of a set of objects. Figure 4 is generated in the first several steps of planning.

We now comment on notation involved in Fig. 4. The diagram refers to goals and actions, and planning relations among them. Goals are shown in hexagons; actions are shown in rectangles. The actions are instances of action schemata that were discussed in Section II. Relations between actions and goals are labeled as prerequisites (prereq), corequisites (coreq), postrequisites (postreq), consequences (conseq), and effects. Recall that a prerequisite must be true before an action can be performed, a corequisite must be kept true throughout performance of an action, and a postrequisite must become true for the action to be completed. A consequence or an effect becomes true as a result of performing the action.

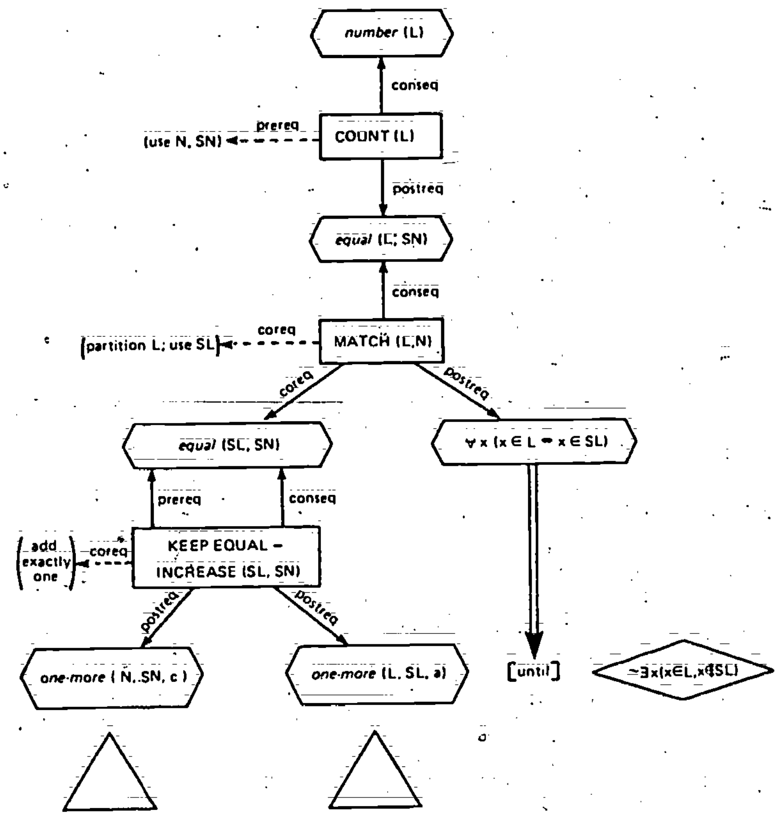

FIG. 4. Portion of planning net for standard counting.

The phrases in Fig. 4 that are in parentheses refer in brief form to conditions that are satisfied in the task setting; propositions in utilizational competence are used to prove that these conditions are satisfied.

The planning process begins when the main goal *number*(*L*) is presented. *L* refers to a specific set of objects that are to be counted; recall that we call it *L* as a reminder that the objects are arranged in a straight line.

The planner tries to prove a theorem that *number*(*L*) is already known. This fails, so it becomes a goal for planning. The planner searches among the action schemata for a schema with a consequence that matches the goal. COUNT(*X*) is found and is tentatively placed in the plan, with *L* identified as its argument.

Next, COUNT(*L*)'s requisite conditions are examined. The prerequisite is an ordered set of numerals with initial segments, and utilizational competence shows that this is satisfied. (For purposes of planning, mental objects that are in memory are considered as part of the task setting.) The planner then notes that the set *N* and its initial segments, *SN*, will be used in the plan. The postrequisite of COUNT(*L*) is a segment *SN* that is equal to *L*. This is not provable in the setting, so the planner sets *equal*(*L*,*SN*) as a goal.

A search is made for a schema with *equal*(*X*,*Y*) as its consequence. Two are found: MATCH(*X*,*Y*) and KEEP-EQUAL-INCREASE(*X*,*Y*). The prerequisite of KEEP-EQUAL-INCREASE cannot be satisfied for the arguments *L* and *SN*; they are not equal, as is required. MATCH requires arguments with initially empty subsets, and this can be satisfied; MATCH(*L*,*N*) is selected and tentatively included in the plan. To include MATCH, the planner is required to designate a partition of *L* that will be used to satisfy its corequisite. The planner notes that *L* is arranged in a straight line and infers (with utilizational competence) that *L* can be ordered. The subset of *L* to be used is designated as the initial segment *SL* formed by the ordering.

The remaining requisites of MATCH(*L*,*N*) are the postrequisite that all members of *L* should become members of *SL*, and that *SL* and *SN* should be kept equal while MATCH is being performed. These are set as goals.

To plan for the goal involving all members of *L*, the planner needs some special knowledge about iterative procedures. There is no action schema that takes a set as an argument and makes it equivalent to another set; however, there are schemata that take individuals as arguments and put them into a set. To enable use of these schemata, the planner converts the goal about *L* into a goal involving members of *L* and a test for completion. The goal is *one-more*(*L*,*SL*,*a*), where *a* denotes some object that will be added to *SL*, and the completion test is the absence of any members of *L* that have not become members of *SL*.

The planner proceeds to work on achieving *one-more*(*L*,*SL*,*a*). This results in a condition that violates the corequisite of keeping *SL* and *SN* equal, because an object will be added to *SL*. A search is made for a way to maintain *equal*(*SL*,*SN*), and KEEP-EQUAL-INCREASE(*SL*,*SN*) is found. Its prerequisite is satisfied in the plan. It has two postrequisites; *one-more*(*L*,*SL*,*a*) has already been included in the plan, and the other, *one-more*(*N*,*SN*,*c*) is set as a goal. The corequisites of KEEP-EQUAL-INCREASE are also set as goals; they eventually are confirmed as being satisfied by properties of actions that are chosen later to satisfy the *one-more* goals.

Figure 5 shows the planning net completed under the goal of *one-more* until there are no more objects to count. There are two schemata with

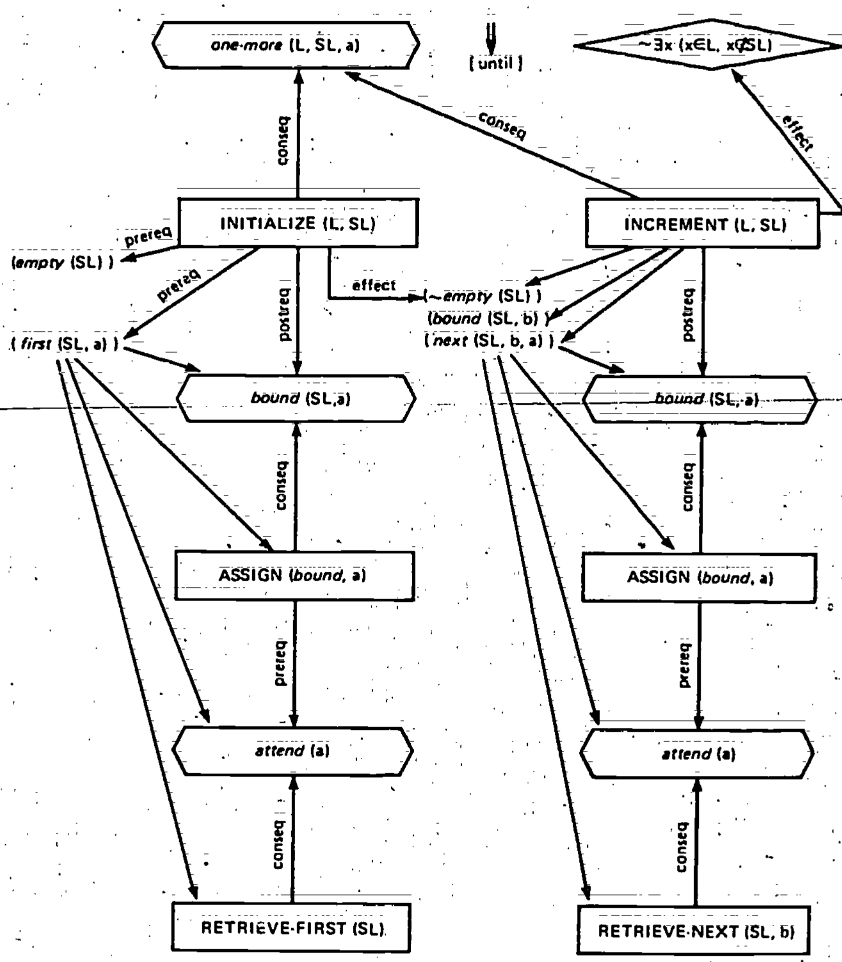

FIG. 5. Completion of planning for one-more (*L,SL,a*).

the consequence *one-more* for a set: INITIALIZE and INCREMENT. A prerequisite of INITIALIZE(*L,SL*) is that *SL* is empty; a prerequisite of INCREMENT(*L,SL*) is that *SL* is nonempty. The planner adopts INITIALIZE(*L,SL*) since its prerequisite is known to be satisfied at the beginning of counting. However, performance of INITIALIZE(*L,SL*) has the effect of making *SL* nonempty, and since *one-more* must be satisfied repeatedly, the planner includes INCREMENT(*L,SL*) in the plan as well.

The theorem prover infers that repeated use of INCREMENT(*L*,*SL*) will eventually reach the last object in *L*, and thus will satisfy the goal of having no objects in *L* that are not in *SL*.

A further prerequisite of INITIALIZE(*L*,*SL*) is that there is a first member of *L*; this is verified by utilization knowledge. This first member is henceforth referred to as *a*. A postrequisite is that *a* should become the upper bound of *SL*, and this is set as a goal. This goal can be achieved by ASSIGN(*bound*,*a*), which requires *attend*(*a*), which can be achieved by RETRIEVE-FIRST(*SL*). A parallel plan, differing only in details, is developed under INCREMENT(*L*,*SL*).

The plan is completed by a network similar to Fig. 5 that is developed for the goal *one-more*(*N*,*SN*;*c*), the goal of increasing the set of used numerals.

The completed planning net for standard counting is shown in Fig. 3, with the diagram abbreviated by the omission of prerequisite conditions that are achieved with utilizational competence and the goals that are achieved by single actions at the base of the network. The actions also have been ordered sequentially as they would be for the procedure to be executed. (We did not analyze the knowledge needed to arrange actions in order for execution, but see VanLehn & Brown, 1980.) The planning net represents constituent units of the procedure, grouping together actions that are included to achieve each goal that is required for counting to be done correctly. It also indicates the relationship between the actions in the procedure and the principles of counting, showing how the procedure satisfies the goal of finding the number in a set by forming a set of ordered numerals that has a one-to-one correspondence with the set of objects.

III.D. Flexibility and Robustness

Generative capability is the hallmark of competence, and a major goal of a competence hypothesis is to give an account of the generative character of knowledge. In this section, we describe analyses of counting procedures that differ from the "standard" case presented in Section III.C. We discuss two forms of generative capability, which we call *flexibility* and *robustness*. Flexibility is the ability to generate procedures for achieving a goal in a variety of task settings. Robustness is the ability to adapt a procedure to accommodate constraints that are not normally imposed in the task.

Flexibility. One indication that an individual has competence, rather than a mechanical skill, is that the individual can perform the task in a variety of settings that require differing procedures. Different settings provide different resources and requirements for implementing a procedure, and different utilizational competence is required for successful planning.

An outcome of analyzing competence for counting in different task settings is identification of the conceptual "core" of competence for counting. The schemata that are used in deriving procedures for counting in all of its settings correspond to the essential principles of counting, and thus can be distinguished from components of competence that are required by task characteristics that can be varied without changing the essential nature of counting.

We have analyzed procedures for three settings that differ from the one discussed in detail in Section III.C. In two of these settings numerals are used to count objects, and different methods are found to maintain a partition in the set of objects between those that have been tagged and those that have not. In the third setting there is a procedure for matching a set of tokens with a set of objects, where the tokens are physical objects rather than numerals or any other stably ordered set of tags. This sharpens the definition of "counting," providing a case that is intuitively outside the domain of counting, enabling a judgment of necessity of some of the competence in our characterization.

In one situation that we have analyzed, a special location is designated for the objects that have been tagged. Initially, all the objects are at a place called the *Source*, and as they are counted they are moved to a place called the *Pile*. The schemata PICK-UP, PUT-DOWN, and ADD-TO provide actions that change the locations of objects and thereby increase the set of objects that have been counted.

In another situation that has been analyzed, the objects cannot be moved, but there are physical markers that can be placed on objects that have been included in the count. For this situation, the schema ADD-MARK is used to accomplish the goal of adding a new member of the counted set. The planner accomplishes the postrequisite of ADD-MARK using PICK-UP and PUT-DOWN, this time changing the location of a marker rather than an object.

A third variant on standard counting that has been analyzed uses many of the components of the conceptual structure of counting, but not the schema COUNT itself. In this situation, a set of objects is presented, and another set is to be constructed that is equal to the presented set. One could imagine a transaction in which a person is buying some large objects—say, used cars or stacks of hay—and the task is to form a set of coins that is equal to the set of objects being purchased. The setting that was analyzed includes a constraint that the objects cannot be moved, but can be tagged with markers. The tokens used for the constructed set can be placed in a special location. The procedure that was derived has the schema MATCH to satisfy the main goal of an equal set, and derives subprocedures that use ADD-TO with ADD-MARK to maintain the partition of tagged objects, and with the special location to maintain the partition of used tokens.

The planning nets for these three procedures are all closely related to the planning net for counting objects in a line, derived in Section III.C. Recall Fig. 4. The nets for counting objects by moving them into a pile and by marking them have all the components of Fig. 4, and also have the same expansion under *one-more(N,SN,c)* as the planning net for objects in a line. They differ in the planning net that is derived under the goal *one-more(L,SL,a)*. In the case of movable objects, the planner uses a proposition (in utilizational competence) that a special location can provide a property that identifies a subset, and includes ADD-TO with PICK-UP and PUT-DOWN to satisfy the goal of *one-more* for the set of tagged objects. When the objects are not movable, but markers are available, the planner uses a proposition that a subset of marked objects can be identified and constructs a net with ADD-TO, ADD-MARK, PICK-UP, and PUT-DOWN.

The schemata that are included in all of the counting procedures can be considered as the conceptual core of counting, distinguished from other schemata that are needed for counting to be accomplished in specific task settings. This provides one way to distinguish between competence and performance. In considering children's competence for counting, it is reasonable to consider the components that vary among task settings as knowledge that enables a child's competence for counting to be applied in the various settings. For example, a child might have the basic cognitive structures that we represent with the schemata COUNT, MATCH, KEEP-EQUAL-INCREASE, and so on, but not have a schema such as ADD-MARK that would enable a subset of objects to be identified by placing markers on them. This would lead to failure in some counting tasks that we would not want to call a lack of competence for counting, but a failure of performance of the kind Flavell, Beach, and Chinsky (1966) called a production deficiency.

Note that this distinction between competence and performance is relative to the choice of a set of tasks taken to involve counting, a choice that is not entirely arbitrary, but depends on a kind of intuitive judgment. An example is provided by the procedure that matches a set of tokens and a set of objects. Our intuitive judgment is that a procedure that forms matching sets is not in the set of procedures that should be called "counting," even though many components of counting are included. Indeed, our analysis provides a precise characterization of the ways in which the one-one procedure is similar to counting, and we have cited as evidence for our analysis Miller's (in press) observation that children choose a matching procedure when asked to produce equal shares of objects. The intuition that a matching procedure without numerals or some other stably ordered set of tags is not counting supports the separation of COUNT and MATCH as separate schemata, since that distinction enables both counting and the related matching procedure to be generated from a set of shared cognitive structures, but also preserves an apparently significant distinction among procedures. The intuition also supports a judgment that competence for counting includes implicit understanding of a requirement to use a stably ordered set of tags—usually the numerals—and the significance of subsets of that ordered set in assigning the property *number* to sets of objects.

The distinction between schemata in competence and performance is not a simple partition. Some schemata, such as COUNT, MATCH, and KEEP-EQUAL-INCREASE, seem to belong clearly in the competence for counting, and some others such as PICK-UP and PUT-DOWN seem to belong clearly in the performance component, since they are not used at all in some counting procedures. On the other hand, there are schemata that are required for counting, such as INITIALIZE and INCREMENT, needed to relate the relation of precedence in an ordered set and subset membership, that also are used in implementing counting procedures in special circumstances, such as a situation where the objects are arranged in a straight line. INITIALIZE and INCREMENT seem to belong in the competence for counting, but use of these schemata is an important element of utilizational competence as well.

Robustness. Another way in which knowledge can be generative involves ability to adapt to new constraints that are imposed on performance. An analysis of conceptual competence should show how successful adaptations depend on general conceptual structures. The analysis also can show how adaptations that are only partially successful can be generated when significant components of conceptual competence are neglected.

We have done an analysis of robustness in counting, using the task of modified counting studied by Gelman and Gallistel (1978), described at the beginning of this article. Recall that the task asks a child to count some objects repeatedly with each count constrained so that a specified numeral is to be paired with a specified object; for example, the experimenter may point to the second object in the row and say, "Make this the *four*."

In Gelman and Gallistel's experiment, most of the 5-year-old children gave nearly perfect performance; that is, they used procedures that complied with all of the counting principles on at least 9 of the 10 trials. Performance by most of the younger children involved less successful adaptations, with violations of one or more of the counting principles. Typically, however, these children used procedures that were partially consistent with the principles. A major goal of our theoretical analysis is to show how a variety of procedures, involving partial compliance with principles, can be understood as results of failures to utilize certain specifiable components of conceptual competence.

To perform the constrained task, a child is required to modify the normal counting procedure. As the analysis of Section III.C shows, the linear arrangement of objects supports a procedure in which the partition between tagged and untagged objects is kept by remembering the last object that was tagged. In the constrained task, either the spatial array itself has to be changed, or a modification of the procedure is needed to avoid violations of counting principles.

Indeed, some of the children responded to the constraint by changing the display. Denote the objects A, B, C, D, and E in their spatial order. When the instruction was to "make B the *one*," they moved B to the front; for "make B the *two*," they put B back in its original position, and so on. This reflects sophisticated knowledge about the procedure, involving understanding of the conditions that enable the procedure to be performed and generation of a method for restoring the needed conditions when they are not made available. We have not analyzed adjustments that restore the conditions for the counting procedure; however, empirical analyses of knowledge for such adjustments was conducted in the task of finding the area of a parallelogram by Morris and Resnick and by Pellegrino and Schadler (reported by Resnick & Glaser, 1976).

The cases that we have analyzed involve modifications of the counting procedure. The features that are required for any procedure to conform to the added constraint are tests to determine whether the object or numeral that is retrieved is the one that is constrained. Procedures differ in the actions that are taken as a result of these tests.

First, we discuss a modified procedure that we call SC-1; part of its flow chart is in Fig. 6. In SC-1, tests for the special object and numeral are included in a very simple way. When either the special object or the special numeral is encountered, the other constrained element is retrieved to accompany it. The sequence of actions is modified from the procedure for standard counting, shown in Fig. 2, by delaying the assignment of *bound* for an object until after a numeral has been retrieved. This is needed for those cases in which the object retrieved first has to be replaced by the special object because the special numeral has been retrieved in the meantime.[9]

[9] The part of the procedure shown in Fig. 6 applies after counting has been initialized. In the procedure for standard counting, shown in Fig. 2, the subprocedure shown in Fig. 6 replaces the four Retrieve and Assign steps in the lower left section of Fig. 2. A similar modification of the standard procedure is required in the initialization, involving tests whether the first object or the first numeral have been designated as special. Similar remarks apply to the subprocedures that are shown in Figs. 7 and 8.

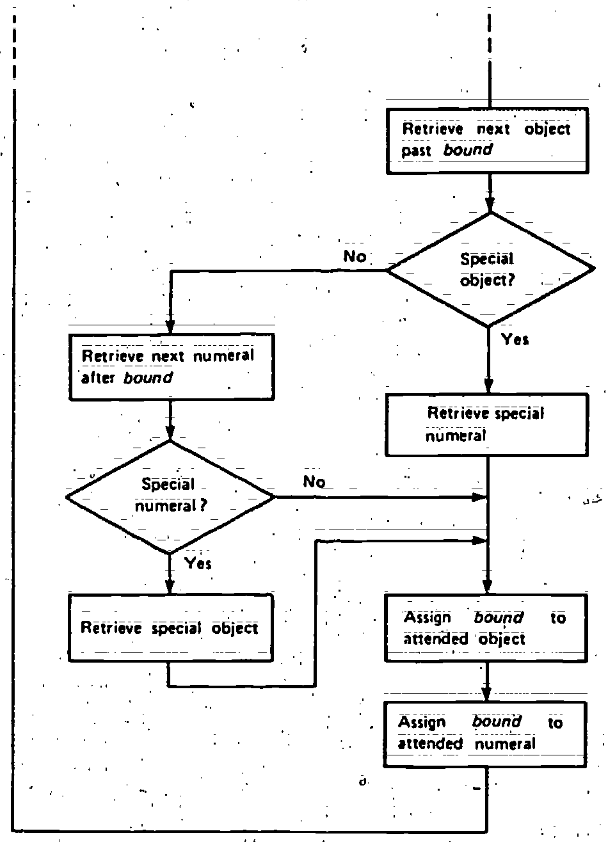

FIG. 6. Modified counting by Model SC-1.

SC-1's performance conforms to the added constraint, but it violates one-to-one correspondence and cardinality. For example, if the constraint is "make B the *four*," SC-1 counts (A,*one*), (B,*four*), (C,*five*), (D,*six*), (E, *seven*). For "make D the *two*," SC-1 counts (A,*one*), (D,*two*), (E,*three*). Note that SC-1 uses the order of numerals and the spatial order of objects in the weak sense that no reversals occur. Violations of one-to-one correspondence and cardinality result from skipping numerals, rather than using them in their standard order, and from not returning to objects that are skipped when the constrained numeral is encountered.

A second modification of counting that agrees with the performance of some children conforms to the principle of one-to-one correspondence, but modifies the order of numerals. We call this modified procedure SC-

2; part of its flow chart is in Fig. 7. SC-2 uses an additional property, *marked*, to remember whether the special numeral has been used. If the special object is encountered first, the special numeral is retrieved and assigned the property *marked*, but the upper bound of used numerals is not changed. When the special numeral is retrieved as the next member of the list, SC-2 skips it.

For "make B the *four*," SC-2 counts (A,*one*), (B,*four*), (C,*two*), (D,*three*), (E,*five*); for "make D the *two*," it counts (A,*one*), (B,*three*), (C,*four*), (D,*two*), (E,*five*). It could be argued that this procedure counts correctly, although it would return an incorrect result if E were the constrained object, a condition not tested by Gelman and Gallistel.

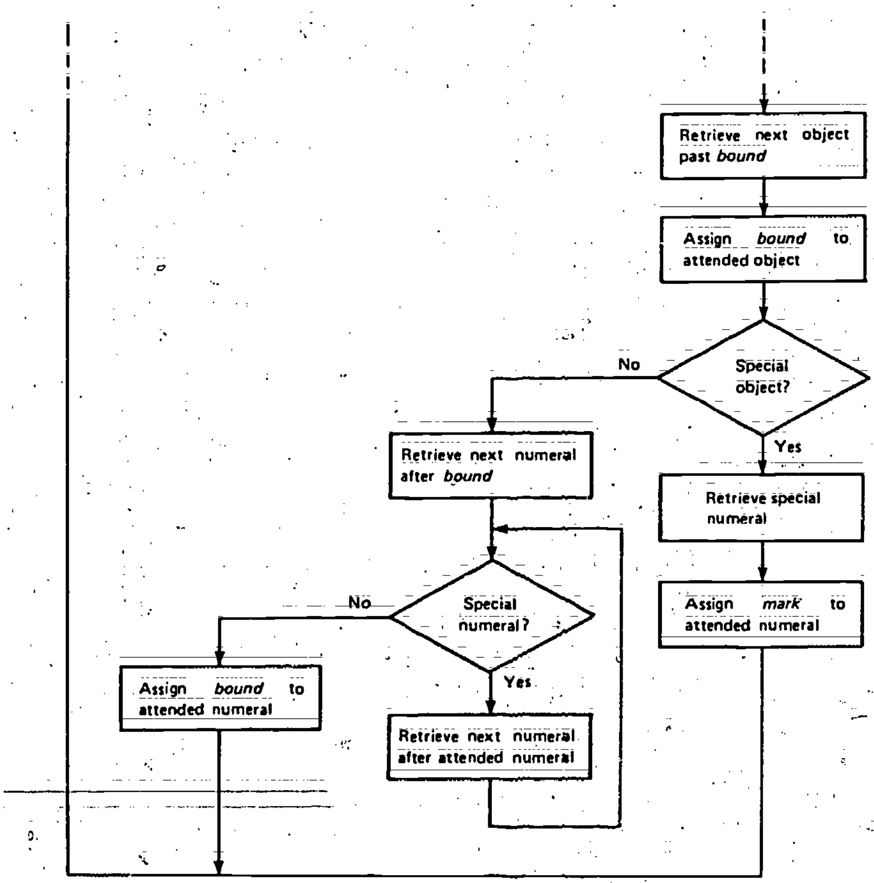

FIG. 7. Modified counting by Model SC-2.

A third modification complies with the added constraint and with the general principles as well. This procedure, which we call SC-3, ignores the position of the constrained object, but otherwise proceeds as in standard counting. Part of the flow chart for this procedure is in Fig. 8. SC-3 is analogous to SC-2, except that priority is given to using the numerals in their standard order. When the special numeral is encountered, SC-3 retrieves the special object and assigns the property *marked* to it. When the special object is next in the spatial sequence it is skipped. Except for the special object, the property *bound* is used to keep the partition of tagged and untagged objects. A feature of SC-3, not shown in Fig. 8, is

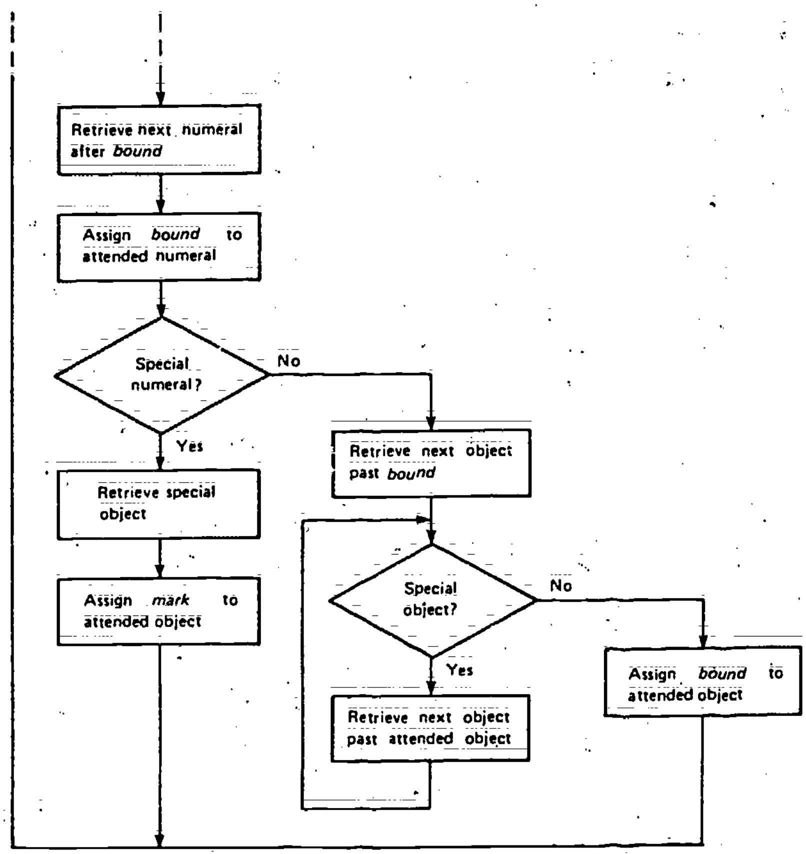

FIG. 8. Modified counting by Model SC-3.

that the test for completion of counting includes determining whether the property *marked* has been applied to the special object.

SC-3's performance conforms to all of the counting principles, including use of the numerals in their standard order. For example, for "make B the *four*," it counts (A,*one*), (C,*two*), (D,*three*), (B,*four*), (E,*five*). For "make D the *two*," SC-3 counts (A,*one*), (D,*two*), (B,*three*), (C,*four*), (E,*five*).

We have derived planning nets for the procedures of modified counting, SC-1, SC-2, and SC-3. The conceptual competence used in these derivations is the same as that used for ordinary counting tasks, described in Section II. Additions are required in procedural and utilizational competence to enable the planner to recognize exceptions and generate changes in its use of setting features as well as procedures for use when the exceptions are encountered.

We briefly describe a derivation of the correct procedure, SC-3. The goal *number*(*L*) is presented to the planner, along with the constraint that a specified object and a specified numeral should be paired. The constraint is interpreted in relation to the schema MATCH's corequisite (recall Schema 10), involving a subset *A* of tagged objects and a subset *B* of used numerals. Let λ be the constrained object and ν be the constrained numeral, so the constraint is stated as, "Make λ be the ν." The interpretation given to the planner is that λ should become a member of *A* and ν should become a member of *B* together—that is, $\lambda \in A \langle = \rangle \nu \in B$.

The first effect of the constraint during planning involves the prerequisite of MATCH, the requirement of a partition of the set of objects. A proposition in utilizational competence suggests using the spatial order of the objects to keep the partition, but the theorem that is needed cannot be proved because of the pairing constraint—the special object λ may have to be tagged out of its spatial order to be paired with ν. The planner's solution is a partition that uses an exception. The partition is based on initial segments of *L*, except for λ, which is in the tagged set when it has the property of being *marked*.

The constraint's other effects occur in planning for the *one-more* goals for increasing the sets of tagged objects and used numerals. Because the retrieved object or numeral may be in the constraint, tests for that are included in the procedure in the way indicated in Fig. 8. Use of the retrieved numeral is given priority because of the ordering prerequisite of the COUNT schema, and the ADD-MARK schema is used to plan inclusion of the constrained object in the tagged subset.

We have derived planning nets for the incorrect procedures SC-1 and SC-2 by assuming that selected components of conceptual competence are not utilized. SC-2 can be derived by neglecting the prerequisite of COUNT that requires use of the ordered set of numerals. Then the planner can decide to partition the objects according to their spatial order and use the special numeral whenever the special object is retrieved. The procedure SC-1 can be derived if the corequisites of KEEP-EQUAL-INCREASE are neglected. The corequisites require that when a member is added either to the tagged objects or the used numerals, exactly one new member is to be added. The corequisites prohibit skipping numerals or objects in the ordered sets, because a skipped object or numeral becomes a member of the initial segment when the upper bound is moved to an item beyond it in the ordered set. When the corequisites are not enforced, numerals or objects can be skipped by moving the upper bound by more than one position, as occurs in the performance of SC-1.

Planning nets for incorrect procedures could also be derived by assuming incomplete conceptual competence, instead of incomplete utilization of conceptual competence. There is unavoidable uncertainty in determining whether a failure of performance is caused by a lack of knowledge or from a failure to use the knowledge appropriately. We consider it more likely that partially correct procedures in the modified counting tasks result from failures of utilization, given the considerable body of evidence that supports attribution of substantial competence to preschool children in the domain of number.

IV. SIMULATION OF PERFORMANCE

As we mentioned in Section I.A., our analysis of the understanding of counting principles began with the development of a model that simulates salient aspects of children's performance in counting tasks. In this section, we describe this model, called SC, and discuss its relation to the analysis of competence in Sections II. and III.

There are some important components of SC that do not appear in the analysis of competence, and vice versa. Components of SC that are not in the analysis of competence can be considered as implementations of general functions that are specified in the competence analysis. (We will note that this distinction depends strongly on the focus of the theoretical analysis.) Components of the competence analysis not present in SC represent structural characteristics of the counting procedure and their relations to the general principles of counting and number.

The task of counting objects is presented to SC in the form of a set of objects, each represented as a label and a pair of spatial co-ordinates. SC has an ordered list of numerals stored in memory, with one of the numerals designated as the *first*, and with adjacent members in the list linked by the relation *next*.

The procedure for counting represented in SC is summarized in the flow chart shown in Fig. 9.[10] Figure 9 may be compared with Fig. 2, the simplified version that was used for our analysis of competence. SC includes two sets of components that were not considered in the competence analysis. One of these involves representation and memory operations involving the goal of counting. The other involves perceptual operations of scanning and forming gestalt groupings of objects in the set to be counted.

The operations involving the counting goal were described in Section I.B. A representation is formed, including a goal stored in memory to find the number of objects in the set; then when counting is completed, the goal is retrieved from memory and an association between the last numeral used and the counted set is stored.

Perceptual operations in SC provide a mechanism for the process of moving through the linear array of objects. The general idea that we used, taken from Beckwith and Restle (1966), is that the partition between tagged and untagged objects is kept by a process of grouping the tagged objects, based on gestalt principles. We implemented a simple version of grouping for the case involving objects in a straight line.

After storing the counting goal, SC identifies a small group of objects at one end of the array. It uses the positions of these objects to determine a direction for scanning, which will be used in moving its attention along the array. Then SC identifies the object in the initial perceptual group that is at the end of the array, and assigns to that object the property of being the upper bound of the subset of tagged objects. SC then retrieves the first member of its stored list of numerals and assigns to it the property of being the upper bound of used numerals.

SC continues to count by repeated execution of a subprocedure: a new object is brought into attention and is made the bound of the tagged subset, and a new numeral is retrieved and made the bound of the used subset. In moving attention to a new object, if there is an object in the current perceptual group that has not been tagged, attention is moved from the current upper bound to the next object along the scanning path. If the current group contains no untagged objects, but there are more objects in the set, the group is extended by including more objects along the scanning path.

Retrieval of the next numeral is simpler in our simulation than retrieval of the next object. We assume that the list of numerals can be retrieved

[10] A detailed description of SC has been given by Riley and Greeno (1980). The sequence of steps in the implemented program differs in some details from the procedure described here. The version presented here can be described more easily; it is computationally equivalent to the version that we programmed, and the discrepancies are irrelevant to the substantive questions that we are addressing.

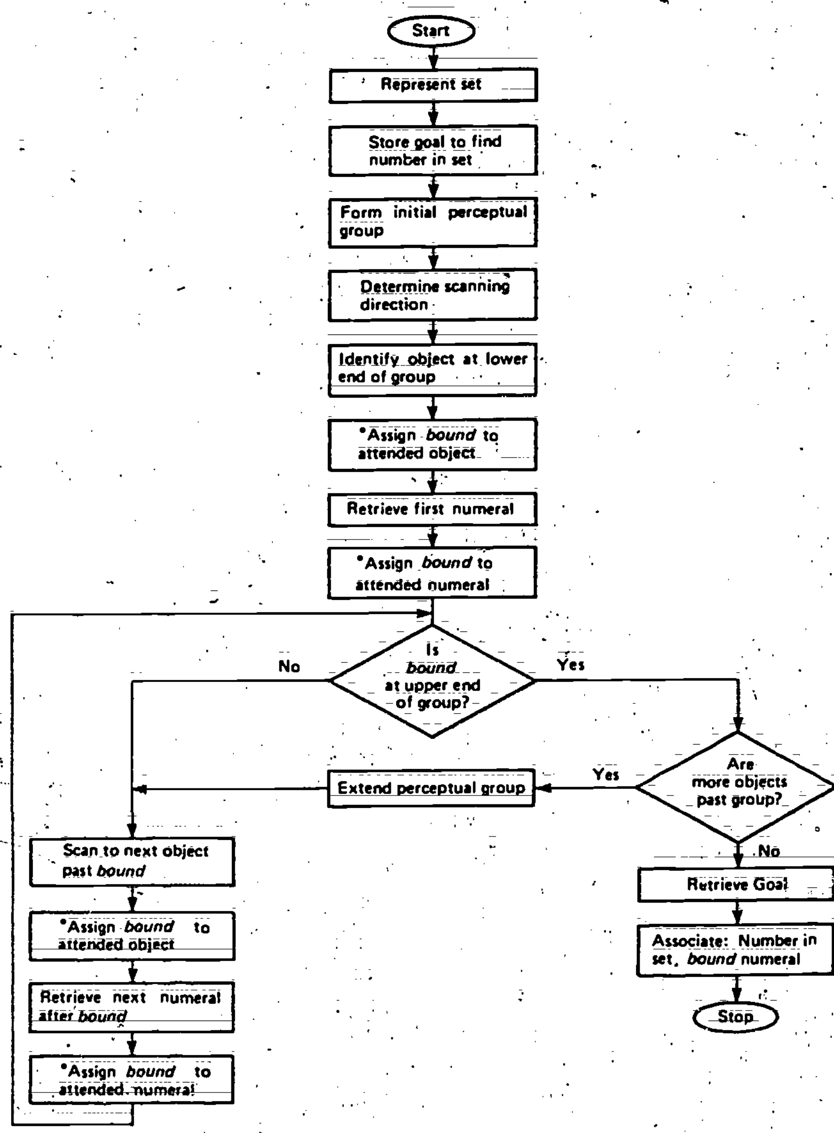

FIG. 9. Procedure for standard counting represented in SC. Prerequisites of components labeled with * may be violated by pairing constraint.

from memory, with the accumulated set of previously used numerals serving as the cue for retrieving the next numeral in the list.[11]

SC determines that counting is finished when it attempts to extend a perceptual group and finds no more objects to include. Then the goal to determine the number of objects is retrieved from memory and is achieved by storing an association between the numeral used last in the process and the symbol that refers to the set of objects.

In addition to simulating standard counting with SC, we also developed simulations of three modifications of the counting procedure for the modified task in which a specified numeral and object are required to occur together. The modified procedures, called SC-1, SC-2, and SC-3, are described in Section III.D. The modifications consist of tests inserted in the procedure when a numeral or an object is retrieved, to determine whether it is the special numeral or object. The procedure includes actions used when the special numeral or object is identified that either retrieve the other item or delay use of this identified item until its mate is retrieved. The three modified procedures differ in these actions. The simplest model, SC-1, just skips to the other special item whenever either the numeral or object is retrieved. SC-2 skips to the special numeral when the object is retrieved, but if the numeral is retrieved SC-2 delays its use until the object is retrieved. SC-3, a correct modification, skips to the special object when the special numeral is retrieved, and if the object is retrieved SC-3 delays tagging it until the special numeral is retrieved.

Many features of our process models are intended to simulate performance that is relevant to the principles of counting. In Section I.B. we discussed the principle of cardinality and described SC's use of a symbol to represent the set of objects to be counted, storage in memory of a goal to find the number of objects, and formation of an association between the last numeral used and the symbol that represents the set. These features provide simulations of children's successful performance of counting, and provide mechanisms to account for two features that can be interpreted as flaws. The explicit representation of a set should be less likely if objects are referred to using a class noun rather than a collection noun (cf. Markman, 1979), and counting of larger sets would make it less

[11] This is a more restricted retrieval process than the one described by Riley and Greeno (1980), where the cue was a single numeral with the property "Current." The main difference is that the model described here cannot retrieve the successor of an arbitrary numeral that is presented. Fuson, Richards, and Briars's (1982) data indicate that young children cannot begin counting at arbitrary points, although they seem to have a few entry points into the counting string. The model we describe here could be considered as an initial knowledge structure, where only the first numeral can be used to enter the list, with additional numerals becoming usable as entry points as a result of further learning.

likely that the goal to associate a numeral with the set would be retained in memory (cf. Gelman & Gallistel, 1978).

SC achieves one-to-one corresondence with a simple device: the procedure that assigns the property *bound* to a new object includes setting a goal to retrieve another numeral, and the scanning procedure ensures that all the objects in the line are included in the count before it is completed. When children make errors regarding one-to-one correspondence, they most frequently skip or double count an object, rather than making errors in the order of numerals. SC provides an interpretation of this finding, in its relatively more complex procedure for spatial scanning and formation of perceptual groups than the simple retrieval of numerals. The more complex procedure regarding the objects should be more prone to errors, in agreement with the empirical result.

Agreement with stable order of numerals is achieved simply by having the numerals stored in memory in an ordered list, with a retrieval process that keeps a memory record of the numeral used most recently to retrieve its successor. Indifference to the order of tagging objects is simulated by use of the spatial arrangement to tag the objects, rather than identities of the objects. Thus, if objects are rearranged and recounted, the order of tagging objects will be changed by SC, as it is by children (Gelman & Gallistel, 1978). Performance in the modified counting task also is relevant to the order of numerals and of objects. SC-3 is a procedure for the modified task in which use of the stable order of numerals is maintained, and the order of tagging objects is modified.

The procedures and data structures implemented in SC provide plausible hypotheses about the cognitive processes and structures of children's performance in counting tasks. The process model does not include hypotheses about the way in which principles of counting are understood, or of the way in which that understanding is related to the performance of counting. This is the contribution of our analysis of competence, presented in Section II, with the derivation of planning nets in Section III that provides an explicit connection between the principles and components of the process model.

The process model includes several components that are not derived in our analysis of competence, involving such processes as storing and retrieving a goal in memory and perceptual grouping and scanning. The memorial and perceptual processes in SC can be interpreted as parts of an analysis of performance, rather than as competence for counting. They provide implementations of processes such as the retrieval of objects in the planning nets. On the other hand, in another analysis dealing with a different set of principles, there would be schemata for processes such as perceptual grouping and scanning, and memorial processes. For example, suppose an analysis were made of competence underlying performance in a visual search task. For such an analysis, schemata would include actions that determine spatial configurations and scanning. These would provide a hypothesis about knowledge of spatial principles underlying performance, comparable to the knowledge of principles of counting that we have analyzed. The implication is that we cannot partition knowledge structures into a set that should be called "competence" and another set of structures called "performance," except in the context of an analysis of competence regarding specific principles and concepts.

V. DISCUSSION

We have presented a proposal for characterizing implicit understanding of principles as a form of cognitive competence. The analysis has three components: conceptual, procedural, and utilizational competence. Conceptual competence, which we have discussed in detail in this paper, is represented as a set of schemata that constitute conceptual structures in the task domain. These are formally equivalent to a set of axioms for the domain, but their formulation as action schemata enables their use as premises for deriving planning nets for procedures. As with axiomatic analyses, the level of abstraction is determined by a choice of where to start. In analyzing counting, we chose a set of schemata that correspond to principles of cardinality, order, and one-to-one correspondence, but a higher or lower level of abstraction could have been used.

The derivations also use procedural competence, in the form of heuristic planning rules, and utilizational competence. The planning heuristics play the role of inference rules in the derivations and correspond to general competence underlying procedural knowledge. Utilizational competence provides connections between features of the setting in which procedures are to be performed and conditions that are required for conceptual knowledge to be applied in that setting.

The analysis has clarified several aspects of the distinction between competence and performance. First, the process of accessing and applying conceptual competence is not simple. A complex interaction among different kinds of knowledge that include general principles about actions and relations between features of a situation and requirements of performance is involved in generating procedures for performance that conform to general principles of a domain.

A second conclusion is that distinctions between competence and performance should be viewed as being relative to the level and focus of a theoretical analysis, rather than reflecting intrinsic differences in structures of knowledge. A distinction between competence and performance can be based on the grain size of a competence analysis, which specifies a structural description of procedures down to a quite arbitrary level of detail. From the point of view of that analysis, additional procedural details included in a model that simulates human behavior can be considered reasonably as procedures needed for performance that implements the specifications in the competence analysis.

Another distinction between competence and performance is obtained if we identify components in a "core" of competence for a family of tasks that are intuitively members of a single class. Then the knowledge structures that vary among procedures within the class can be considered as requirements for performance that conforms with the core principles.

The demarcation between competence and performance would be different in analyses with a different theoretical focus or a different level of detail. A different choice of grain size in the competence analysis would relegate a different set of processes to performance implementations. More significantly, different conceptual principles could be analyzed, for example, spatial principles instead of numerical principles, and this would relocate some of the schemata with respect to the boundary between competence and performance.

A third result involves the way in which formal principles correspond to the schemata that we have developed to represent implicit understanding of the principles. We did not formulate a schema for understanding of order, another schema for one-to-one correspondence, and so on. Instead, it seemed more reasonable to hypothesize schemata that represent different aspects of the various principles, and often include aspects of different principles. If our analysis is accepted, then competence for each of the principles is distributed among several schemata, rather than being located in any single structure. This emphasizes that a child should not be considered as either having or not having competence regarding any of the principles, since it clearly is possible for the child to have developed some aspects of the competence and not others.

We propose our analysis of competence as a hypothesis about principles that children understand implicitly. Our notion of implicit understanding is the same as the idea of tacit knowledge, or cognizing, as used by Chomsky (1980). The idea is also closely similar to Newell's (1982) discussion of the knowledge of a system, which is characterized functionally (that is, by what it does rather than as physical structures and their properties) and includes the implications of components that are specified. along with the components themselves.

Our analysis of competence is generative in that a single set of components of conceptual competence can be used in deriving planning nets for procedures in different task contexts and with different constraints. However, our analysis lacks the important formal property of characterizing the class of procedures that are valid within a domain of possible procedures, analogous to the demarcation between strings that are grammatical and ungrammatical according to the syntactic rules of a language.

The main reason is that we do not have a natural characterization of the domain of possible procedures comparable to the domain of possible strings of the symbols of a language. To develop such a characterization does not seem impossible; it would require specifying some elementary procedural components and general relations of composition. However, such an activity seems premature at present, pending development of at least a few analyses in exemplary specific task domains.

The claim that children have this competence says that they have mental representations of the principles characterized in the analysis, and the principles are used in children's thought and behavior. We view principles included in conceptual competence as constraints on procedural knowledge, in much the same sense that Keil (1981) has proposed. At the same time, it seems unreasonable to claim that the specific forms used in representing the principles and the derivations of procedures correspond in detail to psychological mechanisms. Like Pylyshyn (1973), we consider the model of competence as a formal system that generates sequences of performance (in our case, process models) along with structural descriptions. Newell (1982) remarked that specification of a system at the knowledge level often lacks a definite mechanism for implementing the procedures that use the knowledge. We consider the content of the competence in our analysis a plausible set of hypotheses about children's tacit knowledge, but the way in which the three components of competence are used in deriving planning nets should be interpreted as a formal relation, not necessarily corresponding to cognitive mechanisms.[12]

The formulation of competence that we have developed is generally similar to the one Piaget gave, but also differs in important ways. Like Piaget, we conclude that an understanding of number reflects a cognitive structure that coordinates understanding of sets and understanding of the relation of order. However, there are three important differences. First, we make more generous attributions to children who are able to count sets of objects. Piaget concluded that skill in counting, unlike skill on conservation tasks, does not warrant attribution of understanding of number; but we, following Gelman and Gallistel (1978) conclude that it does. Second, the concept of one-to-one correspondence has a less fundamental role in our analysis than it does in Piaget's formulation and the standard axiomatic analyses. Third, while implicit understanding of set-

[12] The distinction we have in mind is analogous to one between knowing a global property of a physical system and knowing the mechanism that causes that property to be true. For example, in an electrical circuit Ohm's and Kirchhoff's laws describe relations of voltage, resistance, and current without specifying the mechanism of electricity. Consequences of having certain amounts of voltage and resistance can be computed formally—by arithmetic calculation—but these formal computations do not correspond in any simple way to the flow of electric charge through circuits.

theoretic concepts is significant in our formulation of competence, we do not link the development of number concepts to the development of an explicit and general concept of class. Instead, we link it to development of an understanding of counting.

Regarding attribution of understanding, Piaget and subsequent investigators have withheld attributing understanding of cardinality until children succeed on conservation tasks that require explicit use of one-to-one correspondence to define equivalence. Likewise, children are not granted an understanding of ordinal properties of natural numbers unless they can seriate, which requires construction of an ordering in space that corresponds to another ordering that must be perceived or measured, such as length or weight. And understanding of number is not granted until children perform correctly and efficiently in tasks where cardinality and order are jointly present, such as determining the location of a stack of a given size in a staircase display using the ordinal positions of the stacks rather than counting their sizes. This Piagetian conclusion led to the view that very young children's counting is a kind of rote procedure, not based on understanding of what counting or number is about. The view of Gelman and Gallistel (1978), which we developed further in this article, is that young children do understand some important numerical concepts. In particular, the competence underlying children's counting includes implicit understanding of the principles of cardinality, order, and one-to-one correspondence, along with principles involving application of these concepts and significant set-theoretic components of the principles. The competence that we hypothesize provides significant principled understanding of counting, but our assumptions do not imply that children know how to apply the principles in all tasks or situations in which the principles are needed for correct reasoning and problem solving. We hypothesize that sucessful peformance in the more complex tasks used by Piaget requires further conceptual development, in which understanding of quantitative concepts becomes more explicit, flexible, and robust.

Regarding one-to-one correspondence, Piaget followed the logicists' formal analysis of number, in which one-to-one correspondence is used in defining the concept of number. In our hypothesis of competence for counting, one-to-one correspondence is a property that results from understanding of an aspect of cardinality; we postulate knowledge that two equal sets will remain equal if exactly one new member is added to each of them. On this view, cognizance of one-to-one correspondence as an explicit property is not required for a principled understanding of counting. For Piaget, one-to-one correspondence is the psychologically primitive device for determining whether two sets are equal or not. The formulation that we give is consistent with a conclusion of Gelman and

Gallistel (1978), as well as evidence reported by Russac (1978), that a concept of equality of sets that rests on their being equal in number precedes a concept of set equality based on one-to-one correspondence. This implies that young children could determine that two sets are equivalent because they both yield the same enumeration sequence—and thus the same cardinal number—and still fail tests that force a reliance on the use of one-to-one correspondence.

In our discussion here, we have emphasized that significant competence should be attributed to young children in the domain of counting and number. The specific characterization we have given seems appropriate for children approximately 5 years old in industrialized societies. We do not imply that younger children have all of the competence that we have characterized, or that children's competence for number does not become more fully developed as they grow older and learn mathematics in school.

A significant theoretical problem is the characterization of changes in children's competence that correspond to the stronger understanding that they achieve through learning and development. Important aspects of this growth are reflected in performance on tasks used by Piaget (1941/1952) and analyzed by Klahr and Wallace (1976) in terms of process models. Significant information is also provided by performance on more demanding forms of counting (Fuson & Hall, 1983; Fuson et al., 1982; Steffe & Thompson, 1981; von Glasersfeld, 1981) and knowledge of number relations involved in place value and elementary arithmetic (Resnick, 1983; Siegler & Robinson, 1982).

It is widely agreed that cognitive growth includes increasing accessibility and differentiation of conceptual structures. That is, conceptual capabilities can be used in a wider range of task settings, and a richer set of properties and relationships are included in the structure. In terms of our analysis, such changes could take the form of more fully developed schemata of conceptual competence, or they could involve increased capabilities for using conceptual competence, corresponding to growth of procedural or utilizational competence.

Rozin (1976) has proposed that important aspects of cognitive growth can be understood as a process in which implicit knowledge becomes more explicit. One way in which knowledge becomes more explicit was characterized by Piaget as reflective abstraction, in which cognitive operations become objects of thought. The idea that what is implicit in younger children becomes more explicit with development provides interesting suggestions regarding the relationship between early competence and later understanding of quantitative concepts involving both one-to-one correspondence and iterative ordinality. We believe that cardinal and ordinal concepts are present in a unified form in the minds of very young children. Successful performance on tasks such as class inclusion and seriation may require more explicit understanding of these concepts, in more fully elaborated forms or with more skill in applying the concepts in a broad range of tasks. Explication of concepts such as one-to-one correspondence and iterative ordination seem to require a process of reflective abstraction about *processes*, which Karmiloff-Smith (1979) has discussed.

Regarding one-to-one correspondence, we note that our account of the counting principles requires children to coordinate their tagging and partitioning efforts. The consequence is that they establish a one-to-one correspondence between numerals and objects implicitly. However, they may not—indeed, probably do not—have explicit knowledge of doing so as they assign one and only one numeral to each object in the display. That is, they do not have explicit knowledge of the principle of one-to-one correspondence and, thus, that a one-to-one correspondence between sets implies that they have the same number, no matter what that number is. A reasonable conjecture is that number conservation tasks assess explicit knowledge of the principle of one-to-one correspondence. When understanding of one-to-one correspondence has been in an appropriately explicit form, a child can be freed from relying on enumeration when judging equivalence of sets (Gelman, 1982).

In summary, we conclude that the nature of young children's understanding reflects competence that supports the understanding of counting as well as later development such as explicit understanding of the role of one-to-one correspondence in definitions of equivalence. Although we disagree with Piaget as to when the concepts of cardinal and ordinal number emerge, we agree that they do not follow separate lines of development, but rather represent two aspects of a single conceptual system. Tasks can be designed that emphasize one aspect of number, but inferences made from performance on such tasks should be made with caution, taking into account the way in which success requires explicit forms of understanding and knowledge of a concept's applicability.

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