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ABSTRACT

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Solving Algebra Word Problems

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Abstract

Algebra word problems were analyzed in terms of the information integration tasks that are required to solve the problems. These tasks were classified into three levels: value assignment, value derivation, and equation construction. Novices (35 first year algebra students) and experts (13 analytic geometry students) were compared on the proportion of tasks completed at each level in their attempts to solve six word problems. As predicted, the novices showed greatest weakness on the tasks from the second and third levels, which required an appreciation of the structure of the problems. Consistent with this finding, novices performed at chance levels on a task that required them to identify which two problems of three were most similar. Experts performed very well on this task. Instruction focussed on the structure of the problems was successful in improving performance of a group of novices.

Solving Algebra Word Problems

If you haven't repressed the memory, you may recall the sense of frustration and feeling of incompetence that accompanied your first encounter with algebra word problems. Although some students quickly overcome these feelings as they gain a degree of mastery over word problems, many other students are left with the impression that mathematics is beyond them. As a high school algebra teacher some years ago, I had the opportunity to encounter the frustrations with word problems from another point of view. I was struck by the difficulty many students had with translating word problems into equations. Even good students who were expert at solving algebraic equations were often baffled when the same problems were cloaked in a verbal cover story.

The extent of the problem has been well documented. In 1980, a large national survey to assess educational progress (see Carpenter, Corbitt, Kepner, & Lindquist, 1980) posed the following problem to 17 year olds:

Present Figure 1 here (Lemonade problem)

This problem is not especially difficult. One bottle will fill 7 cups. At 20 cents per cup, one bottle will yield $1.40 for a profit of 45 cents on each bottle. Virtually all 17 year old students have had at least one course in algebra where they were taught how to solve problems such as this, and many of the students have taken several mathematics courses. Yet the solution rate for these students was (care to guess?)

only 29%.

The persistent difficulty students have with word problems has attracted the interest of many educators and researchers. An early line of research focussed on the linguistic structure of the word problems (cf. Loftus & Suppes, 1972). While much has been learned about how the phrasing of a question affects problem difficulty, it can be argued that this approach is not very helpful to a teacher. Although comprehension may be facilitated by rewriting problems, teachers naturally are more concerned with how they might help their students develop their problem solving skills.

An alternate approach which emphasizes the role of the student's prior knowledge assumes processing based on schemata. The premise is that solvers are able to retrieve information about the formal structure of a problem upon recognizing the problem's relationship to a familiar prototype. Jill Larkin (1980; Larkin, McDermott, Simon, & Simon, 1980) has applied this approach to the study of problem solving in physics. She concluded that expert problem solvers make use of larger structural units than do novice problem solvers.

Richard Mayer (1981a) made an important and rather heroic contribution to the study of algebra word problems when he compiled and categorized a set of about 1200 problems from major algebra texts used in California public schools. On the basis of underlying source formulas, he identified 25 families, such as time-rate and unit-cost problems. Families were divided into categories which share variables, derivations, and methods of formula construction. For example, the time-rate family was divided into 13 categories, such as motion, current, and work problems. Each category was further divided into templates, defined by the propositional structure of the problems. For example, motion problems have at least 12 templates, including vehicles converging, one overtaking another, and one vehicle making a round trip. Problems within a template differ only in the values that are used.

Mayer (1981b) found that the relevant information in nearly all problems could be described using four types of propositions:

1) assignments (e.g., A cup holds 8 ounces.)

2) relations (e.g., A man is twice as old as his son.)

3) questions (e.g., How much money will the school make?)

4) relevant facts (e.g., The gas tank is full.)

In a series of studies, Mayer (1981b) found relational propositions were harder for students to deal with than assignment propositions. This applied to memory as well as to accuracy in using the information.

The Current Study

The data I will report today were drawn from Jeffrey Wilde's dissertation which I chaired last spring. One goal of Jeff's dissertation was to develop an analysis of the structure of algebra word problems in terms of the information processing tasks required to solve the problems. To do this, he analyzed 50 common word problems selected from Mayer's (1981a) collection. Jeff generated a taxonomy of information integration tasks, which is shown in Figure 2 with some examples.

Present Figure 2 here (Our taxonomy)

The taxonomy consists of nine tasks organized into three levels of information integration with three types of tasks at each level. The levels are value assignment, value derivation, and equation construction. For example, the first problem illustrates equivalence assignment of values, transformation of these values to derive new values, and construction of an equation by applying a function rule. The second problem illustrates assignment of an unknown, construction of a representation of an unknown using the values assigned at level 1, and then application of a source formula to generate an equation. The third problem shows a value assignment based on a relationship with another value, application of a source formula to derive new values, and combination of these values into a final equation.

There is a hierarchical relationship between the three levels of information integration. The value assignments from the first level are often operated upon in the second level to derive new values, which are then used in the third level for the construction of the equations. However, problem solving can begin at any of the three levels. Activities at each level place constraints on activities to be completed at each of the other levels. For example, if one can determine the form of the final equation, the range of possibly appropriate value assignments and derivations may be reduced.

The present research was designed to compare novice and expert problem solvers in terms of their facility with information integration tasks at each level, and on their awareness and use of problem structure. We expected to find novice problem solvers to have more difficulty with aspects of problem solving that require an appreciation for the structure of the problem. This would be reflected in greater difficulty with the second and third levels of information integration than with the first level, which is assignment of values and unknowns. We also examined the effects of instruction on problem structure for novice problem solvers. We were hopeful that the training would improve performance on the higher levels of information integration.

Method

Three groups of High School students were recruited for this study. Volunteers from first semester algebra classes were split into two groups of novice problem solvers, an instruction group and a control group. A third group of experienced problem solvers was comprised of volunteers from analytic geometry classes. Each student was paid $2.00 for participating. Figure 3 shows the research design and tasks performed by each group.

Present Figure 3 here (Research design)

The initial task for all groups was a test set of six word problems, followed by a vocabulary test. About a month later half of the novices were given special instruction on word problems while the other half served as a control group and were given a filler task. A posttest followed for both groups. A special test of ability to identify the structure of word problems was given to the control group of novices at the end of the second session and to the experts at the end of their first and only session.

The six word problems in the initial test were two problems each from Mayer's (1981a) motion, age, and rectangle families. All students had been exposed to these three categories of problems in their classrooms. Students were tested individually. The problems were presented in a booklet, with each problem written on the top of a separate page, leaving space for calculations below. Students were asked to write down each step, and to report their thoughts as they worked. The experimenter recorded all comments. Students who were unable to get started on a problem were prompted with the hint that they should determine what the problem's unknown is. If this failed, they were told to go on to the next problem. No problem could be returned to once the page was turned.

Results

On the six problems, the combined novice groups solved only 9% of the problems compared to 85% for the experts. The problem protocols were analyzed for each group to determine the proportion of tasks completed at each level of information integration. The results of this analysis are shown in Table 1.

Present Table 1 here (Overall Group by Level)

Here, and in other analyses where the dependent variable was a proportion, we used an arcsine transformation prior to conducting an analysis of variance. Both main effects and the interaction were highly significant in this table, all in the expected direction. Experts outperformed the novices, and performance for both groups was progressively poorer as the integration tasks required more structure specific integrations. Value assignments at Level 1 were easiest while formula constructions at Level 3 were the most difficult. This was dramatically true for the novices, who were moderately good at setting up givens but very poor at applying procedures which depend on the problem structure.

<u>Problem solving and verbal abilities.</u> One might expect verbal comprehension to be a good predictor of algebra word problem solving success. To evaluate this notion, we gave all students the first part of Vocabulary Test II from the ETS Kit of Factor Referenced Cognitive Tests (Ekstrom, French, Harman, & Derman, 1976). The correlation between the proportion of information integration tasks completed and verbal comprehension for the entire sample was a highly significant .75. This high correlation was the result of large differences between the groups on both measures. The average score on the 18 item vocabulary test was 14.3 for the experts, and only 8.2 for the novices. The correlation between problem solving and vocabulary for the experts alone was .17 and for the novices it was .01, both nonsignificant. A high level of verbal ability may be required to become established in the high math performance group, but verbal ability does not account for the variability of math performance found within a group.

<u>Comparison of novices and experts on information integration tasks.</u> We next examined performance of the experts and novices on specific tasks. The tasks at Level 1 are assignment of unknowns, relational assignments, and equivalence assignments. Table 2 shows the mean proportion of success for each Level 1 integration.

Present Table 2 here (Group by Level 1)

Both main effects and the interaction were all highly significant. For the novices, performance on unknown assignment and relational assignment did not differ significantly, but both were easier than equivalence assignment. This pattern is somewhat different from Mayer's (1981b) finding that equivalence propositions were easier than relational propositions for college students to remember.

At Level 2 were the value derivation tasks using transformations, construction, and source formulas. Transformations were completely specified by the problem, in that the initial value, a transforming value, and the transforming operation were all stated explicitly. Constructions and source formulas, however, involve combining information based on ideas about the problem structure that were not stated explicitly in the problem. This led to the prediction that transformations would be easier than constructions and source formulas for novices. Table 3 shows the proportion of Level 2 integrations completed for each group.

Present Table 3 here (Group by Level 2)

For the novices, the construction tasks were significantly more difficult than either the source formula or construction tasks, which did not differ significantly from each other. No differences were reliable for the experts.

One might suspect that the poor performance of the novices on using source formulas might be because they do not know the formulas. However, when they were asked to recall the formulas at the end of the experiment, 83% of the novices correctly recalled the area formula and 71% recalled the rate formula. It is the application of the formulas that is not well understood.

Performance of the novices on construction of variables was abysmal. Failure to construct variables that are needed in the final equation is consistent with the hypothesis that novices do not have a good understanding of the structure of the word problems.

The Level 3 integration tasks of equation construction involved the use of a function rule, source formula, or combination of variables. In the six problems used here no source formulas were needed at Level 3. Table 4 shows the proportion of Level 3 tasks completed by each group.

Present Table 4 here (Group by Level 3)

Both groups were more likely to obtain the final equation when a function rule was required than when a combination of variables was needed. These results and the earlier tables should be interpreted with some caution since there were only six problems in the test set. In our problems, the combination of variables was needed only for the two motion problems. Generalization to a wider range of problems has not been established.

Performance on the structure task. Novices did not perform well on those information integration tasks that depend on a knowledge of the problem structure. To test more directly the students' ability to detect and compare the structure of problems, we administered a special structural task.

The structure task consisted of five triads of problems, where each triad was constructed of three problems from the same category, with two from one template and the third from a different template. Students were asked to determine, for each triad, which two problems were most alike. An example of a problem triad is shown in Figure 4. The first two problems here are isomorphs which differ only in values of the variables. The third problem presents the second proposition is a form different from the first two problems.

Present Figure 4 here (Structure Task)

Chance performance on the structure task was 33% correct. The novices performed right at chance, 33% correct, while the experts were correct on 88% of the triads. This is convincing evidence that the novices had little appreciation for the structure of the problems, in contrast to the experts who were able to identify the structure quite consistently.

<u>Analysis of problem protocols</u>. A third source of data was the problem protocols. All students were asked to write down each step of their solution attempt, and to "think out loud" as they proceeded. These protocols showed striking differences between the novices and the experts in terms of their use of the problem structure.

Present Table 5 here (Protocol Analysis)

Strategies that led to solution are listed in Table 5 as Type 1. Just over half of the problems solved by experts showed a "working-down" strategy that started with the Level 3 integration, allowing the solver a relatively clear idea of what the goal path would be. The novices never started with the Level 3 integration. On the 18 problems solved by novices, 17 showed a working-up strategy where all Level 1 integrations were listed first, then the Level 2 integrations, and finally the Level 3 equation emerged.

The second strategy type was an incorrect application of a formula or procedure from a problem thought to be similar. Novices were likely to show formulas, while experts tended to show diagrams. The most common strategy for the novices was Type 3. On 51% of the problems, the novices produced only a simple listing of some or all of the Level 1 value assignments, with little else.

An examination of the 18 successful protocols from novices showed that 13 protocols included a complete labeled diagram. Of the 122 unsuccessful protocols from novices on the same problems, only 3 included such a diagram. It seems likely to us that the diagrams played a role in structuring the problem.

Consistent with other information, the protocol data suggest that the novices generally made little use of the structure of the problems in determining their approach to the problems. Experts, on the other hand, made extensive use of their knowledge of the problem structure to find their path to the solution.

These data suggest that if we wish to train novices to behave more like experts, it may help to teach a thorough understanding of the structure of word problems. Specifically, training on the three levels of information integration and diagram construction should be helpful. We designed a short training program to test this notion.

Effects of Instruction. The novice problem solvers were paired on the basis of their performance on the six word problems, and then were randomly split into two groups, instruction (n=16) and control (n=17). Four weeks after the initial testing, the instruction group received about 30 minutes of individual training.

Students were first given the Hikers problem (see Figure 5), and asked to list the variables, defined as things named in the problem that have a numerical value.

Present Figure 5 here (Hikers Problem)

Examples were given, and the students were helped to produce a list of the rates and times for the two hikers and the initial distance between them. Students were next asked to determine the values for each variable on the list. Particular note was made of the fact that "x" can represent an unknown value and the value of one variable may be defined in terms of another. Next, the students were asked to find the equality and find the values that must be derived to complete the equality. Figure 6 was provided to aid students.

Present Figure 6 here (Training Aid)

The same procedure was repeated for the second motion problem, Cyclists. The final step was to compare the two problems using the figure. The similarity of variables, value derivation, and equation construction was pointed out.

The control group of novices were given the two motion problems and asked to set them up. A posttest for both groups consisted of four problems, three of which were isomorphs of the training problems sharing category and template features. The fourth was a generalization problem which was a motion problem from a different category. The isomorphs involved combining two subdistances to equal a known total, while the generalization problem involved comparing two subdistances to find an unknown total. The mean proportion of problems solved for each group is shown in Table 6.

Present Table 6 here (Instruction vs. Control)

The instruction group outperformed the control group on both the isomorphs and on the generalization problem. An examination of performance at each of the three levels of information integration showed that the Instruction group was better at all three levels on the isomorphs, and better on the first level on the generalization problem. The differences on the second and third levels of information integration were not significant for the generalization problem, although the they were in the expected direction.

Summary

Overall, the clearest lesson to be drawn from our study is that an appreciation of problem structure is a crucial part of expertise in problem solving. Experts are able quickly to identify the form of the equation to be solved and they use this information to guide them on the path to solution. Novices are much more likely to stop after they have generated a list of value assignments, unable to see relationships inherent in the structure of the problem.

Some implications for instruction can also be drawn from the study. Instruction on word problems should give attention to helping students build schemata for the general structure of word problems and the specific structures found within problem categories. Our small training study suggests that detailed side-by-side comparisons of the structure of problems from the same category may be a useful approach. Our data also indicate that diagrams can play an important role in helping students to organize information about a problem and to generate a structural representation of the problem. Perhaps students should be trained to draw figures, at least for some categories of problems.

We are encouraged by the results of our study, and are hopeful that teachers and designers of instructional materials will be able to put information like this to good use.

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Author Notes

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Lemonade costs $.95 for one 56 ounce bottle.

At the school fair Bob sold cups holding 8 ounces for $.20 each.

How much money did the school make on each bottle?

Figure 1: Lemonade Problem

Level of Information Integration

Example Problems	Level 1: Value Assignment	Level 2: Value Derivation	Level 3: Equation Construction
	Equivalence Assignment	Transformation	Function Rule
A man is now 40 years old and his son is 14. How many years will it be until the man is twice as old as his son?	40 = man's age today 14 = son's age today	40+x = man's age x years from now 14+x = son's age x years from now	(40+x) = 2(14+x)
	Unknown Assignment	Construction	Source Formula
A framed mirror is 45 by 55 cm. 1911 square cm of the mirror shows. How wide is the frame?	x = width of the frame	L = 55 - 2x W = 45 - 2x	A = (L)(W) A = (55-2x)(45-2x)
	Relation Assignment	Source Formula	Combination
Two hikers start at the same time from towns 36 miles apart, and meet in 3 hours. One hiker walks twice as fast as the other. What is the rate of each hiker?	(x = rate one) 2x = rate two	$(R)(T) = D$ $(x)(3) = D_1$ $(2x)(3) = D_2$	$D = D_1 + D_2$ $36 = 3x + 6x$

Figure 2: Taxonomy of Information Integration Tasks

Group	Session 1	Session 2
Novice Instruction (n=18)	Problem set Verbal test	Instruction Posttest
Novice Control (n=17)	Problem set Verbal test	Filler activity Posttest Structure test
Expert (n=13)	Problem set Verbal test Structure test	

Figure 3. Research design and tasks performed in each session

Table 1

Mean Proportion of Information Integration Tasks Completed

Group	n	Level of Information Integration 1	2	3
Novice	35	.73	.27	.09
Expert	13	1.00	.92	.85

Table 2

Mean Proportion of Level 1 Tasks Completed: Value Assignments

Group	n	Level 1 Value Assignment Tasks: Unknown	Relation	Equivalence
Novice	35	.81	.75	.64
Expert	13	1.00	1.00	.99

Table 3

Mean Proportion of Level 2 Tasks Completed: Value Derivation

Group	n	Level 2 Value Derivation Tasks: Transformation	Source Formula	Construction
Novice	35	.45	.20	.03
Expert	13	.98	.89	.83

Table 4

Mean Proportion of Level 3 Tasks Completed: Formula Construction

Group	n	Level 3 Formula Construction Tasks: Function	Combination
Novice	35	.19	.04
Expert	13	.92	.81

Structure Task

Which two of the following three problems are most alike?

a) Problem 1 and Problem 2. ( )

b) Problem 1 and Problem 3. ( )

c) Problem 2 and Problem 3. ( )

1. Dana is five times as old as his dog, Texas. In nine years Dana will be twice as old as Texas. What are their ages now?

2. Roger is four times as old as his sister. In six years he will be twice as old as she. How old are they now?

3. Pam is twice as old as her brother. In five years their ages will total 22 years. How old are they now?

Figure 4: Sample Problem from the Struture Task

Table 5

Strategies Used by Novice and Expert Problem Solvers

Strategy Type	Number of Problems			
	Novices		Experts	
1. Successful Strategies	18	(9%)	61	(78%)
a) Work up	17		25	
b) Work from the middle	0		4	
c) Work down	0		31	
d) Ideosyncratic arithmetic model	1		1	
2. Memory for a Similar Problem	20	(10%)	11	(14%)
a) Generate a Formula Table	20		1	
b) Diagram a Familiar Procedure	0		10	
3. List Variables	108	(51%)	4	(5%)
4. Oversimplify Structure	36	(17%)	2	(3%)
a) Simplify Formula	4		0	
b) Simplify Diagram	32		2	
5. Structure Insensitive	28	(13%)	0	(0%)
a) Incorrect Direct Translation	10		0	
b) Incorrect Arithmetic Relations	14		0	
c) No Apparent Strategy	4		0	
Total number of problems	210		78	

<u>Note</u>. This table is based on six problems given to 35 novices and 13 experts. The numbers indicate a count of instances where each strategy or type of strategy was used.

Hikers

Family: Amount-Per-Time

Category: Motion 1

Template: A

Two hikers start at the same time from towns 36 miles apart. The hikers move towards each other and meet in 3 hours. One hiker is going twice as fast as the other. What is the rate of each hiker?

Level	Integration	Variable	Value
1	EQUIVALENCE	Total dist.	= 36 miles
	UNKNOWN	Rate h_1	= x mph
	RELATION	Rate h_2	= 2x mph
	EQUIVALENCE	Time h_1	= 3 hours
	EQUIVALENCE	Time h_2	= 3 ~~hours~~
2	SOURCE FORMULA	Distance h_1	$= R_{h1} * T_{h1} = 3x$
	SOURCE FORMULA	Distance h_2	$= R_{h2} * T_{h2} = 6x$
3	COMBINATION	Total Distance	= Dist. h_1 + Dist. h_2

Equation: $36 = 3x + 6x$

Figure 5: Hikers Problem

	R	* T	= D
Trip made by Hiker 1			
Trip made by Hiker 2			

	Distance covered by H_1	Distance covered by H_2	Total distance covered
Relation between trips			

Figure 6: Diagram used to demonstrate the structure of the rate problems used for instruction

Table 6

Proportion of Post-Test Problems Solved by Instruction and Control

		Isomorphs			Generalization
Group	n	Trains 1	Planes	Trucks	Trains 2
Instruction	16	.91	.83	.88	.79
Control	17	.55	.36	.49	.40