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AUTHOR Gerace, William J.; Mestre, Jose P.
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ABSTRACT

This document reports in detail an investigation of the cognitive processes and learning difficulties of ninth-grade students in Algebra I. A clinical interview approach was used with 14 students. A group of six Hispanic students and a group of five Anglo students were enrolled in the same algebra class; a group of three Hispanic students who had begun algebra one-half year earlier was also included. Probed were students' appreciation of algebra as an abstract logical system, command of the formal operations of algebra, ability to use algebra, and ability to solve problems. Performance in English vs. Spanish was compared, and students' teaching styles were analyzed as they taught peers. It was found that students preferred not to use algebraic techniques in solving problems, were poor at verbalizing definitions and procedures and at translating problem statements into equations, did not use their textbooks very much except as a place to find assigned problems, and treated algebra as a rule-based discipline, not as a concept-based one. Detailed descriptions of student performance are given, and findings are discussed in relation to textbooks, pedagogy, linguistics, and directions for future research. Nine recommendations for improved pedagogy and learning are also presented. Questions asked in eight interview sessions are appended, as is selected background information on student participants. (MNS)

BILINGUAL RESEARCH PROJECT

Department of Physics & Astronomy
Hasbrouck Laboratory
University of Massachusetts
Amherst, Massachusetts 01003

A STUDY OF THE COGNITIVE DEVELOPMENT OF HISPANIC ADOLESCENTS LEARNING ALGEBRA USING CLINICAL INTERVIEW TECHNIQUES

Final Report

William J. Gerace and Jose P. Mestre
Department of Physics and Astronomy
University of Massachusetts
Amherst, Massachusetts 01003

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I. INTRODUCTION

There has been an increased concern in recent years over the poor mathematical preparation of the general populace. Of particular concern has been the decline of mathematical preparedness of entering college students. It is a well-documented fact that standard indicators of mathematical aptitude, such as the Scholastic Aptitute Test (Educational Testing Service, 1948-1982) have shown a monotonic decline over the last decade (Harris, 1976; Harneschfeger and Wiley, 1976; Braswell, 1978). Several explanations, both academic and sociological, have been offered for this decline (Edson, 1976). Perhaps the most cogent academic explanation for the lower mathematical preparation of college students is the decline in the overall enrollment in high school second year algebra (National Science Foundation, 1980), coupled with the attenuation in high school mathematics enrollments as the level of the mathematics course becomes more advanced (National Assessment of Educational Progress, 1979). This situation has given rise to an anomaly which is rapidly gaining acceptance -- college level remedial mathematics programs (Whitesitt, 1982).

The purpose of the current research was to investigate the cognitive processes employed by ninth grade students enrolled in Algebra I. The choice of algebra for the subject of this study was not arbitrary. Algebra is traditionally the first mathematics course where students encounter concepts which are much more subtle and abstract than the simpler arithmetic manipulations of their prior math courses. Further, because algebra is the foundation upon which the majority of advanced mathematics is based, any insights gained from researching this area may eventually make the learning of algebra a more efficient process. Some national mathematical organizations have even gone on record to emphasize the importance of algebra for college level work over other mathematics courses taught in high school (The Mathematical Association of America and the National Council of Teachers of Mathematics, 1978).

A clinical interview approach was employed for data collection.

The use of clinical interviews to study problem solving has met with some success. Unlike any paper-and-pencil assessment, the clinical interview approach is unique in permitting immediate feedback and interaction between the interviewer and subject. This allows the interviewer to probe for the cause of difficulties and misconceptions as they arise during the interview-- something which is not possible with non-interactive data collection techniques.

The focus of this study was on the learning difficulties experienced by Hispanic students. There are several reasons for this focus. As will become evident in a section later in this chapter which reviews past research studies, the number of investigations that have focused on the learning and performance of Hispanics in mathematics is vanishingly small. Further, several statistics published in The Condition of Education for Hispanic Americans (National Center for Education Statistics, 1980) reveal that the educational, as well as socioeconomic situation of Hispanics in the mainland is bleak. For example, even though in 1978, Hispanics in the mainland comprised 5.6% of the total population, the percentages of Hispanics enrolled in 4-year college programs, graduate programs, and professional programs were 2.8, 2.0, and 2.1, respectively, of the total student enrollment. In terms of college enrollment in fields where a strong mathematical preparation is indispensable, such as engineering and physical science, Hispanics are substantially underrepresented; the percentage of white-non-Hispanics enrolled in these fields out of the total white-non-Hispanic undergraduate enrollment is more than twice that of the percentage of Hispanics enrolled in these fields out of the total Hispanic undergraduate enrollment. There is a need for more research efforts which may help in the development of strategies for increasing the number of Hispanics wishing to pursue math/science related professions.

Finally, we would like to comment that the results of this study have a wider applicability than is implied by the title. Even though this study focused on Hispanic students, the majority of the findings apply not only to the Hispanic participants, but also to a control group of Anglo students who participated in the study.

A. Study Site*

The study was conducted in a small city in western Massachusetts. There are approximately 45,000 inhabitants in the city and 6,200 of these are of Hispanic origin (figures taken from 1980 census). The school system has a total of approximately 8,000 students at all levels with 2,300 of them Hispanic. The school at which the study was conducted was a small junior high school (grades 7 through 9) with 572 total students, 128 of whom were Hispanic. The study was conducted during the 1981-1982 academic year.

It should be pointed out that the Hispanic composition of this community has undergone a tremendous increase since the 1970 census. Figures from the 1970 census place the number of persons of Hispanic origin in this city at 1,870. This 230% increase in the last decade was largely due to an influx of Puerto Rican seasonal agricultural workers who opted to make this community their permanent place of residence.

The teacher of the Algebra I class from which the subjects came was an integral part of the study. The classroom style used in the Algebra I course was traditional, with the teacher presenting new material during the first half of each period, and the students working out examples and asking questions during the second half.

B. Review of Relevant Research

This section has been divided into four subsections covering the following topics: 1) Effect of cultural and socioeconomic factors upon performance, 2) Cognitive style, Piagetial level, and

*We will be deliberately vague in identifying the study site by name. The agreement between the principal investigators and the school system in question stipulated that the school system and all persons participating in the study from the school system would remain anonymous.

mathematics, 3) Research in mathematics education, and 4) Research in mathematics education with bilingual subjects.

<u>Effect of Cultural and Socioeconomic Factors Upon Performance</u>

When engaged in research with differing ethnic or cultural groups, special care must be taken to ensure that findings are interpreted appropriately. Often, the criteria used by the experimenter to judge performance when dealing with diverse cultural groups are not the same as the criteria used by the subjects. Careful analyses of non-standard responses in several cross-cultural studies have, in many instances, revealed that the logic used in performing certain cognitive tasks is entirely in keeping with the "logic" of the culture, and that the supposed non-standard responses are indicative of intelligent behavior in that culture.

A well-known example of this difference in standards is a study conducted with the Kpelle tribe of Liberia (Cole, M. and Bruner, J.S., 1971; Cole, M., Gay, J., Glick, J., and Sharp, D., 1971). On a classification task, Kpelle subjects were asked to group all objects that belonged together. There were 20 objects, 5 each from the following categories: food, clothing, tools, and cooking utensils. Many of the Kpelle classified the objects into 10 groups of 2. For example, a knife and a potato were placed together, because the knife cuts the potato. This type of classification, considered useless by Western standards, was considered practical and intelligent behavior by the Kpelle. In fact, when asked how a fool might classify the objects, the "standard" classification of 4 groups of 5 objects each was given.

In the Navajo indian culture, Ohannessian (1967) found that subjects were reluctant to engage in problem solving tasks until they were confident of success. To try too soon and fail was tantamount to being shamed in that culture. According to Wober (1972), certain Ugandan groups associate wisdom with "slowness" rather than "quickness". Wober also notes that the more schooling a Ugandan has, the farther away he/she is from the ability-slowness association. To have interpreted these two behaviors by Western standards would have clearly led to severe misunderstandings.

Another factor which can often have a confounding effect on research findings, if not taken into account appropriately, is socio-economic status (SES). A cross cultural study by Hertzig, Birch, Thomas, and Mendez(1968) illustrates this. This study compared the performance of middle class Anglo and working class Puerto Rican three year-olds at various cognitive tasks, and arrived at conclusions that paint a somewhat bleak picture for the Puerto Rican group, namely:

- Puerto Rican children were more likely to be distracted from the task, and stay distracted, than were middle class Anglo children.

- Middle class Anglo children responded to cognitive demands by verbalization much more frequently than by action or gesture. Puerto Rican children tended to use passive and silent unresponsiveness frequently.

- The "atmosphere" of the middle class families was business-like, and middle class parents considered toys as a source of educational experience. The atmosphere in Puerto Rican working class families was much more social, and Puerto Rican parents regarded their children's toys as amusements.

The researchers claimed that stylistic differences between the groups were sustained when I.Q. level was controlled for, and claimed also to have controlled for other significant variables, such as family stability and ethnicity of the interviewer, implying that the differences found were cultural. Since the Hertzig et al. study was supposed to be comparing middle class children with lower class children, there were substantial differences in SES among subjects. However, the low SES subjects were Puerto Rican while the high SES subjects were Anglo. For example, 87% of the Anglo fathers had had at least some college preparation, whereas 84% of the Puerto Rican fathers had not even completed a high school program. In addition, 75% of the Anglo fathers were professionals or business executives, whereas 61% of the Puerto Rican fathers were unskilled workers. All of the Anglo families lived in suburban or urban apartments and houses, whereas 86% of the Puerto Rican families lived in low-income public housing projects.

Given the large differences in SES factors between the two groups, it is not clear how informative this study really is. In fact, since there were near perfect correlations between ethnic class membership and SES, to attribute the "negative" findings to ethnic group membership without adequately controlling for SES is misleading. It would have been more useful to compare the Puerto Rican group studied with an economically and educationally equivalent group of Anglo families before drawing conclusions about cultural differences, especially in view of the study by Anastasi and deJesus (1953) cited in the Hertzig et al. paper which concluded from observations of children during free play, that Puerto Rican children were not found to be less verbal than Anglo children. Hertzig et al. do point out in their conclusions that if these differences between middle class Anglo and lower class Puerto Rican children persist, a conventional educational setting would be less effective for the Puerto Rican group and would likely result in higher achievements by the Anglo group.

Another well-known study by Lesser, Fifer, and Clark (1965) investigated various cognitive abilities among Chinese, Jewish, Black, and Puerto Rican children. The study concluded that significant ethnic differences existed among the groups, and that socio-economic status affected performance; that is, a lower SES implied a poorer performance. The study also presented Puerto Ricans as the group ending last, or next to last, on the general intelligence tasks in relation to other ethnic groups. Since then, there have been several studies (Reiss, 1972; Leifer, 1972; Burnes, 1968; Flaugher and Rock, 1972) that compared the performance of different ethnic groups on cognitive tasks and found no clear evidence supporting the premise that performance was directly related to ethnicity.

More recent studies have demonstrated a relationship between academic performance and SES. In a study with Hispanic college students majoring in science and engineering, Mestre and Robinson (1982) found statistically significant differences in the obvious direction between low-, and high-SES subjects in college grade point average, vocabulary, and two measures of algebraic problem solving ability. Bender and Ruiz (1974) found in a study with Mexican-American high school students that class membership was more significant than race in determining levels of achievement and aspirations. Yet another study by Buriel and Saenz (1980) found that, in comparison to non college-bound Mexican-American females, college bound Chicanas came from higher income families. It should be pointed out that this positive correlation between SES and academic achievement is not endemic to minority groups, but holds for non-minority groups as well (Sewell, 1971; Sewell and Hauser, 1975).

Cognitive Style, Piagetian Level, and Mathematics

Cognitive style, which refers to individual variations in modes of perceiving, remembering, transforming, and utilizing information, has been shown to differ across different cultural and linguistic groups (Berry and Dasen, 1974; Cole and Bruner, 1971; Witkin and Goodenough, 1976; Witkin, Dyk, Paterson, Goodenough, and Karp, 1962). Cognitive style is also supposed to be a differentiating measure in the analytic domain. More specifically, field-independent individuals allegedly possess more highly developed analytic, and problem solving abilities than field-dependent individuals (Kagan and Buriel, 1977; Witkin, Moore, Goodenough, and Cox, 1977). Field-dependency constructs have been used to differentiate in the personality and social domains as well. For example, field-independence is associated with attributes such as positive self-evalation, assertiveness, and a preference to work independently, whereas field-dependence is associated with low self-evaluation, passivity, and a preference toward group activities (Kagan and Buriel, 1977 and references therein).

The cognitive style construct has evolved into an association between field-dependence/independence and child rearing practices, and since child rearing practices depend upon cultural values, certain cultures "promote" the development of one cognitive style over another. The fact that Hispanics have tested to be more field-dependent than Anglos (Holtzman, Diaz-Guerrero, and Swartz, 1975; Kagan, 1974; Buriel, 1975; Mebane and Johnson, 1970; Witkin, Price-Williams, Bertini, Christiansen, Oltman, Ramirez, and Van Meel, 1974) has been used to explain the lack of "mathematical proclivity" of Hispanics.

Although the evidence is not incontrovertible, there are indications that performance and learning style in mathematics may be related to field-dependency measures. For example, a study by McLeod, Carpenter, McCormack and Romoualdas (1978) resulted in field-dependent (nonminority) students responding better to instructional treatments in mathematics which had substantial guidance; in contrast, field-independent students responded better with minimum guidance. Other studies have resulted in statistically significant correlations between mathematics performance and measures of field-dependency (See Table 2 in Kagan and Buriel, 1977; Erh, 1980).

Another measure of an individual's cognitive prowess concerns his/her Piagetian level. Perhaps less culture dependent than other measures, Piagetian levels supposedly delineate an individual's cognitive ability in a hierarchy of categories (e.g., concrete, formal, etc.). The higher the category of an individual the more likely that this individual will do well at complex cognitive tasks. Since the existing literature on Piagetian assessments concerning bilingual populations is extensive, we will only review a few studies, and point out some of their limitations.

One thing is clear when one reviews the existing literature on Piagetian tasks for bilingual populations, namely, that there is no consistent pattern. Perhaps this is due to faulty control of significant variables. For example, Liedtke and Nelson (1968) compared bilingual and monolingual subjects on a variety of Piagetian tasks. Their results support the hypothesis of a cognitive advantage for bilinguals, but are weakened by failing to control for linguistic variables. The children participating in the study were assigned to bilingual groups solely on the basis of teacher observation. Another study, by Feldman and Shen (1971), compared bilingual and monolingual subjects on object constancy, and found bilinguals to be superior in performing the assigned tasks. However, the designation of children to the bilingual group in this case was based on their answers to several simple questions in Spanish, and on the ability of the child to speak Spanish at home. Finally, a study by Brown, Fournier, and Moyer (1977) examined the performances of Mexican-American children on 10 Piagetian tasks. The study concluded that acquiring a second language leads to a developmental lag, without attempting to measure or control for language proficiency. Moreover, the tasks were administered both in oral and written form, leading to further confounding effects for the bilingual group.

There was one study which attempted to control for factors such as SES, language proficiency, and teacher perception of students. DeAvila and Duncan (1980) studied the performance of 903 children from 9 distinct ethno-linguistic groups on various academic, cognitive, and linguistic tasks. The results from the study are elucidating, though not surprising; it was found that language proficiency predicted academic achievement much better than measures of cognitive style. Further, for all ethnolinguistic groups, language proficiency was the most important predictor of achievement relative to any other factor. DeAvila and Duncan also conclude that there is a positive, monotonic relationship between linguistic proficiency and cognitive functioning, implying that "deficiencies" in the subjects' performances on cognitive tasks were linguistic rather than intellectual in nature. In terms of cognitive style, DeAvila and Duncan state that even though there is support for the contention that bilinguals possess above average ability in the cognitive style dimension in earlier grades, this ability attenuates by fifth grade; in fact, their data showed no strong correlation between math performance and cognitive style.

Mathematics Studies

The number of research studies in mathematics education which are directly or indirectly related to this study is so large that a brief review such as the one we are about to give cannot even begin to cover a small fraction of them. We will therefore only attempt to give the reader a "flavor" for what has been, and is being done.

Much of the research in mathematics education uses the clinical interview approach. In two interviews with a gifted twelve-year-old seventh grader, Davis (1975a, 1975b) explored the processes employed by the student while solving the equations $\frac{3}{x} = \frac{6}{(3x+1)}$ and $10^{2x} - 1,100\ 10^{x} + 100,000 = 0$. Davis identified two modes of mathematical thinking employed by this student -- "one-small-step-at-a-time" thinking and "chunk" thinking. Davis also found that mathematics prerequisites such as definitions and axioms may be substantially different from cognitive prerequisites such as heuristics and creative thinking. Pereira-Mendoza (1979) also found with high school students solving novel problems that the heuristics employed were not applied generally, but instead were problem dependent. In another clinical study by Days, Wheatley, and Kulm (1979) with eighth graders solving problems with both simple and complex structures, it was revealed that formal operational (in the Piagetian sense) students used a larger variety of processes than concrete operational students, and that the formal group was more affected by problem structure than the concrete group.

Several interesting findings have emerged from recent studies on students' understanding of equations and variables. Clement and his collaborators (Clement, 1982; Clement, Lochhead, and Monk, 1981; Rosnick, 1981) have found that college engineering students are prone to use a sequential left-to-right translation of written algebraic statements resulting in erroneous equations where the variables are treated like labels, not unlike the labels for inches and feet in 12I=1F. Findings by Wagner (1981) reveal that some students are not aware that the two equations, 7W + 22 = 109 and 7N + 22 = 109 have the same solution. Herscovics and Kieras (1980) suggest a method for teaching students the concept of equation which starts from a concrete perspective and proceeds toward the abstract.

In an interesting treatise of the mechanisms by which many algebraic errors are committed, Matz (1980) lists 33 common errors sited in numerous studies and proceeds to explain why these errors are not the result of some random process, but rather follow a regular pattern. According to Matz, errors are the result of reasonable, albeit unsuccessful, attempts to adapt previously learned knowledge to a new situation. She identifies two processes as the cause of many commonly made errors: 1. Use of a known rule "as is" in a new situation where it is inappropriate, and 2) Incorrectly adapting a known rule so that it can be used to solve a new problem.

One study by Davis, Jockusch, and McKnight (1978) deserves particular attention. This is a rather comprehensive treatment (300 pages in length) of the cognitive processes employed by 7th, 8th, and 9th grade students learning algebra. We will comment on a few aspects of this work which are very apropos to the findings we will discuss later in this report. Davis et al. comment that students often use sloppy language in mathematics, for example reading X^3 as "X three", and that this poor use of language can produce errors and ambiguities. This study also cites cases of students being able to solve certain types of problems, and yet when confronted with one such problem containing an error, not being able to verbalize or otherwise identify what the error was. Davis et al. advocate giving the students the fullest possible appreciation of the importance of logically identifying every algebraic step, beginning as early as the seventh grade; the "do it this way" approach commonly used in so many schools, Davis et al. claim, is not sufficient to inculcate in the students the notion that each algebraic step requires explicit logical justification. Finally, these authors report being able to teach many remedial low SES seventh graders who scored two years below grade level in standardized achievement tests, ninth grade level algebra by using teaching methods appropriate for these students.

Mathematics Studies with Bilingual Subjects

Few studies have been conducted with bilingual subjects. The oldest and perhaps best known study with Puerto Rican bilinguals was done by the International Institute of Teachers College, Columbia University (1926). This study found that the English mathematics problem solving ability of 12th grade Puerto Rican bilinguals educated on the island of Puerto Rico was significantly below that of U.S. 12th graders. This was surprising, in view of the fact that these students had been receiving mathematics instruction in English (second language) since fifth grade.

Kellaghan and Macnamara (1966) found a similar retardation in problem arithmetic among Irish fifth standard primary students when it was taught in their weaker language (Irish), despite the fact that the components of the problems were separately understood. Kelleghan and Macnamara also found that the problem reading time was longer in the weaker language by a factor of 1.4 to 1.7, even when the English-Irish versions were equated for number of words. Several other studies cited by Macnamara (1967) show similar retardations when the weaker language is the language of instruction. According to Macnamara (1966), the research evidence suggests that bilingual children keep pace with monolinguals in mechanical arithmetic, but fall behind in problem arithmetic. On the other hand, in a study conducted with Mexican-American 14 year olds, Ortiz-Franco (1977) found no significant correlation beyond the .01 level between reading proficiency and problem solving ability. Another interesting finding of this study was the lack of a significant correlation between math achievement and field-dependency.

Recent research with Hispanic college students majoring in engineering and science-related fields also supports the premise that language proficiency plays a very important role in the problem solving process. In a study investigating the translation skills of bilingual Hispanic technical college students in going from a written statement to a mathematical equation, Mestre, Gerace, and Lochhead (1982) found that performance was significantly correlated to language proficiency, and that the types of errors Hispanics made were very different from those made by monolinguals. For example, in the problem,

> Write an equation using the variables C and P to represent the following statement: "At a certain restaurant, for every four people who ordered cheesecake, there were five who ordered pie". Let C represent the number of cheesecakes ordered and P the number of pies ordered,

the "typical" error made by monolinguals (Clement, Lochhead, and Monk, 1981), and by a substantial fraction of the Hispanic group, was to reverse the variables and write 4C = 5P. Several other types of errors surfaced for the Hispanic group, however. Two consisted of writing $\frac{4C}{5P}$ and 4C < 5P. In videotaped clinical interviews, the students explained their rationale for these answers. Those writing the former, $\frac{4C}{5P}$, claimed that this fraction set up a "relationship" expressing the appropriate ratio of cheesecakes to pies sold at the restaurant; those writing the latter, $4C < 5P$, claimed that because of the 4 to 5 ratio of cheesecakes to pies, one could never set up an equation relating the two variables. As evidence, these students pointed out that if 4 people bought cheesecakes, then 5 bought pies; if 8 people bought cheesecakes, then 10 people bought pies, and so on. Hence, the two quantities could not be related via an equation.

Other findings by Mestre (1981, 1982) revealed statistically significant differences in the performance of these Hispanic students between equivalent problem sets, one set requiring little semantic processing, the other requiring substantial semantic processing. Further, there were persistent statistically significant correlations between algebraic problem solving skills and language proficiency for the Hispanic students. The surprising finding was the presence of strong correlations between language skills and algebraic skills, even for problems requiring very little semantic processing, such as "solve for x: $2x - 9 = 3x + 5$". Such statistically significant correlations between mechanical algebraic skills and language proficiency were not present for monolingual college students.

II. METHOD

A. Subjects

Three groups of students participated in the interviews. The first group consisted of all 6 Hispanic students in the participating teacher's Algebra I class. These six (5 male, 1 female) students, which will be called the "participating Hispanics" during this report, were not enrolled in any special bilingual program and received all their instruction in English. The second group consisted of 5 Anglo students (3 males, 2 females) also from the participating teacher's Algebra I class. These students were selected by the participating teacher during the second month of classes as representative of the spectrum of mathematical ability in the algebra class. These students will be called the "participating Anglos". There were a total of 34 students in the participating teacher's algebra class.*

The third group consisted of all 3 Hispanic students (1 male, 2 female) enrolled in an advanced algebra class which was not taught by the participating teacher. Here "advanced" means that these students had started studying algebra one-half year before the participating teacher's Algebra I class. These students will be called the "advanced Hispanics". There were a total of 29 students in the advanced algebra class.

Having these three groups allowed various comparisons. For example, comparisons made between the participating Anglos and the participating Hispanics helped to evaluate whether some of the difficulties encountered were due to linguistic, or semantic factors. Comparisons between the advanced Hispanics and the other two groups provided information on the degree of expertise derived from having been exposed to the subject matter over a longer period of time. We will note in passing that none of the Hispanic students in the school

*To be more precise, the 34 students were divided into two separate classes of approximately 17 students each. Both classes, however, were identical in terms of the material covered and the exams given. For the sake of convenience, we will refer to all of these 34 students collectively as the Algebra I class.

enrolled in the transitional bilingual program was taking algebra.

B. Textbooks

The textbook used in the Algebra I class was "Modern Algebra: Structure and Method-Book 1" by Mary P. Dolciani and William Wooton, Houghton Mifflin Co. Publishers, Boston, MA: 1973. The textbook used by the advanced algebra class was "Algebra" by Richard E. Johnson, Lona Lee Lendsey, and William E. Slesnick, Addison-Wesley Publishers, Reading, MA: 1967. Both texts employ a "traditional" approach.

C. Project Personnel

There were five people intimately involved in the study. They were the two principal investigators, two half-time research graduate assistants, and the participating teacher. Throughout this report, if there is a need to make a reference to the participating teacher's name, the name "Mr. Smith" will be used.

D. Interview Logistics and Question Selection

The research plan during a typical month consisted of four tasks: 1) preparing questions for the interviews, 2) conducting the interviews, 3) analyzing the interviews, and 4) a staff discussion of the interview results. The process of selecting questions for the interviews involved a staff meeting where, guided by the course syllabus, the two half-time research assistants and the participating teacher suggested possible interview questions. These were evaluated and discussed with the principal investigators. Following the staff meeting, the principal investigators met to select the actual set of interview questions to be used for the month's interviews. All questions used during the interview sessions are included in Appendix I.

Once the interview questions were selected, the participating teacher arranged an interview session. Whenever possible, three interviews were conducted simultaneously by the two research assistants, and one of the principal investigators (JPM) and students were rotated through the interviewers so that a student spoke with the same interviewer once every three interviews. All interviews were audio-recorded. The interviews took place during one of the regularly scheduled "study" periods and lasted approximately 45 minutes. The participating Hispanics were interviewed on a monthly basis, while the participating Anglos and the advanced Hispanics were interviewed bimonthly. The participating Hispanics were interviewed a total of 8 times during the academic year, while the students in the other two groups were interviewed 4 times during the academic year. The odd-numbered interview sessions were administered to the participating Hispanic group only. The even-numbered interview sessions were administered to all three groups. The only exception to this was Interview 2 where only 3 out of the 5 participating Anglos were interviewed due to logistical problems.

E. Conduction of Interviews

The method for conducting an interview went as follows: An 11" x 14" "sketch pad" was used during the interview. The interview questions were taped onto the top page of the pad allowing the student to see the questions while he/she worked out the solution on the bottom page.

The interview questions were used as guidelines for discussion rather than as "quizzes" for the students to work out. Each question was read to the student and the interviewer made every effort to make sure that the student knew what was being asked before a solution was attempted. That is not to say that the interviewer cleared up misinterpretations of the problem statements when they occurred. The interviewer's role was to present the problem to the student, clear up questions regarding any vocabulary with which the student was unfamiliar and similar stumbling blocks, until the student <u>thought</u> he/she understood what was being asked and started on the solution.

All participants were told (and quickly adjusted to the modus operandi) that the interviewer was interested in how they solved problems, and would therefore interrupt at various stages along the way to ask for an explanation of why a certain problem was solved in that particular way. The participants were told not to interpret this as their having made a mistake, since the interviewer would "challenge" them to offer an explanation for any answer or procedure, no matter whether it was correct or incorrect.

F. Analysis of Interviews

A two stage analysis procedure was used. The first stage consisted of a brief pre-analysis where the interviewer would write down comments on the actual student protocol sheets (the sheets used by the student to work out the problems). This stage took place immediately following the interview, and the comments written were intended to help clarify any points which would aid in understanding the events that transpired during the interview. The comments made covered areas such as the order of "flow" of the student's protocol, any student reactions which may be missed during an aural review of the taped interview, and general impressions of the students' knowledge level, anxiety level, etc.

The second stage consisted of the more detailed protocol analysis where the interviewer listens to the taped interview while reviewing the student's written protocol and pre-analysis comments. The protocol analysis consists of making detailed comments on the students' performance, including strengths, weaknesses, and possible sources of any difficulties encountered by the student during the course of the interview.

Once the analyses were completed for an interview session, another staff meeting followed during which the entire set of interviews was discussed. These staff meetings also served as "brainstorming" sessions where explanatory hypotheses were discussed which attempted to encompass the behavior, misconceptions, and other findings of the interview set.

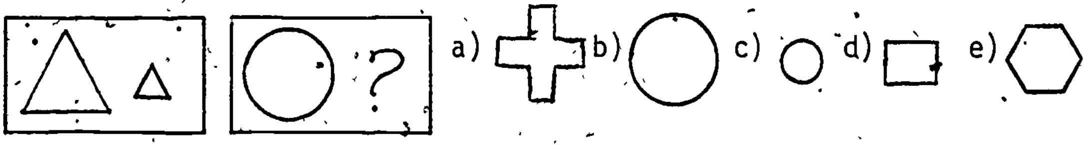
III. STUDENT PROFILE

In this chapter we will present quantitative data, and in so far as possible, give an indication of the academic background of the students. We should state at the outset that due to the nature of this study, the size of the various samples makes it inappropriate to carry out a comprehensive quantitative statistical analysis. The data in this chapter are being presented only to give an indication of how the participating students compare to other students at both the local and national level in several standardized measures.

In April of 1981, all students in the school system in question were given the California Achievement Tests (CAT) (CTB/McGraw-Hill, 1977). This battery consists of eight major divisions: 1) Reading Vocabulary, 2) Reading Comprehension, 3) Spelling, 4) Language Mechanics, 5) Language Expression, 6) Math Computation, 7) Math Concepts & Applications, and 8) Reference Skills. There were three sections which were made up of combinations of the eight major divisions above, namely, Total Reading was made up of the combined scores in Reading Vocabulary and Reading Comprehension, Total Math was made up of the combined scores from Math Computation and Math Concepts & Applications, and Total Battery was made up by combining the first seven categories listed above -- that is, all major divisions except Reference Skills.

In addition, three other measures were administered to the participating teacher's whole Algebra I class, as well as to the three advanced Hispanics. The first of these is a Piagetian exam entitled "An Inventory of Piaget's Developmental Tasks" (Catholic University, 1970) designed by Hans Furth to cover the following 18 areas: 1) Quantity, 2) Levels, 3) Sequence, 4) Weight, 5) Matrix, 6) Symbols, 7) Perspective, 8) Movement, 9) Volume, 10) Seriation, 11) Rotation, 12) Angles, 13) Shadows, 14) Classes, 15) Distance, 16) Inclusion, 17) Inference, and 18) Probability. There were four questions for each of these 18 areas making this battery a 72 question test.

The remaining two measures were the Analogies and Classification subsections of the Test of General Ability-Level 5, Form CE (Guidance Testing Associates, 1962). The Analogies section contained 24 questions which were totally symbolic and asked the student to make one pair of drawings like a given pair, as shown in the following example:



The Classification section was composed of 26 questions which again were totally symbolic and asked the students to identify one drawing in a series of five which was "different". For example,

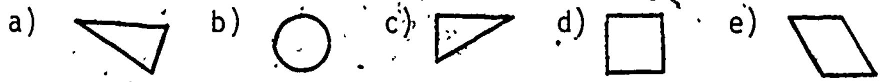


Table 1 gives the results of these tests for the various subgroups of students. The first two entries correspond to the mean and standard deviation. In the CAT scores, the third entry gives the national percentile ranking corresponding to that particular raw score. Also shown in Table 1 are the students' final grades in Algebra. The scores on all standardized measures are based on the total number correct.

Table 2 shows the results of the Piagetian exam broken down by the 18 subsections that make up the test. The two entries correspond to the mean and standard deviation for that subsection, with a maximum possible score of 4 for each subsection.

Finally, Tables 3 and 4 show the Pearson Correlation Coefficients among selected variables for the Algebra I class, and the Advanced Algebra class, respectively.

Several observations can be made from Table 1. First, it is evident that the advanced algebra class is consistently "above average" on the CAT in comparison to the Algebra I class, as well as to national norms; in fact, t-tests between the Advanced Algebra class and the Algebra I class on all the means of the CAT result in statistically significant differences in favor of the Advanced class well beyond the .05 level. Second, it is evident that the 3 advanced Hispanics are very comparable with their Anglo peers in the Advanced Algebra class in all the measures shown on Table 1. Third, it is evident that the five participating Anglos are comparable

TABLE 1

PERFORMANCE ON AVAILABLE MEASURES

	Read. Vocab.	Read. Comp.	Tot Read.	Spell.	Lang. Mech.	Lang. Expr.	Tot. Lang.	Math Cmpt.	Math Cncpts.	Tot. Math	Tot. Batt.	Ref. Skills	Anal.	Class.	Piaget	Alg. Grade
Algebra I Class N=34*	15.1	24.0	39.1	10.7	15.3	24.0	39.4	26.1	23.3	49.3	138.5	18.5	15.8	17.5	49.0	72.1
	5.7	5.7	10.5	2.9	4.1	5.3	8.5	5.5	6.3	9.9	26.1	3.2	3.6	4.8	7.9	15.3
	53	49	52	43	39	44	41	59	45	53	46	43	----	----	----	----
Nonparticipating Anglos N=23*	15.9	25.7	41.6	10.8	16.3	25.7	42.1	26.7	25.7	52.4	146.8	19.4	16.9	17.7	49.8	73.1
	5.6	5.3	9.6	3.2	2.8	4.7	6.3	5.4	5.9	10.0	23.5	2.5	3.1	4.6	8.8	16.0
	54	56	55	43	44	51	48	61	53	57	53	50	----	----	----	----
Participating Anglos N=5	17.4	23.8	41.2	10.8	15.0	20.2	32.5	26.0	19.8	45.8	133.0	18.0	15.2	18.2	49.4	78.8
	4.8	5.2	9.5	2.6	3.7	5.0	8.0	6.2	4.8	7.2	22.8	4.5	1.1	6.0	3.6	13.0
	59	48	54	43	38	29	31	59	34	46	43	39	----	----	----	----
Participating Hispanics N=6	10.5	18.3	28.8	10.3	12.2	21.2	33.3	23.8	17.8	41.7	114.2	16.0	12.2	16.3	45.8	62.3
	5.4	4.2	9.4	2.7	6.9	5.7	11.9	5.4	3.5	6.9	23.6	3.5	4.8	5.5	6.7	10.3
	30	29	29	39	24	32	27	52	27	38	30	30	----	----	----	----
Advanced Algebra Class N=29	23.7	31.3	55.0	15.4	20.7	32.3	53.0	34.0	35.4	69.5	192.9	22.1	----	----	----	81.4
	4.0	4.2	7.3	2.9	2.4	2.9	4.0	3.3	4.2	6.5	16.5	1.6	----	----	----	5.1
	79	79	79	83	78	85	84	81	81	83	86	78	----	----	----	----
Advanced Anglos N=26	23.7	31.6	55.3	15.3	20.7	32.7	53.4	34.0	35.5	69.5	193.5	22.3	----	----	----	81.9
	4.0	3.8	6.8	2.5	2.3	2.2	3.1	3.5	4.3	6.7	14.9	1.3	----	----	----	4.8
	79	81	80	83	78	87	85	81	81	83	86	80	----	----	----	----
Advanced Hispanics N=3	23.7	29.0	52.7	16.0	20.7	29.0	49.7	34.0	34.7	68.7	187.0	20.0	17.3	16.0	50.7	73.3
	4.0	8.2	12.2	6.1	3.8	6.2	9.0	1.7	3.8	5.5	31.2	2.6	2.5	1.0	4.5	7.1
	79	70	76	87	78	67	73	81	79	81	81	55	----	----	----	----

Note: The first twelve measures correspond to the California Achievement Tests. Analogies and Classification are from the Test of General Ability. The last two measures correspond to the Piagetian test and the students' final grade in algebra. The three entries correspond to the mean, standard deviation, and national percentile ranking.

* N for the Algebra I class is 32 for all CAT measures. N for the Algebra I class/nonparticipating Anglo subgroup is 21 for all CAT measures.

TABLE 2

PIAGETIAN EXAM BREAKDOWN

Section	Algebra I Class N=34	Nonparticipating Anglos N=23	Participating Anglos N=5	Participating Hispanics N=6	Advanced Hispanics N=3
1. Quantity	3.15	3.13	3.40	3.00	3.67
	.56	.55	.55	.63	.58
2. Levels	3.68	3.61	3.60	4.00	4.00
	.81	.89	.89	0.	0.
3. Sequence	3.18	3.30	3.20	2.67	3.00
	.97	1.02	.45	1.03	1.73
4. Weight	3.65	3.70	3.80	3.33	4.00
	.81	.76	.45	1.21	0.
5. Matrix	3.74	3.65	3.80	4.00	4.00
	.45	.49	.45	0.	0.
6. Symbols	3.26	3.39	3.00	3.00	3.67
	.71	.66	.71	.89	.58
7. Perspective	2.79	3.26	2.40	1.33	3.33
	1.15	.81	1.14	1.03	1.15
8. Movement	2.74	2.83	2.80	2.33	2.00
	1.08	1.07	1.10	1.21	1.00
9. Volume	2.21	2.39	2.00	1.67	2.00
	.95	.99	.71	.82	1.00
10. Seriation	3.71	3.70	3.60	3.83	4.00
	.63	.63	.89	.41	0.
11. Rotation	2.15	2.30	2.20	1.50	3.00
	1.05	.88	1.64	1.05	0.
12. Angles	2.76	2.74	3.20	2.50	1.67
	1.02	1.01	.45	1.38	1.15
13. Shadows	2.74	2.91	2.20	2.50	3.00
	1.14	1.12	.84	1.38	0.
14. Classes	1.53	1.52	1.80	1.33	0.33
	1.05	1.04	.84	1.37	.58
15. Distance	1.97	2.00	1.60	2.17	1.33
	.83	.85	.55	.98	.58
16. Inclusion	2.29	2.70	1.60	1.33	3.00
	1.31	1.26	.89	1.21	0.
17. Inference	3.24	3.35	2.80	3.17	3.33
	.96	.93	.84	1.17	.58
18. Probability	2.44	2.57	2.20	2.17	1.33
	1.31	1.41	1.10	1.17	.58
Total	51.21	53.04	49.20	45.83	50.67
	6.73	6.55	3.56	6.74	4.51

TABLE 3

PEARSON CORRELATION COEFICIENTS

ALGEBRA I CLASS

	Read. Vocab.	Read. Comp.	Spell.	Lang. Mech.	Lang. Expr.	Math Cmpt.	Math Cncpts.	Tot. Batt.	Anal.	Class.	Piaget
Read. Comp.	70**	---									
Spell.	23	29	---								
Lang. Mech.	45**	47**	14	---							
Lang. Expr.	51**	49**	0	59**	---						
Math Cmpt.	54**	63**	54**	54**	50**	---					
Math Cncpts.	31*	59**	03	40*	51**	43*	---				
Anal.	45**	38*	-19	56**	50**	33*	45**	53**	---		
Class.	25	30*	-06	11	0	21	15	21	54**	---	
Piaget	17	25	-04	34*	20	37*	20	31*	49**	34*	---
Alg. Grade	06	32*	02	12	17	31*	41*	31*	27	36*	28

*p < .05, *p < .005 for one tailed tests of significance.

Note: Analogies and Classification are from the Test of General Ability. All other variables except the Piagetian score and the final algebra grade are from the California Achievement Tests. N=32 for correlations involving CAT variables, and N=34 for the remaining variables. Decimal points have been suppressed from correlation coefficients.

TABLE 4

PEARSON CORRELATION COEFFICIENTS

FOR ADVANCED ALGEBRA CLASS

	Read. Vocab.	Read. Comp.	Spell.	Lang. Mech.	Lang. Expr.	Math Cmpt.	Math Cncpts.	Tot. Batt.
Read. Comp.	58**	---						
Spell.	51**	49**	---					
Lang. Mech.	35*	33*	56**	---				
Lang. Expr.	31	53**	42*	13	---			
Math Cmpt.	14	15	17	53**	23	---		
Math Cncpts.	41*	41*	32*	50**	39*	49**	---	
Alg. Grade	-28	-02	-05	05	33*	08	05	01

*$p < .05$, **$p < .005$ for one tailed tests of significance.

Note: N=29 for the correlations above. All variables except algebra final grade are from the California Achievement Tests. Decimal points have been suppressed from correlation coefficients.

to the 23 other Anglos in the Algebra I class, and that all of the Anglos in the Algebra I class are "average" students as determined by the CAT national percentile rankings. Finally, Table 1 shows that the 6 participating Hispanics are consistently "below average" in the CAT both as compared to their Anglo classmates, and as measured by the national percentiles, except in the Math Computation section where this group scored "average".

There were very few differences between the various groups' performance level in the Piagetian exam, regardless of whether one looks at the performance on the exam as a whole, or at the performance on the individual subsections as shown on Table 2. With few exceptions, the advanced Hispanics outperformed both the participating Anglos and Hispanics, while the participating Anglos performed somewhat, albeit not significantly, better than the participating Hispanics.

The correlation coefficients for the Advanced Algebra class from Table 4 show many fewer statistically significant correlations than the corresponding correlations for the Algebra I class of Table 3. This is not surprising since the larger variance among the measures of Table 1 for the Algebra I class necessarily results in larger correlation coefficients for this group (see for example, Hopkins and Glass, 1978, p139ff for an explanation of this phenomenon). Some interesting results to note are the strong correlations between the Math Concepts section and the verbal sections of the CAT for both the Advanced Algebra class and the Algebra I class. This could be indicative of the importance of language skills in "abstract" mathematics. In terms of the final grades received in algebra, the Advanced class shows only one statistically significant correlation with Language Expression, while the final grade for the Algebra I class is significantly correlated with Reading Comprehension, Math Concepts & Applications, and Math computation.

Some information was gathered on several non-academic background characteristics of the students. This information was obtained in one of two ways. The first method consisted of chatting with the students just prior to the beginning of the interview sessions concerning their interests, family background, future plans, etc. This not only gave us a brief profile on the students' background, but also served to put the student at ease just prior to the start of the problem solving tasks. The other method was to administer a short questionnaire eliciting information on topics such as language spoken with ~~friends~~, language spoken at home, and number of years in the mainstream curriculum. It was left up to the student to volunteer the information. If the student did not wish to volunteer personal information to us, the matter was not pursued further. For this reason, many of the background profiles we were able to compile are incomplete. We have included the information we were able to obtain in Appendix II.

In summary, it appears that the advanced Hispanics are extremely well-prepared academically as measured by their performance in the CAT. The participating Anglos' performance in the CAT indicates an average preparation. The participating Hispanics appear to be academically underprepared, both in comparison to their Anglo classmates, and in comparison to national norms.

IV. INTERVIEW RESULTS

The questions used in the interviews were designed to assess student expertise in four relatively independent areas. The four areas are: 1) Appreciation of algebra as an abstract logical system. 2) Command of the formal operations of algebra. 3) Ability to use algebra. 4) Ability to solve problems. These areas will give an indication of a student's command of mathematics as a formal construct, as well as his/her ability to utilize this construct in a problem solving situation. In the four sections that follow, each of the four areas will be discussed in detail, and accompanied by examples from the students' protocols.

Although the focus of this chapter is the results of the actual interviews, in a study such as this it is hard to dissociate the results from a discussion of what the results mean. We find that the best time to discuss the cause of certain errors is at the time we present them. There will, therefore, be various mini-discussions during the course of this chapter. The following chapter, entitled "Discussion", will offer an overview of the whole study.

A. Appreciation of Algebra as an Abstract Logical System

What we wish to investigate here is the degree to which the student creates and develops an abstract conceptual schema for mathematics that is independent of other logical systems, such as languages. Like language, mathematics has its own formal logical structure, largely composed of definitional constructs which form the "rules of the game." We are hopeful that research in this area will help us to understand how mathematical and linguistic development are related. Bilinguals, who already have two languages and therefore two logical systems, may be able to develop a third logical system more readily than monolinguals. On the other hand, if the language logical structures are poorly developed, bilinguals may find it more difficult to develop new logical systems. Bilinguals constitute a good "laboratory" in which to pursue these questions.

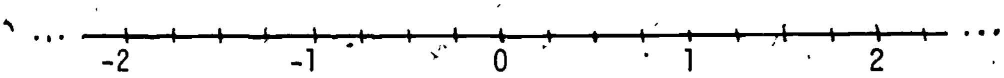
From a practical standpoint, information for this area can be gathered by attempting to find answers to questions such as the following:

- Does the student understand the role of definitions in mathematics? Does the student utilize definitions to resolve points of ambiguity?
- Does the student distinguish between mathematical statements (e.g., equalities and inequalities) and mathematical phrases (e.g., expressions and ratios)?
- Does the student recognize the difference between a variable and a label?

Number Lines

The first interview, in which only the participating Hispanics took part, dealt heavily with the students' understanding of the number line. The number line was not only the first topic covered in the textbook and in class, but also the students' first encounter with a totally abstract topic. From the responses given it was clear that the number line construct was something students had a difficult time grasping. However, it was also clear that many of the difficulties experienced by the students were not due to the number line per se, but could be attributed to incorrect mathematical knowledge, misinterpretations of comments the teacher had made in class, and to semantic difficulties. More often than not, erroneous responses followed a consistent pattern based upon the students' knowledge frame. A few examples will illustrate what we mean.

Various students did not accept number lines which had the first "tick mark" to the right of the origin labeled with anything other than 1. Thus if shown the following number line, these students



insisted on labeling the line with the first tick mark to the right of the origin as 1, the second tick mark as 2, etc. The following section from the protocol of a student discussing the number line above illustrates this phenomenon.

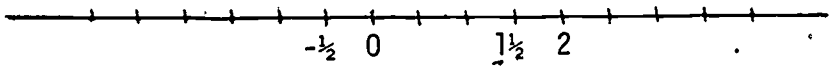
I: Where's four?

S: (Points to tick mark labeled 1)

I: OK. How come this 1 is there? (points to the 1 at the fourth tick mark to the right of the origin.) Is there a mistake?

S: No... Well, Mr. Smith (the teacher) tells us that everything right here, every little thing's a number, so there's zero, one, two, three, and four. (Pointing to the tick marks coresponding to 0, 1/4, 1/2, 3/4, and 1).

A similar response from another student is given below in reference to the following number line in Set C of Interview 1.



I: Where's positive 1?

S: (Puts +1 at the first tick mark).

I: Can you tell me where negative one is?

S: Right there (points to tick mark labeled -1/2).

I: Oops! I've already labeled that point -1/2.

S: Yeah, but that's wrong.

I: I made a mistake?

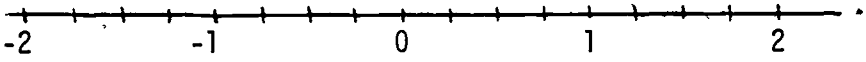
S: Yeah, you probably made a mistake (chuckles).

We can attribute this confusion to two factors. First, the participating teacher told the class that every tick mark represented a number, as stated by the first student above. However, it appears that many students interpreted "number" here to mean "whole number." Second, in all examples and exercises given in the textbook, not one number line is labeled with the first tick mark being anything other than 1. This, needless to say, is somewhat misleading.

It also became evident that much of the confusion exhibited by students in placing fractions on the number line derived from their poor knowledge base concerning relative sizes among fractions. Many students thought that 1/2 < 1/3, 1/3 < 1/4, etc., demonstrating that they were comparing only the denominators in deciding relative sizes among fractions.

Another problem, this one perhaps due to semantic difficulties, was displayed by two students and involved using the numbers "one and one-half" and "one-half" interchangeably. These students would say "one-half" when reading 1 1/2 and conversely, would label a point 1/2 when verbally asked to label "one and one-half." It appears that to these students, the symbol 1/2 means "half" or "one-half", and they use the two terms interchangeably. The problem these students have with 1 1/2 is that the first part, 1, is read as "one", and the second part, 1/2, is read as "half", and hence 1 1/2 becomes "one half."

The last common error we will discuss involved labeling two different tick marks with the same number. For one student, this was caused by having two separate interpretations for labeling whole numbers and labeling fractions on a number line. This is illustrated in the following protocol:



I: Label this first tick mark to the right of 0.

S: (Labels it 1/3).

I: What's the second mark?

S: (Labels is 1/2).

I: What's the third mark?

S: (Labels this one 1/3 also).

I: Let's talk about any number line, not just this one here. If I give you any number at all, like 300, is there one specific location in the number line where that number would go, or can it go in various places in the line?

S: Can go various places on the line.

I: OK. How is that? Can you explain it to me? Or can you give me an example of it, if it's easier to explain it that way?

S: (Pause...) Well, you've got, like, ... negative 300 and you've also got the other side 300 so..like... (pause).

I: So you think that because there are two sides to it, you can put sort of the same number on either side. Does the fact that one side is negative and the other positive make a difference?

S: Yeah.

So it it clear at this point that what this student means by saying it is legitimate to label two different tick marks with the same number is that two different tick marks can have the same absolute value, but one must be negative and the other positive. As the interview continues, we see why the student is not bothered by labeling the number line above with two 1/3's.

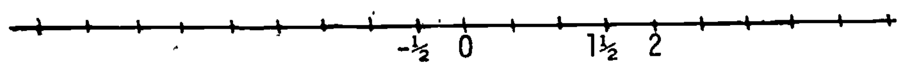
I: If this (pointing to the first tick mark labeled 1/3) is one-third here, could it also be one-third here (pointing to the second tick mark labeled 1/3 at the tick mark corresponding to the value 3/4), or would it have to go someplace else?

S: I suppose it depends, because if you have 1/3 away from, like, the origin, then you got, like, the 1/3 closer to 1.

Here the student has displayed that she attributes different meanings to the representation of whole numbers and the representation of fractions on the number line. To her, the first 1/3 is showing that one-third of the distance between 0 and 1 closest to the origin; the second 1/3 is showing that one-third of the distance between 0 and 1 closest to 1.

The performance of students in number line problems which were somewhat concrete was much better. For example, most students had no difficulty answering the following problem correctly:

1-C4 If I tell you that 1/2 means 1/2 mile east of my house, where would 1/2 mile west of my house be? Where is my house? If my friend Julia lives two miles west from my house, where does she live?



Fractions

In Interview 2, we probed students' understanding of fractions using the following two questions:

2-A4. How would you define a fraction?

2-A5. Which of these are fractions? Why?

2, 1/2, 3/4, 5/3, 4/2, 9/3, 6/1, 10, 52/1.

Responses given to these two questions revealed that students had only a partial understanding of the concept of fractions, and were often inconsistent in applying the definitions given in question 2-A4 to their answers in question 2-A5. A few examples will illustrate the situation.

One student stated that a "fraction is part of any object... a different number on the top and on the bottom." Even though this definition is not self-consistent, it is fairly good for a student of this mathematical sophistication. When asked to select the fractions in the list provided in 2-A5, this student only chose 1/2, 3/4, and 5/3 as being fractions. When asked why 4/2 was not a fraction, the student explained that 4/2 equals 2 and is thus not a fraction. Apparently, even though 4/2 conforms to the second half of the definition, "a different number on the top and on the bottom", it did not satisfy the first half of the definition; that is, 4/2 is equal to 2, and 2 is not "part of any object." Another student claimed that fractions were "numbers that are between whole numbers." However, by stating that every number except 2 and 10 in question 2-A5 were fractions, this student displayed she was either unaware of, or not bothered by, the inconsistency between her definition and claiming that numbers such as 9/3 and 6/1 were fractions. Yet a third student claimed a fraction was "a part of a number -- a piece of something" and answered question 2-A5 consistently with this definition by selecting 1/2, 3/2, and 5/3 as the only fractions among the nine numbers given. However, when the interviewer asked him to explain why 4/2 was not a fraction, he said that, written as 4/2, it was a fraction, but that you could also write 4/2 as 2, which is not a fraction.

In terms of each group's overall performance in these two questions, all of the advanced Hispanics had a good grasp of the concept of fractions. The participating Hispanics and the participating Anglos were somewhat confused as to what constituted a fraction, with the participating Anglos having a slight edge over the participating Hispanics.

Equations

As in defining fractions, we found that students were not able to give correct definitions of the term "equation." In the following question from Interview 2,

2-A6. Which of the following mathematical statements are equations? Why?

a) $3 + 1 = 4$ e) $x - 7 < 10$ i) $\frac{6}{2 + Y}$

b) $3x - 9$ f) $Z = 19$ j) $8 \times 3 = (9 \times 4) - 12$

c) $\frac{3 + Y}{X + Z}$ g) $5(X + Y) = 20$ k) $\frac{W + T}{7}$

d) $10 + 2 = 8$ h) $8 > A$ l) $8/4 = 4/2$

The most popular definition of "equation" given was something that was either manipulated or solved, as evidenced by the following quotes from four of the students interviewed:

An equation is...

- "when you have an operation and you have to figure something out to get the answer"
- "a problem with an answer"
- "you have to look for an answer for something to be an equation"
- "Something that you solve for -- anything that has letters"

One student had an opposite viewpoint and stated,

- "if it has letters, it is not an equation 'cause you don't know what the letters are"

Only four students during this interview session had a correct notion of what an equation was, both in terms of verbalizing a fairly precise definition and in applying their definitions to select the equations among the list of 12 choices given in question 2-A6. Two of these students came from the participating Anglo group and the other two came from the advanced Hispanic group.

It is evident from the answers above that most beginning algebra students view algebra concretely rather than abstractly. The definition "two mathematical expressions separated by an equal sign" is simply too large an abstraction for these students. It appears that, in order to have meaning from these students' perspective, a definition must answer questions such as "what good is it?", "what can you do with it?", and "in what does it result?" Thus a process-product oriented definition such as "an expression you manipulate to get an answer" is something meaningful to these students - after all, it answers the three questions posed above, namely an equation is useful as something you somehow manipulate (the process) and results in an answer (the product).*

More on Definitions

There were various other interview questions probing the students' ability to verbalize their understanding of a concept by defining a term. The performance of all groups was nearly equivalent in the following question from Interview 6:

6-A1. What is the difference between a monomial, a binomial, and a polynomial?

Even though only one student (from the participating Anglo group) offered a correct definition, approximately half from each group thought that these terms referred to how many ____________ were in an expression, and words used by these students to fill in the previous blank were "equations", "numbers", "terms", "problem", and "parts."

In question 5-A1 from Interview 5 (recall that only the participating Hispanics were interviewed in the odd-number interview sessions) where the following was asked,

5-A1. Given these two sets: $A = \{1, 5, 0, -3, 19, 24, 4\}$
$B = \{-1, 19, 21, 29, 4, -40\}$
What is $A \cup B$? What is $A \cap B$?

two out of the six participating Hispanics interviewed were not able to offer any answer to this problem; another student thought "union" meant all the positive numbers and "intersection" meant all negative numbers in the two sets; the remaining three thought "union" meant the numbers that the two sets had in common, while "intersection" meant the numbers in one set which were not in the other set.

The responses to question 5-D1 in this same interview,

5-D1. What is a variable? What are variables used for?

revealed that these students knew what a variable was operationally, but could not define the term correctly. Four of the students said

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* We thus expect students to define a "teacher" as "someone who gives homework" and not as "someone who imparts knowledge."

a variable was a letter, and the remaining two said a variable was a number. However, all of the students that said variables were letters showed that they understood what variables were used for in their responses to the second part of this question; their answers were reasonable facsimiles of the statement "variables are used to represent numbers." This supports the hypothesis adduced in the previous section where we claimed that students are more likely to remember a mathematical term in terms of what its function is, and not in terms of its abstract definition.

Further support for this can be found in questions 1 and 2 of Interview 3 where the participating Hispanics were asked the meaning of the terms "axiom" and "closure." Not one student had any idea of what these terms meant, even though they were covered in both the textbook and in class. Vis-a-vis our hypothesis, the reason these students could not offer a definition was that, unlike terms such as "equation" and "fraction", the terms "closure" and "axiom" are such that answers to the questions "what good is it?", "what can you do with it?", and "in what does it result?" are certainly not readily apparent.

The last definitional question we will discuss is 8-D1 of Interview 8. Here we asked,

8-D1. What is a prime number? Which of the following are prime numbers and why?

a) 2	d) 5
b) 3	e) 6
c) 4	f) 7

Here many of the participating Anglos and advanced Hispanics were unable to accurately verbalize a definition; however all 8 of these students operationally knew what a prime number was and could pick out the prime numbers from the list above. On the other hand, the performance of the participating Hispanic group on this question was somewhat poor. Only 3 of the 6 students were able to pick out the prime numbers from the list, and just as the students from the other two groups, these three students were not able to verbalize a definition. The remaining 3 students thought "prime" had something to do with whether a number was even or odd.

Variables vs. Labels

The fact that variables are treated as labels by many college students has been recently established in various studies (Mestre, Gerace, and Lochhead, 1982; Clement, 1982; Clement, Lochhead, and Monk, 1981; Rosnick, 1981). In the following problem

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

the now-famous "variable-reversal error" where students would write 6S=P, was committed consistently by approximately 35% of non-minority engineering undergraduate students (Clement, Lochhead, and Monk, 1981). Using a population of Hispanic engineering students, the frequency of the variable reversal error was 54% (Mestre, Gerace, and Lochhead, 1982).

In clinical interviews of students solving the above problem, it was discovered that one of the major points of confusion regarding the variable-reversal error derived from treating S and P as labels for "students" and "professors", instead of treating them as variables to represent the number of students and the number of professors. It should be pointed out that the students interviewed displayed that they were aware that there were more students than professors in the problem statement (Clement, 1982). For these students, the meaning of 6S = P was "six students for every one professor".

To investigate whether beginning algebra students are prone to this kind of confusion, we designed the three questions below, given during Interview 6:

6-B1. Mr. Smith noted the number of cars, C, and the number of trucks, T, in a parking lot and wrote the following equation to represent the situation:

$$8C = T$$

Are there more cars or trucks in this parking lot? Why?

Write an expression with variables for the following statements.

6-B10. Six times the length of a stick is 24 feet.

6-B11. If a certain chain were four times as long, it would be 36 feet.

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From the student responses to these questions, the evidence is strong in favor of the students' proclivity to treat variables as labels. In 6-B1, 11 of the 14 students interviewed from the three groups said that there would be more cars in the parking lot as represented in the equation 8C=T due to the factor of 8 in front of C. Of the remaining 3 students, one from the advanced Hispanic group said that there would be more trucks -- her explanation displayed that she was using C and T correctly as variables for the number of cars and trucks. The last two students, both from the participating Hispanic group, gave rather unique answers -- one said that you could not tell whether there were more cars or trucks because the values of C and T were not given; the other said there would be an equal number of cars and trucks because of the "=" in the equation.

A comparison of the students' responses in problems 6-B10 and 6-B11 revealed an interesting phenomenon. All 14 students obtained the correct answer in both of these problems. In problem 6-B10, 8 students wrote a correct equation using the letter "L" for the variable, and the other 6 wrote a correct equation using some other letter for the variable, such as "a", or "X"; none of these 6, however, used the letter "S" for the variable which is not unreasonable to expect since the problem is about a stick. In contrast, the most popular variable name used in 6-B11 was "C" and not "L"; seven students used "C", two used "L", and the remaining used some other variable name. Given that both of these problems asked for a quantity involving length, and that many students were using a variable in 6-B11 to represent "the chain", it is evident that the sentence construction of these two problems triggered (more often than not) specific responses for that particular problem. That is, the manner in which 6-B10 started, "Six times the length..." makes it clear to the student that this is a problem about length, thereby triggering the use of the letter "L" for the variable. However, In 6-B11, the first few words, "If a chain..." make it clear this is a problem about a chain, thereby triggering the letter "C" to be used as the variable. Even though 6-B11 is asking for a length just as 6-B10 is, the student is distracted from this fact by having the references to length via the words "long" and "feet" appearing much later in problem 6-B11.

Thus, although the evidence is not conclusive, there are strong indications that these beginning algebra students, not unlike college technical students, often treat variables as labels. It further appears as though the syntax structure of the problem is largely responsible for the triggering mechanism by which a variable name is chosen, with the first important noun in the problem statement (e,g., "length" in 6-B10 and "chain" in 6-B11) serving as the trigger. This makes it somewhat of a random process whether the student will choose a variable name to mean a label for that noun, or a quantity to be represented by the variable name.

There will be further discussions of the variable reversal error and of the students' performance in other problems, such as the ones above in section D entitled "Ability to Solve Problems."

B. Command of the Formal Operations of Algebra

A complete logical system contains not only a set of definitions and relationships, but also a set of rules for manipulating these relationships. Quite independent of students' knowledge of the structure of algebra is their understanding of the dynamical laws, or permissible operations that may be used in the process of obtaining a solution. We are not concerned here with the students' ability to obtain answers, but rather with their knowledge and comprehension of legal algebraic manipulations. Thus, whereas in the last section we were interested in whether students knew the rules of the game, in this section we are interested in whether students "play by the rules of the game." Investigations in this area may eventually lead to an understanding of what determines a student's approach to learning mathematics. For example, if a beginning student chooses to memorize a set of permissible operations as opposed to more generalizable concepts, then this student may be able to obtain answers more readily, but at the same time, this approach may inhibit the formation of a deeper understanding.

Specific research questions that probe this area are:

- Does the student know what fundamental operations can be performed on an algebraic statement?

- Does the student realize that these operations do not change the solution to an equation?
- Does the student confuse what "rules" he/she may apply in a particular situation?

Interview questions in this area were interspersed throughout the eight interview sessions. This allowed us to monitor progress in mastering previously learned, as well as newly learned concepts.

Commutative and Distributive Properties

In question 4 from Interview 3 we attempted to ascertain whether students had a good grasp of the commutative and distributive properties, and whether they were aware of which of these two properties was applicable to addition, subtraction, multiplication, and division operations:

3-4. Which of the following equations are true and why?

a) $8 \cdot 4 = 4 \cdot 8$	h) $17 \div 4 = 4 \div 17$
b) $a + b = b + a$	i) $X - Y = Y - X$
c) $a \div b = b \div a$	j) $X \div Y = X/Y$
d) $8 - 3 = 3 - 8$	k) $a(7 + b) = 7a + ab$
e) $a(7 - b) = 7a - ba$	l) $a(7 - b) = (7 - b)a$

f) $10(6 + 3) + 4 = 4 + 10 \cdot 6 + 10 \cdot 3$

g) $10 \div (2 + 5) = (10 \div 2) + (10 \div 5)$

The two participating Hispanic students who were successful in answering this problem used numerical substitution often, to check the validity of the equations containing variables. For example, in $X - Y = Y - X$, these students would substitute something like $X=1$ and $Y=2$, and checked the left, and right hand sides of the equation for equality. For the remaining four participating Hispanics, their responses were somewhat random. For example, one student stated that $a(7 - b) = 7a - ba$ was true but that $a(7 + b) = 7a + ba$ was false, while another stated the converse. Upon probing, it became clear that the cause of the problem was a misunderstanding over the role of the parentheses. According to both of these students, the false equation could have been made true by having "b" alone on the right hand side. This response was not consistent with answers given to a question during Interview 1 where we asked "what is $3 \times 6 + 2 = ?$" Here we attempted to confuse the students after all six of them had stated that the answer was 20 by pointing out that the answer could also be 24. All six of them volunteered that if parentheses are placed around the "3x6", then the answer would always be 20, and that placing them around the "6 + 2" would result in 24. These students appeared to have an understanding of parentheses as "grouping operators" in a concrete example using numbers, but not so in more abstract examples using variables such as those above.

Solving Simple Algebraic Equations

The questions below from Interview 4 served as a good evaluation of the students' manipulative skills in solving fairly simple algebraic equations:

4-A1. Solve the following equations:

a) $2 + X = 3$ d) $4 + a = -6$

b) $X + 7 = 10$ e) $r - 3 = -8$

c) $y - 12 = 40$ f) $X + 9 = 4$

4-A2. Solve the following equations:

a) $3X = 6$ d) $9a = -18$

b) $4Y = 10$ e) $X/3 = 33$

c) $-2X = 12$ f) $Y/8 = -4$

4-A3. Solve the following equations:

a) $5Y + 10 - 3Y - Y = 14$

b) $4X + 7 = 6X - 5$

c) $7(X + 3) = 21$

while the following question evaluated the students' ability to multiply a negative and a positive number:

4-A5. a) What is - 7 times 3?

b) What is -3 times 7?

Our discussion of the advanced Hispanics will consist of merely stating that they had an excellent understanding of the procedures involved in solving all of the problems above, and executed these procedures virtually flawlessly. This was not the case for the other two groups. We isolated ten types of errors committed by both the Anglo and Hispanic groups as described below:

1. Failure to reverse the sign of an additive constant when moving it across an equal sign.

   e.g., $2 + X = 3$ or $r - 3 = -8$

   $X = 3 + 2$ $\quad$ $r = -8 - 3$

   $X = 5$ $\quad$ $r = -11$

2. Combining a constant with a variable.

   e.g., $2 + X = 3$ or $X + 7 = 10$

   $2 + 1X = 3$ $\quad$ $8\,X = 10$

   $3X = 3$ $\quad$ $X = 10/8$

3. Treating the factor multiplying a variable as an additive factor.

   e.g., $3\,X = 6$

   $X = 6 - 3$

   $X = 3$

4. Moving a multiplicative factor through an equal sign by multiplying instead of dividing.

   e.g., $4\,Y = 10$ or $X/3 = 33$

   $Y = 10 \times 4$ $\quad$ $X = 33 \times 1/3$

   $Y = 40$ $\quad$ $X = 11$

5. Determining the sign of a product between a positive and a negative number by the sign of the factor with the largest absolute value.

   e.g., $-7 \times 3 = -21$ but $7 \times -3 = 21$

6. Changing the sign of a factor when dividing an equation by it.

   e.g., $3\,X = 6$ $\quad$ corollary: $9\,a = -18$

   $X = 6/(-3)$ $\quad$ $a = 18/9$

   $X = -2$ $\quad$ $a = 2$

7. Trying to perform too many steps at once, resulting in errors along the way.

e.g., $5Y + 10 - 3Y - Y = 14$

$$2Y + 10 = 14 - 10$$

$$2Y = 4$$

8. Failure to respect the transitivity of the equal sign.

e.g., $4X + 7 = 6X - 5$

$$4X = 6X - 5 - 7 = -12 = -2X = 6$$

9. Strange combinations.

e.g., $X + 9 = 4$

$$1X - 9 = 8X$$

$$X = 8 - 4$$

$$X = 4$$

10. Silly computational errors.

e.g., $4X + 7 = 6X - 5$

$$7 + 5 = 6X - 4X$$

$$13 = 2X$$

$$13/2 = X$$

Two of the five participating Anglos exhibited a thorough understanding of the concepts necessary to solve these problems. Of the remaining three, only one committed error 5, while all three committed error types 7 and 10. In the participating Hispanic group, one student showed a thorough understanding; the remainder committed various combinations of the 10 errors above. Our general consensus in evaluating these two groups was that two participating Anglos had a mastery of the "tools" necessary to solve these problems, two had a partial mastery, and one had little mastery. In the participating Hispanic group, one had a mastery, two had partial mastery, and three had little mastery.

Several questions in Interviews 6 and 8 covered polynomials and exponents as shown below:

6-A2. Add the following expressions:

a) $(3Y^2 + Y - 4) + (-2Y^2 + 5Y + 1)$

b) $(7X^2 - 2) + (4X^2 - 3X + 9)$

c) $(6R^2S + 11) + (3R^2S + 2RS + RS^2 + 2R + 1)$

6-A4. Simplify each expression:

a) $Z^2(-3Z^4) =$ d) $(r^2s)^3 =$

b) $(2YZ)(3Y^2) =$ e) $(X^5)(X^4) =$

c) $(ab)^2(ab)^3 =$

6-A5. If X=3 and Y=2, what are the values of the following:

a) X^2Y^3 c) $(X^2Y)^3$

b) $(XY)^2$ d) $(X + Y)^2$

8-D3. Multiply out:

a) $(X + 2)(X - 3)$

b) $(3Y + 1)(Y + 5)$

8-D4. Factor the following:

a) $a^2 + a - 6$

b) $X^2 + 5X + 6$

The additional six errors shown below were made on these five problems during the course of the interviews:

11. Adding bases with different exponents:

e.g., $3Y^2 + Y = 4Y^3$

12. Adding exponents when summing two terms with unequal exponents:

e.g., $3Y^2 + Y = 3Y^3$

13. Multiplying exponents instead of adding them in a product:

e.g., $(X^5)(X^4) = X^{20}$

14. Adding exponents when they should be multiplied:

e.g., $(R^2S)^3 = R^5S^3$

15. Neglecting the middle term in a binomial product:

e.g., $(X + Y)^2 = X^2 + Y^2$

and if X = 3 and Y = 2, $(X + Y)^2 = 13$

or

$(X + 2)(X - 3) = X^2 - 6$

16. Inappropriate factoring of a polynomial:

 e.g., $X^2 + 5X + 6 = (X + 1)(X - 6)$

The frequency of committing these six errors for the three groups was similar to the pattern for error types 1 through 10. Namely, the advanced Hispanics committed these types of errors very infrequently; the participating Anglos committed them with moderate frequency; the participating Hispanics committed these errors frequently.

Before leaving this brief discussion of polynomials and exponents, we would like to point out an amusing result. In Interview 2, question 2-D3 asked "Suppose X = 3 and Y = 4. What is $(X + Y)^2$?" Since at the time of Interview 2 polynomials had not been covered, we were only interested in whether the students could extend their understanding of squaring simple numbers to squaring a relatively complex quantity. In Interview 6, we asked essentially the same question: 6-A5d) If X = 3 and Y = 2, what is $(X + Y)^2$? Again, all three advanced Hispanics got these two equivalent questions correct. However of the 5 students from both the participating Hispanic, and Anglo groups who got the question correct in Interview 2 by taking $3 + 4 = 7$, $7^2 = 49$, all got the question in Interview 6 wrong by using $3^2 + 2^2 = 13$. This we think shows how students can become mind-locked when covering a new topic by attempting to solve all problems using the newly learned procedure, rather than resorting to a previously learned, and perhaps easier procedure.

In terms of the student's overall performance, there was an obvious difference between the participating Hispanic, and Anglo groups in consistency. Whereas the participating Anglo group showed a consistency in error patterns, the participating Hispanic group was much more random in their error patterns. For example, some students in the participating Hispanic group would commit error type 2 in working parts of question 4-A1, but exhibited flawless techniques in the remaining parts; the fact that a student used the correct procedure in part b) for example, was not any indication of the procedure he/she would use later on in part d).

It was also clear that the participating Hispanic group was more likely than the Anglo group to confuse which rules applied to which cases. The participating teacher, in an attempt to help his students learn various algebraic rules, made a chart and posted it at the front of the class. The chart contained reminders of facts such as how to determine the sign of the sum of a positive and a negative number. However, in explaining their procedures for solving some of these problems, some of the participating Hispanics made it clear that they were applying some rules to the wrong cases. For example, for those who said that -3 x 7 = 21 and 3 x -7 = -21, their explanations revealed that they were using the rule for determining the sign of the sum between a negative and a positive number.

C. Ability to Use Algebra

Before algebra can become a useful problem solving tool, the student must be able to formulate a problem in algebra and combine elementary operations into a strategy for solution. These abilities clearly require a deeper assimilation of the principles of mathematics than does learning to solve assigned problems. Knowledge of those factors which influence strategy formation, or of specific learning difficulties which inhibit strategy formation, will permit an optimization of the assimilation process. To extend the metaphor we have been using, this area is attempting to ascertain how well the student can "manipulate the rules of the game to his/her own advantage while still remaining within the confines of legality". Specific questions to be addressed in this area of research are:

- Does the student employ algebraic concepts to solve problems without being prompted, or does the student prefer to use other techniques such a trial and error or guessing?
- Does the student define variables?
- Does the student have a strategy for a solution, or resort to random manipulations in the hope of stumbling onto a solution?
- Is the student able to extend and apply his/her knowledge to a relatively novel problem?

Various questions from Interview 2 were very fruitful in exploring this area. In question 2-C3 of this interview, most students revealed that they preferred not to use variables in solving problems. Question 2-C3 posed the following problem:

> 2-C3. Suppose I told you that the number of students in this school is equal to thirty times the number of classrooms. Can you write an equation that says this?

The result revealed that only one student from the participating Hispanics defined appropriate variables and solved the problem correctly. The remaining 5 attempted to write an equation using actual numbers to represent the classrooms and the students. Of these 5, only 2 used variables appropriately in an equation after a strong prompt. The 3 participating Anglos interviewed (recall that in Interview 2, only 3 participating Anglos were interviewed) could not write a correct equation with variables even after prompting. This, in fact, was one instance where the participating Hispanics outperformed the participating Anglos. Out of the 3 advanced Hispanics, one solved the problem correctly without prompting, one solved it correctly after prompting, and the last one could not solve the problem even after prompting.

These results demonstrate that the large majority of students are either unable to or reluctant to define and use variables in solving problems. They have a much better "feeling" about using concrete numbers rather than abstract variables. What is somewhat surprising is that this symptom was manifested also by the advanced Hispanics, while their overall performance in the interviews would suggest otherwise.

Another good indication of the students' ability to use algebra was their performance on the following problem also from Interview 2:

> 2-C4. Suppose I have a ball which I drop from a certain height. Every time the ball bounces, it only goes up to 1/2 of the highest point reached in the previous bounce. If I drop the ball from 16 feet, how high does it go after the first bounce? How high does it go after the second bounce? Suppose I have another ball that goes up to 3/4 of the highest point reached in the previous bounce ... (repeat problem).

Here, the problem was the students' inability to generalize a mathematical problem. Five of the participating Hispanics could work out this problem with the "bounce factor" of 1/2 without hesitation. However, three could not generalize this procedure for a bounce factor of 3/4. It appeared that they had an excellent intuitive grasp of "halving" but not of "three-quartering". The participating Anglos also suffered from this problem, albeit to a lesser extent; the advanced Hispanics were quite facile with either a 1/2 or 3/4 bounce factor. It is our feeling that the difficulty with the 3/4 bounce factor was due to students not being able to verbalize or understand the procedure they used in the 1/2 bounce factor case --had they realized that all they were doing was multiplying each maximum height times 1/2 to obtain the subsequent maximum height, they could have easily extended this procedure to the 3/4 case. Perhaps the difficulty was that in the 1/2 bounce factor case, students divided by 2, but in the 3/4 bounce factor case, they were not sure whether to divide by 3, 4, or some combination of 3 and 4.

Additional evidence that students are not very aware of the procedures or strategies they follow in solving problems was found in these three questions from Interview 2:

2-D4. Which of the following equations are true statements?

a) $\frac{x}{y+z} = \frac{x}{y} + \frac{x}{z}$

b) $\frac{r+s}{t} = \frac{r}{t} + \frac{s}{t}$

2-D5. Which of the following is $\frac{3 + x}{7 + y}$ equal to?

a) $\frac{3+x}{7} + \frac{3+x}{y}$

b) $\frac{3}{7} + \frac{x}{y}$

c) $\frac{3}{7+y} + \frac{x}{7+y}$

2-D6. If $x = 13$ and $y = 1$, what is the value of $\frac{3 + x}{7 + y}$?

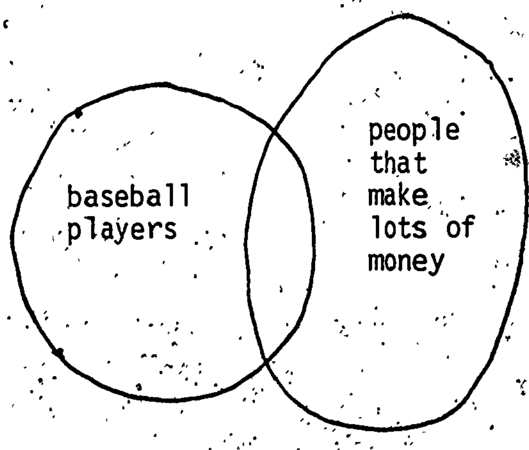
Again, we will not discuss the advanced Hispanics, other than to say that two of the three obtained the correct answers for all cases and demonstrated that their reasoning was not specious. On the other hand, all of the students in the remaining two groups performed poorly in questions 2-D4 and 2-D5. Even those who said 2-D4a) was false and that 2-D4b) was true offered an improper explanation. All but two of the students in these groups chose b) for the answer in question 2-D5; of the remaining two, one said that the answer was not among the choices given, and the other chose the correct answer, but his explanation displayed improper logic. Amusingly, all students from these two groups plugged in the values of question 2-D6 and obtained the correct answer. However, none of those who picked answer b) in question 2-D5 were inclined to plug x = 13 and y = 1 into 3/7 + x/y to see if it resulted in 2 as a consistency check with their answer to question 2-D6. We found it surprising that none of these students realized that answer b) of problem 2-D4 is what they must do to add fractions -- namely, get a common denominator and then add the numerators. Even though they can add fractions properly when couched in concrete terms (i.e., with numbers), they cannot isolate the procedure enough to recognize it when expressed in terms of variables. Before leaving these problems, we would like to point out that the most frequently given reason for question 2-D4b) being false was that there were two t's in the denominator in the right hand side, while only one appeared in the left hand side of the equation.

Several questions from Interviews 5 and 6 also probed the students' ability to generalize their knowledge to problems which went a little beyond what was presented in the textbook or in class. These problems are listed below:

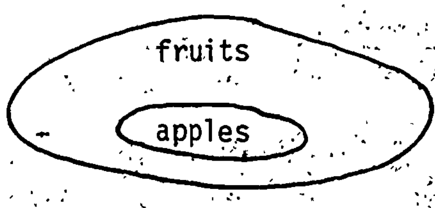
5-A2. Look at this diagram:

a) According to the diagram, are there baseball players that do not make lots of money?

b) Are there people that make lots of money who are not baseball players? If so, what part of the diagram represents these people?



c) What does this diagram tell you?



d) What part of the diagram represents those fruits that are not apples?

5-B2. Suppose we make up a new operation, "*", as follows:

$$A * B = 2A + B$$

What is the value of $3 * 9 = ?$

6-A6. If $f(x) = x + 3$ and $g(x) = x - 2$, what is:

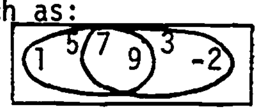
a) $f(x) + g(x) =$

b) $f(x) - g(x) =$

c) $f(x) \cdot g(x) =$

We will first discuss the questions from Interview 5. In 5-A2, we were looking to see if students would extend their knowledge of Venn diagrams about union and intersection from the simple examples using numbers given in the textbook such as:

{1,5,7,9} { 3,-2,9,7} →



The Venn diagram in 5-A2, parts a) and b), above about baseball players and rich people proved confusing to the students. Only one out of the six participating Hispanics answered both parts correctly and appeared to have a good understanding of what the diagrams represented. The remaining five students had no idea how to answer this question. One student answered these two parts by resorting to his knowledge about the situation in the real world -- he said that all baseball players made lots of money. The students performed better in parts c) and d) about "fruits and apples"; four of these six students were able to answer part d) of this question correctly.

The performance in 5-B2 was worse. None of the six participating Hispanics interviewed in this session knew how to interpret the operation "*". Among the erroneous answers given, one student stated that 3*9 = 3A + 3B, another that 3*9 = 2A + B, a third student stated that you could not tell what 3*9 was because you were not told whether "*" meant you added, subtracted, multiplied, or divided the 3 and the 9.

In question 6-A6, the results were much better. Here, four of the participating Hispanics, all five participating Anglos, and two of the advanced Hispanics showed that they were using the correct procedure in solving this problem. At times, however, some of the students who employed the correct procedure were not successful in carrying it out without making errors, e.g., $f(x) + g(x) = x + 3 + x - 2 = x^2 + 1$.

The last three questions we will discuss in this section were suggested by R.J. Jensen, and S. Wagner from the University of Georgia.* These questions were given in Interview 8 and are shown below:

8-A1. Suppose $x = 170$. What is the value of $x - 37$?

8-A2. Suppose $y + 13 = 160$. What is the value of $y + 13 - 15$?

8-A3. Suppose $5(2z + 1) = 10$. What is the value of $\frac{(2z + 1)}{2}$?

Jensen and Wagner, who are conducting one of the three NIE funded studies under this RFP, found in their preliminary analysis that in problem 8-A3, students were likely to solve $5(2z + 1) = 10$ for z, and then substitute into $(2z + 1)/2$ for an answer, instead of using simpler approaches such as dividing $5(2z + 1) = 10$ through by 10, or solving for the quantity $(2z + 1)$ and dividing this by 2.

We found that in 8-A1 and 8-A2, all students had little difficulty in coming up with the correct answer. The procedure used in answering these two questions was to subtract 37 from 170 in 8-A1, and 15 from 160 in 8-A2. Only one student from the participating Anglos actually solved for y in 8-A2, and then evaluated $y + 13 - 15$. The performance in 8-A3, however, was poor across all three groups. Only one advanced Hispanic obtained the correct answer -- her procedure consisted of staring at the problem until she realized that she could divide through by 10 to obtain the desired result, and

* See Jensen, R.J. and Wagner, S. "Three Perspectives on the process uniformity of beginning algebra students." In S. Wagner (Ed.), Proceedings of the Fourth Annual Meeting of the North American Chapter for the Psychology of Mathematics Education, p. 133-139. Athens, University of Georgia, Department of Mathematics Education, 1982.

wrote the answer as 10/10. In contrast to the findings of Jensen and Wagner, no student attempted to solve this problem by solving for z first. Most erroneous answers consisted of either dividing 10 by some number (usually 5), or of leaving the answer as an expression involving z (e.g., $(z + 1)/2$).

D. Ability to Solve Problems

Several aspects of the students' problem solving skills were investigated in order to identify any particular difficulties they might have in learning or using algebra. To conclude the metaphor, this area attempts to assess whether the student can "win the game". The points upon which the discussion will focus are translation skills between verbal and symbolic representations, cultural or linguistic factors that may influence problem solving ability, and problem solving techniques. Some of the specific research questions we will be addressing are:

- Does the student possess the necessary algebraic and arithmetic skills to obtain a solution for a problem without making errors?
- Is the student able to translate word problems into algebraic equations and word phrases into symbolic expressions?
- Does the student misinterpret problems because of language difficulties such as poor vocabulary or reading comprehension, rather than as a result of mathematical misconceptions?
- Does the student's success at solving problems depend upon the amount of linguistic processing that must occur?
- Does the student exhibit good problem solving procedures and habits?
- Are there cultural factors which may affect the student's performance?

Since the four areas into which we have chosen to divide our discussion of the interview findings are only relatively mutually exclusive, many of the above questions have already been discussed in sections A, B, and C of this chapter. In this section, we will draw upon the findings discussed in the previous three sections in our evaluation of the students' problem solving capabilities. At this point, it should be apparent from the previous three sections that the answer to the first question above is often "no". All 11 participating students from the Algebra I class have displayed that they do not have the necessary mastery of arithmetic and algebraic skills that would allow them to confidently obtain the solution to a problem without making some kind of error.

Another impediment to the development of these students' problem solving skills derives from semantic difficulties. We found that all three groups, but particularly the participating Hispanics, often got a problem wrong because they simply misinterpreted a word or phrase in the problem statement. For the sake of fluidity, we will discuss some of the semantic difficulties observed during the course of the study without making reference to the particular interview session or question that elicited the response.

We found two popular interpretations of the phrase "twice a number". The first was "the sum of two numbers", for example, A + B; the second interpretation was "a number times itself", that is, the square of a number. Students from both of the Hispanic groups were more prone to this interpretation. Thus in a problem such as "twice a number equals 16", typical wrong answers were of the form $A + B = 16$ or $x^2 = 16$. The term "reciprocal" used in a question given only to the participating Hispanics was interpreted to mean "additive inverse" in an operational sense; that is when asked to find the reciprocal of 5, the answer -5 would be offered. By far, the most misinterpreted term was also one of the simplest -- "quotient." All but one student from the participating Hispanics misinterpreted this term. Typical meanings that students assigned to "quotient" were "product", "sum", and "answer"; a discussion of possible cause for this interpretation is given in the next chapter where we discuss how students use the textbook. Finally, certain constructions such as "subtract 7 from the quotient of x and y" gave students difficulties; here, students were more likely to write "7-(something)" than "(something)-7". This error is common enough to have been observed by all instructors of algebra at one time or another. It is not so much a semantic difficulty that causes students to write "7-(something)" but rather a sequential left-to-right translation of the statement, that is,

subtract 7 from the quotient of x and y

7 - x + y

or

x^2

or

x/y

Although we have no proof, we expect that they would have had no difficulties in writing a correct mathematical expression for a construction such as "the product of x and y minus 7", since a left-to-right sequential translation will yield the correct result.

The remainder of this section will be devoted to a discussion of how students performed in a variety of word problems given during the last four sets of interviews. Although there were some word problems given during the first four interview sessions, the students had not been working with word problems enough during the fall semester for us to be able to make a fair assessment of their competence. We should, however, comment on three findings that were consistently manifested during the first four interviews. First, there was noticeable anxiety exhibited by the students when asked to solve word problems. Second, students performed extremely poorly on problems that could not be easily solved in their head, or by trial and error. Third, on word problems where students had a reasonable degree of success, the procedures that were used most often were trial and error, or direct solution without use of variables. For example, in problem D1 of Interview 4,

4-D1. In 8 years, Ana will be as old as her sister Sonia is now. If Sonia is 23 years old, how old is Ana?

students would either guess a reasonable answer, and if that did not satisfy the conditions of the problem, would then try another until converging, or they would stare at the problem silently for a while until they figured out that what they needed to do was subtract 8 from 23. Only two students, one advanced Hispanic and one participating Anglo, set up this problem algebraically using variables and equations.

At the start of the second half of the study, we were interested in eliciting the students' opinions on what they found difficult about solving word problems. The following question from Interview 5 was used for this purpose:

5-D3. What do you think is the hardest thing about solving word problems?

The following verbatim quotes were taken from the protocols of the six participating Hispanics interviewed:

Student #1 - "Getting the information you need to make the problem. Like if it says something about how many miles she walks, you gotta get how many miles -- get it into numbers -- take it apart"

Student #2 - "Starting them. First it's hard for me to start a word problem. Like trying to find, you know, what's x"

Student #3 - "Knowing what they want. Sometimes I read it wrong"

Student #4 - "The whole thing. I hate 'em"

Student #5 - "Finding the equation"

Student #6 - "You don't know what to do -- you don't know whether to add or divide -- you don't know which number comes first."

This question was immediately followed by the following question which we hoped would allow the student to tell us, with the aid of a concrete example, the sources of some of the difficulties they experience in solving word problems:

5-D4. Look at the following problem.

The sum of Mary's age and Gary's age is 23. Mary is 3 years older than four times Gary's age. What are the ages of Gary and Mary?

a) How many variables do you need to solve this problem?

b) Is there something you find hard to do when you try to solve this problem?

c) Can you solve it?

The same six participating Hispanics gave the following responses:

Student #

1-a) This student said two variables were needed. After a brief pause, he changed his mind and said that only one was needed. A brief moment later, he changed his answer back to two.

b) Said he found nothing particularly hard about this problem.

c) He wrote the following series of equations:

$$3W+4W=23$$
$$4W=26$$
$$W=26/4$$

2-a) This student stated "None. You don't have any. Just numbers. I think the variables are the numbers".

b) Here he stated, "This part, '3 years older than four times Gary's age', I don't know right here (pause) times 3, times 4, or if you just add this to that".

c) Was not able to attempt a solution.

3-a) This student stated that four variables were needed, namely, A, A+1, A+2, and A+3.

b) Said he found nothing particularly difficult about this problem.

c) Student wrote the following series of equations:

$$A+A+1+A+2+A+3=23$$
$$4A+6=23$$
$$4A=23-6$$
$$4A=18$$
$$A=18/4$$

4-a) This student explained that two variables were needed, one for Gary's age and one for Mary's age.

b) Was not sure what was hard about this problem.

c) Student was not able to start this problem and gave up. The interviewer then specifically asked her what she would do with the phrase "4 times Gary's age", and she wrote "4G". Then said "I don't know what to do with the '3 years older' part--do you add, or multiply, or what?" When asked to write what she thought the problem wanted done, she wrote:

$3+4G=23$

$4G=20$

$G=20/4$

$G=5$

5-a) This student explained that two variables were needed, one for Gary's age and one for Mary's age.

b) This student repeated his answer to 5-D3, and stated that "finding the equation" was the hardest thing about this problem.

c) After the student displayed he was discouraged because he did not know how to proceed, the interviewer asked if he could write an equation for the first part of the problem, "The sum of Mary's age and Gary's age is 23". This prompt allowed the student to write: M+G=23. At this point, he stated that Mary's age was 12 and Gary's age was 11.

6-a) This student said that "about 3 or 4 variables" were needed to solve this problem.

b) Was not sure what was hard about this problem.

c) Said that the only way to solve a problem like this was "to set it up the way Mr. Smith does". After a strong prompt from the interviewer asking him what would happen if he let G = Gary's age, and M = Mary's age, he wrote:

$3M(G-4)=23.$

The answers to 5-D4 above display that these students were able to verbalize rather well what they found hard about solving word problems in their answers to question 5-D3. If we may be so bold as to interpret what these students were saying, we believe that what they found difficult about word problems is that there is no set of rules one can apply which makes the answer miraculously appear. That is, if one asks a student to solve the equation 2+x=8 for x, he/she knows that there is a rule that you follow -- even if he/she does not remember what it is at that precise moment or perhaps even mistakenly applies the wrong rule -- and that this rule guarantees that you get the correct answer. Hence in the above example, "subtract 2 from both sides of the equation", is the temporary panacea for the student's malaise. In word problems, this is no longer the case. Left with no clear-cut rules to apply to a general word problem, we see from the responses above that students will try to do something with the information given in the problem that will lead to some kind of answer.

Perhaps the current emphasis on "teaching students to solve word problems" should be modified to "teaching students sound procedures with which to approach word problems". This would leave the student with something that he/she could use on any word problem rather than force the student to remember "tricks" or "rules" that can be applied specifically to "age problems", "percent problems", etc. That students do try to apply rules to word problems is evident from the answer given to 5-D4, part c), by students #3 and #5. These two students were trying to apply the procedures they were learning in class on "consecutive integer problems" at the time the interviews were given, to the "age problem" of question 5-D4.

We would now like to shift the discussion to a group of short word problems given in Part B of Interview 6. This group consists of a set of 14 problems designed to investigate a) the translation process from a textual to a symbolic representation, and b) the question of whether a letter in an equation is treated as a variable or a label by the students. The variable-label question has already been discussed in Section 1 within the context of problems 6-B1, 10, and 11 of this set, and we will thus restrict our discussion now to the translation process.

It would be helpful to divide the discussion into two parts, the first dealing with problems 6-B2,3,4, and 5, and the second dealing with problems 6-B6,7,8, and 9. The former four problems are listed below:

Write an expression using variables for the following statements:

6-B2. A number added to 7 equals 18.

6-B3. Six times a number is equal to a second number.

6-B4. Nine times a number results in 36.

6-B5. In seven years, John will be eighteen years old.

To aid in the discussion of these problems, we have coded the performance of all 14 students interviewed in Table 5. The letter "C" in Table 5 means that the student worked out the problem correctly; the entries "skipped" and "no idea" meant that the problem was skipped during the interview for that student, and that the student tried to solve the problem but had no idea what to do. Any other entry denotes the student's response in the problem.

Table 5 shows that students had little difficulty working out problems 2, 3, and 4, but that problem 5 caused inordinate difficulties. There is an obvious difference between problems 2, 3, and 4, and problem 5. In 2, 3, and 4, the problem structure is very clear -- the unknown to be represented by a variable always appears near the beginning as the noun "number". The remaining problem structure directs the student on what should be done with this unknown, for example, "a number added to 7...", or "six times a number...". On the other hand, the structure of problem 5 does not make obvious what the unknown is; the unknown, John's current age, must in fact be deduced. Evidently two students, one who wrote "7+11=18" and the other "7+11", understood the problem but were not able to write an equation using a variable. It again appears that variations in the syntax of simple problems has an observable effect on performance. In particular, if the unknown, or quantity being sought in the problem is not readily discernible, there is a higher likelihood for an error than if the unknown appears clearly near the beginning of the problem statement.

Before leaving these four problems, we will like to point out that some of the errors made in problem 6-B3 are very suggestive of semantic difficulties. The participating Hispanic who wrote 6N=N explained that due to the phrase "is equal" in "six times a number is equal to a second number", both numbers must be the same, and therefore N was used to represent both. We conjecture (without evidence) that this was the reasoning that caused one advanced Hispanic to write the equivalent answer, "6X=X". Further, the participating Hispanic who wrote 6X=2 explained that the "2" in this equation represented "the second number"; the "second" was apparently interpreted as "2". Again, we surmise (without evidence) that the participating Anglo who wrote "6A=2" used similar reasoning.

TABLE 5: STUDENT ANSWERS

		Problem #			
		6-B2	6-B3	6-B4	6-B5
	Correct Response	X+7=18	6A=B	9Y=36	J+7=18
Participating Hispanics	1	No Idea	6X=12 (12 was from 6 times second #)	C	Skipped
	2	7=18	6X=2 (second # was 2)	9·4=36	No Idea
	3	C	C	C	C
	4	C	6N=N	C	7+11=18
	5	C	6A=2A	C	7A·18
	6	B+7	6N·2N	9N+6=36	8+9=18
Participating Anglos	1	C	C	C	C
	2	C	C	C	7+A
	3	C	6A=2	C	7+11
	4	11N+7=18	6N=2N	9Ax4N=36	7=18
	5	C	C	C	X-7=18
Advanced Hispanics	1	C	C	C	C
	2	C	6X=X	C	y+7y=18
	3	C	C	C	C

66

The participating Hispanic who wrote 6X=12 explained that the "12" came from multiplying the "second number" by 6. Finally, there were 3 students who wrote variants of the form 6N=2N. Although we do not have first hand evidence to confirm our guess, it appears that these students were representing "six times a number" by 6N, and the "second number" by 2N.

Let us now move to problems 6-B6, 7, 8, and 9, as shown below: Write an expression using variables for the following statements:

6-B6. The number of nickels in my pocket is three times more than the number of dimes.

6-B7. The number of math books on the book shelf is equal to eight times the number of science books.

6-B8. There are four times as many English teachers as there are math teachers at this school.

6-B9. Last year, there were six times as many men cheating on their income tax as there were women.

We have again coded the responses to this question in Table 6. The focus of our discussion will be on the "variable reversal error", coded as "Reversal" in Table 6, which was briefly discussed in Section A of this chapter.* The syntax of problems 6 and 7 are such that a left-to-right sequential translation should yield the appropriate answer, for example,

"the number of nickels in my pocket"	"is (equal to)"	"three times the number of dimes"
N	=	3 D

A similar left-to-right translation of problems 8 or 9 will result in the variable reversal error, for example,

"there are four times as many English teachers"	"as there are"	"math teachers at this school"
4E	=	M

* We urge the reader to review the discussion of the "variable reversal" error from the Variable vs. Labels section of section A, in this chapter.

TABLE 6: STUDENT ANSWERS

		Problem #			
		6-B6	6-B7	6-B8	6-B9
	Correct Response	N=3D	M=8S	E=4M	M=6W
Participating Hispanics	1	No Idea	Skipped	No Idea	No Idea
	2	No Idea†	8·6=48*	4·4=16	2·6=12 (2=women, 6=men)
	3	N·D=	8xS=-8	4xS=-4	6·W=
	4	C	C	Reversal	Reversal*
	5	3N+D	Reversal	Reversal	Reversal
	6	3d·4+N=	64M·8·8=	4MT=20ET	Reversal
Participating Anglos	1	3N+D=X*	M·8S=B* (B was tot. books)	4·E·M=T*	Reversal*
	2	a>3·B	Reversal*	Reversal*	Reversal*
	3	N^3·D	b=S^8	E^4·M	Reversal
	4	3N>10	N=8	Reversal	Reversal
	5	C	C	Reversal*	Reversal*
Advanced Hispanics	1	C	C	C	C
	2	C	C	Reversal	Reversal
	3	X^3 more than N	X^8 more than N	X^4·N=X=1*	X^6 more than N

Note: Students appear in the same order as in Table 5.

†Upon prompting, student displayed an improper understanding of the relative sizes among the two quantities in the problem statement.

*Upon prompting, student displayed a proper understanding of the relative sizes among the two quantities in the problem statement.

We would thus expect more correct answers on 6 and 7 than on 8 and 9, and further, we expect that those who translate problems using the sequential left-to-right method would likely get 6 and 7 correct but 8 and 9 variable-reversed. This was borne out by the results; four students obtained both 6 and 7 correct as shown on Table 6, but of these four students, only one advanced Hispanic was able to get 8 and 9 correct.

We should make one important distinction with respect to the order of occurrence of two separate events. The question is, does the proclivity to treat variables as labels, such as problems like "the students and professors", a problem discussed in section A of this chapter, or problems 8 and 9 above, come <u>before</u> writing a variable reversed equation, or does the proclivity to treat variables as labels come <u>after</u> the student has obtained a variable reversed equation by a sequential left-to-right translation of the problem? We believe the latter to be the case. <u>First</u>, the student translates the textual portions of the problem sequentially and arrives at a variable-reversed equation. <u>Then</u>, the student's focus shifts from the problem <u>to</u> the equation, and henceforth, the "variables" are treated operationally as labels by the student. This is why many students who write variable-reversed equations at both the college level and in this study have an accurate understanding of exactly what is stated in the problem, and yet believe that their equation is also an accurate representation of this. Clement (1982) showed in clinical interviews with college engineering students solving the "students and professors" problem that many of these students are perfectly aware that there are more students than professors, and think 6S=P reflects this situation. Similarly, Table 6 shows that in those cases where the interviewer prompted the student about the relative sizes of the quantities in the problem statement, there was only one instance where a student showed an incorrect interpretation; that is, participating Hispanic student #3 in problem 6-B6 (marked with "†") thought that the problem stated there were more dimes than nickels -- in all other cases where prompting on the relative size of the problem quantities occurred (marked with "*"), the students displayed knowledge of the relative sizes of the two quantities in the problem statement.

There is one surprising result in Table 6. If we group all the variable-reversal errors together with all correct responses for a particular problem, then this corresponds to all those students who at least understood that the problem was asking for an equation relating two quantities in some specified proportion. There were a total of 4 and 6 students in this category in problems 6-B6 and 6-B7, respectively. However, in problems 6-B8 and 6-B9, there were 6 and 10 students in this category, respectively. Thus, if this aggregate correct-answer/reversal-error category is taken as a measure of "understanding" what the problem is asking, it appears that questions 6-B8 and 6-B9 were easy to understand in comparison to questions 6-B6 and 6-B7. We find this surprising since questions 6-B6 and 6-B7, albeit somewhat harder, have the same structure as questions 6-B2, 3, and 4, (see pages 56 and 57), which students found easy to work out. A possible explanation for this result is that the phrasings in problems 6-B6 and 7 are those used in formal mathematics and therefore more arcane than the colloquial phrasings of problems 6-B8 and 9.

Finally we would like to offer a possible explanation for the "unexpected" answers in problem 6-B6. By unexpected answers in 6-B6 we mean the three answers with the inequality "greater than", and the four answers which were not in the form of an equation, e.g., "$N^3 \cdot d$" and "N·D=". It should first be noted that these types of answers did not occur with anywhere the same frequency in problem 6-B7 -- a problem fairly equivalent in structure to problem 6-B6. It appears that the biggest difference between these two problems is that in 6-B6, the word "equal" was never explicitly used as was the case in problem 6-B7; the equality in 6-B6 had to be deduced from context. Those students that wrote their answers as inequalities seem to have interpreted the phrase "is three times more than" as a statement of inequality rather than as a statement of equality. Lochhead (1980) has pointed out that implicit in algebra is the ability to write equations to relate quantities which are in fact not equal, but can be made equal by using appropriate weightings (e.g., in the "students and professors" problem, even though the two quantities, S and P, are not equal, they can be related via the equation P=6S).

Performance as a Function of Word Problem Length

Our assessment of how performance was affected by the amount of semantic processing required by a word problem was made by designing two sets of problems, each set containing four problems. In the first set, the problems were terse, while in the second set, the problems were verbose and often contained information irrelevant to the solution. Algebraically, the structure of the problems were completely equivalent across the terse-verbose sets. It was expected that students would perform better in the terse problems than in the verbose problems. The two sets of problems, given during Interview 8, are shown below:

Terse

8-B1. A number added to 11 equals 23. What is the number?

8-B2. Twice a number is 26. What is the number?

8-B3. Six times a number is equal to a second number. If the second number is 48, what is the first number?

8-B4. A number multiplied by 4 results in 36. What is the number?

Verbose

8-C1. In a recent survey conducted among the teachers of this school, it was discovered that 11 teachers wanted to continue school through the summer. How many more teachers would be needed to raise this number to 23?

8-C2. Different bicycles have different size wheels. Some have very small wheels (only 10 inches high) and some old bicycles have a front wheel over 6 feet tall. Many modern 10 speed bicycles have wheels 26 inches high. What would be the distance from the ground to the center of a 26 inch bicycle wheel?

8-C3. An article in a recent medical magazine states that college women who smoke cigarettes are six times as likely to have lung cancer as college women who don't smoke. If in a large eastern state university there are 48 smoking women with lung cancer, how many non-smoking women with lung cancer are there likely to be?

8-C4. In a small blood sample analyzed at the local hospital, there were 36 blood cells found. It was also found that the number of serum antibodies was smaller than the number of red blood cells by a factor of 4. How many serum antibodies were found in this blood sample?

A summary of the students' performance is shown in Table 7. We have added a different code, "H", to mean that the problem was worked out in the student's head without any algebraic set-up.

Two aspects of Table 7 are surprising. First, even though there were more correct answers in the terse set as was expected, the performance in the verbose set was not significantly worse for any group. This is surprising in view of a recent study with Hispanic college students majoring in the sciences, where we found statistically significant differences in student performance between terse and verbose sets of algebra problems of slightly higher difficulty than the problems given to the ninth grade students of this study (Mestre, 1982). Second, the number of students that used algebra in solving these problems was extremely low. This was perhaps more discouraging than surprising, particularly since this interview was given in May near the end of the school year. It appears that even after completing one academic year of algebra, students still prefer to work out problems using non-algebraic techniques. This finding is consistent with a recent study by Karplus, Tourniaire, Pulos, and Stage (1982) who found that eighth grade students will not use algebraic techniques unless problems are sufficiently difficult that non-algebraic techniques would prove an inefficient procedure for obtaining an answer.

There were some difficulties worth mentioning in the verbose set which we believe were a consequence of the increased verbosity. We mention three such difficulties. The first refers to the response on problem 8-C2 by the second participating Hispanic on Table 7. By writing "10-6=16" this student displayed that he was trying to incorporate all the irrelevant information given in the problem, such as "... some have very small wheels (10 inches high) and some old bicycles have a front wheel over 6 feet tall..." into his

TABLE 7: STUDENT ANSWERS

		Terse Problems				Verbose Problems			
		8-B1	8-B2	8-B3	8-B4	8-C1	8-C2	8-C3	8-C4
	Correct Response	X+11=23 X=12	2X=26 X=13	6X=Y Y=48 X=8	4X=36 X=9	12	13	8	9
Participating Hispanics	1	C,H	C,H	C,H	C,H	C,H	C,H	C,H	C,H
	2	Said "23"	C,H	C,H	C,H	C,H	10-6=16 16-26=10	48-6=42	36+4=40
	3	C,H	C,H	C,H	C,H	C,H	C,H	6x48=288	C,H
	4	C,H	C,H	C,H	C,H	C,H	C,H	48x6=288	C,H
	5	A+11=23 A=23/11	2A=26 A=26/(-2) A=-13	6A=48 A=-8	4R=36 R=36/(-4)	11+t=23 t=23-11 t=12	No Idea	6A=48 A=48/6	Said "32"
	6	C,H	C,H	C,H	C,H	C,H	Said "30"	Said not enough info. given	C,H
Participating Anglos	1	C,H	C,H	C,H	C,H	C,H	C,H	C,H	C,H
	2	C	C	C	C	C,H	C,H	C,H	C,H
	3	C,H	C,H	C,H	C,H	C,H	C,H	C,H	C,H
	4	C	C	C	C	C	10S+26=26 S=0	6+C=48 C=42	C
	5	C,H	C,H	C,H	C,H	C,H	C,H	6x48=288	C,H
Advanced Hispanics	1	C,H	C,H	C,H	C,H	C,H	C,H	Said "48"	C,H
	2	C,H	C,H	C,H	C,H	C,H	C,H	C,H	C,H
	3	6,H	C,H	$48+X^6=N$	C,H	C,H	C,H	C,H	C,H

Note: Students appear in the same order as in tables 5 and 6. An entry of C,H implies a correct answer worked out in the student's head. An entry of C implies a correct answer using algebra.

solution. Being able to sort essential from nonessential information in a problem is a necessary tool for successful problem solving.

The second difficulty concerns the response given by the sixth participating Hispanic in problem 8-C3. This student allowed his personal opinion to bias his understanding of the problem, as we can see from the following quote from his protocol:

"I don't know 'cuz they don't say how many women are at the college. They say 48 of them smoke. It could be, like, 200 girls, or more women than that. Colleges have a lot of women and there are, like, 48 of them that smoke and there's a lot more that are there, so maybe other people don't smoke"

We are neither sure of the cause of the third difficulty we are about to discuss, nor are we sure why there was a difficulty at all. We are referring to the fifth participating Anglo's response to problem 8-C2. A detailed description of the sequence of events leading to this student's answer would be elucidating. Initially, the student wrote "10S+26=26", solved for S and did not like the fact that S turned out to be 0. The interviewer then asked her to draw a picture depicting the situation, and she drew an accurate picture of a wheel and by pointing, displayed that she understood the distance that was being asked for in the problem. Next she rewrote the same equation above, this time using the variable I instead of S, where presumably I stood for inches. Again she solved for I and did not like the fact that I turned out to be 0 for the second time in a row.* At this point, the interviewer asked her to talk about what she thought the problem was saying and she read and explained the problem extremely accurately, even going as far as stating that the information about the old bikes having 10 inch wheels and the part about the 10 speeds had nothing to do with the problem. She finished by saying that what the problem was asking was the "distance to the center of the wheel". After all this, she could not figure out that all she needed to do was to divide the distance from the top of the wheel to the ground by 2. She finally estimated the answer by looking at the picture she had drawn of

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* That students are often not aware that merely changing the variable name in an equation does not change the solution to the equation has been observed by Wagner (1981).

the wheel.

English vs. Spanish Performance

Several questions were constructed to investigate whether there existed observable differences in problem solving style, confidence, anxiety, etc. for the partipating Hispanics when their performance in solving problems in English was compared to their performance in Spanish. Since the Algebra course was taught completely in English, all information, vocabulary, and experience in this new subject will be associated with English. This may mean that these students are "coordinate bilinguals" with respect to algebra.

There are numerous theories as to how language is represented in the brain. One such theory, compound vs. coordinate bilingualism (see Segalowitz, 1979 for an excellent review) suggests that this representation is dependent upon the type of upbringing to which the individual has been exposed. A compound bilingual associates the same meaning for a word, concept, etc. in both languages due to exposure of that word or concept in both languages simultaneously. On the other hand, a coordinate bilingual has separate meanings for the same word, concept, etc. in two languages; this situation arises when the bilingual assimilates information from two very different settings. In the case at hand, we were interested in how the participating Hispanics approach a word problem written in Spanish given that all their previous associations in solving word problems had been in English.

To study this, the following questions were used during Interview 5:

5-E1.	La suma de dos números enteros consecutivos es 51. Cuales son los números?	The sum of two consecutive integers is 51. What are the numbers?

5-E2. Work out the following two problems:

a) Si lápices cuestan 5¢, cuantos puedes comprar con $2.05?

b) How many 15¢ candy bars can you buy with $1.65?

5-E3. Write an equation for the following two statements:

a) Dos veces un número sumado a 3 es igual a 5.

b) Doce mas que el doble de un número resulta en 6.

5-E4. El costo de una casa y un terreno es $40,000. La casa cuesta siete veces mas que el terreno. Cuanto cuesta la casa y cuanto cuesta el terreno?

5-E5. a) John has 6 times more quarters than dimes. How many dimes does he have if he has 18 quarters?

b) En una fiesta hay 8 veces mas botellas de cerveza que de vino. Cuantas botellas de vino habrán si hay 16 botellas de cerveza?

c) The larger of two consecutive even integers is six less than twice the smaller. What are the numbers?

d) El menor de dos números enteros consecutivos es uno mas que el doble del mayor. Cuales son los números?

In the first problem above, the students were asked to read the equivalent problems in English and Spanish and state whether they found the Spanish version hard to understand. Only one student out of the six stated that he could not understand the problem in Spanish well enough to attempt a solution. The rest of this discussion will be restricted to the remaining five participating Hispanics.

The performance of the students in the Spanish problems was not observably different from their performance in the English problems. The styles used in solving the Spanish problems differed somewhat across the students. Two students read the Spanish problems and proceeded to try and solve them directly. One student, who had not had any formal training in Spanish, was not comfortable reading the problems silently; instead, she "sounded out" the problems, that is, she read them aloud, until she aurally understood what the problem was asking. The remaining two students translated the Spanish problems into English, and would not start on the solution until they understood the English translation. The former two students stated that they were equally comfortable reading the problems in either English or Spanish, while the latter three stated a preference for the English versions. All five students encountered some Spanish words they did not recognize, and asked about their meaning; once told, they had no further difficulty understanding what was being asked in the problem. That is not to say that once they understood what was being asked, they were successful at solving the problem.

These differences in approach, and unfamiliarity with certain vocabulary words did not, per se, create any additional difficulties in the ability of students to solve problems. We observed nothing in their overall performance to indicate that they could solve problems in Spanish any better, or worse than in English. We should caution, however, that if these students had been solving word problems in Spanish totally on their own, it is very likely that their performance would have been worse than in English due to vocabulary difficulties, or inappropriate translations. Given that these students have been mainstreamed for some time and receive all their instruction in English, it is not surprising that the majority prefer to work out word problems phrased in English.

E. Students Explaining Mathematical Concepts to Each Other

Up to this point, no mention has been made of Interview 7. This interview session was entirely different in style from the other interviews, and therefore does not fit into the discussions of the previous four sections. In this interview, we were interested in how students communicated mathematical concepts among themselves. It was our hope that by listening to how a student explains a newly learned concept to a peer that has not seen this concept before, we would be able to determine what the student had deemed essential, or important enough to teach to one of his or her peers.

The approach taken was to select 6 topics which we were confident the students had not encountered before. The three interviewers chose two topics each and prepared "mini-lessons" to be presented to the students. A typical interview session consisted of two simultaneous but separate presentations of two different topics by the interviewers to two students. Each presentation took approximately ten minutes. After both interviewers were finished with the mini-lesson presentation, the two interviewers and the two students got together as a group, and each student in turn would teach the other what they had just learned in their mini-lesson. When they were finished, the group split back up into the original interviewer-student pair and a short quiz, both on the topic the student had learned from the interviewer, and on the topic the student had learned from his or her peer, was administered to the student. All six participating Hispanics were interviewed in this fashion.

There were various precautions taken to ensure consistency among the mini-lesson presentations given by the interviewers. For example, each interviewer presented both of their mini-lessons during our weekly staff meeting in order to evaluate and critique the presentation. The mini-lesson approach selected was designed to optimize on clarity of presentation and simplicity of vocabulary. There was a stipulation made that the interviewer have some assurance that the student had grasped the "essence" of the presentation, otherwise it would be pointless to have the student teach something about which he/she has no understanding.

The following were the six topics that were used in the mini-lesson presentation:

1. Bar graphs. Students were shown how to read, interpret, and draw their own bar graphs.
2. Operations. Several operations were defined, such as "*", defined as $A * B \equiv 2A + B$, and students were asked to use and manipulate them. Students were also asked to deduce an operation by giving them the result of the operation on two numbers.
3. Square Roots. How to take simple square roots as well as how to simplify expressions such as $\sqrt{8} = 2\sqrt{2}$ were covered.
4. Circles. The topics of radius, diameter, and circumference, and mathematical relationships between them were covered.
5. Angles. Here the student was taught how to measure an angle using a protractor, and why a ruler could not be used for this.
6. Linear Graphs. The student was taught how to plot a linear graph and the meaning of "y intercept" and slope.

Before proceeding to discuss the results of these interviews, a word of caution is in order. The situation contrived to have these students talk to each other about mathematical concepts was not one which would naturally occur. Further, there were some constraints that could not be removed. For example, the mere presence of the interviewers while the students were teaching each other may have inhibited the students and forced them to communicate in ways "shyer" than they would have otherwise communicated. We are not aware of whether these students ever talk to each other about math, or of the communicative techniques they use if in fact they do talk to each other about math under more "natural" circumstances. The findings from this "experiment" may therefore not be indicative of how a student would tutor a peer. Perhaps this experiment is useful only in allowing "us adults" to perceive how we sound to "them kids".

Summarizing the "experiment" we can state that the teaching style used by the students did not differ appreciably. There were, however, distinct differences between the style used by the students and the style used by the interviewers. The most obvious contrast between the two styles was that whereas the interviewers were attempting to impart knowledge, the students were attempting to impart information. At no time did the students attempt to "motivate" the subject they were teaching to their peers. The students' presentations were geared toward answering the latter two questions of the "three question hypothesis" posed in Section A , namely, "what can you do with it?", and "in what does it result?" At no time did they attempt to answer the first question, namely, "what good is it?" Perhaps the best way to illustrate this contrast is to follow one protocol of an interviewer teaching a student, "Carlos", about circles, and how Carlos teaches this material to another student, "Ana".

I: OK. What we are going to do is--I am going to teach you something which will take me about 5 or 10 minutes. He (referring to another interviewer at the other end of the room) is teaching Ana something different from what I am teaching you, and then we will all get together and you will teach Ana what I taught you, and she will teach you what was taught to her by David. Then we will get separated and I will give you a quiz on everything -- what I taught you and what Ana taught you.

C: (Chuckles) OK.

I: All right, we are going to talk about circles. Let's first draw a circle with this compass (draws circle). That (pointing to the hole left by the compass) is the center of the circle. Let's get this ruler 'cause we are going to need it soon. Now there are various relationships which are important. Before we talk about that, let's talk about the names of some of the parts of this circle. For example, the distance from the center of the circle to any place on the circle itself is the same no matter where you measure it and it is called the "radius". This circle here happens to have a radius equal to 2 inches (the circle radius is being measured with a ruler during this). So let's give that distance from there (pointing to the center of the circle) to there (a poing on the circle) the symbol R. Another length that's important in a circle is called the "diameter". That length is from one side of the circle to the other, but it must go through the center, otherwise you could--sort of go anyplace you wanted. So if I take a line that goes through the center (draws one such line with the ruler) then the distance from there to there (points) is the diameter. You can see that there are many diameters since you can draw a line like this any place you want--they are all the same length. Now you can see that from here to here (points to distance from circle to center along the diameter line) is a radius and from here to here (points to distance from center of the circle to the other side of the circle along the same diameter line) is another radius, so we can write a relationship that says that a diameter is equal to two of these (pointing to the drawing). So you can write a relationship which says that twice the radius is equal to the diameter:

$$2R=D$$

So the radius of this circle is 2. And so the diameter is? (we measure it with a ruler)

C: Four.

I: Right. Now the last thing we are going to talk about is something called the "circumference". The circumference is the distance around the circle and we could measure it with some kind of circular ruler if we had one. So if I start here (points at a point on the circle), it is the distance completely around and back to here again (denotes this distance along the circle with finger). Now I brought one of these tapes (shows Carlos a measuring tape) and we are going to measure the circumference of this circle (points to a circle made out of a block of wood). But first, let me tell you something else. The nice thing about circles is that there is a relationship between the circumference, the distance all around here, and the diameter <u>for any circle</u> no matter how big or small the circle is. I will tell you what that is-- if you take the circumference and divide it by the diameter you always get close to this number:

$$\frac{C}{D} = 3.2^{*}$$

So this is one of those things that are true--sort of a law of nature. I'll show you what that means. (interviewer grabs the flat, wooden, cylindrical "circle" and using the ruler measures the diameter). Now the diameter of this circle is?

C: Six

I: Yes, six inches. Now the circumference (takes measuring tape and wraps it around the wooden circle to measure it) is (showing Carlos) about 19.5 inches. So let's divide that now. The circumference is 19.5 and we divide that by the diameter which is six inches:

$$\begin{array}{r} 3.25 \\ 6\overline{)19.5} \\ \underline{18} \\ 15 \\ \underline{12} \\ 30 \\ \underline{30} \\ 0 \end{array}$$

so you can see that it's close to 3.2. If I ~~had~~ done it a little more carefully, it would have come out closer.

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*We decided to use 3.2 rather than the actual value of π for simplicity, and because the measurements we took at the staff meeting resulted in answers very close to 3.2.

C: Yeah.

I: So you can see it works. And it doesn't just work for this circle. It will work for this one too (points to a smaller wooden circle). What's the diameter of that circle?

C: (measures it) Four.

I: Why don't you try measuring the circumference--you'll probably be able to hold that tape better than I. (Carlos wraps the tape around the circle).

C: Twelve point five.

I: So let's try that again--if we take the circumference which is 12.5 and divide that by the diameter which is 4,

$$\begin{array}{r} 3.12 \\ 4\overline{)12.5} \\ \underline{12} \\ 0\;5 \\ \underline{4} \\ 10 \\ \underline{8} \\ 2 \end{array}$$

So again, you can see that it is not quite 3.2 but it's close. It would be better if we made our measurements very accurately. One last thing. You know that C/D = 3.2. And you know from here (points to eqn. 2R=D) that the diameter is just two "radiuses". So we can write,

$$\frac{C}{2R} = 3.2$$

C: (Nods approval).

I: Now what all this is telling you (points to series of equations:

$$\frac{C}{D} = 3.2$$

$$\frac{C}{2R} = 3.2)$$

is that if I give you a circle, and you measure the diameter, you can tell me the circumference without having to measure it. You just take D times 3.2 to get C (points to the first equation above). If I give you another circle, and I tell you the circumference, and I ask you "what's the diameter?" you can just take the circumference and divide it by 3.2

after putting D on the other side (points to first equation above). So that says that any two of these things that I give you, you can give me the other right away.

C: uh huh.

I: That's what's useful about it. Do you have any questions?

C: No

I: Explain to me then, if I had a circle like that one (points to the original circle drawn on the sheet of paper with the compass) and I tell you the radius is 4 inches, how would you find the circumference without measuring it?

C: By the diameter

I: What would be the diameter?

C: Eight (Carlos now writes $\frac{C}{8}$ = 3.2 and stares at it for a little while)

I: Can you put C by itself?

C: (Carlos now multiplies through by 8 arriving at C = 8 x 3.2 and answers) It's 8 times 3.2.

I: So you understand circles?

C: Yeah

I: O.K. Let's stop here and when David is through teaching Ana, we will get together and you can teach Ana about circles.

The following is the protocol of Carlos teaching Ana the same lesson:

C: This is a compass. (Draws a circle with the compass). We got a circle here, right? We will measure from the thing, like from one side to the other side to get the diameter, and from here to here's the radius, or whatever they call it. Now we measure from... let's see... from here to here (uses ruler to measure the radius of the circle) we get 2 inches, right? That'd be R. And from this side, from one side of the circle to here (points with his finger to a distance corresponding to a diameter) is the diameter...it could be either ~~side~~. It will be 4. So it will be R=2 inches and D=4. And see over here, we just got, like, it's two radius that equal...that make up the diameter. And this (pointing with his finger around the circle to denote the circumference) is, uh.. let's see.. (after a short pause, Carlos looks to his interviewer as if asking for help in remembering the word for that distance)

I: Circumference

C: Circumference is..uh.. measurement for the whole circle. See, you start here (points to a point on the circle) and just measure all the way around the circle, and you will get that. See, if you divide...(at this point Carlos looks at his interviewer and asks) say it that way? Just look at it and say it?

I: Say it as best as you can. Explain it to Ana as best as you can.

C: OK. We divide that, um, the circumference and the diameter and it would equal 3.2. And if you measure this it would come close (pointing to one of the circles), if you measure any of these it would come close to it. Let's see. Six, diameter (He is measuring the diameter of a wooden circle and writing down the diameter as D=6. Then he takes the tape and wraps it around the wooden circle to obtain the circumference. He appears to be engrossed in this process and does not let Ana know what he is doing).

I: Show Ana what you are going, don't just do it.

C: Yeah, I'm just getting it ready (meaning all the measurements before he shows Ana the relationship C/D = 3.2). Measurement's 19.5 (referring to the measurement of the circumference). I'm measuring that (pointing to the distance around the circle) (at this point he turns to the interviewer and adds the parenthetical remark) and I got it easier than you this time (smiles). And you would divide 19.5 by 6. Three..eighteen.. one point five, and six goes into 15.. 12 and you get 3 (he is saying this as he writes:)

$$\begin{array}{r} 3.25 \\ 6\overline{)19.5} \\ \underline{18} \\ 1.5 \\ \underline{1.2} \\ .30 \end{array}$$

It comes close, and you put it down... it comes to whatever it comes to here (pointing to the 3.2 in the equation

C/D = 3.2). You measure that one (gives Ana the ruler, tape, and another wooden circle).

A: It's twelve around (she is using the tape to measure the circumference. Now she takes the ruler and begins to measure the diameter. At this point she is not sure how to read the divisions on the ruler and asks Carlos:) The little ones or the big ones, or...?

C: (points) this one right here.

A: Uh, threeee...

C: Make it as close to the bigger number as you can—that way.

A: Four.

C: All right. 4 into 12 and you get...3? (at this point Carlos is somewhat puzzled because when he did it, the diameter and circumference were measured to be 4 and 12.5, respectively, giving the answer 3.12, instead of the answer 3 which Ana just obtained. Ana senses that something went wrong).

A: A mistake?

C: No. (He does not sound too convinced as he says "No").

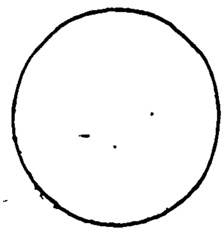
I: Well you're measuring those things kind of casually. If you do it really accurately, it would come out closer.

C: I'm done.

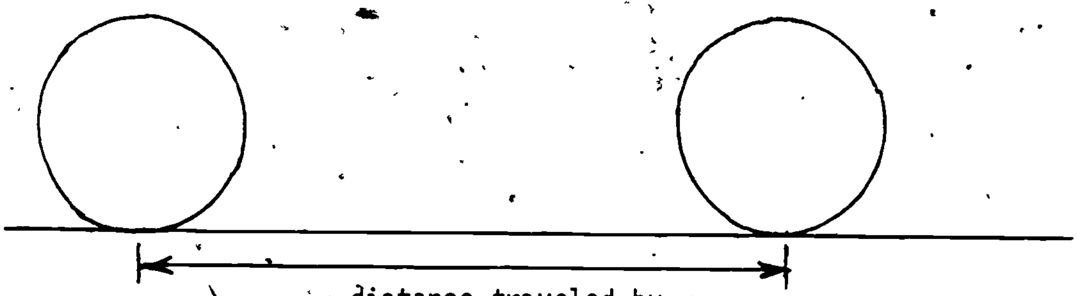
A: Oh, No! (the Oh, No! refers to the fact that now it is her turn to teach Carlos what she was taught).

For the sake of completeness, the quiz administered at the end of the students' teaching session is shown below:

1. If the diameter of a circle is 10 inches, what would be its circumference?
2. If the radius of a circle is 4 inches, what would be its diameter?
3. Find the radius, diameter, and circumference of the following circle:



4. If the radius of a certain circle is 6 inches, what is its circumference?
5. If you buy a 27 inch bicycle, this means that the diameter of the bicycle wheel is 27 inches. This also means that the radius is 27/2 = 13.5 inches and the circumference is 86.4 inches. In the picture below is a wheel of this bicycle. How far does the wheel move after it makes one complete turn?



In this quiz, Carlos could not do any of the problems dealing with circumference. In problem 1, he wrote "$\frac{C}{D} = 3.2$, $D = 10$" but could not solve for C. He obtained the correct answer in problem 2, and measured the diameter and radius correctly in problem 3. His performance in problem 4 was similar to that in problem 1. Problem 5 was put on the quiz to see if students could generalize their knowledge to a relatively "obscure" example; in this particular instance we wanted to see if the student realized that the wheel moved one circumference in making one complete revolution. Carlos looked at the picture, thought for a minute, made a gesture with his hand while looking at the floor as if estimating the distance he thought a bicycle wheel would actually travel in making one complete turn, and answered "about a foot". As far as Ana's performance goes, she was totally lost with respect to any problem dealing with relationships between C, R, and D. She, however, was able to measure the diameter and the radius in problem 3.

We will make one observation before concluding this discussion. From both, Carlos' teaching, and his quiz performance, it appears that in his own mind, he had answered the questions "what can you do with it?" and "in what does it result?" He knew that what you did was measure certain quantities, and plug them into an equation which resulted in something close to 3.2. It appears that Ana was only able to answer the question, "what can you do with it?" after Carlos' presentation, namely, you take a ruler and tape, and measure certain quantities on a circle. Other than that, she had little idea what you did with these quantities once you measured them.

We will offer one explanation to this. In the interviewer's attempt to be pedantic and answer the question "what good it is?" as well as the more practical questions "what can you do with it?" and "in what does it result?", it appears that he was only successful in answering the latter two. The attempts to teach the student that what was good about these relationships was that they held for <u>any</u> circle, and that they allowed you to compute any pair of the quantities C, R, and D, by knowing only one of them, were not successful. Carlos, in his distillation of the lesson, came away with a "working knowledge" of circles, even if he was not successful at carrying out some of the algebraic manipulations. On the other hand, Ana's distillation from Carlos' somewhat terse and imprecise presentation consisted of learning the names for quantities like "radius", "diameter", and "circumference" and being able to measure them. There appears to be a substantial amount of attenuation of information between the original presentation by the interviewer and the final distillation by Ana.

If this "attenuating distillation process" occurs often, then to assure that students come away with at least a working knowledge of a topic, the topic must be presented at a much higher, complete, and rigorous level than the level that the student is actually expected to assimilate. The distillation process itself will guarantee that a good portion of what is presented (in fact, probably the "what good it is?" portions) will be lost. A presentation aimed only at giving a working knowledge may not be successful at imparting this working knowledge after distillation by the presentee, unless perhaps reinforced by substantial drill work.

V. DISCUSSION

The fact that a large fraction of the errors we encountered have also been found in other investigations with non-minority students (Davis, Jocusch, and McKnight, 1978; Matz, 1980) means that Hispanics are not unique in what they find difficult about algebra. It is true, however, that the participating Hispanics were at a disadvantage in language proficiency as evidenced by their poor performance on the language portions of the California Achievement Tests. That this deficiency serves as an impediment in the learning and problem solving processes is, we believe, apparent. On the other hand, the impressive performance of the advanced Hispanics on the CAT and on the interview problems certainly dispels any question of intrinsic mathematical limitations for Hispanics.

For the rest of this chapter we will focus our discussion on specific topics. Before proceeding further, however, it would prove helpful to summarize the salient findings of this study:

- Students prefer not to use algebraic techniques in solving problems.
- Students are extremely poor at verbalizing definitions of mathematical terms, even when they possess a correct operational definition of the term.
- Students can often obtain a solution to a problem, but can seldom verbalize the procedure they used in obtaining the solution.
- Problem syntax is often the most important factor in determining problem difficulty.
- The step that students find most difficult in solving word problems is the translation of the problem statement into the appropriate mathematical equation(s).
- Students do not use their textbooks very much except as a place to find assigned problems.
- Students treat algebra as a rule-based discipline and not as a concept-based discipline.
- When applying algebraic rules, students often do not apply them self-consistently.

A. The Textbook

The criticisms we are about to make of the textbook should not be taken as an indictment of this *particular* textbook. The Dolciani and Wooton textbook that was used in the course is not atypical of beginning algebra books (or mathematics books in general). It has been, and still is, one of the most popular texts used in beginning algebra courses.* As we stated in our summary above, it became apparent soon after the beginning of the study, and was confirmed by the participating teacher, that the majority of all the students in the class did not read the textbook as a means of supplemental instruction--they merely used it as a place to find the problems that were assigned for homework.

The fact that students did not read the text is not something that can be easily blamed on the text. Perhaps the teacher should have made more of an effort to hold the students responsible for reading the book. What we can say is that the language and style of the textbook seems more appropriate for someone who already knows a little about algebra, than for 14 year old students with absolutely no prior training in algebra. The book does attempt to convey the precision necessary to "communicate" in mathematics. We believe, however, that the "incomplete" use of the text by the students may be responsible for some of the difficulties we uncovered, as we hope the following examples will illustrate.

As discussed earlier in Chapter IV, Section D, the most misinterpreted term we encountered was "quotient." Popular interpretations given by the students to the term quotient were "answer" and "product". In the section of the textbook dealing with quotients (Chapter 4, section 4) we find the following instructions given in the quotient exercises:

"Read each quotient as a product. Then state the value of the quotient"

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*We will note in passing that a slightly different edition of the Dolciani and Wooton textbook used in the Algebra I class of this study is the textbook used in the Algebra I class of the study being conducted by Jensen and Wagner in Georgia.

and

"State the value of each quotient".

For students reading these instructions, it is understandable why they might come away with the interpretations of "product" and "answer" for quotient -- the first instruction above can be taken to imply that "product" and "quotient" are interchangeable; the second instruction makes perfect sense if one substitutes the word "answer" for "quotient".

Further, the text often attempted to draw upon "real life" situations for its word problems. This attempt to be relevant may be of questionable pedagogic value for students such as the participating Hispanics of this study. Even though we have no formal measure of socioeconomic status (SES) for the participating Hispanics, the profile in Appendix II indicates that these students are "below average" in SES. The following two problems from the textbook (found on page 76) were used in Interview 3 to assess the understanding that the students had of the vocabulary, and will illustrate the point we are trying to make.

"A stock selling for $30 per share rose 2 dollars per share each of two days and then fell $1.75 per share for each of three days. What was the selling price per share of the stock after these events?"

"On a revolving charge account, Mrs. Dallins purchased $27.50 worth of clothing, and $120.60 worth of furniture. She then made two monthly payments of $32.00 each. If the interest charges for the period of two months were $3.25, what did Mrs. Dallins then owe the account?"

Upon asking the 6 participating Hispanics who were interviewed in Interview 3 to tell us what they thought terms like "stock", "share", "revolving charge account", "monthly payments", and "interest" meant, they displayed that they had little idea as to the meaning of these terms. Several students came close to being able to define "interest", and stated that it was something that banks and stores did to make more money.

What is of questionable pedagogic value in using problems like the ones above is that students were being confused by the jargon, and not necessarily by the mathematics. Several students stated that even though they were not sure what some of the terms meant in the "revolving charge account" problem, they thought they could nevertheless solve the problem. Their attempts to solve this problem consisted of combining the four monetary quantities given in the problem in some fashion to obtain a final, albeit incorrect, answer.

Perhaps more effort should be devoted on behalf of researchers and textbook publishers alike, on the efficacy of using "plain talk" textbooks over "arcane talk" textbooks in teaching students mathematics. The only attempt of which we are aware at using a "plain talk" text in teaching algebra has been somewhat successful in its approach, but has apparently also met with some ambivalence from mathematics textbook publishers (Time, 1981).

B. Pedagogy

We would like to begin this section by briefly summarizing some characteristics displayed by the students in this study which we believe are not conducive toward the learning of mathematics. As has become clear from the interview results, students did not seem to appreciate that when working in a subject like mathematics, a) the slightest degree of imprecision and sloppiness can lead to errors and b) there are logical and legitimate reasons for every stage in any series of algebraic manipulations. It is also evident from the students' unwillingness to use algebra that they had little "faith" in using algebraic procedures to obtain answers. In other words, the notion that algebra allows one to start with a word problem, and by applying certain procedures such as defining variables, translating the relationships among the variables stated in the problem into mathematical equations, and manipulating the equations, one is *guaranteed* of finding a solution *even if one has absolutely no idea what the solution is anywhere along the way*, is something students find quite incredible.

To compound pedagogic difficulties, we found that the students were not very careful listeners, nor were they good at following instructions. One problem given in Interview 3 will exemplify this situation. The problem was the following:

3-7. For each of the following, underline all the operations that mean addition.

$3+7 =$

$4 \div 2 =$

$3+(4 \cdot 8) =$

$9+ (7-19) =$

Here we were not interested in the students' answers to the problem inasmuch as we were interested in whether they followed our instructions. As stated in Chapter II, our normal operating procedure consisted of the interviewer presenting each problem to the student as opposed to asking the students to read a problem silently before offering a solution. In the problem above it was decided at the staff meeting that upon reaching it, the interviewer would merely point to it and say to the student "do this problem". Admittedly, we were attempting to see how easy it was to trick the students, since the response we predicted was that they would solve the left hand side of the equality, and write an answer on the right hand side. Only two of the six (participating Hispanic) students interviewed attempted to read and follow the instructions of the problem; the other four did what we had predicted.

In order to address the situation as described above, what is needed is a pedagogic *approach* which addresses all of the problems in a global fashion, as opposed to remedies which are applicable only in specific situations. To help evaluate the pedagogical suggestions we will make, it would help to have a viewpoint or ideal against which our suggestions can be judged. The viewpoint we will present will not only encompass the role of concepts and skills in learning algebra, but also include a perspective on the importance of communication skills in the educational process. In an oversimplified description of the situation, there appear to be two extremes in the approach used in teaching mathematics. There are those who feel that rigorous formality is indispensable to the learning of concepts and those that feel that the possession of manipulative basic skills is a precondition to learning concepts. We tend to agree and yet disagree with both of these views. If the formal aspects of mathematics are emphasized at the expense of training in basic skills, the student may learn the jargon of mathematics, yet remain quite incapable of solving problems. On the other hand, we will not be the first to point out that although working out lots of problems may be a necessary condition for problem solving proficiency, this does not mean it is a sufficient condition as well (Kilpatrick, 1978).

To perceive that this contention of views regarding formality versus basic skills is not easily reconciled, one need only observe that whereas basic skills are *taught*, concepts are *formed*. Concepts cannot be taught, although they are learned in some sense of the word. The actual formation of a concept, however, is a purely internal process on the part of the student. Although the possession of basic skills can certainly aid the student in this endeavor, no amount of honing of basic skills will necessarily force a student to conceptualize. Our resolution of this dilemma is based on our belief that the single most important ingredient in the educational process is communication. Our recommended *approach*, therefore, focuses on the use of the communicative process.

Communication between the teacher and the student can assist the student in the process of conceptualization. The teacher can not only suggest the existence of concepts and encourage students to grapple with them, but also, through interacting with the student, guide him/her toward the formation of correct concepts. It is imperative that first and foremost, the teacher convey to the student as early as possible the need to be precise when working in mathematics, whether it be in listening, following instructions, communicating, or manipulating mathematical expressions. The emphasis thereafter should be on helping students form generalizable concepts rather than on asking students to memorize algebraic rules and facts.* Further, we are in agreement with Davis et al. (1978) in their advocacy of giving students the fullest possible appreciation of the importance of logically identifying and justifying algebraic

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*We do not mean to imply that having a command of algebraic rules and facts is unnecessary. Although a command of algebraic rules and facts will aid in getting an answer, it does not help the student in designing a strategy for obtaining the answer.

procedures, and in their claim that "do it this way" approaches are not sufficient. If we were to choose one algebraic concept that is often taken for granted by experts, but which is a very difficult concept to grasp for neophytes, it is the notion that algebra is an artform in which *unknown* mathematical quantities can be manipulated via a set of rules in order to extract a *known* answer.

With the recent technological advances, one instructional approach that should be given serious consideration is the use of low-cost microcomputers for supplemental mathematics instruction. That microcomputers are gaining rapid acceptance in mathematics instruction, both as teaching and programming tools, is unarguable. For the types of students in this study, computer instructional modules which combine presentation of material with detailed step-by-step worked-out examples, and drill work would be particularly helpful. We would like to emphasize the phrase "presentation of material" above to distinguish this approach from the "drill" modules which have become the standard product of many software and publishing firms. There are several reasons why this approach may prove very effective for this age group. First, given that students are not inclined to read the textbook, having them read the material on a computer's screen may be a possible solution to this problem, particularly since the students will likely associate this with watching T.V. and not with reading a textbook. Second, the "mystique" surrounding computers can be exploited to have students spend more time working on mathematics. Finally, the fact that computers are so intolerent of sloppy communication, they would help in training these students to work and communicate in mathematics with more accuracy and precision.

C. Linguistics

Whenever a cognitive research study is conducted with a bilingual population, there is always one albatross with which to contend; namely, the question of how language proficiency affected the findings. The idea that language may have an effect on cognitive processes (not necessarily for bilingual populations) is not new. For example, in Vygotsky's (1962) view, many facets of intellectual functioning are intimately related to language acquisition. According to Vygotsky, the internalization of language introduces a restructuring of many mental processes. Of particular relevance to this study is Vygotsky's claim that problem solving strategies become more rational and sophisticated when they can be verbalized.

Another view, that of Whorf (1956), states that the language we speak can set certain limits or constraints on our perception. Perhaps the justification for this view derives more from cultural effects than from linguistic effects; that is, it may well be that cultural experiences are as important as linguistic experiences in forming our perceptions. The difficulty with the Whorfian hypothesis lies in how to dintinguish between these two effects.

One hypothesis adduced by Cummins (1979) deserves particular attention due to its wide range of applicability. Cummins' *linguistic threshold hypothesis* posits that "there may be a threshold level of linguistic competence which bilingual children must attain both in order to avoid cognitive deficits and to allow the potentially beneficial aspects of becoming bilingual to influence their cognitive growth" (1979, p. 229). Cummins does not define the threshold level in absolute terms since it is likely to vary depending on the child's stage of cognitive development, and on the academic demands of the different stages of schooling.

Cummins does define three types of bilingualism. The first, "semilingualism", is characterized by a lower than threshold level of linguistic competence in both languages. In semilingualism, both languages are sufficiently weak to impair the quality of interaction the student can have with his/her educational environment. The negative effects of semilingualism are no longer present in "dominant bilingualism", characterized by an above threshold level of competence in one of the two languages. Dominant bilingualism is supposed to have neither a positive, nor a negative effect on cognitive development. The last category, "additive bilingualism", is one which has positive cognitive effects. Additive bilingualism is characterized by above-threshold competence in both languages.

In our investigations with college level Hispanic engineering students, we have found that, on standardized language proficiency measures, Hispanics score considerably below their Anglo peers

(Mestre, 1981). This below-average performance for the Hispanics holds across both English and Spanish. In terms of Cummins' definitions, these Hispanic engineering majors appear to be "semilingual". Even though we have not made an assessment of the Spanish proficiency of the participating Hispanics of this study, it appears from their performance on the language portions of the CAT that these participating Hispanics are below threshold, at least in English. Although it is extremely difficult to separate language effects from other effects, findings with college Hispanic students (Mestre, Gerace, and Lochhead, 1982; Mestre, 1982), as well as with the participating Hispanic group of this study indicate that this lower language proficiency level has an adverse effect on performance in various mathematical tasks. The facts that the advanced Hispanics of this study appear to be at least "dominant bilinguals" (and perhaps "additive bilinguals"), and that their performance in this study was extremely strong by any measure, lend support to Cummins' hypothesis.

Finally, it appears to us that any effort designed to increase the language proficiency level of bilinguals, at least up to the level of their monolingual peers, is most desirable. It does not appear to be too important which of the two languages is developed, as long as at least one of them is highly developed; however, for the obvious reason, the language used for instruction in the student's school may be the most appropriate to target for development. A word of warning is in order. Although there are strong indications that being highly proficient in language is a necessary condition for cognitive development, it certainly is not a sufficient condition as well.

D. Directions for Future Research

Even though there have been a plethora of research studies conducted in mathematics education, very few have focused on minority populations. In fact, we are not aware of any other systematic study of the algebra acquisition and problem solving skills of Hispanic adolescents. Before serious consideration can be given to the design and implementation of programs which attempt to increase the percentages of minorities pursuing math, and math-related careers to a level commensurate with their national representation, there must be an increase in the number of research programs which attempt to identify, assess, and define the specific problems contributing to the current situation.

We would like to suggest several questions for possible future investigations which would be of particular relevance if researched with minority populations. These questions have been divided into four distinct areas:

1. Effectiveness of Textbooks
   - Why are students inclined not to read their mathematics textbooks?
   - Would their understanding of mathematics improve if students were forced to read their textbooks?
   - Are "plain language" textual presentations more readable for students with language deficiencies? Will these students read a plain language textbook without encouragement?
   - What are some possible changes in textbook style that may make them more effective didactic tools?
   - Does the fact that almost all math textbooks on a particular subject appear to be cloned from some "standard" derive from tradition, logical pedagogic reasons, or the unwillingness on behalf of textbook publishers to take risks with "innovative" approaches?
2. Effectiveness of Computer Assisted Instructional Approaches
   - How effective are CAI approaches as supplemental instruction for minority students?
   - Will students read textual material more willingly if it is presented at a speed and level commensurate with the student's speed, and level of reading comprehension?
   - Will minority students spend more time working in mathematics if they had the chance to work with computers?
   - Are computers effective for use in teaching problem solving as opposed to the traditional use as "drill instructors?"
3. Effect of Language in the Problem Solving Process
   - How do reading speed and reading comprehension affect problem solving performance?

- What is the single best language predictor of word problem solving proficiency for minority students?
- Is an increase in language proficiency followed by an increase in mathematical proficiency?

4. Sociological and Economic Factors
   - How can minority parents be made to involve themselves in the educational process of their children?
   - How do programs that combine career counselling and role models affect the attitude and motivation of minority students to pursue mathematics?
   - What is the effect of family economic level on a student's proclivity to pursue mathematics?

VI. RECOMMENDATIONS

The participating teacher made two points during the last staff meeting when this report was the topic of discussion. It would perhaps be elucidating to convey these two points before moving to the specific recommendations proposed below. The first point made by the participating teacher was that since the "character" of mathematics classes varies from year to year and from class to class depending on the students comprising the class, it is unrealistic to expect that one pedagogic approach which proves very effective for one particular class would have the same result when used with another class. The second point concerned the realism of dealing with 14 year old adolescents. He claimed that, given this age group's maturity level, some pedagogic approaches were likely to prove more effective than others; in particular, approaches which demand the undivided attention of the whole class for a prolonged period of time would not prove fruitful.

In terms of the recommendations below, we believe that they would be of benefit to all students. However, we have designated several which we feel would be particularly beneficial for students such as the participating Hispanics of this study whose language proficiency level is below that of their Anglo peers. Our recommendations are the following:

1. Students should be asked to participate in the process of learning concepts. Although these students are not of an age where they are naturally introspective, encouraging and aiding them in forming general concepts are preferred over passively absorbing rules. Whenever possible, procedures which are generalizable to a wide range of problems should be emphasized over rule-oriented procedures which apply only to a narrow range of problems. Concept formation can be reinforced by presenting the students with both correct and incorrect examples and asking them to recognize valid procedures as well as fallacious logic. It is often the case that the real hint of a concept lies in the path to the answer and not in the answer itself.

2. More important than the pedagogic style of the teacher or textbook is the harmony between the two. The teacher should use a textbook which is consonant with his/her teaching style, so that both text and the teacher emphasize a single approach. The teacher's primary objective should be to impart to the student his/her understanding of the material rather than some supposedly superior way of thinking about the subject -- one which he/she neither uses nor feels fully comfortable with.
3. Students should be held responsible for reading the textbook. The ability to learn from written material is an indispensible tool for self-learning and should therefore be incorporated as early as possible in the educational process.
4. Whenever a new definition or procedure is introduced, it should be compared and contrasted with previous ones. Equally important as telling students what something is, is telling them what something is not. The use of counter-examples or discussions of incorrect applications of rules/procedures would help students assimilate the correct rules/procedures more quickly.
5. Students should be made to realize that two very important ingredients in mathematical reasoning are precision and consistency. These same characteristics should be sought in the communication process itself. Due to the great redundancy present in oral communication, and aided by the context of the situation, most of us can tolerate large doses of imprecision and inconsistency in oral communication. When attempting to teach or learn mathematics, however, imprecision and inconsistency can be very debilitating.
6. The use of microcomputers for supplemental instruction in mathematics should not be underestimated. Microcomputers may serve to motivate students to spend more time on mathematics, force them to communicate more precisely with a machine which is very intolerant of sloppy communication, and provide an opportunity to present material which students

would not otherwise be inclined to read in the textbook. The following three recommendations would be of particular benefit to students with language deficiencies.

7. Students should be asked to verbalize the rules, strategies, definitions, and procedures that they employ in solving problems. This would serve to a) monitor the precision with which the students communicate mathematical ideas, b) encourage students to always have a reason to justify what they are doing, and c) reveal any misconceptions the student has so that the teacher has an opportunity to address them.
8. In word problems, the emphasis should be on teaching students sound procedures in translating the problem statement(s) into mathematical notation. By defining variables and writing appropriate equations to represent the problem statement, students would begin to appreciate more quickly that they do not have to know the answer "all at once", but that the resulting equations are the means by which to obtain an answer.
9. A concerted effort should be made to increase the language proficiency level of "semilingual" students (in the Cummins sense) to at least the "dominant bilingual" level. Although the evidence is not conclusive, indications are that "semilingualism" may have an adverse effect on the communication process which we believe to be crucial in the educational process.

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APPENDIX I

Interview Questions

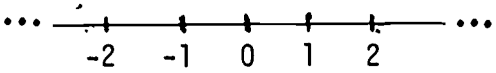
Interview Questions

First Interview Session

Question Set A

1. Where is the origin on this number line?

2. Which is the positive direction? What is the other direction called?

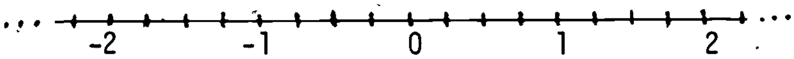


Question Set B

1. Where is -1/4 on this number line Where is 1/2?

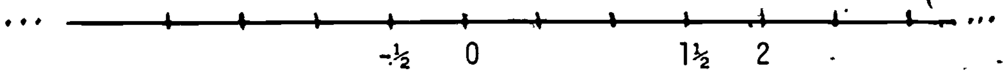
2. Can you label all the marks? Explain

3. How many numbers are represented by each mark?



Question Set C

1. Where is +1?



2. How much distance is between $-1\frac{1}{2}$ and $\frac{1}{2}$?

3. Can you label all the marks?

4. If I tell you that 1/2 means 1/2 mile east of my house, where would 1/2 mile west of my house be? Where is my house? If my friend, Julia, lives 2 miles west from my house, where does she live?

Question Set D

1. What is 3 x 6 + 2 = __________? (If student answers 20, point out it could be 24 and launch into discussion).

2. Are these sets equal? Why?
   {1,2,3} {3,1,2}

3. What is an infinite set?

Interview Questions

Second Interview Session

Question Set A

1. How would you answer the question "What is a number"?
2. How many numbers are in this set? {1, 1/2, 2/3, 4, 1/8, 7}
3. Is a fraction a number? Why or why not?
4. How would you define a fraction?
5. Which of these are fractions? Why? 2, 1/2, 3/4, 5/3, 4/2, 9/3, 6/1, 10, 52/1
6. Which of the following mathematical statements are equations? Why?

a) $3 + 1 = 4$

b) $3x - 9$

c) $\frac{3 + y}{x + z}$

d) $10 + 2 = 8$

e) $x - 7 < 10$

f) $Z = 19$

g) $5(x + y) = 20$

h) $8 > a$

i) $\frac{6}{2 + y}$

j) $8 \times 3 = (9 \times 4) - 12$

k) $\frac{w + t}{7}$

l) $\frac{8}{4} = \frac{4}{2}$

Question Set B

Look at the following sets

I. {the positive integers}

II. {the negative integers}

III. {the non-negative integers}

IV. {the even positive integers}

V. {-5, -3, -1, 0, 1, 3, 5}

VI. {-4, -2, 0, 2, 4}

1. Which of these sets has the fewest elements? Why?
2. Which of these sets has the most elements? Why?
3. Are any of these sets equal? Why?
4. What is the solution to $25 > 5x$

if $x \in$ {the positive integers}
if $x \in$ {the negative integers}
if $x \in \{-5, -3, -1, 0, 1, 3, 5\}$
(ask for explanation in each case)

Question Set C

1. You probably know that there are 12 inches in one foot. If I write the equation

   $$12I = 1F$$

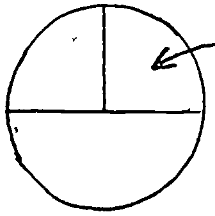
   What does I represent? What does F represent?

2. Write a mathematical expression to represent the statement: "subtract 7 from the quotient of x and y".

3. Suppose I told you that the number of students in this school is equal to thirty times the number of classrooms. Can you write an equation that says this.

4. Suppose I have a ball which I drop from a certain height. Every time the ball bounces, it only goes up to 1/2 of the highest point reached in the previous bounce. If I drop the ball from 16 feet, how high does it go after the first bounce. How high does it go after the second bounce. Suppose I have another ball that goes up to 3/4 of the highest point reached in the previous bounce...(repeat problem)

5. Suppose I have an apple pie and I cut it like this, and then like this.



If I eat this piece, how much of the pie is left? (try to get them to give a fraction)

Question Set D

1. If I write 3^2, what does it mean? What does it equal?

2. Which of the following is $(x + y)^2$ equal to?

   a) $2x + 2y$ b) $x^2 + y^2$ c) $(x + y)(x + y)$ d) $x + y^2$

3. Suppose $x = 3$ and $y = 4$. What is $(x + y)^2$

4. Which of the following equations are <u>true</u> statements?

a) $\frac{x}{y+z} = \frac{x}{y} + \frac{x}{z}$

b) $\frac{r+s}{t} = \frac{r}{t} + \frac{s}{t}$

5. Which of the following is $\frac{3 + x}{7 + y}$ equal to?

a) $\frac{3 + x}{7} + \frac{3 + x}{y}$

b) $\frac{3}{7} + \frac{x}{y}$

c) $\frac{3}{7 + y} + \frac{x}{7 + y}$

6. If $x = 13$ and $y = 1$, what is the value of $\frac{3 + x}{7 + y}$?

Interview Questions

Third Interview Session

1. What is an axiom?
2. What is closure? In other words, what does it mean to say that the set of integers is "closed" under the operation of addition?
3. Can you explain what "absolute value" means?
4. Which of the following equations are true and why:

$8 \cdot 4 = 4 \cdot 8$	$17 \div 4 = 4 \div 17$
$a + b = b + a$	$x - y = y - x$
$a \div b = b \div a$	$x \div y = \dfrac{x}{y}$
$8 - 3 = 3 - 8$	$a(7+b) = 7a + ab$
$a(7-b) = 7a - ba$	$a(7-b) = (7-b)a$

$10(6+3) + 4 = 4 + 10 \cdot 6 + 10 \cdot 3$

$10 \div (2+5) = (10 \div 2) + (10 \div 5)$

5. What is $\frac{3}{2} \div \frac{6}{8}$?
6. What is $\frac{7}{8}$ times $\frac{1}{5}$?
7. For each of the following underline all the operations that mean addition.

$3 + 7 =$

$4 \div 2 =$

$3 + (4 \cdot 8) =$

$9 + (7-19) =$

8. For each of the following real numbers, find the reciprocal, or multiplicative inverse.

$-\frac{2}{3}$, -4, 8, 12, $\frac{1}{2}$, $\frac{3}{4}$

9. Find the numerical value of the following:

$6 - |4| =$ $\qquad$ $|6-4| =$

$6 - |-4| =$ $\qquad$ $|-6| - 4 =$

$|4-6| =$ $\qquad$ $|-6| + |-4| =$

10. If $a = -3$, what is the value of the following:

$7-a =$ $\qquad$ $7 \cdot a =$ $\qquad$ $a - 7 =$

$7 - |a| =$ $\qquad$ $7 + a =$ $\qquad$ $\big|\, |a| - 7 \,\big| =$

11. The temperature at 12:00 noon is 2 degrees. At 6:00 p.m. it is -8 degrees. If between 6:00 p.m. and 11:00 p.m. the temperature drops twice as much as it did between 12:00 noon and 6:00 p.m., what will be the temperature at 11:00 p.m.?

12. A submarine dives to a level 730 feet below the surface of the ocean. Later it climbs 200 feet and then dives another 80 feet. What is then the depth of the submarine?

13. (Explore students' understanding of vocabulary in the following two problems from page 76 of text, "Modern Algebra Structure and Method".)

    a) A stock selling for \$30 per share rose 2 dollars per share each of two days and then fell \$1.75 per share for each of three days. What was the selling price per share of the stock after these events?

    b) On a revolving charge account, Mrs. Dallins purchased \$27.50 worth of clothing, and \$120.60 worth of furniture. She then made two monthly payments of \$32.00 each. If the interest charges for the period of two months were \$3.25, what did Mrs. Dallins then owe the account?

Interview Questions

Fourth Interview Session

Question Set A

1. Solve the following equations:

a) $2 + x = 3$

b) $x + 7 = 10$

c) $y - 12 = 40$

d) $4 + a = -6$

e) $r - 3 = -8$

f) $x + 9 = 4$

2. Solve the following equations:

a) $3x = 6$

b) $4y = 10$

c) $-2x = 12$

d) $9a = -18$

e) $\frac{x}{3} = 33$

f) $\frac{1}{8}y = -4$

3. Solve the following equations:

a) $5y + 10 - 3y - y = 14$

b) $4x + 7 = 6x - 5$

c) $7(x + 3) = 21$

4. Simplify the expression:

$\frac{1}{b}(ab) =$

5. a) What is -7 times 3?

b) What is -3 times 7?

6. If $a = 3$ and $b = 4$, find a numerical value for the following:

a) $ab =$

b) $ba =$

7. Look at the following equation: $b + 3 = 7$

Which of the following changes do not change the answer to the equation above:

a) $4 + b + 3 = 4 + 7$

b) $6b + 3 = 6 \cdot 7$

c) $2(b + 3) = 7$

d) $b + 3 + a = 7$

e) $b + 3 + a = 7 + a$

f) $4(b + 3) = 4 \cdot 7$

8. Pick the correct answer (or answers) to the following problem:

$$\frac{1}{b}(ab) = ?$$

a) b^2a

b) b

c) a

d) $\frac{a}{b}$

Question Set B

1. Write an equation for each sentence below.

a) A number added to 7 equals 12.

b) Six times a number results in 24.

c) Twice a number equals 16.

d) Twelve more than twice a number is 6.

e) The quotient of x and 3 is equal to 21.

f) Add 4 to a number, then subtract 5 from the sum and you get 43.

g) Add 3 to a number, then multiply this sum by 4 to get the answer 50.

Question Set C

1. Use the calculator to find answers to the following problems.

   a) $8-3 =$ c) $\frac{90}{15} =$

   b) $5 \cdot 4 =$ d) $9^2 =$

2. For the following five problems, first find an answer, then use the calculator to check the answer.

   a) $3 - 8 =$ c) $5(3 + 4) =$ e) $(2-4) \cdot 3 =$

   b) $3.4 \div \frac{1}{2} =$ d) $(3 + 4) \cdot 5 =$

Question Set D

Solve the following problems.

1. In 8 years, Ana will be as old as her sister, Sonia, is now. If Sonia is 23 years old, how old is Ana?
2. John spent 6 hours on the lake. If he fished for twice as long as he swam, how long did John spend fishing?
3. A MacDonald's hamburger has 250 calories more than an apple pie. An apple pie has 50 calories less than a milkshake. If 3 MacDonald hamburgers have a total of 1800 calories, how many calories are in one milkshake?
4. Tom rides the school bus part way and walks the remainder. He walks 3 minutes longer than he rides. If it takes Tom 17 minutes to arrive at school, how long does he spend on the bus?

Interview Questions

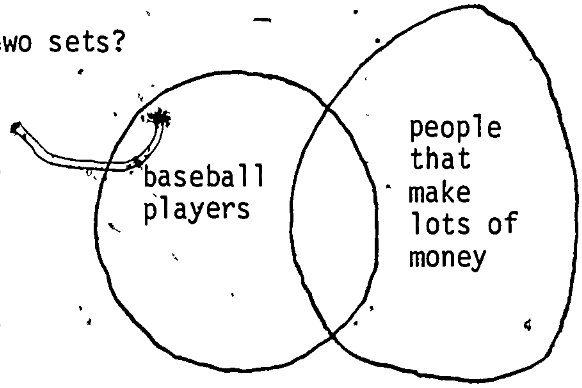
Fifth Interview Session

Question Set A

1. Given these two sets: $A=\{1,5,0,-3,19,29,4\}$ and $B=\{-1,19,21,29,4,-40\}$

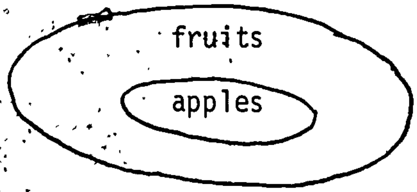
   What is $A \cup B$? What is $A \cap B$?
   Can you draw a diagram showing these two sets?

2. Look at this diagram:



   a) According to the diagram, are there baseball players that do not make lots of money?

   b) Are there people that make lots of money who are not baseball players? If so, what part of the diagram represents these people?



   a) What does this diagram tell you?

   b) What part of the diagram represents those fruits that are not apples?

Question Set B

1. Look at the following function: $f: X \to 4X-2$.
   What is the value of $f(1)$? What is the value of $f(3)$? What is the value of $f(-5)$?

2. Suppose we make up a new operation, $*$, as follows:

$$A * B = 2A + B$$

   What is $3*9=$?

Question Set C

1. Which of these are true statements?

   a) $5 \neq 2$ b) $6 < 3$ c) $-3 > -8$ d) $-5 > 3$ e) $1/3 < 1/2$
   f) $-5/3 < -1/2$

2. Starting with $5 < 7$, how could you multiply both sides by -2 and write the answer as an inequality?

3. If $a < b$, then which of the following are true, which are false, and which can't you tell whether they are true or false?

   a) $a < b+3$ b) $a+3 < b+3$ c) $a+3 < b$ d) $a < -b$ e) $-a > -b$

Question Set D

1. What is a variable? What are variables used for?

2. Circle all the variables below:

   a) 2Y+3 b) 2X+Y=15X c) 7+2=9

3. What do you think is the hardest thing about solving word problems?

4. Look at the following problem:

   The sum of Mary's age and Gary's age is 23. Mary is 3 years older than four times Gary's age. What are the ages of Gary and Mary?

   a) How many variables do you need to solve this problem?

   b) Is there something you find hard to do when you try to solve this problem?

   c) Can you solve it?

Question Set E

1. La suma de dos números enteros consecutivos es 51. ¿Cuáles son los números?

   The sum of two consecutive integers is 51. What are the numbers?

2. Work out the following two problems:

   a) Si lápices cuestan 5¢, ¿cuántos puedes comprar con $2.05?

   b) How many 15¢ candy bars can you buy with $1.65?

3. Write an equation for the following two statements:

   a) Dos veces un número sumado a tres es igual a 5.

   b) Doce mas que el doble de un número resulta en 6.

4. El costo de una casa y un terreno es $40,000. La casa cuesta siete veces mas que el terreno. ¿Cuanto cuesta la casa y cuato cuesta el terreno?

5. a) John has 6 times more quarters than dimes. How many dimes does he have if he has 18 quarters?

   b) En una fiesta hay 8 veces más botellas de cerveza que de vino. ¿Cuantas botellas de vino habrán si hay 16 botellas de cerveza?

   c) The larger of two consecutive even integers is six less than twice the smaller. What are the numbers?

   d) El menor de dos números enteros consecutivos es uno mas que el doble del mayor. ¿Cuales son los números?

Interview Questions

Sixth Interview Session

Question Set A

1. What is the difference between a monomial, a binomial, and a polynomial?

2. Add the following expressions:

   a) $(3Y^2+Y-4) + (-2Y^2+5Y+1)$

   b) $(7X^2-2) + (4X^2-3X+9)$

   c) $(6R^2S + 11) + (3R^2S + 2RS + RS^2 + 2R + 1)$

3. Solve each equation:

   a) $(4-2X) - (X-5) = (X+2) - (3-X)$

   b) $-3Y - (7-5Y) = 15$

4. Simplify each expression:

   a) $Z^2(-3Z^4)=$ b) $(2YZ)(3Y^2)=$ c) $(ab)^2(ab)^3=$ d) $(r^2s)^3=$

   e) $(X^5)(X^4)=$

5. If X=3 and Y=2, what are the values of the following:

   a) X^2Y^3 b) $(XY)^2$ c) $(X^2Y)^3$ d) $(X+Y)^2$

6. If $f(x)=x+3$ and $g(x)=x-2$, what is:

   a) $f(x) + g(x)=$

   b) $f(x) - g(x)=$

   c) $f(x) \cdot g(x)=$

Question Set B

1. Mr Smith noted the number of cars, C, and the number of trucks, T, in a parking lot and wrote the following equation to represent the situation:

$$8C=T$$

   Are there more cars or trucks in this parking lot? Why?

Write an expression using variables for the following statements:

2. A number added to 7 equals 18.

3. Six times a number is equal to a second number.

4. Nine times a number results in 36.

5. In seven years, John will be eighteen years old.

6. The number of nickels in my pocket is three times more than the number of dimes.

7. The number of math books on the book shelf is equal to eight times the number of science books.

8. There are four times as many English teachers as there are math teachers at this school.

9. Last year, there were six times as many men cheating on their income tax as there were women.

10. Six times the length of a stick is 24 feet.

11. If a certain chain were four times as long, it would be 36 feet.

12. The sum of two consecutive integers is 25.

13. If the number of people in China were decreased by a factor of 8, then the number of people in China and the number of people in England would be the same.

14. The sum of the ages of two horses born one year apart is 25.

Interview Questions

Eighth Interview Session

Question Set A

1. Suppose X=170. What is the value of X-37?
2. Suppose Y+13=160. What is the value of Y+13-15?
3. Suppose 5(2Z+1)=10. What is the value of $\frac{(2Z+1)}{2}$?

Question Set B

Solve the following problems:

1. A number added to 11 equals 23. What is the number?
2. Twice a number is 26. What is the number?
3. Six times a number is equal to a second number. If the second number is 48, what is the first number?
4. A number multiplied by 4 results in 36. What is the number?

Question Set C

1. In a recent survey conducted among the reachers of this school, it was discovered that 11 teachers wanted to continue school through the summer. How many more teachers would be needed to raise this number to 23.
2. Different bicycles have different size wheels. Some have very small wheels (only 10 inches high) and some old bicycles have a front wheel over 6 feet tall. Many modern 10 speed bicycles have wheels 26 inches high. What would be the distance from the ground to the center of a 26 inch bicycle wheel?
3. An article in a recent medical magazine states that college women who smoke cigarettes are six times as likely to have lung cancer as college women who don't smoke. If in a large eastern state university there are 48 smoking women with lung cancer, how many non-smoking women with lung cancer are there likely to be?
4. In a small blood sample analyzed at the local hospital, there were 36 red blood cells found. It was also found that the number of serum antibodies was smaller than the number of red blood cells by a factor of 4. How many serum antibodies were found in this blood sample?

Question Set D

1. What is a prime number?

   Which of the following are prime numbers and why?

   a) 2   b) 3   c) 4   d) 5   e) 6   f) 7

2. Simplify: $\frac{9Z^2 + 15Z}{3Z}$

3. Multiply out:

   a) (X+2)(X-3)

   b) (3Y+1)(Y+5)

4. Factor the following:

   a) a^2+a-6

   b) X^2+5X+6

APPENDIX II

Selected Background Information
on Student Participants

Appendix II

Selected Background Information on Student Participants

Participating Anglo Group Student #	Background
1 (female)	Born and raised in the same city where the study was conducted. Father is deceased; mother works as a head nurse. Hobbies include participating in numerous sport activities and playing poker. Plans to attend college and major in nursing.
2 (male)	No information
3 (male)	No information
4 (female)	Was born in a town next to the city where she now has been living for 4 1/2 years. Father not living at home (did not specify reason). Mother is a housewife holding no outside employment. Hobbies include playing soccer and softball. Is not sure whether she plans to attend college.
5 (male)	Born and raised in the same city where the study was conducted. Father works in construction while mother is a mill worker. His hobby is weight lifting. Plans to attend college and study medicine.

Participating Hispanic Group Student #	Background
1 (male)	Born in New York City of Puerto Rican parents. Has lived in New York City, Connecticut, New Jersey, Puerto Rico. Father is deceased, mother is a housewife. Has spent the last 4 years in the city of the study site. The language of instruction used in grades 1 and 2 was Spanish. Grades 3,4, and 5 were spent in a bilingual program where both English and Spanish were used for instruction. Speaks to parents and friends in both Spanish and English. Has been mainstreamed since grade 6. His favorite subject in school is math. Hobbies include participating in numerous sports activities. Plans to join the Army after high school.

Participants in Hispanic Group (Continued)

Student #	Background
2 (male)	Emigrated from Columbia 5 years ago and has been living in the city of the study site since then. Did not specify parents' occupations - only stated they both work. The language of instruction for grades 1 through 4 was Spanish. Has been mainstreamed since grade 6. Speaks to friends in English but to parents in Spanish. His favorite subject in school is math. Hobbies include participating in numerous sports activities. Plans to go to college and major in architecture.
3 (male)	Born in New York City of Puerto Rican parents. Moved to city of study site from New York City 5 years ago. Father does not live with family. Mother works but did not specify occupation. Speaks English at home and both English and Spanish with his friends. His favorite subject in school is math. Has always been in the mainstream curriculum. Hobbies include drawing, dancing, and participating in numerous sports activities. Plans to attend college and major in architecture.
4 (female)	Born in New York City of Puerto Rican parents. Parents are co-directors of a day-care center. Has always been in the mainstream curriculum. Speaks English at home and with friends. Her favorite subject in school is social studies. Hobbies include participating in numerous sport activities. Plans to attend college and major in engineering.
5 (male)	This student is the older brother of student #4. Background information is the same as for his sister. His favorite subject in school is math. Plans to go to college but is undecided on what field to study.
6 (male)	Born in Puerto Rico. Moved to Pennsylvania as a young child and has been living in the city of the study site for the last 5 years. Neither father nor mother were employed. Speaks Spanish at home and both Spanish and English with his friends. Has always been in a mainstream curriculum. His favorite subject in school is math. Hobbies include participating in numerous sports activites. Plans to attend college; he is undecided on what field to study, but is sure he wants to play football while in college.

Advanced Hispanics Student #	Background
1 (female)	Born in Columbia and immigrated to the city of the study site when she was 4 years old. Has always been in the mainstream curriculum. Speaks Spanish at home. Her favorite subject in school is social studies. Plans to attend college and study law.
2 (female)	Born in Puerto Rico and immigrated to the city of the study site when she was 3 years old. Has always been in a mainstream curriculum. Speaks both English and Spanish at home. Her favorite subject in school is English. Plans to attend college and study nursing.
3 (male)	Born and raised in the mainland. Has always been in a mainstream curriculum. Speaks both English and Spanish at home. His favorite subject in school is math. Plans to attend college but is not sure what field to study.