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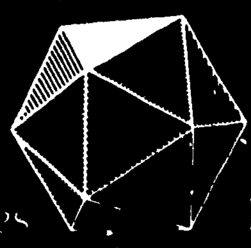
ABSTRACT

This report, prepared for and published by the Mathematical Association of America's Committee on the Teaching of Undergraduate Mathematics, includes a description of the state of the art on problem solving, lists available resources, and makes recommendations regarding the place of problem solving in the college curriculum and ways to teach it. The report recommends (1) an approach to teaching mathematics that fosters an alert and questioning attitude in students and that actively engages them in the process of doing mathematics, (2) a series of problem-solving courses at various levels of sophistication as regular offerings in the standard college curriculu, and (3) a series of texts for problem-solving courses at all levels to be developed and disseminated. Specific suggestions are given on how to teach problem solving, especially pertaining to the role of the teacher and ways of organizing the class. Some typical problems and class discussions are provided. Then follows an extensive annotated bibliography of problem-solving resources, with characterizations of the type of course for which each appears most appropriate, its focus or subject matter, and its level. Journals, books, and articles are listed separately. Finally, the problem-solving questionnaire and responses are briefly presented. (MNS)

PROBLEM SOLVING IN THE MATHEMATICS CURRICULUM

A Report, Recommendations, and An Annotated Bibliography

Alan H. Schoenfeld



MAA Notes Number 1

THE MATHEMATICAL ASSOCIATION OF AMERICA
COMMITTEE ON THE TEACHING OF UNDERGRADUATE MATHEMATICS

Alan H. Schoenfeld

PROBLEM SOLVING IN THE MATHEMATICS CURRICULUM: A REPORT, RECOMMENDATIONS, AND AN ANNOTATED BIBLIOGRAPHY

1983

The Mathematical Association of America
Committee on the Teaching of Undergraduate Mathematics

Acknowledgments

Contributions to this report and bibliography came from many people. It is a pleasure to acknowledge their help, and to thank them for it. The M.A.A. Committee on the Teaching of Undergraduate Mathematics and its Problem Solving Subcommittee developed the Survey of Problem Solving Courses (pp. 134-137), and the M.A.A. distributed it nationwide. After the data were compiled, the Committee suggested that I write the suggestions for teaching problem solving and compile the bibliography. Henry Alder first suggested that the report, recommendations, and bibliography be combined into the volume that you are now reading. He provided encouragement and helpful suggestions throughout its development. Tom Butts wrote the first draft of section 3D. A large number of people provided lists of "favorite" sources in response to question 17 of the Survey. Murray Klamkin provided a long list of books and articles. Johanna Zecker spent endless hours in the library checking bibliographic data. Jerry Alexanderson annotated many of the references. Members of the Committee and Subcommittee, most notably Don Bushaw, vigilantly tracked down flaws in manuscript. Of course, I am solely responsible for the flaws that remain.

AHS

PROBLEM SOLVING IN THE MATHEMATICS CURRICULUM:

A REPORT, RECOMMENDATIONS, AND AN ANNOTATED BIBLIOGRAPHY

Contents

Introduction

> A Teacher of Mathematics has a great opportunity. If he fills his allotted time with drilling his students with routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.
>
> G. Pólya, <u>How to Solve It</u>

In March 1980 the Mathematical Association of America's Committee on the Teaching of Undergraduate Mathematics formed a Subcommittee on Problem Solving, with the following charge:

> to gather information from undergraduate programs, analyze the current literature, and produce a report which
>
> 1. describes the "state of the art,"
> 2. lists available resources, and
> 3. makes recommendations regarding the place of problem solving in the curriculum and ways to teach it.

In early 1981 a "Survey of Problem Solving Courses" was mailed to a group of faculty including all college level mathematics department chairmen in the United States and Canada. A total of 539 departments responded. Of those, 195 indicated that they currently offer problem solving courses, and provided descriptions of them. In addition, there were 86 responses like the following: "We do not have a problem solving course at present but are interested in developing one. Please send a copy of your report and any other useful material." The responses to the questionnaire provide the basis for our description of the state of the art. There were many suggestions and much enthusiasm for teaching problem solving. There were also many requests for help. Specifically, we received repeated requests for two kinds of information: a collection of suggestions for teaching problem solving courses, and a bibliography of resources for such courses. We are pleased to offer this volume in response to those requests.

RECOMMENDATIONS

The full rationale for offering problem solving courses is given in "suggestions for teaching mathematical problem solving," which follows. To put things briefly, we believe that the primary responsibility of mathematics faculty is to teach their students to think: to question and to probe, to get to the mathematical heart of the matter, to be able to employ ideas rather than simply to regurgitate them. As P.R. Halmos argues in "The Heart of Mathematics,"

> The major part of every meaningful life is the solution of problems; a considerable part of the professional life of technicians, engineers, scientists, etc. is the solution of mathematical problems. It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts.

The "problem approach" to teaching mathematics is valuable for all students: those who will simply "appreciate" it, those who will use it, and those who will live it (solving problems is, in essence, the life of the professional mathematician!). In particular,

1. We endorse any approach to teaching mathematics that fosters an alert and questioning attitude in students, and that actively engages students in the process of doing mathematics. We encourage the use of a "problem based approach" wherever possible in standard course offerings, including the participation of students in discussing, solving, and presenting their solutions to problems. (Those worried about subject matter coverage should see section 2D of the teaching suggestions.) We similarly encourage "problem of the week" contests, informal problem seminars, etc.

2. We recommend that a series of problem solving courses at various levels of sophistication be developed and made regular offerings in the

standard curriculum. In particular,

a. Elementary problem solving courses serve as welcome and meaningful alternatives for students who wish to take a college math course but have no need for the calculus; as replacements for the typical "math isn't so bad" liberal arts courses; and as supplements to the calculus for students who wish to be introduced to substantive mathematics at an elementary level.

b. Upper division problem courses, either on specific subject matter or covering a range of general problem solving topics, can introduce students to the spirit of mathematical inquiry in a substantive way long before they would encounter it on their own, whether in professional careers or in doing mathematical research.

c. Special courses for teachers, in modeling, in general literacy, etc. (as in the survey results), all provide access to the mathematical experience for students who might not otherwise experience it.

3. In order to foster the implementation of Recommendation 2, we recommend that a series of texts for problem solving courses at all levels be developed and widely disseminated.

We hope that this volume serves as a step in that direction. The following section offers some suggestions for teaching a problem solving course. These are put forth in the same spirit as recommendations in CTUM's "College Mathematics: Suggestions on How to Teach It." We offer them for your consideration, and hope you find them useful. The section on teaching is followed by an annotated bibliography on problem solving and the results of the survey. As the scope of the bibliography indicates, there are a great variety of available resources. Whatever the particular nature of the course you might like to offer, you will find ample collections of problems appropriate for it and you will find a wide variety of ideas about teaching it. In a sense, the most difficult aspect of giving a problem solving course is making the decision to offer it. We encourage you to do so, and believe that you and your students will benefit from the experience.

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SUGGESTIONS FOR TEACHING MATHEMATICAL PROBLEM SOLVING

There is no one "right" way to teach problem solving, and it would be presumptuous to consider recommending one: there are as many effective ways to teach mathematical thinking as there are talented teachers. Moreover, classroom methods are a matter of personal style. What "works" for one teacher may have to be modified in order for another teacher to use it comfortably, if at all. These suggestions are offered with that understanding. For that reason they are written somewhat informally and in the first person. The suggestions have worked well in the classroom. Please treat them as you would treat the suggestions from a close colleague. Consider them, try the ones that seem appropriate on for size, and then tailor them so that you feel comfortable with them.

1. Background and rationale

There is a huge difference between the way that we do mathematics and the way that our students see it. Doing mathematics is a vital, ongoing process of discovery, of coming to understand the nature of particular mathematical objects or systems. First we become familiar with an area. As we do, our intuitions develop. We begin to suspect that something ought to be true. We test it with examples, look for counterexamples, try to get a sense of why it ought to be true. When we think we know what makes it work, we try to prove it. The attempt may or may not succeed. There may be any number of false starts, reverses, retrenchments, and modifications. With perseverance and luck, the result falls into place. Few experiences are so gratifying or exciting: we have charted unknown territory, and enriched ourselves in the process.

Unfortunately, our students rarely have any idea that doing mathematics can be like this. In a strange way, they are the victims of our professionalism. Because there is so much for them to learn, we present the results of our mathematical explorations in organized and coherent fashion. As a result, they can "master" it better. But this kind of "mastery" has some unfortunate consequences. Students think that all of mathematics is known and, like Latin grammar, must be rehearsed until it is learned. There is no excitement in discovery, but simply the (minor) satisfaction of achieving competence. Doing mathematics looks so easy for us that they feel incompetent when it is difficult for them. They have no idea that we, too, must struggle to understand new mathematics. More importantly, they have no idea that "understanding" mathematics means asking questions until things make sense; instead, it means passively reproducing what they have been shown.

I will argue here that we can, and should, introduce students to the experience of doing mathematics as we know it. Moreover, I believe that we can do so, with some success, reasonably early in their mathematical careers. In a sense, my problem solving course is remedial: it is disturbing that college freshmen do not routinely think to draw diagrams to help them understand problem statements, test hypotheses with special cases, etc. Even more disturbing is the fact that my students rarely if ever realize that they can think, that they can watch themselves thinking, and that they can improve their problem solving performance by reflecting on their successes and failures. What seems perfectly natural in the context of playing tennis, or any other sport, seems completely alien in the context of training one's mind!

At the risk of seeming silly, it may be worth asking just what we want our students to get out of the mathematics courses they take. Nearly half of our students have their last formal exposure to mathematics instruction in a calculus course. "College mathematics" has become synonymous with "calculus," and taking the course is almost a rite of passage. Truthfully, however, I see little value in training such students to (for example) calculate the surface area of a solid of revolution. It isn't that such results lack instrinsic value -- both aesthetic and mathematical -- but that the students will not, in general, see either. The power of the mathematics will not be theirs to apply (this is their last mathematics course) and we do not, for a variety of reasons, focus on the aesthetics. Our majors are short changed as well. In the give-and-take of mathematical exploration, the procedure for calculating surface areas can be "discovered" and "appreciated" as an ingenious application of Riemann sums, a point missed by most of our majors. They see it as a mechanical procedure of (dubious) applied value.

It seems to me that the real service we can offer our students, both our majors and the ones we will never see again, is to provide them with thinking skills that they can use after they take our final exams. I have no doubt that mathematics can serve as an ideal vehicle for this. There is no better discipline for learning what "understanding" means. Mathematical thinking is logical and precise, and the techniques we use for attacking problems are broadly applicable. But students are not likely to get a feel for "understanding" or to profit from those techniques, unless they are made explicit; they are unlikely to develop their mathematical thinking after instruction unless we have served as the catalysts for their doing so. I believe we can. What follows are some of the things I do in my problem solving course, and some of the reasons why I do them.

2. Some issues in teaching problem solving.

A. THE TEACHER AS ROLE MODEL

There is an anecdote about a famous professor whose reasoning was so fast that it often left students in the dark. One day at the beginning of class, a student raised his hand and asked the professor to solve a particular homework problem. The mathematician read the problem, thought for a few seconds, said "Ah, yes, the answer is $\pi/4$," and wrote the answer on the board. The student, who was clever, came up with a way to get more information. "Excuse me, Professor, but could you solve the problem another way?" "That's an interesting question," said his teacher. He went deep into thought for a while, and then said "This one is more straightforward, although the computations are a bit more messy." He turned to the board, wrote another $\pi/4$ neatly next to the first, and asked the class if they had any more questions.

Part of the difficulty in teaching mathematical thinking skills is that we've gotten so good at them (especially when we teach elementary mathematics) that we don't have to think about them; we just do them, automatically. We <u>know</u> the right way to approach most of the problems that will come up in class. But the students don't, and simply showing them the right way doesn't help them avoid all the wrong approaches they might try themselves. For that reason we have to unravel some of our thinking, so that they can follow it. There are three related ways to do this.

(i) GOING THROUGH THE PROCESS, ON A "BLOW BY BLOW" BASIS (EVEN WHEN YOU KNOW THE ANSWER). Consider the following problem, for example:

1. Let P(x) and Q(x) be two polynomials with "reversed" coefficients:

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$

$Q(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-2} x^2 + a_{n-1} x + a_n,$

where $a_n \neq 0 \neq a_0$. What is the relationship between the roots of P(x) and those of Q(x)? Prove your answer.

There is, of course, an elegant solution, which will emerge in a page or two. But I think that something on the order of the following, even if it may seem a bit contrived, is better in the long run.

"What do you do when you face a problem like this? I have no general procedure for finding the roots of a polynomial, much less for comparing the roots of two of them. Probably the best thing to do for the time being is to look at some simple examples, and hope I can develop some intuition from them. Instead of looking at a pair of arbitrary polynomials, maybe I should look at a pair of quadratics: at least I can solve those. So, what happens if

$$P(x) = ax^2 + bx + c, \text{ and}$$

$$Q(x) = cx^2 + bx + a?$$

The roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2c}$, respectively.

That's certainly suggestive, since they have the same numerator, but I don't really see anything that I can push or that'll generalize. I'll give this a minute or two, but I may have to try something else...

"Well, just for the record, let me look at the linear case. If

$P(x) = ax + b$ and $Q(x) = bx + a$, the roots are

$-b/a$ and $-a/b$ respectively.

They're reciprocals, but that's not too interesting in itself. Let me go back to quadratics. I still don't have much of a feel for what's going on. I'll do a couple of easy examples, and look for some sort of a pattern. The clever thing to do may be to pick polynomials I can factor; that way it'll be easy to keep track of the roots. All right, how about something easy like $(x + 2)(x + 3)$?

Then $P(x) = x^2 + 5x + 6$, with roots -2 and -3. So,

$Q(x) = 6x^2 + 5x + 1 = (2x + 1)(3x + 1)$, with roots -1/2 and -1/3.

Those are reciprocals too. Now that's interesting. How about

$P(x) = (3x + 5)(2x - 7) = 6x^2 - 11x - 35$? Its roots are -5/3 and 7/2;

$Q(x) = -35x^2 - 11x + 6 = -(35x^2 + 11x - 6) = -(7x - 2)(5x + 3)$.

All right, the roots are 2/7 and -3/5. They're reciprocals again, and this time it can't be an accident. Better yet, look at the factors: they're reversed! What about

$P(x) = (ax + b)(cx + d) = acx^2 + (bc + ad)x + bd$? Then

$Q(x) = bdx^2 + (ad + bc)x + ac = (bx + a)(dx + c)$.

Aha! It works again, and I think this will generalize....

"At this point there are two ways to go. I hypothesize that the roots of $P(x)$ are the reciprocals of the roots of $Q(x)$, in general. (If I'm not yet sure, I should try a factorable cubic or two.) Now I can try to generalize the argument above, but it's not all that straightforward: not every polynomial can be factored, and keeping track of the coefficients may not be that easy. It may be worth stopping, re-phrasing my conjecture, and trying it from scratch:

Let $P(x)$ and $Q(x)$ be two polynomials with "reversed" coefficients.

Prove that the roots of P(x) and Q(x) are reciprocals.

All right, let's take a look at what the problem asks for. What does it mean for some number, say r, to be a root of P(x)? It means that P(r) = 0. Now the conjecture says that the reciprocal of r is supposed to be a root to Q(x). That says that Q(1/r) = 0. Strange. Let me go back to the quadratic case, and see what happens.

Let $P(x) = ax^2 + bx + c$, and $Q(x) = cx^2 + bx + a$. If r is a root of P(x), then $P(r) = ar^2 + br + c = 0$. Now what does Q(1/r) look like?

$$Q(1/r) = c(1/r)^2 + b(1/r) + a = \frac{c + br + ar^2}{r^2} = \frac{P(r)}{r^2} = 0.$$

So it works, and this argument will generalize. Now I can write up a proof.

Theorem: Let P(x) and Q(x) be as in (1) above. Then the roots of Q(x) are the reciprocals of the roots of P(x).

Proof: Let r be a root of P(x), so that P(r) = 0. Observe that $r \neq 0$, since $a_0 \neq 0$. Further, $Q(1/r) = a_0(1/r)^n + a_1(1/r)^{n-1} + \ldots + a_{n-2}(1/r) + a_n =$

$(1/r^n)(a_0 + a_1r + a_2r^2 + \ldots + a_{n-2}r^{n-2} + a_{n-1}r^{n-1} + a_nr^n) =$

$(1/r^n)P(r) = 0$, so that (1/r) is a root of Q(x).

Conversely, if S is a root of Q(x), we see that $P(^1/_S) = 0$. Q.E.D.

"All right, now it's time for a post-mortem. Observe that the proof, like a classical mathematical argument, is quite terse and presents the results of a thought process. But where did the inspiration for the proof come from? If you go back over the way that the argument evolved, you'll see there were two major breakthroughs.

"The first had to do with understanding the problem, with getting a feel for it. The problem statement, in its full generality, offered little in the way of assistance. What we did was to examine special cases in order to look for a pattern. More specifically, our first attempt at special cases --

looking at the quadratic formula -- didn't provide much insight. We had to get even more specific, as follows: Look at a series of straightforward examples that are easy to calculate, in order to see if some sort of pattern emerges. With luck, you might be able to generalize the pattern. In this case we were looking for roots of polynomials, so we chose easily factorable ones. Obviously, different circumstances will lead to different choices. But that strategy allowed us to make a conjecture.

"The second breakthrough came after we made the conjecture. Although we had some idea of why it ought to be true, the argument looked messy and we stopped to reconsider for a while. What we did at that point is important, and often overlooked: we went back to the conditions of the problem, explored them, and looked for tangible connections between them and the results we wanted. Questions like 'what does it mean for r to be a root of P(x)?', 'what does the reciprocal of r look like?', and 'what does it mean for (1/r) to be a root of Q(x)?' may seem almost trivial in isolation, but they focused our attention on the very things that gave us a solution."

Now the past few pages may seem all too much like flogging a dead horse. The mathematician is primarily interested in the result, which only took a few lines to prove; the thought processes that generated it are pretty much second nature -- to us. My experience, however, is that they are completely alien to students. Elucidating those processes does two things. (1) It demystifies the mathematics, and makes it more accessible. When the students see where the idea comes from, it no longer seems like pulling a rabbit out of a hat. (2) The strategies that were underlined in the above discussion are generalizable, and useful elsewhere. Learning how to use them helps students to become better problem solvers.

My primary objection to the discussion above is that the presentation is still one-way: the teacher is still explaining how he or she approached a problem. If problem solving is a personal experience, then the student needs to be involved. That leads to a second way to serve as a role model.

(ii) SOLVING PROBLEMS WITH THE STUDENTS, USING THEIR IDEAS. The idea here is for the class to solve problems together, with the teacher serving as "moderator," orchestrator of ideas, and as the "alter ego" that raises important questions and keeps things on track. The teacher is not to generate solutions, but rather to help the students make the best of the resources they have. The teacher may have handed out some problems (as homework, or earlier that day), and convened the class to discuss one of them. The following sequence of "executive" questions, as I make my way through a problem, is typical:

"Does anyone have any suggestions? Any others? What made you think of that? What makes you think it's a reasonable thing to do? All right, we have these as plausible ideas. Which one should we do? What makes you think it's a better alternative? What'll you do with it when we're done? Ok, does that sound reasonable? Should I try it?

"Hmm, we've been doing this for five minutes and we haven't gotten anywhere. Are you really sure we understand the problem well enough? What might we consider? Are any of our heuristics appropriate?" etc.

This gives the flavor of the discussion, which will be treated again in section 3.4. With luck (and perseverance on the part of the teacher), these questions eventually become second nature to the students. By the middle of the semester I can ask them "Ok, what question am I going to ask now?" and they can usually tell me; by the end of the semester, they may actually be asking them themselves.

(iii) TEACHER ON THE SPOT: SOLVING PROBLEMS "FRESH." A problem solving course is tough on the students, because there are no "rules;" just when they think they've gotten things down, a new problem throws them for a loop. To give the students a break, and to let them see me in a similar situation, they are allowed to pose problems to me in the same way that I do to them. Class starts with "Any questions?", and if they have one for me, then I work it "out loud" at the blackboard. This way they get to see me use the problem strategies in an unrehearsed form, which removes the "canned" nature of the presentations in (i) above. There are also other consequences: see section 2 G.

B. ON THE TEACHER AS COACH.

I have heard some of my colleagues describe mathematics to their students as a "contact sport." What they meant, of course, is that one has to be involved with doing mathematics; one can't appreciate it from the sidelines. There is another aspect to the sports analogy as well: the teacher, who normally plays the role of dispenser of knowledge, instead takes on the role of coach. Since, in many ways, our teaching of athletic skills is more advanced than our teaching of intellectual skills, the notion of an "intellectual coach" is worth exploring.

Consider the act of coaching a routine skill such as making a foul shot in basketball or a serve in tennis. The coach who said "Watch me do it, and then go practice it by yourself" would be considered derelict, and wouldn't have a job for very long. Of course the given process is demonstrated, and also broken into minute detail: one is told how to stand, how the hands should be positioned, etc. The athlete is generally "walked through" each of the parts of the task -- so far, the parallel of our teaching. Also, the athlete is sent off to practice for a while. But soon after, the coach is back to make corrections, and of a rather detailed nature: "Your shoulder is too low, you're not getting enough loft on the toss," etc. If the performance is important, it is not uncommon for the coach and the athlete to review slow-motion videotapes of that student performing the act in question, to isolate minor points that could stand improvement.

This aspect of coaching has to do with what might be called the "basic skills," or standard procedures. But coaches do far more than that. Much of their job consists of training their charges to make intelligent decisions during the course of play: probably the most commonly heard complaint from a coach, right after a mistake, is "That was a low percentage shot (or play). It just didn't make sense to take it."

Consider the intellectual equivalent, even in routine problem solving. In an exam on techniques of integration, for example, 44 of 178 students evaluated $\int \frac{x\,dx}{x^2-9}$ by partial fractions, and another 17 students evaluated it using the substitution $x = 3 \sin \theta$. It makes no sense to try either of these approaches, both of which are time-consuming; a brief check indicates that the problem can be solved by the far more elementary substitution $u = x^2 - 9$. A piece of standard advice, cutting across virtually all domains, is "Don't do anything hard until you've made sure there are no easy alternatives." It's the kind of advice a coach would give, and it strikes me as far more valuable than simply showing the student the "right" way to solve the problem.

C. <u>THERE'S MORE THAN ONE WAY TO SKIN A MATHEMATICAL CAT.</u> Since most of the "problems" we solve in class are really exercises, we are generally content with the first solution that closely resembles the techniques students have been shown. When that "problem" has been solved, we move on to the next;

the exercise has served its purpose. Our students are left with the impression that they have seen the "right" way to solve the problem -- and that there is one right way.

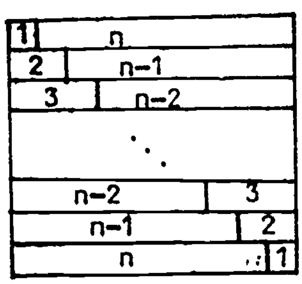
This is nonsense. Consider the large number of proofs we know of the Pythagorean theorem, for example, and how happy any of us would be to discover a "new" one. Part of the joy of mathematics consists of discovering new things, but part is also discovering connections among ones we already know, or finding new ways to see things with which we are familiar. The frequency of articles entitled "A New Proof of Theorem X" in our journals makes that point clear enougn.

More importantly, the notion is dangerous. "Understanding" a mathematical fact or system means having as many "connections" to it as possible. My understanding of Gauss's sum, $\sum_{i=i}^{n} i = \frac{1}{2}n(n+1)$, is all the richer because I think of it:

(1) as the result of n/2 pairings that each add up to (n+1),

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$

(2) pictorially as half of the n x (n+1) figure represented below,



which can be represented symbolically by the arithmetic

$$s = 1 + 2 + 3 + (n-2) + (n-1) + n$$

also, $s = n + (n-1) + (n-2) + \ldots 3 + 2 + 1$

so $2s = \underbrace{(n+1)+(n+1)+(n+1)+\ldots(n+1)+(n+1)+(n+1)}_{\text{a total of } n \text{ times}}$;

(3) as an argument to be verified by induction;

(4) as the special case of a difference equation; and so on.

I could be considered "deprived" if I only knew one of these. But this deprivation is only part of the story: each one of these ways of thinking about the problem embodies a sightly different way of thinking, and can be generalized in different ways. When I'm confronted with a new problem, any one of these approaches (but not necessarily all) might provide the "key" to it.

Also, the knowledge that problems can be solved a lot of different ways has an effect on the way that people work them. The student who thinks that there is one "right way" to solve a problem may work on a particular problem for a while; if he or she makes no progress, the student may then give up and wait to be shown the "appropriate" technique. (This is, after all, the pattern they have learned implicitly in their schooling.) The student who thinks that there is room for exploration in mathematics -- and benefits from it -- is more likely to play with the problem, to make connections for him or herself, and perhaps to stumble upon an unexpected solution.

D. MORE ISN'T NECESSARILY BETTER.

Halmos (1980) makes the following argument*. I agree. "Many teachers are concerned about the amount of material they must cover in a course. One cynic suggested a formula: since, he said, students on the average remember only about 40% of what you tell them, the thing to do is to cram into each course 250% of what you hope will stick. Glib as that is, it probably would not work.

"Problem courses do work. Students who have taken my problem courses were often complimented by their subsequent teachers. The compliments were on their alert attitude, on their ability to get to the heart of the matter

*Halmos, P. R. The Heart of Mathematics. Americal Mathematical Monthly, 87 (1980), pp. 519-524.

quickly, and on their intelligently searching questions that showed that they understood what was happening in class. All this happened on more than one level, in calculus, in linear algebra, in set theory, and, of course, in graduate courses on measure theory and functional analysis.

"Why must we cover everything that we hope students will ultimately learn? Even if (for example) we think that the Weierstrass M-test is supremely important, and that every mathematics student must know that it exists and must understand how to apply it -- even then a course on the pertinent branch of analysis might be better for omitting it. Suppose that there are 40 such important topics that a student must be exposed to in a term. Does it follow that we must give 40 complete lectures and hope that they will all sink in? Might it not be better to give 20 of the topics just a ten-minute mention (the name, the statement, and an indication of one of the directions in which it can be applied), and to treat the other 20 in depth, by student-solved problems, student-constructed counterexamples, and student-discovered applications? I firmly believe that the latter method teaches more and teaches better. Some of the material doesn't get covered, but a lot of it gets discovered (a telling old pun that deserves to be kept alive), and the method thereby opens doors whose very existence might never have been suspected behind a solidly built structure of settled facts. As for the Weierstrass M-test, or whatever else was given short shrift in class -- well, books and journals do exist, and students have been known to read them in a pinch."

E. IF YOU DON'T SAY IT THEY WON'T GET IT (MOST OF THE TIME).

Much of the preceding discussions, and the following section on a "sample" class, may seem like belaboring the obvious. A story told by Mary Grace

Kantowski about a research project on problem solving* may indicate otherwise.

The research consisted of a "teaching experiment" in which students were given special problem solving training and then tested in detail (with interviews, etc.) to determine how the instruction had affected their problem solving performance. The teaching was based on Pólya's notion of problem solving, and there was a great emphasis on his fourth stage of problem solving, "looking back." Nearly forty per cent of class time was spent on reviewing solutions, recapitulating and condensing arguments, generalizing, etc.

The testing brought a shock for the researchers: the students engaged in virtually no "looking back" at all, despite the great emphasis on it in class. Videotapes of classroom sessions provided the reason. After a problem had been solved, the teacher generally stepped aside and said something like "All right, let's look back at the solution and see what we can learn from it." What the teacher <u>meant</u>, and thought was obvious, was something like "Looking back is an important part of the problem solving process. Checking the answer, checking the argument, looking for alternate derivations, placing it in different contexts, using the method or result for different problems, all help us to better understand the solution." What the students <u>saw</u> was the following: "The teacher is reviewing the solution. I understood it, so I don't really have to pay close attention here." If we do not make something explicit -- no matter how obvious it seems to us -- there is always the chance it will go unheard. That, I suspect, is the reason for the classic instructions for writing U.S. Army instructional manuals:

1. Tell them what you're going to tell them.

*This was reported at the 1979 NCTM Annual Meeting, in a research session.

2. Tell them.

3. Tell them what you've told them.*

Needless to say one need not follow these instructions in lock step fashion, especially in a course where students are supposed to discover many of the results for themselves. It doesn't hurt, however, to make sure that they have made the discoveries. How much should one say? We would do well to follow the advice to travelers about how much to tip a cab driver in a foreign country. "Drop the coins in his hand one at a time. When the cabbie's face begins to light up, you can stop your hand in mid-air."

F. TWO NOTES ON PROBLEM DIFFICULTY.

1. "Elementary" problems can be challenging and instructive.

We train our students to do some remarkably sophisticated things in our classes, and it is natural to think that we must "challenge" them in problem courses. There is, then, a temptation to avoid problems that look "simple" to us. Out of context, however, elementary problems can be quite challenging. I taught the first version of my problem solving course to a group of eight junior and senior mathematics majors at Berkeley. All had taken advanced calculus (the analysis kind). Some had seen topology and measure theory, others some sophisticated applied mathematics. Early in the semester I gave the following as a homework problem.

> Prove that in any circle, the central angle that subtends a given arc is twice as large as the inscribed angle that subtends the same arc.

*An alternate version, consisting of three terse rules for making sure that people remember what you've said, is the following:
Rule 1. Repeat yourself.
Rule 2. Repeat yourself.
Rule 3. Repeat yourself.

The problem is routinely solved in high school geometry classes, and I had the distinct feeling when I assigned the problem that the students felt that it was beneath them. Yet only two of them were able to solve it. One managed to reconstruct the classical argument using the special case where one side of the given angle is the diameter of the circle. The other used arc length integrals! Back in tenth grade, when they had been shown the key that unlocks the problem and asked to memorize it, the problem seemed trivial. When left to their own devices, however, this "elementary" problem was quite challenging. A good source of such problems is the collection of straightedge-and-compass constructions at the end of Chapter 1 of Pólya's Mathematical Discovery. Despite their elementary nature, they might well give beginning graduate students (and a rusty professor!) some pause. For example:

> Construct a triangle, given two line segments whose lengths respectively are:
> the length of a side a of the triangle,
> the length of the altitude to a,
> and an angle whose measure is
> the measure of the angle α opposite the side a.

Equally simple problems from other domains are just as rich. The following problem has served as a first-day problem for liberal arts mathematics courses, for my freshman problem-solving courses, and for upper-division students (who have not had the relevant course in number theory).:

> A magic trick
> Take any three-digit number, for example 123. Make a six-digit number by writing it down twice. In our example we get 123,123. I bet you a dollar that the new number is divisible by 7, without leaving a remainder.
>
> Want to get your dollar back? Consider the quotient you just obtained when you divided your six-digit number by 7. I'll bet another dollar that that quotient is divisible by 11, without leaving a remainder.
>
> OK, so I got lucky. Now here's your big chance: double or nothing. Consider the quotient you just obtained in the division by 11. I'll bet the quotient is divisible by 13, without leaving a remainder.
>
> (You know the real question....WHY??)

Similarly, questions such as

> Can you find a simple rule to determine if a number is divisible by 4? (or any other digit),

or

> What is the greatest common divisor of 692,481 and 237,612?

can lead to some very solid mathematics.

2. Long and tedious problems are important too.

One of the unfortunate consequences of our instruction is that our students believe that mathematics <u>should</u> be easy. Because we do our "homework" well and present clear, coherent lectures, they get the impression that the discovery and apprehension of mathematical ideas should be logical and straightforward. Because most of our "problems" are exercises that can be solved by the techniques we have shown them (to be honest, how many problem solutions have we ever shown that took more than 15 minutes?), they get the impression that problems should be solvable within, say, a half-hour or an hour. Many simply give up after that much time, feeling that a problem that they can't solve within an hour simply can't be solved (by them). Those of us who have spent days, weeks, or months just trying to make sense of a problem know how wrong that perspective is. But partly out of kindness and partly because we have so many routine things to cover, we rarely ask students at the elementary level to work long, time-consuming problems. However, it is important for students to learn that (i) drudgework is sometimes necessary, and (ii) long periods of (sensible) exploration are often needed before one can really get a feel for a problem, or a domain. My students in an advanced calculus class were upset with me when, after deriving the formula for the trapezoidal rule and parabolic approximations for definite integrals, I asked them to derive the formula for the best cubic approximation. It took hours of computations (but isn't that what mathematics is like?)! Similarly, in a freshman problem solving course I will assign problem like

> What numbers of the form aaaaaaa...aaa (the same digit a, repeated n times) are perfect squares?

and

> Derive a formula, in general, for the polynomial of degree n+1 that passes through the n points
> $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$.

It may take us a week or two to solve the problem, returning to it periodically to see what progress the students have made. The same is true of my take-home midterm examination problems. A few days after I handed out the exam, desperate students found little consolation in my assurances that "if you play with this for long enough, it will start to make sense." But they did, and it did too.*

G. ON BEING FALLIBLE.

I mentioned in section A that my students have the "right" at the beginning of each class period to pose questions for me to solve. They are often reluctant to do this at the beginning of the semester. By the third or fourth week of the term, however, (1) they know me better and feel more comfortable with this kind of reversal, and (2) they have been consistently frustrated in class (the give-and-take of a problem class is much harder on the students than the passivity of the standard lecture class) and look forward, without malice, I believe, to seeing me on the spot as well. It's only fair to see if I can use my own strategies. In the middle of the semester, I can usually expect a

*One word of political warning: such actions can be dangerous to our health. Students, accustomed to being fed "learning" in small doses, can be resentful of what they perceive to be sadism on our parts. I have found, however, that once I explain to them why I am doing what I am doing to them, they are willing to give me the benefit of the doubt (for a while). By the end of a semester, there are no problems.

positive response to the question that opens class: "Does anyone have any problems for me?"

In truth, the problems rarely cause any difficulty. Many of them are familiar. When a student says "You come to a fork in the road and...," or "You have a square cake whose top and sides are evenly covered with frosting. There are seven people...," I tell them "I'm sorry, but I know the problem. I'll work it at the board (or with the class, if I think it's a good class problem), but I can't make believe that I'm solving it from scratch."

Most of the unfamiliar problems are straightforward. Working problems out loud at the board is a fairly slow process, giving me time to think; I can usually manage to solve them without looking too clumsy. But occasionally a problem will throw me. One day I was asked to solve the following problem:

> Construct a triangle given the lengths of two sides <u>a</u> and <u>b</u>, and the length of the median $\underline{m_c}$ to the third side.

I started off the problem as usual, going through the standard consideration of alternatives, deciding what to pursue, etc. My first approach failed, so I tried a second. It failed as well, and after about fifteen minutes it became clear that I was not going to solve the problem then and there. At that point I said something like the following: "Well, I think I've done everything that I can think of for the moment. Occasionally a problem simply doesn't yield to the standard kinds of techniques. I can't think of anything productive to do right now, and it won't do any good to have you watch me flail around. So let's go on. I'll think about this tonight, and I'll tell you next class meeting how I solved it."

It is difficult to describe the class's reaction without seeming melodramatic. There was a stunned silence, and an atmosphere of complete disbelief.

Of course they had seen teachers stumble on occasion, but such mistakes are usually dismissed as momentary lapses (and we have a variety of ways of extricating ourselves from such situations without looking too bad). And of course they had heard me tell them that, as a student, I had gone through the same difficulties in problem-solving that they were experiencing in class; moreover, that when I worked on difficult problems I used the same strategies that I was teaching them to use. Nonetheless, I don't think that they had really believed any of it: I was simply saying such things to encourage them. This demonstration of fallibility -- that I had difficulties also, and that I really did go through the same struggles to solve problems that they did -- was one of the most valuable lessons in the course.

3: Class format

Students learn by doing, not by watching. The problem solving course uses a variety of class formats, all of which are designed to encourage student participation. Lectures are kept to a minimum: even when there is a point that I want to get across, it can be made much more powerfully if the students have grappled unsuccessfully with a problem before they are shown why another type of approach "makes sense." On the whole, perhaps 10% of class time is spent in lectures.* Perhaps 5-10% of class time is spent with me solving problems "fresh" at the blackboard, as described above. The bulk of class time, in roughly equal parts, is spent in the following three ways.

A. DISCUSSION OF HOMEWORK PROBLEMS.

If a student has solved an assigned problem since the last class meeting, he or she will present the solution. There are two kinds of questions from the class: (1) about the correctness of the solution, and why we should accept it, and (2) about where the solution "came from." What led the problem solver to approach it that way, and why? If a problem has not yet been solved, we may work on it for a while as a group (see the discussions in section 3.4) or I may send them back to work on it, with or without any suggestions. Some of the longer problems, like those in section 3.2F, have been the subject of sporadic discussion for a week or two before they were dealt with to our satisfaction.

B. SMALL GROUP SOLUTIONS OF NEW PROBLEMS.

Roughly a third of the way through class, I hand out a collection of "problems to think about." A typical handout is given in figure 3.1. A discussion of the nature of such problems, and details of classroom discussions

*Most of the points we wish to make can be made naturally, and more effectively, during the class discussion of problem solutions.

A SAMPLE IN-CLASS ASSIGNMENT

1. Suppose that P is any prime greater than 3. Show that P^2 leaves a remainder of 1 when it's divided by 12.

   (First question: what is this related to; what should you be thinking about?)

2. Suppose that P is a polygon with 1001 sides. Can you
   a. never
   b. sometimes but not always
   c. always

   find a straight line which passes through all the sides of P?

3. Pottsylvanian currency uses 7 and 17 dollar bills. Can you buy a 5-dollar book and receive exact change? (using only 17 and 7 dollar bills). An 11 dollar magazine? A 98769876 dollar tank?

4. Transylvanian currency uses 6 and 15 dollar notes. Can you buy a 12 dollar pencil with exact change? A 5 dollar bookmark? A 123456789 dollar lifesize replica of the Goodyear blimp?

5. If A,B,C,D, are given positive numbers, show that

$$\frac{(A^2+1)(B^2+1)(C^2+1)(D^2+1)}{ABCD} \geq 16 .$$

6. Given a line segment of length L, can you construct one of length $L\,((\sqrt{13} - 3)/4)$? Is this easy or hard?

7. Find the sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!}$$

8. For those of you who like "cryptarithmetic:" Each letter is a different digit. Solve

```
 FORTY              SEND
+  TEN    and      +MORE
+  TEN             -----
 -----             MONEY.
 SIXTY
```

-- figure 3.1 --

of them, is given in section 4.*

The problems on the handout will generally occupy us for the rest of the day. Half of that time is spent in small groups. When the students receive the problems, they break into groups of approximately four each, and work together as teams on the problems. While the students are working on the problems, I circulate through the classroom serving as a roving "consultant." The purpose of the consulting is not to guarantee that the students reach the right answers, but rather to guarantee that they are proceeding in a reasonable fashion. If a group is doing well I may pass them by without comment, or I may ask them to justify their actions. They know that I will expect them, at any point in a problem solution, to be able to (a) specify in some detail whatever operation they are engaged in, (b) be able to justify the reason for being engaged in that operation, and (c) be able to say what they will do with the result of it, as the solution progresses. This emphasis on monitoring and assessing the evolution of a problem solution focuses on avoiding the "wild goose chases" that typify so much of the students' problem solving attempts. If the students are genuinely stuck, I may point them back to our list of heuristic strategies or may suggest that they think about some related problems we have solved. In general, the idea is to say the least amount necessary to make sure that the students are moving forward.

*"One third of the way through class" is a rough approximation that works fairly well in a 75 minute class. I prefer to meet twice a week for 75 minutes because meetings of that length allow us to sink our teeth into some fairly complex problems. If one only has 50 minutes for each class, things can be broken up differently. One day can be spent working on new problems, another with students presenting their solutions of homework problems, etc.

to make sure that the students are moving forward.

Breaking the class into small groups to work on problems is an unorthodox use of class time, and it should be justified. The following are some of the reasons I believe the small-group format is useful.

1. This format allows the teacher the unique opportunity to intervene directly in the students' problem solving, rather than being confronted with the "finished product." The impact of that intervention on the students' behavior can be much more dramatic than in any other format.

In spite of the fact that we would consider much of their problem solving almost primitive, the fact is that these students (who have volunteered for a problem solving course after doing well in a calculus class) are the successes of our educational system. They have made it into my course precisely because their problem solving habits, developed over twelve or more years of schooling, have served them well. In a very real sense I am asking them to "unlearn," in the sense that a music teacher or sports coach takes a talented but ill-trained student and has to undo some well-practiced but counterproductive behaviors. This simply cannot be done by talking at the student. I might lecture for hours about "monitoring your solution" and "not going off on wild goose chases," and so on. Such lectures by themselves are almost guaranteed to have no effect on the students' behavior when they are working on homework problems; old habits simply take over and they do what they are used to doing. My interventions in class have an immediate impact, however. The following kind of dialogue has taken place many times in class.*

*The behavior that prompts the dialogue given below is all too typical. The transcript of a twenty-minute problem session in which students spent fifteen minutes in the useless calculation of the area of a triangle, is discussed in my "Episodes and executive decisions in mathematical problem solving."

Me: Why are you calculating the area of that triangle (or...)?

Student(s): I thought that...well, actually, I'm not really sure.

Me: Suppose I told you the area of the triangle. What would you do with it?

S: Well, ummm,...

Me: How long have you been working on that calculation?

S: Five, maybe ten minutes.

Me: What have you gotten out of it?

S: Nothing, I guess.

Me: Wait a second. The problem isn't that I don't see any value to the computation. You might have a perfectly legitimate reason for doing it, perhaps one that I haven't thought of. It's not my "disapproval" that matters. But the idea is that you have already spent some time on this, and might spend quite a bit more, only to wind up with something that turned out to be absolutely useless to you. It seems to me that if you'd stopped to ask yourself something like: "This will take me quite a while to do. What am I going to get out of it? How will I use it? Are there any alternatives?", you could have saved yourself quite a bit of trouble.

2. Solving problems in groups provokes discussions of plausible choices. When a student works on a problem alone, the first "reasonable" option is often the one pursued. When a small number of students work on a problem together, two or three different ways to approach the problem may be suggested. Decisions about the merits of each -- about which should be pursued and why -- are precisely the kinds of decisions that the students should be making. I tell them that, eventually, they should themselves be generating a range of options and choosing among them in the same way that the groups do.

3. Problem solving is not always a solitary endeavor. Students have little opportunity to engage in collaborative efforts, and this does not do them any harm.

4. Students are remarkably insecure, especially in a course of this

nature. Working on problems in groups is reassuring: one sees that his fellow students are also having difficulty, and that they too have to struggle to make sense of the problems that have been thrown at them.

C. THE CLASS CONVENES AS A WHOLE TO WORK NEW PROBLEMS.

After the small groups have had a chance to make progress on some of the problems, we convene as a class to work our way through them. By this time the students are familiar with the problems, so there is a solid base for us to build upon. The class makes suggestions, and I serve as moderator. Full sample discussions are given in the next section. Of course, there are more problems each day than we can solve. The ones we leave unfinished are homework assignments, to be dealt with in the next class meeting.

D. NOTES ON TEACHING PROBLEM SOLVING TO LARGE CLASSES

Ideally all mathematics instruction should take place in small classes, but practical realities sometimes dictate otherwise. Tom Butts has taught problem solving courses with large enrollments ($n > 100$). He makes the following suggestions with regard to planning, running the course, and grading.*

Planning

The larger a class is, the more one must depend on support staff and the less one can rely on the kinds of spontaneous discussions that often make small classes "fly." Also, very large classes tend to fall into the "literacy" end of the spectrum. Students in such courses need more reassurance and structure than most. Despite urging, contact with the professor is minimal for all but a few. Thus planning becomes especially important.

*Taking on a large class and teaching problem solving is a bit like trying to juggle three or more balls while riding a unicycle. I strongly recommend that you develop each skill separately before trying to combine them.

It is important to have high quality, well trained teaching assistants. Ideally, the TAs should (a) attend the lectures (to serve as roving consultants during group sessions, as note-takers, etc.), (b) grade homework problem sets, and (c) hold several office hours per week, more than for standard courses. The TAs have to be sensitive to the problem solving process, both in tutoring and grading. They will also bear the brunt of the unhappiness and frustration felt by the students as they struggle to learn to think mathematically. For these reasons, close coordination with TAs is essential before and during the course. Meeting with them after class, if only briefly, pays dividends. Dividends in the form of extra pay are also appropriate for this kind of extra duty, if that can be managed.

As the term starts, students need some sort of "anchor" to keep them from feeling completely at sea. A text provides this of course, and more problem solving texts are making their way into the marketplace. If you do not use one, you might consider giving students class notes (perhaps taken by TAs or assigned as "term papers" for students), a "strategies handout," sets of sample problems accompanied by several solutions each, or some articles on problem solving. At the very least, such materials can be placed on reserve in the library.

Even if you use a text, you may want to write your own problem sets. Your students' sophistication, and the kind of solutions you expect them to produce, may suggest the level of "guidance" that is appropriate in the problem statements. Consider, for example, students' reactions to these versions of a perennial favorite:

1. How many diagonals has a convex (a) quadrilateral, (b) pentagon, (c) hexagon, (d) 17-gon, (e) 101-gon, (f) n-gon?
2. How many diagonals has a convex 17-gon?

3. How many diagonals has a convex 101-gon?

4. How many diagonals has a convex n-gon? Justify.

5. N people are in a room. Each person shakes hands with everyone except the two nearest people. How many handshakes are there?

Also, "ground rules" should be established. What do we mean by "justify your answer," and what will we accept as justification? In general, time spent in discussions of (a) what constitutes a "legitimate" solution to a problem, and (b) the nature of mathematical language, for example "find all," "for which n," etc., is time well spent.

Running the course

One of the major objectives of this kind of course is to teach guessing. To break the ice and encourage guessing, we can start the day off with non-mathematical problems like the following:

What do each of the following have in common?

a. half, paper, blue
b. grease, room, tennis
c. Christmas, Easter, Melville
d. Victoria, King Edward III, Angel

Students who are ordinarily reluctant to speak up are often eager to guess on problems like this. They can then be encouraged to guess on mathematical problems as well.

Just as in small classes, getting students to work together on problems is important. The small group model can be adapted to lecture classes, although the logistics are generally more complex. After posing a problem, you can allow students time to work on it while you and the TAs serve as roving consultants. Have an extension problem or two ready for early finishers. For example:

In "simplified football" the only scores possible are 7 points for a touchdown and 3 points for a field goal. What values are possible for the number of points one team can score?

Extension 1: In how many ways can a team score 2100 points?

Extension 2: Suppose the game were played with the values of N points for a touchdown and M points for a field goal. If there are exactly 14 scores that cannot be achieved, what values are possible for N and M?

Deciding when to call the group sessions to a halt is a delicate matter. On the one hand, students need time to get involved with the problems and to make some progress on them. On the other hand, there is no need to have them reach polished solutions in class. This give-and-take of working together with students using their partially formulated ideas to solve a problem, is an excellent use of class time (see section 4C). Also, their misconceptions or incorrect proposed solutions are often good points of departure for discussions.

As noted in section 2E, getting students to "look back" over their solutions can be difficult. We can encourage that kind of behavior with some special-purpose problems. For example, the problem:

A. Calculate $\sqrt{(\underbrace{111\ldots1}_{100\ 1\text{'s}})(1\underbrace{000\ldots0}_{99\ 0\text{'s}}5) + 1}$

can be followed by

B. Construct a similar problem.

Most students will guess the answer to A by looking at simpler cases and guessing the result from the pattern. They are usually content with that. However, problem B forces them to take another look at the problem to see what makes it tick. Asking for generalizations, or for an entirely different solution, often does the same thing. In general, they are more likely to re-examine a solution if we ask them to do something they have not thought of before. (They cannot be complacent about their solutions under those circumstances.)

Evaluation

Assigning grades in large classes is always difficult, because we do not get to know students nearly as well as in small classes. This kind of course, where students feel consistently insecure about their abilities, induces particular anxiety in them. Two ways to reduce anxiety are to allow the students to amass some points on "routine" homework problems, and to use a sliding scale that rewards improvement shown by a good score on the final exam.

In large classes, students' abilities often differ widely. Each assignment can offer problems of varying difficulty. The students can be given some latitude, and rewards for devotion above the call of duty. For example:

> Assignment X: Work problems 1 through 5 and at least five of problems 6 through 15. Extra credit will be given for more than 10 solutions, for more than one solution to a problem, or for exceptionally nice solutions.

The grading scheme should reflect the priorities in the course. One quick scheme is to grade problems on a 0 through 4 point basis. A student gets 1 point for the reasonable use of any heuristic, 2 points for making a plausible guess at a solution (even if incorrect), 3 points for a good try, and 4 points for a complete solution (perhaps modulo arithmetic errors).*

Testing becomes especially difficult in a large class. While take-home tests are ideal and may be used in small classes, one hesitates to use them in courses with large enrollments.Butts suggests using frequent in-class testing, with one test for roughly every five class days. His tests are generally open notes, 60-75 minutes long, and contain four problems: (1) a variant of a problem discussed in class (essentially an exercise, to guarantee a "floor" score on the exam), (2) a problem of the form "use heuristic X to solve problem Y," and (3) and (4), original problems. He uses two options to reduce anxiety. First, tests are

*In general, the issue of how to evaluate students' efforts in these circumstances is quite thorny. For an extended discussion and the details of a grading scheme, see my "Measures of problem solving performance and of problem solving instruction," in the January 1982 Journal for Research in Mathematics Education (Vo. 13, No. 1, pp. 31-49).

scored on a weighted scoring system. The student's best effort is graded on a 40-point basis. The student's second best effort is graded on a 30 point basis. The third best is worth a maximum of 20 points, and the worst at most 10. A student who solves, say, two and a half problems thus gets a score of 40 + 30 + 1/2 (20) = 80. Tom also sells hints during an exam: a completely solved problem with a "1/3 cost" hint earns the student 2/3 of a solution, etc.*

Teaching problem solving is difficult, more so in a large class. But, as Tom Butts writes: "You may not reach every student, but you will feel a great deal of satisfaction for those you do."

*In the long run, of course, these options do little or nothing to affect students' rank in class. Other faculty might prefer to use a "straight" scoring system, explaining to students that in "real" problem solving a score of 50% can be very respectable. Two points should be kept in mind here. (1) Large classes tend to be of the "literacy" variety, and expectations should be tailored accordingly. (2) Since so much mathematical performance depends on the problem solver's confidence, it doesn't hurt to induce some--even if a bit artificially, on occasion.

4. Some "Typical" Problems and Class Discussions.

The kinds of problems we discuss in class and the lessons we derive from them vary during the semester. Once again the analogy to learning to play a sport explains the progression. Early in the term the students have little mastery of the problem solving techniques that they will come to use during the semester. They receive training and practice in those basic techniques (looking for inductive arguments, examining special cases, exploiting easier related problems, specializing, generalizing, etc.) in much the same way that (for example) a novice at tennis receives training and practice in how to serve and how to make the basic forehand and backhand volley shots. Once the basic skills have been mastered, they can be used in an increasingly wide variety of situations. The problems we work become more difficult and time consuming. They are no longer "training" problems but simply good, solid mathematics. The issue for the students is now that they must select the appropriate techniques to grapple with the problems, and do so with some efficiency. Classroom discussions shift as well, with a much greater emphasis on planning solutions and evaluating them as they progress. Some representative problems are described below.

A. PROBLEMS TO MAKE A POINT

Occasionally I want to make sure that a particular point is dramatically made to the students. For that purpose there is a small collection of problems that, in my experience, are almost guaranteed to produce certain reactions. The judicious use of these can be quite telling.

For example, it is important to convince the students at the beginning of the term that you really do have something to teach them. The whole nature of the class is unusual and must be justified. You may be the first teacher they have even had who tried to "focus on the problem solving process." The students have done quite well academically up to this point, without ever worrying about such things. Why should they suddenly do so, especially in a course that deals with elementary subject matter, puts them on the spot and makes them feel uncomfortable so often? To deal with that issue, my problem sets for the first few days of class generally include some problems like the following.

4.1. Determine the sum of the series

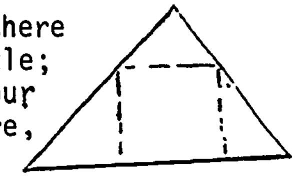
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} .$$

4.2 . For what values of "a" does the system of equations

$$\left\{ \begin{array}{l} x^2 - y^2 = 0 \\ (x-a)^2 + y^2 = 1 \end{array} \right\}$$

have 0, 1, 2, 3, 4, or 5 solutions?

4.3. Consider the triangle to the right. Show that there is a square which can be inscribed in the triangle; that is, show that there is a square with its four corners lying on the sides of the triangle. Here, 2 of the corners will lie on the base.



4.4. If A,B,C, and D lie between 0 and 1, show that

$$(1-A)(1-B)(1-C)(1-D) > 1-A-B-C-D.$$

My experience is that students will generally spend a good twenty minutes on each of these problems, without success. If they do manage to solve them, the solutions are generally contorted and clumsy. For example, we recognize problem 1 as the familiar "telescoping series," in which adjacent terms cancel when each $\frac{1}{i\cdot(i+1)}$ is expressed as $(\frac{1}{i} - \frac{1}{i+1})$. The students who have not seen this are most unlikely to discover it. The students who have acknowledge that it seemed like "pulling a rabbit out of a hat," and that they could never do it on their own.* In problem 2, students will generally jump into an algebraic solution. Keeping track of the multiple solutions is quite difficult, and few if any of the students will be able to solve it. Both problems 3 and 4 can be solved a variety of ways. After problem 4 appeared in a Monthly article, I received a half-dozen non-isomorphic solutions from readers. But my students' behavior on the problem (even junior and senior mathematics majors at Berkeley) is all too predictable: they multiply out the expression on the left, bring all the terms over to that side, and then laboriously try to show that the morass of symbols is positive.

I let the students work on the problems for some time (usually in small group format) and then provide them with "some general mathematical problem strategies that you should be aware of." The suggestions for these problems would be:

1. If there is an "integer parameter," n, in the problem statement, calculate a few special cases for n = 1,2,3,4,5. There may be a pattern that becomes evident. If so, you can verify it by induction.

2. Draw a diagram whenever possible!

3. If the problem in its current form is too difficult, relax one of the conditions. Ask for a little less than the current problem does, while making sure that the problem you consider is of the same nature. Now there should be more than one solution to the new problem. Look at the collection of solutions to the easier problem, and see if the solution to the original is among them.

4. If there are a large number of variables in a problem, all of which play the same role, look at the analogous 1- or 2-variable problem. You may be able to build up a solution from there.

With these hints, the students can generally solve problems 1 and 2

*If the class is at the junior or senior level, this problem will be too familiar to use. However, a problem like "How many subsets containing an even number of elements are there in a set of 87 objects?" has a comparable effect on students -- especially on the ones who jump into complicated combinatoric proofs.

in just a few minutes. Problems 3 and 4 may take a little more time, and are useful for the class to discuss as a whole. But all of the problems have the same effect. The suggestions for solving the problems appear perfectly natural and logical. These are the kinds of things the students should have thought of, but didn't. The students walk out of class convinced that they will learn some useful skills in the course.

Other problems are suitable for driving home certain "morals" as the students become more proficient. For example, the students soon come to recognize the value of suggestion 1 above. At that point in the term, the following problem is useful.

> 4.5. In an elimination tournament in chess, opponents are randomly paired and play one game. Losers are eliminated from the tournament, but winners go on. If we start with 32 players, we get 16 winners, so 16 go on to the next round. If there are an odd number of players in a round, one person does not play but does advance to the next round. With 15 players, one person advances without playing, and 7 winners do.
>
> In general, if there are N players,
>
> If N is even, then N/2 games are played and N/2 players go on to the next round.
>
> If N is odd, $1 + \frac{(N-1)}{2}$ or $\frac{N+1}{2}$ players go on, after $\frac{N-1}{2}$ games have been played.
>
> If N people start in a tournament, how many games must be played before the winner is determined?

The vast majority of students succumb to their training, and impetuously jump into the use of their "integer parameter" strategy. After they have calculated the special cases for N=2,3,4,5,6, and 7, the pattern becomes <u>too</u> obvious: if N people start in the tournament and if one person is eliminated in each game, there will be N-1 losers and therefore N-1 games played! The morals of this problem: make sure you understand the problem fully before you jump into any solution, and do not engage in complex computations unless you are sure there are no simpler options.

One additional "special purpose" problem has to do with the role of proof in mathematics. There are, of course, many ways to obtain an answer to problem 4.6, the least popular of which is the rigorous one.

4.6. Determine the sum of the "geometric series"

$$S = 1/2 + 1/4 + 1/8 + \ldots + 1/2^n \ldots$$

If students are willing to agree that such a series converges, they generally find the following argument most convincing:

"Observe that, multiplying each term by 2, we obtain

$$\begin{aligned} 2S &= 1 + \underbrace{1/2 + 1/4 + 1/8 + \ldots + 1/2^n + \ldots} \\ &= 1 + \qquad\qquad S, \end{aligned}$$

so that S = 1, as we expected."

Once they accept this type of argument, they generally feel that the "epsilon-delta" argument we force upon them is unnecessary. Why go through all that work when you already have a convincing argument? I find problem 4.7 useful.

4.7. Determine the sum of the series

$$T = 1 + 2 + 4 + 8 + \ldots + 2^n + \ldots$$

An argument similar to the one given above yields

$$\begin{aligned} 2T &= \quad 2 + 4 + 8 + 16 + \ldots + 2^n + \ldots = \\ &= \quad T - 1, \end{aligned}$$

so that $T = -1$.

(Q.E.D.?)

(Maxwell's Fallacies in Mathematics is a rich source of such arguments. For more advanced courses, Gelbaum and Olmsted's Counterexamples in Analysis is useful.)

B. "TRAINING" PROBLEMS

Training a student to examine special cases, or to exploit easier related problems, or to use any other problem solving strategy, must be done with the same care and practice as training them to use (for example) the quadratic formula or integration by parts. Generally, I find the following sequence most useful for teaching any particular technique:

i. introducing it with a particularly interesting problem,

ii. having an extensive amount of practice over the next week (say 1/3 of the class problems),

iii. distributing other problems solvable by the same technique randomly through the balance of the term.

C. THE CLASS DISCUSSION OF A DIFFICULT PROBLEM

In this section I shall try to give the flavor of the classroom discussion that took place over the following problem:

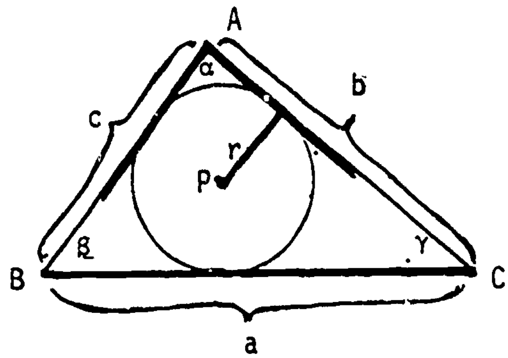
4.8. You are given two line segments of length a and r, respectively, and an angle of measure α. Construct a triangle that has the following properties:

i. one side of the triangle has length a

ii. the radius of the inscribed circle of the triangle is r

iii. the measure of the angle opposite the side of length a is α.

For the sake of easy reference, let us represent the desired triangle, T, as in figure 4.1. The heavily shaded objects represent the three given quantities, from which we are to construct T.



The triangle T

-- figure 4.1 --

The class was familiar with the "basic" straightedge and compass constructions. In addition, (having solved a problem that used it) they knew the construction for the locus of the (variable) vertex A of fixed measure α that lies opposite a fixed side, a. The "standard" procedure for such problems is to try to construct the desired triangle directly. One starts with a given part of T, and then tries to locate -- by means of the intersection of two constructible loci -- a point that uniquely determines the triangle. The class was also aware that an alternate procedure (construct a triangle similar to T, and then scale up or down by a proportionality construction) might be appropriate, and that we should keep an eye out for it. The following is a telegraphic version of the class discussion, which occupied us for a solid forty minutes. My reconstruction of the discussion is an expanded version of class notes written by two students.*

*The class moves at a fast pace, and taking notes during class often proves a distraction for the students. I have found the following alternative quite useful. Each day, two students are designated "official note-takers." Class is audiotaped, and these students use the audiotape to write an "official" description of what happened. These notes are then edited by me or an assistant, typed up, and distributed to the class. The other students are freed of the responsibility of taking notes that day, and wind up with far more comprehensive notes than they would otherwise have. Individual sets of notes are graded and count as term papers.

Planning decision: should we start with

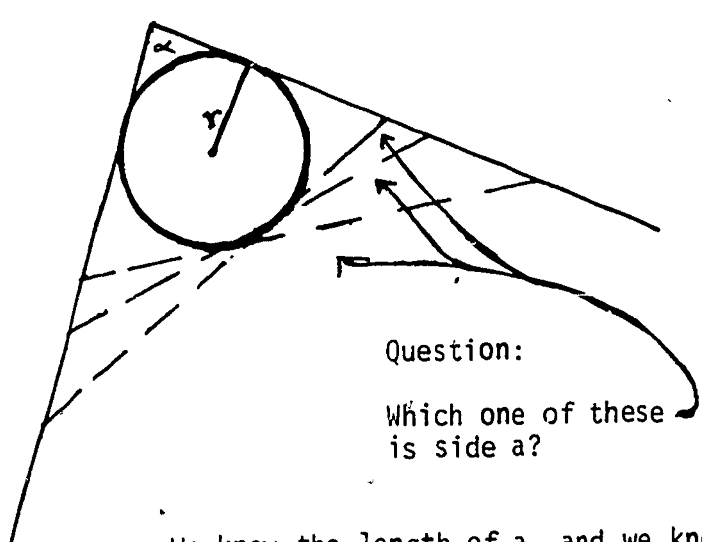
(a) the inscribed circle,

(b) the side a, or

(c) the angle α at vertex A?

Choice (a) is out of the question. If we start with the circle, where does side a go? How is side a related to vertex A? This isn't worth pursuing. Choice (b) is reasonable. If we start with side a, we can (i) construct one locus for vertex A, and (ii) one locus for the center of the circle, P. But how are the two related? Not clear. This may be worth pursuing, but let's look at (c). If we start with the angle α, it looks like we can inscribe the circle. Can we get a solution from there? Maybe, maybe not; but it's worth pursuing.*

We started with the angle α, easily found the point P, and wound up with the dilemma in figure 4.2:



We knew the length of a, and we knew that a must be tangent to the circle.

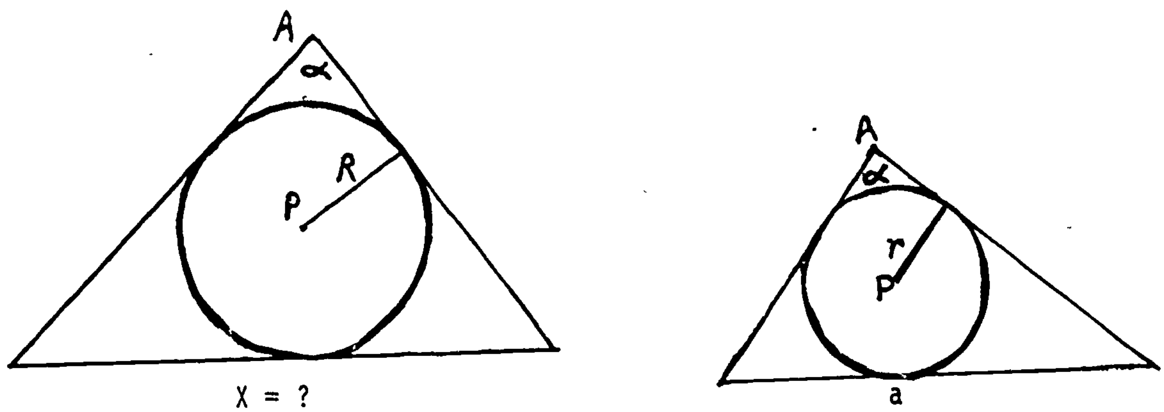
We had two pieces of information about side a, but no way to link them.

-- figure 4.2 --

*Footnote is on next page.

This seemed hopeless. Yet, we might be able to save something with a similarity construction. This was worth a brief try. We drew in an arbitrary tangent at the bottom of the circle, hoping that we could later scale upwards or downwards (figure 4.3). This led nowhere, and we were stuck.

The Similarity Construction



We can get this figure; we can inscribe a circle of any given radius, R.

This is the figure we want.

If the two triangles are similar,

$$\frac{X}{R} = \frac{a}{r}.$$

This doesn't seem to lead anywhere.

--figure 4.3--

*The "Planning decision" summarized on the previous page took about five minutes of actual discussion in the classroom. I played the role of moderator, asking questions like: "All right, what choices do we have? Are there any others? Which of these seem promising? So, it's between (b) and (c). Which one do you want to work on?" Members of the class argued about the relative merits of the two approaches, and decided to try (c) "for a while." If it didn't work out, they would look at (b) again.

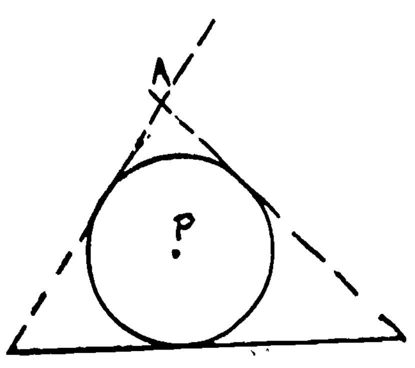
Managerial Decision: Should we pursue this line of thought further, or should we back off and consider another alternative?

Given that we had really seemed to reach a dead end, we decided to look at (b) again. What if we began the construction with side a?

1. We knew that we could construct the locus of points that made a fixed angle opposite a (one locus for the vertex A).

2. We knew we had one locus for the point P.

We needed a third piece of information about the triangle. If we could find another locus for vertex A, that would finish off the construction. Another locus for point P would allow us to construct the inscribed circle, and that would do it also: we construct the tangents to the circle through the endpoints of a (figure 4.4).



Given the side a and the inscribed circle (in place), complete the triangle.

-- figure 4.4 --

Here are the choices:

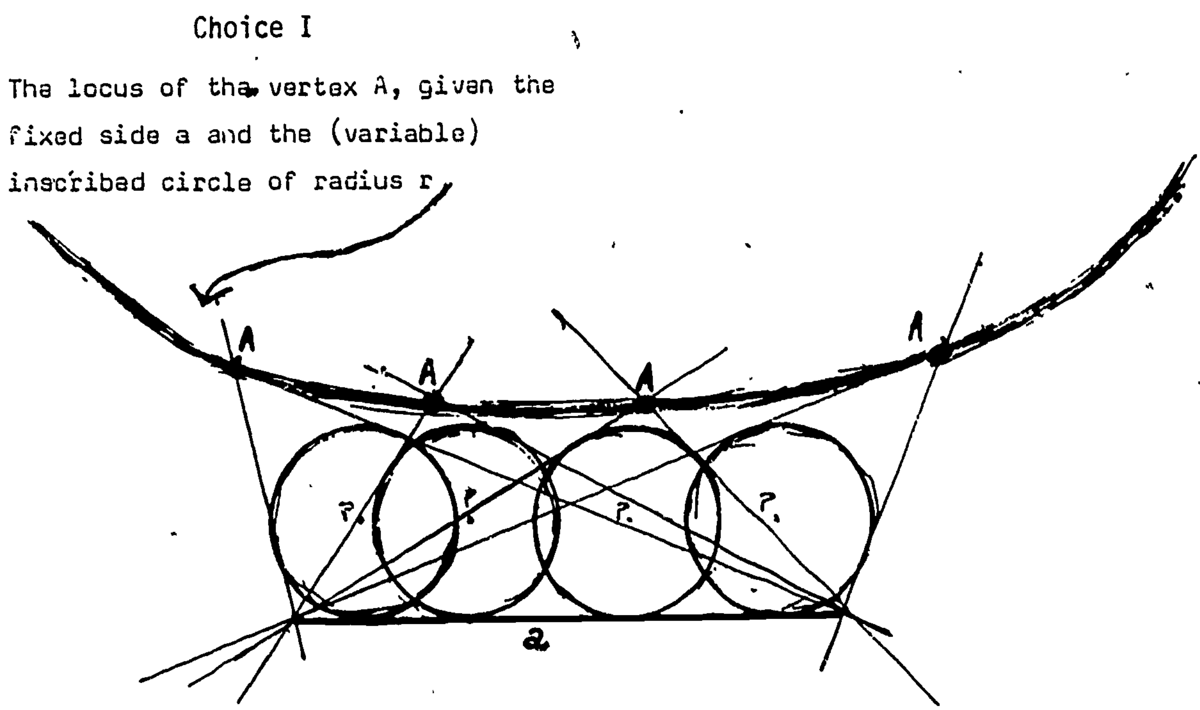
I. Determine the locus of the vertex A, given the side a and the (variable) inscribed circle of fixed radius r.

II. Determine the locus of the center of the inscribed circle, given the side a and the (variable) vertex A that makes an angle α opposite a.

Which one should we pursue?

Suggestion: We're on very shaky ground, and have no foundation for making a good judgment. It may be time for making some rough sketches. The result of some empirical work may suggest that we choose one alternative over the other. It may even suggest an hypothesis.

We try choices I and II respectively, in figures 4.5 and 4.6.



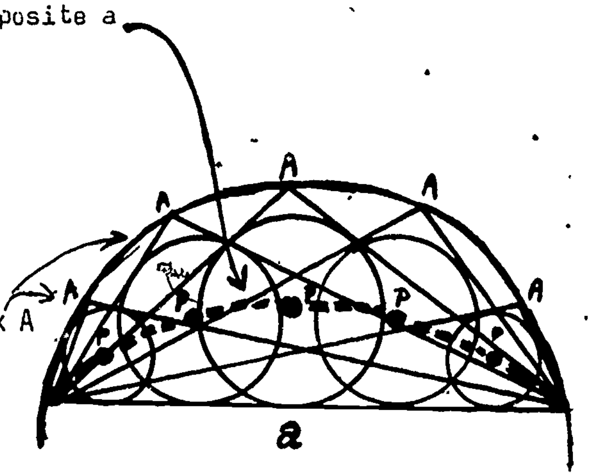
This locus does not appear suggestive....

-- figure 4.5 --

Choice II

The locus of the centers of the inscribed circles, P,
given the side a and the (variable) vertex A
that makes angle α opposite a

The locus of vertex A



The locus is symmetric, passes through the endpoints of a,...
might it be the arc of a circle?

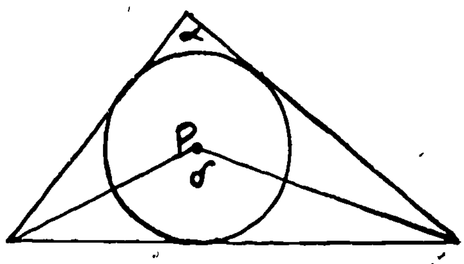
-- figure 4.6 --

Choice I appears to lead nowhere, but choice II might give us something: the rough sketch suggests that the locus of P (given the variable point A) may be a circle that has the side a as a chord. (If we were unsure about the conjecture, we could do a more accurate sketch. Let us not demean empirical exploration.)

<u>Subproblem</u>: Prove that the locus of P, given fixed a and variable A, is a circle that has a as a chord.

<u>Question</u>: How do we prove such things? What do we know about circles and chords? In this context, we know that the set of points that make a fixed angle opposite a given line segment (chord) is a circle.

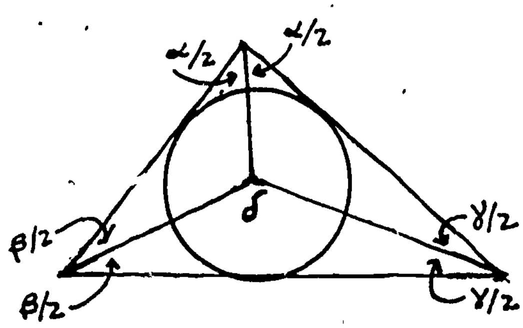
Reformulation of subproblem: Show that, given a and α, the point P makes a fixed angle opposite the side a. Re-reformulation: in figure 4.7, can we show that the angle δ is a function of α alone?



--figure 4.7--

Sub-subproblem: obtain a formula for δ in terms of α.

We knew that the point P lies on the intersection of the three angle bisectors of T, and this led to the argument in figure 4.8.



From the bottom triangle, $\delta + \beta/2 + \gamma/2 = 180^0$,

or $2\delta + \beta + \gamma = 360^0$

From the triangle T, $\alpha + \beta + \gamma = 180^0$.

Thus $\delta = 90^0 + \alpha/2$.

The relationship between δ and α.

-- figure 4.8 --

This solved the problem. We knew that the center of the inscribed circle could be obtained as the intersection of

(i) the circle that made an angle $\delta = 90 + \alpha/2$ with chord a, and

(ii) the line parallel to a at distance r.

Once we had the inscribed circle, we could finish the construction as suggested in figure 4.4.

A brief summary discussion

As I noted above, the solution of this problem took the class some forty minutes to achieve. I could have presented it, completely, in ten minutes. Is that much time on one problem, with false starts, reversals, blind alleys, major strategic decisions, subproblems, etc., really justifiable? I think so, although I am certainly not about to recommend that we solve every problem this way. There are times when we simply need to present information, when students need to master routine procedures, and when (for any of a number of good reasons) we must ask students to learn and discover by themselves. Indeed, our most important function as teachers is to train our students to learn and think by themselves. I believe this kind of classroom problem solving is a catalyst for that kind of learning.

In solving problem 4.8, the class made a completely unexpected discovery: the locus of the centers of the inscribed circles, under the given conditions (fixed side a and variable vertex A of measure α opposite a) is a circle with side a as a chord. The discovery was prompted by need. It was suggested by some empirical work. It was surprising, and it would prove useful in other constructions. To put it simply, the students were doing mathematics in class that day. The experience they had, in discovering that (minor) result, is similar to the experience that we have when we are engaging in real mathematics. It allows them to see mathematics as a living, breathing discipline in which discovery is both possible and enjoyable.

What about the false starts, reversals, blind alleys, etc.? The fact is that doing mathematics involves all of them. Doing mathematics successfully involves overcoming those difficulties: knowing when to "explore," making choices about which avenues to pursue, pursuing leads to see whether they will bear fruit but knowing when to abandon them, etc. Students who know this are more likely to be adventurous when they try to do mathematics themselves. Discussions like the one above provide them with a means of seeing how they can do so in a sensible and efficient way.

Solving problems is the business of mathematicians; it is the excitement of mathematics. We owe it to those who will be the mathematicians of the future, to those who will use mathematics, and to those who would like a "feeling" for mathematics, to introduce them to the problem solving experience. We hope and believe that the problem solving approach to mathematics, throughout the curriculum and through a variety of problem courses, will convey to our students the excitement and beauty of mathematics. To the degree that we train our students to think independently and to use the knowledge at their disposal, we will have succeeded as teachers.

An Annotated Bibliography of Problem Solving Resources

Overview

This bibliography offers a broad sampling of the problem solving literature. We solicited extensive contributions from experts in each of the types of problem solving courses represented in our survey, and received in addition a large number of suggestions from those teachers of problem solving who responded to the survey. Whatever facet of the subject interests you, you will find some useful resources listed below. Of course, our listing a source in this bibliography does not constitute an endorsement in any sense. Our familiarity with and enthusiasm for each reference can best be seen in the annotations. Similarly, exclusion should not be taken as a negative comment: in any sampling, some valuable sources will be overlooked. Our coverage of the literature in languages other than English is particularly sparse. If a reference that you find especially valuable does not appear in the bibliography, please call it to our attention.

The problem solving literature is vast, and may seem overwhelming. We have tried to make "initial entry" as easy as possible. The three sections of the bibliography list journals, books, and articles, respectively. At the beginning of the sections of books and articles, we highlight a small, exemplary collection of references, which serve as a brief introduction to the best in each area. Wherever possible, we have provided detailed information about each source. The characterizations include (a) the types of courses for which the reference appears most appropriate (see the survey results for more detailed descriptions of each type), (b) its focus or subject matter, and (c) its level. The categories are as follows.

(a) Types of courses

Con: Contests
Gen: General
Lit: Literacy
Mod: Modeling
Rem: Remedial
Tch: Teaching Training

(b) Focus or subject matter

AI: Artificial Intelligence
Alg: Algebra
Ana: Analysis
Cre: Creativity
Geo: Geometry
His: History
Num: Number Theory
Phil: Philosophy
Pro: Probability
Psy: Psychology
Rec: Recreational Mathematics
Res: Research in Problem Solving
Top: Topology

(c) Levels

E: Elementary (up to and including freshman-sophomore level)
I: Intermediate (sophomore-junior)
A: Advanced (senior and beyond)

Of course, many of our sources are suitable for more than one category. The format, for example the (Gen, Res, Tch: E,I) listing given to Pólya's Mathematical Discovery, is self-explanatory.

Category I:

Journals

Journals and newletters offer the best ways to keep abreast of developments in the world of problem solving. Whether you have a specialized interest in research or teaching, a general wish to keep up with new developments in a variety of fields, an interest in following contests, or an insatiable thirst for new problems -- for yourself or your students -- there are journals that suit those interests. As noted below, many journals have problem sections, often at varying levels of difficulty.

JOURNALS

The AMERICAN MATHEMATICAL MONTHLY (all categories, all levels)

The MONTHLY is published ten times a year by the MAA. The annual subscription price to a member of the Association is $20.00, included as part of the annual dues of $40.00. Students receive a 40% discount.

The problem section, now edited by G. L. Alexanderson and Dale Mugler, offers a rich variety of problems and solutions from elementary to advanced levels. Articles on all aspects of problem solving periodically appear in the MONTHLY. For a subscription, write:

A.B. Willcox
Executive Director
Mathematical Association of America
1529 Eighteenth Street N.W.
Washington, D.C. 20036

The ARITHMETIC TEACHER (Tch:E)

The ARITHMETIC TEACHER is published by the NCTM. Its primary focus is on classroom suggestions for elementary school teachers. The November 1977 (Vol. 25, No. 2) issue was devoted to problem solving. Write:

National Council of Teachers of Mathematics
1906 Association Drive
Reston, VA 22091

CANADIAN MATHEMATICAL BULLETIN

This is a journal of the Canadian Mathematical Society, published 4 times a year. The editor of the Problem Section is E.J. Barbeau. The cost varies for different categories of membership in the CMS. Write:

C.P. Wright, Executive Secretary
Canadian Mathematical Society
577 King Edward Ave.
Ottawa, ON K1N 6N5
Canada

CRUX MATHEMATICORUM (Gen,con:I)

This is an excellent source of problems, and is very useful for problem seminars. It is a good way to keep on top of current events in problem solving. The editor is Léo Sauvé, and there is an "Olympiad Corner" edited by Murray Klamkin. Bound volumes from 1975 are available. Many of our references come from The Olympiad Corners: #3 (Vol. 5 (1979), 62-69), #4 (Vol. 5, 102-107), #8 (Vol. 5, 220-228), and #21 (Vol. 7, (1981), 11-17). Write:

F.G.B. Maskell
Algonquin College
200 Lees Avenue
Ottawa, Ontario, Canada
K15 OC5

DELTA

Write:

Polskie Towarzystwo Matematyczne
Ul. Śniadeckich 8, 00-950
Warsaw, Poland

The FIBONACCI QUARTERLY (Num:E,I)

This is a magazine devoted to the study of integers with special properties. It is the official journal of the Fibonacci Association. Dues are $20 per year. The problem editors are A.P. Hillman (Elementary Section) and R.E. Whitney (Advanced Section). Write to:

Mr. Richard Vine
Mathematics Department
University of Santa Clara
Santa Clara, CA 95053

FUN WITH MATHEMATICS (Gen:E)

This is an informal publication of the Ontario Institute for Studies in Education. It is published 8 times a year and is sold in sets of 10 copies. A single issue (10 copies) costs $1.50; a year's subscription (10 copies of all 8 issues) costs $10.00. It is especially designed to provide continuous material for children's individual reading in mathematics and to supplement material studied at school by problems, games, and investigations in which the child can be involved on his own. (For students in the grades 5-8 range and also for bright children in grade 4 or for older children taking a general mathematics or remedial arithmetic course.) The editors are Shmuel Avital and Mary Stager. Write to:

Fun with Mathematics
C/O Mary Stager
Ontario Institute for Studies in Education
252 Bloor Street West
Toronto, ON M5S 1V6
Canada

JAMES COOK MATHEMATICAL NOTES

This informal journal appears 3 times a year and the editor is B.C. Rennie. The first 17 issues have been reproduced in a single bound volume available at a cost of $5.75 US (includes postage). Write to:

Professor B.C. Rennie
Mathematics Department
James Cook University of North Queensland
Townsville 4811
Australia

JOURNAL OF CREATIVE BEHAVIOR

Write:

Creative Education Foundation Inc.
State University College
1300 Elmwood Avenue
Buffalo, NY 14222

JOURNAL OF RECREATIONAL MATHEMATICS (Rec:E,I)

This journal is published twice a year. It has a Problems and Conjectures Section edited by Friend H. Kierstead, Jr. Write to:

Baywood Publishing Co.
120 Marine Street
Farmingdale, NY 11735

JOURNAL FOR RESEARCH IN MATHEMATICS EDUCATION (Res:E,I,A)

JRME is the one national journal devoted to research in mathematics education, and is edited by Jeremy Kilpatrick. Research articles on all aspects of problem solving appear with some regularity. Write:

NCTM
1906 Association Drive
Reston, VA 22091

The MATYC JOURNAL (Computers:E,I)

This is the journal of two-year-college mathematics and computer education. It is published 3 times a year at a cost of $8.50 per year or $16 for 2 years. The problem editor is M.J. Brown. Write to:

The MATYC Journal
Department of Math./Stat./Comp.
Nassau Community College
Garden City, NY 11530

The MATHEMATICAL GAZETTE (Gen:E)

This is the journal of the Mathematical Association of Great Britain, an association of teachers and students of elementary mathematics. It is published 4 times a year and is included in the membership fee. Although it does not have a formal problem section, it does have a Problem Bureau, and many problems can be extracted from the papers and notes it publishes. Write to:

Honorary Treasurer, Math. Assoc.
259 London Road
Leicester LE2 3BE
Great Britain

MATHEMATICAL SPECTRUM (Gen:E,I)

This is a magazine for the instruction and entertainment of student mathematicians in schools, colleges, and universities, as well as the general reader interested in mathematics It is published 3 times a year. The editor is D.W. Sharpe. Write to:

The Editor, Mathematical Spectrum
Hicks Building
The University
Sheffield S3 7RH
England

MATHEMATICS MAGAZINE (Gen:E,I)

This is published by the MAA. There are 5 issues a year. Members of the MAA or of Mu Alpha Theta may subscribe at reduced rates. Write to:

A.B. Willcox
Executive Director
Mathematical Association of America
1529 Eighteenth St. NW
Washington, DC 20036

The MATHEMATICS STUDENT JOURNAL (Con:E)

No longer published. This slim newsletter was published 8 times a year by the National Council of Teachers of Mathematics and contained a Competition Corner edited by George Berzsenyi. The individual subscription rate was $2 a year for NCTM members.

MATHEMATICS TEACHER (Tch:E)

The MATHEMATICS TEACHER is published by the NCTM. Its primary focus is on useful ideas for classroom teachers at the secondary school level. Problem solving is a frequent topic of discussion. Write:

NCTM
1906 Association Drive
Reston, VA 22091

NIEUW ARCHIEF VOOR WISKUNDE (Con:A)

This journal is published 3 times a year. The problem editor is M.L.J. Hautus and the problems are generally of an advanced type. Write to:

Adm. of Mathematisch Centrum
Tweede Boerhaavestraat 49
1091 Al Amsterdam
The Netherlands

ONTARIO SECONDARY SCHOOL MATHEMATICS BULLETIN (All categories:E)

The BULLETIN is published 3 times a year at the University of Waterloo. It has a problem section edited by E.M. Moskal. Write to:

Mr. E. Anderson
Faculty of Mathematics
University of Waterloo
Waterloo, ON N2L 3G1
Canada

THE PENTAGON

This is the official journal of the Kappa Mu Epsilon College Honor Society. It is published twice a year. There is a Problem Corner edited by Kenneth M. Wilke. Write to:

Douglas W. Nance
Business Manager, The Pentagon
Central Michigan University
Mount Pleasant, MI 48859

PI MU EPSILON JOURNAL (Gen:con:E,I)

This journal is published twice a year at the South Dakota School of Mines and Technology. It is the official journal of the Pi Mu Epsilon honorary mathematical fraternity. There is an extensive problem section edited by Clayton W. Dodge. Write to:

Pi Mu Epsilon Journal
South Dakota School of Mines and Technology
Rapid City, SD 57701

PROBLEM SOLVING (Gen: E,I,A)

PROBLEM SOLVING serves as a general clearinghouse for information about ongoing research and development in problem solving. Conferences and publications are announced and reviewed. A broad spectrum of interdisciplinary work is covered. Write:

Franklin Institute Press
P.O. Box 2266
Philadelphia, PA 19103

P.S. NEWS: A SHARING OF IDEAS ABOUT PROBLEM SOLVING (Gen, Res:E,I,A)

P.S. NEWS is an informal newsletter edited by Don Woods. It is an interdisciplinary offering, with ideas about problem solving in engineering, medicine, mathematics, etc. Write:

Donald Woods
Dept. of Chemical Engineering
McMaster University
Hamilton, Ontario L8S 4L7
Canada

SCHOOL SCIENCE AND MATHEMATICS (Gen:E)

This is the official journal of the School Science and Mathematics Association, Inc. The Problem Department is edited by N.J. Kuenzi and Bob Prielipp. The March 1978 (Vol. 78, No. 3) issue was devoted to problem solving. Write:

Dale M. Shafer, Executive Secretary
School Science and Mathematics
Stright Hall, P.O. Box 1614
Indiana University of Pennsylvania
Indiana, PA 15705

TWO YEAR COLLEGE MATHEMATICS JOURNAL (Gen: E,I)

This is one of three journals published by the Mathematical Association of America. It is published 5 times a year. The Journal has a Problem Section edited by Erwin Just. Write to:

TYCMJ Subscription Department
The Mathematical Association of America
1529 Eighteenth St., N.W.
Washington, D. C. 20036

UMAP JOURNAL (Mod: E.I.A)

The UMAP project is concerned with disseminating information and classroom materials dealing with applications of mathematics. Many results are published in the UMAP Journal, which is published by COMAP, the Consortium for Mathematics and Its Applications. The UMAP catalogue, which lists hundreds of modules covering a wide range of applications, is available from:

COMAP
Suite #4
271 Lincoln Street
Lexington, MA 02173

Category II:

Books

The literature of problem books, and of books about problem solving, is immense. The best general introduction to problem solving, at virtually any level, comes from the pen of Pólya. How to Solve It is a classic introduction to heuristics at an elementary level; Mathematical Discovery is used almost universally for teacher training and has many interesting problems; Mathematics and Plausible Reasoning is much more substantive; and anyone who can claim to have solved all the problems in Pólya and Szegő's Problems and Theorems in Analysis has already embarked on a very solid problem solving career. In general, problem sources come in a wide variety of shapes and sizes: see the characterizations for the particular ones of interest to you. For those with an interest in contests, the definitive work on the Putnam exam through the mid-1960's is Gleason, Greenwood, and Kelly's The William Lowell Putnam Mathematical Competition: Problems and Solutions, 1938-1964. At the secondary level, see Greitzer's International Mathematics Olympiads, 1959-1977 and Salkind's compilations of the annual MAA high school contests, The Contest Problem Book(s). Modeling is too diverse for us to point to a single source; for the best overview of the area, see the CUPM's Recommendations for a General Mathematical Sciences Program. For a view of research with classroom applications, see Schoenfeld's Mathematical Problem Solving. The best introduction to problem solving at the school level is the NCTM's 1980 Yearbook, Problem Solving in School Mathematics. The NCTM's Research in Mathematics Education has a review of the research literature, mostly at the school level. The intersection of problem solving and remediation is recent and small, but rapidly growing; we look for a forthcoming report from an M.A.A. panel on remediation to help sort things out. A precursor to contemporary work is Bloom's Problem Solving Processes of College Students. A current work is Whimbey and Lochhead's Problem Solving and Comprehension, a Short Course in Analytical Reasoning. For work outside mathematics, see Newell and Simon's Human Problem Solving, or Nilsson's Principles of Artificial Intelligence.

BOOKS

Aaboe, Asger. EPISODES FROM THE EARLY HISTORY OF MATHEMATICS. Washington: Mathematical Association of America, 1964. (His, Lit:E,I)

Abell, P. MODEL BUILDING IN SOCIOLOGY. New York: Schocken, 1971 (Mod) The CUPM Modeling Panel recommends this.*

Adams, J. CONCEPTUAL BLOCKBUSTING, 2nd Edition. (Stanford Alumni Association) New York: W.W. Norton, 1980. (Gen,cre:E,I) This book offers a broad discussion of creativity, with many interesting examples.

Aggarwal, R. and Khera, I. MANAGEMENT SCIENCE CASES AND APPLICATIONS. San Francisco: Holden-Day, 1979. (Mod) The CUPM Modeling Panel recommends this.

Aichele, D. B., and Reys, R.E. (Eds.). READINGS IN SECONDARY SCHOOL MATHEMATICS. Boston, Mass: Prindle, Weber, & Schmidt, Inc., 1974. (Tch:E) This is a source book with a variety of essays on different topics related to mathematics education, including problem solving. The essays can serve as focal points for discussion in a teacher training class.

Aleksandrov, A. N. and Lavrentiev, M.A. MATHEMATICS: ITS CONTENT, METHODS AND MEANING. Boston: MIT, 1964. (Gen,His:E,I) This three volume set is a rich source of material on the history and the background of mathematics. It does not deal with problem solving per se, but serves in encyclopedia-like fashion as an introduction to a broad range of mathematics.

Anderson, B.F. THE COMPLETE THINKER. Englewood Cliffs, NJ: Prentice-Hall, 1980. (Lit,cre:E)

Anderson, Carolyn and Haller, Jackie. BRAIN STRETCHERS BOOK I. Pacific Grove, CA: Midwest Publications Co., Inc. 1975. (Rec:E)

*The phrase "The CUPM Modeling panel recommends this," used often in the sequel, is shorthand for the following: "This reference was listed as a 'reference on modeling' in the Modeling and Operations Subpanel Report in the Committee on the Undergraduate Program in Mathematics' (1981) Recommendations for a General Mathematical Sciences Program."

Anderson, Carolyn and Haller, Jackie. BRAIN STRETCHERS, BOOK 2. Troy, MI: Midwest Publications Co., Inc., 1977.
(Rec:E)

Anderson, R.C., Spiro, R.J. and Montague, W.E. (Eds.). PROCESSES IN ACQUIRING KNOWLEDGE. Hillsdale, NJ: Lawrence Erlbaum Associates, 1976.
(Res,psy:I,A) A collection of papers from the psychological community dealing with the cognitive mechanisms by which humans acquire, store and process knowledge.

Andrew, J. and McLone, R. MATHEMATICAL MODELING. Woburn, MA: Butterworth, 1976.
(Mod) The CUPM Modeling Panel recommends this.

Antonov, N., Vygodsky, M., Nikitin, V., Sankin, A. PROBLEMS IN ELEMENTARY MATHEMATICS FOR HOME STUDY. Moscow: Mir Publishers, 1974.
(Gen,con:E) Many problems from arithmetic, algebra, geometry and trigonometry.

Aris, R. MATHEMATICAL MODELING TECHNIQUES. Belmont, CA: Pitman, 1978.
(Mod) The CUPM Modeling Panel recommends this.

Arnold, B.H. INTUITIVE CONCEPTS IN ELEMENTARY TOPOLOGY. Englewood Cliffs, NJ: Prentice-Hall, 1962.
(Top:I)

Atkinson, R. et al. INTRODUCTION TO MATHEMATICAL LEARNING THEORY. Huntington, NY: Krieger, 1965.
(Mod, psy:I) The CUPM Modeling Panel recommends this.

Averbach, Bonnie and Chein, Orin. MATHEMATICS: PROBLEM SOLVING THROUGH RECREATIONAL MATHEMATICS. San Francisco: W.H. Freeman, 1980.
(Gen,Tch,Lit:E) A non-threatening and well-written introductory text on recreational mathematics, using a "problem solving" format as a means of introducing the subject. The book is designed to serve as the text for an introductory level college course.

Ball, W.W.R. and Coxeter, H.S.M. MATHEMATICAL RECREATIONS AND ESSAYS. 12 ed., Toronto: University of Toronto Press, 1974.
(Gen,Lit,Rec:E,I) A classic collection of problems and entertainment. It should be on everyone's bookshelf.

Barbeau, E. and Moser, W. THE FIRST TEN CANADIAN MATHEMATICAL OLYMPIADS (1969 - 1978) WITH SOLUTIONS. Ottawa: Canadian Mathematical Society (577 King Edward Avenue, Ottawa, Ontario, Canada. KIN 6NS)
(Con:E,I) A good source of challenging problems.

Barbeau, E., Klamkin, M. and Moser, W. 1001 PROBLEMS IN HIGH SCHOOL MATHEMATICS I, II, III, and IV. Ottawa: Canadian Mathematical Society, 1976
(Con:E, I) Some very challenging problems here.

Barnard, Douglas St. Paul. FIGURE IT OUT: 100 PUZZLES. London: Pan Books, 1973.
(Rec:E,I) A large variety of puzzle problems.

Barnard, S. and Child, J.M. HIGHER ALGEBRA. London: MacMillan, New York: St. Martin's Press, 1955; 585 pages.
(Alg, con:E,I) Recommended reading by M. Klamkin for a Mathematical Olympiad Program.

Barr, S. EXPERIMENTS IN TOPOLOGY. New York: Thomas Crowell, 1964.
(Top)

Barr, Stephen. A MISCELLANY OF PUZZLES/MATHEMATICAL AND OTHERWISE. New York: Crowell, 1965.
(Rec:E)

Barr, Stephen. SECOND MISCELLANY OF PUZZLES/MATHEMATICAL AND OTHERWISE. New York: MacMillan, 1969.
(Rec:E)

Bartholomew, D. STOCHASTIC MODELS FOR SOCIAL PROCESSES. New York: Wiley, 1973.
(Mod) The CUPM Modeling Panel recommends this.

Bartlett, M. STOCHASTIC POPULATION MODELS. New York: Methuen, 1960.
(Mod) The CUPM Modeling Panel recommends this.

Barton, R. A PRIMER ON SIMULATION AND GAMING. Englewood Cliffs, NJ: Prentice-Hall, 1970.
(Mod) The CUPM Modeling Panel recommends this.

Bauman, R.P. THE LOGIC OF MATHEMATICS AND SCIENCE. Birmingham: The University of Alabama, 1977.
(Lit,Rem:E) A good resource book for "literacy" and "remedial" courses.

Beck, A., Bleicher, M. and Drowe, D. EXCURSIONS INTO MATHEMATICS. Worth, New York, 1969.
(Tch,Lit:E)

Beckenbach, E. and Bellman. INEQUALITIES. Berlin: Springer-Verlag, 1965. 198 pages.
(Con:E,1) Recommended reading by M. Klamkin for a Mathematical Olympiad Program.

Begle, E.G. CRITICAL VARIABLES IN MATHEMATICS EDUCATION: FINDINGS FROM A SURVEY OF THE EMPIRICAL LITERATURE. Washington, DC: Mathematical Association of America, 1979.
(Tch,Res:E,I) A brief "state of the art" summary of research in mathematics education.

Beiler, Albert, H. RECREATIONS IN THE THEORY OF NUMBERS/THE QUEEN OF MATHEMATICS ENTERTAINS. New York: Dover, 1966.
(Num,Lit,Rec:I) Lots of interesting problems from number theory are described.

Bell, E.T. THE LAST PROBLEM. New York: Simon and Schuster, 1961.
(His,Num,Rec)

Bender, E. AN INTRODUCTION TO MATHEMATICAL MODELING. New York: Wiley, 1978.
(Mod) The CUPM Modeling Panel recommends this.

Benson, R.V. EUCLIDEAN GEOMETRY AND CONVEXITY. Ne York: McGraw-Hill, 1966, 265 pages.
(Gen,con,geo:I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Berlekamp, E., Conway, J.H., and Guy, R. WINNING WAYS. Harcourt Brace Jovanovich: London, NY: Academic Press, 1982.

Biggs, N., Lloyd, E.K., and Wilson, R. GRAPH THEORY 1736-1936. Clarendon Press, Oxford, 1976.
(His,Rec, TOP: I, A.)

Billstein, Rick, Liebeskind, Shlomo, and Lott, Johnny W. A PROBLEM SOLVING APPROACH TO MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS. Menlo Park, CA: Benjamin/Cummings, 1981.
(Tch:E) A text for prospective elementary school teachers with a great (and conscious) debt to Pólya.

Black, M. CRITICAL THINKING. Englewood Cliffs, NJ: Prentice-Hall. 1946.

Block, James H. MASTERY LEARNING IN CLASSROOM INSTRUCTION. New York: MacMillan, 1975.
Discusses a "level of competence" approach to grading, as opposed to curved exam scores.

Bloom, B.S., and Broder, L.J. PROBLEM-SOLVING PROCESSES OF COLLEGE STUDENTS. Chicago: The University of Chicago Press, 1950.
(Gen,Res,Rem:E) The book deals with "problem solving" or "thinking" in a very broad sense - as we might see it on the SAT or GRE exams. Bloom was one of the first researchers to focus on what students actually do when they work on such problems - a far cry from the logical analysis that we expect them to perform. A look at what actually goes on in the students' heads is enlightening.

Boden, M.A. ARTIFICIAL INTELLIGENCE AND NATURAL MAN. New York: Basic Books, 1977.
(AI,Lit) Research on computer simulations of intelligent performance sheds light on thinking processes.

Bogen, J. E. THE OTHER SIDE OF THE BRAIN.
(Gen,psy) Two interesting papers having to do with how the left side of your cerebral cortex differs from your right side. Crudely: One half does algebra - the other half does geometry.

Bottema, O. et al. GEOMETRIC INEQUALITIES. Groningen, Netherlands: Wolters-Noordhoff, 1969, 151 pages.
(Con,Geo:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Bourne, L.E. et al. THE PSYCHOLOGY OF THINKING. Englewood Cliffs, NJ: Prentice-Hall, 1971.
(Res,Psy:E,I)

Bradis, V.M., Minkovshii, V.L., and Karcheva, A.K. LAPSES IN MATHEMATICAL REASONING. Translated by J.J. Schorr-Kon. New York: The Macmillan Company, 1963.
(Res,Psy)

Brams, S. GAME THEORY AND POLITICS. The Free Press, 1975.
(Mod) The CUPM Modeling Panel recommends this.

Brooke, Maxey. COIN GAMES AND PUZZLES. New York, New York: Dover Publications, Inc., 1963.
(Rec,E,I) Don't let the title fool you, these puzzles are not just for kids.

Brooke, Maxey. 150 PUZZLES IN CRYPT-ARITHMETIC. 2nd Rev. Ed. New York: Dover publications, 1969.
(Rec:E,I) These are like "FORTY + TEN + TEN + TEN = SIXTY" where numbers are substituted for letters. Some are difficult. If you like these kinds of puzzles, look here first. The first few pages give a few hints on how to solve this type of problem.

Bittinger, Marvin L., LOGIC, PROOF AND SET THEORY. Reading, MA: Addison-Wesley, 1982.

Braswell, J.S. MATHEMATICS TESTS AVAILABLE IN THE UNITED STATES. Reston, VA: NCTM, 1976.

Brousseau, Brother Alfred. AN INTRODUCTION TO FIBONACCI DISCOVERY. San Jose: The Fibonacci Association, 1965. (Available from the Fibonacci Association, University of Santa Clara, Santa Clara, CA 95053).
Many nice problems involving Fibonacci numbers. Useful for high school students or college freshmen.

Brousseau, Brother Alfred. SAINT MARY'S COLLEGE MATHEMATICS CONTEST PROBLEMS. Palo Alto, CA: Creative Publications, 1972.
(Con:E,I) Some good contest problems are here, grouped as "elementary" and "advanced". Many of the problems here are quite clever and unusual.

Bruner, J.S. THE PROCESS OF EDUCATION. Cambridge: Harvard University Press, 1960.
(Gen,Tch,Res:E) Bruner's influence on curricular development in the U.S. was tremendous. This is one of his most important books.

Bruner, J.S., Goodnow, J.J., and Austin, G.A. A STUDY OF THINKING. New York: John Wiley & Sons, Inc., 1956. (Gen,tch:E)

Bryant, Steven J., Graham, George E., and Wiley, Kenneth G. NON-ROUTINE PROBLEMS IN ALGEBRA, GEOMETRY AND TRIGONOMETRY. New York: McGraw-Hill, 1965.
(Gen,Lit:E,I) The book contains problems ostensibly accessible to 10th and 11th graders. The more thought-provoking problems would keep college freshmen and sophomores busy.

Burkill, J.C. and Cundy, H.M. MATHEMATICAL SCHOLARSHIP PROBLEMS. Cambridge: Cambridge University Press, 1961.
(Con:E,I) A collection of practice problems for the Cambridge University scholarship examination in mathematics. The problems vary from the routine to the unusual, covering algebra, geometry, trigonometry, calculus, mechanics and "misc."

Burns, Marilyn. THE BOOK OF THINK (OR HOW TO SOLVE A PROBLEM TWICE YOUR SIZE). Boston: Little, Brown and Company, 1976.
(Rec:E) A pleasant introductory book for young students.

Bushaw, Donald et al. A SOURCEBOOK OF APPLICATIONS OF SCHOOL MATHEMATICS. Reston, VA: NCTM, 1980 (Gen,Tch:E) A collection of problems prepared by a joint MAA/NCTM committee that offers real world mathematics applications, not just "story problems."

Butts, Thomas. PROBLEM SOLVING IN MATHEMATICS: ELEMENTARY NUMBER THEORY AND ARITHMETIC. Glenview: Scott, Foresman, 1973.
(Gen,Tch,Lit:E) Very good little book. It's out of print now, but worth looking for.

Buzen, T. USE YOUR HEAD. London: BBC Publications, 1974.
(Gen:E) Ideas and suggestions for organizing material, improving your memory and learning.

Carrier, G. TOPICS IN APPLIED MATHEMATICS, VOL. I AND II. MAA summer seminar lecture notes, Mathematical Association of America, 1966.
(Mod) The CUPM Modeling Panel recommends this.

Carroll, L. MATHEMATICAL RECREATIONS OF LEWIS CARROLL. Dover New York: 1958.
(Lit,Rec:E) Carroll's recreations are just as charming as you would expect - and there is interesting mathematics behind them.

Carroll, Lewis. PILLOW PROBLEMS AND A TANGLED TALE. New York: Dover, 1958. (Gen, Lit, Rec:E) This is a nice collection of elementary problems.

Charosh, Mannis. MATHEMATICAL CHALLENGES. Washington, D.C: NCTM, 1965. (Gen,Con:E)

Chinn, W.G. and Steenrod, N.E. FIRST CONCEPTS OF TOPOLOGY: THE GEOMETRY OF MAPPING OF SEGMENTS, CURVES, CIRCLES AND DISKS. Washington: Mathematical Association of America, 1966. (Top)

Churchill, E. Richard and Linda. PUZZLE IT OUT. New York: Scholastic Book Services, 1971.

Clark, C. MATHEMATICAL BIOECONOMICS. New York: Wiley. 1976. (Mod) The CUPM Modeling Panel recommends this.

Coffman, C. and Fix, G., Eds. CONSTRUCTIVE APPROACHES TO MATHEMATICAL MODELS. New York: Academic Press, 1980. (Mod) The CUPM Modeling Panel recommends this.

Coleman, J. INTRODUCTION TO MATHEMATICAL SOCIOLOGY. Fress Press, 1964. (Mod) The CUPM Modeling Panel recommends this.

Collea, F. DEVELOPMENT OF REASONING IN SCIENCE: A COURSE BOOK IN FORMAL REASONING. Fullerton, CA: California State University, 1981. (Gen,Rem,Lit:E) Materials to translate Piaget's ideas about concrete and formal thinking into the classroom.

Conference Board of the Mathematical Sciences. THE ROLE OF AXIOMATICS AND PROBLEM SOLVING IN MATHEMATICS. Ginn, 1966. (Gen,Tch:) A beautiful collection of essays by distinguished mathematicians and educators - on axiomatics, Buck, Gleason, Henkin, Kline, Suppes, Young, among others; and on problem solving, Pólya, Dilworth, P.S. Jones, Lax, Pollak, Rosenbloom and others.

Conrad, S., Ewen, I., Flegler, D., and Sitomer, H. THE PROBLEMS VOL. I. New York City Interscholastic Mathematics League, Senior A Division, Fall 1967 - Spring 1977. (Con:E,I) A good source of challenging problems.

Conway, J.H. ON NUMBERS AND GAMES. Academic Press: New York, 1977. (Num,Rec:E)

Coolidge, J.L. THE MATHEMATICS OF GREAT AMATEURS. Oxford: Oxford University Press, 1949.
(His,Lit)

Cooney, T. (Ed.) TEACHING STRATEGIES: PAPERS FROM A RESEARCH WORKSHOP. Columbus, OH: ERIC, 1976.
(Tch,Res,Edu:E)

Court, Nathan A. COLLEGE GEOMETRY. New York: Barnes and Noble, 1952.
(Gen,con,geo:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program. A good problem source.

Court, Nathan A. MATHEMATICS IN FUN AND IN EARNEST. New York: Dial Press, 1958.
(Lit,rec:E,I) Mainly essays on mathematical topics, but there are many cute problems included.

Court, Nathan A. MODERN PURE SOLID GEOMETRY. New York: Chelsea, 1964. 363 pages. (Gen,Con,Geo:E,I) Recommended reading by M. Klamkin for a Mathematical Olympiad Program.

Courant, Richard and Robbins, Herbert. WHAT IS MATHEMATICS? Oxford: University Press, 1941.
(Gen,Rec:E,I) A classic introduction to the spirit of the discipline.

Coxeter, H.S.M. and Greitzer, S.L. GEOMETRY REVISITED. Washington, DC: Mathematical Association of America, 1967.
(Gen,Tch,Lit,Rec,Geo:E)

Coxeter, H.S.M. INTRODUCTION TO GEOMETRY. New York: Wiley, 1961.
(Gen,Lit,Rec:I)

Crosswhite, F. Joe, Higgins, J., et al. TEACHING MATHEMATICS, PSYCHOLOGICAL FOUNDATIONS. Worthington, Ohio: C.A. Jones Publishing Co., 1973.
(Psy,Tch:E)

Dantzig, Tobias. NUMBER: THE LANGUAGE OF SCIENCE. New York: Free Press, 1967.
(Gen,Lit:E)

Davis, G.A. PSYCHOLOGY OF PROBLEM SOLVING: THEORY AND PRACTICE. New York: Basic Books, 1973.
(Gen,Tch,Res:E)

Davis, P.J. THE LORE OF LARGE NUMBERS. Washington, DC: Mathematical Association of America, 1961.
(Gen,Tch,Lit,Rec:E)

Davis, P.J. and Hersh, R. THE MATHEMATICAL EXPERIENCE. Boston: Birkhäuser, 1980.
(Gen:I) A delightful, broad introduction to the notion of what doing mathematics is all about. There are sections on major mathematical results, on the "mathematical spirit," on philosophical controversies about the nature of mathematics, and much, much more.

de Bono, E. LATERAL THINKING: CREATIVITY STEP BY STEP. New York: Harper and Row, 1970.
(Cre:I)

de Bono, E. PO: BEYOND YES AND NO. New York: Pelican Books, 1972.
(Cre:E) Many interesting examples of creative thinking.

DeGrazia, Joseph. MORE MATH TEASERS. New York: Barnes and Noble, 1973.
(Rec:E) This is a collection of elementary problems in recreational mathematics, ranging from "logic problems" to cryptarithmetic, etc.

Dinesman, Howard P. SUPERIOR MATHEMATICAL PUZZLES. New York: Simon & Schuster, 1968.
(Rec:E)

DiPrima, R.,(Ed.) MODERN MODELING OF CONTINUOUS PHENOMENA. Providence, RI: American Mathematical Society, 1977.
(Mod) The CUPM Modeling Panel recommends this.

Dombrowski, J., Greenes, C., Spungin, R. PROBLEM-MATHICS: MATHEMATICAL CHALLENGE PROBLEMS WITH SOLUTION STRATEGIES. Palo Alto: Creative Publications, 1977.
(Tch,Lit:E) This is a good book which gives problems and a fair discussion of them. The authors list problem solving techniques. It seems to have been written for high school mathematics teachers.

Domoryad, Aleksandr Petrovich; translated by Halina Moss. MATHEMATICAL GAMES AND PASTIMES. Oxford: Pergamon Press (Distributed in the Western Hemisphere by Macmillan, New York), 1964.
(Rec:E,I)

Dorrie, H. 100 GREAT PROBLEMS OF ELEMENTARY MATHEMATICS. Dover, New York, 1965.

"(Gen:E,I,A) "The triumph of mathematics" is the original title (in German) of Dorrie's book. This is a book that deserves to be much better known than it seems to be. It is eclectic, it is spread over 2000 years of history, and it ranges in difficulty from elementary arithmetic to material that is frequently the subject of graduate courses.

"It contains, for instance, the following curiosity attributed to Newton (Arithmetica Universalis,1707). If "a cows graze b fields bare in c days, a' cows graze b' fields bare in c' days, a'' cows graze b'' fields bare in c'' days, what relation exists between the nine magnitudes a to c''? It is assumed that all fields provide the same amount of grass, that the daily growth of the fields remains constant, and that all the cows eat the same amount each day." Answer:

$$\det\begin{pmatrix} b & bc & ac \\ b' & b'c' & a'c' \\ b'' & b''c'' & a''c'' \end{pmatrix} = 0$$

"This is Problem 3, out of a hundred.

"The problems lean more toward geometry than anything else, but they include also Catalan's question about the number of ways of forming a product of n prescribed factors in a multiplicative system that is totally non-commutative and non-associative ("how many different ways can a product of n different factors be calculated by pairs?," Problem 7), and the Fermat-Gauss impossibility theorem ("The sum of two cubic numbers cannot be a cubic number," Problem 21).

"Two more examples should give a fair idea of the flavor of the collection as a whole: "every quadrilateral can be considered as a perspective image of a square" (Problem 72), and "at what point of the earth's surface does a perpendicularly suspended rod appear the longest?" (Problem 94). The style and the attitude are old-fashioned, but many of the problems are of the eternally interesting kind; this is an excellent book to browse in."
(P.R. Halmos, The Heart of Mathematics)

Dudeney, H.E. AMUSEMENTS IN MATHEMATICS. New York: Dover, 1970. (Rec:E) This book has 430 puzzles. Like Dudeney's other collections, it has a variety of puzzles of varying levels of difficulty.

Dudeney, H.E. THE CANTERBURY PUZZLES. New York: Dover, 1958. (Rec:E,I) The 114 puzzles are of every degree of difficulty and varied in character. Read the introduction to this book; it will give you some thoughts of a professional problemist.

Dudeney, H.E. 536 PUZZLES AND CURIOUS PROBLEMS. Scribner's New York, 1954, 1967
(Rec:E,I)

Dunn, Angela. MORE PROBLEMATICAL RECREATIONS. Beverly Hills, California: Litton, 1972.
(Rec:E)

Dym, C. and Ivey, E. PRINCIPLES OF MATHEMATICAL MODELING. New York: Academic Press, 1980.
(Mod) The CUPM Modeling Panel recommends this.

Dynkin, E.B., Molchanov, S.A., Rozental, A.L., Tolpygo, A.K. MATHEMATICAL PROBLEMS: AN ANTHOLOGY. New York: Gordon and Breach, 1969.
(Gen,con:E,I) A solid problem source.

Dynkin, E.B. and Uspenskii, V.A. MULTICOLOR PROBLEMS. Boston: D.C. Heath, 1968.

Dynkin, E.B. and Uspenskii, V.A. PROBLEMS IN THE THEORY OF NUMBERS. Boston: D.C. Heath, 1963.
(Num:E,I)

Dynkin, E.B. and Uspenskii, V.A. Translated by Norman Whaland and Olga Titelbaum. RANDOM WALKS. Boston: Heath, 1963.
(Pro:I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Emmet, Eric Revell. A DIVERSITY OF PUZZLES: NOT ONLY FOR EXPERTS. New York: Barnes and Noble Books, 1977.
(Rec:E,I) "Not only for experts" the subtitles says. Well, they're not only for beginners either. The difficulty of the problems is given in the table of contents. Pick an easy one and work your way up.

Emmet, Eric Revell. 101 BRAIN PUZZLES. New York: Barnes and Noble Books, 1973.
(Rec:E,I)

Emmet, Eric Revel. PUZZLES FOR PLEASURE. New York: Emerson Books, 1972. (Rec:E) A collection of puzzles, some more mathematical than others.

Engel, Arthur (ed.). MATHEMATISCHE OLYMPIADEAUFGABEN AUS DER UDSSR. Stuttgart: Ernst Klett, 1965. (Con:E,I) A discussion of German Mathematical Olympiads.

Erdős, P., and R. L. Graham. OLD AND NEW PROBLEMS AND RESULTS IN COMBINATORIAL NUMBER THEORY. Geneva: L'Enseignment Mathematique (Université de Genève), 1980. (Num:A)

Ernst, G.W. and Newell, A. GPS: A CASE STUDY IN GENERALITY AND PROBLEM SOLVING. New York: Academic Press, 1969. (Res,Psy,Ai:I,A) General Problem Solver was one of the first computer programs successful at non-trivial, broadly-based problem solving. This book describes its evolution. Though technical, the detailed level of discussion is quite interesting.

Eves, Howard. AN INTRODUCTION TO THE HISTORY OF MATHEMATICS, 3rd. ed. Holt, Rinehart, and Winston, New York, 1969. (His,Lit:E)

Eves, Howard and Starke, E.P. THE OTTO DUNKEL MEMORIAL PROBLEM BOOK. Washington: Mathematical Association of America, 1957. (Currently out of print). (Gen,con:E,I,A) The August-September issue of the American Mathematical Monthly, Vol. 74 #7, contains a collection of the 400 "best" problems published in the Monthly from 1918 to 1950. Nothing more than that need be said.

Eves, Howard. SURVEY OF GEOMETRY. Boston: Allyn and Bacon, 1971. (Geo:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Faddeev, D.K. and Sominski, I.S. PROBLEMS IN HIGHER ALGEBRA. San Francisco: W.H. Freeman, 1965.

Famous Problems and Other Monographs. New York: Chelsea Publishing Co., 1962 (Gen,His) "Famous Problems of Elementary Geometry: by F. Klein; "From Determinant to Tensor" by W.F. Sheppard; "Introduction to "Combinatorial Analysis" by P.A. MacMahon; "Three Lectures on Fermat's Last Theorem" by L.J. Mordell.

Fejes-Tóth, L. REGULAR FIGURES. New York: Macmillan, 1964.
(Geo:I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Fishburn, P. THE THEORY OF SOCIAL CHOICE. Princeton, NJ: Princeton University Press, 1973.
(Mod) The CUPM Modeling Panel recommends this.

Fixx, James F. GAMES FOR THE SUPER-INTELLIGENT. Garden City, NY: Doubleday. 1972.
(Rec:E,I) Despite this book's title, it does have some good problems.

Fixx, James F. MORE GAMES FOR THE SUPER-INTELLIGENT. Garden City, NY: Doubleday, 1976.
(Rec:E,I) This is like the first.

Frauenthal, J. INTRODUCTION TO POPULATION MODELING. UMAP Monograph, 1979.
(Mod:E) The CUPM Modeling Panel recommends this.

Friedland, Aaron J. PUZZLES IN MATH AND LOGIC. New York: Dover, 1970.
(Rec:E,I) A number of unusual and interesting puzzles.

Friedman, B. LECTURES ON APPLICATIONS-ORIENTED MATHEMATICS. San Francisco: Holden-Day, 1969.
(Mod) The CUPM Modeling Panel recommends this.

Friedrichs, K.O. FROM PYTHAGORAS TO EINSTEIN. Washington, DC: Mathematical Association of America, 1965.
(His,Lit:E,I)

Frohlichstein, Jack. MATHEMATICAL FUN, GAMES AND PUZZLES. New York, NY: Dover Publications, 1967.
(Rec:E)

Gagné, R.M., THE CONDITIONS OF LEARNING, 3rd ed., New York: Holt, Rinehart, and Winston, 1977.
(Tch,res:E) A delineation of the behaviorist position of how people learn. It's important to know, because these ideas have shaped the curriculum.

Gagné, R.M. ESSENTIALS OF LEARNING FOR INSTRUCTION. New York: Holt, Rinehart, and Winston, Inc., 1974.
(Tch,Psy:E) A noted behaviorist "takes apart" the learning process so that the teacher can structure lessons carefully.

Gamow, George & Marvin Stern. PUZZLE-MATH. New York: The Viking Press, 1958.
(Rec:E) A clever set pf problems for the layman; some are old classics and some are less familiar.

Gardner, Martin. AHA! INSIGHT. San Francisco: Scientific American/W.H. Freeman and Co., 1978.
(Rec:E) This book like all those to follow, written by Martin Gardner, is entertaining and can lead into substantive mathematics.

Gardner, Martin. THE AMBIDEXTROUS UNIVERSE. New York: Basic Books, 1964.

Gardner, M. MATHEMATICAL CARNIVAL. New York: Knopf, 1975.
(Rec:E)

Gardner, Martin. MATHEMATICS, MAGIC AND MYSTERY. New York: Dover, 1956.
(Rec:E)

Gardner, Martin. MATHEMATICAL MAGIC SHOW. New York: Knopf, 1977.
(Rec:E)

Gardner, Martin. THE SCIENTIFIC AMERICAN BOOK OF MATHEMATICAL PUZZLES AND DIVERSIONS. New York: Simon and Schuster, 1959.
(Rec:E)

Gardner, Martin. MORE MATHEMATICAL PUZZLES AND DIVERSIONS. New York: Penguin, 1961.
(Rec:E)

Gardner, M. NEW MATHEMATICAL DIVERSIONS FROM SCIENTIFIC AMERICAN. New York: Simon and Schuster, 1966.
(Rec:E)

Gardner, M. THE NUMEROLOGY OF DR. MATRIX. New York: Scribner's, 1967
(Rec:E)

Gardner, M. THE SECOND SCIENTIFIC AMERICAN BOOK OF MATHEMATICAL PUZZLES AND DIVERSIONS. New York: Simon and Schuster, 1961.
(Rec:E)

Gardner, M. THE UNEXPECTED HANGING AND OTHER MATHEMATICAL DIVERSIONS. New York: Simon and Schuster, 1969.
(Rec:E)

Gardner, M. MARTIN GARDNER'S SIXTH BOOK OF MATHEMATICAL GAMES FROM SCIENTIFIC AMERICAN. San Francisco: W.H. Freeman and Company, 1971.
(Rec:E)

Garvin, Alfred A. DISCOVERY PROBLEMS FOR BETTER STUDENTS. Portland, ME: J. Weston Walch, 1975.

Gelbaum, B. and Olmsted, J. COUNTEREXAMPLES IN ANALYSIS. San Francisco: Holden-Day, 1964.
(Ana,gen:I,A) Each of these counterexamples is the solution to a good problem, for example:

(Does there exist...)

"A convergent sequence of functions $\{f_n\}$ such that

$$\int_a^b \lim_{n\to\infty} f_n \neq \lim_{n\to\infty} \int_a^b f_n \quad (?)$$

Having students work on such problems is an excellent way to have them learn precision and subtlety in rigorous mathematics. There are some exceptionally nice examples here.

Glaeser, Georges. LE LIVRE DU PROBLÈME. (3 vol.) Paris: CEDIC, 1976.

Glaeser, Georges. MATHEMATIQUES POUR L'ELEVE PROFESSEUR. Paris: Hermann, 1971.

Glazman, I.M. and Ljubic, Ju. I. FINITE-DIMENSIONAL LINEAR ANALYSIS: A SYSTEMATIC PRESENTATION IN PROBLEM FORM. Cambridge: MIT, 197-.
"(I,A) (This book) is an unusual one (I don't know of any others of its kind), and, despite some faults, it is a beautiful and exciting contribution to the problem literature. The book is, in effect, a new kind of textbook of (finite-dimensional) linear algebra and linear analysis. It begins with the definitions of (complex) vector spaces and the concepts of linear dependence and independence; the first problem in the book is to prove that a set consisting of just one vector x is linearly independent if any only if $x \neq 0$. The chapters follow one another in logical dependence, just as they do in textbooks of the conventional kind: Linear operators, Bilinear functionals, Normed spaces, etc.

"The book is not expository prose, however; perhaps it could be called expository poetry. It gives definitions and related explanatory background material with some care. The main body of the book consists of problems; they are all formulated as assertions, and the problem is to prove them. The proofs are not in the book. There are references, but the reader is told that he will not need to consult them.

"The really new idea in the book is its sharp focus: this is really a book on functional analysis, written for an audience who is initially not even assumed to know what a matrix is. The ingenious idea of the authors is to present to a beginning student the easy case, the transparent case, the motivating case, the finite-dimensional case, the purely algebraic case of some of the deepest analytic facts that functional analysts have discovered. The subjects discussed include spectral theory, the Toeplitz-Hausdorff theorem, the Hahn-Banach theorem, partially ordered vector spaces, moment problems, dissipative operators, and many other such analytic sounding results. A beautiful course could be given from this book (I would love to give it), and a student brought up in such a course could become an infant prodigy functional analyst in no time.

"(A regrettable feature of the book, at least in its English version, is the willfully unorthodox terminology. Example: the (canonical) projection from a vector space to a quotient space is called a "contraction", and what most people call a contraction is called a "compression." Fortunately the concept whose standard technical name is compression is not discussed.)"
(P. R. Halmos, The Heart of Mathematics)

Gleason, A.M., Greenwood, R.E. and Kelly, L.M. THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION: PROBLEMS AND SOLUTIONS, 1938-1964. Washington, DC: Mathematical Association of America, 1980. (Con:E,I,A) This is the definitive book covering the Putnam exam from 1938 to 1964. One can only hope that a sequel will bring us up to date (annual updates for the competitions can be found in the Monthly; see the "articles" section).

Gold, H. MATHEMATICAL MODELING OF BIOLOGICAL SYSTEMS. New York: Wiley, 1977. (Mod) The CUPM Modeling Panel recommends this.

Goldin, G.A. and McClintock, C.E. TASK VARIABLES IN MATHEMATICAL PROBLEM SOLVING. Columbus, OH: ERIC/SMEAC, 1979. (Res:E,I) This research volume classifies problem solving variables into four categories dealing with (1) syntax, (2) content and context, (3) structure, and (4) heuristic behaviors. Each of these categories is elaborated at length, and the applications of task variables both to research and teaching in problem solving is studied.

Goldberg, S. SOME ILLUSTRATIVE EXAMPLES OF THE USE OF UNDERGRADUATE MATHEMATICS IN SOCIAL SCIENCE. Hayward, CA: MAA Special Projects Office, 1977. (Mod:E,I) The CUPM Modeling Panel recommends this.

Graham, L.A. INGENIOUS MATHEMATICAL PROBLEMS AND METHODS. New York: Dover, 1959. (Rec:E)

Graham, L. A. THE SURPRISE ATTACK IN MATHEMATICAL PROBLEMS. New York: Dover Publishers, 1968.
(Rec:E)

Greenes, C., Gregory, J., Seymour, D. SUCCESSFUL PROBLEM SOLVING TECHNIQUES. Palo Alto: Creative Publications, 1977.

Gregory, John, and Seymour, Dale. LIMERICK NUMBER PUZZLES. Palo Alto: Creative Publications, 1978.

Greitzer, Samuel L. INTERNATIONAL MATHEMATICAL OLYMPIADS 1959 - 1977. Washington, DC: Mathematical Association of America, 1978.
(Con:E,I) A good source of challenging problems, discussed by an able and dedicated problemist.

Grosche, Günter. ELEMENTARGEOMETRIE. Übungen für Junge Mathematike Teilz, Leipzig: Teubner, 1969.
(Geo:E,I) A well-organized collection of construction problems in the plane and space, including the problem of Appolonius, with solutions.

Grosswald, E. TOPICS FROM THE THEORY OF NUMBERS. New York: Macmillan, 1966.
(Num:E,I)

Gruver, Howell, L. SCHOOL MATHEMATICS CONTESTS: A REPORT. Washington: National Council of Teachers of Mathematics, 1968.
(Gen:E)

Guy, Richard K. UNSOLVED PROBLEMS IN NUMBER THEORY. New York: Springer-Verlag, 1981.
(Num:I,A)

Haberman, R. MATHEMATICAL MODELS, MECHANICAL VIBRATIONS, POPULATION DYNAMICS AND TRAFFIC FLOW. Englewood Cliffs, NJ: Prentice-Hall, 1977.
(Mod) The CUPM Modeling Panel recommends this.

Hadamard, Jacques. AN ESSAY ON THE PSYCHOLOGY OF INVENTION IN THE MATHEMATICAL FIELD. New York: Dover, 1954.
(His,Psy:E) A detailed "gestalt" exposition of the problem solving process. This book is of substantial historical interest, though of questionable practical or theoretical value.

Hadwiger, Hugo and DeBrunner, Hans. Translated by Victor Klee with a new chapter and other materials supplied by the translator. COMBINATORIAL GEOMETRY IN THE PLANE. New York: Holt, Rinehart & Winston, 1964.
(Geo:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Haefele, J.W. CREATIVITY AND INNOVATION. New York: Reinhold Publishing, 1962.
(Cre:E) Wider view on creativity than discussed by most texts.

Hall and Knight. HIGHER ALGEBRA. London: Macmillan and Co., Ltd., 1940.
(Alg:E,I) Recommended reading for M. Klamkin's Mathematical Olympiad Program. A wonderful collection of rather old-fashioned but amusing problems, many from old Tripos exams.

Hardy, G., J.E. Littlewood and G. Pólya. INEQUALITIES. Cambridge: The University Press, 1967.
(Ana:I,A) A classic. Recommended by M. Klamkin for a Mathematical Olympiad Program.

Hardy, Godfrey H. and Wright, E.M. INTRODUCTION TO THE THEORY OF NUMBERS. Oxford: Clarendon Press, New York: Oxford University Press, 5th. ed., 1980.
(Num:I,A) Another classic. Recommended by M. Klamkin for a Mathematical Olympiad Program.

Harnadek, Anita. CLASSROOM QUICKIES, BOOKS 1 and 2 and 3. Pacific Grove, CA: Midwest Publications, Co., Inc. 1978.

Harvey, John G. and Romberg, Thomas A. PROBLEM-SOLVING STUDIES IN MATHEMATICS. Madison, WI: University of Wisconsin, 1980.
(Res.tch:E,I) This volume presents the results of nine dissertations conducted at the University of Wisconsin dealing with problem solving in mathematics. These studies, supplemented by a review of thirty-one parallel studies, give a good sense of the mathematics education literature of the 1970's.

Hatfield, L.L. and Bradbard, D.A. MATHEMATICAL PROBLEM SOLVING: PAPERS FROM A RESEARCH WORKSHOP. Columbus, OH: ERIC/SMEAC, 1978.
(Tch,Res:E,I) Five papers discussing research and instruction in problem solving.

Hayes, J.R. COGNITIVE PSYCHOLOGY: THINKING AND CREATIVITY. Homewood, IL: Dorsey Press, 1978.
(Psy,cre:E,I) An introduction to the area.

Heath, Royal Vale. MATHMAGIC. Toronto, ON: The General Publishing Co., Ltd., 1933.
(Rec:E)

Heofford, Phillip. THE MATH ENTERTAINER. New York, NY: Harper and Row, 1959.
(rec:E)

Hilbert, D. and Cohn-Vossen, H. GEOMETRY AND THE IMAGINATION. New York: Chelsea, 1952.
(Geo,Lit,Rec:I,A) A marvelous book.

Hill, Claire Conley. PROBLEM SOLVING; LEARNING AND TEACHING. AN ANNOTATED BIBLIOGRAPHY. New York: Nichols Publishing Company, 1979.
(Gen,Lit:E,I) This volume offers extensive annotations for more than 250 different sources in the problem solving literature. Topics covered include "problem solving in using associations," "problem solving in forming and testing hypotheses", "problem solving as a goal", etc. The coverage is broad and of general interest.

Hill, Thomas, (Ed.) MATHEMATICAL CHALLENGES. Washington, DC: National Council of Teachers of Mathematics.
(Gen,con:E)

Hill, Thomas, (Ed.) MATHEMATICAL CHALLENGES II PLUS SIX. Washington, DC: National Council of Teachers of Mathematics.
(Gen,con:E)

Hillman, Abraham P. and Alexanderson, Gerald L. ALGEBRA THROUGH PROBLEM SOLVING. Boston: Allyn and Bacon, 1966.
(Gen,lit:E,I) An introduction to a number of topics in intermediate and college algebra with little text and many problems, some challenging. Combinatorial topics and inequalities are featured prominently.

Hindman, Darwin Alexander. NINE MEN'S MORRIS AND OVER 800 OTHER INDOOR GAMES, PUZZLES AND STUNTS FOR ALL AGES. Englewood Cliffs, NJ: Prentice-Hall, 1978.

Hlavaty, Julius H. (Ed.) ENRICHMENT MATHEMATICS FOR THE HIGH SCHOOL. 28th Yearbook, Reston, VA: NCTM, 1963.
(Gen:E) 27 enrichment topics for academically talented students in grades 10-14. Each chapter provides a wealth of problems.

Holt, John. HOW CHILDREN FAIL. New York: Pitman, 1964.
(Tch:E) Holt writes about his experience as a grade school teacher, but his descriptions of classroom exchanges raise issues at all levels of instruction.

Honsberger, Ross. INGENUITY IN MATHEMATICS. Washington, DC: Mathematical Association of America, 1970.
(Rec,lit:E,I) Whether it be a discussion of gems, morsels, plums or ingenuity, Honsberger offers fascinating problems and nice discussions of them.

Honsberger, Ross. MATHEMATICAL GEMS I. Washington, DC: Mathematical Association of America, 1970.
(Rec, Lit:E,I) Beautifully written exposition of some classic problems.

Honsberger, Ross. MATHEMATICAL GEMS II. Washington, DC: Mathematical Association of America, 1976.
(Rec,Lit:I)

Honsberger, Ross. MATHEMATICAL MORSELS. Washington, DC: Mathematical Association of America, 1978.
(Rec,lit:E,I) Problems originally posed in the American Mathematical Monthly.

Honsberger, Ross. MATHEMATICAL PLUMS. Washington, DC: Mathematical Association of America, 1979.
(Lit,rec:I) A collection of essays on problems by a variety of authors: Dorwart, Finkbeiner, Rotman, Boas, Stein, Honsberger, and Chakerian.

Hoppensteadt, R. MATHEMATICAL THEORIES OF POPULATIONS: DEMOGRAPHICS AND EPIDEMICS. Philadelphia: SIAM, 1975.
(Mod) The CUPM Modeling Panel recommends this.

Howson, A.G., (Ed.) DEVELOPMENTS IN MATHEMATICS EDUCATION/PROCEEDINGS OF THE SECOND INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION. Cambridge: Cambridge University Press, 1973.
(Tch,rec:E,I) Essays on mathematical education, including problem solving, originating in the 2nd International Congress on Mathematical Education. Piaget and Pólya were featured speakers at the Congress.

Hughes, Barnabas, O.F.M. THINKING THROUGH PROBLEMS--A MANUAL OF HEURISTICS. Palo Alto, Creative Publications, 1976.
(Gen,tch,lit:E) This is a text book for teaching heuristics. As the title suggests, it is a straightforward introduction to the tools of the trade.

Hunter, J. A. MATHEMATICAL BRAIN TEASERS. New York: Dover, 1976.
(Rec:E)

Hunter, J.A.H. and Madachy, J.S. MATHEMATICAL DIVERSIONS. New York: Dover, 1975.
(Rec:E) This book has some puzzles but its content is mainly discussions of some of the popular puzzle types. It's well worth reading for these discussions alone.

Huntley, H.E. THE DIVINE PROPORTION: A STUDY IN MATHEMATICAL BEAUTY. New York: Dover Publications, 1970.

Inhelder, B. and Piaget, J. THE GROWTH OF LOGICAL THINKING FROM CHILDHOOD TO ADOLESCENCE. New York: Basic Books, 1958.
(Psy:E) The book is an absolute classic in psychology. While it's not "directly" related to mathematical problem solving at the high school or college level, it is critical for us to understand that children are not simply miniature versions of adult computers. Understanding the way that children "construct" their own realities is essential if one is to make sense of what goes on in their heads.

Instrument Society of America, 1969-78. PROCEEDINGS OF THE PITTSBURGH CONFERENCES ON MODELING AND SIMULATIONS, VOLS. 1-9.
(Mod) The CUPM Modeling Panel recommends these.

Jacobs, H. MATHEMATICS: A HUMAN ENDEAVOUR. San Francisco: W. H. Freeman, 1970.
(Lit,rec:E) A charming, entertaining introduction to some interesting mathematical ideas.

Johnson, Donovan. TOPOLOGY, THE RUBBER SHEET GEOMETRY. Pasadena, CA: Webster, 1960.
(Top)

Johnson, Rogers A. ADVANCED EUCLIDEAN GEOMETRY. New York: Dover, 1960.
(Geo)

Judson, Horace F. THE SEARCH FOR SOLUTIONS. New York: Holt,Rinehart & Winston, 1980.
(Lit:E) Intended for middle school children, the book is a broad and literate introduction to what science is all about -- not the silly model of the "scientific method" students are forced to memorize, but a real, honest-to-goodness introduction to the excitement of scientific discovery. It's part of a large package including films on the topics and a teacher's resource book. See the films if you can.

Kalomitsines, Spyros P. ATTACK YOUR PROBLEM. Athens: University Press.
(Gen:E)

Kasner, Edward and Newman, J.R. MATHEMATICS AND THE IMAGINATION. New York: Simon and Schuster, 1940.
(Lit,rec:E,I) Easy reading, well written and accessible to practically anyone. A nice introducation to elementary mathematics.

Kazarinoff, N. D. GEOMETRIC INEQUALITIES. Washington, DC: Mathematical Association of America, 1961.
(Lit,rec:I) Beautiful, unusual problems.

Kemeny, J.G., Snell, J.L., and Thompson, G.E. 3rd ed. INTRODUCTION TO FINITE MATHEMATICS. Prentice-Hall, Englewood Cliffs, NJ, 1974.
(Gen,lit:E,I) A pioneer text with clever problems.

Kemeny, J. and Snell, L. MATHEMATICAL MODELS IN THE SOCIAL SCIENCES. Boston: MIT Press, 1973.
(Mod) The CUPM Modeling panel recommends this.

Kennedy, Joe and Thomas, Diane. A TANGLE OF MATHEMATICAL YARNS. Kennedy-Thomas, PO Box 132 Oxford, OH 45056, 1979.
(Lit:E) 50 imaginative and humorous story problems designed to encourage students to read. The level of the mathematics is easy enough that middle school and high school students shouldn't be intimidated.

Kespohl, Ruth Carwell. GEOMETRY PROBLEMS MY STUDENTS HAVE WRITTEN. Reston, VA: NCTM, 1979.
(Geo:E)

Kilpatrick, J. et al (Eds.) SOVIET STUDIES IN THE PSYCHOLOGY OF LEARNING AND TEACHING MATHEMATICS, VOLS. 1-14. Chicago: University of Chicago Press, 1969-1975.
(Tch,res:E) This series of volumes presents translations of research, classroom procedures, and theoretical discussions about mathematical learning and teaching. The range of articles provides insights into the developments in this field over the past several decades and gives an idea of current practice in the USSR. The Soviet "teaching experiments" have had an impact on mathematics education research in the U.S.

Klahr, D. (Ed.) COGNITION AND INSTRUCTION. Hillsdale, NJ: Lawrence Erlbaum Associates, 1976.
(Psy:E) This book offers discussions and applications of modern cognitive psychology, and provides an idea of how psychology might contribute to the development of a theory of instruction.

Klambauer, G. PROBLEMS AND PROPOSITIONS IN ANALYSIS. New York: Dekker, 1979.
(Ana,Gen:I,A) "Its subject is real analysis, and, although it does have some elementary problems, its level is relatively advanced. It is an excellent and exciting book. It does have some faults, of course, including some misprints and some pointless repetitions, and the absence of an index is an exasperating feature that makes the book much harder to use than it ought to be. It is, however, a great source of stimulating questions, of well known and not

so well known examples and counterexamples, and of standard and not so standard proofs. It should be on the bookshelf of every problem lover, of every teacher of analysis (from calculus on up), and, for that matter, of every serious student of the subject.

"The table of contents reveals that the book is divided into four chapters: Arithmetic and combinatorics, Inequalities, Sequences and series, and Real functions. Here are some examples from each that should serve to illustrate the range of the work, perhaps to communicate its flavor, and, I hope, stimulate the appetite for more.

"The combinatorics chapter asks for a proof of the 'rule for casting out nines' (is that expression for testing the divisibility of an integer by 9 via the sum of its decimal digits too old-fashioned to be recognized?), it asks how many zeros there are at the end of the decimal expansion of 1000!, and it asks for the coefficient of x^k in $(1 + x + x^2 + \cdots + x^{n-1})^2$. Along with such problems there are also unmotivated formulas that probably only their father could love, and there are a few curiosities (such as the problem that suggests the use of the well ordering principle to prove the irrationality of $\sqrt{2}$). A simple but striking oddity is this statement: if m and n are distinct positive integers, then

$$m^{n^m} \neq n^{m^n} .$$

"The chapter on inequalities contains many of the famous ones (Hölder, Minkowski, Jensen), and many others that are analytically valuable but somewhat more specialized and therefore somewhat less famous. A curiosity the answer to which very few people are likely to guess is this one: for each positive integer n, which is bigger

$$\sqrt{n}^{\sqrt{n+1}} \quad \text{or} \quad \sqrt{n+1}^{\sqrt{n}} \quad ?$$

"The chapter on sequences has the only detailed and complete discussion that I have ever seen of the fascinating (and non-trivial) problem about the convergence of the infinite process indicated by the symbol

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}$$

Books, p. 25

"Students might be interested to learn that the result is due to Euler; the reference given is to the article *De formulis exponentialibus replicatis*, Acta Academica Scientiarum Imperialis Petropolitanae, 1777. One more teaser: what is the closure of the set of all real numbers of the form $\sqrt{n} - \sqrt{m}$ (where n and m are positive integers)?

"The chapter on real functions is rich too. It includes the transcendentality of e, some of the basic properties of the Cantor Set, Lebesgue's example of a continuous but nowhere differentiable function, and F. Riesz's proof (via the "rising sun lemma") that every continuous monotone function is differentiable almost everywhere. There is a discussion of that vestigial curiosity called Osgood's theorem, which is the Lebesgue bounded convergence theorem for continuous functions on a closed bounded interval. The Weierstrass polynomial approximation theorem is here (intelligently broken down into bite-size lemmas), and so is one of Gauss' proofs of the fundamental theorem of algebra. For a final example I mention a question that should be asked much more often than it probably is: is there an example of a series of functions, continuous on a closed bounded interval, that converges absolutely and uniformly, but for which the Weierstrass M-test fails?"
(P.R. Halmos, THE HEART OF MATHEMATICS.)

Klein, F. FAMOUS PROBLEMS OF ELEMENTARY GEOMETRY. New York: Dover, 1956.
(Geo:E,I)

Kleinfeld, Gerhard. UNGLEICHUNGEN. Übungen Für Junge Mathematikes, Teil 3, Leipzig: Teubner, 1969.
(Ana:E,I) A well-organized collection of problems about inequalities, with solutions.

Kleinmuntz, B. (Ed.) PROBLEM SOLVING: RESEARCH, METHOD, AND THEORY. New York: John Wiley & Sons, Inc., 1975.
(Psy:res) This book presents a series of papers from a conference at Carnegie-Mellon University. It offers a variety of perspectives on problem solving, and presents some unusual and interesting research accessible to non-specialists.

Kletenik, D. PROBLEMS IN ANALYTIC GEOMETRY. Moscow: Mir Publishers, 1969.
(Geo:E,I)

Kline, Morris, Ed. MATHEMATICS AND THE MODERN WORLD. San Francisco: W.H. Freeman, 1968.
(Lit:E)

Kline, Morris. WHY JOHNNY CAN'T ADD: THE FAILURE OF THE NEW MATH. New York: Vantage Books, 1973.

Knopp, P. and Meyer, G., Eds. PROCEEDINGS OF A CONFERENCE ON THE APPLICATION OF UNDERGRADUATE MATHEMATICS IN THE ENGINEERING, LIFE, MANAGERIAL AND SOCIAL SCIENCES. Atlanta: Georgia Tech. School of Mathematics, 1973.
(Mod) The CUPM Modeling Panel recommends this.

Kordemsky, Boris A. THE MOSCOW PUZZLES. Gardner, Martin ed. New York, NY: Charles Scribner's Sons, 1972.
(Rec:E) A pleasant collection of puzzles with mathematical content accessible to the layman.

Kraitchik, Maurice. MATHEMATICAL RECREATIONS. New York: Dover, 1953.
(Rec:E) This book has good explanations of many classic puzzles, and categories of puzzles. Don't miss reading at least parts of this aged but readable book.

Krechmar, V.A. A PROBLEM BOOK IN ALGEBRA. Moscow: Mir Publishers, 1974.
(Alg:E,I) An extensive collection of problems, including many on inequalities, progressions and sums, complex numbers, mathematical induction, limits.

Krulik, S. (Ed.) PROBLEM SOLVING IN SCHOOL MATHEMATICS, The 1980 N.C.T.M. YEARBOOK.
This is probably the best practical source for school teachers. It offers lots of classroom suggestions.

Krulik, S. and Rodwick. PROBLEM SOLVING: A HANDBOOK FOR TEACHERS. Boston: Allyn and Bacon, 1980.
(Tch:E)

Krutetskii, V.A. THE PSYCHOLOGY OF MATHEMATICAL ABILITIES IN SCHOOL-CHILDREN. Chicago: The University of Chicago Press, 1976.
(Res:E) Krutetskii is a major exponent of the "Teaching Experiment." This book has influenced much U.S. mathematics education work in "clinical" studies.

Lakatos, Imre. PROOFS AND REFUTATIONS. Cambridge, 1975.
(E,I) This book is a mathematical philosophical gem, illustrating the discovery and refinement of a profound mathematical theorem through the use of a simulated classroom discussion. The Socratic dialogue is carried to the ultimate - a tour de force. The footnotes give an interesting history of the Euler formula for polyhedra. It is a marvelous book, charming yet serious.

Lancaster, P. MATHEMATICS: MODELS OF THE REAL WORLD. Englewood Cliffs, NJ: Prentice-Hall, 1976.
(Mod) The CUPM Modeling Panel recommends this.

Landa, L.N. ALGORITHMIZATION. Englewood Cliffs, NJ: Educational Technology Publications, 1974.
(Psy,res:I) Together with INSTRUCTIONAL REGULATION AND CONTROL, Landa's books provide an introduction to Soviet perspectives on learning and teaching of the early 1970's.

Landa, L.N. INSTRUCTIONAL REGULATION AND CONTROL. Englewood Cliffs, NJ: Educational Technology Publications, 1976.
(Psy,res:I)

Lave, C. and March, J. AN INTRODUCTION TO MODELS IN THE SOCIAL SCIENCES. New York: Harper & Row, 1975.
(Mod) The CUPM Modeling Panel recommends this.

Leblanc, H. and Wisdom, W. DEDUCTIVE LOGIC. 2nd ed. Boston: Allyn and Bacon, 1976.

Lefart, G. ALGEBRA AND ANALYSIS PROBLEMS AND SOLUTIONS. Translated by Scripta Technica, Inc. Translation editor, Bernard R. Gelbaum. Philadelphia: W.B. Saunders Co., 1964.
(Alg,Ana)

Lehman, Eberhard. ZAHLENTHEORIE. Übungen für Junge Mathematiker, Teil 1, Leipzig: Teubner, 1970.
(Num:E,I) A well-organized collection of easy and not-so-easy problems in elementary number theory, with solutions.

Lesh, R., Mierkiewicz, D., and Kantowski, M.G. (Eds.) APPLIED MATHEMATICAL PROBLEM SOLVING. Columbus, OH: ERIC/SMEAC, 1979.
(Tch,res:E,I) (From the introduction) "The purpose of the papers in this monograph is to review a variety of perspectives concerning the general question, "What is it, beyond having a concept, that enables an average ability student to use the idea in real situations?" The book offers an up-to-date compendium of good ideas and perspectives in mathematics education.

Lesh, R., and Landau, M. (Eds.) ACQUISITION OF MATHEMATICS CONCEPTS AND PROCESSES. New York: Academic Press, 1983.
(Res,Gen,Tch:E,I) A state-of-the-art collection of papers in mathematics education dealing with thinking and learning mathematically. Highly recommended.

Lester, Frank K., and Garofalo, J. (Eds.) MATHEMATICAL PROBLEM SOLVING: ISSUES IN RESEARCH. Philadelphia: Franklin Institute Press, 1982.
(Res,Psy:E,I) This book contains a number of essays from researchers in mathematical problem solving discussing the nature of current research in the field (from psychological, math-ed, and mathematical points of view), and where such research might be going. It's a good introduction to the field.

Lidsky, D. et al. PROBLEMS IN ELEMENTARY MATHEMATICS. Translated by V. Volosov. Moscow: Mir Publishers, 1963.
(Gen,con:E,I) The subject matter may be elementary but not all the problems are. They range from drill and practice to ones which call for some thought; cleverness is often appropriate and desirable.

Lin, C. and Segal, L. MATHEMATICS APPLIED TO DETERMINISTIC PROBLEMS IN THE NATURAL SCIENCES. New York: Macmillan, 1974.
(Mod). The CUPM Modeling Panel recommends this.

Lindgren, Harry. RECREATIONAL PROBLEMS IN GEOMETRIC DISSECTIONS AND HOW TO SOLVE THEM. New York: Dover, 1972.
(Geo,rec:E).

Lochhead, J. and Clement, J. (Eds.) COGNITIVE-PROCESS INSTRUCTION. Philadelphia, PA: Franklin Institute Press, 1979.
(Psy,res:E,I) (From the preface) "COGNITIVE-PROCESS INSTRUCTION is an approach to teaching which emphasizes understanding, learning and reasoning skills as opposed to emphasizing rote memorization of factual knowledge. This book describes some of the most recent and innovative approaches to cognitive process instruction and describes some recent research studies on thinking skills that have direct implications for instruction of this kind."

Logothetti, D. E. DEVELOPMENT AND IMPLEMENTATION OF THE POINCARÉ-HADAMARD CONCEPTION OF MATHEMATICAL PROBLEM SOLVING. Ann Arbor: University Microfilms, 1972.

Lovasz, Laszlo. COMBINATORIAL PROBLEMS AND EXERCISES. Amsterdam: North-Holland, 1979.
(Pro:A)

Loyd, Sam; Gardner, Martin, ed. MATHEMATICAL PUZZLES OF SAM LOYD. New York, NY: Dover Publications, Inc., 1959.
(Rec:E) Loyd's problem solving books are recreational classics.

Loyd, Sam; Gardner, Martin, ed. MORE MATHEMATICAL PUZZLES OF SAM LOYD. New York, NY: Dover Publications, Inc., 1960.
(Rec:E)

Loyd, Sam. SAM LOYD'S CYCLOPEDIA OF 5,000 PUZZLES, TRICKS, AND CONUNDRUMS WITH ANSWERS. New York: Corwin Books, 1976.
(Rec:E)

Lubkin, J. (Ed.) THE TEACHING OF ELEMENTARY PROBLEM SOLVING IN ENGINEERING AND RELATED FIELDS. Washington: American Society for Engineering Education, 1979.

Lucey, R.M. A PROBLEM A DAY. New York: Penguin, 1952.

Ludwig, D. STOCHASTIC POPULATION THEORIES. New York: Springer, 1974.
(Mod) The CUPM Modeling Panel recommends this.

Lyanchenkov, M.S. MATHEMATICAL ANTHOLOGY (MATHEMATICHESKAYA KHRESTOMATIYA). Petersburg: 1922.

Lyusternik, L.A. CONVEX FIGURES AND POLYHEDRA. Translated by Donald L. Barnett. Boston: Heath, 1966.
(Geo:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

McKim, Robert. EXPERIENCE IN VISUAL THINKING. 2nd Ed. Monterey, CA: Brooks/Cole Publishing Co., 1980.
(Gen,psy:E,I) An excellent introduction to "visual thinking".

Maier, N.R.F. PROBLEM SOLVING AND CREATIVITY IN INDIVIDUALS AND GROUPS. Belmont, CA: Wadsworth Publishing Co., 1970.
(Psy,res:E) Maier did some exceptionally clever experimentation on the Gestalt "AHA" experience. His "coatrack" and "pendulum" problems are classics.

Maki, D. and Thompson, M. MATHEMATICAL MODELS AND APPLICATIONS. Englewood Cliffs, NJ: Prentice-Hall, 1976.
(Mod) The CUPM Modeling Panel recommends this.

Maki, D. and Thompson, M. MATHEMATICAL MODELS IN THE UNDERGRADUATE CURRICULUM. Proceedings of conference at Indiana University, 1975.
(Mod) The CUPM Modeling Panel recommends this.

Maron, I. A. PROBLEMS IN CALCULUS OF ONE VARIABLE. Moscow: Mir Publishers, 1973.
(Ana:E,I) A collection of calculus problems, some routine, some not.

Mason, John. MATHEMATICS: A PSYCHOLOGICAL PERSPECTIVE. Milton Keynes: The Open University Press, 1978.

Mason, John; Burton, Leone; and Stacey, Kaye. THINKING MATHEMATICALLY. London: Addison-Wesley, 1982.
(Gen,tch:E) This book is a charming introduction to the problem-solving process, an excellent book for secondary students to read on their own. Few books meet students on their own terms as well, with a great deal of useful advice. Absolutely recommended reading.

Mathematical Association. MATHEMATICS/ELEVEN TO SIXTEEN/A REPORT PREPARED FOR THE MATHEMATICAL ASSOCIATION. London: G. Bell & Sons Ltd., 1974.
(Tch:E) Nice chapters on patterns and space, as well as a chapter entitled "Delight in Mathematics."

Mathematical Association of America, Committee on the Undergraduate Program in Mathematics. RECOMMENDATIONS FOR A GENERAL MATHEMATICAL SCIENCES PROGRAM.
(Gen:E,I,A) This major report contains recommendations for a general mathematical sciences program. Included are specific subpanel reports on calculus, "core mathematics," computer science, modeling and operations research, and statistics. There are detailed and extensive curricula suggestions, and an extensive bibliography.

Proceedings of the International Conference at the University of Southampton, 1976. MATHEMATICAL MODELS FOR ENVIRONMENTAL PROBLEMS.
(Mod) The CUPM Modeling Panel recommends this.

Mathematics Methods Project. EXPERIENCES IN PROBLEM SOLVING. Reading, MA: Addison-Wesley, 1976.

Mauldin, R. Daniel (Ed.) THE SCOTTISH BOOK: MATHEMATICS FROM THE SCOTTISH CAFÉ. Boston: Birkhaüser, 1981.
(Gen,Ana,Top:A) Selections from the famous problem book, kept in a Polish café between the two world wars, in which leading mathematicians of the period challenged one another. Includes solutions, annotations, references, etc.

Maxwell, E.A. FALLACIES IN MATHEMATICS. Cambridge: Cambridge University Press, 1959.
(Ana,gen:E,I) This book is a little gem. The fallacies and paradoxes are themselves beautifully presented; many can be taken directly into the classroom as entertainment. Moreover, asking our students to figure out what's wrong with these arguments can introduce them to careful and subtle reasoning.

Mayer, R.E. THINKING AND PROBLEM SOLVING: AN INTRODUCTION TO HUMAN COGNITION AND LEARNING. Glenview: Scott Foresman, 1977.
(Psy,res:E) A general introduction to psychological theories that bear on instruction.

Books, p. 31

Maynard-Smith, J. MODELS IN ECOLOGY. Cambridge: The University Press, 1974.
(Mod) The CUPM Modeling Panel recommends this.

Mbili, L.S.R. MATHEMATICAL CHALLENGE: 100 PROBLEMS FOR THE OLYMPIAD ENTHUSIAST. Department of Mathematics, University of Cape Town, South Africa, 1978.
(Gen,Con:E,I) A good source of challenging problems.

Melzak, Z.A. COMPANION TO CONCRETE MATHEMATICS VOLS. I and II. New York: Wiley, 1973.
Recommended by M. Klamkin for a Mathematical Olympiad Program.

Menny, Dagmár, R. OPEN QUESTIONS IN MATHEMATICS (II).

Meyer, Jerome. PUZZLE QUIZ AND STUNT FUN. New York: Dover, 1956.
(Rec:E) This book is different from most of the others in that it makes statements and asks you to determine why. There are of course some of the usual puzzles but there are enough different puzzles and conumdrums to make it worth reading.

Miller, D.W. and Starr, M.K. THE STRUCTURE OF HUMAN DECISIONS. Englewood Cliffs, NJ: Prentice-Hall, 1967.
(Psy:E,I) Includes a chapter on goals identification, three chapters on decision theory, and a chapter and review on problem solving.

Miller, George A. MATHEMATICS AND PSYCHOLOGY. New York: Wiley, 1964.

Mitrinovic, D.S. ANALYTIC INEQUALITIES. Berlin: Springer-Verlag, 1970.
(Ana:I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Mitrinovic, D.S. ELEMENTARY INEQUALITIES. Groningen, Netherlands: P. Noordhoff, 1964.
(Ana:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Mordell, L.J. REFLECTIONS OF A MATHEMATICIAN. Cambridge: University Press, 1959.
(Lit:E)

Mosteller, F. 50 CHALLENGING PROBLEMS IN PROBABILITY WITH SOLUTIONS. Reading, Mass: Addison-Wesley, 1965.
(Pro:I)

Mott-Smith, Geoffrey. MATHEMATICAL PUZZLES FOR BEGINNERS AND ENTHUSIASTS. Philadelphia: The Blackiston Co., 1946. (Rec:I)

National Council of Teachers of Mathematics. RESEARCH IN MATHEMATICS EDUCATION. Shumway, Richard J., Ed. Reston, Va: NCTM, 1980. (Res:E.I) This is the first volume in the NCTM's professional reference series, and gives a good sense of the state of the discipline. A chapter on problem solving covers the literature.

National Council of Teachers of Mathematics. (S. Krulik, Ed.) 1980 NCTM Yearbook, PROBLEM SOLVING IN SCHOOL MATHEMATICS. (Gen,tch,lit:E,I) This is probably the best practical source for school teachers. Mostly pragmatic in flavor, it offers a solid general introduction to the area for K-12 teachers, and lots of classroom suggestions.

National Council of Teachers of Mathematics. THE LEARNING OF MATHEMATICS: ITS THEORY AND PRACTICE (Twenty-First Yearbook) Reston, Va:NCTM, 1953.

National Council of Teachers of Mathematics, 1969. Booklet 17. HINTS FOR PROBLEM SOLVING.

National Council of Teachers of Mathematics, 1976. Braswell, J.S. MATHEMATICS TESTS AVAILABLE IN THE UNITED STATES.

National Council of Teachers of Mathematics. A SOURCEBOOK OF APPLICATIONS OF SCHOOL MATHEMATICS. (D. Bushaw, Ed.) Reston, Va: NCTM, 1980 (Gen,Tch:E) A collection of problems prepared by a joint MAA/NCTM committee that offers real world mathematics applications, not just "story problems."

Newell, A., and Simon, H.A. HUMAN PROBLEM SOLVING. Englewood Cliffs, N.J: Prentice-Hall, Inc., 1972. (Ai,psy,res:I,A) Several decades of work by the authors in the field of information processing have resulted in this extensive discussion of a theory of problem solving. Several empirical studies are described (chess, symbolic logic, cryptarithmetic) in which thinking "or that subspecies of it called problem solving" is analysed. One of the cornerstones of the literature of information processing and artificial intelligence. The bibliography should be of great value to those interested in this field.

Nilsson, N. PROBLEM SOLVING METHODS IN ARIFICIAL INTELLIGENCE. New York: McGraw-Hill, 1971. (Ai,res:I,A) This book is somewhat specialized, but offers an introducation to techniques used in artificial intelligence for simulating intelligent thinking in problem solving.

Nilsson, N. PRINCIPLES OF ARTIFICIAL INTELLIGENCE. Palo Alto: Tioga Publishing Co. 1980. (Ai,Res:I,A) Ditto.

Niven, I., and Zuckerman, H. AN INTRODUCTION TO THE THEORY OF NUMBERS. 3rd ed. John Wiley and Sons, New York, 1972.
(Num:I,A) The problems here are interesting but they are generally harder than in other books with similar titles.

Niven, Ivan. NUMBERS: RATIONAL AND IRRATIONAL. Washington, DC: Mathematical Association of America, 1976.
(Num, lit:E,I)

Noble, B. APPLICATIONS OF UNDERGRADUATE MATHEMATICS IN ENGINEERING. Washington, DC: MAA, 1967.
(Mod) The CUPM Modeling Panel recommends this.

Northrop, Eugene Purdy. RIDDLES IN MATHEMATICS: A BOOK OF PARADOXES. Huntington, NY: R.E. Krieger Publishing Co., 1975.
(Rec:E)

O'Beirne, T.H. PUZZLES AND PARADOXES. Oxford: University Press, 1965.
(Rec:E)

Ogilvy, Charles S. TOMORROW'S MATH: UNSOLVED PROBLEMS FOR THE AMATEUR. New York: Oxford University Press, 1962.
(Lit,rec:E,I) There are some really cute problems here, many now solved, of course.

Olinik, M. AN INTRODUCTION TO MATHEMATICAL MODELS IN SOCIAL AND LIFE SCIENCES. Reading, MA: Addison Wesley, 1978.
(Mod) The CUPM Modeling Panel recommends this one.

Olson, Alton T. MATHEMATICS THROUGH PAPER FOLDING. Reston, VA: National Council of Teachers of Mathematics, 1975.
(Rec:E)

Ore, Oystein. GRAPHS AND THEIR USES. Washington, DC: Mathematical Association of America, 1963.
(Top:E,I)

Ore, Oystein. INVITATION TO NUMBER THEORY. Washington, DC: Mathematical Association of America, 1967.
(Num,lit,rec:E)

Ore, Oystein. THEORY OF GRAPHS. Providence, Rhode Island: American Mathematical Society, 1962.
(Top:I)

Ore, Oystein. THE-FOUR COLOR PROBLEM. New York: Academic Press, 1967.
(Top:A)

Papert, Seymour. MINDSTORMS/CHILDREN, COMPUTERS, AND POWERFUL IDEAS. New York: Basic Books, 1980.
(Psy,res,lit:E) This is the single most important book to read about the "computer revolution". Brash, provocative, and compelling, it raises serious issues about how children learn and how technology can foster that learning.

Parnes, S.J. CREATIVE BEHAVIOR GUIDEBOOK. New York: Scribner's, 1967.
(Cre:E) First four review chapters on creativity. Then the details of a 16-unit course on creativity.

Pedersen, Jean J. and Armbruster, Franz O. A NEW TWIST/DEVELOPING ARITHMETIC SKILLS THROUGH PROBLEM SOLVING. Menlo Park, CA: Addison Wesley Publishing Company, 1979.
(Tch:E) Arithmetical problems that can be used in the classroom.

Pedoe, Daniel. THE GENTLE ART OF MATHEMATICS. New York: Dover, 1973.
(Lit:E)

Pedoe, Daniel. GEOMETRY AND THE LIBERAL ARTS. New York: St. Martin's Press, 1978.
(Geo,lit:E)

Perkins, D.N. THE MIND'S BEST WORK. Cambridge, MA: 1981.
(Gen,Cre:E,I,A) This book attempts to demystify "creativity" by arguing that the most creative people use the same skills as the rest of us, only better.

Perlman, A. DAMNABLE PUZZLES FROM INTELLECTUAL DIGEST. Communications/Research/Machines, 1973.
(Rec:E)

Phillips, H. ("Caliban") MY BEST PUZZLES IN MATHEMATICS. New York: Dover, 1961.
(Rec:E,I) In general, Caliban's puzzles require little mathematical knowledge but do require a bit of skill in logical thought and analysis.

Phillips, H. ("Caliban"). MY BEST PUZZLES IN LOGIC AND REASONING. New York: Dover, 1961.
(Rec:E,I)

Pielou, E. MATHEMATICAL ECOLOGY. New York: Wiley, 1977.
(Mod) The CUPM Modeling Panel recommends this.

Poincaré, Henri. THE FOUNDATIONS OF SCIENCE. New York: Science Press, 1913.
(Gen,lit:I) This is a fascinating, opinionated survey of the "state of the art" by a premier scientist of his time. Poincaré's story of how he discovered Fuchsian functions was the major impetus behind the 4-stage gestalt model posed by Wallas in THE ART OF THOUGHT, and later memorialized in Hadamard's THE PSYCHOLOGY OF INVENTION IN THE MATHEMATICAL FIELD.

Pollard, H. MATHEMATICAL INTRODUCTION TO CELESTIAL MECHANICS. Reston, VA: MAA, 1977.
(Mod) The CUPM Modeling Panel recommends this.

Pollard, H. MATHEMATICAL MODELS FOR THE GROWTH OF HUMAN POPULATIONS. Cambridge: The University Press, 1973.
(Mod) The CUPM Modeling Panel recommends this.

Pólya, George. HOW TO SOLVE IT. Princeton: Princeton University Press, 1945.
(All categories:E) The source on mathematical problem solving. In this book Pólya reintroduced the word "heuristic" to the literature. He codified useful problem solving strategies, or what he called "mental operations typically useful for the solution of problems": for example, analogy, decomposing and recombining, generalization, induction, etc. This pioneering work and his other books are must reading for anyone interested in the way we think when we solve mathematical problems.

Pólya, George. MATHEMATICAL DISCOVERY. 2 Vols. New York: Wiley, 1962, 1965 New combined paperback edition, 1981.
(All categories:E,I) These two volumes, now reissued as a single paperback, elaborate the themes first raised in How to Solve it. There's a wealth of challenging and interesting problems; there are ideas well worth pondering. No prospective teacher should be allowed in the classroom without having thought about the issues raised in this book.

Pólya, George. MATHEMATICAL METHODS IN SCIENCE. Washington: Mathematical Association of America, 1977.
(E,I)

Pólya, George. MATHEMATICS AND PLAUSIBLE REASONING. 2 vols Princeton: Princeton University Press, 1954.
(Gen,res:E,I) Volume I -- Induction and Analogy in Mathematics
Volume II - Patterns of Plausible Inference
These two volumes explore what their titles suggest. They explore a great deal of substantive mathematics, always with an eye towards the reasoning involved in uncovering it. They are most valuable reading.

Pólya, George and Szegö, G. PROBLEMS AND THEOREMS IN ANALYSIS. Berlin: Springer, 1972, 1976.
(Ana:I,A) "Perhaps the most famous and still richest problem book is that of Pólya and Szegö, which first appeared in 1925 and was republished (in English translation) in 1972 and 1976. In its over half a century of vigorous life (so far) it has been the mainstay of uncountably many seminars, a standard reference book, and an almost inexhaustible source of examination questions that are both inspiring and doable. Its level stretches from high school to the frontiers of research. The first problem asks about the number of ways to make change for a dollar, the denominations of the available coins being 1, 5, 10, 25, and 50, of course; in the original edition the question was about Swiss francs, and the denominations were 1, 2, 5, 10, 20, and 50. From this innocent beginning the problems proceed, in gentle but challenging steps, to the Hadamard three circles theorem, Tchebychev polynomials, lattice points, determinants, and Eisenstein's theorem about power series with rational coefficients." (P. R. Halmos, The Heart of Mathematics)

Pólya, George and Kilpatrick, J. THE STANFORD MATHEMATICS PROBLEM BOOK WITH HINTS AND SOLUTIONS. New York: Teacher's College Press, 1974.
(Con:E) For twenty years Stanford University conducted a competitive competition for high school seniors. The test aimed at determining aptitude rather than achievement and for that reason the problems were chosen to disclose originality and insight rather than routine competence. The book gives all the problems used, and hints for solving them. It is a good problem source.

Posamentier, Alfred S. and Salkind, Charles T. CHALLENGING PROBLEMS IN GEOMETRY. 2 vols. New York: Macmillan, 1970.
(Tch,geo:E) Euclidean geometry problems.

Rademacher, H. and Toeplitz, O. THE ENJOYMENT OF MATHEMATICS. Princeton: Princeton University Press, 1957.
(lit,rec:I) A great introduction for the interested amateur. A classic.

Rapaport, Elvira, trans. HUNGARIAN PROBLEM BOOK. 2 vols. Washington, DC: Mathematical Association of America, 1963.
(Con:E,I) The problems are taken from the EOTVOS competitions, and solutions are given. This is a good source of challenging problems.

Raudsepp, Eugene. CREATIVE GROWTH GAMES. New York: Perigee Books, 1980.
(Rec,cre:E)

Read, R. C. TANGRAMS, 330 PUZZLES. New York: Dover, 1965.
(Rec:E) Tangram puzzles, including 7-, 14- and 15- piece challenges. The organization and historical references make this an entertaining as well as challenging book.

Resnick, L. B. (Ed.) THE NATURE OF INTELLIGENCE. Hillsdale, NJ: Lawrence Erlbaum Associates, 1976.
(Res,psy:I) This book offers a series of studies by psychologists on the various components of intelligence. The essays are interesting and cover a lot of territory.

Riggs, D. THE MATHEMATICAL APPROACH TO PHYSIOLOGICAL PROBLEMS. New York: Macmillan, 1979.
(Mod) The CUPM Modeling Panel recommend this.

Riordan, John. INTRODUCTION TO COMBINATORIAL ANALYSIS. Princeton, NJ: Princeton University Press, 1980.
(Pro:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Roberts, F. DISCRETE MATHEMATICAL MODELS. Englewood Cliff, NJ: Prentice-Hall, 1976.
(Mod) The CUPM Modeling Panel recommends this.

Romanian Ministry of Education. 43 PROBLEMS (ROMANIAN MATHEMATICAL OLYMPIAD - 1978.) Bucharest, 1978.
(Con:E,I)
A good source of challenging problems, in English and Romanian.

Rubinstein, M. F. PATTERNS IN PROBLEM SOLVING. Englewood Cliffs, NJ: Prentice-Hall, 1974.
(Gen:I) A broad overview of general problem solving. Contains some excellent material particularly in problem solving via probabilistic models. Designed for graduate students, but could be used by well-prepared undergraduates. Some overlap with introductory computer science textbooks. Possibly might work as a textbook for an undergraduate operations research course.

Rubinstein, Moshe F. and Pheiffer, Kenneth. CONCEPTS IN PROBLEM SOLVING. Englewood Cliffs, NJ: Prentice-Hall, 1980.
(Gen:E,I)

Rucker, R. GEOMETRY, RELATIVITY AND THE FOURTH DEMINSION. New York: Dover, 1977.

Ryser, H. J. COMBINATORIAL MATHEMATICS. Washington, DC; MAA, 1963.
(Pro:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Saaty, T.L. LECTURES ON MODERN MATHEMATICS. New York: Wiley, 1965.
(Lit:E,I)

Saaty, T.L. THINKING WITH MODELS. AAAS Study Guides on Contemporary Problems No. 9, 1974. Oxford, NY: Pergamon Press.
(Mod) The CUPM Modeling Panel recommends this.

Saaty, T.L. TOPICS IN BEHAVIORAL MATHEMATICS. MAA summer seminar lecturer notes, Reston, VA: MAA, 1973.
(Mod) The CUPM Modeling Panel recommends this.

Sacerdoti, E. A STRUCTURE FOR PLANS AND BEHAVIOR. New York: Elsevier North-Holland, 1977.
(Ai,res:I,A) For those of you interested in artificial intelligence, this is a nice elucidation of planning mechanisms for problems which have to be done in "real time".

Salkind, Charles T. and Earl, James M. THE CONTEST PROBLEM BOOKS (3 volumes). Washington: The Mathematical Association of America, 1973.
(Con:E) Collections of problems from the MAA's annual High School Mathematics Contests.

Salmon, Wesley. ZENO'S PARADOXES. Indianapolis, IN: Bobs-Merrill, 1969.

Samples, Robert. THE METAPHORIC MIND: A CELEBRATION OF CREATIVE CONSCIOUSNESS. Reading, Mass: Addison-Wesley Publishing Co., 1976. (Cre:E)

Sawyer, Walter Warewick. VISION IN ELEMENTARY MATHEMATICS. Baltimore: Penguin Books, 1964. (Tch,Lit:E) This is about how to use pictures to understand and explain elementary mathematics. Very helpful.

Scandura. J.M. PROBLEM SOLVING: A STRUCTURAL/PROCESS APPROACH WITH INSTRUCTIONAL IMPLICATIONS. New York: Academic Press, 1977. (Res,psy:I) A rigorous, structuralist attempt to develop a synthetic theory of problem solving.

Scarf,H.et al. NOTES ON LECTURES ON MATHEMATICS IN THE BEHAVIORAL SCIENCES. Washington, D.C: MAA, 1973. (Mod) The CUPM Modeling Panel recommends this.

Schaaf, W.L. A BIBLIOGRAPHY OF RECREATIONAL MATHEMATICS. 4 vols. National Council of Teachers of Mathematics, Reston, VA: 1973. (rec) If you're looking for something in a particular area of recreational mathematics look here first. From the abacus to Zeno it's here.

Schoenfeld, Alan H. MATHEMATICAL PROBLEM SOLVING. To appear, 1983. (Gen,tch,res,psy:E,I) The book covers the theory and practice of teaching problem solving at the college level. There is a literature review, a discussion of what works and doesn't in the classroom, a discussion of ways to test problem solving performance, and of experiments that "back up" the theory. There are also broad philosophical discussions on what students learn and why, and of what we know about how the mind works.

School Mathematics Study Group, Studiesin Mathematics, Volume XVIII. PUZZLE PROBLEMS AND GAMES PROJECT/FINAL REPORT. Stanford, Ca: SMSG, 1968. (res,tch:E) Puzzles for classroom.

Schuh, Fred. THE MASTER BOOK OF MATHEMATICAL RECREATIONS. New York: Dover, 1968. (Rec:E)

Schuster, Seymour. ELEMENTARY VECTOR GEOMETRY. New York: Wiley, 1962.
(Geo:E) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Seymour, Dale. SUM PUZZLES. Palo Alto, Creative Publications, 1979.
(Rec:E)

Shanks, Daniel. SOLVED AND UNSOLVED PROBLEMS IN NUMBER THEORY. Washington: Spartan Books, 1962.
(Num:I,A)

Shklarsky, Chentzov and Yaglom. (I. Sussman, ed.) USSR OLYMPIAD PROBLEM BOOK. San Francisco: Freeman, 1962.
(Con:E) There are scads of challenging problems in this book.

Sierpiński, Wacław. ELEMENTARY THEORY OF NUMBERS. Translated by A. Hulanicki. Warsaw: Panstwowe Wydawnictwo Naukowc, 1964.
(Num:E,J) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Sierpiński, Wacław. A SELECTION OF PROBLEMS IN THE THEORY OF NUMBERS. London: Pergamon, 1964.
(Num:A) A description of solved and unsolved problems in number theory. An erudite author. Some of the statements may now be out-of-date.

Sierpiński, Wacław. 250 PROBLEMS IN ELEMENTARY NUMBER THEORY. New York: Elsevier, 1970.
(Num:I,A) Clever, interesting problems.

Simon, H.A. THE SCIENCES OF THE ARTIFICIAL. Cambridge, MA: MIT Press, 1969.
(Ai:E) A good overall introduction to the philosophy and methodology of AI (of the information processing kind, at least), and for its implications for our understanding of how the mind works.

Sinkov, Abraham. ELEMENTARY CRYPTANALYSIS: A MATHEMATICAL APPROACH. Washington: Mathematical Association of America, 1966.

Skemp, Richard R. THE PSYCHOLOGY OF LEARNING MATHEMATICS. Baltimore: Penguin, 1971.
(Res,tch:E)

Smullyan, Raymond M. WHAT IS THE NAME OF THIS BOOK? THE RIDDLE OF DRACULA AND OTHER LOGICAL PUZZLES. Englewood Cliffs, NJ: Prentice-Hall, 1978.
(Gen,Lit:E) An entertaining collection.

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Sobel, Max; Maletsky, Evan. TEACHING MATHEMATICS: A SOURCEBOOK OF AIDS, ACTIVITIES, AND STRATEGIES. Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975.
(Tch:E)

Solow, Daniel. HOW TO READ AND DO PROOFS. New York: Wiley, 1982.
(Gen:E,I) In this "introduction to the mathematical thought process," Solow tries to train students to "read between the lines" in a proof, to understand what a proof is, and to write one. Many of the things mathematicians take for granted (but students are not usually shown) are explained in detail. An interesting book for students.

South African Academy of Arts and Sciences. THE SOUTH AFRICAN MATHEMATICS OLYMPIAD. Cape Town: Nasou, 1976.
(Con:E) A good source of challenging problems.

Stein, Sherman K. MATHEMATICS: THE MAN-MADE UNIVERSE. Third edition. San Francisco: W. H. Freeman Co., 1976.
(Gen,lit,rec:E,I) Ostensibly a liberal arts text, this book is chock-full of very solid and very interesting mathematics. Starting with simple examples, Stein leads the reader into discussions of number theory, graph coloring, rational and irrational numbers, tiling problems, and much more. The problem sets are extensive and imaginative. Lots of goodies here.

Steinhaus, H. MATHEMATICAL SNAPSHOTS. Oxford, NY: University Press, 1960.
(Gen:I)

Steinhaus, H. ONE HUNDRED PROBLEMS IN ELEMENTARY MATHEMATICS. New York: Basic Books, 1964.
"(Gen,con:E,I) Steinhaus' book has exactly 100 problems, and they are genuinely elementary and good solid fun. When someone says 'problem book' most people think of something like this one, and, indeed, it is an outstanding examplar of the species. The problems are, however, not equally interesting or equally difficult. They illustrate, moreover, another aspect of problem solving: it is sometimes almost impossible to guess how difficult a problem is, or, for that matter, how interesting it is, till after the solution is known.

"Consider three examples. (1)Does there exist a sequence $(x_1,x_2,\ldots,x_{10})$ of ten numbers such that (a) x_1 is contained in the closed interval [0,1], (b) x_1 and x_2 are contained in different halves of [0,1].(c) each of x_1,x_2, and x_3 is contained in a different third of the interval, and so on up through $x_1,x_2,\ldots x_{10}$?

(2) If 3,000 points in the plane are such that no three lie on a straight line, do there exist 1,000 triangles (meaning interior and boundary) with these points as vertices such that no two of the triangles have any point in common? (3) Does there exist a disc in the plane (meaning interior and boundary of a circle) that contains exactly 71 lattice points (points both of whose coordinates are integers)?

"Of course judgments of difficulty and interests are subjective, so all I can do is record my own evaluations. (1) is difficult and uninteresting, (2) is astonishingly easy and mildly interesting, and (3) is a little harder than it looks and even prima facie quite interesting. In defense of these opinions, I mention one criterion that I used: if the numbers (10,1000,71) cannot be replaced by arbitrary positive integers, I am inclined to conclude that the corresponding problem is special enough to be dull. It turns out that the answer to (1) is yes, and Steinhaus proves it by exhibiting a solution (quite concretely: $x_1 = .95$, $x_2 = .05$, $x_3 = .34$, $x_4 = .74$, etc.). He proves (the same way) that the answer is yes for 14 instead of 10, and, by three pages of unpleasant looking calculation, that the answer is no for 75. He mentions that, in fact, the answer is yes for 17 and no for every integer greater than 17. I say that's dull. For (2) and (3) the answers are yes (for all n in place of 1,000, or in place of 71)." (P.R. Halmos, THE HEART OF MATHEMATICS)

Straszewicz, S. MATHEMATICAL PROBLEMS AND PUZZLES FROM THE POLISH MATHEMATICAL OLYMPIADS. London: Pergamon Press, 1965.
(Con:E,I) A good source of challenging problems, and an excellent source of ideas if one is making up contests.

Szasz, G., Geher, L., Kovacs, I., and Pinter, L. CONTESTS IN HIGHER MATHEMATICS, HUNGARY, 1949-1961. Budapest: Akademiai Kiado, 1968.
(Con:I) A good source of challenging problems.

Thompson, M.(Ed.) DISCRETE MATHEMATICS AND ITS APPLICATIONS. Proceedings of a conference at Indiana University, 1976.
(Mod) The CUPM Modeling Panel recommends this.

Thompson, M. (Ed.) EXPERIENCES IN PROBLEM SOLVING. Reading, MA: Addison-Wesley, 1976.
(Gen,lit:E)

Books, p. 43

Tietze, Heinrich. FAMOUS PROBLEMS OF MATHEMATICS: SOLVED AND UNSOLVED MATHEMATICAL PROBLEMS FROM ANTIQUITY TO MODERN TIMES. New York: Graylock Press, 1965.
(Gen:I,A)

Trigg, C. W. MATHEMATICAL QUICKIES. New York: McGraw-Hill, 1967.
(Rec,con:E,I) This is an attractive collection of 270 little problems.

Tucker, Alan. APPLIED COMBINATORICS. New York: Wiley, 1980.
(Pro:I) A good source of challenging problems.

Tuma, D. T. and Reif, F. PROBLEM SOLVING AND EDUCATION: ISSUES IN TEACHING AND RESEARCH. Hillsdale, NJ: L. Erlbaum and Assoc., 1980.
(Res,Psy:I) Papers presented at a conference held at Carnegie-Mellon. It gives an excellent overview of general trends, with a noteworthy summary by Allan Newell.

Ulam, S.M. PROBLEMS IN MODERN MATHEMATICS. New York: Science Editions, Wiley, 1964.
(Gen:A)

Uspenski, J. and Heaslet, M. A. ELEMENTARY NUMBER THEORY. New York and London: McGraw Hill, 1939.
(Num:I) Recommended by M. Klamkin for a Mathematical Olympiad Program. A source of wonderful problems hard to locate elsewhere. A great classic, unfortunately out-of-print.

Uspenski, J.V. INTRODUCTION TO MATHEMATICAL PROBABILITY. New York: McGraw-Hill, 1965.
(Pro:I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Uspenski, J.V. THEORY OF EQUATIONS. New York: McGraw-Hill, 1948.
(Gen,alg:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program.

Vilenesin, N.Y. COMBINATORICS. Translated by A. and S. Shenitzer. New York: Academic Press, 1971.
(Pro:E) An introduction to combinatorics containing over 400 problems with solutions.

Volkovysky, L.; Lunts, G.; Aramanovich, I. PROBLEMS IN THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE. 2nd edition. Moscow: Mir Publications, 1977.
(Ana:A) Many good problems in basic complex variables, some routine and some less commonly encountered.

Von Lanzenauer, C. CASES IN OPERATIONS RESEARCH. San Francisco: Holden-Day, 1975.
(Mod) The CUPM Modeling Panel recommends this.

Walberg, Franette. PUZZLE THINKING: STEPS TO LOGICAL THINKING AND PROBLEM SOLVING. Philadelphia: Franklin Institute Press, 1980.
(Rec:E)

Wang, P.,Ed. INFORMATION LINKAGE BETWEEN APPLIED MATHEMATICS AND INDUSTRY. New York: Academic Press. 1976.
(Mod) The CUPM Modeling Panel recommends this.

Wason and Johnson-Laird (eds.) THINKING AND REASONING. New York: Penguin Books, 1968.
(Res,psy:E) An introduction to the psychology of thinking.

Wenniger, Magnus J. POLYHEDRON MODELS. Cambridge: Cambridge University Press, 1971.
(Geo:E,I)

Wenniger, Magnus J. SPHERICAL MODELS. Cambridge: Cambridge University Press, 1979.
(Geo,lit,rec:E,I)

Wertheimer, Max. PRODUCTIVE THINKING. New York: Harper & Row, 1959.
(Res,psy:E,I) This classic exposition of the Gestalt interpretation of thinking argues that productive thinking can be understood only in terms of a fundamental structural understanding of problems and solutions. Wertheimer's perceptive analysis of the parallelogram problem is found here, and a number of other interesting problems also. There is a long résumé of Wertheimer's interviews with Einstein regarding the "birth" of relativity theory.

Weston, J.C. and Godwin, H. J. SOME EXERCISES IN PURE MATHEMATICS WITH EXPOSITORY COMMENTS. Cambridge: University Press, 1968.

Whimbey, A. and Whimbey, L. INTELLIGENCE CAN BE TAUGHT. Stamford, CT: Innovative Sciences, 1978.

Whimbey, Arthur and Lochhead, Jack. PROBLEM-SOLVING AND COMPREHENSION, A SHORT COURSE IN ANALYTICAL REASONING. Philadelphia, PA: The Franklin Institute Press, 1980.
(Rem,lit:E) This is one of the few texts at the "remedial" level that actually focus on the problem solving process, rather than repeating the same old stuff in slightly different words. It's an interesting approach to getting students to think rationally, and is worth taking a look at.

Whitworth, W.A. CHOICE AND CHANCE, WITH 1000 EXERCISES. New York: Hafner Pub. Co., 1965.
(Pro:E,I) Recommended by M. Klamkin for a Mathematical Olympiad Program. Subject: combinatorics.

Wickelgren, Wayne A. HOW TO SOLVE PROBLEMS: ELEMENTS OF A THEORY OF PROBLEM SOLVING. San Francisco: W. H. Freeman, 1974.
(Psy,tch,lit:E) In this volume Wickelgren tries to make applications of psychological research on problem solving in a useful volume for the average reader. He discusses a variety of general problem solving techniques including hill climbing, means-ends analysis, etc; these are exemplified with a range of interesting problems, going from "Instant Insanity" to cryptarithmetic.

Williams, H. MODEL BUILDING IN MATHEMATICAL PROGRAMMING. New York: Wiley, 1978.
(Mod) The CUPM Modeling Panel recommends this.

Winston, P.H. ARTIFICIAL INTELLIGENCE. Reading, MA: Addison Wesley, 1977.
(Ai,psy,res:I,A) A solid and accessible introduction to computer models of human thought processes.

Yaglom and Yaglom. CHALLENGING MATHEMATICAL PROBLEMS WITH ELEMENTARY SOLUTIONS. (2 vol.) San Francisco: Holden-Day, 1967.
(Gen,con:E) A splendid set of books with nicely grouped sets of problems. An excellent source of problems for high school students or college freshmen.

Yaglom, I.M. and Boltyanskii, V.G. CONVEX FIGURES. New York: Holt Reinhart & Winston, 1961. Translated by Paul J. Kelly and Lewis F. Walton.
(Lit,geo:I) Recommended by M. Klamkin for a Mathematical Olympiad Program. A beautiful book, full of interesting problems.

Yaglom, I.M. GEOMETRIC TRANSFORMATIONS. Translated (Vol. I and II) by Allen Shields and (Vol. III) Abe Schenitzer. Washington: Mathematical Association of America, 1967, 1973.

Yeshurun, Shraga. THE COGNITIVE METHOD: A STRATEGY FOR TEACHING WORD PROBLEMS. Reston, VA: NCTM, 1979.
(Tch,lit,rem:E) Details a method for transforming word problems into equations. The author claims that it works equally well with all ability levels.

Category III:

ARTICLES

Contest enthusiasts will want to note the following: A long series of articles on the USA and International Olympiads appear under the authorship of Samuel Greitzer, the extensive series of articles on the Putnam exam appears alphabetically under P, and a series of articles on contests in eastern European countries was written by Izaac Wirszup. A broad range of other articles appears here, with enthusiasm again reflected in the annotations. There are articles that deal directly with classroom instruction in problem solving (Halmos; Schoenfeld), collections of problems of all levels of difficulty (Erdos; Gardner; Hilbert), articles that have shaped curricula (Bruner; Piaget), and that offer insights into the way the mind works (Brown & Burton; Miller). Have fun!

Aczel, J. A LOOK AT MATHEMATICAL COMPETITIONS IN HUNGARY. American Mathematical Monthly, 67, (1960), 435-437.
(Con:E,I) A good source of challenging problems.

Agnew, J. and Keener, M. A CASE-STUDY COURSE IN APPLIED MATHEMATICS USING REGIONAL INDUSTRIES. American Mathematical Monthly, 87, (1980).
(Mod) Recommended by the CUPM Modeling Panel. On how to teach the modeling process and related pedagogy.

Alder, H.L. THE HIGH SCHOOL MATHEMATICS CONTEST. American Mathematical Monthly, 66 (1959), 138-139.
(Con:E)

Alexander, J.W. SOME PROBLEMS IN TOPOLOGY. Contained in Verhandlungen des Internationalen Math. Kong., Zurich, 1932, pgs. 249-57.
(Gen,con:E,I)

Alexander, Ralph. A PROBLEM ABOUT LINES AND OVALS. American Mathematical Monthly, 75, (1968), 482-7.

Appel, K. and Haken, W. THE SOLUTION OF THE FOUR COLOR MAP PROBLEM. Scientific American, October, 1978.
(His,lit:E,I)

Baddian, Martin. MATHEMATICS FOR CLOCK WATCHERS. The Mathematics Teacher, 72 (1979), pgs. 355-356.
(Lit:E)

Barnes, R. APPLIED MATHEMATICS: AN INTRODUCTION VIA MODELS. American Mathematical Monthly, 84, (1977), 207-210.
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Bauersfeld, H. RESEARCH RELATED TO THE MATHEMATICAL LEARNING PROCESS. Contained in Athen, Kunle (eds.) Proceedings of the Third International Congress of Mathematical Education, ICME, 1977, pgs. 231-245.
(Tch,res:E)

Beaumont, C. and Wieser, R. CO-OPERATIVE PROGRAMMES IN MATHEMATICAL SCIENCES AT THE UNIVERSITY OF WATERLOO. Journal of Co-operative Education, 11, (1975)
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Becker, J., Borrelli, R. and Coleman, C. MODELS FOR APPLIED ANALYSIS. Harvey Mudd College, 1976 and revised annually.
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Birkhoff, G. THE WILLIAM LOWELL PUTNAM COMPETITION: EARLY HISTORY. American Mathematical Monthly, Vol. 72, (1965), #5, pgs. 469-474.
(His,con:E,I)

Bleicher, M.N. SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN. American Mathematical Monthly, 72, (1965), pgs. 412-416.
(Con:E,I) A good source of challenging problems.

Boas, R.P. TRAVELER'S SURPRISES. Two Year College Mathematics Journal Vol. 10, (1979), #4, pgs. 255-258.

Borrelli, R. and Spanier, J. THE MATHEMATICS CLINIC: A REVIEW OF ITS FIRST SEVEN YEARS. UMAP Journal, 2 (1981).
(Mod:E)The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Botts, Truman. PROBLEM SOLVING IN MATHEMATICS, I AND II. I: The Mathematics Teacher, October, 1965, pgs. 496-500; II: The Mathematics Teacher, November, 1965, pgs. 596-600.
(Tch:E)

Boughn, E. MATHEMATICAL CONTEST. School Science and Mathematics, 17, (1917), pgs. 329-330.
(Con:E)

Bourne, L.E. and Dominowski, R. L. THINKING. Annual Review of Psychology, 1972, pgs. 105-130.
(Res:E) An overview of the topic as of 1972, from the psychologist's perspective.

Branca, Nicholas A., and Kilpatrick, J. THE CONSISTENCY OF STRATEGIES IN THE LEARNING OF MATHEMATICAL STRUCTURES. Journal For Research in Mathematics Education, 3, 1972, pgs. 132-140.
(Res:E)

Brookshear, J. A MODELING PROBLEM FOR THE CLASSROOM. American Mathematical Monthly, 85, (1978). pp. 193-196.
(Mod:E) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Brown, J.S. and Burton, R.R. DIAGNOSTIC MODELS FOR PROCEDURAL BUGS IN BASIC MATHEMATICAL SKILLS. Cognitive Science, 1978, 2, pgs. 155-192.
(Res,ai:E) An exemplary introduction to the role of artificial intelligence in mathematical instruction. Students' mistakes in arithmetic are so consistent that the authors are able to predict their incorrect answers to addition and subtraction problems! A knowledge of these "bugs" in students' arithmetic allows us to teach them much more effectively. This is must reading for anyone who wonders if AI has useful applications to human intelligence.

Brown, J.S., Collins, A., and Harris, G. ARTIFICIAL INTELLIGENCE AND LEARNING STRATEGIES. Contained in O'Neil, H. Learning Strategies, New York: Academic Press, 1978.
(Ai,res:I,A,) A discussion of the cognitive implications of research in AI.

Brown, Stephen I. RATIONALITY, IRRATIONALITY, AND SURPRISE. Mathematics Teaching, Summer, 1971.
(Tch,gen:E) Choose any two points in the plane and draw the line connecting them. Do it again. What can you say about the point of intersection? And how can you make this an interesting problem? Brown plays with an elementary problem, to show what can be done with it.

Brown, S.I. FROM THE GOLDEN RECTANGLE AND FIBONACCI TO PEDAGOGY AND PROBLEM POSING. The Mathematics Teacher, March, 1976, Vol. 69, pgs. 180-188.
(Tch,res:E) A collection of interrelated settings for problem solving, problem posing, and pattern searching.

Brownell, W.A. PROBLEM SOLVING. Contained in The Psychology of Learning: The Forty-first Yearbook of the National Society for the Study of Education, Part II. Chicago: The Society. 1942, pgs. 415-443.
(Res,psy:E)

Bruner, J.S. THE ACT OF DISCOVERY. Harvard Educational Review, 31, pgs. 21-32.
(Tch:E) A "pro-discovery" argument, by one of America's most influential educators.

Bruner, J.S. ON LEARNING MATHEMATICS. In D. B. Aichele and R. E. Reys (Eds.), Reading in Secondary School Mathematics. Boston, Mass: Prindle, Weber, & Schmidt, Inc., 1971, pp. 166-177.
(Tch:E)

Buck, R.C. A LOOK AT MATHEMATICAL COMPETITIONS. American Mathematical Monthly, Vol. 66, (1959), pgs. 201-212.
(Con:E,I) A good source of challenging problems.

Bush, L.E. THE WILLIAM LOWELL PUTNAM COMPETITION: LATE HISTORY AND SUMMARY OF RESULTS. American Mathematical Monthly, Vol. 72, (1965), #5, pgs. 474-483.
(Con:I,A)

Clark, E. HOW TO SELECT A CLINIC PROJECT. Harvey Mudd College, 1975.
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Clarke, E.H. PRIZE PROBLEMS FOR PRIZE STUDENTS. The Mathematics Teacher, (1930), pgs. 30-34.
(Con:E)

Conrad, Steven R. THE WIDENING CIRCLE OF MATHEMATICS COMPETITIONS. Mathematics Teacher, Vol. 70, (1977), pgs. 442-447.
(Con:E)

Crawford, and Long. GUESSING, MATHEMATICAL INDUCTION, AND A REMARKABLE FIBONACCI RESULT. The Mathematics Teacher, November, 1979, pgs. 613-616.
(Tch,Num:E)

Croft, H.T. SOME PROBLEMS OF COMBINATORIAL GEOMETRY. Contained in Combinatorial Structures and Their Applications, Proceedings of the Calgary International Conference on Combinatorial Structures and Their Applications. Calgary: Gordon and Breach, 1970.
(Top,pro:I)

Crowe, D.W. SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN, III. American Mathematical Monthly, Vol. 74, (1967), pgs. 855-858.
(Con:E,I) A good source of challenging problems.

Daniels, P. STRATEGIES TO FACILITATE PROBLEM SOLVING. Cooperative Research Project No. 1810, Provo, Utah: Brigham Young University, 1964.
(Res:E)

DeFrancis, J. MATHEMATICAL COMPETITION IN CHINA. American Mathematical Monthly, Vol. 67, (1960), pgs. 756-762.
(Con:E,I) A good source of challenging problems.

Dickenson, J. A REPLY TO J.M. SCANDURA ON MATHEMATICAL PROBLEM SOLVERS American Mathematical Monthly, Vol. 83, (1976), #3, pgs. 196-197.
(Res:E)

Dodson, J. CHARACTERISTICS OF SUCCESSFUL INSIGHTFUL PROBLEM SOLVING. No. 71-13,048, Ann Arbor, MI: University Microfilms, 1970.
(Res:E)

Duncker, Karl. ON PROBLEM SOLVING. Psychological Monographs, Vol. 58, No. 5 (1945): whole No. 270.
This is one of the classic "gestalt theory" studies of problem solving. It makes for interesting reading.

Elgarten, Gerald H. A MATHEMATICS INTRAMURALS CONTEST. The Mathematics Teacher, Vol. 69, (1976), pgs. 477-478.
(Tch:E)

Engel, Arthur. TEACHING PROBABILITY IN INTERMEDIATE GRADES. International Journal of Mathematics Education in Science and Technology, 1977, pgs. 243-294.
(Pro,tch:E)

Erdös, Paul. SOME UNSOLVED PROBLEMS. Michigan Mathematics Journal, Vol. 4, (1957), pgs. 291-300.
(A) Erdos has been a long-time problemist: collector, disseminator, solver, encourager. His collection of problems - a number of which follow - are notoriously difficult and interesting.

Erdös, Paul. REMARKS ON NUMBER THEORY IV. EXTREMAL PROBLEMS IN NUMBER THEORY. I. Mat. Lapok, Vol 13 (1962), pgs. 228-255.
(Num:A)

Erdös, Paul and Selfridge, J. L. SOME PROBLEMS ON THE PRIME FACTORS OF CONSECUTIVE INTEGERS. Illinois Journal of Mathematics, Vol. 11, (1967), pgs. 428-430.
(Num:A)

Erdös, Paul. LIST OF UNSOLVED PROBLEMS, 1962-1967. American Mathematical Monthly, Vol. 76, (1969), pg. 711.
(A)

Erdös, Paul. FINAL RESEARCH PROBLEMS (18 of them). Bull. Amer. Math. Soc., Vol. 76, (1970), pgs. 971-979.
(A)

Erdős, Paul and Hajnal, A. UNSOLVED PROBLEMS IN SET THEORY. Contained in Axiomatic Set Theory: Proc. Symp. Pure Math., Vol. 13, Providence, R.I.: American Mathematical Society, 1971, pgs. 17-48.
(A)

Erdős, Paul. LIST OF UNSOLVED PROBLEMS. American Mathematical Monthly, Vol. 78 (1971), pg. 1033.
(A)

Erdős, Paul. LIST OF UNSOLVED PROBLEMS. Fibonacci Quarterly II, (1973), pg. 77.
(Num:A)

Fagerstrom, W.H. FOURTH ANNUAL MATHEMATICAL CONTEST SPONSORED BY METROPOLITAN NEW YORK SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA. The Mathematics Teacher, Vol. 47, (1954), pgs. 211-212.
(Con:E)

Fagerstrom, W.H. and Lloyd, D.B. THE NATIONAL HIGH SCHOOL MATHEMATICS CONTEST. The Mathematics Teacher, Vol.51, (1958), pgs. 434-439.
(Con:E)

Feltges, Edna M. PLANNING A MATHEMATICS TOURNAMENT. The Mathematics Teacher, Vol. 43, (1950), pgs. 268-270.
(Con:E)

FIFTH ANNUAL WILLIAM B. ORANGE MATHEMATICS PRIZE COMPETITION. Mathematics Magazine, Vol. 29, (1955), pgs. 77-82.
(Con:E) A good source of challenging problems.

Flener, Frederick. MATHEMATICS CONTESTS AND MATHLETES. The Mathematics Teacher, Vol. 69, (1976), pgs. 45-46.
(Con:E)

Friedman, B.A. A MATHEMATICS TOURNAMENT. School Science and Mathematics, Vol. 42, (1942), p. 523.
(Con:E)

Freudenthal, Hans (ed.) ICMI REPORT OF MATHEMATICAL CONTESTS IN SECONDARY EDUCATION I. Educational Studies in Mathematics, Vol 2, (1969), pgs. 80-114.
(Con:E) A good source of challenging problems.

Gardner, M. MATHEMATICAL GAMES COLUMN. Scientific American, Vol. 196, (1957-1980).
(Rec:E,I,A,) Gardner's column needs no introduction or comment. Many of the columns were republished in collections; see the books section of this bibliography.

Getzels, J.W. CREATIVE THINKING, PROBLEM SOLVING AND INSTRUCTION. Chapter 10, Theories of Learning and Instruction, Ernest Hilgard, Ed. Yearbook of the National Society for the Study of Education. Chicago: University of Chicago Press, 1964.
(Cre:E)

Gnedenko, B.V. MATHEMATICAL EDUCATION IN THE USSR. American Mathematical Monthly, Vol. 64, (1957), pgs. 389-408.
(Con:E,I) A good source of challenging problems.

Gold, B. LOS ANGELES CITY COLLEGE MATHEMATICS PRIZE COMPETITION. The Mathematics Teacher, Vol. 47, (1954), pgs. 129-131.
(Con:E)

Goldberg, D.J. THE EFFECTS OF TRAINING IN HEURISTIC METHODS IN THE ABILITY TO WRITE PROOFS IN NUMBER THEORY. Unpublished doctoral dissertation. Teacher's College, Columbia University, 1974. Dissertation Abstracts International, 1975, Vol. 35, 4989B (University Microfilms #75-7836)
(Tch,Res:I)

Gottlieb, A. PUZZLE CORNER Column. Technology Review, Vol. 69--(1966--)

Greeno, J.G. INDEFINITE GOALS IN WELL-STRUCTURED PROBLEMS. Psychological Review, Vol. 83, (1976), pgs. 479-491.
(Res:E) Not all well-structured problems have well-structured "search spaces".

Greitzer, Samuel L. THE FIRST USA MATHEMATICAL OLYMPIAD. American Mathematical Monthly, Vol. 80, (1973), pgs. 276-281.
(Con:E,I) The USA mathematical olympiads, and the international olympiads, pit the country's and the world's most talented secondary students against a range of very challenging problems. All of these tests are good sources of problems, and the reports of students' performance are interesting.

Greitzer, Samuel L. THE SECOND USA MATHEMATICAL OLYMPIAD. The Mathematics Teacher, Vol. 67, (1974), pgs. 115-119; also in American Mathematical Monthly, Vol. 81, (1974), pgs. 252-255.
(Con:E,I)

Greitzer, Samuel L. THE THIRD USA MATHEMATICAL OLYMPIAD. The Mathematics Teacher, Vol. 68, (1975), pgs. 4-9; also in American Mathematical Monthly, Vol. 82, (1975), pgs. 218-221.
(Con:E,I)

Greitzer, Samuel L. THE FOURTH USA MATHEMATICAL OLYMPIAD. The Mathematics Teacher, Vol. 69, (1976), pgs. 28-32; also in American Mathematical Monthly, Vol. 83, (1976), pgs. 119-121.
(Con:E,I)

Greitzer, Samuel L. FIFTH USA OLYMPIAD SOLUTIONS AND EIGHTEENTH INTERNATIONAL MATHEMATICAL OLYMPIAD SOLUTIONS. Mathematics Magazine, Vol. 49, (1976), pgs. 261-263.
(Con:E,I)

Greitzer, Samuel L. THE FIFTH USA MATHEMATICAL OLYMPIAD. The Mathematics Teacher, Vol. 70, (1977), pgs. 220-221.
(Con:E,I)

Greitzer, Samuel L. THE SIXTH USA MATHEMATICAL OLYMPIAD. American Mathematical Monthly, Vol. 85, (1978), pgs. 353-356.
(Con:E,I)

Greitzer, Samuel L. THE SEVENTH USA MATHEMATICAL OLYMPIAD. The Mathematics Teacher, Vol. 71, (1978), pgs. 589-590.
(Con:E,I)

Greitzer, Samuel L. THE SEVENTH USA MATHEMATICAL OLYMPIAD: A REPORT. American Mathematical Monthly, Vol. 86, (1979), pgs. 195-197.
(Con:E,I)

Greitzer, Samuel L. THE SIXTEENTH INTERNATIONAL MATHEMATICAL OLYMPIAD AND SOME IMPLICATIONS. The Mathematics Teacher, Vol. 68, (1975), pgs. 420-424.
(Con:E,I)

Greitzer, Samuel L. 1977 INTERNATIONAL MATHEMATICAL OLYMPIAD. Mathematics Magazine, Vol. 50, (1977), p. 222.
(Con:E,I)

Greitzer, Samuel L. SOLUTIONS TO THE 1977 USA MATHEMATICAL OLYMPIAD AND THE 1977 INTERNATIONAL MATHEMATICAL OLYMPIAD. Mathematics Magazine, Vol. 50, (1977), pgs. 278-280.
(Con:E,I)

Greitzer, Samuel L. 1978 USA MATHEMATICAL OLYMPIAD AND 1978 INTERNATIONAL MATHEMATICAL OLYMPIAD. Mathematics Magazine, Vol. 51, (1978), pgs. 312-315.
(Con:E,I)

Greitzer, Samuel L. SOLUTIONS TO THE 1978 USA MATHEMATICAL OLYMPIAD AND THE 1978 INTERNATIONAL MATHEMATICAL OLYMPIAD. Mathematics Magazine, Vol. 51, (1978), pgs. 312-315.
(Con:E,I)

Greitzer, Samuel L. THE TWENTIETH INTERNATIONAL MATHEMATICAL OLYMPIAD. Mathematics Teacher, Vol. 72, (1979), pgs. 67-68.
(Con:E,I)

Hachigian, J. APPLIED MATHEMATICS IN A LIBERAL ARTS CONTEXT. American Mathematical Monthly, Vol. 85, (1978).
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Hall, C. INDUSTRIAL MATHEMATICS: A COURSE IN REALISM. American Mathematical Monthly, Vol. 82, (1975).
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Halmos, P.R. THE TEACHING OF PROBLEM SOLVING. American Mathematical Monthly, Vol. 82, (1975), #5, pgs. 446-470
(Tch,gen:E,I,A) To put it simply, anything Halmos writes about problem solving is top priority reading.

Halmos, P.R. THE HEART OF MATHEMATICS. American Mathematical Monthly, Vol. 87, (1980), pgs. 519-524.
Halmos argues that problem solving is the "heart of mathematics," and that a large percentage of our classroom time should be devoted to developing problem solving skills in our students. He reviews a number of important problem sources, and presents a solid discussion of the rationale for teaching problem solving. This article is "must" reading.

Hansen, Viggo P., Greitzer, Samuel L., Berger, Emil J., and McNabb, William K. MATHEMATICS PROJECTS, EXHIBITS AND FAIRS, GAMES, PUZZLES, AND CONTESTS. Contained in Instructional Aids in Mathematics, the Thirty-fourth Yearbook of the National Council of Teachers of Mathematics, Washington, DC, 1973, pgs. 347-399.
(Tch.,gen,con:E)

Hayes, J.R. MEMORY, GOALS, AND PROBLEM SOLVING. Contained in Kleinmuntz, B. (Ed.) Problem Solving: Research, Method and Theory. New York: Wiley, 1966.
(Res,psy:E) An "information processing" decomposition of components of the problem solving process.

Hayes, J.R. and Simon, H.A. UNDERSTANDING WRITTEN PROBLEM INSTRUCTIONS. Contained in Gregg, L.W. (Ed.) Knowledge and Cognition. Hillsdale, NJ: Lawrence Erlbaum Assoc., 1974.
(Res:I)

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(Con:E,I)

Herlands, Charles W. THE MATHEMATICAL CONTEST AT STOCKTON STATE COLLEGE. American Mathematical Monthly, Vol. 87, (1980), #4, pgs. 300-302.
(Con:E,I)

Hersee, J. THE TWENTIETH INTERNATIONAL MATHEMATICAL OLYMPIAD. Mathematical Spectrum. Vol. 11, (1978 - 1979), pgs. 33-35.
(Con:E,I)

Higgins, Jon L. A NEW LOOK AT HEURISTIC TEACHING. The Mathematics Teacher, October, 1971, pgs. 487-495.
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Hilbert, D. MATHEMATICAL PROBLEMS. Bull. Amer. Math. Soc., Vol. 8, (1902), pgs. 437-479.

"(AA) The most risky and possibly least rewarding kind of problem collection to offer to the mathematical public is the one that consists of research problems. Your problems could become solved in a few weeks, or months, or years, and your work would, therefore, be out of date much more quickly than most mathematical expositions. If you are not of the stature of Hilbert, you can never be sure that your problems won't turn out to be trivial, or impossible, or, perhaps, worse yet, just orthogonal to the truth that we all seek - wrongly phrased, leading nowhere, and having no lasting value.

"A list of research problems that has had a great effect on the mathematical research of the twentieth century was offered by Hilbert in the last year of the nineteenth century at the International Congress of Mathematicians in Paris . The first of Hilbert's 23 problems is the continuum hypothesis: is every uncountable subset of the set R of real numbers in one-to-one correspondence with R? Even in 1900 the question was no longer new, and although great progess has been made since then and some think that the problem is solved, there are others who feel that the facts are far from fully known yet.

"Hilbert's problems are of varying depths and touch many parts of mathematics. Some are geometric (if two tetrahedra have the same volume, can they always be partitioned into the same finite number of smaller tetrahedra so that corresponding pieces are congruent? -- the answer is no), and some are number-theoretic (is $2^{\sqrt{2}}$ transcendental? -- the answer is yes). Several of the problems are still unsolved. Much of the information accumulated up to 1974 was brought up to date and collected in one volume in 1976, but the mathematical community's curiosity did not stop there -- a considerable number of both expository and substantive contributions has been made since then."
(P.R. Halmos, The Heart of Mathematics)

Hlavaty, Julius H. THE CZECHOSLOVAK NATIONAL MATHEMATICAL OLYMPIADS. Mathematics Teacher, Vol. 61, (1968), pgs. 80-85. (Con:E,I)

Kantowski, M.G. PROCESSES INVOLVED IN MATHEMATICAL PROBLEM SOLVING. Journal for Research in Mathematics Education, 1977, Vol. 8, 163-180. (Tch,res:E) A research study based on a "teaching experiment," exploring the frequency with which students use certain problem solving heuristics, and the contributions of that heuristic usage to the students' success.

Kaplansky, Irving. PROBLEMS IN THE THEORY OF RINGS. American Mathematical Monthly, Vol. 77, (1970), pgs. 445-454. (A)

Katchalski, M., Klamkin, M.S. and Lin, A. AN EXPERIENCE IN PROBLEM SOLVING. American Mathematical Monthly, Vol. 88, (1981), #8, pgs. 551-556. (I)

Kilpatrick, J. PROBLEM-SOLVING AND CREATIVE BEHAVIOR IN MATHEMATICS. Contained in Wilson, J.W. and Carry, L.R. (Eds.) Review of Recent Research in Mathematics Education, Studies in Mathematics Series, Vol. 19, 153-187. Stanford, CA: School Mathematics Study Group, 1969. (Res:E) This article provided a "state of the art" discussion of research in mathematical problem solving, as of 1969. A briefer but more accessible version, "Problem Solving in Mathematics," appeared in the Review of Educational Research: 1970, Vol. 39, pgs. 523-534.

Kinsella, John J. PROBLEM SOLVING. Contained in the 33rd yearbook of the National Council of Teachers of Mathematics, 1970, pgs. 241-266. (Res,tch:E)

Klahr, D. and Siegler, R.S. THE REPRESENTATION OF CHILDREN'S KNOWLEDGE. Contained in Reese, H. and Lipsitt, L.P., (Eds.) Advances in Child Development, New York: Academic Press, 1977. (Res:E)

Klamkin, Murray S. VECTOR PROOFS IN SOLID GEOMETRY. The American Mathematical Monthly, Vol. 77 (1970), pgs. 1051-1065. Also reprinted in Selected Papers in Precalculus.

Klamkin, Murray S., and D.J. Newman. THE PHILOSOPHY AND APPLICATIONS OF TRANSFORM THEORY. SIAM Review, Vol. 3 (1961), pgs. 10-36. (Gen,Ana:E,I) Transform theory is "illustrated by a series of problems starting off with some very simple ones in arithmetic and geometry, then some in probability, number theory, differential equations, and finally ending with a rather involved boundary value problem which is solved by first determining the proper integral transform to resolve the boundary conditions.

Klamkin, Murray S. THE TEACHING OF MATHEMATICS SO AS TO BE USEFUL. Educational Studies in Mathematics, Vol. 1 (1968), pgs. 126-160. (Gen:E)

Klamkin, Murray S. ON THE IDEAL ROLE OF AN INDUSTRIAL MATHEMATICIAN AND ITS EDUCATIONAL IMPLICATIONS. American Mathematical Monthly, Vol. 78 (1971), pgs. 53-76. Also reprinted, with additional footnotes, in Educational Studies in Mathematics, Vol. 3 (1970-71), pgs. 244-269.

Klamkin, Murray S. MATHEMATICAL MODELING: DIE CUTTING FOR A FRESNEL LENS. Mathematical Modeling, Vol. 1 (1980), pgs. 63-69. (Mod:E,I)

Kneale, B. A MATHEMATICS COMPETITION IN CALIFORNIA. American Mathematical Monthly, Vol. 74 (1966), pgs. 1006-1010. (Con:E) A good source of challenging problems.

Koch, E. MATHEMATICS CONTESTS. The Mathematics Teacher, Vol. 9 (1917), pgs. 179-186. (Con:E)

Koch, E., and McCormick, T. MATHEMATICAL RELAYS FOR HIGH SCHOOLS. School Science and Mathematics, Vol. 16 (1916), pgs. 530-536. (Con:E)

Landa, L.N. THE ABILITY TO THINK: HOW CAN IT BE TAUGHT? Soviet Education, Vol. 18, 5, March, 1976. (Res,tch:E) A major example of the Soviet approach to teaching thinking.

Larkin, J.H. and Reif, F. UNDERSTANDING AND TEACHING PROBLEM SOLVING IN PHYSICS. European Journal of Science Education, Vol. 1 (1979), pgs. 191-203 An application of "cognitive engineering" to teaching in physics.

Larkin, J., McDermott, J., Simon, H., and Simon, D. EXPERT AND NOVICE PERFORMANCE IN SOLVING PHYSICS PROBLEMS. Science, 1980, Vol. 108, pgs. 1335-1342.
(Ai,res:I,A) This is a major review paper summarizing the ways that research in artificial intelligence has made progress in elucidating the problem solving skills of experts in physics.

Lenat, D. THE NATURE OF HEURISTICS. Paper CIS-12, Xerox Palo Alto Research Center, 3333 Coyote Hill Road, Palo Alto, CA 94304.
(Ai,res:I,A) Can a computer program generate interesting mathematics? Given the definitions of elementary set theory, Lenat's AM discovered arithmetic, primes, and a number of interesting concepts. It conjectured a number of well known theorems, and also one that was new... or so it was thought, until it was discovered that Ramanujan had also conjectured it. Not bad company for a machine; eh? EURISKO, the sequel to AM, is an attempt at providing what people in artificial intelligence would call a "computational theory of heuristics." Interesting reading.

Lesh, R, Landau, M., and Hamilton, E. CONCEPTUAL MODELS IN APPLIED MATHEMATICAL PROBLEM SOLVING. In Acquisition of Mathematical Concepts and Processes, R. Lesh and M. Landau (Eds). NY: Academic Press, 1983.
(Res,Tch,Gen:E,I) Teaching students about "real" problem solving means knowing about the "real world," about mathematical models, and most importantly, about how students think and learn. This is a perceptive discussion of all of those issues, with the beginnings of a theoretical framework for dealing with them.

Lesh, R. APPLIED MATHEMATICAL PROBLEM SOLVING. In Educational Studies in Mathematics, 1981, Vol. 12, No. 2.
(Res,Tch,Gen:E,I) What does classroom mathematics instruction have to do with "real" problem solving? What skills do students rely upon when confronted with problems that use mathematical thinking, but don't fall into the "cookbook" category? An interesting, provocative paper.

Lloyd, D.B. A NEW MATHEMATICAL ASSOCIATION CONTEST. The Mathematics Teacher, Vol. 48, (1955), pgs. 469-472.
(Con:E)

Lucas, J.F. THE TEACHING OF HEURISTIC PROBLEM-SOLVING STRATEGIES IN ELEMENTARY CALCULUS. Journal for Research in Mathematics Education, (1974), Vol. 5, pgs. 36-46.
(Res:E,I)

Luchins, A.S. MECHANISATION IN PROBLEM SOLVING. Psychological Monographs, Vol. 64, 6 (whole no. 248), Americal Psychological Association, (Res,psy:E) A classic monograph on the "einstellung" or "mental set" phenomenon.

Mattson, Robert J. MATHEMATICS LEAGUES: STIMULATING INTEREST THROUGH COMPETITION. The Mathematics Teacher, Vol. 60, (1967), pgs. 259-261.
(Con:E)

Mayer, R.E. INFORMATION PROCESSING VARIABLES IN LEARNING TO SOLVE PROBLEMS. Review of Educational Research, 1975, Vol. 45, pgs. 525-541.
(Res,psy:E)

Melzak, Z.A. PROBLEMS CONNECTED WITH CONVEXITY. Canadian Mathematical Bulletin, Vol. 8, (1965), pgs. 565-573.
(Geo:I)

Melzak, Z.A. MORE PROBLEMS CONNECTED WITH CONVEXITY. Canadian Mathematical Bulletin, Vol. 11, (1968), pgs. 489-494.
(Geo:I)

Merrill, H. and Stark, M. A MATHEMATICAL CONTEST. American Mathematical Monthly, Vol. 49, (1942), pgs. 191-192.
(Con:I)

Miller, G.A. THE MAGICAL NUMBER SEVEN, PLUS OR MINUS TWO. Psychological Review, Vol. 63, 2, pgs. 81-97.
(Res,psy:I) An example of "information processing" psychology at its best: experimental evidence that humans can hold 7 ± 2 "chunks" of information in short-term memory, with resulting implications for the way humans can process information (i.e.think).

Newell, Allen; Shaw, J.C.; and Simon, Herbert A. ELEMENTS OF A THEORY OF HUMAN PROBLEM SOLVING. Psychological Reviews, Vol. 65, 3 1958, pgs. 151-166.
(res,psy:I,A)

Niman, John (Ed.) PROBLEM SOLVING. School Science and Mathematics, March, 1978.
(tch,res:E) A special issue of SS&M containing a dozen articles on the role of problem solving in the mathematics curriculum; from history and research findings to practical classroom techniques.

Ouellette, Hugh, and Bennett, Gordon.THE DISCOVERY OF A GENERALIZATION: AN EXAMPLE IN PROBLEM SOLVING. Two Year College Mathematics Journal, Vol. 10, (1979), #2, pgs. 100-106.

Papert, S.M. TEACHING CHILDREN TO BE MATHEMATICIANS VERSUS TEACHING ABOUT MATHEMATICS. International Journal of Mathematical Education in Science and Technology, 1972, Vol. 3, pgs. 249-262.
Interesting philosophical ideas about doing mathematics, even for young children.

Perfect, Hazel. UNSOLVED PROBLEMS. Contained in Recent Progress in Combinatorics, Tutte, W.T. Ed. Academic Press, 1969, pgs. 341-347.
(Pro:A)

Piaget, Jean. HOW CHILDREN FORM MATHEMATICAL CONCEPTS. Scientific American, November, 1953.
If you want to know how children think and how they develop an understanding of complex structure, you must read Piaget. This is a readable introduction.

Pólya, George. GUESSING AND PROVING. Two Year College Mathematics Journal, Vol. 9, (1978), #1, pgs. 21-27.
(Gen:E)

Pólya, G., and Kilpatrick, J. THE STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS. American Mathematical Monthly, Vol. 80, (1973), pgs. 627-640.
(Con:E) A good source of challenging problems.

Post, T.R. and Brennan, M.L. AN EXPERIMENTAL STUDY OF THE EFFECTIVENESS OF A FORMAL VERSUS AN INFORMAL PRESENTATION OF A GENERAL HEURISTIC PROCESS ON PROBLEM SOLVING IN TENTH GRADE GEOMETRY. Journal for Research in Mathematics Education, 1976, 7(1), pgs. 59-64.
(Res,geo:E)

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITIONS. American Mathematical Monthly, 45 (1938) 64-66, 332,339; 49(1942) 348-351; 53 (1946) 482-485; 54 (1947) 400-403; 55 (1948) 630-633; 56 (1949) 448-452; 57 (1950) 467-470; 58 (1951) 479-482; 59 (1952) 538-542; 60 (1953) 539-542; 61 (1954) 542-549; 62 (1955) 558-564; 64 (1957) 21-27, 649-654; 68 (1961) 18-33, 629-637; 69 (1962) 759-767; 70 (1963) 712-717; 71 (1964) 634-641; 72 (1965) 469-473, 474-483, 732-739; 73 (1966) 726-732; 74 (1967) 771-777; 75 (1968) 732-739; 76 (1969) 909-915; 77 (1970) 721-728; 78 (1971) 763-770; 80 (1973) 170-179, 1017-1028; 81 (1974) 1086-1095; 82 (1975) 905-912; 83 (1976) 701-708; 85 (1978) 26-33; 86 (1979) 168-175, 749-757; 87 (1980) 634-640; 88 (1981) 605-612.
The Putnam exam is the national competition at the college level. The problems are tough and interesting. Must reading for any serious problemist.

Rabinowitz, Stanley. THE INTERCOLLEGIATE MATHEMATICS LEAGUE. American Mathematical Monthly, Vol. 73, (1966), pgs. 1004-1006.
(Con:E,I) A good source of challenging problems.

Reif, F., Larkin, J.H., and Brackett, G.B. TEACHING GENERAL LEARNING AND PROBLEM-SOLVING SKILLS. American Journal of Physics, 1976, Vol. 44, pgs. 212-217.

Resnick, L.B. and Glaser, R. PROBLEM SOLVING AND INTELLIGENCE. Contained in The Nature of Intelligence, Resnick, L.B., Ed. Hillsdale, NJ: Erlbaum, 1976.
(Res,Psy:I)

Rodin, E. MODULAR APPLIED MATHEMATICS FOR BEGINNING STUDENTS. American Mathematical Monthly, Vol. 84, (1977).
(Mod:E) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Rubin, R. MODEL FORMULATION USING INTERMEDIATE SYSTEMS. American Mathematical Monthly, Vol. 86, (1979).
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Scandura, J.M. MATHEMATICAL PROBLEM SOLVING. American Mathematical Monthly, Vol. 81, (1974), #3, pgs. 273-280.
A structuralist approach to the decomposition of complex problems.

Scheerer, Martin. PROBLEM SOLVING. Scientific American, April, 1963.
(Gen:I)

Schoenfeld, Alan H. CAN HEURISTICS BE TAUGHT? Contained in Lockhead, J. Ed. Cognitive Process Instruction, Philadelphia: Franklin Institute Press, 1979.
(Gen,tch,res:E,I) A practical discussion of what it takes to teach problem solving skills at the college level. Numerous examples are given along with a bit of theory.

Schoenfeld, Alan H. EXPLICIT HEURISTIC TRAINING AS A VARIABLE IN PROBLEM SOLVING PERFORMANCE. Journal for Research in Mathematics Education, May, 1979.
(Res:I) Do heuristics make a difference? A small-scale laboratory study indicates that they do.

Schoenfeld, Alan H. TEACHING PROBLEM SOLVING SKILLS. American Mathematical Monthly, Vol. 87, (1980), #10, pages 794-804.
(Gen,res:I) This is a "nuts and bolts" discussion of teaching problem solving at the college level. It contains a number of nice problems.

Schoenfeld, Alan H. EPISODES AND EXECUTIVE DECISIONS IN MATHEMATICAL PROBLEM SOLVING. Contained in Lesh, R. and Landau, M., Eds. Acquisition of Mathematics Concepts and Processes, New York: Academic Press, 1983.
(Res:I) What accounts for success or failure in problem solving? This paper argues that "executive" or "strategic" decisions make a difference; it presents a framework for examining the decisions that "make or break" problem solutions.

Schoenfeld, Alan H. MEASURES OF PROBLEM SOLVING PERFORMANCE AND OF PROBLEM SOLVING INSTRUCTION. Journal for Research in Mathematics Education, 13(1), January 1982, pp. 31-49.
(Tch,gen,res:E,I) Suppose you've gotten up the nerve to teach a problem solving course. How do you design a test that reflects the importance of the problem solving strategies you've taught? And how do you find out whether the students can "transfer" their learning to problemsnot quite like the ones you've shown them? Some suggestions are given here, along with a large collection of test problems.

Schoenfeld, Alan, H. BEYOND THE PURELY COGNITIVE: METACOGNITION AND SOCIAL COGNITION AS DRIVING FORCES IN INTELLECTUAL PERFORMANCE. Cognitive Science, 1983.
(Gen,res:E,I) If we could rely on students to use the information we give them in the classroom, things would be nice and straightforward. Unfortunately, they don't: a large range of conditions determines what they do, and why, when they are placed in problem solving situations. This paper examines some of them.

Shepard, G.C. TWENTY PROBLEMS ON CONVEX POLYHEDRA I and II. Mathematical Gazette, I:52(1968), pgs. 136-156; II: 52(1968) pgs. 359-367.
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(Rec:E)

Sherry, D.L. and Weaver, J.R. THE MATHEMATICS OLYMPIAD AT THE UNIVERSITY OF WEST FLORIDA. American Mathematical Monthly, Vol. 86, (1979), pgs. 125-126.
(Con:I)

Shulman, L.S. PSYCHOLOGICAL CONTROVERSIES IN THE TEACHING OF MATHEMATICS. In D.B. Aichelle and R.E. Reys (Eds) Readings in Secondary School Mathematics. Boston, Mass: Prindle, Weber & Schmidt, Inc., 1971, pgs. 178-192.
(Res,psy:E)

Shulman, L.S. PSYCHOLOGY AND MATHEMATICS EDUCATION. Contained in Begle (Ed.) 69th Yearbook of the NSSE, Chicago: University of Chicago Press, 1970, pgs. 23-71.
(Res, Psy: E) An overview of contemporary psychological theories about teaching mathematics.

Silver, Edward A. PROBLEM PERCEPTION, PROBLEM SCHEMATA, AND PROBLEM SOLVING. In Journal of Mathematical Behavior, 1982.
(Res,Tch,Psy:E,I) What we "see"in a situation often determines how we react to it. The same is true in problem solving: recognizing particular classes of problems by stereotypical features helps proficient problem solvers to classify and solve them, while misclassifying problems and acting inappropriately may harm students. This paper explores what people "see" in problem situations, and some of the ramifications of that.

Silver. Edward A. KNOWLEDGE ORGANIZATION AND MATHEMATICAL PROBLEM SOLVING. In Mathematical Problem Solving: Issues in Research, F. Lester and Joe Garofalo (Eds.) Philadelphia: Franklin Institute Press, 1982
(Res,Tch,Psy:E,I) Problem solving success depends not only on having certain knowledge, but on having it accessible, and on choosing to access and use it at appropriate times. Among the factors that determine knowledge organization and its usage are (1) the presence of problem schemata, (2) the use of elaboration, and (3) the role of metacognition in the selection of resources. These are described and discussed here.

Simon, Herbert A., and Newell, A. COMPUTER SIMULATION OF HUMAN THINKING AND PROBLEM SOLVING. In W. Kessen and C. Kuhlman (Eds.), Thought in the Young Child. Chicago: The University of Chicago Press, 1970.
(Ai,psy:E)

Simon, D.P. and Simon, H.A. INDIVIDUAL DIFFERENCES IN SOLVING PHYSICS PROBLEMS. Contained in Siegler, R. (Ed.) Children's Thinking: What Develops? Hillsdale, NJ: Lawrence Erlbaum Associates, 1978.
(AI,psy:E)

Simon, H.A. and Hayes, J.R. THE UNDERSTANDING PROCESS: PROBLEM ISOMORPHS. Cognitive Psychology, 1976, Vol. 8, 165-194.
(psy:E,I)

Skinner, B.F. TEACHING THINKING. Chapter 6 of The Technology of Teaching, New York: Appleton-Century-Crofts, 1968.
The behaviorist position should be reflected at least once in this bibliography. Here it is.

Smart, J.F. SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN, II. American Mathematical Monthly, Vol. 73, (1966), pgs. 401-407.
(Con:E) A good source of challenging problems.

Smith, D. A SEMINAR IN MATHEMATICAL MODEL-BUILDING. American Mathematical Monthly, Vol. 86, (1979).
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Smith, J.P. THE EFFECT OF GENERAL VERSUS SPECIFIC HEURISTICS IN MATHEMATICAL PROBLEM SOLVING TASKS. Unpublished doctoral dissertation, Columbia University, 1973. Dissertation Abstracts International, 1973, Vol. 34, 2400A. University Microfilms #73-26,637.

Spanier, J. THE MATHEMATICS CLINIC: AN INNOVATIVE APPROACH TO REALISM WITHIN AN ACADEMIC ENVIRONMENT. American Mathematical Monthly, Vol. 83, (1976).
(Mod) The CUPM Modeling Panel recommends this. On how to teach the modeling process and related pedagogy.

Swetz, F. THE CHINESE MATHEMATICAL OLYMPIADS: A CASE STUDY. American Mathematical Monthly, Vol. 79, (1972), pgs. 899-904.
(Con:I) A good source of challenging problems.

Troutman, A.P. and Lichtenger, B.P. PROBLEM SOLVING IN THE GENERAL MATHEMATICS CLASSROOM. The Mathematics Teacher, November, 1974, pgs. 590-597.
(Tch:E)

Turner, Nura D. THE NATIONAL CONTEST IN HIGH SCHOOL MATHEMATICS IN UPPER NEW YORK STATE. American Mathematical Monthly, Vol. 67, (1960), pgs. 73-74.
(Con:E)

Turner, Nura D. WHITHER MATHEMATICS CONTEST WINNERS. American Mathematical Monthly, Vol. 71, (1964), pgs. 425-426.
(Con)

Turner, Nura D. NATIONAL ASPECTS OF THE MAA-SA CONTEST IN THE DEVELOPMENT OF TALENT. American Mathematical Monthly, Vol. 74, (1967), pgs. 60-67.
(Con)

Turner, Nura, D. AN HISTORICAL SKETCH OF THE OLYMPIADS, NATIONAL AND INTERNATIONAL American Mathematical Monthly, Vol. 85, (1978), pgs. 802-807.
(Con)

Van Lehn, Kurt. BUGS ARE NOT ENOUGH: EMPIRICAL STUDIES OF BUGS, IMPASSES AND REPAIRS. Paper CIS-11, Xerox Palo Alto Research Center, 3333 Coyote Hill Road, Palo Alto, CA 94304.
A sequel to the Brown & Burton "buggy" paper, this paper continues to unfold the epistemology of "simple" arithmetic. If anyone needs to be convinced of the remarkable complexity of such processes, and the subtlety required to explain them, these papers are the place to start.

Walter, Marion & Brown, Stephen. WHAT IF NOT? Mathematics Teaching, Spring 1969, pgs. 38-45.
Given the proper perspective, a plaything for elementary school children (the geoboard) can give rise to interesting mathematical questions, and to good pedagogy.

Walter, M. and Brown, S.I. PROBLEM POSING, AND PROBLEM SOLVING: AN ILLUSTRATION OF THEIR INTERDEPENDENCE. The Mathematics Teacher, January, 1977.
(Gen:E)

Warren, T.F. and Davis, G.A. TECHNIQUES FOR CREATIVE THINKING: AN EMPIRICAL COMPARISON OF THREE METHODS. Psychological Reports, 1969, Vol. 25, pgs. 207-214.
(Res,cre:E)

Watson, F.R. ON MATHEMATICAL COMPETITIONS. Mathematics Teaching, Vol. 43. (1968), pgs. 4-10.
(Con:E)

Wirszup, I. THE SEVENTH MATHEMATICAL OLYMPIAD FOR SECONDARY SCHOOL STUDENTS IN POLAND. The Mathematics Teacher, 51, (1958), pgs. 585-589.
(Con:E,I) A good source of challenging problems.

Wirszup, I. THE FIRST TWO INTERNATIONAL MATHEMATICAL OLYMPIADS FOR STUDENTS OF COMMUNIST COUNTRIES. American Mathematical Monthly, Vol 69, (1962), pgs. 150-155.
(Con:E,I) A good source of challenging problems.

Wirszup, I. THE SCHOOL MATHEMATICS CIRCLE AND OLMPIADS AT MOSCOW STATE UNIVERSITY. The Mathematics Teacher, Vol. 62. (1963), pgs. 194-210.
(Con:E,I) A good source of challenging problems.

Wirszup, I. THE THIRD INTERNATIONAL MATHEMATICAL OLYMPIAD FOR STUDENTS OF EUROPEAN COMMUNIST COUNTRIES. American Mathematical Monthly, Vol. 67, (1960), pgs. 73-74.
(Con:E,I) A good source of challenging problems.

Wirszup, I. THE FOURTH INTERNATIONAL MATHEMATICAL OLYMPIADS FOR STUDENTS OF EUROPEAN COMMUNIST COUNTRIES. American Mathematical Monthly, Vol. 71, (1964), pgs. 308-316.
(Con:E,I) A good source of challenging problems.

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(Gen:E,I)

THE STATE OF THE ART

The "survey of problem solving courses" (Appendix) was mailed to all department chairmen in the United States and Canada, and in addition to all colleagues who responded to notices in a variety of journals about the survey's existence. 539 departments responded. Of them, 195 described problem solving courses currently being offered. Those courses fell into five categories, as follows.

Category 1: General Mathematical Problem-Solving Courses (78 responses)

Two responses to question 11 (Do you have a particular rationale or set of goals?) typify this group:

a) "to develop the students' problem solving ability and to provide experience in problem solving."

b) "to train students to think creatively and to provide problem-solving experience. I specially encourage students to generate problems themselves."

Within this category there were two broad camps. First, there are general courses offered to broad audiences (liberal arts majors, science majors, etc.--and a few high school courses), usually at the freshman-sophomore level and focusing on Pólya-type heuristics. Second, there are courses at the upper division level for math or science majors, with eclectic collections of problems (often from contests but not directed at contest-taking) and an emphasis more on doing problems then on the strategies for solving them. There was an average enrollment of 30 students per course, and the courses met for an average of 36 hours per term.

Of the 78 courses in this category,

76 are offered for credit, 2 not;

66 are offered for a grade (2 with P-F option), 12 on P-F basis only;

20 can be repeated for credit.

27 are offered each term, 25 each year, 5 every other year, 21 sporadically.

Enrollments average 30:

33 @ 4-15 students, 31 from 16-45, 8 from 46-100, 6 exceed 100.

Hours per term average 36:

1 @ 6 hours, 15 from 12-19 hours, 22 from 20-39 hours, 35 from 40-50, 5 exceed 50.

Category 2: Contest-Related Problem Solving Training (34 responses)

Many of these are only slightly formal offerings. One suspects that there are many more informal offerings ("problem of the week" contests, occasional afternoon work sessions in preparation for the Putnam, etc.) that were not considered "serious" enough to be labeled as "courses" for questionnaire responses. In response to question 14 (Do you use contests to motivate the course? If so, which?), all responded "the Putnam," 4 the high school Olympiads and 3 the MAA contests. The primary problem source was previous Putnam exams (all 34) with the Monthly second (15) and Math Magazine third (8). Enrollments average 8 per course, with an average of 19 contact hours per course.

Of the 34 responses in this category,

16 are offered for credit, 18 not;

12 are offered for a grade, 4 P-F, 18 no response (no credit)

12 can be repeated for credit, 3 not; 19 did not respond.

5 are offered each term, 24 each year, and 5 sporadically.

Enrollments average 8:

16 @ 3-6 students, 15 @ 7-12, 1 each @ 15,20,24.

Hours per term average 19 (excluding one course @ 75 hours):

11 from 8-15 hours, 13 from 16-25 hours, 4 from 26-35, 5 from 36-40, 1 @ 75.

Category 3: Problem Solving in Teacher Training (~ responses)

These courses are designed for both inservice and preservice teachers (about equal proportions of each), with the vast majority (27) aimed at secondary teachers. The use of at least one of Pólya's books is nearly universal; the NCTM 1980 Yearbook is frequently mentioned, as are a fair number of recreational books. Enrollments average 18 per course, and the courses met for an average of 43 hours per term.

Of the 36 responses in this category,

all 36 are offered for credit;

all 36 are offered for a grade (1 with P-F option)

2 can be taken again for credit.

23 are offered each year, 5 every other year, 8 sporadically.

Enrollments average 18:

9 courses @ 5-10 students, 18 from 11-20, 5 from 20-40, 4 exceed 40.

Hour per term average 43:

2 @ 15-24, 2 @ 25-35, 26 @ 36-49, 6 at least 50.

Category 4: Specialized Problem-Solving Instruction (28 responses)

The defining characteristic of these courses is that they are narrow in focus, directed at one aspect of problem solving or one particular topic. Of the 28 responses, 18 were in applied mathematics or mathematics modeling, 4 were in computer science, 2 in algebra, and 1 each in probability, finite mathematics, calculus, and operations research. Enrollments average 21, at 43 hours per term.

Of the 28 responses in this category,

all 28 are offered for credit;

all 28 for a grade (1 with P-F option)

4 can be repeated for credit.

8 are offered each term, 11 each year, 7 every other year, 2 sporadically.

Enrollments average 21, excluding 1 @ 1200:

10 from 4-10 students, 7 from 11-20, 3 from 21-30, 7 from 40-60, 1 @ 1200.

Hours per term average 43:

4 from 24-30, 0 from 31-39, 20 from 40-48, 3 from 56-58, 1 @ 72.

Category 5: Other Problem-Solving Courses (19 responses)

This last category splits into two parts. The first (10 responses) seems to focus on basic mathematics through a "problem solving" approach. This group includes remedial courses and courses for those who have avoided math ("Excursions" courses). The second, somewhat overlapping group (9 responses) is much broader in scope, while also mostly elementary or remedial in level. It focuses on "critical reasoning" or "analytical thinking" skills. These courses may not fall within the mathematician's "standard" view of problem solving. However, the number of responses points to a phenomenon of interest and importance. Enrollments averaged 20, for 34 hours per term.

Of the 19 responses in this category,

all 19 are offered for credit,

17 for a grade, 1 of those with a P-F option; 2 P-F only.

2 can be repeated for credit.

5 are offered each term, 9 every year, 4 every other year, 1 on request.

Enrollments average 20:

4 with 5-10 students, 9 with 11-20, 5 with 21-30, 1 @ 38.

Hours per term average 34:

4 @ 15-20, 4 from 28-30, 9 from 37-45, 1 @ 50, 1 @ 60.

Appendix

MAA Committee on the Teaching of Undergraduate Mathematics
Subcommittee on Problem Solving

SURVEY OF PROBLEM SOLVING COURSES

We would like to know about instruction designed to teach students "mathematical problem solving skills" or "how to think mathematically." Courses like Putnam Preparation Seminars or "Techniques of Problem Solving" are appropriate. A regular calculus class is not, but a one-hour "add-on" seminar might be. A course which teaches how to model mathematically would be, but one which simply demonstrates the use of mathematical models would not.

We would like one copy of the questionnaire for each course in your curriculum. Feel free to copy the questionnaire if your department offers more than one course--or contact me at the address below and I'll send more. Thank you in advance for providing this information. It will be most helpful to us.

1. Your Name ______________________

   Department ______________________

   Institution ______________________

   Address ______________________

2. What are the number and title of this problem solving course?

   Number ____________ Title ______________________

3. How often is the course offered?

   a. Each term ______ c. Every other year ______

   b. Every year ______ d. Sporadically (how often) ______

4. In what year was this course first introduced? ______________

5. How often does the course meet? ____hours per week, for ____weeks, or

   other ______________________

6. Is the course designed for any audience in particular (for example: for liberal arts majors, science or math majors, in-service teachers, students preparing for problem-solving competitions, etc.)?

   ____

7. For which students, if any, is the course required? ______________

8. Approximately what percent of the students who take the course are
   a. freshmen______ c. juniors______ e. Other (explain)__________
   b. sophomores_____ d. seniors______

9. What is the average enrollment in the course? __________

10. Is the course offered for credit? Yes___No___ If so, is it graded pass-fail____ or offered for a grade ___? Can it be repeated for credit? Yes____No____.

11. Do you have a particular rationale or set of goals for the course (e.g., "to provide problem solving experience," "to teach a strategy for solving problems," "to train students to think creatively," etc.)? Please explain.

12. Please characterize the class structure.
   a. Lecture ____%
   b. Discussion of problem sets ____%
   c. Problem solving in class, ____%, individually ____%, in groups_____%
   d. Any comments?

13. How many hours, total, are students expected to spend on homework problems?______

14. Do you use contests to motivate the course? If so, which of these:
   a. The Putnam Exam______
   b. The MAA High School Contest______
   c. The International Mathematical Olympiads______
   d. State____, regional____, or local____contests.

   Please name the contest(s) and specify if they are for high school (H) or college (C) students.

15. Do you have a formal course description_____, a set of notes_____, or a set of problems_____ you would be willing to share? If so, please enclose any such materials with the questionnaire or forward them to me at the address below.

16. Do you use a text, perhaps any of those listed below?

a. Problems from Journals:

The Monthly ____ TYCMJ ____ MATYC Journal ____

Mathematics Magazine ____ School Science and Mathematics ____

Other (please list) ____________________

b. The Hungarian Problem Books ____

c. The MAA High School Contest Books ____

d. Polya's How to Solve It ____, or Mathematical Discovery ____

e. Collections of Putnam Problems ____

f. The USSR Olympiads ____

g. The International Mathematical Olympiads ____

h. Wickelgren's How to Solve Problems ____

i. Other (please list)

17. Are there any references or problem sources you recommend? We may compile an annotated bibliography and would appreciate any suggestions you might offer. (Feel free to append as large a list as you would like.)

The space below can accommodate only brief responses. Feel free to attach additional sheets if you wish to answer in more detail.

18. In teaching problem solving, what do you find works well?

19. What recommendations do you have for someone who wishes to start offering a problem solving course?

20. In teaching problem solving, what do you find problematic?

21. Is there any information CTUM could provide you which would be useful to you?

22. What additional comments do you have?

Please send the completed questionnaire, <u>and any other material you think might be helpful</u>, to

Alan H. Schoenfeld
Chair, CTUM Subcommittee on Problem Solving
Mathematics Department
Current Address: The University of Rochester
Rochester, NY 14627

We are <u>most</u> appreciative of your help.