DOCUMENT RESUME

ED 229 218 SE 041 298

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TITLE	Children's Quantitative Notion of Rational Number.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	83
GRANT	SED-81-12643
NOTE	43p.; Paper presented at the annual meeting of the American Educational Research Association (Montreal, Canada, April, 1983).
PUB TYPE	Reports - Research/Technical (143) -- Speeches/Conference Papers (150)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	*Cognitive Processes; Educational Research; *Elementary School Mathematics; *Fractions; Grade 5; Intermediate Grades; Interviews; Manipulative Materials; *Mathematics Instruction; *Number Concepts; *Rational Numbers
IDENTIFIERS	*Mathematics Education Research

ABSTRACT

This study was undertaken to gain insights into children's understanding of rational numbers as quantities; that is, the extent to which they associate a size with a fraction like 2/3. Eight children in an experimental group in DeKalb, Illinois, chosen to reflect the range from low to high ability, were observed during 30 weeks of experimental instruction during grades 4 and 5. A classroom-sized group of 34 middle-ability children in grades 4 and 5 in Minneapolis simultaneously took part in the same teaching experiment, providing children with manipulative-oriented instruction. Seven interview assessments, each preceded by about 4 weeks of instruction, were videotaped with each DeKalb child and with eight Minneapolis children. Written tests were also given. Data from the three fifth-grade assessments are included in this report. The three tasks are described, and children's reactions are reported in detail. They had varying success on the tasks. It appeared that three knowledge structures are essential for the development of a quantitative understanding of rational number: estimation, fraction equivalence, and rational-number order. These structures appeared to develop somewhat independently, but need to be coordinated for success with rational number situations. (MNS)

CHILDREN'S QUANTITATIVE NOTION OF RATIONAL NUMBER[1]

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The research was supported in part by the National Science Foundation under grant number SED 8112643. Any opinions, findings, and conclusions expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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The authors are indebted to the following people who assisted during the research: Adrian Banas, Mark Bergstrom, John Bernard, Kathleen Cramer-Post, Laura Crowley, Debra Hartline, Nancy McQue, Thao Nguyen and Roberta Oblak.

Paper presented at the 1983 Annual Meeting
of the American Educational Research Association
Montreal, Canada, April 1983

Printed in the United States of America

CHILDREN'S QUANTITATIVE NOTION OF RATIONAL NUMBER[3]

This study was undertaken to gain insights into children's understanding of rational numbers as quantities, that is, the extent to which children associate a size with a fraction like 2/3 or 4/6 which represents a (positive) rational number. The aim was to identify levels of children's conceptions and misconceptions about rational number size that are observable across a variety of task situations.

CONTENTS

[3] The names of all children appearing in this paper were changed.

1. INTRODUCTION

1.1 The issue

While schools put emphasis on children's acquisition of fraction algorithms, that is, on how to operate with fractions as numbers, there is indication that a vast majority of children across grade levels have poor understanding of the number concept of fractions (see an earlier discussion in Behr et al, Note 2). A good understanding of fraction size, however, seems important not only in the context of fraction operations, but also in a variety of contexts including the number line and ratio and proportion. Insights into children's difficulties with a quantitative notion of rational number thus would be relevant in a larger context than only for computation.

1.2 The approach taken in this study

The task of assessing children's quantitative understanding of rational numbers appears to be difficult. This is true because a quantitative notion evolves from, and is relevant for, numerous and diverse situations. Clearly, since a (positive) rational number can be characterized as the property common to all fractions in an equivalence class, an assessment of children's notion of fraction equivalence would contribute to insights into their quantitative conception of rational number. Other situations relevant to the number conception would include order of fractions (as representatives of positive rational numbers), and estimation; for example, an estimate of the location of a fraction on the number line, or, an estimate of the outcome of an operation with fractions, requires that a size is associated with a fraction symbol.

Oversimplified, the underlying assumption in the present study was the following: Children who do not have a well-internalized, stable conception of fractions as numbers can be expected to exhibit substantial differences in their performance _across a set of tasks that vary the context in which the number concept of fraction is involved_. That is, the observations made about the number concept for a particular individual across tasks would be expected to be inconsistent. On the contrary, children who do have a well-internalized, stable conception of fractions as numbers can be expected to exhibit consistent success across such tasks. The approach taken to obtain insights into what cognitive structures are required for an individual to exhibit consistent success with different task situations, and what components can be identified as important precursors of such cognitive structures, was to look at subject performance on a variation of tasks.

2. THE STUDY

The present study was conducted by the Rational Number Project during 1982-83 (Behr _et al._, 1980). The Rational Number Project is a multi-site effort funded by NSF from 1979 through 1983. One focus of the project is to assess the development of the number concept of fraction in children.

2.1 Subjects

Subjects in this investigation were eight children in an experimental group in DeKalb, Illinois, that were chosen to reflect the full range from high through low ability and were continually observed throughout 30 weeks of experimental instruction during their 4th and 5th grade. In addition, a classroom size group of 34 4-th/5-th grade children took part in the same teaching experiment conducted simultaneously at the Minneapolis site. This class consisted of a more or less homogeneous group of middle ability children.

2.2 Instruction

The teaching experiment provided children with manipulative-oriented theory-based instruction (Behr, et al, 1980). At the time of the assessments from which data were taken for this study the children had dealt with the following manipulative aids: Colored fractional parts of circular and rectangular models, paper folding, centimeter rods, a discrete model using counting chips, and the number line. Based on the multiple-embodiment principle (Dienes, 1971), instruction had included the rational number constructs of part-whole, quotient, measure, and ratio. Students had learned to translate between different physical representations and between different modes of representation. They had associated fraction symbols and symbolic rational-number operations and relational sentences with embodiment displays. In some lessons near the assessments relevant for the present study, class activity included that children were given a fraction for which they in turn were to give fractions that were successively closer.

2.3 Assessments

Seven major (video-taped) interview assessments, each preceded by about 4 weeks of instruction, were given on a one-on-one basis to all subjects in the DeKalb group, and to 8 of the 34 subjects in the Minneapolis group. In addition, written tests were given to all subjects in both experimental groups at each time an interview assessment was scheduled. The first four assessments (I-IV) were administered during children's 4-th grade, and the other three (V-VII) during children's 5-th grade. The data relevant for the present study were gathered during these last three assessments. Additional supportive data are available from classroom observations made on a daily basis throughout the teaching experiment.

2.4 Tasks

Three specific tasks were utilized to obtain an across-task assessment of individual subjects' performance in situations involving a quantitative understanding of fractions.

2.4.1 Estimate-the-sum task

The first of two different versions of this task consisted of numeral cards on which the whole numbers 1,3,4,5,6,7 were written and a form board as shown in Figure 1.

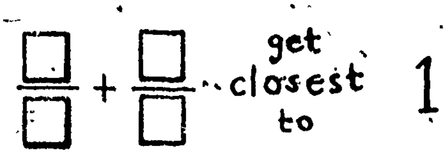

Figure 1

The second version used the same form board but numeral cards with the following numbers: 11,3,4,5,6,7. Version 1 was presented as part of Assessment V, and both versions were presented during Assessment VI. In each case, subjects were directed to "put number cards inside the boxes to make fractions so that when you add them the answer is as close to one as possible, but not equal to one." To discourage the use of computational algorithms, subjects were encouraged to estimate, and a time limit of one minute was imposed on the task. After completing the task subjects were asked to "tell me how you thought in solving this problem."

2.4.2 DARTS

The DARTS tasks were set up as a video game on an APPLE II computer (Apple Computer, 1979) and were presented as part of Assessment VI. Each screen in the game consisted of a vertical number line with randomly generated begin and end marks and a further mark at some point on the number line. At three random positions balloons were attached to the number line. The task was to pop the balloons by keying in a fraction or a mixed number to shoot a dart at the corresponding location on the number line. A sample task is shown in Figure 2.

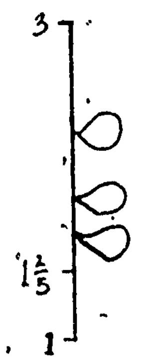

Figure 2

The number lines generated by the DARTS program by random choice consisted of one or more units. Presented to each subject was a sequence of three screens. Recorded were the subjects' attempts to pop the balloons and the unstructured dialog between interviewer and subject. Subjects were encouraged to "think aloud" throughout the game.

2.4.3 Gray-levels task

The gray-levels task was a complex problem-solving task that was given in the final assessment (VII). Subjects were presented a gray-level scale that showed 11 distinct gray levels increasing in darkness from 0% (white) to 100% (black) in steps of 10%. Subjects were then given 12 fraction cards with the fractions 0/20, 1/5, 2/7, 6/20, 2/5, 4/10, 6/15, 2/4, 4/8, 4/6, 6/9, and 12/15. These fractions were to be understood as representing concentrations of mixtures consisting of black ink and water in a way previously explained to the subject (e.g., 2/4 means "2 of 4 parts is black ink" which results in a mixture that is "two-fourths dark"). The task was to order the fraction cards from lightest to darkest and put each at a corresponding gray level in the scale; permitted was placement between two gray levels to allow for finer discrimination. The correct placement of all 12 cards is shown in Figure 3.

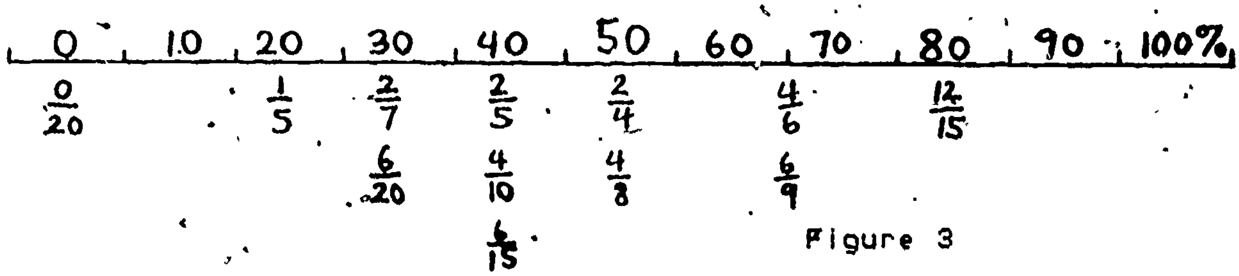

Figure 3

Recorded were the subjects' placement of cards along the gray-level scale, and anecdotal data from observations during the problem-solving process and from follow-up questions.

3. DICUSSION AND RESULTS OF THE SPECIFIC TASKS

3.1 Estimate-the-sum

The Key idea in this task was that the children were made to think in ranges rather than computing unique answers by using fraction alogrithms. Because of the imposed time constraint, trial-and-error methods in the sense of choosing any two fractions and working out the addition algorithm would not have been successful. The children were informed accordingly: "You won't have time to work out the addition. What you have to do is think about how big each fraction is and then think about how big the answer will be." In the first version of the task it was possible to make exactly 1 (4/6 + 1/3): Some subjects soon found this solution. But the difficulty imposed by the second constraint, to get close to, but not exactly to 1, required that the subjects really were to deal with the number size of fractions. For example, if 1/3 cannot be added on to 4/6, then a judgement is to be made as to what can replace 1/3, (1/5?, 1/7?, 3/7? ...) so that the result of the addition still is approximately 1. Consequently, it was expected that subjects who had a good notion of rational-number size would succeed in this task, whereas subjects lacking such a quantitative notion would exhibit considerable difficulty.

Table 1 gives the percentage deviations of the sum from 1 for Task I and Task II. The percentage deviations from 1 of constructed responses varied from 2.38 percent to 285.71 percent with an overall average percentage deviation over all subjects on all tasks of 42.36 percent.

Table 1. Overall performance on the Estimate-the-sum task

SUBJECT	d_0	d_1	d_2	$\bar{d}$
BERT	2.38	2.38	3.10	2.62
joan	3.33	= 1	nr	3.33
brett	5.00	13.33	3.64	7.32
andy	nr	8.33	nr	8.33
KRISTY	16.67	11.69	2.60	10.32
JESSIE	9.52	16.67	15.58	13.93
erica	25.00	nr	nr	25.00
JEREMY	5.00	46.67	a	25.48
margret	45.00	= 1	13.64	29.32
TERRI	29.17	45.71	232.81	30.15
MACK	58.33	23.33	9.52	30.39
TED	40.48	40.48	15.58	32.18
richard	50.00	15.00	38.64	34.55
tricia	= 1	58.33	nr	58.33
JEANNIE	285.71	26.67	250.00	187.46
till	66.71	19.00	491.67	192.79

[a] Child gave up in frustration.
nr = no response given, " = 1" = given response equal to 1.

DEKALB average 41.57

minneapolis av. 44.87

high: average deviation less than 11%
middle: average deviation less than 30%
low: average deviation more than 30%

Based on the explanations which children gave about how they solved the tasks, the responses were partitioned in 4 categories plus an "other" category. The categories together with a description of the responses in that category and one or more subject responses to exemplify the category are given. The responses indicate the type of thinking and the cognitive structures the subjects exhibited in responding to the tasks.

CATEGORY ER (Estimate by correct comparison to a standard Reference point). Response explanations in this category indicate a successful attempt to estimate the constructed rational number sum by using one-half or one as a point of reference. The spontaneous use of fraction equivalence and rational number order is evident in the subject's response.

BERT: [Using 1 3 4 5 6 7, constructed 5/6 + 1/7] .... well uh, ... five ... five-sixths [pointing to 5/6] is one piece away from the unit, and a seventh is just a little bit smaller, so that could fit there (i.e., between 5/6 and 1).

KRISTY: [From 11 3 4 5 6 7, constructs 6/11 + 3/7 and changes to 5/11 + 3/7] ... Well five and a half is half of eleven [pointing to 5/11] and [pointing to 3/7] three and a half is half of seven, so it would be one (i.e., one what is not clear) away from ... (and I changed 6/11 to 5/11) ... because [pointing to 6/11] that would be a little more and that's [pointing to 3/7] is less than one (-half) ... I was afraid they'd get exactly one. (Recall the direction to get close to one.)

BERT: [From 11 3 4 5 6 7 makes 3/6 + 5/11.] ... Three sixths is half a unit and ... if it was five and a-half-elevenths [Pointing to 5/11] that would be half; and a-half (i.e., one-half-elevenths) would be very thin.

CATEGORY MC (Mental algorithmic Computation). Response explanations placed in this category indicate that the subject did mental computation to carry out a correct standard algorithm (e.g. common denominator) to determine the actual sum of the generated fractions. The spontaneous use of fraction equivalence and rational number order is evident in the subject's thinking.

KRISTY: [Using 1 3 4 5 6 7 , makes □/3 + □/4, then changes to 1/3 + 4/5] ... If you find the common denominator, twelve; but ..., and then four times one would be four [explaining the change of □/4 to 4/5], but then three times .... I didn't have a two or anything (among the number cards given and remaining) and I used up my three so ... (Observe what Kristy is apparently doing: 1/3 is equivalent to 4/12. How many more twelfths to get close to one? This is determined from □/4 or 3□/12, so realizing that she has only 5, 6, or 7 to choose for the box, each of which gives too many twelfths, she changes the denominator to 5 and now must do the same type of thinking with fifteenths.)

CATEGORY ERI (Estimate by Incorrect, gross, or uncertain comparison to a standard Reference point). Response explanations placed in this category indicate that the subject attempted to estimate the constructed rational number sum by using one-half or one as a point of reference. Little or constrained understanding of fraction equivalence and rational number order is evident in the subjects thinking.

JESSIE: [From 11 3 4 5 6 7 makes 11/3 + 4/7, but during discussion after reading 11/3 as three-elevenths, changes to 3/11 + 4/7] ... (Is this close to one?) ... I think so, ... you take three of eleven things, that's less than a half and take four out of seven things, it's (i.e., 4/7) more than a half, I think so ... .

MACK: [From 1 3 4 5 6 7 makes 5/6 + 3/4] I just thought about equivalent fractions ... like ... wait ... like you take closest to one you can get, three-fourths 'cause it's only one (i.e. one-fourth) away (from 1), and the same with this one [pointing to 5/6].

MACK: [From 11 3 4 5 6 7 makes 4/6 + 3/7] ... Well [pointing to 4/6] it had two (-sixths) to get ... it would take two ... uh ... to equal one and I thought [pointing to 3/5] and this takes two (-fifths) ... to get to one ... and the less they (difference between each fraction addend and 1) are the greater they'd be (fraction addends), so I said (the sum) would be a little bit less (than one) ... [pause] ... a little bit more than one.

CATEGORY MCI (Mental algorithmic Computation based on Incorrect alogrithm). Responses in this category indicate that the subject used mental computation based on an incorrect algorithm to compute the actual sum.

Ted: [From 1 3 4 5 6 7 makes 5/6 + 4/7] ... Well first I thought, I tried to figure out what would come closest to one and I found out that five-sixths and four-sevenths would come the closest ... 'cause I used the top number ... (If I added them) nine-thirteenths.

JESSIE: [From 1 3 4 5 6 7 makes 3/7 + 4/6] ... Three plus four is seven and that's [pointing to 7 and 6] thirteen; it would be one-thirteenth close to one.

JEANNIE: [From 1 3 4 5 6 7 makes 6/7 + 4/3, changes to 6/7 + 3/1] ... this [pointing to 6 and 3] would be 9 and this [pointing to 7 and 1] would be 8; that's [pointing to 8] the whole and this (9) is one after it (i.e., 1 greater), so it's (i.e. 9/8) close, but not right on the dot.

CATEGORY G (Gross estimate) Response explanations placed in this category suggest that the subject made a gross estimate of each rational-number addend, but did not make a comparison to a standard reference point, and did not use fraction equivalence or rational-number ordering.

TED: [From 1 3 4 5 6 7, makes 3/11 + 4/7] ... the same thing ... I wanted to use up the little pieces for the top ... then use the highest number of pieces for the bottom ... Well, if I ever thought if it was equal, or one's less or greater and stuff, I always have to be greater than the top number.

Discussion

The children in the experimental classes had received some instructions on estimation of whole-number sums by rounding, but no formal instruction on strategies that might be used in estimating the sum of two rational-number addends. The children had received extensive experience with rational number order and equivalence. The level of understanding about order and equivalence of fractions and rational numbers exhibited by these children in tasks limited specifically to these concepts will be available in a forthcoming paper (Post et al, Note 3).

Children who displayed a spontaneous use of fraction equivalence and order concepts (i.e., those whose responses appeared in Categories ER and MC) display the highest performance on these estimation tasks as measured by the deviation of the constructed sum from 1. Responses categorized in Category ER had an average deviation of 2.98 percent, those in Category MC, 14.18 percent and the percentage across Categories ER and MC was 6.28 percent. For responses categorized in Category ERI which indicated little or a constrained understanding and application of order and equivalence concepts, the average deviation of responses was 24.68 percent. The average deviation of responses which exhibited some spontaneous understanding of fraction order and equivalence concepts, i.e., responses in Categories ER, MC, and ERI was 16.11 percent. This is contrasted to the average deviation of responses in categories other than ER, MC, and ERI; those in which concepts of order and equivalence were not applied, which was 67.55 percent.

While one must be careful about broad generalizations given the small sample sizes, it is useful to make some observations about the cognitive structures of children who exhibited Category ER and MC responses as compared to cognitive structures of children whose reponses fall in other categories. Bert (see the first and third subject responses in Category ER above) exhibits considerable imagual memory. There is evidence in his explanation (i.e., "is one *piece* away", "so that could *fit* ..." "one-half-elevenths would be very *thin*") which suggests that this child imagines episodic experiences associated with the manipulative based instruction. Moreover, he exhibits, in his ability to associate the oral symbols for mathematical entities ("*five-sixths* is *one piece* away ...") with the imagual units, the ability to translate between ideas expressed via manipulatives to ideas expressed in oral and written mathematics symbolism. This same child (see the third response in Category ER) has, in addition, excellent ability for spontaneous application of fraction order and equivalence concepts.

An example of Kristy's ability to store a long sequence of memory units together with tremendous mental symbol manipulation capability is evidenced in her response given for Category ER. She also displays considerable imagual memory for symbolic manipulations. It appears that she has excellent ability to "pre-view" an entire algorithm sequence. The order of events in the algorithm are obviously automatized so that her memory load needs to deal only with numerical entries of the algorithm while the process is automatic.

Jeannie gave Category MCI and "other" responses. This child was chosen to participate in the experimental group to represent somewhat the lower segment of high achieving children. During the teaching experiment participant-observers on numerous occasions observed that she showed reluctance to work with manipulative aids, frequently short-cutting such activity, and seek-

-ing the algorithm or rule to obtain answers. One might conjecture that her concepts of order and equivalence, rather than abstracted from manipulatives, is more likely based on given or self-generated rules or procedures.

Ted (see Category G) displayed a very gross method of estimating fraction size. He seemed firm in his understanding that fractions with denominators greater than their numerators have a value less than one. He apparently generalized this incorrectly to believe that the sum of two fractions, both of this form, would be less than one.

Again we observe in many of the students the inability to use concepts of order and equivalence in an application type of task. Data from a forthcoming paper will show that most of these children were quite capable with symbolic order and equivalence tasks in a setting for which the question was for two given fractions, are they equal or is one less (Post, Behr, and Wachsmuth, Note 3). Further elaboration of the results of the Estimate-the-sum study will be presented in (Behr and Wachsmuth, Note 1).

3.2 DARTS

The DARTS game is a nice task to assess children's quantitative notion of rational number since it offers a challenging situation that requires associations of fractions and mixed numbers with points on a (vertical) number line, that is, the size of a rational number is embodied in a length. (The unit size randomly varied from screen to screen so memorization of the unit size was unlikely.) If a subject's attempt at popping a balloon was unsuccessful, the actual location of the attempted rational number was displayed on the screen as a label at the number line. That is, an immediate feed-back to an attempt was given. As almost always a subject's next attempt would build on this feedback, the DARTS task is a powerful means for eliciting behavior that gives insights into the cognitive structures acquired by the individual subjects about rational numbers.

Presented in this section are selected segments of episodes in which children were responding to the micro-computer-presented tasks and the interviewer's questions. The episodes were selected to exemplify different levels of children's thinking in the context of rational number order and fraction equivalence.

The first episode involves Kristy after she was presented with the number line: (5, 5 1/9, 5 1/3 , 5 5/8, 5 4/5, 8), that is, a 5 - 8 number line with a further label at 5 1/3, and balloons attached at (non-labeled!) points 5 1/9, 5 5/8, and 5 4/5. Especially notable about Kristy's thinking in this excerpt is the flexibility with, and automatic generation of equivalent fractions. It appears that when Kristy thinks of a fraction she is aware of an unlimited set of equivalent fractions and is able to think about a number of them automatically. In some cases she gives evidence, especially with pauses in the explanations, that she is using some computation to generate an equivalent fraction. She appears to be completely comfortable in this excerpt to use different fraction names for the same point on the number line. She uses equivalent fractions to move the number line within self-specified bounds.

KRISTY: Oh boy, that's one-third [iterates the distance from 5 to 5 1/3 along the number line] and that [pointing to the balloon at 5 5/8] would be five and two-thirds (The dart is projected and misses) ... [Taking aim at the same balloon] ... about 5 3/6.

INTERVIEWER: How did you think to come up with five and three-sixths?

KRISTY ... Well, I thought it (pointing to 5 2/3 on the number line] would be equal to four-sixths; and then, you want it to be lower (but) I didn't want to take a third lower (dart misses) ... Ok, five and two-thirds is equal to ... six-ninths ... . I'm going to take it (i.e. 2/3) equal to eight-twelfths, then how about seven-twelfths (i.e. 5 7/12 for the next shot) because, that's a little bit less (than 2/3) [shot misses]. Ok, two (-thirds) is equal to ... ten ... ten-fifteenths and so nine-fifteenths (i.e. for the next shot) [shot hits balloon at 5 5/8].

At this point 5 3/6, among other fractions, is marked on the number line.

KRISTY : [Takes aim at the balloon at 5 1/9 by iterating the distance from 5 to 5 1/9 up to 5 3/6] That [pointing to the balloon at 5 1/9] will be five and one-eighth ... because that [pointing to 5 3/6] was one-half and that took about four (i.e. iterations of the distance from 5 to 5 1/9) to get there, so that would be eight all across.

The next episode involves Bert at screen (1, 1 1/3, 1 1/2, 1 3/5, 1 3/4, 2). Bert had made shots (1 3/5, 2), (1 3/4, 1 2/3), (1 1/3, 1 2/6)* (popped balloon is indicated by *) and was taking aim at the balloon at 1 3/5 and he explains:

BERT: One and two-thirds is more than one and two-sixths ...... [points to the balloon at 1 3/5]. What's between a-half [using the fixed point 1 1/2] and two-thirds, it'd be one and three-fifths.

Bert gave no overt indication of how he arrived at the fact that 1 3/5 is between 1 1/2 and 1 2/3. Since he earlier referred to 2/6, one conjecture is that he thought of 1/2 as 3/6 and then chose 3/5 because it is greater than 3/6.

The next episode also involves Bert; he was presented with screen (1, 1 1/5, 1 1/3, 1 1/2, 1 3/4, 2). The following shots had been made (1 1/5, 1 1/6)*, (1 1/2, 1 3/5). We noted in the above that Bert, as Kristy, makes spontaneous use of equivalent fractions; he also displays a good application of fraction-order to the number line. Bert seems to have order on the number line clearly associated with the order of fractions via symbolic interpretations.

In the very next episode we get a feel for Bert's sense of rational number size; in the episode he orders three fractions after indicating that one of them is just over (i.e., just a little more) than the least of the three while another one is more (i.e., more than the little more). In the excerpt that follows we observe his strong imagual base for his fraction concept. This is evident through the imagual language (i.e., pieces are smaller).

BERT: [Taking aim at the balloon at 1 1/2]... something between one-third (i.e. 1 1/3) and one and three-fifths ... one and 3-fifths is just over one and one-half and two-thirds is more than a half, so un... one and three-sixths, same as one and one-half.[6]

Next Bert measures on the number line, since 1 1/6 has been marked on the number line he iterates the distance from 1 to 1 1/6 up the number line and finds that 1 5/6 takes him above the target.

BERT: It couldn't be one and five-sixths; one and five-sevenths.
INTERVIEWER: Tell me how you chose one and five-sevenths.
BERT: ... Since the pieces are smaller ... one and five-sevenths would be a little more down (i.e. than 1 5/6).

In Jessie we see a level of functioning with the concept of fraction equivalence which might be called latent. We say that Jessie's level of thinking with respect to fraction equivalence is latent because her use, generation, and recognition of equivalent fractions occurs only after she is prompted by some external source such as the interviewer or the computer screen to consider equivalent fractions. The following protocol deals with DARTS screen (3, 3 1/9, 3 3/8, 3 1/2, 3 3/5, 4).

JESSIE: [Aims at the balloon at 3 1/9, shoots (3 1/9, 3 3/6)].

INTERVIEWER: [After the 3 3/6-dart hits and Jessie can observe that 3 3/6 hits the same point as 3 1/2] What can you say about this (3 3/6)?

JESSIE: It's equal to three and one-half. ... [Indicates shot (3 1/9, 3 2/3)].

....

INTERVIEWER: [Points to 3 1/3 and balloon at 3 1/9] Can you name a mixed number less than three and one-third?

JESSIE: Three and ... three and two-fourths.

INTERVIEWER: That would be below three and one-third?

JESSIE: Wait ... wait, three and one-fourth ... wait three and one-seventh [laughs].

INTERVIEWER: Why do you say three and one-seventh?

JESSIE: Because the pieces are smaller. [Shot (3 1/9, 3 1/7) misses above the target] [Indicates (3 1/9, 3 2/4) as the next shot.]

INTERVIEWER: Where do you think it will go?

JESSIE: [Points to the balloon at 3 1/9] Right there. [Shot misses and records 3 2/4 at same point with 3 1/2 and 3 3/6].

INTERVIEWER: Why do you think it hit the same point as three and one-half?

JESSIE: 'cause they are equal.

Jeremy shows behavior which suggests a level of thinking on fraction equivalence which would be classified as latent. Moreover, we observe in Jeremy a weak concept of fraction order; he is very doubtful about the order of 1/2 and 4/13. The screen display is (0, 2/5, 2/3, 4/5, 1, 2). The following episode begins after Jeremy has made these three shots (2/5, 2/5)*, (2/3, 3/10), (2/3, 9/15)*.

JEREMY: [Takes aim at balloon at 4/5] one hundred-nineteenths.

INTERVIEWER: I can't key that in, will it be above 3 or below 1?

JEREMY: Above. [Indicates next shot] Twelve-twenty-fifths.

INTERVIEWER: Oh, I cannot use that.

JEREMY: Six-twelfths.

INTERVIEWER: [After shot hits and marks the same spot as 1/2] Why will six-twelfths go through the one-half?

JEREMY: One-half, two-fourths, six ... (twelfths) ... [Shoots, (4/5, 6/12), (4/5, 4/9), (4/5, 6/18), (4/5, 1/2), then suggests (4/5, 4/13)].

INTERVIEWER: Would this (4/13) be more or less than one-half?

JEREMY: It would be a little bit more, I have the feeling, I hope, wait ... wait, wait, wait; five-thirteenths (shot misses).

In the following episode Mack displays his ability to apply concepts of fraction ordering as it relates to a quantitative concept of rational number. The task dealt with the following screen: (1, 1 2/5, 1 2/3, 1 8/9, 2 1/3, 3); the following shots had been made (1 2/3,1 4/5), (1 2/3, 1 3/5)*, (1 8/9, 2) when the following took place:

INTERVIEWER: You have to get closer to two.

MACK: I know, ... wait ... one and ... seven-eights.

INTERVIEWER: Why did you say ...

MACK: ... I thought, it's got to be one away (i.e. one fractional part away from 2) ... one thing away from something ... I thought it was small pieces (i.e. the eights are small pieces).

In the next episode Mack exhibits considerable knowledge about the number line structure and of fraction order. The task is (7, 7 1/3, 7 1/2, 7 4/7, 7 5/7, 8).

MACK: [Measures the line with his fingers] Holy smokes! For the top one (i.e. the balloon at 7 5/7) it'd be seven and five-sevenths ...[measures line between 7 5/7 and 7 4/7]. That's got to be one-seventh so go up [measures number line from the bottom]. seven and three-sevenths [points to the balloon at 7 1/3] seven and three-sevenths ... (shot misses) At least that gets me somewhere.

That Mack is using the fact that the balloon at 7 1/2 is bracketed by shots marked at 7 3/7 and 7 4/7 is evidenced by his shot (7 1/2, 7 1/2) which is midway between 7 3/7 and 7 4/7. It might be possible to infer that Mike is thinking of one-half as three and one-half-sevenths, or is thinking about 3/7 and 4/7 as 6/14 and 8/14, respectively and then chooses 1/2 as 7/14.

Subjects' overall performance on the DARTS task is compiled in Table 2.

Table 2. Overall performance on the DARTS task

Subject	Shots/screen	Average/screen
KRISTY	(3, 3, 3)	3
BERT	(5, 4, 5)	4.7
JEANNIE	(4, 3, 7)	4.7
andy	(5, 4, 6)	5
brett	(6, 4, 6)	5.3
richard	(3, 4, 11)	6
tricia	(9, 4, 5)	6
joan	(8, 6, 5)	6.3
TED	(4, 9, 6)	6.3
MACK	(7, 7, 6)	6.7
erica	(11, 9, 5)	8.3
margret	(5, 4, 8)	8.4
JEREMY	(9, 13, 6)	9.3
JESSIE	(10, 12, __)	11
till	(9, 15, 9)	11
TERRI	(11, 16, __)	13.5
	DEKALB average	7.4
	minneapolis av.	7.0

high: average/screen ≤ 5
middle: average/screen ≤ 9
low: average/screen > 9

3.3 Gray-levels task

The key idea in embodying rational numbers in gray levels will be described briefly using the rational number 1/2 as an example. One-half is the property common to an equivalence class of fractions: [1/2] = {1/2, 2/4, 3/6, 4/8, 5/10, ...}. The property common to the fractions in this class is that for each the comparison of numerator to denominator is reflected in the ratio 1:2. In the common part-whole embodiment the interpretation is that "half" of the total number of parts into which a unit is partitioned are shaded; see Figure 4a.

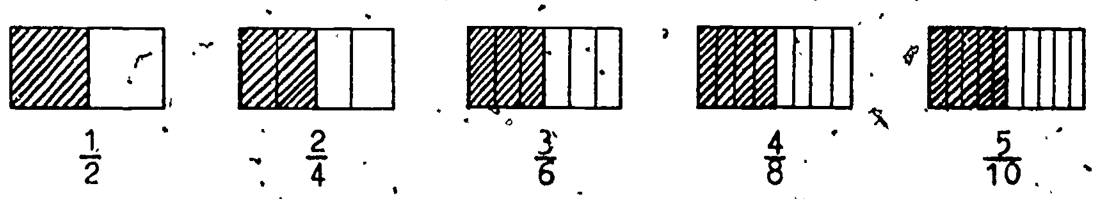

Figure 4a

Continued to the extreme, this way of embodying 1/2 would still require that an agreed-upon unit is shown, "half" of which is shaded; see Figure 4b.

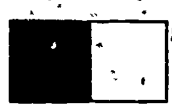

Figure 4b

The numerator/denominator comparison of the fractions in the class [1/2] is also reflected in the following way of shading half of the total number of parts into which a unit is partitioned; see Figure 5a.

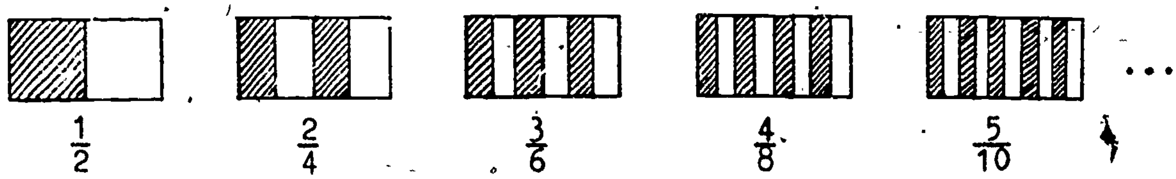

Figure 5a

Continued to the extreme, this way of embodying 1/2 would lead to a shading as shown in Figure 5b, that is, an average gray shading.

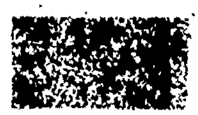

Figure 5b

This makes the perception of the embodiment for 1/2 somewhat independent of the reference to an actual size of a unit (much in the same way as the rational number 1/2 is independent from a unit) since 1/2 is embodied in <u>any subsection</u> of the unit, namely, in the darkness of the shading.

The gray-levels task was presented to the subjects in the context of the following (ficticious) situation: Black ink and water are mixed together to make lighter ink in a way printers might do it for their printing machines. Then mixtures where 1 of 2 parts is black ink, or 2 of 4 parts, or 3 of 6 parts, etc., would be equivalent in the sense that in each case the resulting gray level is the same (no matter how much liquid is produced, i.e. which unit is chosen). Consequently, mixtures where, for example, 2 of 5 parts, or 4 of 10 parts, etc., is black ink would yield a gray color less than "half dark." A pilot assessment (during Assessment VI), ensured that subjects understood this manner of embodying a rational number. Upon this, it was decided to use gray levels to assess the quantitative notion of rational numbers (in the range from 0--clear--through 1--totally black).

Even though the task was presented to the subjects embedded in a situation they were able to grasp (and the gray scale was prepared with considerable care to show perceptually distinguishable stages), limitations of human visual perception and imagination restrict the association of a rational number with a unique gray level. Were this not the case, subjects would be able to associate the fractions 2/5, 4/10, and 6/15, for example, with a single gray level without symbolic-level realization that these are equivalent fractions. Since visual perception and imagination is probably not sufficiently sensitive in this situation, subjects' solutions would necessarily draw upon their individual knowledge about the fractions. That is, because the ratios of black ink to total liquid are equivalent in 2/5, 4/10, and 6/15, these fractions would have to be associated with the same gray level.

The question of which gray level is the appropriate one for each of the 12 fractions in the task would involve subjects' knowledge of the (order and equivalence) relationships between the fractions. In associating the fraction cards to the gray scale, subjects could use the fact that the left border of the scale was white ("no ink, clear water") and the right border was black ("all ink, no water"), and the center gray level "half black" ("half ink, half water").

We now present results of the DeKalb and Minneapolis interview subjects' performance on the gray-levels task. A rough "performance index" of subjects' placement of the cards along the gray-level scale is compiled in Table 3. Shown is the average percent deviation, d, of each subject's placement of the cards:

$$d = 1/12 \sum_{i=1}^{12} |\text{correct location card } i - \text{subject's location card } i|$$

Also shown is the maximal percent deviation, dmax, by which each subject's placement of cards deviates from the correct locations:

$$dmax = \max_{i} |\text{correct location card } i - \text{subject's location card } i|$$

The performance index as shown in Table 3 reflects subjects' ability to associate a quantitative value with the fractions involved, but it does not convey the causal relationship between subjects' understanding of the order and equivalence relationships among the fractions and their size perception of the corresponding rational numbers. For example, one subject recognized the equivalence of 2/5 and 4/10 but misplaced the pair by 10 % ; another subject did not recognize this equivalence and misplaced only 4/10 by 10 % while placing 2/5 correctly.

Recognition of equivalences certainly is an important variable in this task. Four distinct equivalences were involved: 2/4 and 4/8, 2/5 and 4/10, 6/15 and 2/5 (or 4/10), and 4/6 and 6/9. Not always did recognition of an equivalence result in a subject's placement of the corresponding cards at the same gray level as will be documented below. The sets of equivalent fractions which were attached at the same gray levels by individual subjects are shown in Table 4.

Table 3. Overall performance on the gray-levels task
Average and maximal deviation in card placement

Subject	d	dmax
KRISTY	2.1	10
BERT	3.3	10
erica	4.2	10
till	5.8	20
joan	6.3	30
richard	7.1	25
brett	8.3	25
JEANNIE	10.0	30
andy	12.1	30
tricia	13.5	35
JESSIE	14.2	30
margret	16.7	70
JEREMY	19.2	70
TED	22.5	70
TERRI	23.3	70
MACK	29.6	90
DEKALB average	15.5	
minneapolis av.	9.3	

high: averaged less than 10% off
middle: averaged less than 20% off
low: averaged more than 20% off

Table 4. Placement of equivalent fractions at the same gray level by individual subjects

Subject	2/4 & 4/8	2/5 & 4/10	4/6 & 6/9	6/15 & 2/5 or 4/10
KRISTY	+	+	+	-
JESSIE	+	+	+	-
erica	+	-	-	+
richard	+	+	-	+
brett	+	-	-	+
BERT	+	-	-	-
till	+	-	-	-
joan	+	-	-	-
JEANNIE	+	-	-	-
andy	+	-	-	-
margret	+	-	-	-
TED	+	-	-	-
tricia	-	-	-	-
JEREMY	-	-	-	-
TERRI	-	-	-	-
MACK	-	-	-	-

high: recognized 2 or more equivalences
middle: recognized equivalence of 2/4 and 4/8
low: recognized no equivalence

Further clarification about subjects' thinking is provided in short descriptions of some subjects' behavior while performing the task; this is complemented with dialogue to exemplify some of the conceptions and misconceptions elicited (the full presentation of results is deferred to a forthcoming paper). The presentation follows the organization of Table 3 in groups of high, middle, and low performers.

The highest subject, Kristy, showed superior performance both in recognizing equivalences of fractions and placing them at the correct level of darkness on the scale. The only fraction she did not associate with its equivalents was 6/15 which she placed only 5 % off (left) of 2/5 and 4/10 which she had placed correctly at 40 %. Besides coordinating her knowledge about fraction equivalence with the placement of fraction cards at appropriate gray levels, Kristy also made strong use of the length embodiment for fractions that was implicitly present in the gray level scale by associating lengths with the position of gray levels. This is demonstrated in her explanation of why 6/9 should be between 60 % and 70 %: She first observed that 6/9 is equal to 2/3, then said "you can't divide it (the scale) into thirds," but then she observed the following partitioning (conjectured from her behavior and comments): Consider 0 % - 80 % (i.e., the corresponding positions at the scale); 60 % would be at about the 2/3 point, however, adding on the 90 % and 100 % levels makes the scale larger to the right so the location of 2/3 would move to the right as well. Further, Kristy's discrimination for fraction size was so exact that she even put 2/7 slightly left of 6/20.

Although Bert's overall performance on the gray-levels task was nearly as high as Kristy's, a striking difference was observed. Kristy coordinated her knowledge about fraction equivalences with a good perception about the location of (lowest-term!) fractions on the scale; on the other hand, Bert's behavior suggests that he possesses both of these relevant knowledge structures, but the connections to be made are latent in his performance on the task. This is further commented on in the following anecdote.

BERT: [Early-on, sorts the cards and puts 2/4 and 4/8 together on table.]

INTERVIEWER: You put two-fourths and four-eighths together?

BERT: [picks them up] They're equal.

INTERVIEWER: I see... Would you put them on the same card (i.e. gray-level)?

BERT: Yeah... [now puts 6/9 together with 4/6] These two are equal...

That is, before Bert starts putting cards at the gray scale, he makes some observations about the fractions and only then starts putting them, one-by-one, at the gray scale. In so doing, he first puts 4/6 at the 60 % level then and 6/9 at the 70 % level. Similarly, he puts 2/5 at 40 %, 4/10 at 45 %, and 6/15 at 35 %. That is, with respect to placement on the gray-level scale, Bert rates these (equivalent) fractions as very close in size but has lost sight of their equivalence.

INTERVIEWER: [after the whole task has been completed] You put six-ninths right of four-sixths, why did you do that?

BERT: Because four-ninths and a half (ninths) would be half a unit...

Bert apparently talks about 4 1/2-ninths which explains why he placed 6/9 right of, but still not far away from, the "half-dark" position; this kind of flexible and fine-tuned thinking Bert also displayed in his explanations for his placement of other cards.

INTERVIEWER: ... Before, you mentioned that they are equal ... four-sixths and six-ninths ...

BERT: Oh yeah, they are! [picks up 6/9 and 4/6] I think they'd be right there [puts both cards on 60 %].

It may be of interest noting here that Bert did a very similar thing in a parallel version of this task that involved ratio cards (to be reported in a different context in Wachsmuth et al, 1983); there he also placed 2:3, 4:6, and 6:9 at different but adjacent gray levels.

This phenomenon of a good sense of fraction size independent of recognition of equivalences is displayed in similar ways in most of the other "high" subjects' (Table 3) performance on the gray-levels task: Except for Brett (and of course Kristy), they all placed 4/6 and 6/9 at different but adjacent gray levels close to the correct position.

From Till, a lower subject in the group of high performers (Table 3), we see indication that during exposure to the experimental instruction he developed at least a rough feeling for the size of a fraction; about 12/15, 6/9, and 4/6 which he placed at the 80 (4/6) and 70 percent levels he remarks:

TILL: Because that looked like three-fourths, and that looked like three-fourths, and that looked like three-fourths [in pointing to the 12/15, 6/9, and 4/6 cards].

INTERVIEWER: And three-fourths to you is what?

TILL: Like four-sixths and six-ninths and twelve-fifteenths.

The subjects in the middle group (Table 3) are characterized by generally lower performance in positioning the fraction cards at appropriate gray levels on the one hand, and on the other by making mistakes in ordering the fractions. In many cases the "more familiar" fractions like 4/8 and 1/5 were placed correctly, while the less familiar ones could be placed incorrectly. For example, all three lower middle subjects (Table 1) placed 6/15 left of 6/20.

In particular the lowest middle subject, Jeremy, in several cases arranged fraction pairs in the wrong order within the string of all twelve fractions, for example, his arrangement reflected that 4/6 should be less than 4/8. When Jeremy was asked which of the two fractions is less, he answered that 4/8 is less, that is, knew it "in some sense." But as can be seen from the following anecdote, Jeremy had to be prompted to the insight that in this case 4/8 should go with a lighter gray level than 4/6; his "in-some-sense" knowing was inconnected with, and did not apply to, the task situation.

INTERVIEWER: Now, Jeremy, what about four-sixths and four-eighths [Jeremy starts to move the cards] no, don't move them.... Tell me why you put 4/8 here [points to 60 %].

JEREMY: I don't know.

INTERVIEWER: Tell me why you put 4/6 here [points to 4/6 at 30 %].

JEREMY: [shrugs shoulders].

INTERVIEWER: ... If we look at four-sixths and four-eighths, which one is less?

JEREMY: [squirming] Four-eigths.

INTERVIEWER: Now let's see, it's four-sixths over here [points to it at 30 %] and four-eighths over there [points to 4/8 at 60 %], which one is less?

JEREMY: [points to 4/8].

INTERVIEWER: Now if we put a fraction with a lighter one, does it take a smaller fraction or a larger one?

JEREMY: Smaller.

INTERVIEWER: OK, so in what order should four-sixths and four-eighths go?

JEREMY: Umm... that! [switches 4/6 and 4/8, i.e. 4/8 to 30 % and 4/6 to 60 %].

INTERVIEWER: I see, why?

JEREMY: Because fourths [pointing to 4/8 and to 0 % level], and four-sixths is more [points to 4/6 and then to 100 % level at end, i.e. to suggest the direction for greater].

The behavior of all three subjects in the low group (Table 3) is characterized by their treating the fractions as ordered pairs and putting them in a strict "lexical" order by increasing numerators/denominators. Ted and Terri ordered like 0/20, 1/5, 2/4, 2/5, 2/7, 4/6, 4/10, etc., while Mack used the denominators as first and the numerators as the second order criterion. All three subjects in general put one fraction card at each gray level and dealt separately with the left-over card (there were twelve cards for eleven gray levels). Terri just added on (postulated) a twelfth gray level right of 100 % . Mack squeezed in one fraction between two gray levels (as was permitted). Ted, when realizing that "there's going to be one left" attempted to deal with that situation in several ways and finally decided to put 4/8 together with 2/4 (on 20 %).

Mack early-on had grouped 2/4 and 4/8 together at about 60 % but indicated that he did not understand what is meant by getting the fraction cards in order. In the follow-up discussion he demonstrated better understanding for the fraction size than is documented in his lexical ordering, for example, he grouped 2/4 with 4/8 at 50 %, put 0/20 on 0 % and 4/10 on 40 % "because it's a little bit less than a half." That is, the low overall performance recorded for Mack presumably reflects his misunderstanding of the task more than misconceptions in fraction size.

Terri's performance probably more adequately reflects the inconsistencies and misconceptions in her knowledge about fraction order and equivalence. There seemed to be two conflicting "frames" that were relevant for her judgment about particular fractions as is supported by the following anecdote. From earlier observations Terri was known to consider two fractions as equivalent if (and only if) they had the same denominator. In the present task, she attached 6/15 and 12/15 at different gray levels (90 % and right of 100 %). Although her lexical ordering of the fractions raised doubts over whether she understood the connection between fractions and gray levels at all she seemed to understand something, for about her placing of 0/20 on the white (0 %) level she explains:

TERRI: Because there'd be no black ink, no black ink so it would be clear water.

Later, Terri is asked what she thinks about the two fractions 6/15 and 12/15.

TERRI: They're equal, like [laughs].

INTERVIEWER: OK, but you put them in different positions, though, why did you do that?

TERRI: Because! That's the way I thought I should do it! [moves and messes up chart].

....

INTERVIEWER: I would still like to know--you say six-fifteenths and twelve-fifteenths are equal?

TERRI: Right.

INTERVIEWER: But you put them on different parts...

TERRI: 'Cause six comes before twelve so I thought that's the way you do it....

INTERVIEWER: OK, did you think in terms of darkness when you did that?

TERRI: Yeah, sorta like...

INTERVIEWER: Which would be darker? Six-fifteenths or twelve-fifteenths?

TERRI: Twelve-fifteenths.

INTERVIEWER: OK, and which fraction would be bigger?

TERRI: Twelve-fifteenths.

INTERVIEWER: And if I ask you. six-fifteenths, twelve-fifteenths, are they equal or is one less?

TERRI: It's less.

INTERVIEWER: Which one is less?

TERRI: Six... um... fifteenths.

INTERVIEWER: And why did you say it's less?

TERRI: 'Cause it... oh! [puts head in hand and sighs] No, they're equal. Because they have the same denominator.

4. RESULTS OF ACROSS-TASK OBSERVATIONS

At this time an initial look at the data shows the following: There exist subjects that were consistently successful with all three tasks (e.g., Kristy and Bert). Secondly, there exist subjects that were consistently unsuccessful with all three tasks (e.g., Terri). Finally, there exist subjects that exhibited high performance on one task, and middle or low performance on the others (e.g., Jeannie and Richard). In Table 5 is shown the ranking of all subjects in groups of high, middle, and low performers for each task as obtained from Tables 1, 2, and 3; the original rank orders within each group were kept.

Table 5. Comparison of high, middle, and low performers by specific tasks (compiled from Tables 1, 2, and 3).

average perf.	Estimate-the-sum	DARTS	Gray levels
high	BERT	KRISTY	KRISTY
	joan	BERT	BERT
	brett	JEANNIE	erica
	andy	andy	till
	KRISTY		joan
			richard
			brett
middle	JESSIE	brett	JEANNIE
	erica	richard	andy
	JEREMY	tricia	tricia
	margret	joan	JESSIE
		TED	margret
		MACK	JEREMY
		erica	
		margret	
low	TERRI	JEREMY	TED
	MACK	JESSIE	TERRI
	TED	till	MACK
	richard	TERRI	
	tricia		
	JEANNIE		
	till		

5. CONCLUSIONS

The quantitative notion of rational number is a concept that is too general to be assessed by a single type of task. The fact that subjects were identified in the study that showed inconsistent success across the variety of task situations involving the number concept of fraction graphically supports that. Only for subjects who exhibited high performance on all three tasks that where utilized, could one assume that a general, flexible conception of number size has been developed which can be expected to apply to an even broader set of situations involving rational numbers.

From the observations made in this early evaluation stage of the present study it appears that three knowledge structures are essential for the development of a quantitative understanding of rational number: Estimation, fraction equivalence, and rational-number order. It appears that these three knowledge structures develop somewhat independently but need to be coordinated for success with rational number situations. Levels of development seem to exist including, for example, the latency of access of relevant knowledge in an applicational situation.

Further substantiation of these first conclusions will be provided upon full evaluation of the data that were acquired and be presented in a forthcoming paper.

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