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ABSTRACT

To better understand the nature of problems in math learning, a case study was conducted of the specific strengths and weaknesses of a child with a severe math-learning disability. The subject, Adam, was nearly 11 years old when interviews began and had been officially classified as learning disabled due to organic brain dysfunction. A semistructured clinical interview method was employed in the study: tasks or problems, often followed by flexible questioning, were posed to the subject. Interviews lasted from 45 to 90 minutes and were conducted over a 14-month period. Initial clinical interviews revealed that the subject had informal skills and concepts upon which to build, but that his formal math skills were quite deficient. Many weaknesses in formal skills could be traced to a poor grasp of part/whole relationships and base ten notions. Remedial efforts making extensive use of games and activities were implemented. Linking understanding with procedural knowledge, these efforts focused on basic place value and base ten concept skills and resulted in some improvement. Gains made in math ability were perhaps facilitated by affective factors. Over the course of the study, Adam appeared more willing to assert himself, was less defensive, and seemed to have more confidence in his abilities. (RH)

The Case of Adam: A Specific Evaluation of a Math Learning Disability

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Abstract

Using semi-structured clinical interviews and teaching experiments, a long-term case study was undertaken with a boy with severe math learning disabilities. Despite organic brain dysfunction, Adam had many informal math skills, including an ability to add by counting-on. He differed from his peers, however, in efficiently deploying some informal skills, and this helped to account for some school math difficulties. In terms of formal math ability (e.g., execution of the addition algorithm involving carrying), Adam was quite deficient. Many weaknesses in formal skills could be traced to a weak grasp of part-whole relationships and base ten notions. Instruction which built upon informal strengths and which explicitly linked understanding with procedural knowledge appeared to be helpful. Over the course of the case study, there seemed to be considerable improvement in affect, and this may have played an important role in Adam's cognitive growth.

The Case of Adam: A Specific Evaluation of a Math Learning Disability

Math learning problems are widespread, and P.L. 94-142 mandates individual educational plans including specific learning objectives for children with (math) learning disabilities. Yet, little is known about the specific nature of math learning difficulties or math learning disabilities. While there have been—in recent years—significant advances in our knowledge of the normal development of mathematical thinking (e.g., see Gelman & Gallistel, 1978; Ginsburg, 1982; and Resnick & Ford, 1981), insufficient attention has been devoted to a psychological analysis of children's difficulties with school mathematics (Ginsburg & Allardice, in press). Unfortunately, research which directly studies deficiencies in academic knowledge—especially in the area of mathematical thinking—is sparse (Allardice & Ginsburg, 1983; Torgesen & Dice, 1980). This case study, then, was undertaken to examine the specific strengths and weaknesses of a child with a severe math learning disability in order to better understand the nature of math learning problems.

A cognitive psychology of mathematics is a rapidly expanding field of research. This approach holds the promise of providing a foundation for a theory of math instruction which considers the interaction of the structure of the subject matter and the nature of human thinking (Resnick & Ford, 1981). It is this cognitive approach to analyzing children's mathematical thinking which provides the theoretical framework for this case study.

It appears that mathematical learning begins quite early in children of normal intelligence and that most preschoolers have a surprising range of mathematical competencies (Gelman & Gallistel, 1978; Ginsburg, 1980). Knowledge of math may be conceptualized in terms of three systems, which in older children and adults operate concurrently (Ginsburg, 1982). Intuitive knowledge is often perceptually based and does not employ counting or symbolic mathematics. It is our earliest form of mathematical thinking. For example, children as young as three years and from various social classes and cultures are capable of using perceptual cues such as area, density, and length to compare sets and judge which has "more" (Estes & Combs, 1966; Ginsburg & Russell, 1981; Posner, 1978). This strategy is often successful because perceptual cues frequently covary with numerosity (e.g., the longer row of candies frequently has the greater number).

Quite early, the child begins to develop a more reliable, informal means of coping with quantitative problems: counting-based procedures and concepts. Various models of the development of counting skills and principles have been described (Gelman & Gallistel, 1978; Klahr & Wallace, 1976; Schaeffer, Eggleston, & Scott, 1974). Preschoolers even learn counting strategies to add and subtract (e.g., Ilg & Ames, 1951; Starkey & Gelman, 1982). Initially, this involves using concrete supports such as blocks, fingers or marks (e.g., Steffe, Thompson, & Richards, 1982). Later, children use increasingly sophisticated mental counting procedures to compute sums and differences (e.g., Baroody in press-a; Carpenter and Moser, 1982; Woods, Resnick, & Groen, 1975). A particularly economical mental addition procedure is counting-on from the larger addend (e.g., "2 + 3: 3; 4 [1 more], 5 [2 more]—so the answer is 5") (Groen & Resnick, 1977). Bley and Thornton (1981) note that counting-on may be extremely difficult for learning disable children because, after locating the starting point, they may have great difficulty figuring out the next number. In school, children are introduced to symbolic, codified (formal) mathematics. This includes written work and is more efficient for dealing with problems involving large numbers. According to Ginsburg (1982), children often learn formal math in terms of their informal knowledge. A gap between school or formal material and a child's informal knowledge may result in rote learning or learning problems.

Resnick (1982) notes that school mathematics learning can proceed on two—though not necessarily interconnected—levels. Children can learn (1) computational procedures or algorithms and (2) mathematical concepts. Learning procedures without an adequate conceputal basis (rote learning) sometimes leads to learning problems. On the other hand, conceptual learning promotes problem solving (e.g., Wertheimer, 1945). For example, even first graders will use such principles as commutativity (the order in which addends are added does not affect the sum) and the addition-subtraction complement principle (if $6 + 6 = 12$, then $12 - 6 = 6$) to short-cut computation effort (Baroody, Berent, & Packman, 1982; Baroody, Ginsburg, & Waxman, 1983). Conceptual knowledge, then, gives meaning to procedures and permits flexible use of procedures.

Base ten representation provides a conceptual base for much of elementary school math (cf. Resnick, 1982, 1983). That is, it provides the underlying rationale or meaning for many procedures. Moreover, it gives children flexibility and facility in dealing with a wide range of mathematical tasks: writing numerals, comparing or ordering larger numbers, computing (especially that involving carrying or borrowing), and estimating (Payne & Rathmell, 1975). Yet, it seems that many—especially learning disabled—children have difficulty making the transition from counting-based (informal) views of math to base ten (formal) representations. The consequences are inefficiency, rote learning of formal skills, and/or math learning problems (cf. Beardslee, 1978; Hazekamp, 1978). Indeed, the cognitive and emotional handicaps that result may plague some individuals throughout their school careers and adult lives.

A deficiency in base ten representation may manifest itself in a variety of ways. Russell (1981) found that math impaired children had difficulty dealing with larger values including difficulties with their multiples (e.g., "How many tens in 100?" or "Hundreds in 1000?"). Deficiencies in base ten representation may, moreover, result in problems with our written number symbols—in writing numerals the way they are spoken (e.g., writing 203 for "23") (Ginsburg, 1982). One of the most significant deficits displayed by math impaired children is weakness in basic number fact knowledge, especially with larger problems (Kraner, 1980; Russell, 1981; Smith, 1921). Difficulty with the larger sums may be related to a failure to discover heuristics. This may be due, in part, to not seeing teens as composites of ten and units. For example, a child who appreciates that ten 1s are equal to one 10 can take a problem such as 9 + 7 = 16, decompose 16 into 10 + 6, and over a number of 9 + N problems, "see" that the sum is always 10 + (N - 1). As a result of this process, the child abstracts the "9 + N = 10 + (N - 1)" heuristic (cf. Resnick, 1983). Furthermore, base ten notions provide the underlying rationale for (written) addition and subtraction computational procedures. A failure to appreciate or connect this conceptual knowledge to procedural rules often results in difficulty with written computation—especially that involving zeros, carrying, and borrowing (cf. Hazekamp, 1978). This difficulty is usually manifested as systematic errors or "bugs" (see Brown & Burton, 1978; Buswell & Judd, 1925; Ginsburg, 1982; Resnick & Ford, 1981). Finally, the abilities to compare larger numbers and make estimates are enhanced by base ten concepts (Trafton, 1978). A lack of conceptual knowledge may produce wild guessing or a refusal to estimate (and hence the possibility of being wrong).

Various authorities (e.g., Briars & Larkin, 1981; Resnick, 1982; Greeno & Heller, 1983) hypothesize that the part-whole schema (an understanding that the whole is the sum of its parts) underlies a base ten concept and other basic principles such as addition–subtraction complement principle. For example, among the terms 6, 4, and 10, 10 is always the whole and 6 and 4 are parts. This holds whether the problems is written $6 + 4 = ?$, $6 + ? = 10$, $? + 4 = 10$, $10 - 6 = ?$, $10 - 4 = ?$, $10 - ? = 4$, $10 - ? = 6$. Resnick notes that part-whole schema is available at least in its primitive form before school and that systematic application to quantity is the focus of early school math.

From case study work, Ginsburg (1982) concludes that school math is more likely to make sense to children it if builds upon their informal knowledge and skills. Case studies—even with children experiencing great math learning difficulties (Baroody, in press-c; Ginsburg, 1982)—and formal studies with lower class children (Ginsburg & Russell, 1981) have found informal strengths upon which to build. Resnick (1982, 1983) undertook intensive, long-term teaching experiments with a number of children to demonstrate that an informal approach was useful in promoting an understanding of base ten. The training focused on representing written numerals in concrete form (using Dienes blocks, chips, bundles of sticks, or money). Instruction on computational routine using informal methods was also undertaken. This involved representing addition and subtraction problems in written and concrete form. Most of the children benefitted from the base ten instruction. Several children demonstrated deep understanding of the written code. For example, one boy demonstrated insight when comparing 9 with 90 indicating that the latter was larger because the 9 "doesn't even have a ten." Interestingly, there was little correlation between knowledge of the base system and learning of the calculational procedures. Resnick concluded that instruction need to explicitly link the concepts of the base system with the procedures for calculation. This mapping technique was undertaken in a new case study (Resnick, 1982, 1983; Resnick & Ford, 1981). Leslie, a nine-year-old, had difficulty with borrowing. As a result of instruction, which in part modeled the borrowing algorithm with Dienes blocks, Leslie learned and remembered the algorithm. Moreover, she then quickly learned a subtraction procedure using expanded notation.

The case study of Adam permitted an extensive and longitudinal investigation of a child suffering from a severe math learning disability. Since other case studies had uncovered informal strengths among children having math learning difficulties (Ginsburg, 1982), the first objective of this case study was to check for informal skills and concepts. For example, was a child with a severe disability capable of the fairly sophisticated mental addition procedure of counting-on? A second major goal was to gauge the extent of the formal skills and concepts the child had acquired and—more importantly—was capable of acquiring. For instance, could he use conceptual knowledge to short-cut computational effort or appreciate and apply base ten and part-whole notions?

Method

Procedure

The case study used the semi-structured clinical interview method. Tasks or problems were posed to Adam and often followed by a flexible method of questioning. Such a procedure is especially well suited to exploring the richness of a child's mathematical thinking (Buswell & Judd, 1925; Ginsburg, 1981). Ginsburg (1981, p. 5) notes that, while such a method has its limitations, it is the most appropriate means "to discover the cognitive processes actually used by children in a variety of contexts." To enhance motivation, most of the testing was done in the contexts of games and activities. Feedback or instruction was usually given during these games or activities. The test information and training procedures were shared with Adam's parents and teachers and incorporated in his IEP. Thus, training procedures were often reinforced at home and in school.

This report summarizes the first 28 sessions with Adam—spanning about 14 months. Each interview lasted between 45 to 90 minutes. Some were conducted in Adam's school; most were at the author's home. Most of the interviews were videotaped in total or in part and then transcribed. Notes were taken on non-video taped interviews or interview portions. The interviews began in February, 1981. There were a total of 14 sessions during the spring of 1981—the last on June 25. The interviews resumed on August 15, 1981. There were a total of 6 sessions during the fall of 1981. From January to March, 1982, there were a total of 7 sessions.

Adam

Adam was 10 years - 11 months (birthdate of 3/23/70) when the interviews began in February, 1981. He comes from an intact family of high socio-economic status. His father has a professional occupation, and his mother is a former teacher. He is the eldest of three sons. His first brother is apparently quite bright, and the second is a baby, without any apparent developmental disabilities.

When Adam was 5 years old, his parents became concerned about his slowness, lack of coordination, and sullenness. Examination revealed a normal verbal IQ (96) and a depressed performance IQ (74)—producing a full scale score of 84. Organic brain dysfunction was indicated: Adam's EEG was abnormal. The neurological disturbance was focused in the right temporal lobe. Indeed, Adam suffered from minor psychomotor seizures, which is usually related to a circumscribed temporal lobe disturbance. Adam continues to receive medication to control the seizures.

Adam was officially classified as learning disabled. He was severely disabled in reading (G.E. = 2.0, Woodcock Reading Mastery Test, Form A, 5/18/81 at grade 4.9) as well as in math. For example, he had comprehension difficulties including temporal sequences and the passive voice. His last two KeyMath grade equivalent scores were 2.4 (5/19/80, grade 3.9) and 2.7 (5/28/81, grade 4.9). The most recent achievement test (the Stanford Achievement Test administered 10/81) yielded a total math score in the 2nd percentile (concepts, 2nd percentile; computation, 1st percentile; applications, 18th percentile). The most recent IQ test (WISC-R, 3/29/80) produced results consistent with earlier results: verbal IQ = 100, performance IQ = 77, and full scale IQ = 87. He appeared to have age appropriate vocabulary and abstraction ability. Finally, his learning difficulties were not primarily the result of emotional problems.

When I first saw Adam, he seemed quite reticent, unhappy, and very passive. He seemed to have difficulty learning the math games and often appeared confused and unsure of himself. He appeared to have little enthusiasm or interest in anything. Indeed, he played our math games rather mechanically.

Informal Math Ability

Results and Discussion

Given the apparent severity of Adam's math learning disability, competence in informal math concepts and skills could not be taken for granted. Therefore, a systematic examination of informal abilities was undertaken. The preliminary sessions focused on an ability to generate count sequences, count (enumerate) objects, make numerical comparisons, and use counting algorithms to perform arithmetic operations.

Testing revealed important strengths and weaknesses in Adam's ability to generate count sequences. He was proficient in generating the standard count sequence at least to 101. This is a basic skill which is necessary for more sophisticated informal skills (e.g., enumerating objects) and performing informal arithmetic computations (e.g., adding via counting-on). Adam could count backwards from 20, but he was not consistently correct or fluent. This inefficiency disruped his informal substraction ability (discussed below). He had mastered some skip counting, but other repetitive patterns were difficult for him (cf. Bley & Thornton, 1981). Adam could efficiently count by fives at least to 100. He could count by tens to 100, but not beyond. An ability to count by tens is a prerequisite skill for efficient adding by tens (e.g., "30 and 10 is 30, 40" or "42 and 10 is 40, 50 and 2") and estimating. His inability to count by tens beyond 100 helped to account for the tremendous difficulty he had with problems.

involving numbers greater than 100. For example, for a problem such as 100 + 30, counting by tens (100; 110, 120, 130) permits a child to deal with the problem in an efficient manner. Adam, however, had to count by ones (100; 101, 102, 103...130). This is a cognitively taxing approach—tedious and subject to error. Adam could count by twos up to about 10 or 12. Thereafter, the process was not automatic. That is, he would produce a term (e.g., "12"), count by ones silently to himself ("13"), and then announce the next term ("14"), etc. However, he could count by threes to only 6 and by fours to only 8. Limited ability in such count sequences has important consequences for informal multiplication. Specifically, it limits the use of "skip counting" to multiply (e.g., for 4 x 3, counting by fours three times: 4, 8, 12) (Rathmell, 1978).

Adam was competent in enumerating objects and in determining the larger of two numerals (e.g., 2 vs. 1, 5 vs. 6, 3 vs. 4, 9 vs. 8, etc.). Especially with quantities greater than ten, neither skill was entirely automatic. To compare, for example, 16/17, 23/22, 35/36, or 108/107. Adam appeared to first ponder the relevant portion of the count sequence and then made a judgement. Both enumeration and numerical comparison are basic to learning and executing informal arithmetic (e.g., adding via counting-on from the larger requires a judgment of which addend is larger).

In terms of informal arithmetic, Adam did use a counting-on algorithm proficiently to solve addition problems. For a problem such as 5 + 7, he would start with 7 and continue the count sequence as he enumerated or kept track of the smaller addend (7; 8[is 1], 9[is 2], 10[is 3], 11[is 4], 12[is 5]). This double count was often facilitated by using his fingers to keep track of the smaller addend (the second count).

Subtraction was handled by several informal strategies with varying success. Adam occassionally used a "separating from" strategy, the most basic informal subtraction strategy. For the problem 9 - 5, for example, he drew 9 marks, crossed out 5 (1111~~11111~~), and counted the remainder to determine the difference. Note that the "separating from" strategy models a "take away" notion of subtraction (Carpenter & Moser, 1982).

On other problems (e.g., 12 - 6), he used a "counting-down" strategy (counting backwards while using a second count to keep track of the subtrahend): 12; 11[1 less], 10[2 less], 9[3 less], 8[4 less], 7[5 less], 6[6 less]—6. For larger problems (e.g., 22 - 11 or 19 -15), he attempted to use a counting-down strategy, but appeared overwhelmed by the difficulty of the double count involved. Note that the number of steps in the double count of the counting down procedure increases as the subtrahend get larger (6 steps in the case of 12-6), 11 steps in the case of 22-11 and 15 steps in the case of 19-15). In the case of the larger problems then, Adam was faced with generating a long backward count—a procedure that was less than automatic. He was faced with simultaneously keeping track of a long subtrahend count that ran in the opposite direction! The demands of generating this large and difficult double count simply overtaxed Adam's working or short-term memory.

Indeed, Adam later mentioned to his parents that subtracting by counting backwards was too hard and that he would rather count forward. He did, sporatically, use a "counting-up" strategy (e.g., 10-7: 7; $8^{(1)}$, $9^{(2)}$, $10^{(3)}$ — 3). This strategy is often discovered after the counting-down algorithm (Woods, Resnick, & Groen, 1975) and models the missing addend definition of subtraction (Carpenter & Moser, 1982). Eventually, children realize that the various subtraction strategies are interchangeable (Carpenter & Moser, 1982) and use them selectively. Note that the counting-up strategy is more economical than counting-down for basic problems in which there is a small difference between minuend and subtrahend (e.g., for 9 - 7, counting-up requires a two step double count while counting down requires one of seven steps). However, for basic problems in which there is a large difference between minuend and subtrahend, counting-down remains the more efficient strategy (e.g., for 9 - 3, counting down requires a double count of three steps while counting up requires one of six steps). By third grade, many children discover this pattern and choose the most economical strategy to solve assigned subtraction problems (Woods, Resnick, & Groen, 1975). Adam had not yet shown this selectivity in the service of economy. Attention to such details may depend on executing the strategies efficiently. That is, as execution of the counting-down and counting-up strategies becomes more automatic, more attention can be given over to examining the results and comparing one strategy with another. Adam, in fact, might well benefit from explicit instruction on the interchangeability of the two strategies and when best to employ them. In any case, Adam's dislike of subtraction, which seems to be shared by children in general, may stem in part from the difficulty of executing the more meaningful and often used counting-down algorithm.

Initially, Adam did not appear to understand multiplication. He was, therefore, introduced to multiplication by means of an informal strategy. The problem 4 x 3, for example, was solved by putting out four fingers and placing three blocks before each of these fingers. The blocks were then tallied for the answer. With practice, Adam mastered this informal procedure. Moreover, soon after it was introduced, Adam began to short-cut the multiplication procedure. For instance, for 3 x 6 he might put out three fingers, put out six blocks for the first finger only, count the blocks ("1-6"), and then counted the spaces where the other blocks would have been placed ("7-12," "13-18). For problems like 4 x 3, he very soon abbreviated the strategy by using a known addition fact (4 + 4 = 8) in combination with counting-on (8; 9, 10, 11, 12). For 5 x N, Adam very quickly realized that he could count by fives, which he was quite capable of, to generate the answer.

While he appears to have learned an informal procedure for multiplication, it is not clear—even now—that he really appreciates the equivalence of multiplication and repeated addition (e.g., 4 x 3 = 4 + 4 + 4). Thus his understanding of multiplication remains suspect and needs continued emphasis.

Conclusions

In sum, the initial clinical interviews revealed that even a child with a severe math learning disability had informal skills and concepts upon which to build (cf. Ginsburg, 1982). In terms of informal skills, Adam differed from others his age primarily in terms of the efficient deployment of a number of these skills. In the case of counting backwards, the lack of automaticity contributed to difficulties with his informal (counting-down) subtraction procedure. Indeed, while he had several informal subtraction strategies, all required considerable effort and consumed much of his attention. This may help to account for the fact that Adam has not yet reached the point where he chooses the most economical strategy for a given problem. Adam appeared to have a major weakness in more advanced informal knowledge. It did not appear that Adam had developed an informal procedure or understanding of multiplication. Taught a multiplication procedure in an informal way, Adam not only learned the procedure, but intelligently invented short cuts for it. The connections between his informal multiplication procedure and his informal semantic knowledge of multiplication and addition do not—as yet—seem well established.

Formal Math Ability

Addition

Basic Facts. To date, Adam knows very few basic addition combinations automatically. For example, some doubles (3 + 3, 4 + 4, 6 + 6, 7 + 7, 9 + 9) and sums to ten (6 + 4, 7 + 3, and 8 + 2) are still not automatic. N + 0 and N + 1 facts are produced quickly as are a few doubles (2 + 2, 5 + 5, 8 + 8, and 10 + 10). The doubles 5 + 5 and 10 + 10 may have been learned through familiarity with our money system. The double 8 + 8 was apparently learned because of a nursery rhyme song by his mother.

Several attempts have been made to help Adam see patterns in the number facts and learn heuristics (e.g., the doubles + 1, 10 + N = N + teen, etc.). While he usually picked up the pattern during a training session, lasting improvement in number facility has yet to be achieved.

The evidence of this case study is consistent with the view that difficulty in learning the number facts is not due to a deficiency in long-term memory ability (competence), but rather to performance factors (e.g., Hallahan, Lloyd, Kosiewicz, Kauffman, & Graves, 1979; Torgesen, 1980). When important to him, Adam appears to remember information well. For example, Adam was taught a procedure for using the Apple computer to add. It was pointed out that the first step was to type in a "?", which—as an upper case letter requires a "shift" key. Adam was then given the problem 1 + 1. He typed in 1, located the "+" key, and asked "Do I need to 'shift'?" Asked what he thought, Adam demonstrated transfer by responding, "Yes." Eleven more trials followed, in which Adam occassionally forgot to use to "shift" key but spontaneously corrected himself. One week later—without any further practice or reminders from myself—Adam <u>remembered</u> the procedure for having the microcomputer compute addition sums.

Several things may contribute to Adam's number fact deficits. First, he may not have a rich network of rules and principles to produce or to permit the discovery of heuristics for producing number combinations economically (Baroody, in press-b; Baroody & Ginsburg, 1982). Faced with the burden of memorizing many apparently isolated facts, Adam may not "see" sufficient merit in undertaking such a chore.

<u>Commutativity Principle</u>. Adam not only appreciated the commutativity principle but, when given the opportunity, used the principle to short-cut computation effort. Shown pairs of problems such as 9 + 4 and 4 + 9, 42 + 9 and 9 + 42, 5 + 4 and 8 + 5, and 7 + 45 and 23 + 8 very briefly and in random order, Adam—without computing—concluded that commuted pairs (such as the first two pairs) produced the same sum and problems such as the last two produced a different sum. Afterward, while checking the

correctness of his judgments, Adam noted for several commuted pairs that they produced the same answer because they were the same but were just in different order. Another task measured the use of principles to short-cut computational effort. Adam was presented with a series of addition problems (e.g., 4 + 2, 2 + 4*, 6 + 4, 4 + 6*, 4 + 3, 3 + 4*, 3 + 1, 3 + 0, 2 + 7, 7 + 2*, 13 + 6, 6 + 13*, 15 + 0, 6 + 15). After a problem was solved and its sum recorded, it was put in a "used" pile abut 10 cm. to the right of where the next problem was presented. Thus, the previous problem and solution could be viewed simply by shifting the eyes or turning the head slightly. In the case of commuted trials (starred problems), the effort of computation could be avoided by looking at the previous problem and using its answer. Such a response would suggest use of the commutativity principle. For the first two commuted trials encountered (2 + 4 and 4 + 6) Adam did not use the short-cut, but counted to determine the solution. Thereafter, however, he used the commutativity principle consistently to short-cut computation effort—even on problems involving two digit addends (e.g., 13 + 15). These data are consistent with other research (Baroody, 1982; Baroody & Gannon, 1983; Baroody, Ginsburg, & Waxman, 1983) which has found that commutativity is readily abstracted from informal adding experience and is widely appreciated by young children.

Written Computation. Initially, Adam did not know the standard algorithm for addition with carrying. For example, in interview #5 (3/9/81), Adam was given the problem 66 + 4 and produced an answer of 610. Note that he did not carry but simply wrote down the sum of the one's place addition. Given a verbal problem "66 stamps and 4 more is," Adam used a counting-on procedure to arrive at the correct answer of 70. Over the course of the case study, a number of informal approaches were used to teach the carrying algorithm. One technique involved juxtaposing his own familiar, informal (counting-on) efforts with attempts to use the unfamiliar written, carrying algorithm.

Other approaches were done in connection with teaching base ten representation, described below. Adam became quite proficient in executing the standard carrying algorithm with addends of two and three digits.

Subtraction

Basic facts. Few basic subtraction combinations were automatic. The first interviews during the spring of 1981 revealed that Adam often even responded to N - 0 problem (e.g., 8 - 0) incorrectly— with an answer of "0" (N - 0 = 0 bug). A year later, Adam had mastered the N - 0 = N rule (and N - N = 0 rule), but otherwise had few automatic subtraction facts—including differences of one (e.g., 5 - 4, 6 - 5, 7 - 6, etc.).

Addition-Subtraction Inverse and Complement Principle. Except for the N + 1/N-1 problems, the addition-subtraction inverse principle (the addition of N can be undone by the subtraction of N and vice versa) does not seem to be entirely secure. For example, in a modified version of Gelman's (1972) magic task, Adam was shown a pan with 7 blocks. The interviewer ("Mr. Magic") then covered the pan with a cardboard sheet and surreptitiously removed two blocks.

A (Adam): Five

I (Interviewer): I thought there were seven in there. What did Mr. Magic do?

A: He took two.

I: I took away two. And how would I restore what was there originally? How would I get back to 7?

A: In a magical way.

I: How can I get from 5 back to 7 again?

A: 6, 7, [fingers move two times] two.

Adam seemed to know that in a general sense, addition could be undone by subtraction. However, except for additions (reductions) of one, he often compute to determine how many had to be taken away (added) to restore an original set. These results are comparable to those obtained by Gelman (1977) with preschool children, who appeared to appreciate that addition and subtraction cancel each other and who could accurately make "repairs" of +1, but who were imprecise about larger inverse problems.

Until the inverse principle becomes clear, checking by using the inverse operation (e.g., 35 - 17 = 18 by adding the difference and the subtrahend) may only be performed perfunctorily. That is, Adam may learn the mechanics of checking, but until the inverse principle is firmly understood, he will not understand why this procedure works and he may not apply the procedure to real-life situations.

A lack of basic addition fact facility might contribute to a weak grasp of the addition-subtraction inverse and complement (since 3 + 2 = 5, then 5 - 3 = 2 and 5 - 2 = 3) principles (cf. Baroody, Berent, & Packman, 1982; Baroody, Ginsburg, & Waxman, 1983). These deficiencies in turn might account for the paucity of automatic subtraction facts. A child who can efficiently call to mind addition combinations is more likely to discover and use the inverse and complement principles. For example, a child who computes 10 - 7 and arrives at a difference of 3 and who can quickly call to mind that 7 + 3 = 10 may, over the course of computing various subtraction problems, abstract the complement principle—i.e., see the relationships among the parts (addends) and whole (sum) of addition combinations and their corresponding elements of related subtraction problems (cf. Resnick, 1983). The child who does not mentally have readily available the addition facts is much less likely to make this comparison and discovery. This is especially true for a child whose attention is absorbed in executing a less than automatic counting algorithm for subtraction. Moreover, a child who appreciates the complement principle and who has immediate command of the addition facts (e.g., 6 + 6 = 12 or 6 + 4 = 10) can very quickly construct answers for basic subtraction combinations (e.g., 12 - 6 or 10 - 6) (Baroody, in press-b; Baroody & Ginsburg, 1982).

With a weakness in one or both abilities, the child must resort to less efficient (counting) or less economical (rote) means of producing the basic subtraction facts. Thus, Adam, may have little success "learning" the basic subtraction facts until the basic addition combinations and the complement principle are mastered.

<u>Written Computation</u>. Adam did not appear to interpret "difference" problems in terms of subtraction or to appreciate that his written subtraction algorithm was applicable to such problems. Asked in interview #27 (3/3/82) what the difference between our game scores of 30 and 8 were ("By how many did you beat me?"), Adam responded, "20" (a rather good approximation by the way). Asked to figure out exactly by how many points he beat me, and encouraged to use pencil and paper, Adam wrote 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30 and announced, "30." Apparently he had attempted to count-up but did not engage the second count necessary to compute the difference: $9^{(1)}, 10^{(2)}, 11^{(3)}, \ldots 29^{(21)}, 30^{(22)}$–
–<u>22</u>. Upon questioning, he contemplated the problem and revised his strategy. As he counted-up to himself he made marks to keep track of the second count (the difference). After arriving at 30, he counted the 22 marks to obtain the difference. The interviewer then asked Adam about solving the problem by means of written subtraction.

I: If we said . . . the difference . . . is [writes $\begin{array}{r}30\\ \underline{-\ 8}\end{array}$]. Would that give the right answer?

A: No.

I: Your score is 30 and my score is 8.

A: Oh, I thought it was 0.

I: Will this tell us the difference between our scores?

A: It's 8.

Adam, then, did not see the subtraction algorithm as relevant to "difference" problems.

Adam—like most children—assimilated his written subtraction procedure to his informal notion of subtraction ("taking away") and the related informal procedures of separating-from and counting-down. "Difference" problems may not be viewed as "subtraction" problems but as a separate category of problems to be dealt with by the informal counting-up procedure. Hence, Adam's conclusion that the written subtraction algorithm was not relevant to the difference problem. Indeed, primary school children, in general, only gradually see the interchangeability of subtraction problems and informal strategies (Carpenter & Moser, 1982). Future work with Adam should focus on fostering these connections. For example, Adam might be asked to solve "difference" via his written subtraction algorithm and compare the results with his informal counting-up strategy.

In terms of the written algorithm involving borrowing, Adam has demonstrated considerable improvement in a year's time. When the interviews began, Adam had a borrowing algorithm, but was unsure of when to use it. He sometimes borrowed when it was not necessary. In the problem below, the minuend in the one's place was already larger than the subtrahend.

$$\begin{array}{r} {}^{0}\cancel{1}{}^{1}9 \\ -15 \\ \hline 04 \end{array}$$

Nevertheless, he borrowed from the ten's place, used a separating from strategy to compute 19 - 5, put down the 4, carried the 1 and subtracted 1-1 in the ten's place. With minuends in the teens, he sometimes engaged a borrowing algorithm:

$$\begin{array}{r} {}^{0}\cancel{1}{}^{1}2 \\ -9 \\ \hline 3 \end{array}$$

Indeed, he sometimes tried to borrow when he added. Moreover, his work was often replete with "bugs" or systematic errors. The solution below involved two bugs. The one's place subtraction involves subtracting the smaller N from the larger N, and the ten's place involves the 0 - N = 0 bug. Overall, his initial performance did not inspire optimism.

$$\begin{array}{r} 203 \\ -\ 17 \\ \hline 204 \end{array}$$

Nevertheless, efforts were made—as in the case of addition—to relate formal subtraction procedure to Adam's informal notions. This included using Dienes blocks to model the borrowing algorithm (re: Resnick & Ford, 1981). Over the period of a year, he learned to deploy his borrowing algorithm appropriately and effectively. Indeed, given several addition problems and then a subtraction problem, Adam shifted easily from a carrying to a borrowing algorithm. At the beginning of the case study, such a change in problems often resulted in preservation—continued use of the first procedure. "Bugs" appeared much less frequently—usually only when Adam was tired or not concentrating. Under such circumstances, it appeared that Adam engaged a relatively undemanding but incorrect "buggy" procedure rather than have no answeror expend the effort to use his borrowing algorithms. Finally, Adam began to take pride in his computational ability.

Base Ten and Numeration

Initial Diagnosis. When the case study began, Adam could not even label correctly the one's and ten's place of a two-digit numeral. Except for 10 + 10, the addition (subtraction) of ten to (from) any two-digit number including other decades (20, 30, etc.) was computed via counting-on (counting-down). Writing two digit numerals did not appear to be a problem, but writing larger terms is still not automatic. In the eighteenth session (9/2/81), Adam wrote 1013 for "113," 1017 for "117," and 2002 for "202." More recently (interview #22, 1/13/82), Adam wrote 1003 for "103."

A: That's not the right way, right...?

I: One-zero-zero-three. That's the way it sounds, doesn't it?

A: Hmhmm.

I: Which one of these is correct [writes 103] 1-0-3 or 1-0-0-3?

A: 1-0-3.

Moreover, Adam was a reluctant estimator. For example, during session #20 (12/9/81), Adam punched into a microcomputer a series of addition problems (displayed on a monitor). Before he was permitted to hit the "return" key to see the answer, he was asked to estimate the answer. Adam made up the first problem 356 + 896.

I: What do you think it will be?

A: It will have to be 12 and some other numbers. [Note: The "12" referred to the sum of the one's column.]

I: How many?

A: Four. [Note: Adam may have realized that the sum of the ten's and hundred's place would each produce two-digit sums.]

Adam made up another problem 56 + 7093, but could not make an estimate. He attempted to mentally align the numerals in order to employ the standard addition algorithm, but could not. With 57 + 25, he did not attempt to estimate the answer, but employed the standard carrying algorithm and announced: "Eighty and some number." A similar procedure produced 33 for 28 + 16 and 41 for 32 + 19. (The carried value was not considered.) When asked to estimated 65 + 38, Adam responded, "10." This evidence suggested that (1) Adam preferred to give exact answers rather than engage in the inexact process of estimating, (2) he has very little sense for large numbers, and (3) he had great difficulty with mental arithmetic involving two-digit terms.

<u>Remedial Efforts</u>. Remedial efforts focused on basic place value and base ten concepts skills. Initial efforts were directed at (1) identifying the place names and value of numerals up to three digits, (2) thinking of ten elements as <u>a</u> group (one 10), (3) viewing double digit numbers as composites of ten(s) and ones (4) connecting concrete representations of base ten and written numerals including the use of zero as a placeholder, (5) performing mental addition with two digit terms starting with the addition and subtraction of ten, and (6) appreicating the structure or patterns of base ten written numbers systems. For the most part, the instruction built on informal knowledge and made extensive use of games and activities (see, for example, Baroody, in press-c). This instruction has had some success. Adam now can readily identify the one's ten's, and hundred's place, spontaneously translate ten ones into one ten, decompose double digit numbers into ten(s) and ones, and relate this to written symbolism.

For example, in interview #17 (8/31/81), Adam was introduced to Egyptian hieroglyphics (1 = I, ∩ = 10, and ℮ = 100) (re: Bunt, Jones, & Bediant, 1976). After some work representing one digit numbers (e.g., translating 3 into lll), Adam was asked to translate 10 into hieroglyphics. He proceeded to make ten marks. Later, he did write ("10") for the problem lllll + lllll ("5 + 5"), but made 13 marks for "10 + 3." Each time he erred the interviewer pointed out that there was an easier way to represent the answer and wrote the correct symbol. Thereafter, at least for numbers and sums to 99, Adam picked up the system quickly. This training went on for three successive weeks and then discontinued until session #22 (1/13/82)—four months later. During session #22, after some preliminary review of the basic symbols, Adam correctly translated 6 and 24 into hieroglyphics. More importantly, he solved problems 7 + 6 and 12 + 9 and translated their sums (13 and 21) immediately into the correct hieroglyphics (∩lll and ∩∩l). The terms 13 and 21 were not treated merely as 13 or 21 units, but as composites of ten(s) and ones.

Another example from interview #17 (8/31/81), involves a scoring procedure for a miniature bowling game (adapted from the Wynroth curriculum, 1975). The scoring procedure, in which strikes and spares are simply scored as ten, had been taught and mastered the previous spring but not practiced in over four months time when the following transpired. With a score of 13 already (represented by 3 Dienes-like blocks in the "one's dish" and a ten-bar in ten's dish"), Adam knocked down eight more pins. He counted out eight blocks and added attached two blocks from his one's dish while saying, "8, 9, 10." Then he announced, "Now I have two tens, twenty." (Note that he equates base ten and count representations.) He put the new ten-bar in the ten's dish and related the dish with a numeral "2" and the "one's dish" with a numeral "1." Hence he readily connected the concrete representation with its written symbol. Then later with a score of 26, he knocked down 9 more pins. He obtained the appropriate number of blocks and made a new (third) ten-bar. Holding the new (third) ten-bar in his hand, he announced, "Thirty...[counted the remaining one's blocks] five." The following frame he went through the scoring procedure to add 7 to his total announcing afterward: "42."

I: You have 42 there? How many tens?

A: 4.

I: How many ones?

A: 2

Adam, moreover, could operate in the opposite direction. Given a numeral, he could indicate how many tens and ones it represented as well as make the appropriate concrete model with blocks.

While the skills described above are an important basis for base ten representation, they do not guarantee a "deep" knowledge of this concept or its effective application to mental addition, estimation, etc. (cf. Resnick, 1982). For instance, Adam does not have a strong sense of the <u>structure</u> of our (base ten) number system. Consider the following exchange during interview #26 (2/24/82).

I: What's the smallest one-digit numeral in our number system?

A: 0

I: What is the largest one-digit number?

A: 10

I: One digit.

A: 100

I: How many digits are in 100?

A: 3

After several more guesses including 11, 12, and 21, the interviewer guided Adam to the correct answer: 9. Asked then what the smallest 2-digit numeral was, he responded correctly: "10." However, he again needed help to conclude that 99 was the largest 2 digit numeral. Adam had been exposed to the same kind of questions and help, each of three previous weeks (sessions #23, 24 and 25). Apparently, seeing the organization of the base ten system is a difficult step.

Using base ten representation in the service of, for example, addition also appears to be a major step. While he could count by tens with facility and decompose numerals into tens and units, adding and subtracting by ten was still handled by counting-on. Even problems such as 10 + 6 or 10 + 7 are still usually solved by counting. During interview #15 (8/25/81), Adam was taught a version of the card game "99." Briefly, the object of play is to avoid discarding a card that puts the discard total over 99. Three cards are dealt to each player. Cards have their face value except A (which adds 1 or 11 to the discard total), 4 (adds 0, but reverses the direction of play), 9 (automatically makes the discard total 99), 10 (deducts 10 from the discard total), J (adds nothing), K or Q (adds 10). The player to the left of the dealer starts by discarding, announcing the new total, and drawing a replacement from the deck. Play continues until someone loses (puts the discard total over 99). Adam used a counting-on strategy for nearly all his calculations—including plus or minus 10 calculations (e.g., 40 + 10, 53 + 10, 99 - 10). However, over five months time, there has been significant improvement. During a game of "99" in interview #22 (1/13/82), Adam responded to 13 + 10, 32 + 10, 86 + 10 by counting-on. In each case, he held the +10 card. When asked by one of the interviewers what, for example, 10 + 10, 83 + 10, or 96 - 10 was, Adam quickly responded with the correct answer. Thus when expected to respond quickly, Adam used MN + 10 and MN - 10 rules effectively. The fact that he seemed to prefer counting-on suggests that the rules are not yet completely automatic and trusted.

Affective Factors

Not only have there been important gains in Adam's mathematical thinking over the course of the case study, there appeared to have been striking changes in his personality. Adam appeared to have much more spark and seems much more capable of enjoying himself. He seemed more willing to give others a chance to teach him and to give himself a chance to learn. In brief, he seemed more comfortable with himself, others, and mathematics. For example, in a recent interview (#28, 3/10/82), the interviewer explained to Adam the rules for an estimation game, which involved the addition of two digit addend and a time limit of two seconds. Adam's response was: "Let her rip!"

More specifically, Adam appeared more willing to assert himself, he was less defensive, and he seemed to have more confidence in his abilities. In interview #22 (1/13/82), for example, Adam explained that adding 20 + 20 (horizontally and recently introduced in school) is harder than adding 10 (vertically).

I: Why is it harder to do it this way [horizontally] than this way [vertically]?

A: It's, um, it's like harder to carry and you get mixed up. And I don't know how to carry it so good.

I writes: 38

$$\begin{array}{r} 38 \\ \underline{+17} \end{array}$$

A: Oh that's easy. [Adam proceeds to correctly use the carrying algorithm.]

Thus, Adam demonstrated a willingness to discuss a problem that was bothering him, accurately described his difficulty, and confidently employed a now familiar procedure.

Conclusions

While Adam's informal math ability were not greatly dissimiliar from other children his age, formal math ability was obviously and extensively different. Even basic applications of the part-whole schema to mathematics—the focus of early school math (Resnick, 1983)—had not been mastered (e.g., basic addition combinations, the addition-subtraction inverse and complement principles). Not surprising, then, Adam had considerable difficulty with more advanced notions of the part-whole schema such as base ten representation. Lack of this semantic basis, in turn, helps to account for the pervasive deficiencies in formal procedural skills (e.g., written addition and subtraction computation procedures, writing three^{+} digit numerals, mental addition with two^{+} digit addends, estimation). Indeed, this case study again demonstrates Ginsburg's (1977) observation that written formal procedure are initially more troublesome than children's mental, informal procedures (e.g., the response of "610" to 66 vs. "70" to the verbal problem "66 & 4 more"). There were a few exceptions to the general lack of formal math knowledge. For example, Adam clearly appreciated the principle of commutativity. This, however, was probably derived from his informal addition experience.

Formal instruction which built upon informal ability (Ginsburg, 1982) and which explicitly linked semantic and procedural knowledge (Resnick, 1982) appears to be a useful approach. It has permitted Adam to invent short-cuts (e.g., for his multiplication procedure), achieve insights (e.g., "42" is a composite of four tens and two ones)—as well as learn basic procedures (e.g., carrying and borrowing algorithm).

While there are undeniable organic factors, much of Adam's forgetting, inattention and perserverance may be symptoms rather than causes of his learning problems (cf. Allardice & Ginsburg, 1983). These characteristics are reduced when Adam is actively involved in instruction which is meaningful and interesting.

Though the progress Adam has made in the past year has been considerable, much remains to be done. Two problems, which seem to characterize much of Adam's mathematical knowledge, need to be addressed. The first is the apparent nonconnectedness of many math concepts and procedures (e.g., different concepts of subtraction with each other, with various informal procedures and with the formal written procedure). The second is the weak sense of the base ten number system. For example, while Adam can readily decompose a two digit numerals into tens and ones, he must learn the mathematical significance of the transition from one to two digit numerals, two to three digit numerals, etc. His number sense of three digit and larger values may depend on this conceptual knowledge. More recent work on mental addition and estimation suggests that Adam is making strides in connecting his formal (basic base ten) knowledge with his formal procedures. Progress in (mental and written) computation, in turn, may enrich his conceptual knowledge (cf. Resnick & Ford, 1981). In other words, growing computational and conceptual competence may feed each other.

Finally, though not the focus of this report, affective factors appeared to have played an important—perhaps the most important—role in Adam's improvement. The recognition and acceptance of his informal strengths by others may have been an important basis for Adam accepting himself. Moreover, as he developed competence in formal math, his estimate of himself and his abilities grew giving him confidence for new challenges (cf. Ginsburg, 1982). Adam's affective as well as cognitive growth were made possible by a mutual effort—the cooperation of learning specialists, school teachers, and parents.

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