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ABSTRACT

A study was undertaken to develop guidelines for the interpretation of the parameters of three multidimensional item response theory models and to determine the relationship between the parameters and traditional concepts of item difficulty and discrimination. The three models considered were multidimensional extensions of the one-, two-, and three-parameter logistic models. The three models were found to have quite distinct definitions of difficulty and discrimination, making it imperative that the user take care to ensure that the characteristics of the model used match the characteristics of the data to which it is applied. The guidelines developed as a result of this study provide information that should be helpful in ensuring that the appropriate model is used for a particular application. (Author)

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The Definition of Difficulty and Discrimination for Multidimensional Item Response Theory Models

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Multidimensional item response theory (MIRT) models have been available for at least twenty years (Bock and Aitkin, 1981; Mulaik, 1972; Rasch, 1961; Samejima, 1974; Sympson, 1978), but little use has been made of these procedures. The reasons for the lack of use are threefold: (a) practical computer programs for estimating the parameters of the models were unavailable, (b) examples of the application of the models were not present in the literature to stimulate interest, and (c) information was not available on how to interpret the results of the procedures. Some gains have been made in the first two of the deficient areas. Programs for the estimation of the parameters of at least one class of MIRT models are now available. (Bock and Aitkin, 1981; McKinley and Reckase, in press) and several applications of the models have appeared in the literature (Bock and Aitkin, 1981; McDonald, 1967; McKinley, 1983; McKinley and Reckase, 1983). However, guidelines for the interpretation of the results of MIRT models are still not available in the literature. The purpose of this paper is to present some initial guidelines for the interpretation of the item parameters and to relate the parameters to the traditional measurement concepts of item difficulty and discrimination.

Three approaches will be taken in presenting the guidelines for the interpretation of the results of the application of the MIRT models. First, the form of the item response surface (IRS) for three different MIRT models will be presented and the effect of the model parameters on the shape of the surfaces will be indicated. Second, the concepts of item difficulty and item discrimination will be defined for the MIRT models and will be related to the IRS. Finally, the interpretation of the model parameters will be discussed and guidelines will be given for the use of parameter estimates.

The Multidimensional Item Response Theory Models

Three different MIRT models will be used as examples in this paper to show the generalizability of the concepts presented. The models used are: (a) a multidimensional extension of the two-parameter logistic model (M2PL) (McKinley and Reckase, 1982); (b) a multidimensional extension of the three-parameter logistic model (M3PL) (Sympson, 1978); and (c) a multidimensional extension of the Rasch model (M1PL) (Mulaik, 1972).

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Although these three models are generalizations of a single class of IRT models, they are all distinctly different models and no one of them is a special case of any of the others.

The M2PL model is a special case of a general model presented by Rasch (1961). The equation for the model is given by

$$P(x_{ij} = 1 | \underline{a}_i, d_i, \underline{\theta}_j) = \frac{e^{d_i + \underline{a}_i' \underline{\theta}_j}}{1 + e^{d_i + \underline{a}_i' \underline{\theta}_j}} \qquad (1)$$

where $P(X_{ij} = 1 | \underline{a}_i, d_i, \underline{\theta}_j)$ is the probability of a correct response to Item i by Person j, $\underline{a}_i$ is a vector of discrimination parameters for Item i, d_i is a scalar difficulty parameter for Item i, and $\underline{\theta}_j$ is a vector of ability parameters for Person j. Both the $\underline{a}_i$- and $\underline{\theta}_j$-vectors have the same number of elements which is dependent on the dimensionality of the data being modelled. Note that there is only one difficulty parameter in this model, but a number of discrimination parameters. The model has this form because it was found to be impossible to estimate multiple difficulty parameters when using this model (McKinley and Reckase, 1982).

The M3PL is an extension of the three-parameter logistic model presented by Birnbaum (1968). This model is given by

$$P(x_{ij} = 1 | \underline{a}_i, \underline{b}_i, c_i, \underline{\theta}_j) = c_i + (1 - c_i) \prod_{k=1}^{n} \frac{e^{Da_{ik}(\theta_{jk} - b_{ik})}}{1 + e^{Da_{ik}(\theta_{jk} - b_{ik})}} \qquad (2)$$

where $\underline{b}$ is a vector of difficulty parameters, and c_i is a scalar parameter indicating the lower asymptote of the IRS. The other symbols are defined above.

The M1PL model is an extension of the unidimensional Rasch model (Rasch, 1960). This model is of the form

$$P(x_{ij} = 1 | \underline{e}_i, \underline{\theta}_j) = \frac{\sum_{k=1}^{n} e^{(\theta_{jk} + e_{ik})}}{1 + \sum_{k=1}^{n} e^{(\theta_{jk} + e_{ik})}} \qquad (3)$$

where $\underline{e}_i$ is a vector of item easiness parameters. The elements of this vector are of opposite sign to the usual difficulty parameter.

The Item Response Surface Defined by the Models

In order to gain a better understanding of the characteristics of each of these three models, two plots of the item response surface defined by each of the models will be presented for cases limited to two dimensions. The plots have been produced using item parameters that have been selected to accentuate the effects of the parameters on the shape of the surface. By contrasting the two surfaces that are being presented for each model, an intuitive grasp of the effect of the parameters can be obtained.

M2PL

The IRS's for the M2PL model are shown in Figure 1. The parameters used to generate these plots are given in Table 1. Notice that the surface in Figure 1a tends to rise closer to the (-3,-3) point than the surface in Figure 1b. This is a function of the difficulty of the two items. For an examinee population in which θ_1, and θ_2 are distributed N(0,1) with $\rho_{\theta_1\theta_2} = 0$ a larger proportion of the examinees will obtain a correct response to Item 1a than to Item 1b. This fact is reflected in the d-parameter for the item. The d-parameter for Item 1a is much larger than that for Item 1b, indicating that Item 1a is easier for the population described above. Item 1a may not be easier for every population, however. For example, for a population concentrated at (3,-3) on the θ plane, Item 1b is easier than Item 1a. A definition of item difficulty that addresses this issue will be presented later in this paper.

Table 1

Item Parameters Used to Generate the Surfaces for the M2PL Model

	Parameter		
Item	a_1	a_2	d
1a	.5	1.5	1.0
1b	1.75	.25	-1.5

A second characteristic of the IRS for the M2PL model can be noticed in Figure 1. In Figure 1a, the surface increases more quickly parallel to Dimension 2 than to Dimension 1. For that item, a_2 is much larger than a_1. These parameters control the steepness of the surface. For Item 1b, the surface increases very slowly parallel to Dimension 2. For that item, a_2 is much smaller than a_1.

M3PL

The IRS's for the M3PL model are shown in Figure 2. The parameters used to generate the plots for this model are given in Table 2. One obvious

Figure 1

Examples of Item Response Surfaces
for the M2PL Model

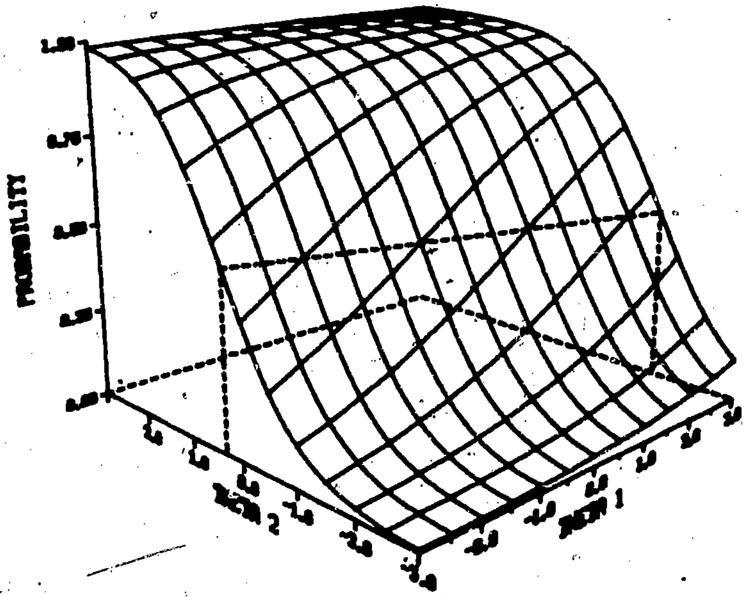

Figure 1a

Parameters

$a_1 = .5$

$a_2 = 1.5$

$d = 1.0$

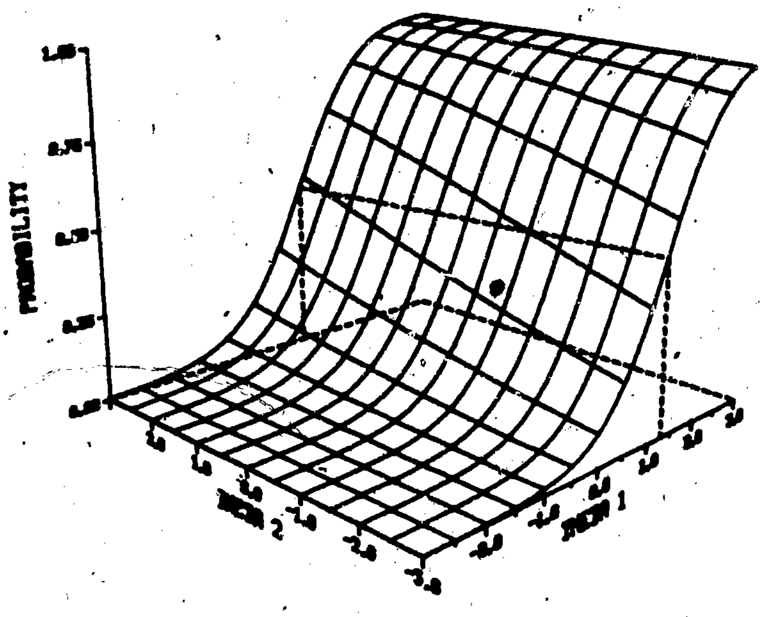

Figure 1b
Parameters

$a_1 = 1.75$

$a_2 = .25$

$d = -1.5$

Figure 2

Examples of Item Response Surfaces
for the M3PL Model

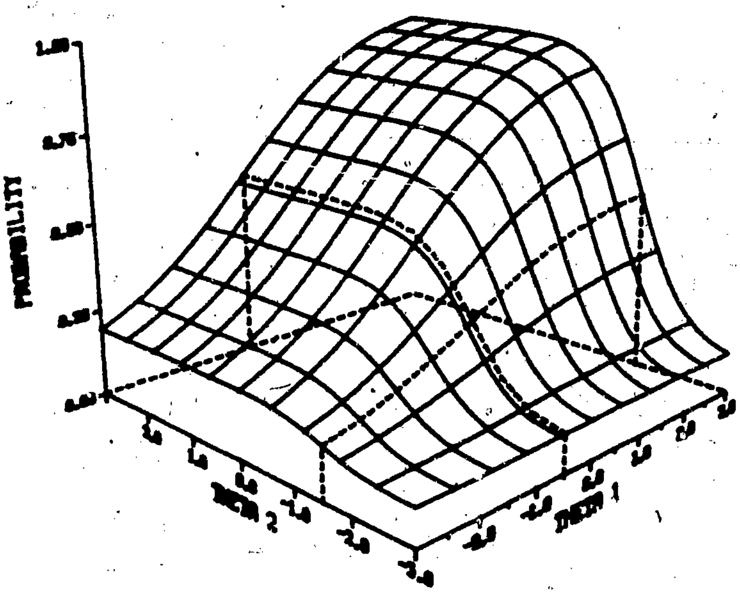

Figure 2a

Parameters

$a_1 = .5$ $b_1 = -.5$

$a_2 = 1.5$ $b_2 = -1.5$

$c = .1$

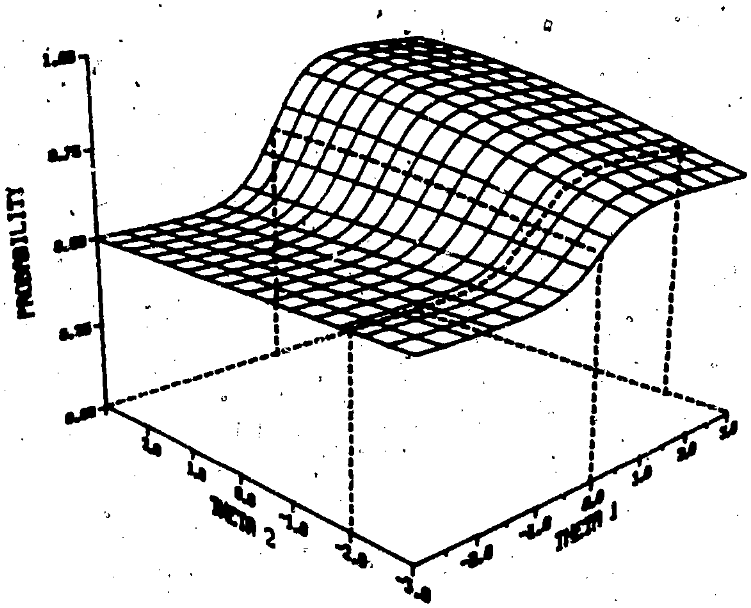

Figure 2b

Parameters

$a_1 = 1.75$ $b_1 = 0$

$a_2 = .25$ $b_2 = -2.0$

$c = .5$

difference between the two plots shown in Figure 2 is that the lower asymptote for the two surfaces are quite different. The lower asymptote is controlled by the c-parameter, which has a value of .1 for Item 2a and .5 for Item 2b. The surface cannot drop below the value of the c-parameter on the probability scale.

Table 2

Item Parameters Used to Generate the Surfaces for the M3PL Model

	Parameters				
Item	a_1	a_2	b_1	b_2	c
2a	.5	1.5	-.5	-1.5	.1
2b	1.75	.25	0	-2.0	.5

A second, quite obvious difference in these two plots is the general shape. The surface for Item 2a increases fairly quickly along both the θ_1 and θ_2 axes, while the surface for Item 2b increases mainly along the θ_1-axis. These results are due to a combination of the effects of the a- and b-vectors. The a-parameters control the rate of increase of the surface in the area around the point defined by the b-values. For Item 2a, the curve increases more quickly along the θ_2 dimension than the θ_1 dimension because a_2 is much larger than a_1. The opposite is true for Item 2b.

M1PL

The IRS's for the M1PL model are shown in Figure 3. The parameters used to generate the plots for this model are given in Table 3. These two plots almost look like mirror images of each other because of the reversal in the signs of the e-parameters for Item 3a and Item 3b. Although the curves along the θ_1- and θ_2-axes are of the same shape, they are shifted to different locations by the selection of the e-parameters. Figures 3a and 3b are merely showing a different segment of the same surface. The point on the surface at (-1,.5) on Figure 3a is the same as the point on the surface at (1,-1.5) on Figure 3b. The two surfaces differ only by a translation.

Figure 3

Examples of Item Response Surfaces
for the M1PL Model

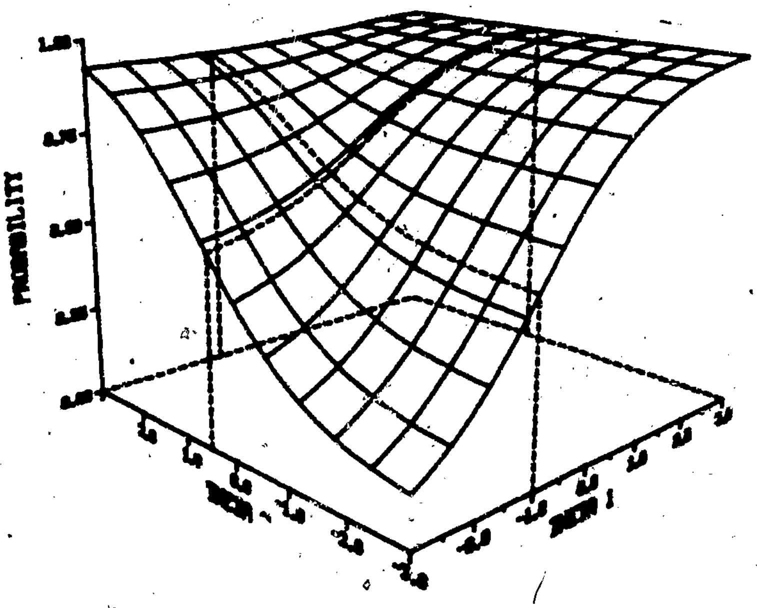

Figure 3a

Parameters

$e_1 = 1.0$

$e_2 = -.5$

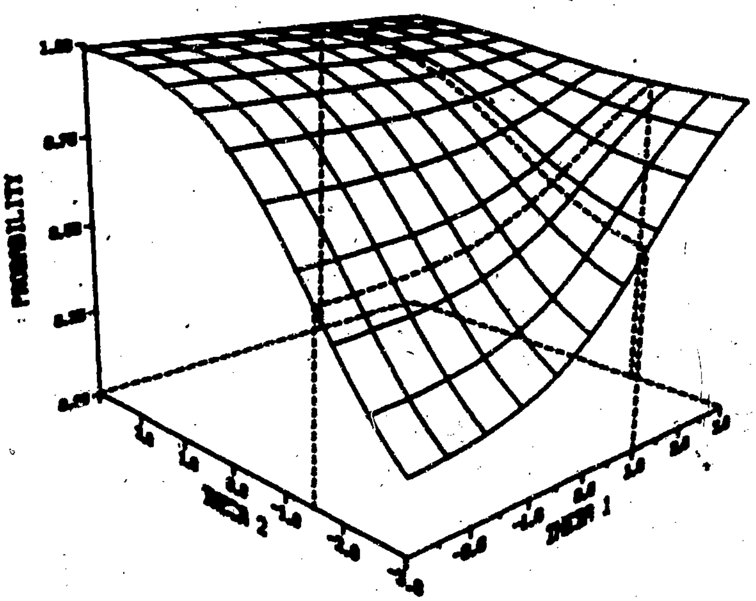

Figure 3b

Parameters

$e_1 = -1.0$

$e_2 = 1.5$

Table 3

Item Parameters Used to Generate the Surfaces for the M1PL Model

	Parameters	
Item	e_1	e_2
3a	1.0	- .5
3b	-1.0	1.5

Definition of Item Difficulty and Discrimination

From an analysis of Figures 1 thru 3, it is clear that the parameters of the models have a direct influence on the shape of the IRS's. However, the relationship between the shape of the surface and the parameter values is not always simple. In order to clarify the relationship, the common psychometric concepts of item difficulty and discrimination will be generalized to the MIRT models. The relationship between these concepts and the parameters will clarify the meaning of the parameters.

Definition of Item Difficulty

For unidimensional item response models, the difficulty of an item is defined as the point on the ability scale below the point of inflection of the item characteristic curve. This point can be determined mathematically by solving the second derivative of the item response function with respect to θ for zero. For the MIRT models, item difficulty will be defined in the same way--as the root of the second derivative of the item response function. However, in the multidimensional case the second derivative yields a matrix of the form

$$\begin{bmatrix} \frac{\delta^2 P}{\delta\theta_1^2} & \frac{\delta^2 P}{\delta\theta_1\theta_2} & \frac{\delta^2 P}{\delta\theta_1\theta_3} & \cdots \\ \frac{\delta^2 P}{\delta\theta_2\theta_1} & \frac{\delta^2 P}{\delta\theta_2^2} & \cdots & \\ \vdots & \vdots & & \end{bmatrix}$$

and a solution may not exist for which all the elements are equal to zero. This is equivalent to saying that the point(s) of inflection may be different depending on the direction relative to the axis that is taken along the surface. Therefore, the second directional derivative (Kaplan, 1952; p. 124) will be used to define the multidimensional difficulty of an item. For the two-dimensional case, the second directional derivative in direction α, with respect to the θ_1-axis is given by:

$$\nabla_\alpha^2 P = \frac{\delta^2 P}{\delta\theta_1^2}\cos^2\alpha + 2\frac{\delta^2 P}{\delta\theta_1\theta_2}\sin\alpha\cos\alpha + \frac{\delta^2 P}{\delta\theta_2^2}\sin^2\alpha \quad (4)$$

In most cases this definition yields one or more functions as a result, rather than a single value. Examples of the use of this definition will now be given using the three models presented above.

M2PL If $P(X_{ij} = 1|\underline{a}_i, d_i, \underline{\theta}_j)$ is set equal to P for the sake of convenience, the second directional derivative of P with respect to the $\underline{\theta}$-vector in the two dimensional case is equal to

$$\nabla_\alpha^2 P = a_1^2 P(2P^2 - 3P + 1)\cos^2\alpha + a_1 a_2 P(2P^2 - 3P + 1)\sin\alpha\cos\alpha + a_2^2 P(2P^2 - 3P + 1)\sin^2\alpha \quad (5)$$

for the M2PL model. To determine the difficulty of the item, the second directional derivative is set equal to zero and solved for the appropriate values of P. The solution yields P = 0, .5, and 1. Since P = 0 and 1 are degenerate cases that occur when $\theta = -\infty$ and $+\infty$ respectively, only P = .5 defines the difficulty of the item using this model. Thus, the difficulty of an item for the M2PL model is defined by the intersection of the IRS with a plane parallel to the $\underline{\theta}$-plane at P = .5. The equation for the intersection for the two-dimensional case of this model is given by

$$d_i + a_{i1}\theta_{1j} + a_{i2}\theta_{2j} = 0 \quad (6)$$

This is the equation for a straight line. The dashed line on Figures 1a and 1b shows the difficulty line for the two items shown. When a person's position in th θ-plane is behind that line, the probability of a correct response is greater than .5. If it is in front of that line, the probability is less than .5.

M3PL If $P(X_{ij} = 1|\underline{a}_i, \underline{b}_i, c_i, \underline{\theta}_j)$ is set equal to P and the first term in the product is set equal to P_1 and the second term is set equal to P_2, the second directional derivative of P with respect to P for the two-dimensional case is given by

$$\nabla_\alpha^2 P = (1 - c)a_1^2 P_1 P_2 (2P_1^2 - 3P_1 + 1)\cos^2\alpha + (1 - c)a_1 a_2 P_1 P_2 (1 - P_2)(1 - P_1)\sin\alpha\cos\alpha + (1 - c)a_2^2 P_1 P_2 (2P_2^2 - 3P_2 + 1)\sin^2\alpha \quad (7)$$

for the M3PL model. To determine the difficulty of the item, the second directional derivative is set equal to zero and solved for the appropriate values of $\underline{a}_i$, $\underline{b}_i$, and $\underline{c}_i$. The solution is much more complicated for the

M3PL model than for the M2PL model. The solution for the line of inflection is dependent on the angle of approach to the slope relative to the θ-axes. If the line of inflection is determined parallel to the θ_1-axis ($\alpha = 0°$), the solution is given by the equation $\theta_2 = b_2$. The projection of this line on the θ_1, θ_2 plane is a straight line parallel to the θ_1-axis. If the line of inflection is determined parallel to the θ_2-axis ($\alpha = 90°$), the solution is given by the equation $\theta_1 = b_1$. The projection of this line on the θ_1, θ_2 plane is a straight line parallel to the θ_2-axis. In both cases, the lines of inflection divide the θ_1, θ_2 plane into two regions, one having a relatively low probability of a correct response, and the other having relatively high probability of a correct response. The lines of inflection for Items 2a and 2b are shown as dashed lines on Figure 2.

M1PL The second directional derivative for the MIPL model is given by

$$\nabla_\alpha^2 P = e^{(\theta_1 + e_1)}\left(1 + e^{(\theta_2 + e_2)}\right)\cos^2\alpha - 2e^{(\theta_1 + e_1)} e^{(\theta_2 + e_2)} \sin\alpha\cos\alpha + e^{(\theta_2 + e_2)}\left(1 + e^{(\theta_1 + e_1)}\right)\sin^2\alpha \tag{8}$$

To determine the difficulty of an item using this model, the second directional derivative is set equal to zero and solved for the values of $\underline{e}_i$. Different functions are obtained as solutions for this model depending on the angle of approach to the surface relative to the θ_1-axis. If the direction is taken as parallel to the θ_1-axis, ($\alpha = 0°$), the solution is given by the function, $\theta_2 = -e_2$. If the direction taken is parallel to the θ_2-axis ($\alpha = 90°$), the solution is given by the funtion $\theta_1 = -e_1$. If the direction of solution is halfway between the θ_1 and θ_2 axes ($\alpha = 45°$), the solution is

$$e^{(\theta_1+e_1)} + e^{(\theta_2+e_2)}. \tag{9}$$

The solution for $\alpha = 0°$ and $\alpha = 90°$ are shown as dashed lines in Figure 3. Note that just as with the other two models, the lines of inflection divide the $\underline{\theta}$-plane into a region with low probability of response and a region with a high probability of response. In this case, the low probability region is the quadrant nearest (-3, -3) and the high probability region is the quadrant near (+3, +3).

Definition of Item Discrimination

In the unidimensional IRT models, the item discrimination is related to the slope of the ICC at the point of inflection. In the multidimensional case, the discrimination can be defined in the same way, but the direction of the slope relative to the θ axes must be specified. The slope may be quite different depending on the direction in which it is determined. The value of the slope at the line of inflection can be determined by evaluating the first directional derivative at the points on the line of inflection. The first directional derivative for the two dimensional case is given by

$$\nabla_\alpha P = \frac{\delta P}{\delta \theta_1} \cos \alpha + \frac{\delta P}{\delta \theta_2} \sin \alpha \tag{10}$$

where α is the angle with the θ_1-axis. Generally the slope is of greater interest when α equals 0° or 90° than for other angles, since these cases indicate the usefulness of the item for measuring ability on the θ_1 and θ_2 dimensions, respectively. Any other direction can also be used, however, to determine the discriminating power of the item for weighted composites of θ_1 and θ_2.

<u>M2PL</u> The directional derivative for the M2PL model is given by the equation

$$\nabla_\alpha P = a_1 P(1 - P) \cos \alpha + a_2 P(1 - P) \sin \alpha. \tag{11}$$

When the derivative is determined for $\alpha = 0°$, the slope is given by $a_1P(1-P)$. Since the line of inflection is defined by $P = .5$ for this model, the slope in a direction parallel to the θ_1-axis is equal to $a_1/4$ all along the difficulty line. This fact shows that the discrimination of the item relative to θ_1 is dependent on a_1. Likewise when $\alpha = 90°$, the slope at the line of inflection is $a_2/4$. The a_2 parameter controls the discrimination of the item relative to the θ_2-dimension.

<u>M3PL</u> The directional derivative for the M3PL model is given by the equation

$$\nabla_\alpha P = (1 - c)Da_1 P_1 P_2 (1 - P_1) \cos \alpha + (1 - c)Da_2 P_1 P_2 (1 - P_2) \sin \alpha \tag{12}$$

where all the symbols have been defined at Equation 7. The slope of the IRS at this line of inflection when $\alpha = 0°$ is given by $(1-c)Da_1P_2/4$. Thus, for this model, the slope along the line of inflection, $\theta_1 = b_1$, is dependent on θ_2, the parameter that controls the value of P_2. The slope at the line of inflection, $\theta_2 = b_2$, when $\alpha = 90°$ is given by $(1-c)Da_2P_1/4$, which is dependent on θ_1. This means that, according to the M3PL model, the discriminating power of an item for a particular dimension changes depending on the level of ability on the other dimensions. As the level of ability on the other dimensions increases, the discriminating power on the first dimension increases. For the two-dimensional case, as ability on the second dimension increases, the slope parallel to θ_1 approaches $(1-c)Da_1/4$. Similarly, the slope at the line of inflection approaches $(1-c)Da_2/4$ for θ_2 as θ_1 increases.

M1PL. The directional derivative for the M1PL model is given by the equation

$$\nabla_\alpha P = \frac{e^{(\theta_1 + e_1)}}{\left(1 + e^{(\theta_1 + e_1)} + e^{(\theta_2 + e_2)}\right)^2} \cos\alpha$$

$$+ \frac{e^{(\theta_2 + e_2)}}{\left(1 + e^{(\theta_1 + e_1)} + e^{(\theta_2 + e_2)}\right)^2} \sin\alpha \qquad (13)$$

The slope of the IRS at the line of inflection, $\theta_1 = -e_1$, when $\alpha = 0°$ is given by $1/(2+e^{(\theta_2+e_2)})^2$. When $\alpha = 90°$, the slope of the surface along the line $\theta_2 = -e_2$, is given by $1/(2+e^{(\theta_1+e_1)})^2$.

Thus, for the M1PL model, as with the M3PL model, the discriminating power of an item on a particular dimension is dependent on the level of ability on the other dimensions. However, for the M1PL model, all items are equivalent in their discriminating power for a particular combination of $\theta_1 + e_1$. No discrimination parameter is present in the model. Note that for the M1PL model, the point of greatest discrimination on the θ_1-dimension occurs when $\theta_2 = -\infty$. At that point, the slope is equal to $\frac{1}{4}$. As the value for θ_2 increases, the discriminating power for the θ_1 - dimension decreases. The same result can be obtained for the θ_2-dimension.

Discussion

The definitions of difficulty and discrimination presented in this paper can be used in much the same way that their unidimensional counterparts are--items can be compared on their difficulty and items can be selected for a test based on their discrimination. However, before those uses can be made of these characteristics of the items, a decision must be made concerning which of the MIRT models to use. The definitions of difficulty and discrimination for these models can be used to assist in making that decision.

The definition of difficulty for the M2PL model is quite different from that for the M3PL or M1PL models. In both of these latter cases, there is essentially a separate difficulty parameter for each dimension being measured. In the case of the M3PL model, this fact is a result of the non-compensatory nature of this model. No matter how high the ability is on one dimension, it cannot compensate for the lack of ability on the other dimension. The M2PL model is compensatory, and as a result, the difficulty of the item depends on the ability on all dimensions.

The M1PL model yields two difficulty functions for a different reason than the M3PL model. The dimensions of this model affect each other in an unusual way. The level of ability on one dimension fixes the range of effect of the other dimension. For example, if the sum of θ_1 and e_1 is 0 for Dimension 1, the minimum probability of a correct response for the person on the item is .5. The second dimension can only determine how much above .5 the probability will fall. The surface has the same shape at that point as at any other point, but the range of the probabilities has been reduced. Thus, the definition of difficulty at all levels of ability stays the same.

The definition of discrimination for the three models differ considerably. For the M2PL, the discrimination of the item is defined as a constant for each dimension. For the M3PL model, the discriminating power of an item on one dimension is dependent on the ability on the other dimensions. For this model, the discriminating power of an item on a dimension increases as the ability on the other dimensions increase. For the M1PL model, the discriminating power of an item on one dimension is also dependent upon the ability level on the other dimensions, except for this model, the discriminating power on a dimension declines as ability on the other dimensions increases.

These three models yield clearly different definitions of discrimination. Before any of the models is applied, the user should be sure that the characteristics of the model match the characteristics of the data being analyzed. The information presented here should help in insuring that the appropriate model is used for a particular application.

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