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ABSTRACT

This booklet provides an overview of the Elements of Mathematics Program used in Project MEGSSS (Mathematics Education for Gifted Secondary School Students). It was designed for gifted students in grades 7-12 with excellent reading and reasoning ability. It attempts to explore the upper content limits of the mathematics that such students can understand and appreciate. An innovative feature is that some of the more advanced parts are not to be teacher-taught in the classical mode; students read several books on their own, with the teacher then leading a discussion of what was read. The contents of the program are listed, with a detailed description in an appendix. The role of logic and intuition, enrichment and acceleration, teacher preparation, and evaluation are considered, followed by some questions and answers about the Project. (MNS)

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Mathematical Education for the Gifted Secondary School Student

MEGSSS in Action

a project of CEMREL, Inc.

MEGSSS in Action

by

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ELEMENTS OF MATHEMATICS PROGRAM

Introduction

We begin our discussion of the Elements of Mathematics program (EM) by presenting very briefly our general views on mathematics and its role in the modern world.

A very popular argument for the importance of mathematics in the school curriculum runs like this: "The modern world is permeated by the consequences of science and technology; hence people must know some mathematics to cope with the world in which they live."

This argument was always rather weak because it focuses attention on one aspect of the problem, excluding many other equally if not more important aspects. It is especially weak in an age when the importance and glamour of science and technology are no longer unquestionably accepted. Developers of mathematics curricula must not be content with an essentially "technocratic" answer to the question: what is the role of mathematics in human affairs? We must not think that the role of mathematics in the school is essentially the same as the role of "driver training." The latter is included—and rightly so—because it is useful and important in one sense; the former is included because it is useful and important in a very different sense. We might ask ourselves why the Greeks, too, thought mathematics essential to an educated person—yet theirs was *not* a technological society.

If we are to serve the young well in our teaching of mathematics, we must ourselves have a thorough, humanistic, encompassing view of our discipline. Surely we have an obligation to the students we teach not simply to seize on the first straw that blows our way for an explanation of why they should, and indeed *must,* spend on the order of 12 years studying mathematics. The EM view is that the justification for teaching mathematics in the schools is indeed that it is "useful," but that we must remember to interpret the word "useful" in its very broadest sense. The point is that the mathematical mode of thought is one of the most powerful and elegant ways we have for organizing our experience and our ideas. This is not to say that it is the only way or even, for many purposes, the best. However, it is a way that is so important that no one should be denied the opportunity to explore mathematics beyond rudimentary arithmetic.

What then are the qualities of mathematics that are of such importance? In the first place, we have in mathematics the best and richest example of the power (and limitations) of human *reasoning*. If we wish to illustrate what we mean by a correct argument, a subtle argument, an elegant argument, we can easily turn to mathematics for our examples. Indeed, for 2000 years mathematics has been a model of "rational" thought, and we believe that students should understand the kinds of reasons that made people think this was so. Phrases such as "mathematical precision" and "mathematical certitude" are often found in common parlance. However, one cannot appreciate what these phrases mean without having first studied some good mathematics.

In mathematics we also find new languages—languages which differ from our everyday English in very important respects. There is, for one thing, the precision of mathematical languages. It is much easier to understand the grammar of mathematical languages than to understand the grammar of English. On the other hand, the languages of mathematics also display remarkable flexibility. Mathematicians appreciate the power of apt notations and they abandon natural language readily whenever they find that it is not well adapted to expressing an idea. For example, imagine using only natural English to state the following simple proposition from high school algebra:

$$(\forall x \in R)[x^2 - 5x + 6 = 0 \Leftrightarrow (x = 3 \vee x = 2)]$$

Of course it can be done, but at a tremendous cost in clarity and manageability. Mathematicians have been remarkably successful in inventing symbols, diagrams, and graphs to express and work with the ideas that concern them. The specially created languages of mathematics are not just a jargon to keep outsiders confused; they really make intensive and extensive study of ideas possible. We would argue then that mathematics can help to develop a creativity and freedom with the use of language on the one hand, while on the other, it keeps this freedom and flexibility within the bounds imposed by a very rigorous and demanding discipline.

Further, there is the power of mathematics in its application to the physical, biological, and social sciences. But the true nature of these kinds of applications must be understood. The examples of "applications" that students are often given in traditional courses simply miss the point. At their very worst, they look like this:

The distance traversed by a particle in free fall *(in vacuo)* is given by

$$s = \frac{1}{2}gt^2$$

where $g = 32$ ft/sec and t is the time (in seconds) that the body has been falling. Find the speed of a particle that has been in free fall for 1 minute.

Such problems have, of course, nothing to do with genuine application; they merely exercise the mechanical activity of substituting into formulae. Not much better are most of the typical "story problems" in which students are usually presented a contrived situation to which they must apply the mathematics they have been taught in the preceding section of the book. True application has to do with looking at a real situation, abstracting from it those elements one wishes to study more closely, or which appear to be tractable, devising a mathematical model, making inferences within the model, periodically checking to see whether one's results are borne out by observation and experiment, tinkering with the model to try to make it correspond better with reality, and so forth. An example of such an application is presented in *Book A* of the EM program. By working with genuine applications, we expect our students to become aware that for the power gained by constructing a mathematical model there is, of course, a price to pay: much of the richness and subtlety of the original situation may no longer be present in the model.

Finally, we should like to point out that mathematics, for better or worse, has pervaded the thinking of our whole civilization. We will not be able to understand or appreciate our cultural heritage if we have not studied some of its mathematics. The subtle influence of mathematics appears in the most unexpected situations. We read, for example, in our Declaration of Independence, "We hold these truths to be self-evident . . . ." It surely can be no accident that those who wrote these words considered Euclid part of their basic intellectual equipment. We are all users of mathematics—not merely when we count our change at the grocery store, or calculate our income tax return, or interpret statistical data in our newspapers, or become engineers or carpenters or actuaries. We are users of mathematics whenever we reason, whenever we organize a body of information, whenever we search for patterns, whenever we try to cope with abstractions.

Objectives of the EM Program

The EM program is a seventh- through twelfth-grade mathematics program designed for students with excellent reading and reasoning ability who are in the upper 5% of the school population. (With a very exceptional group, the program may be started in the sixth grade.) A somewhat less select (top 10%) population may be chosen to begin the program, but a higher attrition rate should be expected. One of the experimental aspects of the EM program is to explore the upper content limits of the mathematics that such students can understand and appreciate. Another innovative feature is that some of the more advanced parts of the program are not to be teacher-taught in the classical mode. The students read several of the upper-level books on their own, with the teacher then leading a discussion of what was read, handling any problems the students may have had understanding the material. The ultimate goal of the EM program is simple enough to state: it is to familiarize the student with as much good mathematics as we can. In order to assure breadth of knowledge, the mathematical topics presented in the EM program are chosen from a wide variety of mathematical fields.

It is surely a sad state of affairs that in the traditional high school curricula, the student encounters very few if any mathematical ideas that postdate the seventeenth century. This state of affairs is not altered when some of the "modern" curricula embellish Greek and medieval ideas with a bit of set language. The set language currently in favor in high school may indeed have the merit of making the ideas under consideration somewhat clearer and easier to teach, but this does not alter the fact that the only ideas considered date from antiquity. Even worse is the pretense that mathematics becomes modern when an ancient idea is applied in a "modern" setting: to compute the surface area of a jet runway is not more modern a mathematical problem than to compute the surface area of the pyramid of Cheops. Old ideas do have a place in modern curriculum. The ancient mathematicians thought deeply and beautifully about many nontrivial problems, and in many cases we cannot improve on what they did. However, it is surely the duty of curriculum developers in the twentieth century to bend every effort to make accessible to high school students some of the mathematics of the past 200 years. It would be ludicrous if an English curriculum for the high school never contemplated confronting the student with a piece of literature written after Shakespeare. Nevertheless, mathematics curricula generally do just that; indeed, students aren't even given the best of Greek mathematics. It is as if we decided in the

English curriculum that we could trust a student with only Lamb's *Tales from Shakespeare*, rather than with the original.

It is, then, the aim of the EM program to familiarize the student with important mathematical problems, ideas, and theories that have at any time engaged the attention of serious mathematicians and serious users of mathematics. Therefore, it is necessary to bring the student as close as possible to the kinds of things that are of interest to contemporary mathematicians and contemporary users of mathematics. These basic goals immediately suggest a number of subsidiary goals:

> Students should be familiar with, and indeed comfortable with, some of the basic ideas and techniques that are typical of mathematics. Similarly, they should be familiar and comfortable with some of the basic language and notation of mathematics.
>
> Students should be able to follow a mathematical argument; they should also have had experience in trying to invent and report such arguments themselves.
>
> Students should have a familiarity with the axiomatic method in mathematics and have an appreciation of what this method does and does not provide. For example, students should be aware that a mathematical proof does not guarantee the "truth" of a given proposition, but does serve to show how this proposition is logically related to other propositions.
>
> Students should have a familiarity with abstraction and its role in the development of a mathematical theory. They should be made aware of the power of apt abstractions as well as some of the ways in which mathematicians are led to such abstractions.
>
> Students should have experience with nontrivial, relevant applications of mathematics; in particular, students should become familiar with the notion of model building and have the opportunity to develop mathematical models of "real" situations.

Content of the EM Program

The titles of the books comprising the EM program are shown below. The chapters of *Book 0* are really short books.

Book 0: Intuitive Background

Chapter 1: "Operational Systems"
Chapter 2: "The Integers"
Chapter 3: "Sets, Subsets, and Operations with Sets"
Chapter 4: "Ordered n-tuples"
Chapter 5: "Mappings"
Chapter 6: "The Rational Numbers"
Chapter 7: "Decimals and an Application of the Rational Numbers"
Chapter 8: "Introduction to Probability"
Chapter 9: "Introduction to Number Theory"
Chapter 10: "Algebra in Operational Systems"
Chapter 11: "Algebra of Real Functions"
Chapter 12: "Geometry: Incidence and Isometries"
Chapter 13: "Geometry: Similitudes, Coordinates and Trigonometry"
Chapter 14: "Topics in Probability and Statistics"
Chapter 15: "Topics in Number Theory"
Chapter 16: "Introduction to Programming"

Book 1: Introductory Logic
Book 2: Logic and Sets
Book 3: Introduction to Fields
Book 4: Relations and Sequences
Book 5: Functions
Book 6: Number Systems
Book 7: Real Analysis: Calculus of One Variable (2 volumes)
Book 8: Elements of Geometry
Book 9: Linear Algebra and Geometry (with Trigonometry)
Book 10: Groups and Rings
Book 11: Finite Probability Spaces
Book 12: Introduction to Measure Theory

Supplemental Books:

Book A: Short Course in Mathematization: A Theory of Voting Bodies
Book B: EM Problem Book

A detailed description of each of these books appears in Appendix I.

Logic and Intuition

The roles of logic and intuition in mathematics have been discussed by mathematicians and philosophers for a very long time. Recently, these roles have become the subject of controversies among mathematics educators. One thing is clear: both logic and intuition, both formalism and informality, both rigor and heuristics, both abstraction and concreteness play roles in how mathematics should be taught to students. The question is: what is the proper mix and sequencing of these various elements of mathematical activity? The answer to this question will surely depend to a large extent on individual preferences and tastes. Unfortunately, not much useful knowledge concerning how students learn mathematics has thus far been accumulated by the researchers. Hence, a curriculum project like EM can only use intuition in developing its curriculum. We believe that the decisions we have made in these regards are sound ones—both from a mathematical and pedagogical point of view. However, in view of how little is really known about these questions, we are cautious enough to say only that we view the program as an experiment—one designed to tell us more about how the various elements of mathematical thought can best be taught to bright students.

As we have said, we hope to prepare our students to think, read, write, and talk as contemporary mathematicians and contemporary users of mathematics do. The style of writing in the later EM books and the problems presented in them have been modeled on the kind of style we find in good mathematics books and survey articles. To get the student to this level of mathematical sophistication, we have begun the program with a two-fold attack. In *Book 0,* we discuss all the topics in an intuitive, loose, heuristic way. This is not to say that the discussions are slovenly or false. We try, using English and a limited number of special symbols, to say things clearly and accurately. When we cannot say something correctly, we don't say it at all. In *Books 1—3* on the other hand, we very carefully introduce a formal language in which to develop mathematics. The reasons for this will be discussed below.

Very briefly, the philosophy underlying *Book 0* may be stated as follows: we take the view that mathematics has subject matter—namely abstractions that we develop from everyday experience with things. These abstractions, such as numbers, geometric points, sets, functions, and groups, are objects about which we can think and make discoveries. Indeed, we can argue about mathematical objects more or less the same way that we reason about the things in our everyday life. Since mathematical objects are abstractions from "real" objects, we can even use our experience with "real" objects as a guide to our conclusions about their mathematical counterparts. In *Book 0* we use concrete objects (especially drawings) extensively in coming to grips with their mathematical counterparts, and we are able, by using observations, experimentation, intuition, imagination, and informal argumentation to develop some of the basic results of mathematics. So, to summarize, *Book 0* has as its aim to develop a number of topics from this point of view: from where do mathematical ideas come and what helps us to understand and utilize these ideas? While much of mathematical activity, especially the discovery of new mathematics, is of the type explored in *Book 0,* even the Greeks had standards of rigor in demonstration far surpassing those of *Book 0.*

Indeed, while it is true that the *Book 0* approach enables one to cope quite readily with relatively simple and unsophisticated mathematics, exclusive use of this mode of thinking fails one as the mathematics becomes more complex. The situation is quite nicely illustrated by the real-number line. The assumptions made about the real number line seem to be naturally drawn from experience with physical lines (*e.g.,* pencil lines, light rays, etc.) None of these seems terribly surprising, or bizarre, or difficult. While some may doubt that between any two points there lies another, many will readily accept this assumption as a natural extrapolation from experience. But from this and other very simple and natural assumptions one derives very startling results, *e.g.,* the nonrational points on the line are nondenumerable; between any two rational points there is an irrational point *and* between any two irrational points there is a rational point, etc. Surely, one is hard put to argue that one can have a clear and distinct intuitive idea of such states of affairs. It is because our intuitive assumptions together with convincing, informal arguments lead us to such complexities that mathematicians have been motivated to make their demonstrations more formal and rigorous. It is the same for the students in the EM program. We believe that we will be able to carry our students much farther in mathematics if we have prepared the way by giving them a firm foundation in logic. Most of this job is done in *Books 1* and *2.* In the work in logic, we insist that the arguments must be formal, *i.e.,* they must no longer depend on the *meaning* of the words used but rather on the *form* of the sentence. We derive sentences from other sentences, not by pleading that the derivation is plausible, but rather by quoting previously agreed-upon inference schemes. Thus *arguments* are transformed into *proofs* (in the logical sense). The authority for the correctness of a proof is no longer one's own feeling of security with an argument, or someone else's agreement that it is convincing; it is simply whether one has "played" with the sentences according to the clearly prescribed rules of the game—a matter one can almost always settle with certainty for oneself if one is just willing to muster a sufficient amount of attention to detail. Indeed, it is fair to say that *checking* a proof written in this mode is really just a matter of "calculation," rather than of thought in the usual sense. The advantages derived from a program in which students learn to read and write such proofs are obvious.

Of course, one discovers rather quickly that certain kinds of steps in formal proofs recur very often, and so one learns to abbreviate steps, or even blocks of steps. Students will take to such abbreviations more or less rapidly as a function of personal predilection and taste. As formal proofs are more and more abbreviated they take on more and more the look of informal arguments. But for the student there is a vast psychological difference between just an informal argument and an abbreviated formal proof. For an informal argument the students have no canon of correctness other than their own "feel" or the authority of the teacher. In the case of an abbreviated formal proof, on the other hand, students who have lingering doubts about its correctness may, if they wish, supply that which has been "abbreviated out."

We believe that it is wise and useful at the outset to keep informal and formal argumentation quite clearly distinct. Thus, the methods of *Book 0* are surely the essence of mathematical exploration, discovery, and creation, while the methods of *Books 1* and *2* are the essence of mathematical demonstration. We use the former methods to decide which problems to attack, how to attack them, and the kinds of solutions we expect to get. We use the latter methods to organize and establish what we have discovered. As we have said before, it is a different thing to say "I believe," or "I am convinced" than to say "I have proved," or "I have demonstrated." The former comes from informal arguments; the latter from formal proofs (even if these proofs have been severely abbreviated).

Some of the language and symbolism of the books may appear to be unusual, difficult to learn, or difficult to manage. But, our experience has shown that this is simply not the case. The notation is not unusual for someone who has not previously become accustomed to some other notation. Students find it easy to learn and easy to manage. Further, and most important, the students have no difficulty in later shifting to a more conventional notation when this becomes desirable. The cases where we adopt "in-house" notation are only those where the usual notation can be misunderstood, or blurs a distinction, or is ambiguous. Once a clear and unambiguous notation has been understood and mastered, it is not difficult for the students to change over to an ambiguous notation if they are told that this ambiguous notation is what most other people use. But it *is* extremely difficult to make clear an idea encountered for the first time when the language used to introduce it contains equivocation or deception for the unwary.

In summary, *Books 0–3* are essentially "proto-mathematical" books. Neither *Book 0* alone nor *Books 1–3* alone gives a complete picture of the spirit of mathematics. The former is very intuitive and does not adequately represent the logical concerns of mathematics; the latter is primarily concerned with the logical machinery and does not do justice to the mathematical ideas. However, we firmly believe that the two sets of books together provide a firm foundation on which a great deal of really good mathematics can be built. To do this is the task we have set for ourselves in the remaining books of the EM program.

Project MEGSSS

Project MEGSSS (Mathematics Education for Secondary School Students) is a response to two problems facing many school districts today. Due to the relatively small size of many school districts and schools, the cost of adequate programs for their correspondingly small number of gifted students is out of line with that of other programs. Project MEGSSS removes the economic burden of providing a specialized mathematics curriculum for a very small number of gifted students from the school or district, yet provides the school or district with a means of meeting the real needs of these students.

This project attends not only to the content but also to the social needs of gifted secondary school students by providing an opportunity for gifted students to come together in an intellectually stimulating setting without removing them completely from the social setting of junior high and high school. To meet the challenge of the gifted, this project focuses on outstanding abilities and exceptional needs, resulting not only in improved social relationship and study habits, but also in positive attitudes toward school.

In September, 1978, Project MEGSSS was established at CEMREL, Inc., a regional educational laboratory. Due to limitations of classroom space at CEMREL, Inc., the offices and classrooms for Project MEGSSS were moved to the campus of North Kirkwood Middle School, the Project's present location. During the first year of operation, approximately 120 students were nominated for the program; of these 47 were selected to begin the program. These students represented seven school districts and three parochial schools. At the present time, the number of nominations has increased to over 500 and selected students will represent more than 14 school districts and 30 private or parochial schools.

The MEGSSS Center has two major areas of administration: day-to-day operation and student selection. During the first year of participation, students attend classes two mornings per week for six weeks during the summer. Since they start in the summer, students have an opportunity to see if this demanding program is really something they want to do. If a student decides that this program is not the challenge he or she wants, the student can return to regular classes in September. If a student decides to meet the challenge of this program, he or she has a good beginning for the continuation of the program. Starting in September, the new students attend classes two afternoons per week, while the older students attend three classes per week. Classes meet from 3:00 p.m. until 5:00 p.m. and are taken in lieu of the student's regularly scheduled mathematics class. On the days in which they do not have classes at the MEGSSS Center, students are scheduled for a study period in their schools.

By following the schedule suggested by the table below the entire EM Program will be completed in six years.

Grade Level	1st Hour Class	2nd Hour Class
7	*Book 0,* Chapter 1	*Book 0,* Chapter 1
	Book 0, Chapters 2–5	*Book 1*
	Book 0, Chapters 6–7	*Book 0,* Chapters 8–9
8	*Book 0,* Chapters 10–11	*Book 2*
	Book 0, Chapters 11–13	*Book 3*
9	*Book 0,* Chapters 14–16	*Books 4–5*
	Books 5–6	*Books 5–6*
10	*Book 6*	*Book 8*
	Book 6	*Book 9*
	Book 7	*Book 9*
11	*Book 7*	*Book 9*
	Book 7	*Book 10*
12	*Book 7*	*Book 10*
	Book 11	*Book 12*

Finish with *Books A* and *B*

There are usually two or three hours of homework given for each two-hour class. This homework consists of reading and studying the texts and doing written assignments. These assignments are corrected, graded, and returned to the students. Students who do not turn in assignments on a regular basis are informed that continuation of such practice will result in their being dropped from the program. Tests are administered at the end of each chapter. The tests are corrected, graded and must be taken home by the students, signed by parents and returned. Quarterly progress reports are mailed to parents and to schools. These reports indicate the number of assignments turned in, the average of these assignments, test grades, and quarter averages, as well as individual comments by the teachers. Conferences between staff, students, and parents are held as needed to assess student progress.

At various times during the six-year program, various standardized tests are administered to assess students on standard high school mathematics content. Our experience indicates that MEGSSS students compare very favorably on norms used to measure achievement of national samples of high school mathematics students.

Student identification and selection begins in early December with an invitation to all school districts and private and parochial schools in the St. Louis metropolitan area to nominate sixth-grade (and a few exceptional fifth-grade) students from their schools to participate in Project MEGSSS. Officials are advised to nominate students who are at or above the 98th percentile in mathematics and the 95th percentile in reading on the school's regularly administered standardized test and who display such traits as persistence, curiosity, perception, superior reasoning ability, originality, motivation, independence, social maturity, emotional stability, and enthusiasm for mathematics.

After the nominations are received, parents are invited to an orientation meeting at which time the program is explained in detail and questions are answered. Those nominees wishing to be considered for admission take a three-hour test battery that includes the mathematics subtest of the Scholastic Aptitude Test, the Watson-Glaser Critical Thinking Appraisal, a test on logical reasoning, and a test involving complicated arithmetic word problems. Nominees are also asked to complete a questionnaire that probes their interest in and suitability for Project MEGSSS.

Upon evaluation of the test results and completed questionnaires, students under consideration are requested to come in with their parents for a personal interview and are informed of their acceptance status either during the interview or shortly thereafter. Schools are also notified at this time so that schedule adjustments can be made.

Enrichment and Acceleration in the EM Program

The EM program is an important departure from the usual method of providing for the mathematics education of the gifted student. The usual approach is through a so-called "accelerated" curriculum, but this is actually a misnomer. Typically what is done in these programs is to identify a group of capable seventh graders and simply shift down the traditional curriculum by one year for these students: ninth-grade algebra given in the eighth grade, and so on, so that seniors are taught freshman college calculus, often in a watered-down version. The results are often disappointing.* Even when supplemented with enrichment materials, such programs hardly capitalize on the more able students' ability to absorb good and interesting mathematics. Indeed, if we look closely we see that acceleration actually occurs at only one time. It occurs when the usual eighth-grade mathematics course is skipped and replaced by ninth-grade algebra. Thereafter the student is once again placed in the lock-step of the traditional curriculum, but it surely stands to reason that a student who can accelerate at one point in his or her mathematical studies may be able to do so again and again.

The EM program is a continuous acceleration program which provides advanced and continually advancing material throughout the secondary years. There is also a horizontal enrichment which provides materials suitable for the gifted but perhaps not for the average student, even if the material were to be postponed for the latter.

There is little available in the way of curriculum materials for the mathematically gifted. Several mathematics education projects, such as SMSG** and UICSM*** have developed much excellent instructional material which in practice ~~is often~~ used with the best students. Nevertheless, these materials are not generally written expressly for the gifted population. Little is known about how far the upper 5% of students can go using materials designed expressly for them. We believe we can contribute to mathematics education by finding out.

We conclude with a couple of brief comments on student interaction. Students of all ability levels are capable of learning from each other and this is particularly so with the gifted. While students should be brought to realize that an important part of mathematical activity occurs when a person struggles with a problem individually, it also should be recognized that mathematics is a social enterprise — a matter of discussion, argumentation, thesis, and antithesis. For these reasons and others, all modes of interaction should be encouraged *between the students themselves.* Another point is that students need to learn to read critically. Much of what students read in their various textbooks may be read and absorbed very quickly and without much thinking on the part of the student. The richness and rigor of the EM material, however, require careful, meticulous reading, usually with paper and pencil readily available. EM students learn to think about and question what they read to be sure they understand how the various ideas presented follow one from the other.

*Ruth Lefkowitz. *The Advanced Placement Program in a Large New York City High School.* Teachers College, Columbia University, 1966.
**School Mathematics Study Group.
***University of Illinois Committee on School Mathematics.

Teacher Preparation for the EM Program

The use of the EM curriculum in a MEGSSS center requires teachers adequately prepared in mathematics, experienced in dealing with young students highly motivated to mathematics, and able to assume a teaching role suitable for the materials. No matter how extensive the mathematical background of the prospective EM teacher, some briefing in the procedures and objectives of the program will still be necessary for that teacher.

A teacher of the EM curriculum will need a more substantial background in mathematics than that of most secondary school teachers. A teacher with a good undergraduate background in mathematics may be able to teach the first year of the program provided ample time in advance is given to become familiar with the content. More time will be required to prepare to teach the content of the succeeding year.

It is strongly urged that prospective EM teachers have opportunities for observing students working with the materials. The nature of the program in *Books 1–3* is usually quite unfamiliar to a teacher accustomed to the usual classroom environment.

The basic ingredients in any EM teacher preparation program include:

- study of EM materials
- discussion of philosophy and methodology of the EM program
- first-hand experience observing students working with EM materials

In conjunction with its MEGSSS center, the State University of New York at Buffalo has mounted a full program of internship for prospective EM teachers. Part of the internship, at the election of the intern, may be associated with graduate education coursework. Area classroom teachers observe classes, participate in some small group instruction, and assist individual students for a full year before they are qualified to teach classes at that level in the program. They also have full opportunity to discuss teaching strategies and mathematical problems related to course content. It is important to mention that one year of internship is required for each year of the EM program that is taught. For example, someone who teaches the third year of the EM program will have had three years of internship working with first-, second-, and third-year students.

Evaluation of the EM Program

The formative evaluation of the EM program is essentially complete. The EM program has had the benefit of a tryout system with developmental classes that has brought authors, students, and teachers together. Much of the time, the authors themselves were available to do the teaching. The inventory of EM books has been maintained at a level low enough to permit a policy of continual revision. Input for these revisions has come from ongoing work with local developmental classes as well as from the pilot test classes in various locations throughout the country. Comparative evaluation studies have been carried out in various pilot test sites. Two such studies investigated EM students' progress in earlier versions of *Books 0* and *1* and resulted in evaluation reports.* These studies indicated that EM students performed as well as other students at similar ability levels on tests covering the mathematical content usually taught at that grade level, and did *much* better on tests reflecting the EM content.

Formal evaluation studies with EM students who have completed the program have not yet been conducted, though data are now being collected from a fairly large number of students who will have completed the third year of EM in 1981. In addition to this study of student achievement, parent and student reactions to the program (from both present and former students) are periodically solicited and empirical studies are in progress to determine better methods of selection of students for the program. From time to time, reports from these studies will be written and will be made available.

*EM Pilot Study Report. *First Year Comparisons Pilot Trial of Book 0 of the Elements of Mathematics in Baltimore County, 1972–73:* Summary Report and Recommendations

Some Questions and Answers about Project MEGSSS

Q: What is the purpose of Project MEGSSS and what are its special advantages?

A: The purpose of Project MEGSSS (Mathematics Education for Gifted Secondary School Students) is to provide an instructional program for gifted students utilizing the Elements of Mathematics (EM) program, a curriculum specifically designed for students with superior reasoning ability in mathematics. To this end, the project operates a center to which qualified students from the St. Louis metropolitan area come for their mathematics education. Incorporated as a not-for-profit corporation in 1980, Project MEGSSS is staffed by highly qualified instructors with many years of experience in developing curriculum and teaching gifted students at this level. Project MEGSSS accepts students solely on the basis of ability as evidenced by a battery of tests administered to students nominated by their respective schools and does not descriminate with respect to race, color, sex, or national origins..

Project MEGSSS offers special advantages to each of its constituencies:

for students: to study high-quality mathematics and to accelerate their study of mathematics

to interact with other gifted students and yet remain within the geographic, social, and chronological environment of their peers

to meet an intellectual challenge beyond that which they are accustomed to find in their school program

to have the opportunity to earn up to thirty-two semester hours of college credit

for parents: to enrich the academic program of their gifted children

to provide a substantial portion of their children's college education at far less expense

for schools: to provide a strong program for an important group of students whom they are not able to serve as well by other means

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for society: to respond to the serious shortages in scientific personnel projected for the decade ahead by a national manpower commission appointed by the President of the United States

Q: Why can't this program be taught as part of the regular curriculum in a school?

A: In most schools and districts, there usually are not enough gifted students to sustain this type of program over a period of six years. It is also extremely difficult to find qualified teachers who have experience teaching youngsters ages 12–18 and at the same time have the appropriate mathematical background to handle such a sophisticated curriculum as EM. It appears to be more cost-effective to have the small number of gifted students from each school come to a centralized location than it is to try to provide this level of mathematics education in each individual school that desires to offer this program to its qualified students.

Q: How often do MEGSSS students attend classes?

A: Normally, students begin their participation in Project MEGSSS in September, and during their first year they attend classes two afternoons per week from 3:00–5:15 p.m. In the St. Louis program, however, students begin the program during the summer prior to their first full year of participation. During the summer, students attend classes two mornings per week (9:00 a.m.–12:00 noon) for six weeks. The main reasons for this early start are twofold: students are under less academic pressure during the summer and therefore can begin the program in a more relaxed atmosphere; furthermore, the early start gives students (and the project staff) an opportunity to determine if this rigorous program is suited to their needs and abilities and is something they really wish to pursue. Because this early start has been so successful, it has become a regular feature of the St. Louis-based MEGSSS program.

In the second and third year of participation in the St. Louis program, students attend classes three afternoons per week. At the end of the first and second years, students are given an option of attending or not attending a four-week, three mornings per week, summer program. If they choose to attend during the summer, then the third day of classes during December, January, and February is cancelled. It is anticipated that classes for years four through six of the program will be conducted early in the morning with the frequency and length of the class meeting to be coordinated with the schedules of the various high schools the students will be attending.

Q: How is student achievement measured? How well do MEGSSS students achieve on standardized tests?

A: Students are tested on each unit of material studied. The tests used are designed to measure student understanding of the material most recently covered. High standards are established and maintained throughout the program. Students unable to meet these standards are counseled and tutored. If these approaches fail to improve achievement, parental conferences are held to assess the desirability of the student's continued participation in the program.

In addition to the tests on specific EM content, students are also tested on standard high school topics. Various standardized tests are administered from time to time during the six years of MEGSSS participation. The results of these tests compare MEGSSS students with a national sample of high school mathematics students.

In particular, the *Cooperative Algebra I* test (form B) published by the Educational Testing Service (ETS) is administered during the first semester of the second year. The *average* score of the most recent MEGSSS class to take this test was the 95th percentile.

During the first semester of the third year, an alternate form of the math subtests of the *Scholastic Apptitude Test* (SAT—normally administered to college-bound juniors and seniors), are readministered to MEGSSS students who originally took this test 2½ years earlier as part of the admission process. The most recent administration of the SAT produced a class average of 647 (out of a possible 800). This represented an average increase of 205 points during the two years of MEGSSS participation. To put this result in proper context, it should be noted that in 1980, the *national* average for high school seniors was 468.

Q: May parents enroll their children in Project MEGSSS independently of their school?

A: It is our desire to work closely with the schools in our area. The school must agree to set up the schedule so that students can attend MEGSSS classes; the school must also agree to grant high school credit for these classes. Therefore, if a parent requests that a child be tested for Project MEGSSS, we will advise the parent to contact the school and obtain permission for the child to attend MEGSSS classes in lieu of his/her regularly scheduled math, class provided the student satisfies the criteria for selection.

Q: What is the school's responsibility with regard to having some of its students in Project MEGSSS?

A: To nominate students for Project MEGSSS

to schedule first-, second- and third-year MEGSSS students so that they can leave school early enough to get to the MEGSSS classrooms at North Kirkwood Middle School by 3:00 p.m.

to schedule mathematics for fourth-, fifth-, and sixth-grade MEGSSS students at the appropriate times dependent on when these classes meet. A decision regarding the schedule for the fourth year of the program will be made in cooperation with the students and their parents during their third year of participation.

to schedule first-, second-, and third-year MEGSSS students for a study period on days they do *not* have MEGSSS classes; appropriate facilities for study should also be provided.

to accept the MEGSSS math class in lieu of the regular math class for MEGSSS students. This includes reporting the achievement grade the student receives from Project MEGSSS as the mathematics grade on his/her school report card. It also includes granting high school credits for MEGSSS courses (a total of seven Carnegie Units of high school credit should be granted for completion of the full EM curriculum), awarding honors credit if the school has such a policy for special accelerated courses, and including MEGSSS grades in computing grade point averages, rank in class and academic award winners.

Q: Since much of the EM material is college level, may students receive college credit for their work?

A: Webster College (a St. Louis area college) has agreed to grant up to 32 semester hours of credit in mathematics to students completing the EM program. Two credits may be earned during the first year.*

Following is a list of EM courses for which Webster College will grant college credit:

Course Title	EM Books Involved	Semester Hours of College Credit	Year (within Six-Year Program) When Normally Studied
1. "Introduction to Formed Logic"	*Book 1*	2 *	1st
2. "First Order Theories"	*Book 2*	3	1st–2nd
3. "Structure of Elementary Algebra"	*Book 3*	1	2nd
4. "Relations and Functions"	*Book 4–5*	2	3rd
5. "Number Systems"	*Book 6*	2	3rd–4th
6. "Calculus I"	*Book 7, Vol. I*	4	4th–5th
7. "Calculus II"	*Book 7, Vol. II*	4	5th–6th
8. "Modern Geometry"	*Book 8*	1	4th
9. "Linear Algebra"	*Book 9* (Ch. 1–2)	2	4th–5th
10. "Linear Geometry"	*Book 9* (Ch. 3–5)	2	4th–5th
11. "Abstract Algebra"	*Book 10*	3	5th–6th
12. "Probability"	*Book 11*	3	6th
13. "Lebesque Integration"	*Book 12*	3	6th
	Total Credits	32	

Because these courses are offered on the MEGSSS campus and taught by MEGSSS staff, the cost per credit hour is about one-half that which is charged by Webster College. At the completion of each course, Project MEGSSS grants full or partial scholarships to approximately 10% of the students in each class based solely on achievement. Additional scholarships based on need are also available. Students who attend classes at the Gifted Math Project in Buffalo, New York receive college credit for EM courses from the State University of New York at Buffalo.

*For students who begin their first year during the summer, there is a possibility of earning five college credits by the end of the first year.

Q: Are Webster College credits transferable?

A: This depends on the operable policy regarding transfer students at the specific college or university in question. Most American colleges and universities accept up to a certain number of credits earned elsewhere. Some institutions accept all credits that correspond to courses offered in their own catalogues. Often, the number of credits that are transferable depends on a student's major and/or minor areas of concentration. One university indicated willingness to accept all college credits earned in Project MEGSSS with the expectation that students who choose to major in mathematics take at least two math courses on their campus.

Our first graduating class will not be entering college until September, 1984. Therefore, it will be several years before we can say with certainty which colleges and universities accept which credits earned in Project MEGSSS courses. We have made some preliminary inquiries in this direction and are encouraged by the responses. Most college officials that we've contacted have indicated a strong desire to have students of this caliber and with this mathematical background matriculate at their institutions, and appeared to be willing to do everything possible to encourage their enrollment. One prominent mathematician at a leading university indicated that he would like to see his school approach the recruitment of MEGSSS students in the same way they go after athletes.

It is our intention to keep careful data on how various colleges and universities treat MEGSSS graduates, with respect both to credits accepted and to the more important issue of proper placement in advanced mathematics courses. We will use this information in the future to counsel our students in helping them make their future educational decisions. We will also be in direct contact with leading American colleges and universities in order to make them aware of the existence of Project MEGSSS and to encourage them to recruit our graduates. We will communicate with individual schools to which our students apply in order to inform such schools of the depth of the mathematical education of the specific student as well as to recommend particular mathematical placement of that student, if he or she decides to matriculate there.

Q: Why do students drop out of Project MEGSSS? When they do drop out, how well do they fit back into their school's math program?

A: Attrition from this program occurs for several reasons: unsatisfactory performance after counseling and tutoring efforts, transportation difficulties; changed student (or parent) goals and commitments, outmigration, and misidentification despite a reasonably careful selection process. In all cases, MEGSSS staff will assist schools in determining the appropriate point of reentry into their mathematics curriculum for students leaving Project MEGSSS. Such students should have little or no difficulty fitting into standard (advanced placement directed) school acceleration programs for talented students. Our general plan of counseling is as follows:

Leaving Project MEGSSS	Entering Accelerated School Program
during grade 7 (1st year)	grade 7
end of grade 7	grade 8 (algebra I)
during grade 8 (2nd year)	grade 8 (algebra I)
end of grade 8	grade 9 (geometry)
during grade 9 (3rd year)	grade 10 (algebra II and trigonometry)
end of grade 9	grade 11 (pre-calculus).
at any later point	grade 12 calculus

Q: What kind of follow-up does Project MEGSSS do on students who have left the program?

A: Data are obtained on an annual questionnaire from students and parents for the purpose of improving the operation of the project in all of its aspects. Follow-up data are also collected from students who have dropped out of Project MEGSSS as well as their parents. These data are used to better understand why students drop out and to help us refine our initial selection process; in addition, we are also interested in knowing how former MEGSSS students are performing in their present mathematics courses and what influence (if any) their participation in Project MEGSSS is having on their present work.

The responses to these questionnaires also indicate that there appears to be an improvement in students' intellectual habits as a result of participation in Project MEGSSS; for example, the development of self-discipline in regard to study, better management and organization of time, a transfer of the logical thinking and problem-solving techniques acquired in MEGSSS classes to other academic areas, and an overall more positive approach to school. While these are ancillary and difficult to measure, we believe that they are very important benefits that students acquire by participation in Project MEGSSS.

Q: What are the parents' responsibilities with regard to having their child participate in Project MEGSSS?

A: It is the parents' responsibility to provide transportation for the child to Project MEGSSS and to see to it that he/she is on time for class. The transportation is usually handled by means of car pool arrangement. The parents also have the responsibility of providing encouragement for their child as well as a quiet place in which to study and do written assignments. Parents are not expected to help the child with his/her math homework. To help with the homework, one would have to have a strong mathematics background, some experience in the pedagogy of mathematics, and time to read carefully the texts. Rather than trying to help explicitly with the content of the program, parental efforts will be more productive if they are focused in aiding the child in developing good habits of scholarship. Many of them need regular encouragement and guidance regarding proper study habits.

Q: Is there any cost to the parents or schools for children who participate in Project MEGSSS?

A: There is no cost to the school in having some of its students enrolled in Project MEGSSS. At the present time, the only cost to the parents is a one-time testing fee and an annual book/materials fee. The students keep the textbooks and compile a mathematics library.

APPENDIX I

Detailed Description of *Books 0–12, A, B of the Elements of Mathematics (EM)*

EM BOOK LIST OF TITLES

Book 0: Intuitive Background

Chapter 1: "Operational Systems"
Chapter 2: "The Integers"
Chapter 3: "Sets, Subsets, and Operations with Sets"
Chapter 4: "Ordered n-tuples"
Chapter 5: "Mappings"
Chapter 6: "The Rational Numbers"
Chapter 7: "Decimals and an Application of the Rational Numbers"
Chapter 8: "Introduction to Probability"
Chapter 9: "Introduction to Number Theory"
Chapter 10: "Algebra in Operational Systems"
Chapter 11: "Algebra of Real Functions"
Chapter 12: "Geometry: Incidence and Isometries"
Chapter 13: "Geometry: Similitudes; Coordinates and Trigonometry"
Chapter 14: "Topics in Probability and Statistics"
Chapter 15: "Topics in Number Theory"
Chapter 16: "Introduction to Programming"

Book 1: Introductory Logic
Book 2: Logic and Sets
Book 3: Introduction to Fields
Book 4: Relations and Sequences
Book 5: Functions
Book 6: Number Systems
Book 7: Real Analysis: Calculus of One Variable (2 volumes)
Book 8: Elements of Geometry
Book 9: Linear Algebra and Geometry (with Trigonometry)
Book 10: Groups and Rings
Book 11: Finite Probability Spaces
Book 12: Introduction to Measure Theory

Supplemental Books:

Book A: Short Course in Mathematization: A Theory of Voting Bodies
Book B: EM Problem Book

Description of *Book 0*

The EM texts are designed to communicate mathematics of highest quality to very able and motivated students (top 5%). The mathematics content of the full program is essentially all of that included in high school and at least three years of undergraduate university. In response to the greater ability of the gifted student to master abstractions, much of the content is formal, focusing on careful language, underlying concepts and proof. In order to provide a bridge to this exposition, *Book 0* (whose 16 chapters are themselves each of book length and are separately published as such) was written for use during the first three years of the program (grades 7–9) parallel with the early books of the more formal series. In this way, the EM series integrates school- and college-level mathematics rather than simply accelerating the mathematics instruction of the standard curriculum. By teaching these two modes of the EM curriculum in parallel, students are doing college-level work even as early as grade six or seven.

Book 0 plays a specific role in the EM series. An examination of the detailed description of *Books 1–12* in this appendix will make it clear that, for a student to succeed at such a high level of abstraction and independence, work must be based on prior mathematical experiences that are just as rich but are also informal, intuitive, and experimental. Since the abstract structures encountered in the EM series cannot be studied in a vacuum, *Book 0* is designed to provide students with a variety of intuitive experiences and the concrete examples from which the rest of the EM books draw motivation, illustrations, and interpretations. The mathematics in *Book 0* is sound and relevant; it differs from the rest of the EM books only in style and degree of rigor. The language is correct, the concepts significant, the experiences meaningful, but no attempt is made at giving formal proofs in the context of an axiomatic or logical system. One freely uses local deductive argumentation and implicit logic, but all concepts are presented in an empirical setting of real situations and practical problems. Well-designed stories, games, and experiments offer new ways of presenting problems and situations that are attractive, challenging, and highly motivating. Diagrams, tables, and graphs, which play an important role in mathematical communication at all levels, are used abundantly as appropriate tools to sharpen concepts and to extract mathematical content from real situations.

Chapters 1–4 provide initial experiences from which notions of structure naturally arise. Chapter 1, for instance, suggests situations that involve a great number of operational systems other than the familiar ones of ordinary arithmetic. Students learn to accept operational systems as mathematical objects and begin to talk about and analyze their properties. Chapter 2 quickly develops the two-fold operational system of the integers. Using some combinatorial problems for motivation, Chapter 3 gives a concise and informal description of sets and set operations. Chapter 4 extends the notion of structure to include sets of ordered pairs.

Chapters 5, 6, and 7 comprise a development of the rational numbers via mappings, with special emphasis on the decimals as a subring of the rationals. This development of the rational number system is done with stretcher and shrinker mappings and their composites. Composition of these mappings simulates the mutiplicative group of the rationals, while the additive group of the rationals corresponds to addition and subtraction of these mappings. In Chapter 7, the decimal numbers are developed, with emphasis placed on the arithmetic of decimals. Also in this chapter, an application of the rational numbers is described involving the income tax structure of a hypothetical country.

In Chapter 8, students encounter non-trivial probablistic problems involving single and multi-stage experiments, as well as experiences with random digits and the Monte Carlo simulation method. In Chapter 14, the concept of expectation is introduced. Various interesting applications such as random walks, operations research problems, and reliability of systems are studied, and there is also an exploration of situations involving Markov chains.

Chapters 9 and 15 present the usual topics of elementary number theory, but in an unusual story setting.

Chapter 10, "Algebra in Operational Systems," begins a process of reviewing and organizing the experiences of the earlier chapters, using mappings as an organizing concept. This process motivates and constitutes the introduction to intuitive algebra (Chapters 10 and 11) and intuitive geometry (Chapters 12 and 13), with these four chapters interwoven to build on acquired skills and understandings.

The first chapter on algebra (Chapter 10) classifies the many operational systems already encountered according to their group, ring, and field properties. Relations and their properties are then introduced, and mappings are described as special types of relations. A student learns to solve linear and quadratic equations by means of mappings. The notion of an ordered field grows out of the study of relational systems, and some inequalities are solved in the ordered field of the rational numbers.

Chapter 11 starts with an intuitive introduction of the ordered field of the real numbers in preparation for an exploration of real functions and their Cartesian graphs. Solutions of equations and systems of equations and inequalities are first handled by graphical methods: then algebraic techniques are suggested by the graphical methods. Many of the usual skills of computation with real polynomial functions are learned in the process. Besides the polynomial functions, this chapter treats other real functions, such as power and root functions, exponential functions, absolute value function, greatest integer function, and rational functions with emphasis on their graphs and with some consideration of their special properties.

Chapters 12 and 13 develop Euclidean geometry through a series of experiments, conjectures and arguments based on a dynamic approach in terms of mappings rather than on the traditional static approach of synthetic methods. Chapter 12 begins with intuitive descriptions of incidence, the real numbers, and distance, with simple, informal discussions of interior, exterior, and boundary points of figures, as well as open sets, closed sets, and convex sets. Line reflections of a plane are then introduced with mirrors and paper folding. Experiments lead to conjectures about composites of reflections, namely, translations, rotations, and glide reflections. The resulting group of isometries is used to described congruence and symmetry as well as area and volume of figures. Polygons are classified in terms of their symmetry groups. Later, in Chapter 13, the ideas and skills usually associated with similar figures arise from experiments with magnifications of the plane and space. This then leads to the group of similitudes, including properties of right triangles, and the trigonometric ratios. The plane is then coordinatized and the same study of mappings is reviewed in a coordinate setting, leading to the idea of a vector space and providing techniques for such procedures as finding equations of lines. Each type of mapping is first developed in the plane and then is quite naturally extended to space as a regular feature in the course.

Chapter 16 is an introduction to programming with an emphasis on flow charts and algorithms. The chapter is organized as a problem book and by solving the problems the student will learn, step by step, some of the various aspects of programming. The problems are drawn mainly from number theory and probability. The chapter is currently designed to be used with programmable calculators. Additional material to be used with microcomputers may be added in the future.

Book 0 is designed to provide a balanced mixture, for the first three years, between formal and informal experience, between group discussions and independent study, and between local argumentation and formal proof; therefore *Book 0* is studied in parallel with *Books 1–3*, with one hour of each two-hour class session being devoted to each course.

The following are the summaries of contents for the chapters of *Book 0.*

Chapter 1: "Operational Systems"

Codes Using Modular Arithmetic: Multiplication and division module twenty-nine. **Modular Arithmetic:** Addition, Subtraction, Multiplication. **An Application of Modular Arithmetic:** Time Zones. **Examples of Operational Systems:** Modular systems, Maximum, Minimum, Greatest common divisor, Least common multiple, Midpointing, Triple-pointing, Reflection through a point. **Some Properties of Operational Systems:** Commutativity, Associativity, Neutral elements, Invertibility.

Chapter 2: "The Integers"

The operational system $(Z, +)$, Additive inverses, Ordering the integers, Positive and negative integers, Inequalities, the Operational system $(Z, -)$, the Operational system $(Z, \cdot)$.

Chapter 3: "Sets, Subsets and Operations with Sets"

Sets and Subsets: Equality, Membership, the Empty set, Venn diagrams, Subsets, Power sets, Number of elements of a power set, Number of k-element subsets of a set with m-elements (Pascal's formula). **Operations with Sets:** Intersection, Union, Difference.

Chapter 4: "Ordered n-tuples"

Ordered n-tuples compared to n-elements sets, Cartesian products, Open sentences in two variables, Componentwise arithmetic, Operations on subsets of operational systems.

Chapter 5: "Mappings"

Mappings from A to B, Mappings from A *onto* B, One-to-one mappings, Permutations on a set, the Image of an element under a mapping ["f(x)" notation], Composition of mappings, Arithmetic mappings, Circular slide rules. **Magnification Mappings on Lengths:** "Stretchers," "Shrinkers," Composites of stretchers and shrinkers, Addition of lengths, Comparison of lengths. **Applications of Magnification Mapping:** Photography, Map-making, Scale drawing, Solution of problems on weight, time, and money. **Percent Mappings.**

Chapter 6: "The Rational Numbers"

The rational number line, Positive and negative rational numbers, Ordering rational numbers, Density of the rational numbers, Multiplication of rational numbers, Division of rational numbers, Addition of rational numbers, Distributivity of multiplication over addition, Using mappings to solve equations, Subtraction of rational numbers, Absolute value.

Chapter 7: "Decimals and an Application of the Rational Numbers"

Decimals: Addition, Subtraction, Multiplication, Approximation, Positional notation. **Operations Using Positional Notation:** Addition, Subtraction, Multiplication, Division, Approximation, Decimals and percent. **Application:** Income tax in a hypothetical country.

Chapter 8: "An Introduction to Probability"

One-Stage Random Experiments: Outcomes, Outcome sets, Relative frequencies, Events, Probability of an event. **Multistage Experiments:** Tree diagrams, the Product rule, Drawings with and without replacement. **Counting:** Product rule, Counting with restrictions, Factorials, Total number of subsets of an n-element set, Number of k-element subsets of an n-element set, Arrangements. **Random Digits and Simulation:** Properties of random digits, Drawing samples with random digits, Simulation, Applications.

Chapter 9: "An Introduction to Number Theory"

Multiples and Divisors: Multiples, Divisors (or factors), Primes, Composites, Prime factorization, Relatively prime pairs, Division theorem, Unique factorization theorem, Euler phi-function (the totient mapping), Least common multiple, Greatest common divisor, Factors and primes in other operational systems. **Numeration Systems:** Digits, Radial expansions, Number base theorem.

Chapter 10: "Algebra in Operational Systems"

Groups: Review of one-fold operational systems, Examples of noncommutative groups, Cancellation property and uniqueness of inverses, Solving equations in groups, Notational conventions for inverses, the Opposite operation of a group. **Rings:** Review of two-fold operational systems and distributivity, Definition of a ring, Notation for rings, Ring properties, Iterated sums and products. **Fields:** Zero divisors, Definition of a field, Solving equations in fields, Fractional notation, Field formulas. **Relations:** Definition and graphs, Set operations with relations, Converse of a relation, Properties of relations, Equivalence relations. **Mappings and Equations over Fields:** Mappings, Mappings as relations, Converse of a function, Composition of functions, Invertible mappings, the Mappings [+a] and [·a], Additive and multiplicative inverse mappings, the Power mappings. **Relational Systems:** Definition and examples, Linearly ordered sets. **Ordered Groups, Rings and Fields.**

Chapter 11: "Algebra of Real Functions".

The Real Numbers: Decimal sequences of nested intervals, Finding the rational number with a given periodic nest, Historical note about Π. **Real Functions:** Kinematics, Coordinatizing the plane, Functions on the reals, Functions whose graphs are lines, the Ring of real functions, Some functions whose graphs are not lines, Integer exponents, Rational exponents, Converses of real functions, Root functions, Composites of real functions, the Absolute value function, Rational function, Greatest integer function, Growth and decay functions. **Polynomial Equations and Inequalities:** Graphical solutions, Equivalent equations, Some physical problems, Equations and inequalities, Equations with two variables, Systems of equation, Systems of inequalities, Graphs of quadratic functions, Solutions of quadratic equations and inequalities, Factoring.

Chapter 12: "Geometry: Incidence and Isometries"

Geometric Objects and Incidence: Points, Lines, Planes, Space, Subsets of space, Incidence agreements. **Measurement:** Length, Unit distance, Betweenness, Angles, Area, Volume. **Isometries of Planes and Space:** Reflections, Composition of reflections, Parallel lines and planes, Translations, Rotations, Congruence, Operational systems of isometries of a plane, Fixed points, Glide reflections, Isometries of space, Symmetries of plane figures, Symmetries of space.

Chapter 13: "Geometry: Similitudes, Coordinates, and Trigonometry"

Similitudes: Magnifications, Ratio and proportion, Similarity and similitudes of a plane, Similarity in space, Right triangles, Pythagorean theorem, 30°-60°-90° triangles, 45°-45°-90° triangles, Trigonometric ratios. **Coordinates:** Plane coordinates, Euclidean distance, Taxi distance, Two-dimensional real vector space, Space coordinates, Isometries in terms of coordinates, Equations of lines in a plane and in space, Linear mappings, Matrices, Orthogonal mappings, the Complex number field, Trigonometric functions.

Chapter 14: "Topics in Probability and Statistics"

Review Problems: Random walks. **Expectation:** Definition of expectation, Practical significance of expectation, Applications of expectation. **Operations Research:** Inventory problems, Group testing. **Problems and Applications.**

Chapter 15: "Topics in Number Theory"

Linear Diophantine Equations: Existence theorem, Set of solutions theorem, Euclidean algorithm. **Arithmetic of Congruences:** Casting out nines, Congruence modulo m, Residue classes. **Pythagorean Triples. Fermat's Little Theorem and Euler's Generalization.**

Chapter 16: "Introduction to Programming"

Programming: Flow charts, Calculations in memories, Asking and answering questions on the calculator, DSZ, Loops and subroutines, Indirect and direct addressing. **Variety of Problems:** Summing various sequences, Compound interest, Reversing digits of a number, Problems involving recursion, Converting minutes to hours and minutes, Problems involving the Fibonacci sequence, Determining records in sequence, Making a systematic search for numbers satisfying certain conditions, etc. **Probability:** Constructing a random number generator, Performing random walks, Simulating various random experiments, Estimating expectations. **Number Theory:** the Nichomachas algorithm, the Euclidean algorithm, Generating prime numbers, Pythogorean triples, the Period of the Fibonacci sequence modulo m.

Book 1

Introductory Logic

The material in *Book 1* has been written with the following specific aims in mind:

to develop an accurate and efficient symbolic language suitable for expressing mathematical statements and names

to develop essential notions of a proof theory;

The EM materials contain a far more complete treatment of logic than can be found in current school mathematics materials. The idea of a more explicit treatment of formal logic did not, of course, originate with EM. Earlier pioneering projects dealt with formal logic to a degree that seemed unusual at the time. From today's viewpoint it appears that—perhaps with the exception of UICSM*—the logic was approached quite gingerly. It is hard to say whether these projects were responding to, or setting, a trend. In any event, even a cursory check of school mathematics texts confirms that most of them have at least one chapter on logic and set theory. Indeed, this practice extends up through college-level texts and into early graduate level. It is reasonable to conclude that the writers of such higher-level texts regard this attention to logic and set theory as worthwhile, and assume that their readers do not have the necessary competence in these areas, or at least need a review.

Textbook writers have a special attitude toward language, for they know that they will not be there to mediate their expositions when they reach the reader. They welcome any device that makes communication easier. No modern writer would think of giving up the precision and space economy of mathematical symbolism, and it is clear that many wish to use the precision and accuracy of logical symbolism as well. Most texts are written with the expectation that a teacher will play a role in the use of the exposition, with the greater that role, the lower the academic level of the materials. In view of the instructional procedures and objectives of the EM program, it is not surprising that we want to experiment with the use of a strong, formal language for logic and set theory.

In view of the recurrent, repetitive treatment of logic from at least the seventh grade up to graduate school, it is strange that a course in logic does not have some place in the mathematics curriculum. At the college level it is still more commonly found in philosophy departments than in mathematics departments. Certainly, most present treatments of logic must be considered unsatisfactory, both as presentations of logic itself and as tools for mathematics. As one mathematician states, the student "must read the usual canonical chapter on logic and sets." At the school mathematics level it must be read again and again, for although the student matures, the successive canonical chapters do not show a like development. They deal with the same dreary truth tables, inadequate semantic discussion of proof, with syllogisms and an occasional bit of predicate language. The set theory treats the same dreary Venn diagrams.

There is nothing inherently dreary about truth tables or Venn diagrams the first time through. They become dreary with repetition and when the student discovers that, generally, the subject may be left in the first chapter, since the rest of the book makes no essential use of it or is even inconsistent with it. In the EM program we want to teach logic to students with the expection that they will learn it and use it immediately and continually.

A second role of logic is to present some sort of proof theory. For years we have taught students about proof in a way that is analogous to the so-called "direct method" for teaching languages, *i.e.*, by exposing them to lots of proofs and requiring them to construct replicas. In the geometry class there is often some analysis of what a proof is, but the analysis and description of proof is invariably inconsistent with the actual proofs that appear in the text. If students are to go beyond replicating models, they must abstract for themselves some concept of proof from the examples seen, from inadequate descriptions, and always subject to the authority of the teacher. This is hard on the student, since teachers do not all agree on what a satisfactory proof form is. Of course, many students do eventually attain a workable concept of mathematical proof, but we believe the length of time spent to reach this goal is far too long, and the number of students succeeding far too small.

To be consistent with our aim to develop the students' resources to check the correctness of their own work, we must provide them with some criteria for judging their proofs other than approval by a teacher. The plan is to break the large sequence of inferences down into very small, immediate inference rules, leaning very heavily on the students' command of language for acceptance of the reasonableness of the basic rules. Since at this stage we are necessarily concerned with language and form, the mathematical content of *Book 1* is low compared, with that of later books. Also, we do not want to complicate the problem of learning what a proof is with the much harder problem of learning how to find proofs. Thus, most of the proofs that the students see and write are quite long, but, as the students gain experience, immediate inferences are grouped in larger chunks, are abbreviated, or are left tacit. It is the aim that, eventually, the overt mainfestations of the logical machinery will whither away from the students' written proofs, leaving the stage to the mathematical content. The students learn several colloquial styles of proof, and eventually they write proofs that look like those of any well-trained mathematics student able and inclined to use logical symbolism.

To illustrate this process, we present the following three proofs. The first, by a seventh-grade student, from *Book 1;* the second, by an eighth-grade student, from *Book 3;* the third, by a ninth-grade student, from *Book 5* of the EM program. Each of these proofs is given in the student's handwriting. These are fairly typical examples of the form of written proofs at each stage, although each student develops an individual style and some prefer to abbreviate proofs at an earlier stage than this.

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*University of Illinois Committee on School Mathematics.

Prove: $P \Rightarrow T, T \Rightarrow (Q \Rightarrow \sim S), \sim S \Rightarrow R \vdash (P \wedge Q) \Rightarrow R$

Demonstration Outline	[illegible] Outline
1) $P \Rightarrow T$	Hyp
2) $T \Rightarrow [Q \Rightarrow \sim S]$	Hyp
3) $\sim S \Rightarrow R$	Hyp
4) $\sim [[P \wedge Q] \Rightarrow R]$	Dis. Hyp
5) $\sim [[P \wedge Q] \Rightarrow R] \Leftrightarrow [[P \wedge Q] \wedge \sim R]$	Taut
6) $[P \wedge Q] \wedge \sim R$	GSP, For 5, For 4
7) $\sim R$	LCS, For 6
8) $P \wedge Q$	RCS, For 6
9) P	RCS, For 8
10) Q	LCS, For 8
11) T	MP, For 1, For 9
12) $Q \Rightarrow \sim S$	MP, For 2, For 11
13) $\sim [\sim S]$	MT, For 3, For 7
14) $\sim S$	MP, For 12, For 10
15) $\sim [\sim S] \wedge \sim S$	CI, For 13, For 14
16) $[P \wedge Q] \Rightarrow R$	Contra. Dis, For 4, (4)-(15)

Figure 1

Prove: $(\forall x,y)[^-(x+y) = {}^-x + {}^-y]$

Proof Outline

1. $^-(x+y) + (x+y) = 0$	1. AIP
2. $(^-x + {}^-y) + (x+y) = (^-x+x) + (^-y+y)$	2. APA and CPA
3. $= 0 + 0$	3. AIP
4. $= 0$	4. PANE
5. $^-(x+y) + (x+y) = (^-x + {}^-y) + (x+y)$	5. SAE, Ior 4, Ior 1
6. $^-(x+y) = {}^-x + {}^-y$	6. Theorem 7(ii)
7. $(\forall x,y)[(^-(x+y) = {}^-x + {}^-y]$	7. PGU (twice) Ior 6

Figure 2

Let φ be a homomorphism from a group $(G,*)$ to a group $(G',*')$. Prove that for all $u \in G$

$$\varphi(u^{-1}) = [\varphi(u)]^{-1}$$

Assume $u \in G$. Since φ is a homomorphism we know that $\varphi(u*u^{-1}) = \varphi(u)*'\varphi(u^{-1})$. We also know that $\varphi(e)=e'$ where e and e' are the neutral elements of G and G' respectively. We get $\varphi(u*u^{-1})=e'$, since $\varphi(u*u^{-1}) = \varphi(e)=e'$. Since G' is a group we get $\varphi(u*u^{-1}) = \varphi(u)*'(\varphi(u))^{-1} = e'$. SAE gives $\varphi(u)*'\varphi(u^{-1}) = \varphi(u))*'(\varphi(u))^{-1}$. Cancellation yields $\varphi(u^{-1}) = [\varphi(u)]^{-1}$. The rest is routine.

Figure 3

Book 1

Introductory Logic

Summary of Contents

The Formal Language: Introduction, Negation, Conjunction and Disjunction, Sentences and Well-Formed Formulas, Truth Tables, Implication, Tautologies, Equivalence of Formulas, Some Useful Equivalences, the Substitution Principle, Instantiation of Formulas by Formulas, the Tautology Principle.

Introduction to Demonstrations: Modus Ponens, a First Look at Demonstrations, Conjunctive Inference and Conjunctive Simplication, Contrapositive Inference and Modus Tollens, Syllogistic Inference and Inference by Cases, Subroutines Involving the Biconditional, the General Substitution Principle, Are All the Connectives Necessary?, The Deduction Theorem, Using the Deduction Theorem, Object Language and Metalanguage, the Principle of Indirect Inference.

The Propositional Calculus: A Review, the Propositional Calculus and Open Sentences, the Limitations of the Propositional Calculus.

Book 2

Logic and Sets

In *Book 2* we continue the study of logic that we began in *Book 1.* In this book, we develop the first order predicate calculus, a system that contains all of the logical connectives of the propositional calculus but which contains, in addition, quantifiers and rules for their use.

To make clear why this new logical system is needed, we draw upon the reader's background and experience with sets as they are introduced and discussed in *Book 0.* Fourteen carefully selected set properties are discussed and *Book 0* style proofs are given. We then pose the problem of developing a formal logic in which these fourteen proofs can be subjected to a detailed analysis. Here we draw upon the students' experiences with *Book 1* to help them understand the problem we have posed. This problem is solved in the following way. We begin with a review of the steps taken to develop the propositional calculus. With this as a guide we set out to develop a system in which we can formally analyze the fourteen proofs of set properties. These fourteen results were chosen because their proofs presuppose the rules of inference of the predicate calculus. Consequently, as we examine what is needed to formalize each argument, we discover the predicate calculus.

Having completed a formal analysis of each of the original fourteen results, we observe certain differences between the original intuitive arguments and their counterpart formal demonstrations. In particular we note that demonstrations meet the highest standards of rigor, but they contain details that are not essential for human communication. Indeed, those details frequently obscure the basic argument. We then outline procedures for converting a formal demonstration into a paragraph proof. Here we take a major step in our program to prepare students to design sound mathematical proofs. To provide practice with paragraph proofs, we discuss the additional elementary properties of sets that are needed to define "function" and "relation." We develop those properties of functions and relation that will be needed in the next book of the EM series. We then conclude our study of sets with a discussion of the need to know whether or not a given term denotes a set. In this context we have a natural way to introduce all the remaining axioms of a Zermelo-style set theory except an axiom of infinity and an axiom of regularity. These axioms are introduced in *Book 6* where they are needed.

What we have done for the study of sets can, of course, be done for the study of many other subjects. As a second example, we develop a system of the study of commutative groups. These two formal systems have much in common. Indeed, what is common to the two systems we would want in any formal system. This common component is the predicate calculus.

At the completion of *Book 2,* it is intended that the students:

- can read and write set language easily and accurately and can understand it without first translating into natural English
- can prove theorems of the type that occur in the book, can use abbreviated proof patterns for dealing with proofs that consist of a sequence of equivalences or equalities, and have learned how to discuss proofs with one another and with teachers
- have deepened their understanding of axiomatic organization
- have an understanding and reasonable facility with the usual Boolean operations involving intersection, union, empty set, set difference, and the subset relation

Book 2

Logic and Sets

Summary of Contents

Logic and Sets: Introduction, Set Theory, Primitive Notions, Elementary Properties of Sets.

A Logic for Set Theory: the Formal Language, Rules of Inference, Demonstrations, the Comprehension Principle and Properties of Equality, Rules of Inference for the Existential Quantifier, Proofs from Demonstrations, the Deduction Theorem Revisited.

Set Theory: Relations and Functions, Zermelo Set Theory.

Logic: Another Formal System, First Order Logic.

Book 3

Introduction to Fields

(Note: Do not begin *Book 3* until Chapter 11 of *Book 0* has been completed.)

Book 3 is designed to:

- formalize and extend some of the students' experiences from *Book 0* by presenting in detail the development of an axiomatic system (within set theory) as a means of organization
- deal rigorously with some basic concepts of algebra
- continue the development of skills, begun in *Book 0,* normally learned in high school algebra

Two major objectives of the EM series are to develop in each student a deep understanding and appreciation of the real number system and to provide for each student an opportunity to master those algebraic skills that are essential in applications and in the study of many areas of higher mathematics. To provide insight into the basic algebraic properties of the real numbers, we look at those properties from the perspective of field theory. We begin by defining an abstract field within set theory and then applying the general results to specific fields such as the finite modular fields, the field of rational numbers, and the field of real numbers.

Techniques for solving systems of simultaneous linear equations in two variables are presented building on the student's *Book 0* experiences and their facility with the proof theory now available. Needless to say, the algebraic skills developed are not conceived of as just algorithms dealing with rational numbers, as is the case in a traditional algebra course, but rather the students are able to apply these skills in any system which satisfies the definition of a field.

Integral exponents, which were introduced in *Book 0*, are developed formally as iterated multiplication. Iterated addition expressed by "nx" for n (an integer) and x (a field element) is also systematically developed. Both of these notions are defined recursively, and the Principle of Mathematical Induction is introduced to prove theorems concerning iterated addition and multiplication.

We define the notion of ordered fields and note that the finite modular fields are *not* ordered fields. There is full treatment of absolute value, including proof of the triangle inequality. Square roots and quadratic equation in fields, and, in particular, ordered fields, are fully treated.

The kinds of idealizations and assumptions that must be made to apply algebra are dealt with in detail, and problem sets include many exercises dealing with time-rate-distance, mixture, compound interest and annuities, as well as some other less traditional problems.

On completion of *Book 3*, it is expected that the students:

> know what a field and an ordered field are and be able to carry out for any field the processes normally associated with elementary algebra, including a start on developing competency in handling inequalities comparable to that for equations
>
> will be able to apply algebra to simple problems
>
> are virtually independent of a teacher for checking the results of routine algebraic computations
>
> can prove theorems comparable to the ones they have been studying involving the four quantificational inference schemes in reasonably abbreviated form
>
> understand the two different uses of variables in mathematics, exemplified by free and bound occurrences

Book 3

Introduction to Fields

Summary of Contents

Fields: Introduction, Some Elementary Field Properties.

Equations in a Field: Equation Solving in a Field, Further Equation Solving.

Multiples and Powers in a Field: Multiples in a Field, Standard Notation for Multiples, Powers in a Field.

Elementary Polynomial Functions over a Field: Introduction, Constant Functions and First Degree Polynomial Functions over a Field, Second Degree Polynomial Functions over a Field, Properties of Quadratic Functions, Quadratic Equations over a Field, Properties of Quadratic Functions, Quadratic Equations and Quadratic Forms, Procedures for Finding the Zeros of Quadratic Functions, Problems and Puzzles.

Ordered Fields: Total Orders; Ordered Fields; Maxima, Minima, and Absolute Value; the Geometry of the Absolute Value Function; Square Roots in an Ordered Field.

Applications: Geometry of Elementary Polynomial Functions over the Field of Real Numbers; the Intersection of Elementary Polynomial Functions; Problems Concerning Distance, Work, Mixture, and Interest.

Book 4

Relations and Sequences

The notion of relation was introduced in *Book 2* and total ordering relations were studied in *Book 3.* In this book, the previous work is reviewed and then extended to include compositions of relations, the converse of a relation, the image of a set under a relation, etc. Particular attention is devoted to order relations, including quasi-orders, partial orders, rank orders, total orders, dense orders, and well orderings. Particular care is taken in presenting equivalence relations, equivalence classes, and partitions. These, of course, are important matters in any event, and are vital preparation for *Book 6.*

There is an extended section of sequences, including the usual arithmetic and geometric sequences and series. Finite sequences are used to develop various counting models, including much of the material referred to in college algebra books under the tag "Permutations and Combinations".

Mathematical induction is indispensable in dealing with sequences, and is further discussed in this book. Its use in proofs is limited to cases where the induction hypothesis is not overtly quantified. Only weak induction is used at this stage. Sigma notation ("Σ") is defined recursively and the basic properties proved by induction.

At the completion of this book, the student is expected to have made a substantial start on the important ideas of relations, sequence, order, equivalence relations, equivalence class, and to be able to read and use the various ways such matters are expressed. The notion of relation is a unifying concept, as is function, and we expect students to view some of their earlier work in a more comprehensive way from this viewpoint. Finally, it is expected that students will have a sound preliminary grasp of recursive definition and of mathematical induction as a proof form.

Book 4

Relations and Sequences

Summary of Contents

Review of Relations: Definitions, Theorems, Examples, Graphs, Relations Restricted in Domain, Image of a Set under a Relation; the Converse of a Relation, Composition of Relations.

Properties of Relations: Reflexivity, Transitivity, Symmetry, Asymmetry, Antisymmetry, Connectivity.

Order Relations: Quasi-Order, Partial Order, Total Order, Ordered Sets, Smallest and Largest Elements, Well Ordering, Dense Ordering, Upper Bound, Lower Bound, Greatest Lower Bound.

Partitions and Equivalence Relations.

Manifold Unions and Intersections.

Sequences: Definitions, Examples, Arithmetic Sequences and Series, Geometric Sequences and Series, Arithmetic Families, Subsequences.

Mathematical Induction: Induction Principle, Theorems, Starting an Induction from Any Integer, "Σ" Notation.

Finite Sequences and Counting: Definitions, Theorems, Occupancy Models, the Binomial Theorem, Finite Terms and Products.

Book 5

Functions

This book builds naturally on the material of *Book 4*. The definition of a function, presented in *Book 2*, is reviewed together with concepts such as domain, range, etc. The common ways of looking at functions are dealt with, as well as the various standard notations and ways of talking about functions. Every effort is made to use the notation and concept of function that are appropriate to the job at hand.

Operational systems and more general systems with structure are treated in a preliminary way, with notions of group and semigroup used as examples. The concept of structure-preserving map (*e.g.*, homomorphisms, order preserving) is introduced.

Matrices are defined as finite double sequences, and the basic matrix operations treated. They are applied to solutions of systems of equations, but no general theory is attempted at this stage; rather, matrices are treated as useful in dealing with certain types of problems.

An extensive development of the ring of polynomial functions over an arbitrary field is included. Applications of these functions to various problems are explored. Step functions are introduced, including the greatest integer function and applications to approximating areas under curves and to odd and even functions are discussed.

On completion of this book, students are expected to be comfortable with the various languages and notations used to treat functions, and the various different ways of using functions that are needed in the books immediately to follow. It is expected that they will have accepted the notion of a function as a mathematical object and that sets of functions may be fruitful objects of investigation.

In addition, students will have made substantial progress in the matters of problem solving and applications of mathematics, and will have increased their repretoire of techniques.

Finally, they are expected to have matured in the matter of proof style and in the matter of finding proofs. They also are expected to have matured in taking responsibility for their own learning.

Book 5

Functions

Summary of Contents

Structures and Systems: Examples, Properties, One-Fold Operational Systems, Permutation Groups.

Structure-Preserving Mappings: Induced Mappings, Homomorphisms, Isomorphisms.

Matrices: Examples, Definitions, Addition, Multiplication, Matrix Algebra.

Polynomial Functions: Sum and Product of Functions, Zeros of Polynomial Functions, Remainder Theorem, Factor Theorem, Synthetic Division, Rational Root Theorem.

Book 6

Number Systems

In this book, the natural numbers are modeled within set theory. The usual properties of whole numbers are developed, as well as the basic theorem on induction. Induction within set theory and its analogue as a method of proof are treated fully, including the matter of quantified induction hypotheses, and strong induction.

The usual extension to the integers and rational numbers is carried out. The ordered field of real numbers is developed via Dedekind cuts. The notions of least upper bound and convergent sequences are explored. Decimal expansions of real numbers are treated in detail.

The usual extension to the complex number field is made, presenting its algebraic aspects but not pursuing its topology.

Preliminary notions of finite and infinite sets, of cardinal similarity and the cardinal number of a set are introduced in this book.

Book 6

Number Systems

Summary of Contents

The Whole Numbers: Completing the Axiom System for Set Theory, Whole Number Addition, Whole Number Multiplication; the Natural Order Relation on the Whole Numbers, Subtraction and Division of Whole Numbers.

The Integers: Integers As Ordered Pairs of Whole Numbers, Addition, Subtraction, Order, Multiplication, the Ordered Ring of Integers, Induction Theorems.

Fundamental Theorem of Arithmetic: The Divides Relation, Common Multiples and Common Divisors, Fundamental Theorem of Arithmetic, Prime Integers, Euclidean Algorithm, Congruence Relations on the Intergers.

Set Theory: Finite Sets, Infinite Sets.

The Rational Numbers: Rationals As Ordered Pairs of Integers, Multiplication, Addition, Order, the Ordered Field of Rationals, Density, the Archimedean Property.

The Real Numbers: Some History of the Thinking about Real Numbers, Dedekind Cuts, Total Order, Least Upper Bounds, Greatest Lower Bounds, Density, Completeness, Addition, Multiplication, Archimedean Property, Knaster Fixed Point Theorem, Decimal Expansions of Reals, Decimal Expansions of Rationals, Real Sequences and Convergence, the Metric Space of Reals, Rational Exponents.

The Complex Numbers: Complex Numbers as Ordered Pairs of Reals, Addition, Multiplication, a Metric on the Complex Numbers, Polynomial Functions over the Complex Numbers.

The Cardinal Numbers: Counting, Cardinality, Denumberability, Countability, Schroder-Bernstein Theorem, Partial Order on the Cardinal Numbers.

Book 7

Real Analysis
Calculus of One Variable

(two volumes)

Essential to the understanding and appreciation of mathematics and science is mastery of *The Calculus* (of functions of a single real variable). In this book the basic material of *The Calculus* and real analysis is developed in a thorough and rigorous fashion, but the approach is unusual for three reasons:

> continuity is introduced topologically in terms of deformation of an interval; then the idea of limit is developed in the process of extending the notion of interval continuity to the point-wise concept of continuity
>
> the calculus of finite differences is developed, with applications, before *The Calculus* is introduced

approximation is emphasized as an important concept and is used often as a motivating factor

Anyone who has studied *The Calculus* is aware of the central role of the limit concept, and it was decided that the approach indicated in the first of the three reasons mentioned earlier could be intuitively and conceptually clearer to the student than the usual presentation. The calculus of finite differences, which is usually developed haphazardly and *sub rosa* in calculus books, is also basic to *The Calculus* but is independent of the limit concept. It is treated as a separate topic here, to make the situation clearer to the student and to develop a useful theory. Then the idea of approximation is used to motivate the combining of the limit concept with the calculus of finite differences to produce *The Calculus.*

Book 7

Real Analysis
Calculus of One Variable
Summary of Contents

Continuity and Limits.
Differences and Summations.
Differentiation.
Antiderivatives and Integration.
Exponential and Logarithmic Functions.
Lengths of Arrows and the Circular Functions.
Methods of Integration: Integration by Parts, Integration by Composition, Partial Fraction Expansions.
Applications of Integration.
Infinite Sequences and Series.
The Weierstrass Approximation Theorem.

Book 8

Elements of Geometry

The mathematical and methodological education that students have had before *Book 8* puts us in a very favorable position with respect to the treatment of geometry. The students have had some experiences in distinguishing between, and arguing about, real and ideal situations, mathematical theories as formal systems, and interpretations or applications of theories inside or outside mathematics. This allows us from the beginning to deal explicitly with the full spectrum of geometrical considerations, from the intuitive and empirical to the most abstract formal representations. One of the greatest difficulties in traditional geometry teaching is the transition from a purely intuitive (and hence incomplete) understanding of geometrical concepts and statements, to the mathematical theory and its axiomatic anchoring. Students in the EM Program should be able to comprehend from the outset in *Book 8* what it means to begin with a phenomenological analysis of the intuitive and empirical space and get to one or more theories about certain aspects of this space. They should also be prepared to understand different methodological viewpoints in the history of geometry.

Book 8 gives an introduction to three-dimensional Euclidean geometry, with emphasis on the affine structure. It has an introductory chapter with some history of geometry. It then proceeds to a mathematization of basic phenomena in the intuitive space and plane geometry. At first, this concentrates on concepts like point, line, plane, and the incidence relations. Much emphasis is laid on "model building": this is done by discussions and many problems within finite geometries. At the beginning, the attitude is such that points, lines, and planes are considered as three independent kinds of objects. This so-called "three-sorted" approach allows a greater number of more natural interpretations of the incidence axioms, as well as a natural way to discuss the projective extension of affine planes and spaces.

The mathematization of incidence phenomena in the plane and in space gives many opportunities for exercises in translating statements from natural English into formal language, and vice versa, and in analyzing the meaning of concepts depending on the state of the axiomatization process.

The axiomatic analysis of the order concept includes the important results that ordered affine planes must be infinite.

Congruence is not treated in *Book 8*, but is left for *Book 9*. The intent of *Book 8* is to present a very careful axiomatic development of a fragment of geometry, and to insure that the students have a good historical perspective. Brief concluding chapters on projective and hyperbolic geometries are included.

In connection with the investigation of models, especially of finite models for two- and three-dimensional incidence and affine geometry, some fundamental concepts of the axiomatic method, especially semantical ones, are explained or exemplified, such as consistency, independence, and the soundness and completeness of the underlying logic.

A final chapter on the history of set theory is included. In this final chapter, Russell's paradox is explained and parallels are drawn to the ancient crises in Greek mathematics. Also, the various approaches of Russell, Zermelo-Frankel, Neuman-Bernays-Gödel, and Quine to axiomatic set theory are explained. The chapter concludes with a brief discussion of the Axiom of Choice and the Continuum Hypothesis.

Book 8

Elements of Geometry

Summary of Contents

Some History of Mathematics: Ancient Civilizations, Greece to Euclid, Euclid and the Elements, History of the Pralles Postulate, Cantorian Set Theory and Paradoxes, Alternate Set Theories, Axiom of Choice and Continuum Hypothesis.

Affine Geometry: Formal Language, Axioms of Incidence, Parallel Postulate, Theory of Affine Planes, Ordered Affine Planes, Independence and Consistency, Models.

Projective Geometry: Axioms, Models, Duality, Theory of Projective Planes, Extensions of Affine Planes and Reduction of Projective Planes.

Book 9

Linear Algebra and Geometry

(with Trigonometry)

The main objective of the book is a thorough investigation of plane and space geometry using the language and tools of linear algebra. Thus emphasis is on the development of two- and three-dimensional vector spaces over the field $\mathbb{R}$ of real numbers and the use of these vector spaces to study geometry. EM students have studied fields other than the field of real numbers; therefore, vector spaces are defined over arbitrary fields with no restriction on the dimension. Examples and exercises are given about vector spaces over the field of complex numbers, the field of rational numbers and finite fields; nevertheless, the main investigation is restricted to two- and three-dimensional vector spaces over the field of real numbers.

Chapter One presents the standard concepts of linear algebra. Vector spaces are defined and basic investigations are made into such notions as subspaces, direct sums, independence, bases, dimension, linear transformations, matrices, systems of linear equations, and determinants.

Chapter Two studies affine geometry using the tools developed in Chapter One. The fundamental notion of an affine space is introduced as a set of points, together with "translations" acting on those points. Then much traditional material on coordinate geometry, analytic geometry and parallelism is presented. Also, other fundamental but non-traditional topics such as orientation, affine transformations, and the affine group are included.

The third chapter returns to linear algebra and develops the notion of an inner product space. Orthogonality is defined and discussed. Isometries are studied and the Witt and Cartan-Dieudonne theorems are proven. These results are then used to study in detail any orthogonal group of Euclidean two-and three-dimensional vector spaces.

Chapter Four uses the inner products developed in the third chapter to define distance and perpendicularity. The Euclidean group is studied and the Carten-Dieudonne theorem is extended to Euclidean spaces. Many of the traditional theorems of geometry are proven. For example, congruence is studied and the usual "Side-Angle-Side," "Side-Side-Side" results are obtained. In addition, Euclidean transformations are classified as translations, rotations, reflections, or glide reflections.

Chapter Five presents a concise treatment of trigonometry. The sine and cosine functions are defined in terms of the inner product and all the basic theorems are easy consequences of these definitions. The other trigonometric functions (tangent, cotangent, secant, cosecant) are defined in terms of the sine and cosine and the development of trigonometry is straightforward.

Book 9

Linear Algebra and Geometry
(with Trigonometry)

Summary of Contents

Vector Spaces and Linear Transformations: Definition of a Vector Space, Linear Subspaces, Systems of Equations, Intersection and Sum of Subspaces, Direct Sums of Subspaces, Bases and Dimension, Linear Transformations, Isomorphism of Vector Spaces, Kernel and Image of a Linear Transformation, Matrix Representation of Linear Transformations, Change of Basis, First Degree Equations and Vector Spaces, Theory of Determinants.

Affine Spaces and Affine Transformations: Intuitive Affine Geometry, Definition of n-dimensional Affine Space, Affine Subspaces, Coordinates and Coordinate Systems, Analytic Geometry, Parallelism, Affine Spaces Spanned by Points, Betweeness and Line Segments, Separation of the Plane, Oriented Line Segments, Tangent Spaces, Affine Transformations, Coordinate Representation of Affine Transformations.

Inner Product Spaces: Inner Products, Isometries, Subspaces, Euclidean Vector Spaces, Orthogonality, the Group of Isometries, the Orthogonal Group, the Witt Theorem, the Cartan-Dieudonne Theorem, the Group of Isometries of the Euclidean Plane, the Group of Isometries of Euclidean Three-Space, Similitudes, Oriented Vector Spaces.

Euclidean Geometry: Introduction, Euclidean Spaces, Perpendicularity, Circles and Spheres, Conics, Angles, Rigid Motions, Congruence in the Euclidean Plane, Standard Theorems of Euclidean Plane Geometry, the Cartan-Dieudonne Theorem for Euclidean Spaces, Similarities.

Intuitive Trigonometry: Sine and Cosine, Properties of the Sine and Cosine Functions, Other Trigonometric Functions, Applications of Trigonometry, Circular Functions and Their Graphs.

Book 10

Groups and Rings

Algebraic structures play a major role throughout the entire EM program. The notion of operational system is fundamental to all parts of this program and hence is introduced in the very first chapter of *Book 0* and grows spirally throughout the curriculum.

Groups, rings, fields, and vector spaces are also introduced and studied on an intuitive level in *Book 0*. They are presented as special types of operational systems which occur frequently in the study of various branches of mathematics, particularly algebra and geometry. The geometry of *Book 0* and later of *Book 9* takes full advantage of the students' familiarity with the group concept. Ordered groups, rings and fields, are fundamental in the EM treatment of classical high school algebra both in *Book 0* and *Book 3*. These structures also play a substantial role in *Book 5* in the development of the general notion of structure preserving mappings such as homomorphisms.

The purpose of *Book 10* is to synthesize the knowledge of groups and rings accumulated by the students since the beginning of their

EM studies and then to look more deeply at the structures themselves, thus acquiring a more mature and sophisticated understanding of their anatomy.

Having successfully completed *Book 10*, EM students will have reached a level of algebraic maturity that should stand them in good stead in any applied field (including other branches of mathematics) that requires substantial background in abstract algebra. An examination of the following table of contents of *Book 10* should make it clear that this course is similar to an upper undergraduate course for mathematics majors or a beginning graduate level course.

It is suggested that *Book 10* be taught in a seminar mode with students reading the book on their own and class time devoted to student presentation of the problems given at the end of each chapter.

Book 10

Groups and Rings

Summary of Contents

Groups: Basic Properties, Order of a Subgroup, Permutation Representations, Sylow Theorems, Generators and Cyclic Subgroups, Sylow Subgroups, Finite Abelian Groups, Normal Subgroups, Isomorphism Theorems, Composition Factors.

Rings: Introduction to Commutative Rings, Homomorphisms and Ideals, Principal Ideal Domains, Polyomial Domains, Gauss' Lemma, Unique Factorization.

Fields: Quotient Field of an Integral Domain, Root Fields and Splitting Fields, Galois Theory, Classical Geometric Construction Problems, Finite Fields.

Book 11

Finite Probability Spaces

This book is devoted to probability theory restricted to finite outcome sets with some hints at the end as to how the theory can be extended to deal with infinite outcome sets. The emphasis is on the mathematics of probability. In *Book 0* the students have already been exposed to probability in a concrete, problem-oriented course. Here it is shown how probability theory can be taught within the framework of mathematics consistent with the EM approach.

There are many reasons for including books on probability in the EM series. Probability theory is an important field of mathematics with numerous applications. At present, much research work is done in this area, so this book will give the students contact with a field of mathematics that is presently expanding. This book also gives the student the opportunity to realize the power of his or her mathematics knowledge and skills developed so far. Furthermore, *Book 11* will give the students the opportunity to apply their knowledge of logic, sets, relations, and functions. Since most of the concepts in probability theory are motivated by "real-world" situations, this book will also give the students concrete experiences of the interplay between the real world and mathematical model building (mathematization). Applications from various fields such as statistics, operations research, science, technology, and so on, are included.

After an introductory chapter dealing with some history of probability theory, and some practical activities serving as empirical background, the concept of a finite probability space is defined. The definition is not restricted to deal only with uniform probability measures, which has often been the case at the high school level. Basic theorems concerning probability measures are proved and it is shown how a probability measure can be uniquely defined with the aid of a point probability function. The uniform probability measure case is studied carefully and some useful combinatorial results from *Book 4* are reviewed using a somewhat new approach. Conditional probabilities are defined and the "total probability formula" and Bayes' Theorem are proved. Independence among events is then treated in the language of relations established in *Book 4*.

Multistage experiments and their representation as weighted tree diagrams are fundamental in the approach to probability taken in *Book 0*. Here, multistage experiments are treated by showing how a probability measure can be defined on a Cartesian product with the aid of a transition probability measure. As an introduction, the notion of a product probability measure is introduced.

Usually, real-valued mappings of probability spaces under the name of "random variables" are given a central role in the development of probability theory; here, general mappings are studied first, and later, after real probability spaces have been introduced, real-value mappings are studied. As an application, the binomial distribution is derived.

Concepts especially related to real finite probability spaces, like expectation, variance and standard deviation are defined. The Chebyshev inequality is proved and is used to prove the weak law of large numbers. Thus, the empirical phenomenon of stability of relative frequencies is brought within the theory.

The last chapter contains a comprehensive review of the entire book, followed by a set of review problems. Problems of different degrees of difficulty are presented in two distinct sections. This will strengthen the students' problem-solving ability and will also expose them to important applications from operations research, reliability theory, search theory, etc. Finally, the book discusses the extension of the theory to infinite countable outcome sets and to outcome sets which are intervals in IR or IR X IR (geometrical probability).

An important problem is the "real-world" interpretation of probability concepts. In *Book 11*, this is done mostly in terms of the "relative frequency-limit" interpretations. Other interpretations, however, are also mentioned.

Book 11

Finite Probability Spaces

Summary of Contents

Finite Probability Spaces: Outcome Sets, Events, Relative Frequencies and Probabilities, Finite Probability Spaces, Point Probability Function, Probability Theory, Mathematical Models and Applications.

Uniform Probability Measures: Uniform Probability Measures, Fundamental Principle of Counting, Number of Arrangements and Number of Subsets, Sampling a Finite Population.

Conditional Probability Measures and Independence.

Multistage Experiments: Product Probability Measure, Tree Diagram Representation, Independent Trials, Transition Probability Measure.

Mappings of Finite Probability Spaces, Expectation: Function Induced by a Relation, Image of a Finite Probability Space, Frequency in Independent Trials of Bernoulli Experiments, Applications of the Binomial Distribution, Real Finite Probability Spaces, Expectation, Expectation of Real-Valued Functions, Variance, the Weak Law of Large Numbers.

Book 12

Introduction to Measure Theory

This book gives an introduction to measure theory in the language of probability theory. Part of *Book 7* and most of *Book 11* are necessary prerequisites. Essentially, this book treats the generalization of the theory developed in *Book 11* to the case when the outcome set is an arbitrary set Ω.

First the notion of an algebra of subsets is defined and basic theorems concerning such algebras are proved. Particularly, finite algebras are studied and it is shown how a finite algebra is generated by its atoms. Limit operations with sequences of sets are introduced by the definition of "lim sup A_n" and "lim inf A_n."

Probability measures are first defined on an algebra and it is discussed how such a probability measure can be extended to a probability measure on the generated σ-algebra. It is also shown how a σ-additive normed real valued function on a semialgebra can be extended to a probability measure on the generated σ-algebra. These extension theorems are then used when a probability measure on $\mathbb{R}$ is defined with the aid of a distribution function and when probability measures on product spaces are treated. Compactness of a class of sets is discussed in terms of the finite intersection property and it is shown how compactness can be used to establish that an additive function on a semialgebra is σ-additive.

Measurable functions are introduced and it is shown how a measurable function is the limit of a sequence of stepfunctions. Integration of measurable functions, that is, defining expectation of measurable functions, is treated by taking limits of integrals of stepfunctions. Probability measures on products of measurable spaces is treated and the Fubini Theorem is proved.

This book gives both a generalization of the mathematical theory of probability given in *Book 11* and an introduction to modern real analysis.

Book 12

Introduction to Measure Theory

Summary of Contents

Events and Algebras: Algebras, Finite Algebras, Mappings and Algebras, Limits and Sets.

Probability Measures on Algebras and σ-Algebras: Probability Measure on an Algebra, σ-Algebras, Probability Spaces.

Extension of Probability Measures: Basic Theorem on Extension of Probability Measures, Semialgebras and Probability Measures, Probability Measures and Compactness.

Probability Distributions on the Real Numbers: Borel Sets in the Reals, Distribution Functions, Distribution Functions and Probability Measures.

Integration of Measurable Functions: Indicator Functions and Stepfunctions, Measurable Functions, Measurable Functions as Limits of Sequences of Stepfunctions, Integration of Stepfunctions, Integration of Measurable Functions, the Indefinite Integral.

Product Probability Spaces: Sections of Relations and Functions, Product of Measurable Spaces, Product of Probability Spaces, Fubini's Theorem.

Book A

Introduction to Mathematization: A Theory of Voting Bodies

The purpose of this book is to let the students actually participate in a mathematization process which leads from situations outside of mathematics to a mathematical theory. This process, which is a special type of mathematical model building, plays an important role in modern applications of mathematics, particularly in fields other than physics, such as economics, the social sciences, biology, cosmology, etc.

The situations to be mathematized, which we have selected for *Book A*, occur in human society when a group of people must make decisions by voting. It is possible to develop a small and elementary yet nontrivial theory of voting bodies in such a way that many significant steps of a mathematization process can be properly exemplified.

The students are kept engaged in actually performing these and other steps by interspersed exercises, by problem sets, and by a detailed analysis of different possibilities for each step. One activity, for example, consists in finding a definition for power concepts such as "powerless member," "dictator," "veto-power," "chairman," etc. These definitions are to be given by the students within the framework of the theory already developed. From classroom teaching, we know that different students are inclined to use different characteristics of the intuitive power concepts to make their own definitions.* The fact is simulated in the book by the presentation of three or four different suggestions for the definition of each concept. Comparison of these with their own suggestions, and proving or disproving logical equivalence between the suggested definitions is most challenging to the students. The role of definitions ceases to be the boring part of mathematics, as is usually the case in traditional school mathematics.

The basic mathematical tool in carrying out the mathematization of voting situations is set theory. Voting bodies are considered as structured sets. The fact that a set of voters—such as the Security Council of the United Nations—can get a voting structure by only determining certain subsets to be winning coalitions, without talking about vote distribution and a majority rule, leads to two distinct voting-body concepts. On an elementary level this is a striking analogy to the difference between topological and metric spaces in advanced mathematics. The methodological meaning of this difference can be explained fully to the students by the unexpected result that there are Security-Council type voting bodies for which no such vote distribution and majority rule exist that determine the given winning coalitions.

After the students have been involved in the development and full construction of the theory, they are asked to rewrite the whole body of results in a strict axiomatic-deductive way; appropriate examples are familiar to the students from the preceding books. Here they may, and are urged to, rearrange the order of definitions and theorems according to economical or aesthetic viewpoints. This is another important activity of mathematicians to which students are hardly ever exposed. In most mathematics courses, the final theoretical product as a sequence of axioms, definitions, theorems, and proofs is presented with no reference to the motivations which come from the analysis of the original situations and of the related intuitive concepts.

One of the goals of *Book A* is for the students to learn, through personal experience, the difference between the creative development of a theory and its presentation as a finished piece of mathematics.

In the last part of *Book A* another important feature of mathematical model-building is exhibited. One discovers cases which the theory surprisingly does not cover. The problem then arises of finding an appropriate extension of the theory.

At the completion of *Book A*, it is expected that the students will:

> have a notion of some characteristic steps of a mathematization process and understand especially that axiomatization can be considered a creative act of concept formation

understand the possible role of a definition as a formulation of an intuitive concept within a given framework; understand the problem of adequacy of a definition and the distinction between the "explicans" and the "explicandum"

understand the difference in style and meaning between developing a theory and presenting it as a finished product

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*Steinar. H. G., "Examples of Exercises in Mathematization on the Secondary School Level," *Educational Studies in Mathematics, Vol. I, No. 1-2 (1968), pp. 181-201.*

Book A

Introduction to Mathematization:

A Theory of Voting Bodies

Summary of Contents

Mathematization from a Situation to a Theory: Familiarization, Idealization, Specialization, Organization, Axiomatization, Application.

Voting Bodies: Sets of Voters, Coalitions, Voting Distributions, Powerless Member of a Voting Body, Dictator, Veto Power, Voting Body Theory, Models of Voting Body Theory, Measurable and Nonmeasurable Voting Bodies, Proposal-Related Voting Bodies.

Book B

EM Problem Book

This problem book is a collection of 200 mathematical problems requiring no special prerequisites beyond the most basic mathematical skills for their solution. However, the problems are nonroutine and nontrivial.

The difficulty of each problem is rated on a scale from one to five with level one the easiest and level five the most difficult. Problems related by content, analogy, or method of solution are placed near each other so that solving one often helps in the solution of others. However, the problems need not be solved in sequence.

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Most of the problems allow experimentation where the experimental results reveal patterns which help in arriving at the solution. If these efforts fail, there are short hints given for most problems at the end of each chapter.

Book B

EM Problem Book

Summary of Contents

Miscellaneous Problems, Sets of Weights, the Box Principle, Parity Proofs and Coloring Proofs, Game Problems, General Geometric Problems, Tiling with Dominoes, Pick's Formula for the Area of a Simple Lattice Polygon, Number Theoretic Problems.

APPENDIX II

A Letter to EM Students

Richard Hillegas, the author of the following letter, was a member of the third class of students to participate in the Elements of Mathematics program for the entire six years of his secondary education. He finished the series by the time he graduated and then enrolled at Harvard University. At Harvard, in recognition of his background, he was allowed to enroll in the most advanced course in the honors calculus sequence during his freshman year. Similar courses are offered to juniors and seniors at most universities. While it is also an upper level course at Harvard, some very advanced freshman are accepted. As a sophomore he was scheduled for even more advanced courses which most students would not meet until the graduate level.

In June of 1976, Richard wrote to us of his academic experiences during his freshman year. We asked him to rewrite the letter as an open letter to EM students. It is that letter which follows:

July 10, 1976

Dear Fellow EM students,

I just finished my freshman year at Harvard and thought you might like to know how I felt EM prepared me for it. I hope what I say won't sound like a catalogue of my own strengths and weaknesses, because I do think EM has shaped my outlook and abilities.

I took the last course in the honors calculus sequence, according to math faculty and students the most difficult mathematics course at Harvard. Of the around sixty dedicated persons who started, only twenty-two of us stuck it out to the end. I managed to pull an A—, which I think indicates how EM prepared me. I'll elaborate:

The course handled extremely abstract material. Above all, it required an ability to deal quickly with non-intuitive concepts. I don't say I always succeeded, but my previous experience with abstract algebra in EM certainly helped. Then too, it demanded that the student be capable of clear, correct reasoning, in particular, that he be able to turn out valid, tidy proofs. Again, the EM sequence certainly develops this skill.

Except for a couple questions on the shorter exams, where time limits forced shallower problems, the course had little use for the "crank in, crank out" mentality fostered by a lot of high school math programs. The professors assumed that if you knew the theory, you could do the mechanical problems, while the converse was not true.

I'd say the EM sequence excellently prepares someone who, like myself, plans to major in math. I'd think also that in any field, facility with non-intuitive forms and clear, precise reasoning cannot be overvalued. In addition, EM offered me a far richer mathematics curriculum than most high schools in this country could have. Few secondary mathematics programs treat introductory logic, set and function theories as well, or even teach the already mentioned abstract algebra, nor cover in greater depth any topic except perhaps Euclidean Geometry — which EM does cover as a special case of a broader, more useful geometry. At least, so I gathered from my Harvard friends, most of whom attended large, good high schools or private schools.

Of course, quick crank-in, crank-out technique and Euclidean geometry have important applications, which I don't want to slight. And when it comes down to the line, I don't think anyone needs to go into the supermarket knowing either how to cope with non-intuitive concepts or rapidly compute a matrix determinant. I do think EM gave me a rich math background and outlook, and I hope you stay with the program, enjoying it as much as I did.

Very much yours,

Richard Hillegas

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