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ABSTRACT
                  This document looks at specific applications of
calculus to engineering. It is noted that for an automobile traveling
at constant speed, the mathematical relationship between the
directional heading and the angular deflection of the steering wheel
can be calculated with respect to time. An analysis is presented
which derives the relationship from basic geometrical and kinematic
principles. The derivation is seen as an example of calculus use in
the mathematical modeling of dynamic systems. This module contains
both exercises and a model exam, with answers provided for both at
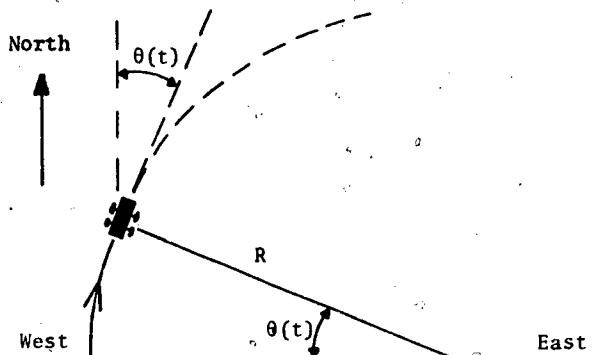the conclusion of the document. (MP)

umap

UNIT 506

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

THE RELATIONSHIP BETWEEN DIRECTIONAL HEADING
OF AN AUTOMOBILE AND STEERING WHEEL DEFLECTION

by John E. Prussing

APPLICATIONS OF CALCULUS TO ENGINEERING

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THE RELATIONSHIP BETWEEN DIRECTIONAL HEADING

OF AN AUTOMOBILE AND STEERING WHEEL DEFLECTION

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Intermodular Description Sheet: UMAP Unit 506

Title: THE RELATIONSHIP BETWEEN DIRECTIONAL HEADING
OF AN AUTOMOBILE AND STEERING WHEEL DEFLECTION

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Review Stage/Date: III 6/5/80

Classification: APPL CALC/ENGINEERING

Prerequisite Skills:
1. Know the definition of even function.
2. Be able to calculate with the series $\tan\delta = \delta^3/3 + 2\delta^5/15 + 17\delta^7/315 + \ldots$
3. Be able to calculate with the approximation $\tan\delta \simeq \delta$ (radian measure, small angles).
4. Be able to use l'Hôpital's rule in simple cases.
5. Be able to integrate and differentiate elementary functions.

Output Skills:
1. Know what assumptions lead to the equation that related $\theta(t)$ and $\phi(t)$, and how the equation is derived.
2. Find automobile headings for given wheel deflection functions and initial conditions.

Other Related Units:

1. INTRODUCTION

For an automobile travelling at constant speed, the mathematical relationship between the directional heading of the auto, as measured for example by a magnetic compass on the dashboard, and the angular deflection of the steering wheel can be calculated by an integration with respect to time. To see this, consider the example of a constant steering wheel deflection. As the auto travels in the resulting circular turning path over the ground, the heading changes linearly with time, i.e., the heading varies at a constant *rate*. Since the time integral of a constant is a linear function of time, this suggests that the heading is in some way related to the integral over time of the steering wheel deflection. In the analysis that follows we will derive this relationship from basic geometrical and kinematic principles. The derivation is an example of the use of calculus in the mathematical modeling of dynamic systems.

2. THE MATHEMATICAL MODEL

2.1 The Directional Heading and the Angular Deflection of the Steering Wheel

Let the *directional heading* of the auto be denoted by $\theta(t)$. A convenient reference direction for $\theta = 0$ is the direction of magnetic north. Thus $\theta = \pi/2$ corresponds to the auto's heading east, $\theta = \pi$ to its heading south, and so forth. The range of values of heading is $0 \leq \theta < 2\pi$.

Let the *steering wheel angular deflection* be denoted by $\phi(t)$, where $\phi = 0$ corresponds to the neutral position of the steering wheel (no turning), a positive value denotes a turn from the neutral position to the right, and a negative value a turn to the left. The range of values of steering wheel deflection is $-\phi_{max} \leq \phi \leq \phi_{max}$, where the value of ϕ_{max} depends on the particular type of auto, but is typically greater than 2π. As one would expect, the value ϕ_{max} is related to the minimum turning radius R_{min} for the auto. For sports cars the value of ϕ_{max} is usually smaller than it is for passenger cars. For a 1965 Porsche, ϕ_{max} is about 7 radians (400°) and R_{min} is 5.5m. For a 1974 Volvo sedan, ϕ_{max} is about 4π radians (720°) and R_{min} is 4.8m.

Note that the steering wheel deflections $\phi = 0$ and $\phi = 2\pi$ are not equivalent (unlike $\theta = 0$ and $\theta = 2\pi$ which are). The auto's response to $\phi = 2\pi$ (a circular turning path to the right) is greatly different from its response to $\phi = 0$ (a straight line path).

2.2 The Angular Deflection of The Front Wheels

The effect of a steering wheel deflection ϕ is an angular deflection of the front wheels of the auto. This angle, measured with respect to the fore-aft center line of the auto, is denoted by δ.

Because of the mechanical linkage between the steering wheel and the front wheels, the value of δ is proportional to the steering wheel deflection ϕ,

$$(1) \qquad \delta = G \phi .$$

The constant of proportionality G, is called the *gear box ratio*. Its value depends on the gear ratio of the steering system for a particular auto. A typical value of G is 1/20, that is, a 20° change in the steering wheel deflection ϕ results in a 1° change in front wheel angle δ. Thus a typical value of δ_{max} corresponding to $\phi_{max} = 720°$ is 36°. Equation (1) permits us to calculate ϕ by dividing δ by G.
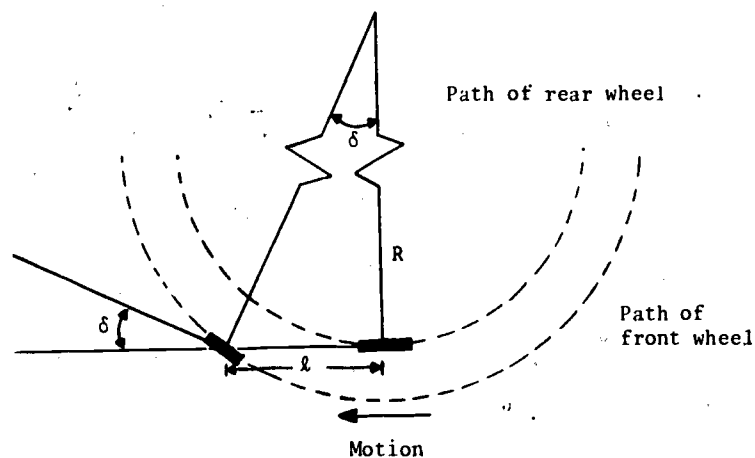
Figure 1. The "bicycle model" of a turning automobile has a single front wheel and a single rear wheel. The angular deflection δ of the front wheel is measured against the fore-aft center line of this automobile. The wheel base is the length ℓ, the turning radius is R.

Figure 1 shows a simplified geometrical model of an auto traveling in a circular turning path of radius R. Because the turning radius R is typically much larger than the width of the auto, the outside wheels and the inside wheels experience essentially the same conditions. For this reason one can analyze the problem using a "bicycle model" for the auto, having a single front wheel and single rear wheel as shown. The angular deflection δ of the front wheel is shown in Figure 1. The *wheelbase* of the auto is the distance between the front and rear axles.

From the geometry of Figure 1 it is evident that

(2) $\qquad \tan\delta = \dfrac{\ell}{R}$.

If the turning radius R is large compared with the wheelbase ℓ, which is usually two or three meters, then δ is a

small angle and one can calculate it reliably the approximation

(4) $\qquad \tan\delta \simeq \delta$.

Exercises:

1. Investigate the error in the approximation $\tan\delta \simeq \delta$ by defining a relative error $e(\delta) = (\tan\delta - \delta)/\tan\delta$.

    a) Show that $e(\delta)$ is an even function of δ, that is, $e(-\delta) = e(\delta)$.

    b) Using l' Hôpital's rule, calculate $\lim_{\delta \to 0} e(\delta)$.

    c) Show that $e > 0$ for $-\pi/2 < \delta < \pi/2$ by using the infinite series $\tan\delta = \delta + \delta^3/3 + 2\delta^5/15 + 17\delta^7/315 + \ldots$, which converges for $|\delta| < \pi/2$.

    d) Calculate numerical values for $e(\delta)$ in $5°$ increments for $0 \leq \delta \leq 30°$.

2. The maximum steering wheel deflection ϕ_{max} corresponds to the minimum turning radius R_{min}.

    a) Use Equations (1) and (2) to show that

    $$G = \frac{\ell}{\phi_{max}R_{min}} \quad .$$

    b) For the data for ϕ_{max} and R_{min} given previously for a 1965 Porsche, calculate the value of G for $\ell = 2.1$ m.

    c) Calculate G for a 1974 Volvo sedan having $\ell = 2.6$ m.

    d) If you have access to a car, calculate G for it.

2.3 Relating the Change in Heading to the Steering Wheel Deflection

The angle δ defined by Equation (2) is called the *Ackerman Angle*. For turns at speeds low enough to avoid side slipping so that we don't have to oversteer, (which are the only speeds we shall consider), the Ackerman Angle is simply equal to the angle of the front wheel

as shown in Figure 1. For high speed turns centrifugal force effects become important and the front wheel angle is equal to the Ackerman Angle plus a correction for the effect of centrifugal force.
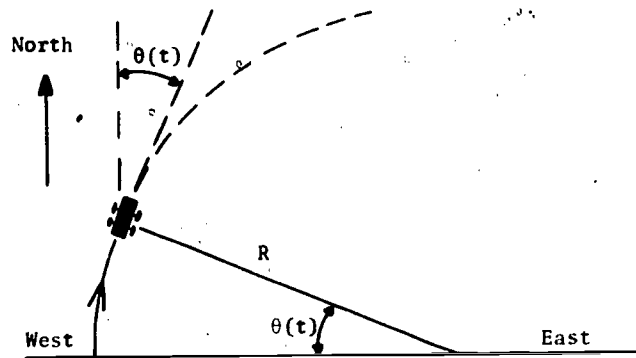

Figure 2. The directional heading $\theta(t)$ of an automobile is its "compass" heading, measured clockwise in radians from magnetic north. The value of $\theta(t)$ shown above is about 0.6 radians.

Figure 2 shows the directional heading angle $\theta(t)$ of the auto as it travels at constant speed V in a circular arc of radius R, corresponding to a constant front wheel angle δ.

Since the steering has no effect on the rear wheels the fore-aft centerline of the auto is always normal to the radius of the turning circle (See Figure 1) and the angle between the radius of the turning circle and the East-West reference line is also $\theta(t)$. In a small time interval Δt the heading changes by an amount $\Delta\theta$ and the distance travelled along the circumference of the circle is simply $R\Delta\theta$. The constant speed V is the distance travelled $R\Delta\theta$ divided by the time interval Δt:

$$R \frac{\Delta\theta}{\Delta t} = V ,$$

By taking the limit as $\Delta t \to 0$ one obtains the instantaneous time rate of change of the heading as

$$(4) \qquad \frac{d\theta}{dt} = \frac{V}{R}$$

Equation (4) makes physical sense: the larger the speed V the faster the heading changes, the smaller the turning radius R the faster the heading changes.

By combining Equations (1) through (4) we can relate the rate of change of heading (called the *yaw velocity* in vehicle dynamics) to the front wheel angle δ and hence to the steering wheel angular deflection ϕ:

$$(5) \qquad \frac{d\theta}{dt} = \frac{V}{R} = \frac{V}{\ell} \tan\delta = \frac{V}{\ell} \delta = \frac{VG}{\ell} \phi .$$

The constant of proportionality VG/ℓ depends on the speed, gear box ratio, and wheelbase of the auto. To simplify the notation, let us denote the constant VG/ℓ by K. Since G is dimensionless, V has units of, say, meters per second, and ℓ has units of meters, the constant K has units of inverse seconds. One can interpret $1/K$ as the time it takes the heading to change by r radians when the steering deflection ϕ is held constant at r radians.

When we rewrite Equation (5) in terms of K, we obtain

$$(6) \qquad \frac{d}{dt} \theta(t) = K\phi(t) .$$

If we then integrate both sides of Equation (6) from 0 to t we obtain the equation

$$(7) \qquad \theta(t) - \theta_0 = K \int_0^t \phi(\tau)d\tau .$$

where $\theta_0 = \theta(0)$ is the heading at an initial time $t = 0$, and a dummy variable of integration τ has been used in Equation (7) so that we will not confuse the variable of integration with the upper limit of integration t.

For any steering wheel deflection function $\phi(t)$, Equation (7) tells us that the *change* in heading $\theta(t) - \theta_0$

is proportional to the *area* under the curve $\phi(\tau)$ from $\tau = 0$ to $\tau = t$.

3. APPLICATION OF THE MATHEMATICAL MODEL

Consider the case in which the auto is initially travelling at a constant heading θ_0, with $\phi = 0$. This satisfies the basic relationship of Equation (6) since θ is constant for $\phi = 0$. Assume that at $t = 0$ the steering wheel is suddenly deflected to an angle ϕ_0 and held at that angle for t_0 seconds. Then the steering wheel is suddenly restored to $\phi = 0$. The graph of $\phi(t)$ for this case is shown in Figure 3.
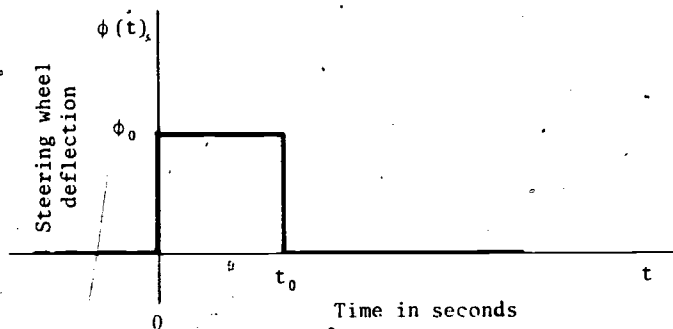

Figure 3. A steering wheel held in neutral position is turned abruptly to the right at $t = 0$, held at a deflection of ϕ_0 radians for t_0 seconds, and returned abruptly to the neutral position.

What heading $\theta(t)$ corresponds to the deflection $\phi(t)$?

To express θ in terms of ϕ we first write down Equation (6) for the two time intervals $0 \leq t \leq t_0$ and $t > t_0$. For the given deflection function, this results in the pair of equations:

$$(8) \qquad \frac{d\theta}{dt} = k\phi_0, \qquad 0 \leq t \leq t_0,$$

$$(9) \qquad \frac{d\theta}{dt} = 0, \qquad t > t_0.$$

We then integrate both sides of Equation (8) from 0 to t to get

$$(10) \qquad \theta(t) - \theta(0) = K\phi_0 t,$$

or

$$(11) \qquad \theta(t) = K\phi_0 t + \theta_0, \qquad 0 \leq t \leq t_0.$$

Finally, we integrate both sides of Equation (9) from t_0 to t to obtain

$$(12) \qquad \theta(t) - \theta(t_0) = 0,$$

or

$$(13) \qquad \theta(t) = \theta(t_0), \qquad t > t_0.$$

The value of $\theta(t_0)$ in Equation (13) may be found by substituting t_0 for t in Equation (11). This yields

$$(14) \qquad \theta(t_0) = K\phi_0 t_0 + \theta_0,$$

so that Equation (13) becomes

$$(15) \qquad \theta(t) = K\phi_0 t_0 + \theta_0, \qquad t > t_0.$$

As a pair, Equations (11) and (15) tell us that

$$(16) \qquad \theta(t) = \begin{cases} K\phi t + \theta_0, & 0 \leq t \leq t_0, \\ \\ K\phi_0 t_0 + \theta_0, & t > t_0. \end{cases}$$

Example 1. An automobile travels initially at a constant heading $\theta_0 = \pi/4$, with $\phi = 0$. At $t = 0$, the steering wheel is suddenly deflected to an angle of $\phi_0 = \pi/2$, and is held there for $t_0 = 2$ seconds. Then the steering wheel is restored to $\phi = 0$. If the automobile travels at a constant speed $V = 13$ m/s, (29 mph), and if $\ell = 2.6$ m and $G = 1/20$, find $\theta(t)$ for $t > 0$.

Solution. We first calculate

$$K = \frac{VG}{\ell} = \frac{1}{4} \text{ s}^{-1}.$$

Equation (16) then yields

$$\theta(t) = \begin{cases} \dfrac{\pi}{8} t + \dfrac{\pi}{4} \ , & 0 \le t \le 2 \\[2ex] \dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2} \ , & t > 2 \ . \end{cases}$$

Thus the heading $\theta(t)$ changes linearly from its initial value of $\pi/4$ (northeast) to a final value of $\pi/2$ (east). The change in heading ($\pi/4$ radians) is the area under the curve $\phi(t)$ of Figure 3 (π radian-seconds) times K (equal to $1/4$ s^{-1}).

Exercises:

3. An automobile travels initially at a constant heading $\theta_0 = \pi/2$, with $\phi = 0$. At $t = 0$, the steering wheel is suddenly deflected to an angle of $\phi_0 = \pi/4$, and is held there for $t_0 = 6$ seconds. Then the steering wheel is abruptly restored to $\phi = 0$. If the automobile travels at the constant speed of $V = 14$ m/s, and if $\ell = 2.1$ m and $G = 1/20$, find the heading $\theta(t)$ for $t > 0$.

4. We wish to change the value of t_0 and ϕ_0 in Example 1 so that the automobile will have a final heading of $\theta = \pi$. Find a relationship between t_0 and ϕ_0 that will achieve this.

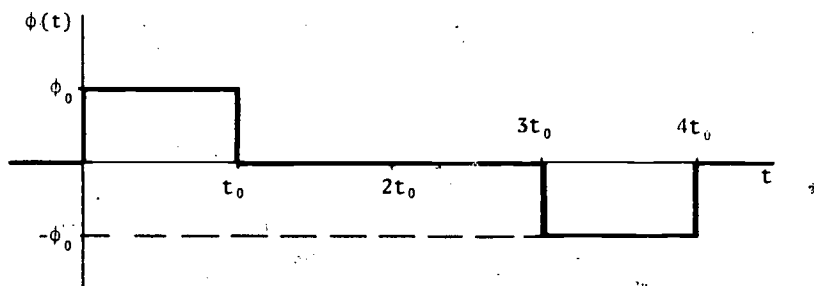5. For a steering wheel deflection function shown in Figure 4,

find the final value of the heading if its initial value is $\theta(0) = \theta_0$.

6. Suppose that the steering wheel deflection function of an automobile is $\psi(t) = \phi_0 e^{-\alpha t}$ for $t \ge 0$.

a) What are the physical dimensions of the constant α?

b) What is the value of t at which $\phi(t) = \phi_0/e$ which is approximately $0.368\phi_0$?

c) Use Equation (7) to obtain an expression for $\theta(t)$ for $t \ge 0$.

d) Find the limiting value $\lim\limits_{t\to\infty} \theta(t)$.

7. An auto travelling at $V = 20$ m/s has a gearbox ratio $G = 1/20$ and a wheelbase $\ell = 3$ m. What value of the steering wheel deflection ϕ_0 will make the auto travel in a circular arc of radius 40 m?

Figure 4. The steering wheel deflection function of Exercise 5.

9

10

14

15

4. MODEL EXAM

1. Automobile turning specifications are sometimes given in terms of a *steering factor*, defined as the number of turns of the steering wheel necessary to establish a 30 foot (9.14 m) constant radius turn. For a 1980 Datsun 210 wagon the steering factor is 0.67, the wheelbase ℓ is 2.33 m. and R_{min} = 4.88 m.

   a. Calculate the gearbox radius G.

   b. Calculate the maximum front wheel deflection angle δ_{max}.

   c. Calculate the maximum steering wheel deflection angle ϕ_{max}.

2. For an automobile travelling at constant speed the steering wheel deflection $\phi(t)$ is given by:
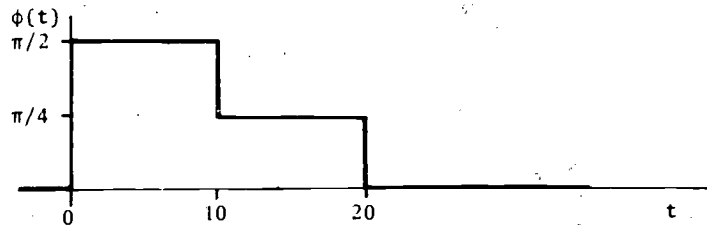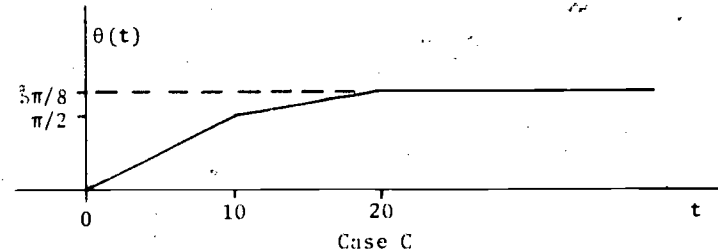

   a. By using the functional relationship of Equation (6) determine which of the following graphs corresponds to the heading $\theta(t)$.

   b. If G = 1/20 and ℓ = 3 m., what is the speed V of the auto?

Case A

Case B

Case C
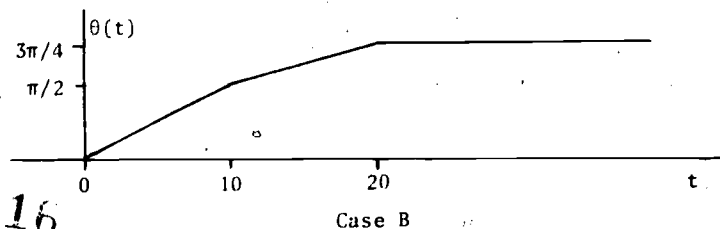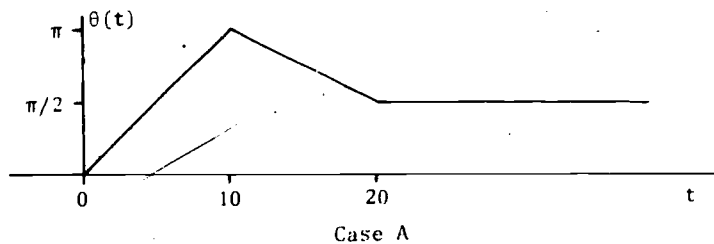
3. During a testing run of the auto the heading $\theta(t)$ is recorded to be:

$$\theta(t) = \begin{cases} \dfrac{K\phi_0}{t_0}\,\dfrac{t^2}{2} + \theta_0 & 0 \le t \le t_0 \\[2ex] \dfrac{K\phi_0}{2}\,(2t - t_0) + \theta_0 & t_0 < t \le 2t_0 \\[2ex] \dfrac{3K\phi_0 t_0}{2} + \theta_0 & t > 2t_0 \end{cases}$$

What is the corresponding steering wheel deflection $\phi(t)$ during each time interval?

4. An auto is travelling at an initial constant heading θ_0. Let us model a realistic "right turn at the next corner" by the steering wheel deflection shown, where t = 0 marks the beginning of the turn and $t = t_0$ the end of the turn.
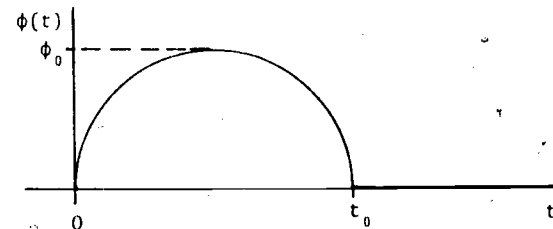
$$\phi(t) = \begin{cases} \phi_0 \sin \pi\,\dfrac{t}{t_0} & 0 \le t \le t_0 \\[1ex] 0 & t > t_0 \end{cases}$$

   a. Determine the heading $\theta(t)$ during the interval $0 \le t \le t_0$ using the relationship of Equation (6).

b. What is the final heading of the auto?

c. If $\phi_0 = \pi$ and $K = \frac{1}{4} s^{-1}$. what is the required value of t_0 for a 90° ($\pi/2$ radian) change in the heading?

1. a) $e(-\delta) = [\tan(-\delta) - (-\delta)]/\tan(-\delta)$. Since $\tan(-\delta) = -\tan\delta$,
    $e(-\delta) = e(\delta)$

    b) $\lim_{\delta \to 0} e(\delta) = \lim_{\delta \to 0} (\sec^2\delta - 1)/\sec^2\delta = 0$

    c) $e(\delta)$ is an even function of δ and for $\delta > 0$, $\tan\delta > \delta > 0$.

    d)

δ	0°	5°	10°	15°	20°	25°	30°
$e(\delta)$	0	.0025	.0102	.0230	.0409	.0643	.0931

2. a) $G = 0.055$; b) $G = 0.043$

3. $$\theta(t) = \frac{\pi}{12} t + \frac{\pi}{2} , \qquad 0 \le t \le 6$$

    $$\theta(t) = \frac{\pi}{2} + \frac{\pi}{2} = \pi . \qquad t > 6 .$$

4. $$\theta_0 + t_0 K\phi_0 = \pi ; \qquad t_0\phi_0 = 3\pi$$

5. Final value is $\theta = \theta_0$ since the area under the curve $\phi(t)$ is zero.

6. a) α has dimensions of inverse time since the exponent must be dimensionless.

    b) $t = 1/\alpha$

    c) $\theta(t) = \theta_0 + \frac{K\phi_0}{\alpha} (1 - e^{-\alpha t})$

    d) $\lim_{t \to \infty} \theta(t) = \theta_0 + \frac{K\phi_0}{\alpha}$; the area under the curve $\phi(t)$ is finite.

7. $$\phi_0 = 1.5 \text{ radians} \approx 86°$$

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19 14

6. ANSWERS TO MODEL EXAM

1. a. $G = \dfrac{\ell}{\phi R} = \dfrac{2.33}{2\pi(.67)(9.14)} = 0.06$

   b. $\delta_{max} = \dfrac{\ell}{R_{min}} = \dfrac{2.33}{4.88} = 0.48$ radians $\simeq 27^{\circ}$

   c. $\phi_{max} = \dfrac{\delta_{max}}{G} = 8$ radians $\simeq 458^{\circ}$

2. a. Case B. Since $\dot{\theta} = k\phi$ the slope of the heading $\theta(t)$ in the interval $10 \le t \le 20$ must be half of what it is in the interval $0 \le t \le 10$.

   b. $K = \dfrac{VG}{\ell} = \dfrac{1}{10}$ from the proportionality between $\dot{\theta}$ and ϕ. Thus $V = 6$ m/s.

3. By differentiating the heading $\theta(t)$ one finds:

$$\phi(t) = \begin{cases} \phi_0 \dfrac{t}{t_0} & 0 \le t \le t_0 \\ \phi_0 & t_0 < t \le 2t_0 \\ 0 & t > 2t_0 \end{cases}$$

4. a. $\theta(t) = \theta_0 + \dfrac{K\phi_0 t_0}{\pi}\left(1 - \cos \pi \dfrac{t}{t_0}\right)$

   b. $\theta(t_0) = \theta_0 + \dfrac{2K\phi_0 t_0}{\pi}$

   c. $t_0 = \pi$ sec $\simeq 3.14$ sec.

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STUDENT FORM 1

Request for Help

Return to:
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55 Chapel St.
Newton, MA 02160

<u>Student:</u> If you have trouble with a specific part of this unit, please fill
out this form and take it to your instructor for assistance. The information
you give will help the author to revise the unit.

Your Name_____   Unit No._____

Page_____
  ○ Upper
  ○ Middle   OR
  ○ Lower

Section_____

Paragraph_____

OR

Model Exam
  Problem No._____

Text
  Problem No._____

Description of Difficulty: (Please be specific)

<u>Instructor:</u> Please indicate your resolution of the difficulty in this box.

  ○ Corrected errors in materials. List corrections here:

  ○ Gave student better explanation, example, or procedure than in unit.
     Give brief outline of your addition here:

  ○ Assisted student in acquiring general learning and problem-solving
     skills (not using examples from this unit.)

Instructor's Signature_____

Please use reverse if necessary.

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STUDENT FORM 2

Unit Questionnaire

Name_____ Unit No._____ Date_____

Institution_____Course No._____

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

_____Not enough detail to understand the unit
_____Unit would have been clearer with more detail
_____Appropriate amount of detail
_____Unit was occasionally too detailed, but this was not distracting
_____Too much detail; I was often distracted

2. How helpful were the problem answers?

_____Sample solutions were too brief; I could not do the intermediate steps
_____Sufficient information was given to solve the problems
_____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

_____A Lot   _____Somewhat   _____A Little   _____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much	Somewhat	About	Somewhat	Much
_____Longer	_____Longer	_____the Same	_____Shorter	_____Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

_____Prerequisites
_____Statement of skills and concepts (objectives)
_____Paragraph headings
_____Examples
_____Special Assistance Supplement (if present)
_____Other, please explain_____

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

_____Prerequisites
_____Statement of skills and concepts (objectives)
_____Examples
_____Problems
_____Paragraph headings
_____Table of Contents
_____Special Assistance Supplement (if present)
_____Other, please explain_____

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)