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AUTHOR	Thompson, Maynard; And Others
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ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to mathematical topics for the elementary teacher. In addition to an introduction to the unit and some perspectives on problem solving, the text has six activity sections titled: In Search of a Strategy; Looking for Another Strategy; To Find a Third Strategy; Now Try Your Hand; Reflection on Your Experiences; and Helping Children Solve Problems. (MP)

Alice Hart

Experiences in

PROBLEM SOLVING

Mathematics-Methods Program
units written under the direction of

John F. LeBlanc or Donald R. Kerr, Jr. or
Maynard Thompson

MATHEMATICS EDUCATION DEVELOPMENT CENTER

INDIANA UNIVERSITY

John F. LeBlanc, *Director*
Donald R. Kerr, Jr., *Assistant Director*

The following is a list of faculty and project associates who have contributed to the development of the Mathematics-Methods Program.

Mathematics Education Faculty

Frank K. Lester, Jr.
Sally H. Thomas
Paul R. Trafton
Ronald C. Welch

Project Associates — Mathematics Education

Gertrude R. Croke
Carol A. Dodd-Thornton
Nancy C. Fisher
Fadia F. Harik
Kathleen M. Hart
Tom S. Hudson
Calvin J. Irons
Graham A. Jones
Charles E. Lamb
Richard A. Lesh
Barbara E. Moses
Geraldine N. Rossi
Thomas L. Schroeder
Carol L. Wadsworth
Barbara E. Weller
Larry E. Wheeler

Mathematics Faculty

George Springer, *Co-principal Investigator*
Billy E. Rhoades
Maynard Thompson

Project Associates — Mathematics

Glenn M. Carver
A. Carroll Delaney
Alfred L. LaTendresse
Bernice K. O'Brien
Robert F. Olin
Susan M. Sanders
Barbara D. Sehr
Karen R. Thelen
Richard D. Troxel
Karen S. Wade
Carter S. Warfield
Lynnette O. Womble

Resource Teacher

Marilyn Hall Jacobson

Continued on inside back cover

EXPERIENCES IN PROBLEM SOLVING

Written under the direction of
MAYNARD THOMPSON

EXPERIENCES IN PROBLEM SOLVING

MAYNARD THOMPSON

Substantial contributors

JOHN F. LE BLANC
DONALD R. KERR, JR.
Tom S. Hudson

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

<u>Numeration</u>
<u>Addition and Subtraction</u>
<u>Multiplication and Division</u>
<u>Rational Numbers with Integers and Reals</u>
<u>Awareness Geometry</u>
<u>Transformational Geometry</u>
<u>Analysis of Shapes</u>
<u>Measurement</u>
<u>Number Theory</u>
<u>Probability and Statistics</u>
<u>Graphs: the Picturing of Information</u>
<u>Experiences in Problem Solving</u>

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugene, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

CONTENTS

INTRODUCTION TO THE EXPERIENCES IN PROBLEM SOLVING UNIT

Mathematical problem solving is obviously important to mathematicians, scientists, and engineers. In addition to these individuals for whom mathematics plays an important professional role, many other people are fascinated with mathematical problems and enjoy puzzling over them. The "Games Passengers Play" pages of airline magazines, the "Mathematical Games" section of Scientific American, and the regular features of several mass circulation periodicals such as the Sunday magazine and comic sections of newspapers are all evidence that mathematical problems, puzzles, and games have a substantial audience in the general public.

Accepting that problem solving has an intrinsic appeal for some adults, consider some possible roles for problem solving in the school curriculum:

- Enrichment activities of a problem-solving nature can be used as supplementary material.
- Standard mathematical topics can be approached in a problem-solving mode.
- The problem-solving process can be discussed as a content area; that is, an attempt can be made to understand what a problem solver does when confronted with a problem.

It should not be overlooked that for many children as for many adults there is an aspect of sheer enjoyment in solving problems. Children seem to be particularly intrigued by cleverly posed problems with ingenious or unexpected solutions.

Most problem solvers, young or old, experience a sense of achievement when they discover a successful attack on a problem which had resisted their initial efforts. It may happen that we can "almost" solve a problem, but that the last step is a very difficult one. Other times it may be that even the first step is far from obvious. One of the purposes of this unit is to provide you, and ultimately your students, with some means other than luck of "getting started" on a problem.

The unit begins with a section which introduces one point of view toward problem solving by sketching the outlines of a problem-solving process and by giving an example which illustrates it. This introductory section will gain in meaning as you proceed through the activities of the unit.

Activities 1 through 3 will provide you an opportunity to try to solve problems and to identify some useful methods for solving problems. In each of these activities you will work on a problem for a while, and then you will be given a handout which shows one method which is applicable to the problem. The first problem in each activity is selected to lead to the method on the handout; however, it may well happen that the method you develop will be different from the one on the handout. The method discussed in the handout can also help in solving the other problems of the activity. However, other methods will work on these problems too.

Activity 4 focuses on the <u>reflection</u> and <u>introspection</u> aspect of problem solving. Many students and teachers have found reflection to be a significant aid in improving problem-solving skills, and we hope you will develop a continuing habit of reflecting on the solution of a problem.

In Activity 5 you will have an opportunity to solve problems, on your own or in groups, without hints. Finally, in Activity 6 you will be presented with certain child-teaching situations in which the experiences and insights you gained in Activities 1 through 5 may be helpful. The primary purpose of the entire unit is to provide you as a prospective elementary teacher with some ideas which may enable you to help your students become better problem solvers. Actual, not vicarious, participation in problem solving is an indispensable part of your experience. Along the way you may find your own problem-solving skills improving. This is not unexpected since one of the best ways to become a better problem solver is to try to solve many and varied problems and to think about what you have done.

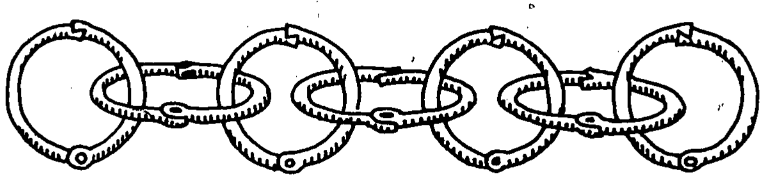
SOME PERSPECTIVES ON PROBLEM SOLVING

Before you begin the activities which form the body of the unit, we will present a framework for the systematic approach to problem solving taken in this unit. This framework can be pursued in much greater depth than is done here. Those of you who would like to do so will find an entry to the literature in the books and articles listed in the references on pages 47-48.

In this unit we concentrate on problems which are not directly connected to specific mathematical topics. Yet they are problems which require some creativity and/or originality. Such problems do not have an obvious niche in the usual mathematics content units. More than upon specific content knowledge, though such knowledge may be needed, the solution of these problems may depend upon selecting a useful approach or upon a particular insight. We have chosen such problems in order to emphasize certain aspects of the problem-solving process. An example of a problem of this type is the famous Golden Chain Problem.

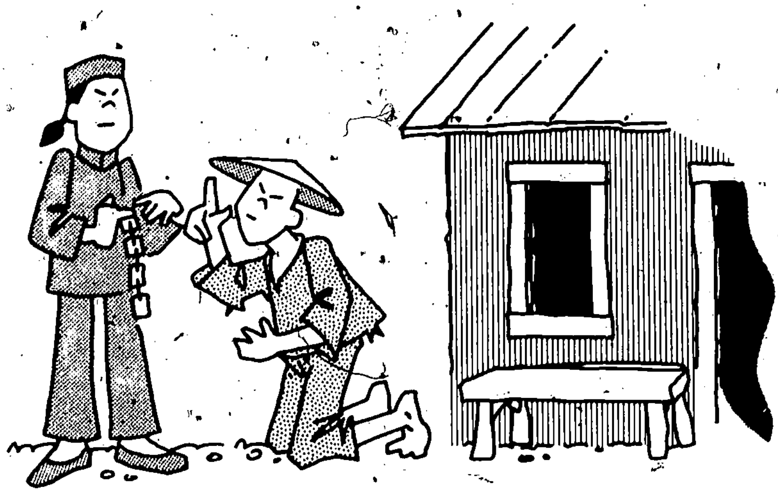
Take a moment to read the Golden Chain Problem, which appears on page 5. This problem is easily understood. Very little thought is needed to understand what it is that you are to do. Yet it is a problem which most students are unable to solve immediately. The usual method of solution is a sort of trial and error that leads (sooner or later) to a solution.

From the standpoint of the elementary school, this problem is also attractive because it is easy to find physical embodiments for it. The metal or plastic rings which are available in most bookstores will do. Children (and adults, too) find such physical representations very useful in understanding and solving problems.



Golden Chain Problem

A Chinese prince who was forced to flee his kingdom by his traitorous brother sought refuge in the hut of a poor man. The prince had no money, but he did have a very valuable golden chain with seven links. The poor man agreed to hide the prince, but because he was poor and because he risked considerable danger should the prince be found, he asked that the prince pay



him one link of the golden chain for each day of hiding. Since the prince might have to flee at any time, he did not want to give the poor man the entire chain, and since it was so valuable, he did not want to open more links than absolutely necessary. What is the smallest number of links that the prince must open in order to be certain that the poor man has one link on the first day, two links on the second day, etc.?

In this unit we will concentrate on problems of the same general character as the Golden Chain Problem, with the goal of identifying some general methods of attack. Remember that the primary objective is to provide you with a small collection of suggestions which you will be able to pass on to your students when they encounter difficulties in problem solving.

Organizing Your Work: A Framework for Solving Problems

The purpose of the next few paragraphs is to provide a framework for the problem-solving activities which comprise the bulk of this unit.*

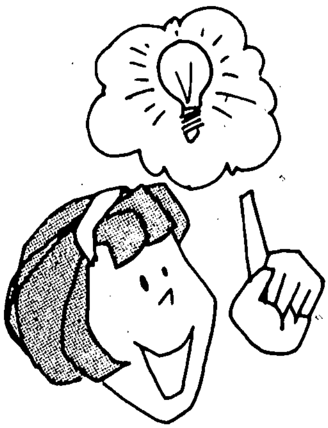
In most problems, the process of finding a solution can be divided into a number of steps or phases. Each of the steps has a somewhat different focus, but their common goal is the solution of the problem. The four steps which we will introduce are

- Understanding the problem
- Getting started with a plan
- Carrying out the plan
- Thinking back

In seeking to understand the problem, you may ask yourself: Is the statement of the problem clear to you? Are the words and mathematical symbols familiar ones whose meaning you know? It may be helpful, even at this early stage, to sketch some figures or construct diagrams or tables summarizing the information given. You may find yourself rereading the statement of the problem several times as you attempt to solve it. Such rereading may be particularly helpful after an unsuccessful attempt. At that point you may find more meaning in the statement of the problem than you did originally. (Understanding of a problem, particularly by children, may be enhanced by stating it in terms which are as interesting as possible.)

*Although there is some overlap with Section II of the Number Theory unit of the Mathematics-Methods Program, the discussion given here is self-contained.

The second step in problem solving is getting started with a plan, i.e., deciding what to do. The task of getting started is frequently the most difficult part of the whole problem-solving process. Putting the first worthwhile mark on a piece of paper can be a very hard job! It is in this second phase that the methods or strategies which are introduced in Activities 1-3, will be most helpful.



The plan of attack may change several times as attempts are made to carry it out, and a successful plan may develop only after a number of failures. Other times, a "bright idea" may give an essential clue very early in the process. In either case it usually happens that our understanding of the problem deepens as we develop a plan of attack. There is no way to be certain of stimulating these bright ideas or of deepening understanding, but experience in problem solving and thinking about the problem-solving process certainly help.

Although the solutions of many problems do not depend on specific mathematics, there is a relation between getting good ideas and knowledge of mathematics. Frequently, one can exploit similarities between problems, and these similarities can best be recognized through common mathematical characteristics. It is not necessary to know a lot of mathematics to be a good problem solver. But mathematical knowledge is rarely a hindrance (although it occasionally leads to overlooking an obvious approach, i.e., to making the problem too hard), and it is frequently a considerable aid to problem solving.

After a plan of attack has been devised, the next step is carrying it out. Following through on an idea is generally easier than creating the idea in the first place, and this phase of the problem-solving process may be more routine than the others. Perseverance is an important characteristic for a problem solver to have at this point. While proceeding with the solution it is important to keep the goal in mind and to be sure that the work is leading toward this goal. It is easy to become sidetracked with details. Check occasionally that what you are doing is contributing to a solution. It may happen that as you carry out your plan of attack you will see that it will not solve the problem. Then you must start over again, building on the experience gained.

Finally, the problem-solving process ought not end immediately when the problem is solved. The inclination to go on at once to other things should be resisted, and you should devote a few moments to <u>thinking back</u> and reflecting on your efforts. Now that you know an answer, is there an easier way to obtain it? Have you taken any detours in your approach to the problem that you now recognize as unnecessary? Does your approach seem as logical and natural now as it did when you began? Can you identify what it was that stimulated a particularly useful idea? It is a good exercise, and many times a difficult one, to try to formulate another problem that you could solve using the same method. Such activity may help you to identify useful similarities in problems in the future.

In summary, the problem-solving process has four phases:

- Understanding the problem
- Getting started with a plan
- Carrying out the plan
- Thinking back

Taken together they form the framework for our view of problem solving. We should emphasize that although there is a top-to-bottom order implied in this list, one rarely proceeds through the phases exactly once in solving a problem. Typically there is some "cycling back" to repeat some steps two or more times before the problem is solved. We return to the notion of cycling later on. The rest of the unit consists of examples and opportunities for you to utilize this framework. Activities 1, 2 and 3 provide some specific suggestions for getting started with a plan and Activity 4 focuses on thinking back.

In order to remind you of this framework for problem solving we introduce the logo

Understand
Plan
Carry out
Think back

which will be displayed at various points in the unit. If we wish to emphasize a particular step in the framework we will highlight it as shown below.

[redacted]

Think back

This should call your attention to the think-back step in the framework.

Before turning to the activities, we return to the Golden Chain Problem and view that problem in the light of framework.

Solution of the Golden Chain Problem

Almost any problem could have been used to exemplify the four phases identified above. Each problem has its individual features and the relative difficulties of the four phases will vary. Consequently, you should not expect the solution process for other problems to resemble this one except in its most general features. Before continuing, reread the statement of the problem on page 5. You may also want to try to solve the problem yourself.

The phrase "he did not want to open more links than absolutely necessary" in the statement of the problem is a crucial one. It tells us that we are to determine how many links must be cut in order that the prince may provide the poor man with one link on the first day, two links on the second day, and so on, until he has given seven links to the poor man on the seventh day. It is understood that there may be an exchange; that is, the poor man may return some links to the prince so long as he has the requisite number of links on each day. Indeed, if each link of the chain is cut, then the condition can obviously be met. Superficially, however, this approach seems to be very wasteful and we expect to be able to meet the condition with fewer cuts. The problem is to determine the smallest number of cuts which must be made to satisfy the condition.

We adopt one of the simplest possible approaches to the problem: organized trials. This is a systemized version of what you probably know as trial and error.

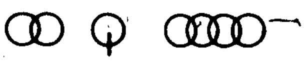
Obviously, it is impossible to satisfy the condition of the problem without making any cuts. In fact, it is impossible to satisfy the condition on day one. We continue by checking whether the condition can be met with one cut. Then with two cuts, and so on. In each case, we will consider systematically the results of cuts of specific links.

Before we consider the case of one cut it will be helpful to agree on some symbols for cut and uncut links.

cut:  uncut:  connected links (3): 

So, for example:

represents the chain with the link on one end cut, and



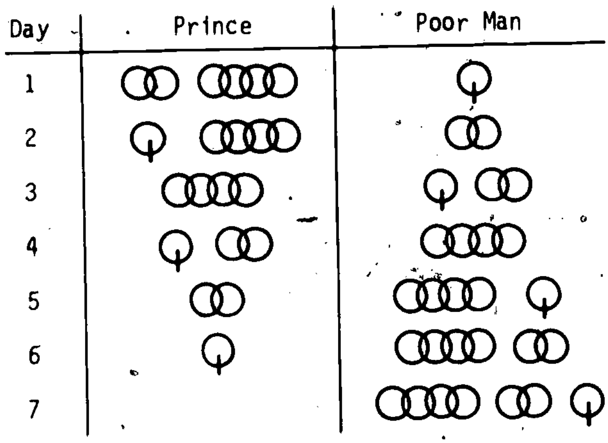
represents the chain with the third link from one end cut.

Case I: One Cut

With the notation introduced above, the pieces which result from cutting the link at one end of the chain are  and  With this cut it is impossible for the prince to give the poor man exactly two links on day two.

Next suppose that the link in second position is cut. The resulting pieces are . It is easy to see that the condition can be met on days one and two, but it is impossible for the condition to be satisfied on the third day. Try it.

To continue, we suppose that we cut the link in third position. The resulting pieces are . The table below indicates that it is now possible to meet the condition on all seven days.



Note that the problem has been solved, and we did not even have to consider Case II (making two cuts) or Case III (making three cuts).

With this systematic approach the problem was remarkably easy to solve. It was necessary to decide how to organize the trials and how to keep track of the results. Our trials were organized first according to the number of cuts and secondly according to which links were cut. In fact, it turned out to be unnecessary to proceed beyond one cut since the solution of the problem utilized a single cut. The notational device of representing uncut links by  and a cut link by  proved convenient. One might conclude from this that <u>organization</u> and a <u>convenient notation</u> facilitate problem solving.

There are other approaches which are equally effective in solving this problem, and some of them are preferable because they are generalizable. However, this approach has been adequate for our present purpose, that of illustrating the problem-solving process.

Summary

In this section the problem-solving process has been analyzed into four phases and illustrated. The reader who is interested in delving further into our analysis of the process of problem solving is referred to the sources cited in the references, especially the book, How To Solve It, by G. Polya.

DIRECTIONS:

Before continuing to Activity 1, identify the four phases of the problem-solving process in the solution of the Golden Chain Problem given above. Discuss as a class the relative importance of the various phases in this example.

Understand
Plan
Carry out
Think back

ACTIVITY 1

IN SEARCH OF A STRATEGY

Understand
Plan
Carry out
Think back

FOCUS:

This activity and the two immediately following it focus on methods for getting started with a plan in solving a problem. The idea is for you to discover a method of attack, a <u>strategy</u>, by working on your own or in small groups. Ideas whose utility you discover on your own will be much more a part of your mathematical tool kit than those communicated to you by others.

DIRECTIONS:

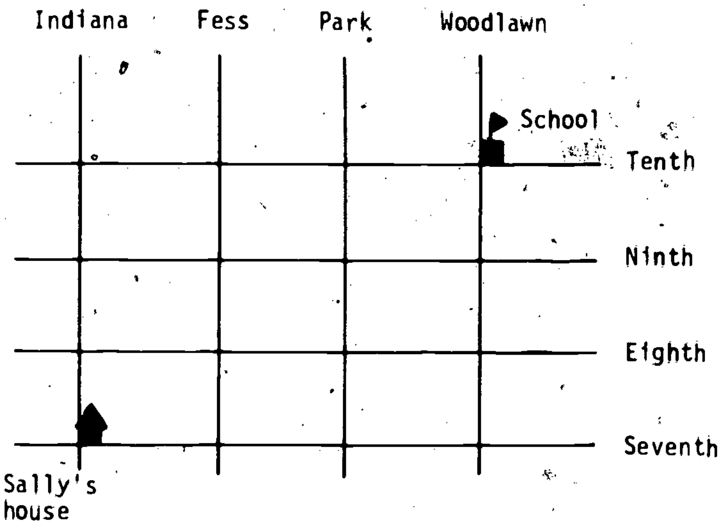
Immediately following these directions there is a list of problems. Begin working problem 1 and continue for the length of time determined by your instructor. After solving this problem, or after working on it for the specified period of time, read the Handout for Strategy 1. After reading the handout, continue with the other problems in this list, as directed by your instructor.

Although each of the problems in the list can be attacked in more than one way, there is a method which is applicable to all of the problems. It is this method that is developed in the handout. Be sure to engage in some thinking back about the way that Strategy 1 as described in the handout is applicable to each of the problems.

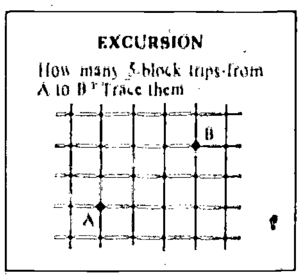
Problems for Strategy 1

1. Riding to School.

Every day Sally rides her bicycle to school along the city streets. She likes very much to ride a different way each day. Since she does not want to ride further than necessary, she rides just six blocks each day; that is the shortest possible ride from her house to school. How many days can she ride to school before she has to take a route that she has used before?

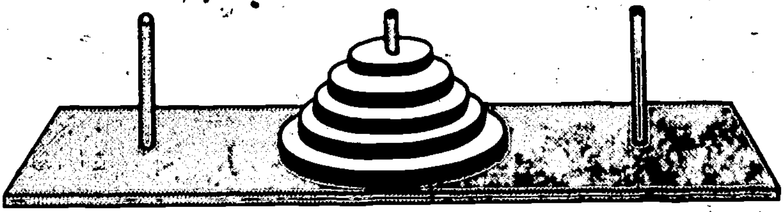


Here is a similar problem which appears in a current elementary school mathematics text.*



*Clyde A. Dilley, Walter E. Rucker, and Ann E. Jackson, Heath Elementary Mathematics, Level 5, teacher's edition, (Lexington, Massachusetts: D. C. Heath and Company, 1975), p. T277.

2. Tower of Hanoi



There are five discs stacked on a post in ascending order from largest to smallest. The discs are to be transferred to one of two empty posts using only certain kinds of moves. A move consists of taking the top disc on one post and transferring it to another post. The basic rules are the following:

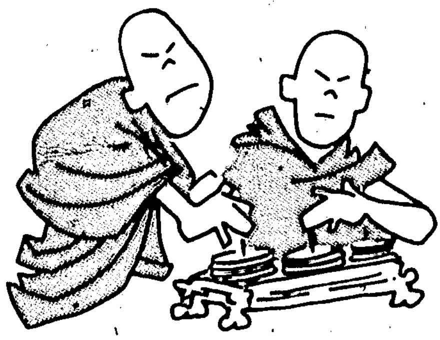
1) A disc may always be transferred to an empty post.

2) A disc may be transferred to a post occupied by another disc if and only if the disc which is transferred is smaller than the topmost disc already on the post.

What is the smallest number of moves necessary to transfer the five discs?

This is, of course, a special case of a more general question. If there were n discs arranged with the smallest on the top on one of three posts, how many moves would be necessary to move them to another post?

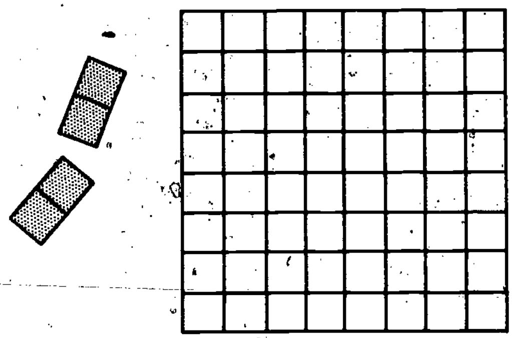
Legend has it that at the beginning of time Buddhist monks were given the task of transferring 64 golden discs from one diamond



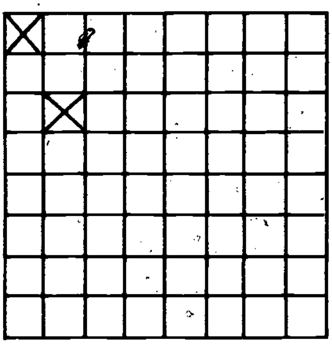
needle to one of two others. It was said that the world would end when they had completed their task. If the monks began with 64 discs and moved one a second, how long would it take them to move the discs to another needle?

3. Laying Tiles

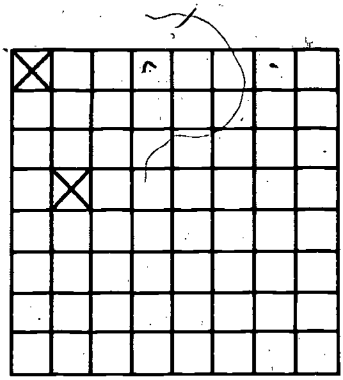
An 8 x 8 square grid with two squares removed is to be tiled with 1 x 2 tiles. Find an easy way of determining whether or



not such a tiling is possible for an arbitrary pair of omitted squares. For example, it is possible to tile the grid marked A below (X's denote squares to be omitted), but it is impossible to tile the grid marked B.

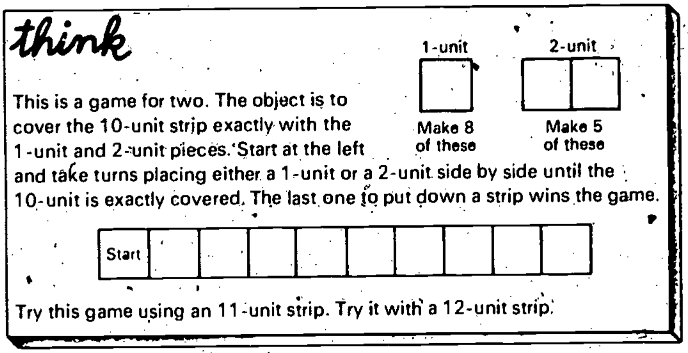


A



B

Here is another sample problem from an elementary school textbook.*


think

1-unit — Make 8 of these

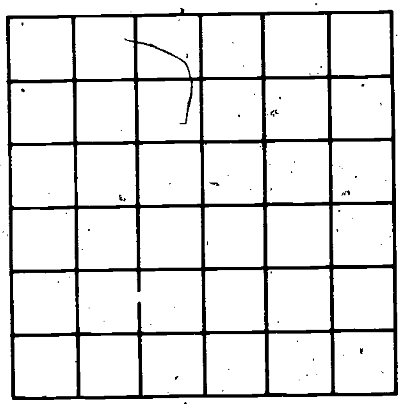
2-unit — Make 5 of these

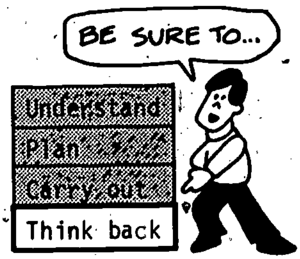
This is a game for two. The object is to cover the 10-unit strip exactly with the 1-unit and 2-unit pieces. Start at the left and take turns placing either a 1-unit or a 2-unit side by side until the 10-unit is exactly covered. The last one to put down a strip wins the game.

Start

Try this game using an 11-unit strip. Try it with a 12-unit strip.

4. How many squares are there in the figure below?





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*Robert E. Eicholz, et al., Investigating School Mathematics, Book 3, teacher's edition (Menlo Park, California: Addison-Wesley Publishing Company, 1973), p. 239.

TYPES OF PROBLEMS

In our use of the term problem we are referring to problems which have mathematical content. These problems may involve mathematics explicitly, or they may only call on the precise logical thinking that is associated with mathematics. In the elementary school curriculum, the more familiar of these types of problems are those which occur in exercise sets or in end-of-chapter reviews in textbooks. Usually, such problems are directly connected with the mathematics being discussed in the text. There are problems whose main objective is accurate computation

$$1\frac{2}{3} + 2\frac{5}{6} = \square$$

and there are story or word problems which involve the translation of a verbal problem into mathematical terms. An example of this type of problem is

> There are 68 cookies and 9 boys. If the cookies are distributed evenly among the boys, how many cookies does each boy receive? How many are left over?

At the other end of the spectrum from textbook problems are problems which require investigation into a situation which arises in another academic discipline or in the real world. Such problems may be studied as group activities for several weeks. Some problems of this sort which are appropriate for elementary school children have been prepared for school use by USMES (Unified Science and Mathematics in the Elementary School).

The problems in this unit lie between the two types identified above. These are problems which are not closely related to specific mathematical topics and which do not require a major investigation into a nonmathematical situation. They are, however, problems which highlight the steps in the problem-solving process and which tend to hold high interest for children.

ACTIVITY 2

LOOKING FOR ANOTHER STRATEGY

Understand
Plan
Carry out
Think back

FOCUS:

In Activity 1 we identified one way of getting started with a plan, namely using the simplification strategy, i.e., working a simpler but related problem. In this activity you will identify a second strategy.

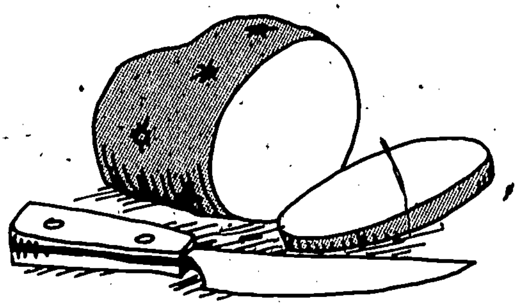
DIRECTIONS:

Proceed as in Activity 1. That is, work on problem 1 of the list which follows until you solve it or until your instructor asks you to stop. Next read the Handout for Strategy 2. Finally, work the remaining problems in the list as directed by your instructor. Be sure to think back over your work after each problem. Try to use the strategy developed in the handout, at least on some of the problems.

Problems for Strategy 2

1. Dicing a Potato

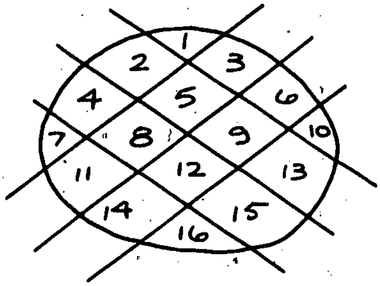
   A slice of potato, taken to be a circular slice for convenience, is to be cut up into small pieces. Although the term "dicing" usually refers to making small cubes, we will not be so restrictive and we admit pieces of any shape. Suppose that the dicing is to be carried out by placing the potato slice on a table and



cutting it up by making (straight) knife cuts through it. The pieces are not to be rearranged or piled up between cuts.

What is the greatest number of pieces which can be produced with a given number of cuts?

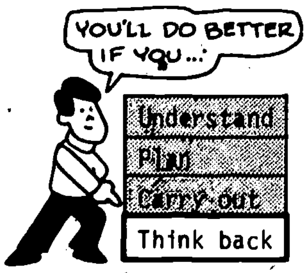
Let us ask a more specific question. The illustration shows how sixteen pieces can be obtained with six cuts. This can be substantially improved by placing the six cuts more judiciously. What is the best you can do?



Hint: In order to get started it may be helpful to adopt Strategy 1 and begin by considering a simpler problem. For example, suppose that instead of a potato slice one had a piece of string that was to be cut into a number of pieces. How many pieces of string are produced by one cut? By two cuts? By n cuts?

2. Connect the Dots

Given n points in a plane no three of which are in a straight line, how many line segments must be drawn to connect all pairs of points?



3. How many diagonals are there in a regular n-sided polygon?

4. By a <u>number pattern</u> in an array of numbers we mean a regularity in the array which can be used together with certain entries in the array to predict other entries, or to extend the array. For example, suppose we are given the array

1 2 3

One might observe that these are the first three natural numbers, and predict that the next two are 4 and 5. In this case the set 1 2 3 is viewed as a part of the larger array 1 2 3 4 5. Also, one might observe that since 1 + 2 = 3, the third entry is the sum of the preceding two. Therefore one might view the given array as a portion of the larger array 1 2 3 5 8 (5 = 2 + 3, 8 = 3 + 5).

i) Find at least five number patterns in the array

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

ii) Continue the array for two more rows using one or more of the patterns you found in i).

iii) Continue the array shown below for two more rows using a number pattern you find in the displayed rows.

```
        1
      1   1
    1   3   1
  1   5   5   1
1   7   13  7   1
```

iv) Same as iii) for the array

				1				
			1		1			
		2		3		2		
	4		8		8		4	
8		20		26		20		8

v) Continue the array shown below for one more row using a number pattern you find in the displayed rows.

				1				
			1		2			
		1		4		6		
	1		6		16		22	
1		8		30		68		90

vi) Create a triangular five-row array using a number pattern that you invent on your own. Ask a classmate to find the pattern and to extend the array by one or two rows.

5. Solve problem 2 of Activity 1 using the methods developed in this activity.

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AN EXAMPLE FROM THE CLASSROOM

In order to illustrate the usefulness of selected hints in helping a child solve a problem, we recount a problem-solving experience of a group of fifth graders. The problem was posed to them as follows:

> Fifteen couples have been invited to a birthday party. The host has several small card tables that can seat one person on a side. He plans to set the small tables end to end to make one long table to seat all the guests. How many of the small tables will be needed to seat the fifteen couples?

When the problem was first posed, the students guessed wildly with no concern for justifying their guesses. One student misinterpreted the problem and tried to find the number of tables needed to seat 60 guests. The teacher aided the student by suggesting that they look at a simpler problem. He asked them, "How many people could be seated at one table?" "... at two tables set end to end?" "... and at three tables set end to end?" Although there were some initial disagreements among the children, they finally settled on the numbers 4, 6, and 8.

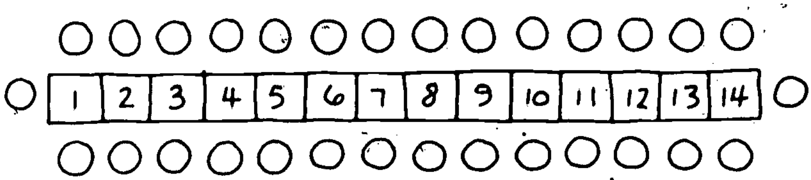
Some of the children had difficulty in understanding that the tables seat one person on a side and that the tables were to be put together to make one long table. The teacher drew some pictures, then suggested that the students organize their findings in a table of information:

Number of Tables	1	2	3	4	5	...
Number of People	4	6	8			

When the teacher asked about 4 tables, the children replied 10 people. They gradually noticed that as one table was added, two additional people could be seated. That is, they noticed the "1 more table = 2 more people" pattern in the table of information. With this clue they were able to complete the table of information to the entry corresponding to 30 people.

Number of Tables	1	2	3	...	14
Number of People	3	6	8	...	30

The children then checked their answer by drawing a diagram of 14 tables and verifying that in fact 30 people could be seated at a long table formed from 14 small tables.



In working with children on this problem the teacher pointed out one method of attacking the problem, or one <u>strategy</u>. The strategy, that of trying a simpler but related problem, was the topic of Activity 1.

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ACTIVITY 3
TO FIND A THIRD STRATEGY

FOCUS:

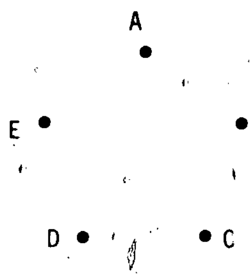
We have identified the strategies of working a simpler but related problem (Activity 1) and of using number patterns and formulas (Activity 2). In this activity you will identify a third strategy which will be useful in getting started with a plan in solving a problem.

DIRECTIONS:

Proceed as in Activity 1. That is, work on problem 1 of the list below for a period of time determined by your instructor. After solving the problem, or when your instructor asks you to, read the Handout for Strategy 3. After reading the handout, continue working the other problems in the list. An important part of the activity is thinking back about how the strategy was used on each problem.

Problems for Strategy 3

1. Connect the Dots



There are five dots arranged in a pentagon. In how many different ways can four or fewer straight line segments connecting the dots be drawn?

Before we can hope to solve the problem we must decide what the phrase "different ways" in the statement of the problem means. This is an important part of the first phase in problem solving, i.e., understanding the problem. Let us agree that two ways of connecting dots with line segments are not "different" if the dots in one figure can be relabelled in such a way that the connections are the same as in the other figure. Remember that it is the connections that are important, not the way the figure is drawn. For example, the connections illustrated in figures 1, 2 and 3 are not different.

Understand
Plan
Carry out
Think back

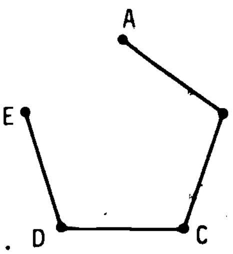


figure 1

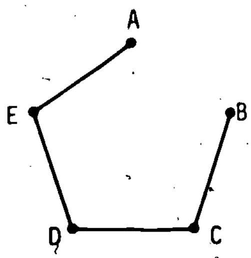


figure 2

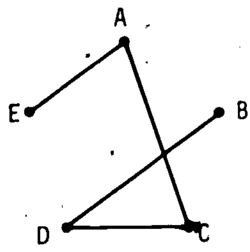
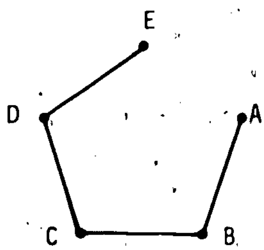


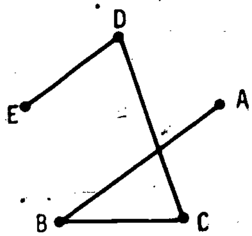
figure 3

Indeed, if we relabel figure 2 by changing B to A, C to B, D to C, E to D, and A to E, then the relabelled figure 2, shown below, is the same as figure 1. You should actually cross out the labels on figure 2 and relabel the points to verify that the relabelled figure is correct.



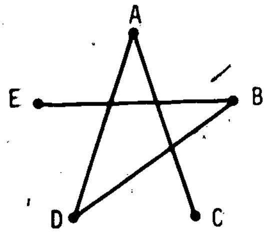
relabelled figure 2

Also, if we relabel figure 3 by changing B to A, D to B, A to D, and leaving C and E unchanged, then the relabelled figure 3, shown below, is the same as figure 1. By this we mean that dots with the same letters are connected in figures 1 and 3; i.e., A is connected to B, B is connected to C, C is connected to D, and D is connected to E.



relabelled figure 3

Can you show by relabelling that the figure given below is not different from figures 1, 2 and 3?



On the other hand, the two connections illustrated in figures 4 and 5 are different. There is no way to relabel the dots in figure 4 so as to obtain the connections shown in figure 5. The reason is that there is a dot in figure 4 which is connected to three other dots, and there is no such dot in figure 5.

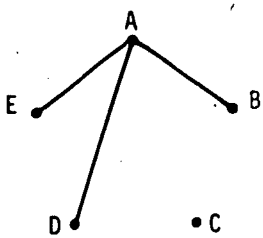


figure 4

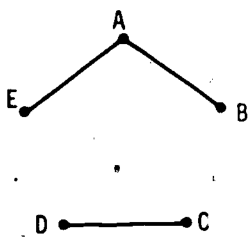


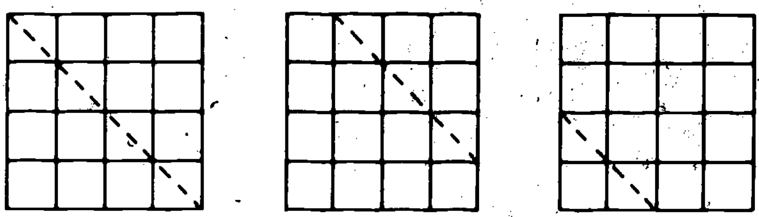
figure 5

With this preliminary discussion, you should be ready to begin work on problem 1. Also, as a result of this discussion you should realize that understanding the problem may not always be a completely straightforward phase of problem solving. Even a simple phrase such as "different ways" may require thoughtful analysis.

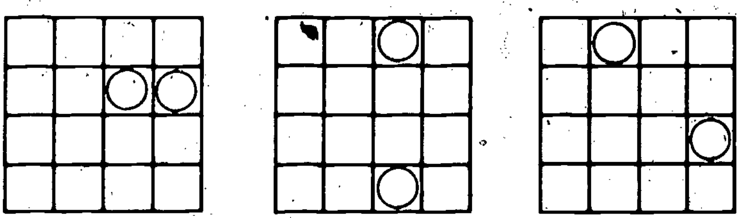
2. Placing Pennies

Place four pennies on a 4 x 4 grid in such a way that no two are in the same row, column, or diagonal.

Two pennies are in the same row if they are in the same horizontal line; they are in the same column if they are in the same vertical line. Three of the five upper left to lower right diagonals are shown by the dashed lines below. There are also five upper right to lower left diagonals.

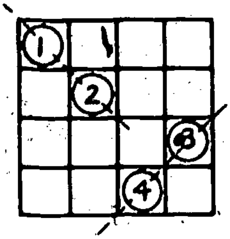


For example, the two pennies in figure A are in the same row, the two pennies in figure B are in the same column, and the two in figure C are on the same diagonal.

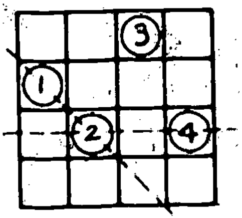


A B C

The arrangements shown below do not satisfy the conditions for the reasons given:

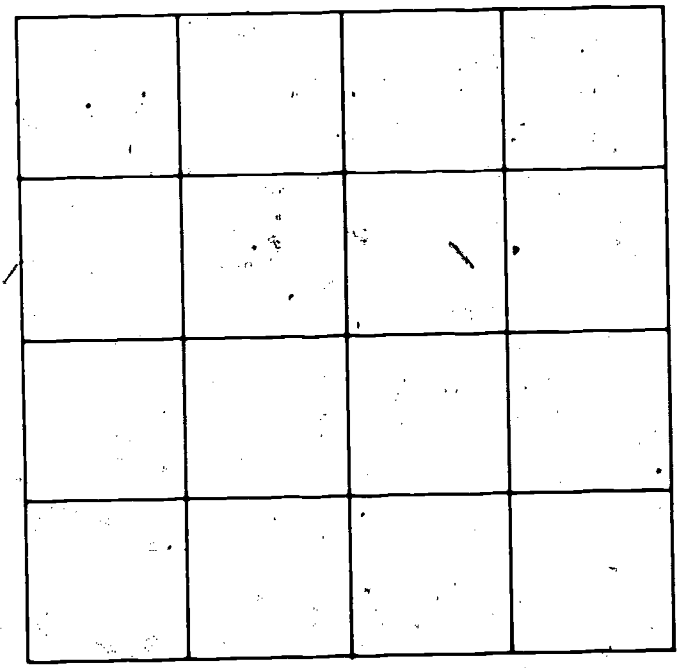


Pennies 1 and 2 are on the same diagonal, as are pennies 3 and 4.



Pennies 1 and 2 are on the same diagonal, and pennies 2 and 4 are in the same row.

Here is a grid which will accommodate pennies, if you wish to experiment.

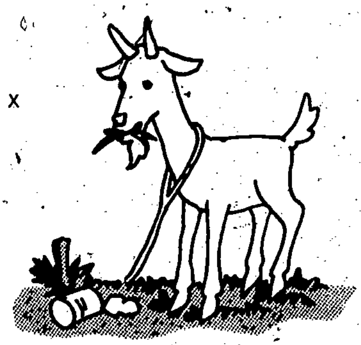


3. Sums Divide Products

Are there two different prime numbers m and n such that m + n divides m x n? If so, give examples; if not, then explain why not.

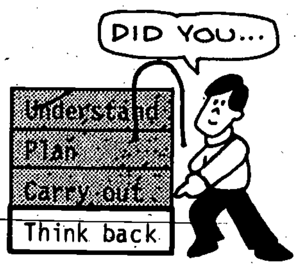
4. The Grass is Greener

A goat is tied at the corner of a 20' x 40' barn with a 50' rope. If it can graze at any spot outside of the barn to which its rope can reach, what is the size of its grazing area?



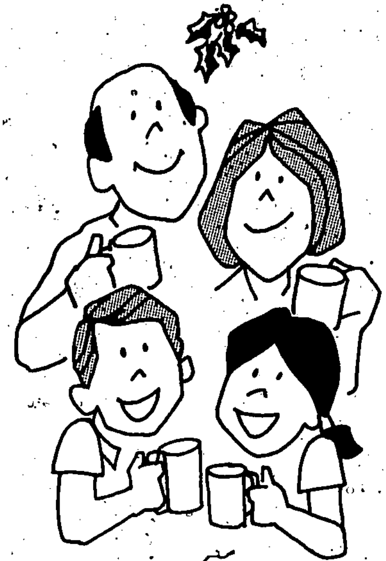
5. Vegetables

Several of the twelve teachers in the local elementary school plan to try to grow vegetables in their science classes. If six teachers plan to grow beans, eight plan to grow corn, and three plan to grow both, how many plan to grow neither beans nor corn?



6. A Cup of Good Cheer

A family of four--two parents, a son and a daughter--have a set of Christmas cups. One evening they observed that Mother had the cup decorated with candles, Father had the one decorated with holly, the son had the one decorated with carolers, and the daughter had the one decorated with angels. In how many ways can the cups be distributed the next evening so that no individual has the same cup?



ACTIVITY 4

NOW TRY YOUR HAND

FOCUS:

In Activities 1-3 three specific strategies were introduced and illustrated: work a simpler but related problem, use number patterns and formulas, decompose the problem into special cases. In this activity you will have an opportunity to try your hand at problems without knowledge of which strategy (if any) may be helpful. Some of these problems can be solved by using any of several strategies, and some are best solved using methods which are somewhat different than those introduced in this unit. The main purpose of the activity is for you to engage in problem solving with the experience of the first three activities behind you.

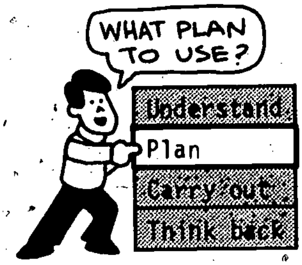
DIRECTIONS:

Try to solve those problems in the following list suggested by your instructor. You will be given directions regarding the preparation of your work, either to hand in or for oral reporting. The problems vary considerably in difficulty so you should anticipate encountering some that are quite easy and others that appear more difficult.

1. Missionaries and Cannibals

Three missionaries and three cannibals wish to cross a river. There is a boat which can carry three people, and either a missionary or a cannibal can operate the boat. It is never permissible

for cannibals to outnumber missionaries, neither in the boat nor on either shore.



- What is the smallest number of trips necessary to make the crossing?
- How many trips are necessary if the boat holds only two people?

2. Three Bears*

Once upon a time there were three bears: Lester, (a large bear), Mildred (a middle-sized bear), and Bashful (a small bear), and three pots of honey: a large pot, a middle-sized pot, and a small pot. One day they found themselves carrying the pots of honey down a path. Lester was carrying the middle-sized pot, Mildred was carrying the small pot, and Bashful was carrying the large pot. Since this was clearly an inequitable arrangement, they proceeded to pass the pots around until each had the pot proportional to his (her) size.

*Adapted from an idea of Richard Hayes.

The rules of pot-passing in Bearland are quite complicated. They are:

i) Only one pot can be passed at a time.

ii) If a bear is holding two pots only the larger of the two may be passed.

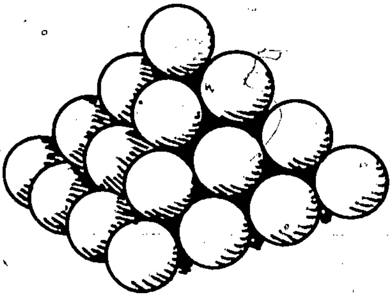
iii) A pot may not be passed to a bear who is holding a larger pot.

By what sequence of passes of pots can the bears solve their problem?

3. Marking a Die

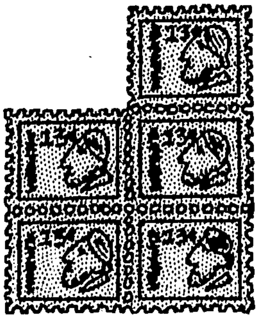
On an ordinary die the numbers 1 and 6, 2 and 5, and 3 and 4 are on opposite sides. In how many different ways can a die be marked subject to this condition?

4. Stacking Marbles



A number of marbles are stacked in a triangular pyramid. How many are in the nth layer? How many marbles are in the top n layers?

5. Buying Stamps



When stamps are purchased at the post office, they are usually attached to each other. In how many ways can five stamps be attached to each other?

6. Making a Calendar

Can single digits be painted on the faces of two cubes so that the cubes can be placed so as to show each of the first 31 numbers? If so, determine how; if not, determine how many cubes are needed.

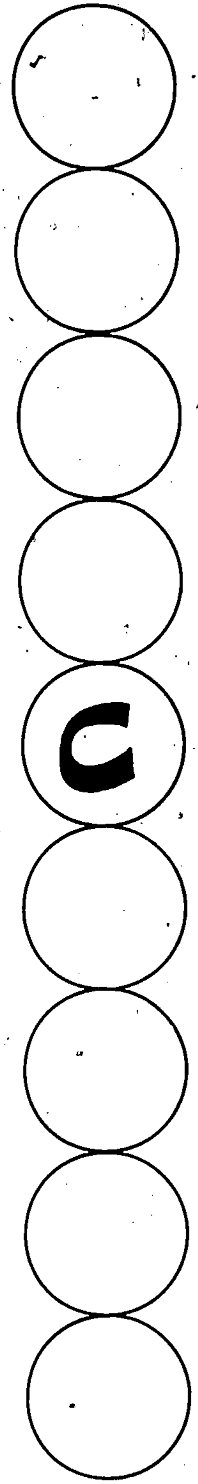
7. Coin Puzzle

Place four pennies on the circles above the center circle (marked C), and place four dimes on the circles below the center circle. The object of the puzzle is to move all the coins on the upper circles to the lower ones and all the coins on the lower circles to the upper ones. The moves of the coins are subject to the following restrictions:

i) Coins can only move forward--that is, away from their starting position.

ii) A coin can only move to an adjacent empty circle or jump over an individual <u>enemy</u> piece onto an empty circle.

What is the smallest* number of moves necessary to accomplish the objective?



8. Bouncing Ball

A Sooper-Dooper Ball is dropped from a window which is 16 feet from the sidewalk. Each time the ball bounces it travels half as high as on the previous bounce, or half its initial height in the case of the first bounce. The ball is caught when it bounces exactly one foot from the sidewalk. How many times did the ball bounce and how far did it travel?

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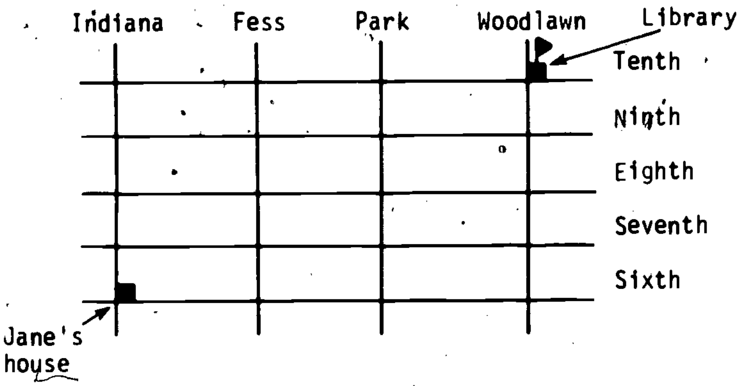
*The number of moves needed to solve the puzzle is unique for a given number of coins on each side of the center. That is, if you and your neighbor both solve the puzzle (legally) then you both used the same number of moves.

9. Across the Desert

Several tourists are stranded at a camp in the desert, and they estimate that it will take six days for one of them to reach the nearest outpost. Each person can carry a supply of food and water sufficient for four days travel, and consequently no individual can make the trip to the outpost alone. However, it is possible for more than one person to start out from the camp and after a time transfer food and water to a person (or persons) who will continue while the rest return to camp. What is the smallest number of people who should start out if one of them is to make it to the outpost and the others return safely to camp? Where should the transfer of food and water take place and how much should be transferred?

10. Riding to the Library

Suppose that Jane's house and the library were located as shown below. How many days can she ride her bicycle to the library before she is forced to take a route she has taken before?



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ACTIVITY 5

REFLECTION ON YOUR EXPERIENCES

FOCUS:

It has been mentioned and repeated several times for emphasis that reflection (or thinking back) on your work is valuable to the development of problem-solving skills. In this activity you will be asked to focus on this aspect of problem solving.

DISCUSSION:

Understand
Plan
Carry out
Think back

In addition to the value gained by thoughtfully reviewing the ideas and methods you used to solve a problem, it is also helpful to organize your thoughts in such a way that they can be communicated to others. Most of us have had the experience of having a very clear picture in our own minds as to how we solved a problem, but being unable to communicate our ideas to a friend or colleague. As prospective teachers, it is important to develop communication skills, both verbal and written; and these skills are helpful in sharing problem-solving experiences. For most beginners, verbalizing the problem-solving process is a demanding task.

DIRECTIONS:

As a homework assignment prepare answers to questions 1 and 2 below. You may be asked to present your analysis of a problem in a class discussion on the value of thinking back.

1. Select one of the problems you worked in Activities 1-4 other than those discussed in the handouts. Analyze carefully the process you used to solve this problem and organize your work using the framework outlined in the section on perspective. Summarize your analysis in a one-page

Understand
Plan
Carry out
Think back

description which would be useful to someone who has thought about the problem less than you have. Be prepared to present your summary orally to the class.

2. Create a new problem which can be solved using a process very similar to that summarized in your answer to 1).

ACTIVITY 6

HELPING CHILDREN SOLVE PROBLEMS

FOCUS:

The methods introduced in this unit and related ones can be used in the elementary school to help children solve problems. The focus of this activity is to help you prepare for this use.

DIRECTIONS:

Prepare responses to the following questions. These questions will provide the central theme for a class discussion of how elementary school students can be helped to become better problem solvers.

1. Consider the following situation.

   Jimmy, a fourth-grader, is working on problem 8 on page A (p. 41). He has found that ten blocks are needed for a staircase with four steps, but he is stumped on part (b) of the problem. What specific suggestion(s) could you make which might help Jimmy solve the problem?

2. What hint could you give to a student who is having trouble answering question 8 on page B (p. 42)? Is your hint related to any of the three problem-solving strategies focused on in this unit?

3. On page C, what sort of strategy is used to obtain a formula for the area of a right triangle? Do you think you could encourage problem solving in your classroom by asking some of your students to find the area of a specific right triangle earlier in the year before the topic was discussed in the text?

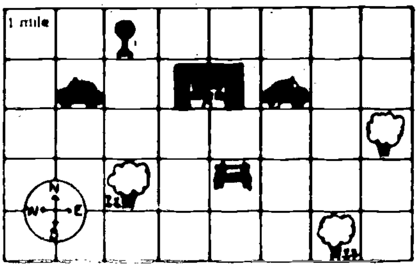
4. What strategy (or strategies) does the text on page D use to guide the student to find the number of straight lines connecting the 12 points? How is the pattern on page E related to the

pattern on page D? What suggestion could you make to a child if he does not see a pattern even after making a table?

5. What sort of strategy might be useful in solving the "think" problem of page F?

6. There is value in student's "floundering around" a bit while working on a problem, and premature hints can detract significantly from the child's learning experience. Write a one or two paragraph essay on the use of hints as a teaching device. (Think back on your own experience with hints.)

Scott, Foresman and Company, Mathematics Around Us; Skills and Applications, Grade 4, page 251. Reproduced with permission.



4. A blue taxi left the garage and went:

3 miles west
1 mile north
4 miles east
3 miles south
2 miles east

Now the blue taxi is miles south and miles east of the garage.

5. A red taxi left the same garage and went:

4 miles east
3 miles south
6 miles west
4 miles north
1 mile west

a. Now the red taxi is miles (east, west) and miles (north, south) of the garage.

b. The red taxi is miles (east, west) and miles (north, south) of the blue taxi.

6. At the end of a race:

Felix was 6 yards behind Ben.
Larry was 5 yards behind Felix.
Steve was 8 yards in front of Larry.

a. Who won the race? How far was the winner ahead of each of the others?

b. Who came in last? How far was he behind each of the others?

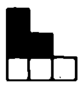
7. At the end of another race:

Carol was 7 yards ahead of Meg.
Ramona was 5 yards behind Carol.
Meg was 10 yards behind Jean.

a. Who won the race? How far was the winner ahead of each of the others?

b. Who came in last? How far was she behind each of the others?

8. Six blocks are used in a staircase that has 3 steps.



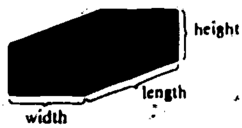
a. How many blocks are needed for a staircase that has 4 steps?

b. Can you make a staircase with 28 blocks? With 34 blocks?

Volume of a rectangular solid

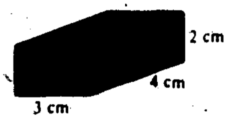
You can find the volume of a rectangular solid by multiplying the **length**, the **width**, and the **height**.

Formula.



$V = l \times w \times h$

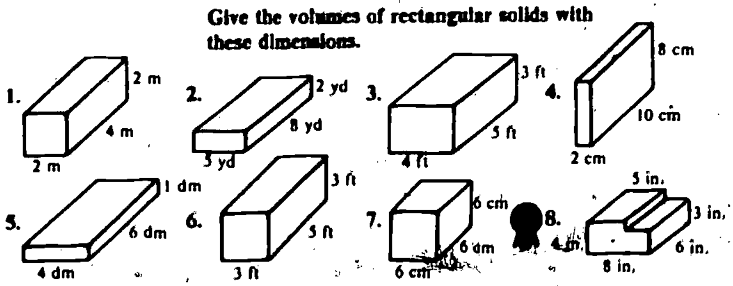
Example.



$V = l \times w \times h$
$V = 4 \text{ cm} \times 3 \text{ cm} \times 2 \text{ cm}$
$V = 24$ cubic centimeters

EXERCISES

Give the volumes of rectangular solids with these dimensions.



Give the volumes of these rectangular solids.

	LENGTH	WIDTH	HEIGHT	VOLUME
9.	5 ft	7 ft	4 ft	
10.	10 yd	10 yd	10 yd	
11.	6 m	9 m	7 m	

140

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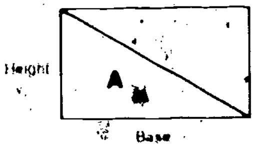
Finding the Area of a Right Triangle

Ed drew triangle A on a sheet of paper with square corners.



The rectangle and triangle A have the same base and the same height.

The area of triangle A is one-half the area of the rectangle.

Area of rectangle = base × height

Area of triangle = $\frac{1}{2}$ × base × height

Ed measured his triangle and found its area.

Base 15 in.
Height 9 in.

$\frac{1}{2} \times 15 \times 9 =$

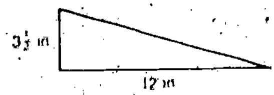
$\frac{1}{2} \times \frac{15}{1} \times \frac{9}{1} = \frac{135}{2} = 67\frac{1}{2}$

Area of Ed's triangle = $67\frac{1}{2}$ sq. in.

Take a sheet of paper with square corners.
Draw a right triangle.
Find the area of your triangle.

Find the area of each right triangle.

1.

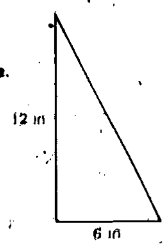


$\frac{1}{2} \times 12 \times 3\frac{1}{2}$

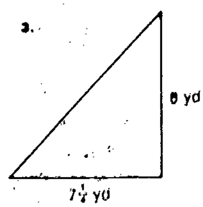
$\frac{1}{2} \times \frac{12}{1} \times \frac{7}{2} = \frac{84}{4}$

Area = ___ sq. in.

2.



3.



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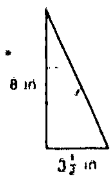
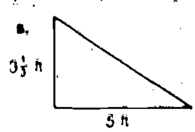
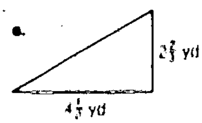


5.



6.



PAGE D

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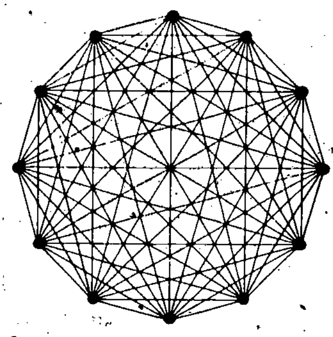
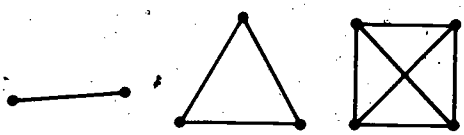
Using Pictures to Solve Problems

Start with 12 points. Draw straight lines to connect each point with all the other points.



1. Without counting, guess the number of lines.

To find the exact number of lines, first look at some simpler problems.



2 points 1 line

3 points 3 lines

4 points 6 lines

2. Draw points and count lines
   a. for 5 points.
   b. for 6 points.
   c. for 7 points.
3. Record your results in a table like the one shown at the right.
4. Find a pattern. Complete the table to find the number of lines for 12 points.
5. How many lines for 15 points? For 20 points? For 24 points?

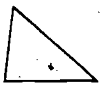
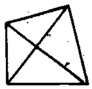
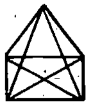
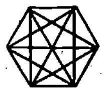
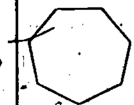
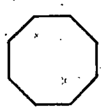
Number of points	Number of lines
2	1
3	3
4	6
5	
6	
7	
8	
9	
10	
11	
12	

2
3
4

Scott, Foresman and Company, Mathematics Around Us; Skills and Applications, Grade 5, page 299. Reproduced with permission.

SIDE TRIP Diagonals of Polygons

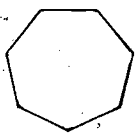
Mr. Le Blanc's class recorded the number of diagonals that certain polygons have.

Number of sides	3 sides	4 sides	5 sides	6 sides	7 sides	8 sides
						
Number of diagonals	0	2	5	9		

Roger thought he could guess the next number without drawing the diagonals. He guessed "14."

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22

1. Copy this polygon and draw all the diagonals. Was Roger's guess correct?



Find the pattern. Use it to answer these questions.

2. How many diagonals does an 8-sided polygon have?
3. How many diagonals does a 9-sided polygon have?
★4. How many diagonals does a 10-sided polygon have?

PAGE F

Addison-Wesley Publishing Company, Investigating School Mathematics, Book 3, page 306. Reproduced with permission.

Keeping in Touch with Addition, Subtraction, Multiplication, Division, Story problems

1. Find the sums, products, quotients, and differences.

A $94 + 39$	B 68×3	C $78 - 52$	D $81 + 79$	E 63×5	F 79×6
G $27 + 88$	H 56×8	I $65 + 99$	J 93×7	K $142 - 80$	L $125 - 52$

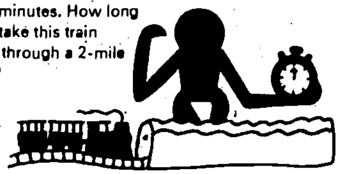
M $65 - 23$ N $350 \div 7$ O $350 \div 5$ P $540 \div 9$ Q $120 \div 3$

2. Tell what operation (+, −, ×, ÷) you think of for:
   - A putting 2 sets together and finding the total number.
   - B finding how many are left after some have been taken away.
   - C finding how many sets of a certain size we get from a set.
   - D finding how many in a certain number of rows of the same number.
   - E finding how many more one set has than another.
   - F finding how many ways we can pair objects in 2 sets.
   - G finding how many rows when we put a set into rows having the same number.

3. Solve the equations.
   - A $n + 6 = 11$
   - B $8 + n = 15$
   - C $3 \times n = 18$
   - D $n \times 8 = 24$
   - E $50 \div 5 = n$
   - F $8 - n = 6$
   - G $10 - n = 2$
   - H $42 \div n = 6$
   - I $n - 8 = 6$
   - J $n \div 6 = 5$
   - K $18 + n = 24$

think

A train that is 1 mile long is traveling 1 mile each 3 minutes. How long does it take this train to pass through a 2-mile tunnel?



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REFERENCES

The literature on problem solving is vast and rapidly expanding. We have listed a few items which may serve as an entry into the field.

George Polya is the acknowledged master of mathematical problem solving. His most accessible book is

> How to Solve It, 2nd edition, Princeton, New Jersey: Princeton University Press, 1957.

His other works on problem solving include

> Mathematics and Plausible Reasoning, two volumes: Introduction and Analogy in Mathematics (Vol. I) and Patterns of Plausible Inference (Vol. II), Princeton, New Jersey: Princeton University Press, 1954.

> Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving, two volumes, New York: John Wiley & Sons, Inc., 1962 (Vol. I), 1965 (Vol. II).

Other references:

NCTM. Mathematical Thinking, Unit 6 of Experiences in Mathematical Discovery. Washington, D.C.: NCTM, 1971.

Seymour, Dale and Margaret Shedd. Finite Differences: A Problem Solving Technique. Palo Alto, California: Creative Publications, Inc., 1973.

Walter, Marion I. Boxes, Squares and Other Things. Washington, D.C.: NCTM, 1970.

Butts, Thomas. Problem Solving in Mathematics. Glenview, Illinois: Scott Foresman, 1972.

There are many excellent collections of problems. One of the classics is the following collection by Dudeney:

Dudeney, H. E. Amusements in Mathematics. New York: Dover Publications, 1958.

Several of the books listed below contain references to other sources.

Frohlichstein, Jack. Mathematical Fun, Games, and Puzzles. New York: Dover Publications, 1962.

Gardner, Martin. Scientific American Book of Mathematical Puzzles and Diversions. New York: Simon and Shuster, 1959.

Gardner, Martin. 2nd Scientific American Book of Mathematical Puzzles and Diversions. New York: Simon and Shuster, 1961.

Ranucci, Ernest R. Four by Four. Boston: Houghton Mifflin Co., 1968.

REQUIRED MATERIALS

ACTIVITY	HANDOUTS	OTHER MATERIALS (OPTIONAL)
Perspectives		7 metal or plastic notebook rings per group.
1	Handout for Strategy 1 (available in the Instructor's Manual).	Rectangular grid paper, commercial Tower of Hanoi puzzles and/or disks or blocks which could be used for that purpose, 8 x 8 square grids and sets of 1 x 2 tiles.
2	Handout for Strategy 2 (available in the Instructor's Manual).	Straightedges or rulers.
3	Handout for Strategy 3 (available in the Instructor's Manual).	4 pennies or counters per student, compasses and rulers, blocks or markers in four colors.
4		Blocks or markers in three colors or sizes, several cubes on which numbers can be marked, marbles or blocks for building a triangular pyramid, pennies and dimes or counters in two colors, rectangular grid paper.

Continued from inside front cover

Project Associates — Audio-Visual

Stephen B. Walter, *Design Coordinator*
Wallace S. Goya
John W. Hiner
Deane W. Hutton
Jean A. Lescohier
Carrie Hendron Lester
James L. McKittrick
Alexander C. Millar
Kyu-Sun Rhee
Vicki J. Semler
Donna J. Toler
Anne S. Walker

Staff

Susan E. Coté, *Administrative Assistant*
Susan J. Bakshi
Joyce A. Ford
Jane Kaho
Linda A. Svrcek

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. . . .

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Addition and Subtraction, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Transformational Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Number Theory,* and *Probability and Statistics.*

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