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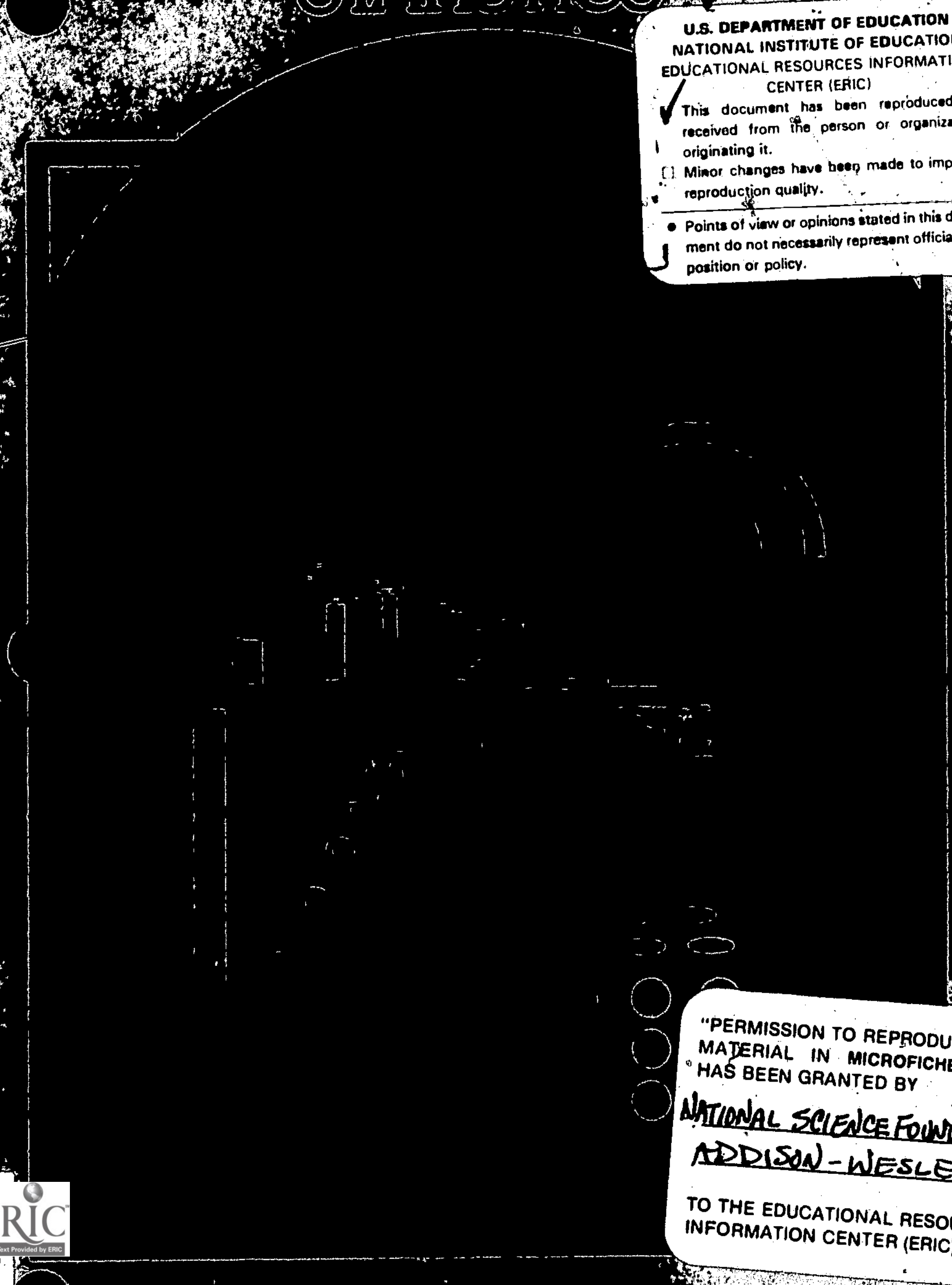
ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to mathematical topics for the elementary teacher. In addition to an introduction to the unit and an overview, the text has sections on basic probability and its role in the elementary school, basic statistics and its role in the elementary school, and winding up (a review and extensions). (MP)



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PROBABILITY AND STATISTICS

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PROBABILITY AND STATISTICS

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

Numeration

Addition and Subtraction

Multiplication and Division

Rational Numbers with Integers and Reals

Awareness Geometry

Transformational Geometry

Analysis of Shapes

Measurement

Number Theory

Probability and Statistics

Graphs: the Picturing of Information

Experiences in Problem Solving

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugene, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

CONTENTS

INTRODUCTION TO THE PROBABILITY AND STATISTICS UNIT

Children of elementary school age encounter probabilistic phenomena in the world of nature, in their social contacts, and in the games they play. It is both useful and interesting for them to consider these situations as examples of mathematics in the real world and to study them in a systematic way. Although most of the activities used in the school deal primarily with actual experiments, it is necessary that the teacher understand the basic mathematical concepts in order to guide the children's development in fruitful directions.

The unit begins with two essays. The first is an overview which gives examples of the occurrence of probabilistic concepts in the real world and instances of related elementary school activities. The second focuses directly on teaching probability in the elementary school.

Section I is concerned with the basic concepts of probability and with the role of the subject in the elementary school. It begins with experiences involving materials and techniques which can be used to generate data with various random features. Many of these experiences can be transferred almost directly into the elementary classroom. The fundamental concepts of <u>sample space</u>, <u>event</u>, and <u>probability</u> are introduced in Activities 2 through 6. Activity 7 focuses on the child's perspective of randomness. Activity 8 includes examples of games involving probabilistic notions, and Activity 9 considers the important idea of the simulation of random phenomena. A seminar on the pedagogical issues which arise in teaching probability concludes Section I.

The basic concepts of statistics are introduced in Section II. The tasks of collection, organization, and analysis of data are the central topics of these activities. Section II concludes with a seminar on the teaching of statistics.

Three of the mathematical ideas introduced in Sections I and II are pursued in Section III. Counting problems, which arose in computing probabilities in Section I and which are of significant independent interest, are discussed in more detail in Activity 15. In Activity 16 the ideas of conditional probability and independence are discussed. These ideas are frequently helpful in understanding simple experiments. Finally, the concept of expected value, a generalization of the simple mean or average, is introduced in Activity 17.

> *"Our system of education tends to give children the impression that every question has a single answer. This is unfortunate because the problems they will encounter in later life will generally have an indefinite character. It seems important that during their years of schooling children should be trained to recognize degrees of uncertainty, to compare their private guesses and extrapolations with what actually takes place--in short, to interpret and become masters of their own uncertainties."*
>
> --John Cohen

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OVERVIEW

FOCUS:

This overview contains a selection of examples of the use of ideas with a probabilistic or statistical basis to help understand the real world. It also identifies some of the ways in which probability and statistics occur in the elementary school.

MATERIALS:

(Optional) The Mathematics-Methods Program slide-tape presentation entitled "Overview of Probability and Statistics."

DISCUSSION:

This overview serves two functions. First, it introduces in an informal manner some of the basic ideas which are investigated in this unit. Second, it provides appropriate background and an advance assignment for Activity 10.

DIRECTIONS:

Read the two essays "Overview of Probability and Statistics," which begins on page 5 (or view the slide-tape with the same title), and "Teaching Probability in the Elementary School", p. 9. Next, engage in a brief discussion of some of the points raised. The following questions can serve as a basis for the discussion. They should be read before reading the essays (or viewing the slides).

1. What are some of the benefits to children of developing a clearer understanding of uncertainty and random events?

2. What are some of children's out-of-school activities in which ideas of probability occur?

3. Discuss the influence of the hand calculator on the range of statistical problems accessible to elementary school children.

4. What other topics from the elementary school mathematics curriculum might be reinforced through a study of probability and statistics?

5. Discuss in detail one real-world application of probability or statistics.

6. Check to see if two people in your class have the same birthday. Your instructor will lead a discussion on this famous problem.

ASSIGNMENT:

Using the essay "Teaching Probability in the Elementary School," pp. 9-12, or the references provided after the essay as a guide, prepare brief answers to the following questions. This assignment will be discussed in connection with Activity 10.

1. Why is the playing of games important in early probability activities?

2. The collection, organization, and interpretation of data frequently serve as the focus for the study of statistics in the elementary school. Is this approach an appropriate one for elementary school children? Give an example of an activity involving the collection, organization, and interpretation of data which is set in the child's real world. You may find the USMES materials a useful reference.

3. After teaching a unit on probability in the sixth grade you are approached by a concerned parent who asks about the role of dice and spinners in a mathematics program. What points would you raise to support the teaching of probability and statistics in the elementary school and, in particular, the use of dice and spinners as instructional aids?

4. In a faculty meeting a teacher expresses the opinion that although statistics is a subject with which every child should have some familiarity, it is more appropriate for middle school or high school and should not be included in the elementary curriculum. How would you respond?

OVERVIEW OF PROBABILITY AND STATISTICS

A great many events in the world around us involve uncertainty. Examples from the social, life and natural sciences, the professions--business, education, law and medicine, and from everyday experience can easily be cited. The subjects of probability and statistics were developed to enable us to discuss situations involving uncertainty and chance in a precise and objective manner. In order to gain a feeling for the breadth and importance of the use of probability it is helpful to cite some examples and raise some questions as to how the conclusions were reached and how they are to be interpreted.

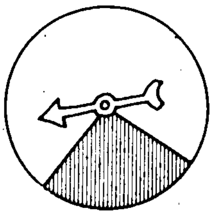
- The weatherman on the evening news says, "There is a 70% chance of rain tomorrow." Assuming that the broadcast reaches people in a fairly large area, what does the "70% chance of rain" mean? How was the 70% figure determined?

- Every cigarette advertisement and package of cigarettes carries the statement: The Surgeon General Has Determined That Cigarette Smoking Is Dangerous To Your Health. How can such an assertion be supported?
- Before all the votes cast in an election are counted, newscasters are able to project winners and the final percentages of votes quite accurately. How can such accuracy be attained with so little information?
- Higher-yielding strains of agricultural crops have been developed using genetic theories depending on probability. Also, crop yields can be accurately forecast long before the harvest actually takes place.

These examples from everyday life, medicine, political science and agriculture illustrate only a few of the many ways that probability and statistics influence our lives. Other examples from business (e.g., quality control), consumerism (e.g., testing of products), and education (e.g., evaluations of curricula or students) could also be given. All of these examples have in common uncertainty or unpredictability of outcomes or results. We will use the term random to refer to experiments or phenomena the specific outcomes or details of which are not predictable in advance.

Uncertainty is as common in the experience of elementary school children as it is in the lives of adults, although its presence may not be as widely recognized. One of the primary goals of the study of probability in the elementary school is to make children aware of the nature of phenomena involving chance. Initial activities in this direction would have children determine the possible outcomes for random experiments and those events which have the greatest chances of occurring.

Game situations provide natural vehicles for the study of probability in the elementary school. For example, games which are played with spinner devices can help develop intuitive ideas of likelihood. There is no guarantee that probabilistic ideas are already a part of every young child's background. It may take careful questioning by a teacher to help children realize that red is more likely to occur than blue on spinner A and that red and blue are equally likely to occur on spinner B. This is especially true for a child whose favorite color is blue or who has obtained a number of blue outcomes on spinner A. After developing an initial feeling for the



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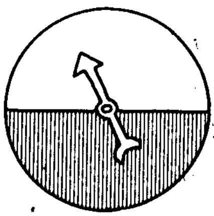


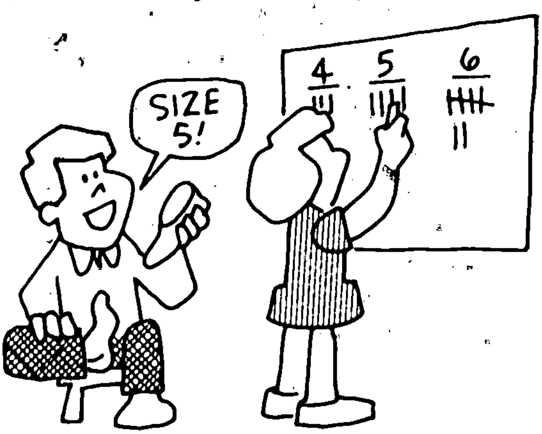
B

situation, children can quantify it in terms such as, "With spinner A we are twice as likely to obtain red as blue." They might then try a large number of experiments in an attempt to verify this conjecture.

In addition to spinners, there are many other random devices which can be used in probability activities. For example, coins and dice are familiar to most children, although they have probably not thought about their experiences with them in an organized way.

Some recent curriculum developers have advocated the inclusion of real-world applications of probability in the elementary school. For example, children might toss three coins to estimate the probability that a family with three children has all boys. Questions such as "Why should we use fair coins?" and "How should the results of the experiments (coin tossing) be interpreted?" serve to focus the children's attention on the use of mathematics to obtain answers to real-world questions. Similar activities can be carried out with sports events. For example, a world series can be simulated with an appropriately constructed spinner.

The major focus of work in statistics in the elementary school is on collecting, organizing, and interpreting data from the real world. Such activities clearly involve arithmetic and graphing



skills, and they also frequently involve measurement. For example, the children in a class might collect data on their shoe sizes. The data could be organized in a table which shows the number of children with each shoe size, or the data could be presented as a graph. An interesting question is that of determining a "typical" shoe size for a student in the class. One possible notion of typicalness is that shoe size which occurs most frequently. Another is that shoe size which occurs in the middle when the set of shoe sizes is arranged from smallest to largest. A discussion of which notion is better and why it is better provides an opportunity to identify strengths and shortcomings of information given in statistical terms.

Through the use of such data-collection experiments some of the important ideas of statistics can be communicated to children. These ideas enable children to better organize their knowledge about themselves and about the world around them.

TEACHING PROBABILITY IN THE ELEMENTARY SCHOOL

Just as many situations and phenomena encountered by adults in their everyday lives involve uncertainty in some form, so also is uncertainty a common element in the everyday experience of children; only the examples are different. Many parlor games are games of chance; most sports involve chance as well as skill; and even school work is sometimes viewed as an activity of chance. The concepts and techniques of probability provide a means of studying uncertainty from a mathematical point of view.

The issues which arise in a consideration of the role of probability in the elementary school have been discussed by many mathematics educators. The short list of references at the end of this essay is only a small sample of the literature available on the subject. This discussion is intended as a brief survey, and the interested reader should explore the references for more details.

It is useful to begin with an identification of several levels through which learning of the subject usually proceeds. This particular classification is due to Alfred Renyi and is discussed in more detail in his article, "Remarks on the Teaching or Probability," in The Teaching of Probability and Statistics, edited by Lennart Råde. The levels are:

1. Experiments and observation of statistical regularities in games of chance, nature, sports, etc.

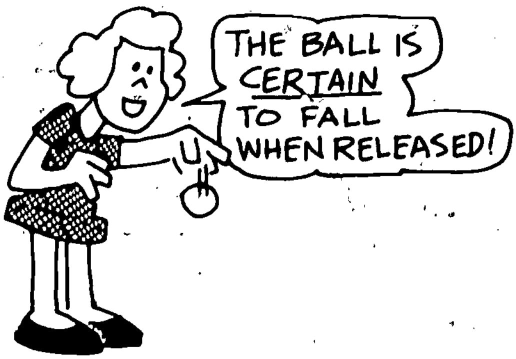
2. The formulation of a mathematical system in which these experiments and observations can be usefully discussed.

3. The use of this mathematical system in description and prediction for problems involving random phenomena.

4. Axiomatic formulation of probability theory.

Level 1 is the most appropriate for the elementary school. Topics from levels 2 and 3 may be discussed in secondary school, and a systematic study of the subject is a part of college mathematics. Some familiarity with levels 2 and 3 is desirable for an elementary teacher.

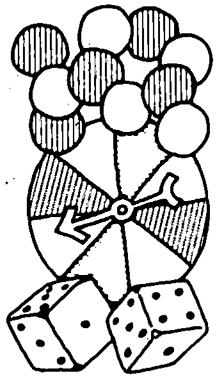
Early work in probability with elementary school children should focus on the notions of certainty and uncertainty. In the primary grades, where the ideas are first presented, children should be led



to distinguish between knowing something "for sure," i.e., with certainty, and knowing that something is "probably true," i.e., without certainty. With the usual meanings of the words, a ping-pong ball will float on water "for sure," while a piece of wood will probably float. (It may not if it is very dense or if waterlogged.) It is easy for children and adults to confuse a correct guess with knowledge, and many examples of certain and uncertain events should be discussed. It should be recognized that very young children, those in levels K and 1, are frequently unaccustomed to dealing with situations in which more than one outcome is possible. They tend to avoid responses which admit alternatives, e.g., "It might be red or green," and usually exhibit a definite preference for unambiguous assertions, "It will be red." It may take several tries at drawing a white ball from a bag containing red and white balls to convince a child that it is impossible to predict in advance which color ball will be drawn. If a child draws a white ball on the first try, he may conclude that he will draw a white ball every time even if he knows there are balls of both colors in the bag. On the other hand, if a child draws a white ball on the first draw, he may conclude that he is certain to draw a red ball on the next draw since there are balls of both colors in the bag. These fundamental ideas develop slowly over a period of time and are best taught through concrete examples.

Many probability experiments involve tallying, counting, and graphing data. At first, simply developing methods to keep track of information may be the primary objective. As the children become more adept, the emphasis may be shifted to displaying the information in an attractive and useful form.

The basic concepts of experiment, trial, outcome, outcome set (or sample space), and frequency should be introduced and developed in game situations. Games using spinners with backgrounds of equal or unequal sectors, colored balls in bags, or dice are readily available or easily constructed. The notion of equally likely outcomes can be introduced naturally through the use of dice, spinners with equal sectors on the background, or a bag with equal numbers of balls of two different colors. These ideas should be reinforced by application to a large number of situations drawn from the experience of the children.



In the intermediate grades (4 through 6) the notions of certainty and uncertainty should be reviewed, extended, and quantified. The concept of probability can be connected with fractions in the interval from 0 to 1. Experiments in which the outcome is certain, e.g., drawing a green ball from a bag containing only green balls, and uncertain, e.g., drawing a green ball from a bag containing both green and yellow balls, should be continued. The degree of uncertainty can be quantified through the notion of the relative frequency of a specific outcome:

$$\frac{\text{number of times that outcome occurs}}{\text{total number of trials}}$$

This idea can be explored through a variety of situations, e.g., bags of balls with various color compositions, spinners with background sectors of various sizes, and attribute cards. Children should learn that the physical characteristics of the apparatus are reflected in the sizes of the relative frequencies of the various outcomes. The ability to recognize this connection and to use it effectively is a significant goal.

More complicated experiments can be analyzed with the aid of tree diagrams. Multistage experiments, e.g., tossing two coins or tossing a coin and rolling a die, have more complicated outcome sets. As the situations become more complex, the need for symbols and terminology becomes more apparent. However, it is the ideas and not the vocabulary and symbolism which are of first importance.

By the fifth and sixth grades students have the computational capability to carry out arithmetic operations with probabilities; in particular, they can add and multiply fractions. The association between fractions in the interval 0 to 1 and probabilities can be pursued further. The fact that the assignment of probabilities is the assignment of numbers to outcomes or to elements of the sample space can be made more explicit. In particular, the probability that none of the elements of the outcome space occurs is 0, the probability that one of the elements of the outcome space occurs is 1, and an outcome with probability $\frac{2}{3}$ is "more likely" to occur than an outcome with probability $\frac{1}{3}$.

In summary, the study of probability provides a quantitative description of a very familiar facet of the student's experience--namely, uncertainty. Also, it provides a connection between several other strands of school mathematics: sets, graphs, rational numbers and rational number operations. The study of probability is most fruitful for children when carried out in a real-world setting.

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Section I

BASIC PROBABILITY AND ITS ROLE IN THE ELEMENTARY SCHOOL

Life inside and outside of the classroom is full of games, experiments, and other activities in which the outcome is uncertain. The use of probability as a means of describing this uncertainty is the primary concern of this section.

Activity 1 includes several experiments which provide motivation for the basic concepts which are developed in Activities 2 through 6. Activities 7 and 8 are primarily of a pedagogical nature and are concerned with the development of probabilistic thinking in children, particularly through games. In Activity 9 we return to the use of probability models to describe real-world phenomena, and we introduce the important idea of simulation as a modeling device. The final activity, Activity 10, is a seminar which reviews the section from the point of view of teaching probability in the elementary school.

MAJOR QUESTIONS

1. Discuss in your own words the statement, "The concept of probability gives a quantitative meaning to the idea of uncertainty."
2. What are the advantages and disadvantages of using only game situations to develop the basic ideas of probability?
3. Comment on the following statement, "The notions of possibility and impossibility are 'pre-probability' concepts."

4. To what extent is precise terminology a matter of importance in the study of probability? Is the possibility of misunderstanding greater here than in other mathematical topics?

5. To what extent do elementary school children need the notion of sample space (not necessarily explicitly) before they can understand the concept of probability?

ACTIVITY 1

EXPERIMENTS

FOCUS:

This activity is designed to provide experiences in actually carrying out experiments involving uncertainty. The major emphasis here is on providing motivation for the basic concepts of probability and on organizing and recording data. The data obtained in these experiments will be used in later activities.

MATERIALS:

1. Several circular cylinders with different cross section-to-length ratios, for example, a section of the cardboard center of a paper towel roll, the center of a cellophane tape roll, several pieces cut from wooden dowels, etc.

2. A bag of 25 chips of four different colors (red, white, blue, green).

3. A spinner which has three equal colored sectors (blue, green, red).

4. A fair coin.

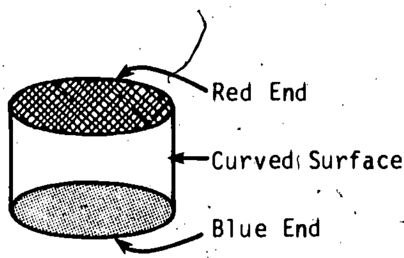
DIRECTIONS:

PART I

Perform each of the three experiments described below. It is useful for three or four students to work together on each experiment and divide the effort. If this procedure is adopted, then each experiment should be replicated twice in order to have more data for Part II. Record the results of the experiments in the tables or diagrams provided. These results will be used in the class discussion which concludes this activity and also in later activities.

A. <u>A Cylinder Experiment</u>

1. Mark (with paint, marker or colored pencil) the faces of the cylinders as illustrated below. The purpose of the marking is to enable you to tell one end of the cylinder from the other. A trial of the experiment consists of flipping the cylinder and recording how it lands. Perform several preliminary flips to determine the best technique for obtaining random landing positions. Once a flipping technique has been selected, use the same technique throughout the experiment.



2. In this case an experiment consists of 30 trials. Select a cylinder, perform an experiment, and record the results in Table 1. Select a cylinder with a different diameter-to-height ratio, perform an experiment, and record the results in Table 1.

Table 1

	Outcome	Number of Times Outcome Occurs in 30 Trials
First Cylinder	Curved Surface	
	Red End	
	Blue End	
Second Cylinder	Curved Surface	
	Red End	
	Blue End	

(Hint: In recording the data, use a tallying system to help you keep track of the results.)

3. Which outcome occurred most frequently for the first cylinder? For the second? If the experiments were repeated, would the same outcomes occur most frequently again?

4. What differences do you see in the data for the two cylinders? What differences would you expect in the data for two cylinders with the same diameter but with different heights?

B. <u>An Experiment with Chips</u>

1. The bag contains 25 chips. Do not examine the contents of the bag or otherwise attempt to determine the distribution of colors among the chips. A trial consists of shaking the bag, selecting a chip (without looking), recording its color, and replacing the chip in the bag.

2. In this case an experiment consists of 25 trials. Perform an experiment and record the results in Table 2.

Table 2

Outcome (Color)	Number of Times Outcome Occurs in 25 Trials

3. Using the fact that there are 25 chips in the bag, provide estimates on the following quantities based on the evidence obtained in your experiment.

The percent of red chips in the bag.

The percent of white chips in the bag.

4. Do you think your estimates in 3) would be the same if you performed another experiment and combined the new results with the old results? Why? (In the class discussion which follows, this question will be considered in more detail and the actual composition of the bag will be examined.)

C. A Spinner and Coin Experiment

1. This experiment utilizes a spinner with three equal sectors and a fair coin. A trial consists of spinning the spinner and tossing the coin, and then recording the results.

2. Decide how to record the results of each trial. Remember a result consists of a color (where the pointer on the spinner ends) and either "heads" or "tails" depending on how the coin lands.

3. An experiment consists of 24 trials. Perform an experiment and record your results in Table 3.

Table 3

Outcome	Number of Times Outcome Occurs in 24 Trials

4. Use the results to estimate the number of blue-tails outcomes in an experiment consisting of 100 trials.

5. Suppose that you had to estimate the proportion of blue-tails outcomes which would occur without performing any experiments. What estimate would you give and why? (Hint: Use the symmetry of the spinner and coin to obtain a theoretical estimate.) Does the relation between your experimental and theoretical results seem reasonable?

PART II

As a class, discuss the results of the experiments. The following topics can serve as a guide for the discussion.

A. A Cylinder Experiment

1. Did the same outcome occur most frequently in every repetition of the experiment using the same cylinder? What about experiments using different cylinders?

2. Compare the geometry of the cylinder used in an experiment with the results. Suggest a relation (as precise as you can) between the geometry of the cylinder and the frequency of the three outcomes. Does this correspond to your intuition?

B. An Experiment with Chips

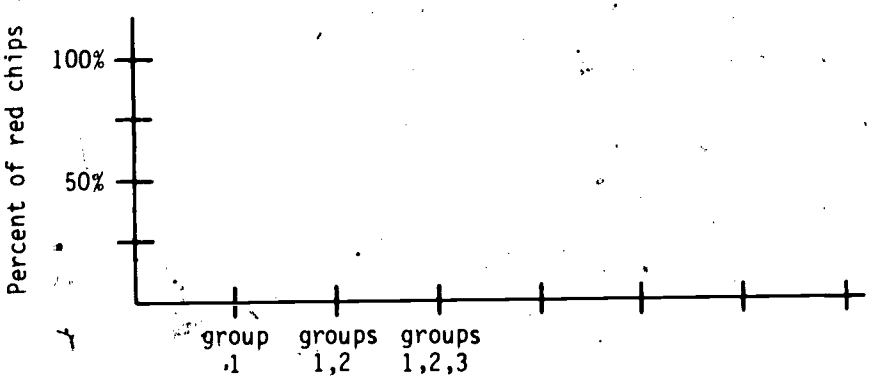
Each group has constructed a Table 2. Combine the results of these tables in the following manner.

i. Identify the experiments performed by the different groups by assigning each group a number: 1, 2, ..., etc.

ii. Graph the percent of red chips obtained by group 1 on the graph below.

iii. Combine the results obtained by groups 1 and 2 and find the percent of red chips in the combined experiment. Plot this information on the graph below.

iv. Continue this process until the results of the experiments performed by all groups have been combined and graphed.



Cumulative Percent of Red Chips Drawn

Using the summary of the experiments provided by the graph on the preceding page, discuss the following questions.

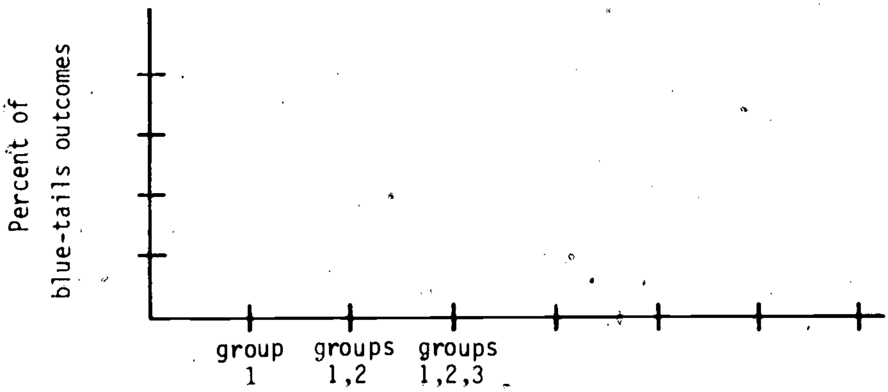
1. Is there a trend in the value of the percent of red chips as the results of more and more experiments are lumped together?

2. If you had to estimate the percent of red chips in the bag now, what estimate would you give? How many red chips do you predict are in the bag?

3. Open the bag and determine the number of red chips exactly. How does the estimate you made in the preceding question compare with the actual number?

C. A Spinner and Coin Experiment

Combine the results of Experiment C for all the groups and complete a graph as in the chips experiment. What is an appropriate vertical scale for the graph?



Cumulative Percent of Blue-Tails Outcomes

Use the graph to discuss the following questions.

1. Is there a trend in the proportion of blue-tails outcomes as more and more experiments are lumped together?

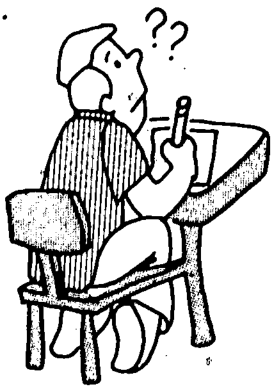
2. If you had to estimate the proportion of blue-tails outcomes which would occur in 1000 repetitions of the experiment, what estimate would you give?

3. How does the estimate you made earlier using the symmetry of the spinner and coin compare with the estimate determined in the preceding question?

General Question:

If you wanted to estimate the chance of a particular basketball player making a free throw on his next trip to the foul line, what data would be most useful to you? In view of your experiences in this activity, what cautions would you need to observe in making an estimate from the data?

TEACHER TEASER



Many teachers feel that since a student is likely to answer some questions on a true-false quiz correctly just by guessing, the score on such a quiz should be obtained by subtracting some fraction of the number of wrong answers from the number of right answers. What is the "best" fraction?

That is, if R is the number of right answers, W is the number of wrong answers, p is some number between 0 and 1, and S is the score to be recorded, then

$$S = R - (p \times W).$$

What is the best choice of p?

ACTIVITY 2

SAMPLE SPACES AND EVENTS

FOCUS:

The concepts of sample space and event are fundamental to a quantitative discussion of the notion of chance. In this activity these concepts are defined and related to the experiments of Activity 1.

DISCUSSION:

> The sample space associated with an experiment is the set of all possible outcomes of the experiment.

EXAMPLES

A. In the experiment consisting of rolling a single ordinary die and noting the number of dots on the face which lands uppermost, the outcomes may be identified with the integers 1 through 6. Therefore, the sample space may be defined as the set {1, 2, 3, 4, 5, 6}.

B. The sample space for the experiment of tossing a coin may be defined as {H, T} where H denotes the outcome "heads" and T denotes the outcome "tails."

C. In the experiment of rolling a die and tossing a coin, each outcome consists of an integer between 1 and 6 (inclusive) which gives the number of dots on the uppermost face of the die, and a letter H or T which identifies the side of the coin which lands up. Therefore, the set of outcomes, the sample space, may be defined by {1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, 6T}.

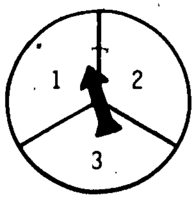
It may happen that we are more interested in the set of those outcomes with some particular characteristic than in the set of all outcomes. For example, suppose the result of rolling a die was an even number of dots on the uppermost face. What outcomes could have occurred? The subset of the sample space satisfying the condition is the set {2, 4, 6}, and this is referred to as the <u>event</u> of an even number of dots on the uppermost face of the die.

> In an experiment, an <u>event</u> E is a subset of the sample space of the experiment. An event E is said to occur if the outcome of an experiment corresponds to one of the elements of E.

EXAMPLES

A. A first-grade class consists of six- and seven-year-old boys and girls. An experiment consists of selecting a child at random and noting the age and sex. The sample space may be defined by {6B, 6G, 7B, 7G}. The event of selecting a boy occurs if the child selected is in the set {6B, 7B}. The event of selecting a six-year-old occurs if the child selected is in the set {6B, 6G}.

B. The experiment of spinning the spinner shown at the right two times and noting the outcomes (in order) has the sample space



{(1,1), (1,2), (1,3), (2,1),
(2,2), (2,3), (3,1), (3,2), (3,3)}.

The event "at least one 2" occurs if the outcome is one of the elements of the set

{(1,2), (2,1), (2,2), (2,3), (3,2)}.

The event "exactly one 2" occurs if the outcome is one of the elements of

{(1,2), (2,1), (2,3), (3,2)}.

DIRECTIONS:

Read the Discussion above. As a class, work and discuss the exercises which follow.

1. Specify a sample space for each of the following (one-trial) experiments:

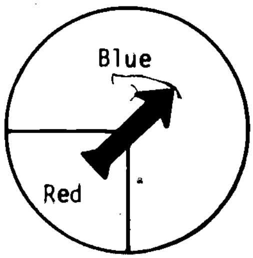
   a) A basketball is thrown at a goal.

   b) The last digit on the license plate of a passing car is noted.

   c) A card is selected randomly from a bridge deck.

   d) A student is selected randomly from your class.

   e) The spinner shown below is spun three times.



2. In the experiment described in 1b), determine the events:

   a) The digit is less than 5.

   b) The digit is even.

   c) The digit is not 7.

3. In the experiment described in 1c), determine the events:

   a) The card selected is a black queen.

   b) The card selected is a queen.

   c) The card selected is black.

4. In the experiment described in 1e), determine the events:

   a) The first spin lands on blue.

b) The first two spins land on blue.

c) No spin lands on blue.

Complete the following exercises as a homework assignment.

5. A cylinder experiment similar to that of Activity 1, Part I. A. (pp. 18-19) consists of three trials. Determine the sample space of this experiment.

6. For the experiment described in exercise 5, describe in words any two events. Define the event using set notation. Be sure that each event is a <u>subset</u>. The subset may be the empty set or it may be the entire sample space.

7. An experiment consists of rolling a red die and a green die simultaneously and noting the number of dots on the uppermost face of each.

   a) Specify a sample space for the experiment. How many elements are there in your sample space?

   Express each of the following events as a subset of the sample space; i.e., list the outcomes in each event.

   b) The same number appears on the face of each die.

   c) A 6 appears on the face of the green die.

   d) The number on the red die is a multiple of 3.

   e) The sum of the numbers of dots on the two dice is 7.

   f) The sum of the numbers of dots on the two dice is 1.

   g) The sum of the numbers of dots on the two dice is less than 13.

8. *OPTIONAL: Select a mathematics text from grade 6, 7, or 8. Describe one activity in the text which involves the concepts of sample space and event. (Note: The words "sample space" and "event" may not be used even if the concepts are introduced.)*

ACTIVITY 3

ASSIGNING PROBABILITIES

FOCUS:

After the set of possible outcomes is determined, the next question is, "How does one evaluate the likelihood that a specific outcome will occur?" This is the question of how one assigns probabilities to outcomes, and it is the topic of this activity. An appropriate assignment of probabilities is crucial to the usefulness of probability as an aid to decision-making.

MATERIALS:

Two ordinary dice; a bridge deck; the results of Activity 1.

DISCUSSION:

Although in some situations you may be provided with the probability of an outcome (e.g., the weather service gives the chance of rain tomorrow as 60%), at some point each probability must be determined. We will discuss two fundamentally different methods of accomplishing this. One method is based on actual experiments, and the other is based on logical argument.

The probability of an outcome is a number associated with that outcome. If A denotes an outcome then Pr[A] will denote the probability of that outcome.

If A is certain to occur, that is, if A occurs every time the experiment is performed, then Pr[A] = 1.

If A never occurs, then Pr[A] = 0.

If all cases Pr[A] is a number between 0 and 1, inclusive.

The sum of the probabilities of all of the outcomes of an experiment is 1. That is, if there are m outcomes, say A_1, A_2, ... A_m, exactly one of which must occur, then

$$Pr[A_1] + Pr[A_2] + \ldots + Pr[A_m] = 1.$$

The two methods by which Pr[A] can be determined for an outcome A will be referred to as the method of symmetry and the method of relative frequencies.

Method of Symmetry or Equal Likelihood

If one opens a new game and finds an ordinary six-sided die, then the normal assumption is that the die is fair. This assumption is based on a superficial examination of the die--all faces appear to be approximately the same (other than the number of dots), and on previous experiences with dice (except in very unusual situations, no one side of a die seems to be favored). Thus, given a new die, one assumes that it is fair, or equivalently, one assumes that when it is rolled each side will land uppermost about the same fraction of the time. Since there are six sides, this means that we expect a 1 about $\frac{1}{6}$ of the time, a 2 about $\frac{1}{6}$ of the time, etc. We express this by making a formal assumption that the probability of the outcome 1 is $\frac{1}{6}$, the probability of the outcome 2 is $\frac{1}{6}$, etc. In the case of a fair die the set of outcomes, the sample space, is

$$S = \{1, 2, 3, 4, 5, 6\}$$

and the assignment of probabilities based on symmetry is

$$Pr[1] = \frac{1}{6}, Pr[2] = \frac{1}{6}, Pr[3] = \frac{1}{6}, Pr[4] = \frac{1}{6}, Pr[5] = \frac{1}{6}, Pr[6] = \frac{1}{6}.$$

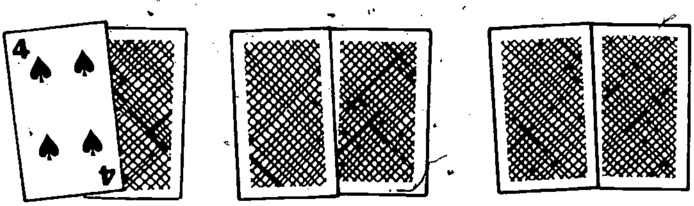
This assignment of probabilities can be expressed more concisely as $Pr[A] = \frac{1}{6}$ for each outcome A in S. Notice that for each outcome A, $0 \leq Pr[A] \leq 1$ and $Pr[1] + Pr[2] + Pr[3] + Pr[4] + Pr[5] + Pr[6] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$. The term symmetry is used here because the basic assumption regarding equally likely outcomes is frequently based upon the symmetry of a physical apparatus, situation, or experiment. It follows that you should be less willing to use this method with a coin which is badly abraded on one side than with a new coin.

Another example of the use of the method of symmetry is that of flipping a quarter selected randomly from the change in your pocket. The faces of the quarter appear to be approximately the same, and consequently we assume that when the coin is flipped each face is equally likely to land up. The sample space for the outcome of the experiment of flipping the quarter is $\{H,T\}$, and the assignment of probabilities based on symmetry is $Pr[H] = \frac{1}{2}$, $Pr[T] = \frac{1}{2}$.

Of course, if experience indicates that the assignment of probabilities based on symmetry does not yield results consistent with observations, then the basic assumption of equally likely outcomes must be re-examined.

Finally, it is important to be sure that the symmetry is actual and not illusory. There are some situations which appear to be symmetric, but which are not. A fairly straightforward example of this is the following.

Suppose that there are three pairs of ordinary bridge cards turned face down in front of you. You know that one pair consists of two

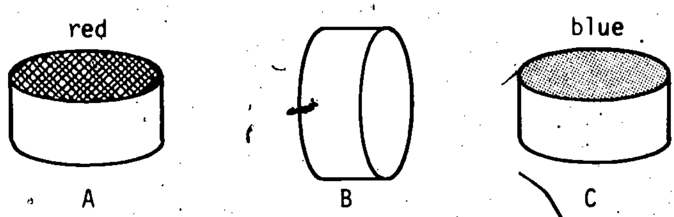


black cards, one pair consists of two red cards, and one pair consists of one black card and one red card. You select one card at random and turn it face up. It is black. What is the probability that the remaining card of that pair is black?

At first glance you might be inclined to argue using symmetry as follows: Since the card is black we know that it came from either the black-black pair or the black-red pair. Since it could just as well have come from either pair (the choice at random) the two are equally likely and consequently the probability that the remaining card in the pair is black is $\frac{1}{2}$. This superficial argument is <u>false</u>, and the error will be pointed out in Activity 16. (Incidentally, the probability that the remaining card in the pair is black is $\frac{2}{3}$.)

Method of Relative Frequency

The cylinder tossing experiment of Activity 1 is an example of an experiment in which there is no easy method using reasoning alone to assign probabilities to the three outcomes:



Depending upon the physical features of the cylinder, it may be possible to use symmetry to argue that Pr[A] = Pr[C]. However, no symmetry argument can be used to determine Pr[B]. In a situation of this sort it is convenient to use the relative frequency of the occurrence of outcome B as an estimate for Pr[B]. That is, toss the cylinder several times and use the relative frequencies

$$\frac{\text{number of times red lands up}}{\text{total number of tosses}}$$

$$\frac{\text{number of times blue lands up}}{\text{total number of tosses}}$$

$$\frac{\text{number of times cylinder lands on its side}}{\text{total number of tosses}}$$

as estimates of (or approximations to) Pr[A], Pr[C], and Pr[B] respectively. The larger the number of tosses the more likely that the resulting estimates of the probabilities are good ones; recall the graph constructed in Activity 1 for Experiment A. The number of repetitions needed depends upon the accuracy desired of the estimates and the desired degree of certainty that this accuracy is achieved. More advanced mathematics can be used to determine the number of repetitions needed.

The probability of an event is a number associated with that event. Once a probability has been assigned to each outcome in a sample space, then we can define the probability of any event as follows:

> The probability of an event E is the sum of the probabilities of all outcomes in that event.

(Remember that an event is a subset of the sample space and therefore is a set of outcomes.)

For example, the event of selecting a queen from a bridge deck is the set {♠Q, ♥Q, ♦Q, ♣Q}. If we use the method of symmetry and assign the probability $\frac{1}{52}$ to the outcome of drawing any specific card on a random draw, then the probability of selecting a queen is

$$Pr[\spadesuit Q] + Pr[\heartsuit Q] + Pr[\diamondsuit Q] + Pr[\clubsuit Q] = \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{4}{52}.$$

DIRECTIONS:

After reading through the Discussion (either individually or as a class) work the following problems.

1. Use the relative frequencies computed by the class in Activity 1 to assign probabilities to the following outcomes.

   a) A specific cylinder lands on its side.

   b) The chip drawn from the bag is red. (Remember, use the <u>frequencies</u>, not your knowledge of what is in the bag.)

   c) The spinner points to blue and the coin lands with tails up.

2. If we shuffle a bridge deck and draw one card at random, what is the probability of drawing

   a) a spade?

   b) a six?

   c) the six of spades?

3. If we roll two dice and note the numbers on the uppermost faces, what is the probability that

   a) they are both 6?

   b) the sum is 12?

   c) the sum is 7?

4. It is a consequence of the properties of probabilities of outcomes (p. 28) that each event has a probability between 0 and 1. What is the significance of an event E with Pr[E] = 0? with Pr[E] = 1?

5. Use the method of symmetry to

   a) Compute the probability of each of the events described in exercise 3 of Activity 2, p. 26.

   b) Compute the probability of each of the events described in homework exercise 7 of Activity 2, p. 27.

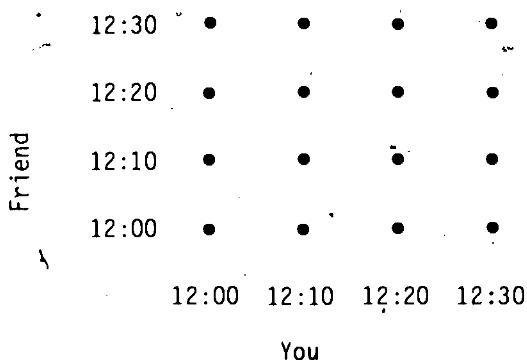
TEACHER TEASER

A geoboard can be used to help you solve the following problem:

You and a friend agree to meet at a bookstore during your lunch hours. You know that each of you has about a ten-minute walk from his or her office to the bookstore and that each will take about thirty minutes to eat lunch, so that each of you will spend only about ten minutes at the bookstore. Assume that you are willing to take the time to eat in two intervals if necessary. If you don't know for sure when your friend plans to go to the bookstore, what time is the best for you to go and what is the probability that you will meet?

You may want to simplify the problem by assuming that both you and your friend leave your offices only on the hour or 10, 20, or 30 minutes after the hour.

Hint: Think of the geoboard as being set up as follows, and identify the region on the geoboard that corresponds to a meeting.



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ACTIVITY 4

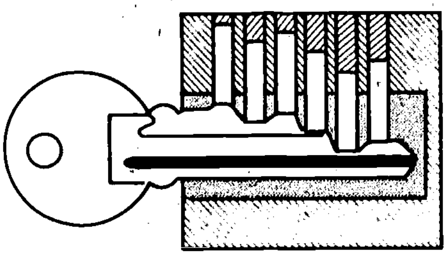
COUNTING I

FOCUS:

In assigning probabilities using the symmetry method one usually has to determine the total number of possible outcomes of an experiment. The counting techniques introduced in Activities 4 and 5 will help you in this determination. In addition, the problems studied in these activities--known as combinatorial problems--are of independent interest and arise in many different contexts. The methods introduced in these activities will be used in computing probabilities in Activity 6.

Setting the Stage

In 1973 General Motors manufactured approximately five million automobiles. Each of these cars was equipped with the same type of lock, but of course they are not all keyed alike. Are there enough different arrangements of the tumblers in the locks to have a different key for each car? This question is answered later in the activity.



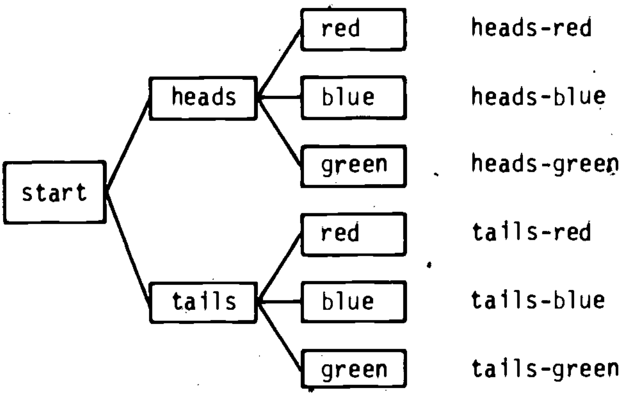
Note: The tumblers in an ordinary lock are small metal cylinders whose height is adjusted by the depth of notches cut into the key. If the correct key is inserted into the lock, the tops of the tumblers are aligned in such a way that the cylinder can be turned and the lock opened.

DIRECTIONS:

Read the examples and answer those questions designated by your instructor.

EXAMPLE 1

In Experiment C of Activity 1 there were six possible outcomes. The set of possible outcomes of each trial is illustrated by the tree diagram shown below. The beginning of the experiment is indicated by the "start" box. The outcomes for the coin toss are indicated by the "heads" and "tails" boxes and the outcomes of the spin of the pointer are "red," "green," and "blue." The six outcomes are listed to the right of the diagram.



For each of the two ways the coin can land, there are three colors on which the pointer of the spinner can stop. Therefore, there are 2·3 or 6 possible outcomes for each trial of the experiment. (Note that there are 6 items listed in the final column of the tree diagram.) The process of constructing a diagram in this manner may be helpful in answering the questions below.

Questions

1. Suppose that a college dormitory keeps records by sex and year in school (F, S, J, Sr, Grad). How many classifications are needed?

2. A class is to make a weather chart on which each day is to be classified with respect to precipitation--none, rain, and snow--and temperature--hot, warm, cool, and cold. How many classifications are needed? (Suppose that there is an agreement which distinguishes between hot, warm, cool and cold days, and days with various sorts of precipitation. For example, we might adopt the convention of recording the temperature at a fixed time and assigning hot, warm, etc. on that basis. A similar convention can be adopted for precipitation.)

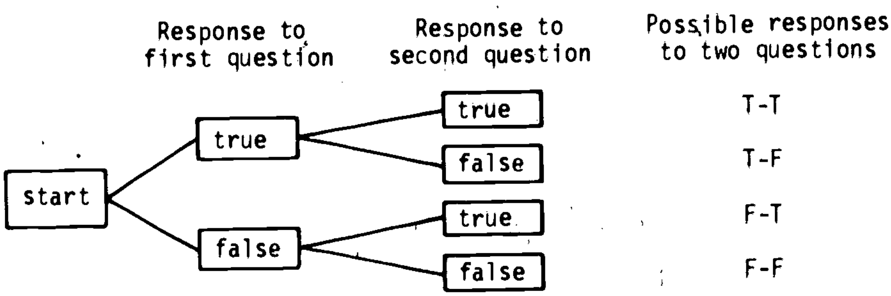
3. A school basketball team has its choice of red, blue or green T-shirts with white or yellow numbers or white or yellow T-shirts with blue or green numbers. From how many alternatives must a choice be made?

You will find tree diagrams very useful in working the problems which follow Example 2. Many problems which appear quite complex when stated in words are shown to be much simpler when a tree diagram is constructed.

EXAMPLE 2

Unprepared students sometimes answer questions on true-false or multiple-choice examinations by guessing. If a student answers five true-false questions by guessing, in how many ways can he fill in the answers? (If you see how to answer this question, go on to question 4.)

Before answering this question, we consider a simpler one. Suppose two questions are answered by guessing. Possible responses can be diagrammed as follows.



For each of the two possible responses to the first question there are two responses to the second question, and consequently there are 2·2 or 4 ways of responding to the two-question true-false quiz.

This result can be used to determine the number of possible responses to a three-question quiz. Indeed, to each possible sequence of responses to the first two questions, there are two possible responses to the third question. Therefore, there are 4·2 or 8 possible ways of completing the three-question quiz. They are

TTT TFT FTT FFT

TTF TFF FTF FFF

It may be helpful to construct a tree diagram similar to the one on page 37 in this case.

In how many ways can a five-question true-false quiz be completed?

Questions

4. Suppose that a multiple choice quiz is given with five possible responses to each item. How many different answer sheets can there be for a two-item quiz? For a five-item quiz?

5. In education, as in most other fields, pompous jargon is sometimes used to conceal a lack of understanding of the issues. In fact, one sometimes has the feeling that phrases are coined independently of the matter being discussed. One such method for creating pompous phrases is to choose any word from the first column below, any word from the second column, and any word from the third.

homogeneous	curricular	resources
unstructured	instructional	activities
individualized	conceptual	approach
evolutionary	developmental	laboratory
accelerated	societal	philosophy
spiraling	sequential	program
solidly-based	cognitive	

For example, an overly enthusiastic textbook writer could assert that his "individualized developmental approach to learning and his use of homogeneous instructional activities provide every student with a solidly-based curricular program."

How many three-word phrases can be created in this way?

6. Why are area codes needed for phone numbers in the U.S.A.? Could the telephone industry get by with two-digit area codes today? (Hint: How many different sequences of seven digits are available for use as phone numbers? On the other hand, the population of the U.S.A. is over 200 million, and there are over 100 million telephones in use.)

7. Summarize the general counting method that you have used to solve the problems in this activity.

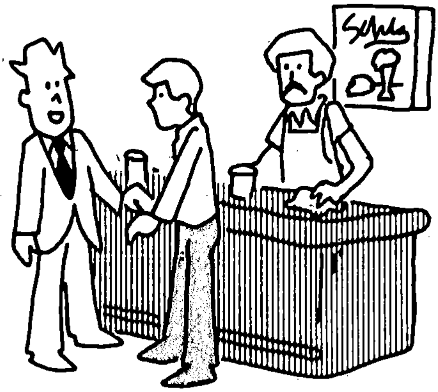
8. Find two additional problems of the type appearing in questions 1 through 6 which might arise in the elementary school.

We conclude this activity by examining the question posed in "Setting the Stage." Keys for 1973 General Motors cars were designed with six notch positions and six depths of notch at each position. How many different keys could be made?

Two Identical Cars Land Mechanics in Jail*

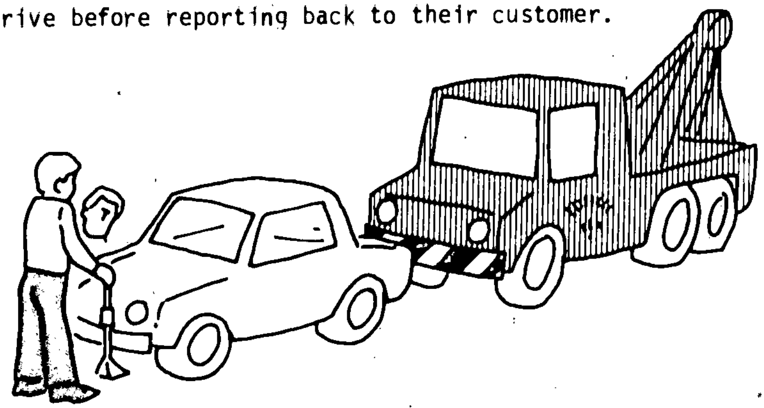
There were two identical cars parked in the same block, two sets of ignition keys fitting both vehicles, and a request to take one of the cars to a service station for repairs. It all added up to two confused mechanics in Tulsa, Oklahoma, being jailed mistakenly for auto theft.

David Johnson, 26 and Larry Cowher, 25, answered a call shortly after midnight to repair a flat on a 1966 white Ford Mustang parked downtown.



They met the owner at a bar and he gave them the keys. When they returned to the bar after fixing the flat, the owner asked them to try to start his car which he had been unable to do.

While Johnson and Cowher were receiving those instructions, however, someone else drove up in an identical white 1966 Mustang and parked in the same block. The mechanics returned to the scene, but got into the wrong car. The keys fit the ignition, however, and when the vehicle started without any trouble they decided to give it a test drive before reporting back to their customer.

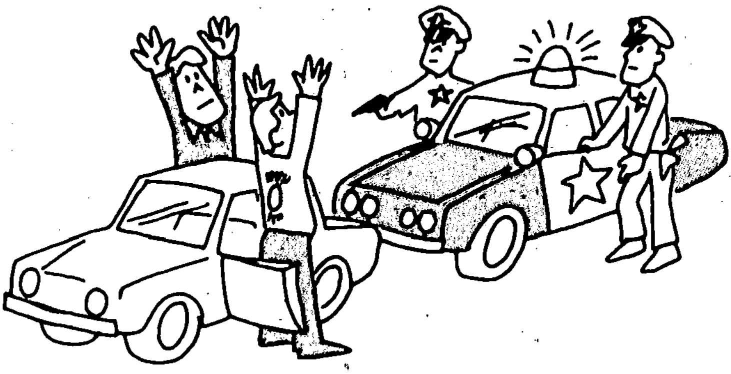


* "Two Identical Cars Land Mechanics in Jail," Daily Herald-Telephone (Bloomington, Indiana), November 6, 1975, p. 14.

The owner of the second car, returning at that moment, saw them drive away and called police, who stopped the vehicle a few blocks away. Johnson and Cowher swore the owner had given them the keys to the car. The owner claimed he had the only keys and had never seen the mechanics.

Police Capt. Ralph Duncan tried to resolve the conflicting stories by sending a patrol car to the scene of the "crime," where they discovered the unhappy customer and his still disabled car. The owner of the first car verified the mechanics' story and Johnson and Cowher were released from jail about 4 a.m.

"I guess those keys fit both cars," Duncan said. "It's just one of those one-in-a-million deals that I seem to get every night or so."



ACTIVITY 5

COUNTING II

FOCUS:

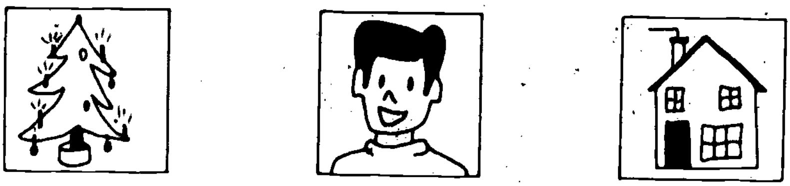
The topic of Activity 4 is continued, and other types of counting problems are considered.

DIRECTIONS:

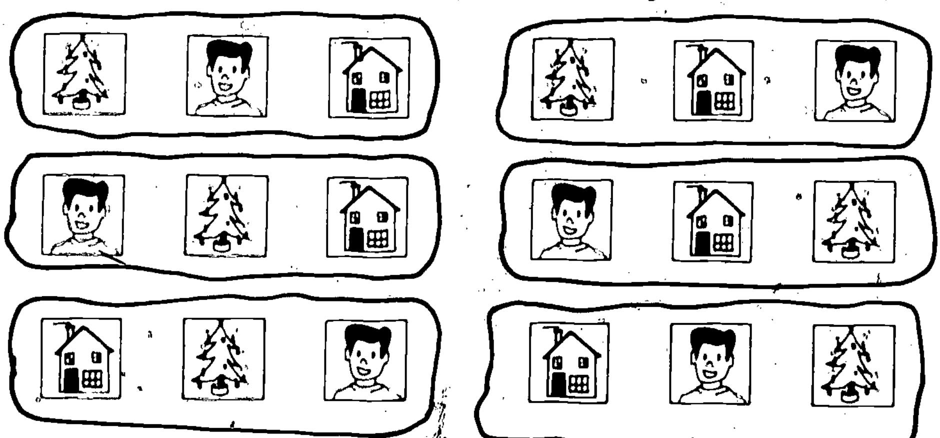
Read the examples and answer those questions designated by your instructor.

EXAMPLE 1

Three children wanted to place their drawings in a row on the bulletin board. The pictures were of a tree, a man and a house:



To be fair, the children decided to change the order of the drawings each day and to leave the drawings up until each arrangement had been used. The children sketched the possible arrangements as follows:



The children concluded that they must leave their display up for six days. After the drawings had been displayed for the six days, one child invented a way to determine the number of arrangements without having to actually sketch them. She reasoned: "Any of the three drawings can be placed at the left end of the row. After the first drawing has been selected, either of the two remaining drawings can be placed next. The remaining drawing must be put last at the end of the row. There are 3·2·1 or 6 possible arrangements of the drawings."

Questions

1. Construct a tree diagram for the situation described above.
2. How many arrangements would have been possible in the situation described above if there had been four drawings instead of three?
3. Anagrams are words which use exactly the same letters but in different orders. For example, plates, staple, petals, pastel and pleats are anagrams.

   a) There are two possible arrangements of the word NO:

   NO ON

   There are six possible arrangements of the word EAR:

   AER (an alternate spelling of the word air)
   ARE
   EAR
   ERA
   RAE (a deer)
   REA (Railway Express Agency or Rural Electrification Administration)

   Note that the six possibilities can be listed systematically by writing the words in alphabetical order. What are the possible arrangements of the letters in the "word" ATP (adenosine triphosphate)? Don't forget TPA (Traveler's Protective Association).

b) How many possible arrangements are there of the four letters NOPE? (Try to figure out how many there are without actually writing out all the possible arrangements. It is not necessary for the "word" to have a meaning--appear in a dictionary--for the arrangement to be a legitimate one.)

4. If a nectarine, an orange, a pear and an eggplant are placed in a row, how many arrangements of the items are there?

5. a) Would it be possible for the manager of a baseball team (nine persons) to try out each of the possible batting orders in order to determine the best one? (How many batting orders are possible? Use a hand calculator or use rounding as you multiply.)

   b) If the manager wants the pitcher to bat last and his best hitter to bat in the clean-up (fourth) position, how many batting orders are still possible?

6. The answers to questions 3b) and 4 are the same. Find a third problem which has the same answer. How would you describe a situation which has the basic features of these problems?

EXAMPLE 2

In how many different ways can the top three positions of the Big Ten basketball standings be filled at the end of the season? (Remember there are 10 teams. Assume no ties.) To analyze the situation we observe that any one of the teams could be in first place, any of the nine remaining teams could be in second place, and any of the eight other teams could be in third place. It follows that there are $10 \cdot 9 \cdot 8$ or 720 ways for the top three positions in the standings to be filled at the end of the season.

Questions

7. Five different floats are to be lined up for a parade. In how many ways can this be done?

8. There are ten floats entered in a contest. The best five are to be selected and lined up for a parade.

   a) How many different parades can there be? What do you mean by different parades?

   b) If a different parade began every half hour, 24 hours a day, how long would it be before the last one started out? (Assume that the floats can appear in as many parades as needed.)

EXAMPLE 3

In the previous problems and examples of this activity the order in which the items were arranged has been important. For instance, in Example 2 the arrangement

1. Indiana
2. Michigan
3. Purdue

is quite different in the eyes of a Big Ten fan than the arrangement

1. Michigan
2. Purdue
3. Indiana

On the other hand, there are some problems for which the order in which the items appear is not important. For example, if there were a post-season tournament involving the top three teams, then each of the following six rankings

1. Indiana	1. Indiana	1. Michigan
2. Michigan	2. Purdue	2. Indiana
3. Purdue	3. Michigan	3. Purdue
1. Michigan	1. Purdue	1. Purdue
2. Purdue	2. Indiana	2. Michigan
3. Indiana	3. Michigan	3. Indiana

would result in the same set of teams being in the final playoff, namely

{Indiana, Michigan, Purdue}.

Since each of the above ordered arrangements involves the same set of three teams, there are six times as many <u>ordered</u> arrangements of three teams as there are unordered sets of three teams. There are 720 possible rankings of the top three teams, and consequently there are $\frac{720}{6}$ or 120 possible sets of teams in the final playoff for the championship.

Questions

9. If a group of seven students wishes to select a Committee of Four to negotiate with the principal, how many committees can be selected from the group? Hint: It helps to break the problem into two steps:

   a) First calculate the total number of <u>ordered</u> arrangements of four students.

   b) Then observe that each Committee of Four corresponds to twenty-four of the ordered arrangements in a) above. That is, there are twenty-four times as many ordered arrangements as there are committees.

   How many Committees of Four can the seven students select?

10. a) If each pair of points in Figure 1 is connected by a straight line, how many lines will there be? (Hint: One possible way of approaching the problem is to note that the number of lines is the same as the number of sets of pairs of distinct points, which is half the number of ordered pairs of points. Each line is associated with two ordered pairs of points, the end-points in either order.) Check your answer by drawing the lines.

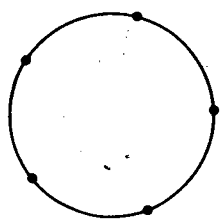
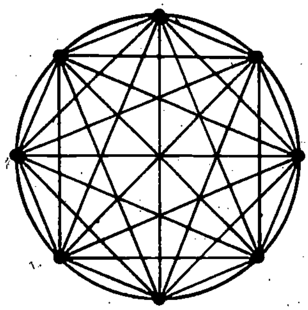


Figure 1

b) How many lines are there in Figure 2 below?

Figure 2



11. How many three-element subsets are there in the six-element set?

12. Five students want to play a round-robin tennis tournament among themselves; in such a tournament, each student plays each other student exactly once. How many matches will there be in the tournament?

13. How are problems 10 a) and 12 related? Create a third problem which is related to them both in the same way.

14. a) Create a "how many possible ordered arrangements"-type problem which has meaning for elementary school students.

    b) Create a "how many possible (unordered) sets"-type problem which has meaning for elementary school students.

15. (Continuation of 11) How many one-element subsets are there in a six-element set? How many two-element subsets? How many four-, five-, six-element subsets?

Pascal's Triangle (see also the Number Theory unit p. 82) is the triangular array of numbers whose first seven rows are reproduced below.

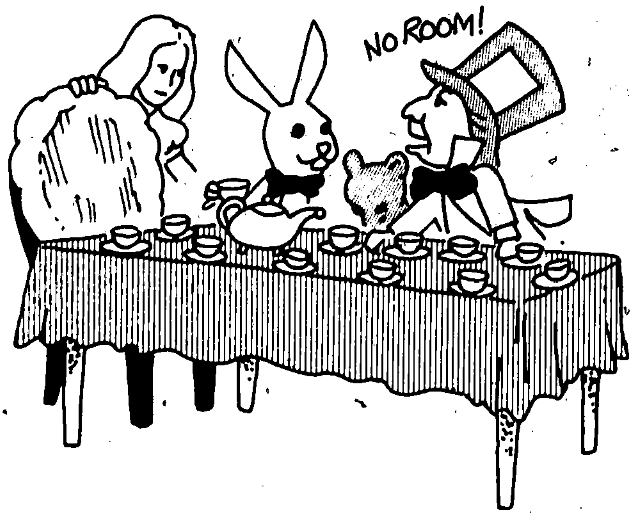
```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
```

Compare the entries in the seventh row with your answers to this problem.

Can you quickly determine the number of two-element subsets of a four-element set using the triangle?

TEACHER TEASER

Even though the March Hare, Dormouse, and Mad Hatter cried, "No room! No room!" when they saw Alice coming, it appears



that there were actually 9 places still available at the tea table. So if we assume that there were 12 places in all at the table, in how many ways can the four creatures be seated around the table?

ACTIVITY 6

COMPUTING PROBABILITIES

FOCUS:

In assigning probabilities using the symmetry method discussed in Activity 3, one frequently encounters the problem of determining how many equally likely outcomes can occur and how many of these outcomes have some desired property. In this activity we make use of the counting techniques introduced in Activities 4 and 5 in solving such problems.

DISCUSSION:

Suppose that we are interested in determining the probability that a specific outcome occurs when an experiment is performed. If we use the method of symmetry to determine this probability, then we need to determine a set of equally likely outcomes which includes the specific outcome which is of interest to us. It is customary to refer to those outcomes which have a desired property as favorable outcomes. This usage is simply a convention, and no preference or evaluation is implied. If there are a total of n equally likely outcomes and m of them are favorable, then the probability assigned to the set of desired outcomes is $\frac{m}{n}$. If the set of desired outcomes is E, then

$$Pr[E] = \frac{\text{number of outcomes favorable to E}}{\text{total number of outcomes}} = \frac{m}{n}.$$

This is an immediate consequence of the definition given on page 32 in Activity 3 and the fact that each outcome has probability $\frac{1}{n}$: There are m outcomes in E and each has probability $\frac{1}{n}$. Remember that this assumes that the outcomes are equally likely.

EXAMPLE 1

Suppose that we ask for the probability that a card drawn at random from a completely shuffled deck is a black face card. Then there are

8 favorable outcomes and 52 total outcomes. Therefore, the probability of a randomly selected card being a black face card is $\frac{8}{52}$ or $\frac{2}{13}$.

EXAMPLE 2

A student draws two cards at random from a completely shuffled deck. What is the probability that they are both face cards?

We view this as an experiment which consists of drawing two cards at random from a completely shuffled deck. By symmetry each outcome is assumed to be equally likely. How many outcomes are there? Using the method introduced in Activity 5 to count the number of ways in which a set of two cards can be selected from 52 distinct cards, we find that the total number of outcomes is

$$\frac{52 \cdot 51}{2} = 1326.$$

How many of these outcomes are favorable? I.e., how many consist of two face cards? There are 16 face cards, and again using the method of Activity 5 to determine the number of ways in which a set of two cards can be selected from 16, we conclude that there are

$$\frac{16 \cdot 15}{2} = 120$$

ways of selecting two face cards. That is, 120 of the outcomes are favorable.

Finally, the probability that a student draws two face cards in a random selection of two cards from a completely shuffled deck is

$$\frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{120}{1326} = \frac{20}{221},$$

which is approximately .09.

EXAMPLE 3

What is the probability that a student who is just guessing will answer correctly at least two of three questions on a true-false quiz?

The total number of possible responses to three questions each of which can be answered in two ways is $2 \cdot 2 \cdot 2 = 8$. The outcomes which are favorable, that is, those in which at least two answers are correct can be determined as follows:

a) Two questions can be answered correctly. This can occur in three ways: Either the first and second, second and third, or first and third can be answered correctly with the remaining question answered incorrectly.

OR

b) Three questions can be answered correctly. This can occur in one way. Therefore,

$$\text{Pr[at least two correct responses]} = \frac{3+1}{8} = \frac{4}{8} = \frac{1}{2}$$

Note that the numerator in the expression $\frac{3+1}{8}$ gives the total number of favorable outcomes, namely 3 + 1 or 4.

TEACHER TEASER

A sports-page editor for a newspaper criticizes a coach for "not trying out every combination and sticking to the best." Assume a football squad of 40 players, and suppose that the coaches work 24 hours a day every day of the year. Let each combination be tested for five minutes. How many years are required to carry out the suggestion?

Warren Weaver, Lady Luck: The Theory of Probability, (Garden City, N. Y.: Doubleday & Co., Inc., 1963), p. 101.

DIRECTIONS:

Answer the questions on the following page designated by your instructor.

1. Determine the probability that on a five-question true-false quiz a student will answer all five questions correctly by guessing.

2. Three cards are drawn at random from a completely shuffled bridge deck. What is the probability that all three are diamonds?

3. The combination to a vault consists of three numbers. The vault is opened by turning the dial to the right and stopping at the first number, then turning to the left and stopping at the second number, and finally by turning to the right to the third number.

   a) There are 100 numbers on the dial of the vault. If a person makes a guess at the combination, what is the probability that he will be correct?

   b) Suppose that a teacher finds such a safe at an auction and has each of his 25 students try out a different combination each school day--five days a week. After about three school years or 100 weeks, what is the probability that the class will have opened the safe? (In a case similar to this one, the class opened the safe after spending a year and three months trying 59,000 combinations.)

4. A bag contains three red chips and three white chips. An experiment consists of reaching in the bag and selecting two chips at random. What is the probability that both chips selected are red?

5. A committee of three students is to be selected at random (by drawing straws) from a group consisting of three fifth-graders and five sixth-graders. What is the probability that the committee will consist entirely of sixth-graders?

6. A wealthy gentleman (perhaps the Grand Duke of Tuscany) knew from long experience that in playing with three dice a sum of 10 is easier to get than a sum of 9. On the other hand, when he

listed the combinations which add to 9 and 10, he found six of each:

Combinations Which Add to 9			Combinations Which Add to 10		
3	3	3	4	3	3
4	3	2	4	4	2
4	4	1	5	3	2
5	2	2	5	4	1
6	2	1	6	3	1
5	3	1	6	2	2

Find where the Duke's reasoning goes astray, and calculate the correct probability of getting a sum of 10 and the correct probability of getting a sum of 9. (You'll be in good company if you can set matters straight; things were cleared up for the Grand Duke by the remarkable Galileo.)

7. OPTIONAL: *What is the probability that at least two people in a room of 25 have the same birthday--that is, the same month and day? (Hint: Find the probability that* <u>no</u> *two people in the room have the same birthday. Then subtract this probability from 1.) A calculator or computer will be handy if you want to obtain a single decimal number as the answer.*

ACTIVITY 7

A CHILD'S VIEW OF PROBABILITY EXPERIMENTS

FOCUS:

The concepts involved in a mathematical study of uncertainty are more complex that those, say, of the whole number operations. Moreover, personal preferences and other subjective influences play a larger role in the child's view of probabilistic situations. This activity is concerned with the way in which children respond to probability experiments.

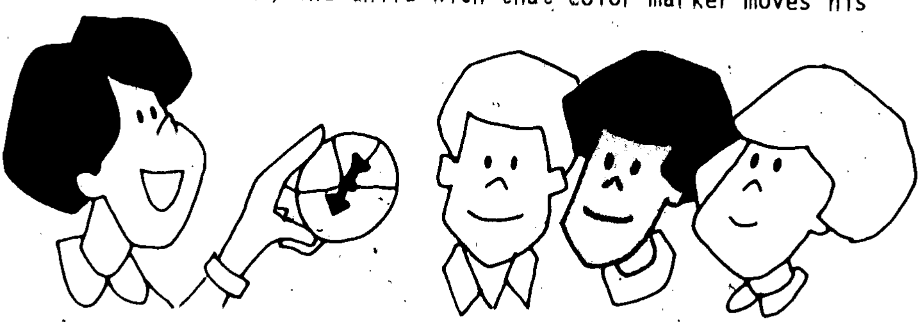
DISCUSSION:

Certain assumptions about probability experiments may seem completely reasonable to adults and yet quite unreasonable or even false to children. Also, there is the ever-present problem of communication: an adult and child may have the same idea and express it so differently that neither realizes the intentions of the other.

DIRECTIONS:

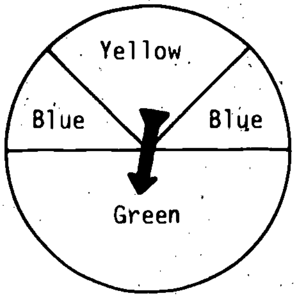
1. Read the following dialogue between a teacher and children named Pam and Pat, and then answer the questions which follow it.

   <u>Teacher</u>: We are going to play a race game. Each of you will play on the game board with a colored marker. Each time the pointer of this spinner lands on a certain color, the child with that color marker moves his



marker one space forward on the game board. The winner of the game is the first one to reach the finish space.

Finish	Finish	Finish
Start	Start	Start



Game Board

If you want to win the game by playing with this spinner, which color would be best for you?

<u>Pat</u>: I would take yellow.

<u>Teacher</u>: Why?

<u>Pat</u>: Because yellow is my lucky color.

<u>Pam</u>: I would take blue.

<u>Teacher</u>: Why would you choose blue?

<u>Pam</u>: Because there are two different places for the pointer to land with that color.

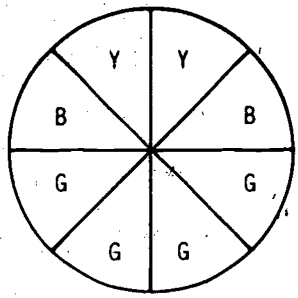
Questions

a) What is Pat's view of the game?

b) Suppose the pointer were spun three times and landed on yellow each time. How would you expect Pat to react? What would be an appropriate reaction for the teacher?

c) What is Pam's difficulty in analyzing the situation?

d) Would a spinner of the form on the following page help her to avoid the mistake?



e) Design an activity in which a child is assigned a color and asked to pick the spinner that is best for him.

2. Read the following dialogue between a teacher and a child named Ellen, and then answer the questions which follow it.

<u>Teacher</u>: Here is a bag containing red and white balls. If you draw a ball from the bag without looking, what color could you draw?

<u>Ellen</u>: White?

<u>Teacher</u>: Any other color?

<u>Ellen</u>: Red?

<u>Teacher</u>: Is that all?

<u>Ellen</u>: (Silence)

<u>Teacher</u>: Could you draw a blue ball?

<u>Ellen</u>: If you put one in.

Questions

a) Does Ellen have a feeling for outcomes which are possible and those which are impossible?

b) Does Ellen understand the concept of sample space?

c) Was the teacher's choice of the word "could" a good choice? Do you have a better choice?

d) How would you improve this learning experience for Ellen?

3. Discuss the relative merits of spinners and balls in bags as devices for generating random outcomes in experiments with elementary school children. Modify the experiment described in question 1 for use with balls in a bag.

4. How would you handle consistently highly-subjective responses from a child in probability game situations? Are such responses so different from those of adults? (Remember the basketball player who always wears orange shoelaces and the football coach who always carries a certain coffee cup along the sideline during a game.)

ACTIVITY 8

PROBABILITY IN CHILDREN'S GAMES

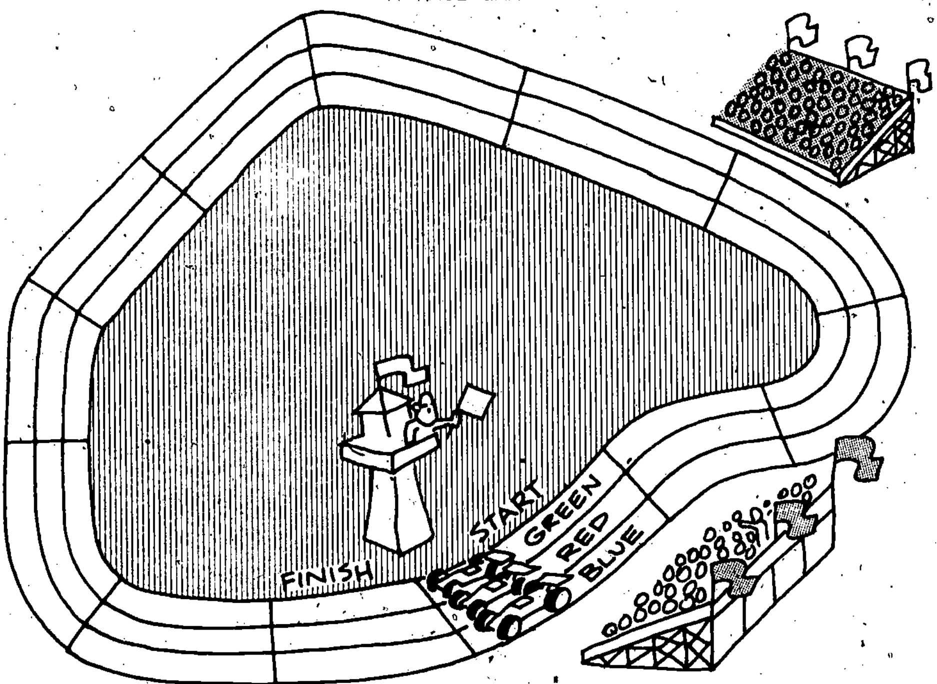
FOCUS:

A large part of children's exposure to chance and uncertainty arises in game situations. In this activity the role of games in introducing probability concepts to children is examined.

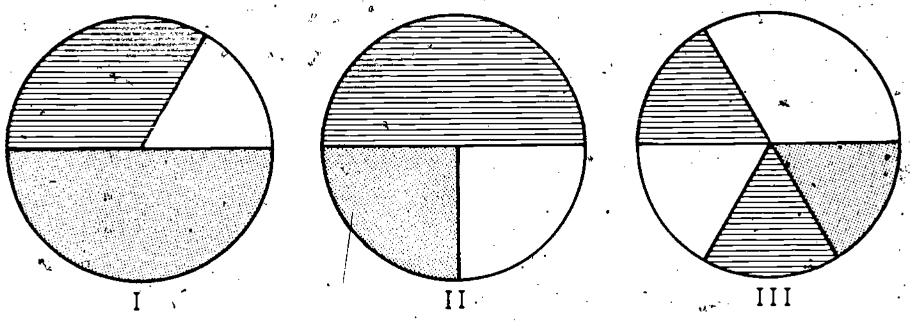
DISCUSSION:

The following game, which is suitable for the primary grades, can be used to introduce certain probability concepts.

A RACE GAME



The game requires a race track (a sample is illustrated above), three paper cars colored blue, green and red, and a set of spinners similar to those shown on the following page.



The game is played by three children as follows:

a) One child picks the spinner to be used in the game. If the game is to be played several times, then the children might take turns in selecting a spinner.

b) Each child picks a colored car; the child who picks the spinner makes the last selection.

c) The children take turns spinning the spinner (or an impartial fourth party may play this role). After each spin the child with the car whose color is the same as the region where the pointer lands moves his car forward one space. The winner is the child whose car crosses the finish line first.

DIRECTIONS:

1. In small groups (three or four students) discuss the following questions.

   a) Identify three ideas connected with probability which could be introduced using this game setting. Formulate these ideas as objectives to be achieved by the child.

   b) Suggest appropriate questions for helping the child to focus on each of the ideas identified in a).

   c) *OPTIONAL: In the previous activity it was suggested that children have certain difficulties in dealing with proba-*

bilistic concepts. How might these difficulties affect the way they play this game? How might the game be played to overcome these difficulties?

2. Construct another game which could be used to introduce basic ideas in probability to elementary children. What are some of the ideas which could be introduced using your game?

TEACHER TEASER

The God Sambu (Siva) has 10 hands. In each hand he holds one of the following symbols: a rope, an elephant's hook, a serpent, a tabor, a skull, a trident, a bedstead, a dagger, an arrow, and a bow. If these symbols can be exchanged from hand to hand, how many different variations in the appearance of the God Sambu are possible?

A Hindu Problem Adapted from Bhaskara (ca. 1150) in Howard Eves, An Introduction to the History of Mathematics (New York: Holt, Rinehart, Winston, 1964), p. 201.

ACTIVITY 9

PROBABILITY MODELS OF REAL-WORLD SITUATIONS

FOCUS:

The use of probability models to aid in understanding real-world situations can be introduced in the intermediate grades. Two typical situations are presented in this activity to provide some experience with these models.

MATERIALS:

One spinner per group, paper and colored pencils.

DISCUSSION:

Many real-world situations involving randomness can be modeled using a spinner or a set of spinners. Here we consider two situations which can be modeled by using a single spinner. Spinners with background sectors of any desired sizes can be constructed by fitting a sheet of paper shaded in an appropriate way over the background of the spinner provided.

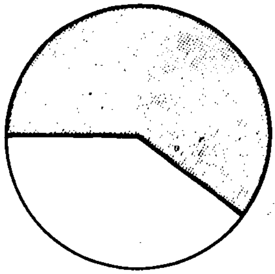
DIRECTIONS:

Each group will be assigned either Problem A or Problem B. Each group will construct an appropriate spinner, perform the indicated experiment, and answer the questions.

PROBLEM A

Tom has a free-throw shooting success probability of .600. He is on the line for a one-and-one free throw. Use a spinner to estimate the probability that he will score 0 points, 1 point, and 2 points. Remember that "one-and-one" means that the player must make the first free throw in order to be eligible to attempt the second.

1. Construct a spinner with .6 of the area shaded. The background should appear as sketched below



2. Perform an experiment consisting of 20 one-and-one trials. Be sure to abide by the rules! Record the results of your experiment in the table below.

Points	Frequency	Relative Frequency
0		
1		
2		

3. Based on your experiment, how do you expect Tom to perform at the free-throw line?

4. Pool the data obtained by all the groups working Problem A to obtain a better estimate. Use the approach of Activity 1.

5. In what ways is this model a realistic one? What are some of its shortcomings?

6. *OPTIONAL: A model such as this may be helpful in decision-making. For example, suppose that Tom could be expected to score 10 points from the field while Sam could be expected to score 8 points on field goals in a typical game. Also, suppose that Sam has a free-throw shooting average of .750. If the coach expects that whichever player is used in the game will be at the free-*

throw line for 5 attempts on one-shot fouls and 5 attempts on one-and-one fouls, which player should he use if his only concern is scoring the maximum number of points?

PROBLEM B

A breakfast cereal company tries to increase its sales by offering small plastic animals, one animal in each box of cereal. If there are five kinds of animals, estimate the number of boxes of cereal you would have to purchase in order to obtain a complete set of animals.

1. Construct an appropriate spinner. Assume that the various animals occur in equal numbers and are uniformly distributed in boxes of cereal. That is, assume that the probability that a box of cereal contains an animal of a certain type is $\frac{1}{5}$.

2. Perform experiments that help you estimate a typical number of boxes to be opened to obtain a complete set of animals.

3. *OPTIONAL: Pool the data obtained by all the groups working Problem B to obtain a better estimate.*

General Questions:

1. Construct two real-world problems suitable for children in the intermediate grades in which probability modeling of this sort is helpful.

2. List briefly the positive effects that you feel could result from providing activities like these for children in the intermediate grades.

ACTIVITY 10

SEMINAR

FOCUS:

This seminar will be concerned with two topics: first, the rationale for teaching probability in the elementary school will be discussed; and second, a summary of suitable probability topics and activities for various grade levels will be developed.

MATERIALS:

Answers to questions 1 and 3 of the assignment given on page 4 of the Overview.

DIRECTIONS:

1. In a classroom discussion led by your instructor consider the following questions posed in the Overview assignment.

   (1.) Why is the playing of games important in early probability activities?

   (3.) After teaching a unit on probability in the sixth grade you are approached by a concerned parent who asks about the role of dice and spinners in a mathematics program. What points would you raise to support the teaching of probability and statistics in the elementary school and, in particular, the use of dice and spinners as instructional aids?

2. As a class summarize by grade level suitable topics and activities involving probability in the elementary school. Take into consideration prerequisite skills, interest and motivation, and connections with other academic work.

Section II

BASIC STATISTICS AND ITS ROLE IN THE ELEMENTARY SCHOOL

Most people (including children) regularly encounter situations in which understanding and decision-making is facilitated by organizing, processing and evaluating data. These activities, which we summarize by the term statistics, are the topic of this section.

> *"Statistical thinking will one day be as important for efficient citizenship as the ability to read and write."*
>
> --H. G. Wells

Activities 11 and 12 introduce some basic statistical ideas and provide examples of the use of these ideas in the elementary school. Activity 13 provides an example of inference based on statistics. This section concludes with a seminar which focuses on some methodological issues that arise in teaching statistical concepts.

MAJOR QUESTIONS

1. What aspects of mathematical thinking are cultivated in children by a study of statistics which are not a part of the study of arithmetic and geometry?
2. What roles does statistical thinking play in decision-making in everyday life by children? By adults?
3. How are the subjects of probability and statistics related? How should this relationship be conveyed to children?

ACTIVITY 11

USING STATISTICS TO SUMMARIZE DATA

FOCUS:

In this activity some of the basic concepts of statistics are introduced. Examples are given to show how these ideas can be used to help describe and summarize data.

DISCUSSION:

Preceding the activity there is a list of definitions of the statistical terms used in this unit. These definitions should be examined briefly and then used as a reference in the remainder of the section. This activity consists of several examples and questions involving basic statistical concepts.

DIRECTIONS:

Read through the definitions of the terms. In small groups read the examples and discuss the questions. Your instructor will identify those questions for which written answers are desired.

STATISTICAL TERMINOLOGY

To illustrate the basic concepts we use the following simple set of data: 7, 6, 10, 7, 4, 6, 10, 10. This data could arise, for example, as scores on a mathematics quiz.

Organizing Data

It is frequently helpful to organize and represent data in a chart, table or graph as shown on the following page.

a) Tally chart

4	6	7	10
I	II	II	III

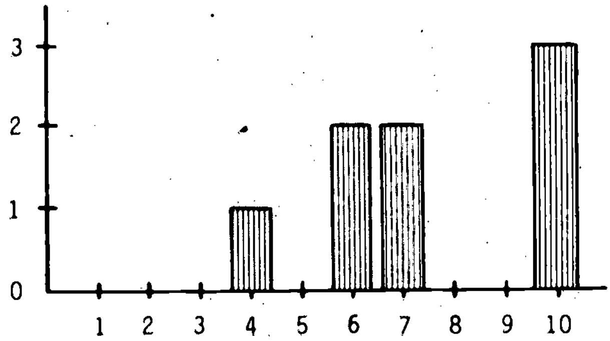
or

1	2	3	4	5	6	7	8	9	10
			/		//	//			///

b) <u>Frequency table</u>

Score	Frequency
4	1
6	2
7	2
10	3

c) <u>Bar graph</u>



<u>Summarizing Data</u>

In summarizing data one looks for a "typical value" which will serve as an appropriate representative of the entire set of data. The manner in which this value is computed usually depends on the situation and the use to which the value will be put. In addition, it is sometimes useful to have a measure of how much the data is spread out or dispersed. Such information provides a measure of how much an actual data value may differ from the typical value selected.

<u>Typical Values</u>

a) <u>Mean</u>. The mean of a set of data is the quotient of the sum of all the data values divided by the number of data values.

The mean of the data 7, 6, 10, 7, 4, 6, 10, 10 is

$$\frac{7 + 6 + 10 + 7 + 4 + 6 + 10 + 10}{8} = \frac{60}{8} \text{ or } 7\tfrac{1}{2}.$$

Notice that the mean need not be one of the data values, and in fact need not even be a possible data value. In this example the mean is $7\frac{1}{2}$ even though the quiz papers may be given only whole-number marks. As another example of this, the mean number of members in a family in a city might well be 4.3 although clearly no family can contain 4.3 members.

You may have encountered the notion of a mean value previously under the name of average. Some authors and textbooks use the term "average" in a generic sense, as we have used "typical value."

b) Median. Order or rank the data in increasing order. If the number of data values is odd, then the median is the middle data value. If the number of data values is even, then the median is the mean of the two middle data values. The median of the data 4, 6, 6, 7, 7, 10, 10, 10 is 7: $\frac{(7 + 7)}{2}$. The median of the data 4, 6, 7, 10 is $6\frac{1}{2}$: $\frac{(6 + 7)}{2}$. The median of the data 4, 6, 7, 8, 10 is 7.

c) Mode. The mode(s) of a set of data are those data values which occur with the highest frequency. The mode of 7, 6, 10, 7, 4, 6, 10, 10 is 10. The modes of 7, 6, 10, 7, 4, 6, 10 are 6, 7, and 10.

EXAMPLE

The set of data 3, 3, 3, 5, 8, 8 has mean 5, median 4 and mode 3.

Measures of Dispersion

a) Range. The range of a set of data is the difference between the greatest and least data values. The range of the data 7, 6, 4, 10 is 10 - 4 or 6.

It is sometimes useful to divide the range into quartiles, deciles or percentiles. The lowest quartile, for example, contains the bottom quarter of the data values; the lowest decile contains the lowest tenth of the data values; and the lowest percentile contains the lowest one percent of the data values. These ideas will not be pursued in this unit, but the interested reader may find more information in the references. The results of standardized tests are frequently reported in terms of percentiles or deciles.

b) Mean deviation. To compute the mean deviation of a set of data one first computes the mean and then the differences or deviations between each data value and the mean. This set of differences contains some positive and some negative numbers. Form the set of absolute values or magnitudes of the differences. That is, just consider the magnitudes of the numbers and not their algebraic signs. The mean deviation is the mean of this set of nonnegative numbers.

For the set of data 7, 6, 10, 7, 4, 6, 10, 10 considered above, the mean was determined to be $7\frac{1}{2}$. Therefore, the differences or deviations are $-\frac{1}{2}$, $-1\frac{1}{2}$, $2\frac{1}{2}$, $-\frac{1}{2}$, $-3\frac{1}{2}$, $-1\frac{1}{2}$, $2\frac{1}{2}$, $2\frac{1}{2}$. The magnitudes of the deviations are $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $\frac{1}{2}$, $3\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $2\frac{1}{2}$, and the mean of these magnitudes is

$$\frac{\frac{1}{2} + 1\frac{1}{2} + 2\frac{1}{2} + \frac{1}{2} + 3\frac{1}{2} + 1\frac{1}{2} + 2\frac{1}{2} + 2\frac{1}{2}}{8} = \frac{15}{8} = 1\frac{7}{8}.$$

Therefore, the mean deviation of this set of data is $1\frac{7}{8}$.

The range of a set of data is determined by the largest and smallest value and does not depend on the distribution of the remaining data between the largest and the smallest values. It follows that any two sets of data which have the same largest and smallest values have the same range. (In fact, even more data sets have the same range.) On the other hand, the mean deviation of a set of data does depend on the way in which the data are distributed between the largest and smallest values. The mean deviation gives greater weight to the data which are far from the mean, and consequently it is a more sensitive measure of dispersion than the range.

There are other measures of dispersion which are particularly useful when dealing with data having certain characteristics. One common measure is the standard deviation which is discussed in detail in the references.

EXAMPLES AND QUESTIONS

The mean, median and mode are all measures of the location of the center of a set of data values. As such they are frequently referred to as measures of central tendency. Any one of them might be used to describe a typical member of a set of data. The choice depends on what use is to be made of this typical value.

EXAMPLE 1*

Suppose 25 families are selected at random, and the numbers of children in 1965 and 1970 are determined. The results are tabulated below:

Family	Number of Children 1965	1970	Increase in 1965-1970
1	3	3	0
2	3	5	2
3	1	2	1
4	3	4	1
5	0	3	3
6	3	3	0
7	1	3	2
8	0	0	0
9	0	0	0
10	3	4	1

*Adapted from Statistics by Example: Exploring Data, ed. Frederick Mosteller, et al. (Reading, Mass.: Addison-Wesley, 1973), p. 23.

Family	Number of Children 1965	1970	Increase in 1965-1970
11	0	0	0
12	2	2	0
13	3	4	1
14	5	5	0
15	1	1	0
16	4	5	1
17	6	7	1
18	3	4	1
19	1	1	0
20	2	3	1
21	1	3	2
22	4	5	1
23	0	0	0
24	4	5	1
25	3	4	1
	56	76	20

"The typical number of children in one of the sample families in 1965 was 3; and in 1970, the typical number of children was again 3. However, the typical increase in family size during the period 1965-1970 was one child."

If "typical number" means median, then the above is a true statement as the frequency table below shows. (Where does the data associated with the thirteenth family lie?)

Number of Children in 1965	Frequency
0	5
1	5
2	2
3	8
4	3
5	1
6	1
7	0

Number of Children in 1970	Frequency
0	4
1	2
2	2
3	6
4	5
5	5
6	0
7	1

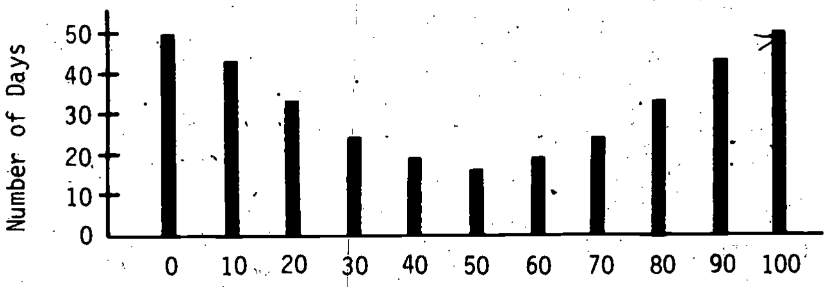
Increase between 1965 and 1970	Frequency
0	10
1	11
2	3
3	1

Therefore, we conclude that the median number of children per family in 1965 plus the median increase in family size is not necessarily equal to the median number of children per family in 1970.

Questions

1. If the typical number used in the above example were the mode, would the same conclusion hold?
2. Would the same conclusion hold if the typical number used were the mean?
3. The owner of a business has an annual income of $50,000, and his 14 employees each have incomes of $5000. How should a typical income for the 15 individuals be determined?
4. A student has test scores of 95, 75, 70, 100, and 75. How should her typical score be computed? Why?
5. A developer is planning to build a small apartment house, and to save costs he plans to construct a building in which all apartments are exactly the same. He conducts an informal telephone survey, and finds that of 100 families, 10 are interested in a studio apartment, 12 in a one-bedroom apartment, 18 in a two-bedroom, 5 in a three-bedroom, and 55 are not interested in apartment living. What size apartment should he build? What sort of "average" apartment preferences has he used?

6. Many American cities have a distribution of cloudiness similar to the following.*



How should the typical cloud cover in such an American city be described?

7. Which measure of central tendency takes every data value into account and always changes when a single data value is changed? Which measure tends to ignore extreme values?

8. Create a situation and a set of data for which one special interest group could give one picture of the situation by using the median to summarize the data, and another group could give another picture by using the mean.

9. OPTIONAL: *Suppose you are given a set of data values* $\{x_1, x_2, \ldots x_n\}$ *with mean* m.

   a) *What is* $(m - x_1) + (m - x_2) + \ldots + (m - x_n)$? *That is, what is the sum of the deviations from the mean?*

   b) *Interpret the result of a) in terms of a balance beam.*

   c) *Convince yourself that for an odd number of data values the median is that data value such that the sum of the magnitudes of the deviations of the remaining data values from it is smaller than the corresponding sum for any different data value.*

---

*S. K. Campbell, Flaws and Fallacies of Statistical Thinking (Englewood Cliffs, N.J.: Prentice-Hall, 1974), p. 71.

d) *Let D_1 and D_2 be two sets of data values with means m_1 and m_2 respectively. Show that the mean of the set of data $D_1 \cup D_2$ (that is, the set of values consisting of all values from D_1 and all values from D_2) can be obtained easily from m_1 and m_2. Give two examples to make your point.*

In concluding this discussion of typical values, it should be mentioned that there are certainly cases in which a person is not interested in the typical or average value of a set of data, but rather in one of the extreme values. A prospective college student with little money would be interested in the minimum annual expenses for attending various colleges--not the average annual expresses. Although a bridge designer is interested in the typical traffic load for the bridge, the bridge builder will want to make the bridge strong enough so that it can handle the peak or maximum load--not just the average load! A hungry soldier in the field isn't satisfied by the knowledge that the average food consumption per soldier is adequate. Similarly, in regard to gasoline conservation, a car's actual speeds are more important that its average speed. In summary, saying that an average can be computed is quite different from saying that the average is useful and significant.

EXAMPLE 2

In choosing between two basketball players with the same scoring average (usually defined as the mean), a basketball coach may be interested in the dispersion of the scores about the mean value. For example, he might prefer to use the player who is most consistent. On the other hand, the coach might start the player who has some very high-scoring games (but also some very low-scoring games) in the hope that he will have a good game.

Questions

1. How might your attitude toward two occupations with the same average pay be influenced by the dispersion of actual salaries about the average?

2. Consider the set of test scores {55, 85, 60, 98, 72, 80}.

   a) Find the range and the mean.

   b) Find a set of scores with the same mean, but with a smaller range.

   c) Find a set of test scores with the same range, but with a higher mean.

3. A student has taken three examinations of 100 points each. Her mean score is 85 and the range of test scores is 10.

   a) Find two sets of test scores satisfying these conditions which have different mean deviations.

   b) Find two different sets of test scores satisfying these conditions with the same mean deviation.

   c) What do you conclude about the mean deviation as a measure of dispersion?

4. How would you communicate information on the dispersion of test scores to the parents of students in your classes?

ACTIVITY 12

USING STATISTICS IN DECISION-MAKING

FOCUS:

This activity is concerned with the use of statistics as an aid in decision-making. An outline of a general approach to decision-making is included.

DIRECTIONS:

Read the discussion and example, and answer the questions posed at the end of the activity.

DISCUSSION:

The following format may prove helpful in analyzing a situation with the aid of statistics.

1. Recognize and clearly formulate a problem.
2. Collect relevant data.
3. Organize the data appropriately.
4. Analyze and interpret the data.
5. Relate the statistics obtained from the data to the original problem.

In Activities 1 and 3 we used data and the relative frequency method of assigning probabilities to determine the probability of a certain outcome. Viewed in the context of the decision-making process outlined above, this would fall in step 4. Thus we see that the determination of probabilities, which was the primary concern of Activity 3, is only one step or a portion of one step in the process considered here. In fact, it sometimes happens that the explicit determination of probabilities is unnecessary and that the data may be used in the decision-making process in other ways.

EXAMPLE

This example is presented in a manner which illustrates the steps identified above.

1. There is to be a softball-throwing contest between representatives of each of three fourth-grade classes. Three children volunteer in Mr. Iron's class. How should the class representative be selected?

2. It was decided that each of the three volunteers would make five throws, and that the class representative would be selected on the basis of these throws. From a specified spot each child threw the ball as far as possible.

   The distance from the throwing spot to the point of impact was measured with a trundle wheel to the nearest one-tenth meter. The data collected are given below:

Volunteers	Distance of Throws (nearest $\frac{1}{10}$ meter)				
Randy	27.7	23.1	22.1	23.8	26.8
Becky	24.0	23.3	27.4	23.9	27.1
Tony	23.1	26.7	28.8	17.8	25.6

   It is difficult to select a class representative simply by looking over these data (try it!); so we proceed to organize the data.

3. We organize the data in a tally chart and in bar graphs.

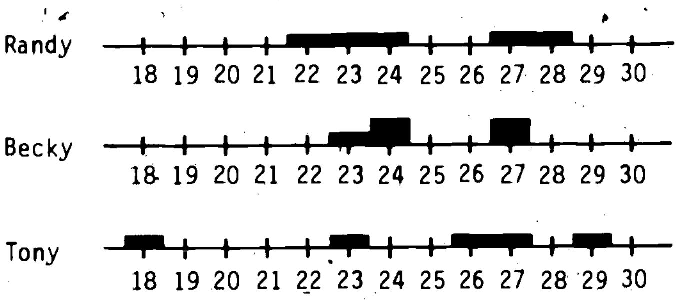
   Tally Chart:

   Distance of Throws (Nearest Meter)

	18	19	20	21	22	23	24	25	26	27	28	29
Randy					/	/	/			/	/	
Becky						/	//			//		
Tony	/					/			/	/		/

Bar Graphs:

Distance of Throws (Nearest Meter)



Simply using the tally chart and the bar graphs we can draw some rough conclusions. For example, Randy and Becky appear to be more consistent than Tony. On the other hand, Tony has the best throw of all.

4. Analyzing data frequently involves asking some questions, and that is the approach we shall follow here. The fundamental question is, of course, "What criteria should be used to select a class representative?" Before confronting this basic question, we consider a number of subsidiary questions.

   a) Which child is the best thrower? In particular, which child has the best "typical" throw? How should a typical throw be determined?

   b) What role should consistency play in the determination of who is the best thrower?

   c) Are five throws for each child an appropriate number? Should very short throws be excluded?

   d) Was important information lost in rounding the distances of the throws to the nearest meter?

Each of these questions raises an important statistical issue.

Question a) is concerned with finding a typical distance for each child. The standard statistical measures of typicalness, the measures of central tendency, were introduced in Activity 11.

Question b) is concerned with the spread or dispersion of the data. Measures of dispersion were also introduced in Activity 11.

Question c) asks whether the data collected were appropriate for the basic question.

Question d) asks whether the data were organized adequately.

a) To determine a typical distance, we compute the mean and median for each child. Carry out the computations and enter the results in the table below.

	Mean distance	Median distance
Randy		
Becky		
Tony		

Using these statistics, discuss which child has the best typical throw and why.

b) A common measure of dispersion of data is the range. What are the ranges for the data for each of the three children? How does this information help in determining who is the best thrower?

c) What sort of considerations enter into this question? Identify and discuss three relevant points. Do the rules of the final competition play a role here?

d) In organizing the data as was done on the tally chart, it was essentially classified in unit intervals: 18.6-19.5, 19.6-20.5, ... 28.6-29.5. Such intervals are known as class intervals and the choice of class intervals of length 1 was an arbitrary one. For example, class intervals of length 5: 15.1-20.0; 20.1-25.0; 25.1-30.0 could also have been used.

Whenever data is organized into class intervals for easier treatment, it is possible that information is lost. In this case did the choice of class interval distort the data in an unfair way?

Finally, use all the information you have to make the best selection of a class representative. Support your choice as much as you can. Would you prefer to have different data on which to base your selection? If so, what data would be most useful to you?

Questions

1. As a class study the following question: How much change does a typical member of the class carry?
2. Identify two other situations similar to the example of this activity where statistical thinking would be useful in the elementary school.

ACTIVITY 13 (OPTIONAL)

BASING INFERENCES ON STATISTICS

FOCUS:

This activity is designed to introduce the basic ideas of statistical inference and to indicate how data can be used in the estimation of probabilities.

MATERIALS:

*Mosteller, Frederick, and David L. Wallace, "Deciding Authorship," in Statistics: A Guide to the Unknown, edited by Judith M. Tanur, et al. San Francisco: Holden-Day, 1972, pp. 164-174.**

DIRECTIONS:

The article by Mosteller and Wallace cited above should be read as an outside assignment. The activity consists of a brief discussion of the article and the ideas introduced in it. In preparation for this discussion you should consider the following questions.

1. *What is the meaning of the term "statistical inference"?*
2. *How are statistics and probability related?*
3. *What do you view as the strengths and weaknesses of the argument presented in this article?*
4. *Give another example of the use of statistical inference in a situation arising in the real world.*

*This collection of essays was prepared by the Joint Committee on the Curriculum in Statistics and Probability of the American Statistical Association and the National Council of Teachers of Mathematics. This same Joint Committee prepared the series Statistics by Example noted in tne references.

ACTIVITY 14

SEMINAR

FOCUS:

This activity is concerned with methodological issues which may arise in teaching statistical ideas at the elementary level.

DIRECTIONS:

In a group discussion consider the following questions:

1. Is it preferable to use student-generated data rather than data that relates to a matter of interest to children, but was obtained by others?

2. The organization and presentation of data is an important component of statistical work and one which poses problems for inexperienced children (or adults). The <u>Graphs</u> unit discusses some means of presenting data, and others are introduced in Activities 1 and 3. Discuss the instructional issues which arise when an elementary school teacher seeks to teach children to organize data.

3. What measures--height, weight, cost of articles, scores in games, etc.--might be used as sources of data in the elementary school? How might the collection of data be handled to avoid individual sensitivities? List several attributes whose measurement is not likely to cause embarrassment to any children.

4. Obtain three articles from recent newspapers which are concerned with matters of interest to children and which have statistical content. How might these articles be used to motivate the study of statistics?

5. Carefully identify those arithmetic skills which must precede the introduction of various statistical concepts and tools, for example, the range, mean, median, tally chart, etc.

Section III

WINDING UP: A REVIEW AND EXTENSIONS

The topics which have been discussed in a rather intuitive way in Sections I and II can be organized and presented as a part of the mathematical disciplines of probability and statistics. In the first activity of this section we will return to a topic (counting) which was introduced in Section I, and we will review some of the results of Activities 4 and 5 in order to identify some general principles. In the next activity we will consider sequential experiments in which the second outcome depends on the first. The final activity of the unit introduces the important and useful concept of <u>expected value</u>. This notion is particularly helpful in the quantitative study of games or game-like situations.

TEACHER TEASER

1	2	3	4
5	6	7	8
9	10	11	12

Suppose that you have 12 postage stamps joined as shown in the diagram. A friend asks for four stamps which are joined in such a way that they will hang together. That is, they could be stamps 1, 2, 3, 5 but not 1, 2, 3, 8. How many sets of four stamps have this property?

ACTIVITY 15

SOME PRINCIPLES OF COUNTING

FOCUS:

In Activity 5 we introduced some techniques for counting or enumerating the elements of a set. That discussion is continued in this activity and some general principles are introduced. It should be reemphasized that although counting techniques have been introduced in this unit primarily to aid in computing probabilities, they are of considerable interest in their own right.

DISCUSSION:

(Read the examples presented below and then discuss them as a class.)

EXAMPLE 1

In Indiana automobile license plates appear as pictured below:

The first two digits identify the county of the owner's residence; the remaining letter and four digits are used to distinguish between plates issued within a county. How many different plates can be issued by each county?

If we assume that all 26 letters can be used (actually I and O are frequently omitted to avoid confusion with 1 and 0) and that all 10 digits can be used (normally the set of four digits each of which is zero is excluded), then the total number of plates can be found by arguing as follows:

There are 26 choices for the letter; for each choice of the letter there are 10 choices for the first digit; for each choice of letter and first digit there are 10 choices for the second digit; for each choice of letter and first two digits, there are 10 choices for the third digit; and finally, for each choice of letter and first three digits, there are 10 choices for the fourth digit. Therefore there are

26	x	10	x	10	x	10	x	10	= 260,000
choices for letter		choices for digit		choices for digit		choices for digit		choices for digit	

possible license plates in each county.

In question 7 of Activity 4 you were asked to summarize the general counting method that you used to solve the problems posed there. One possible method is the following, which we refer to as the Fundamental Principle of Counting:

> Suppose that there are m sets, the first containing n_1 elements, the second containing n_2 elements, ..., and the mth set containing n_m elements. Then the number of ways to form an arrangement of m elements by selecting in order one element from each of the m sets is
>
> $$n_1 \times n_2 \times \dots \times n_m.$$

In the above example we were interested in choosing one letter and four digits. Thus, there were five choices to make, $m = 5$. The first set is the set of 26 letters, and consequently $n_1 = 26$. The remaining four sets each consist of 10 digits, and consequently $n_2 = n_3 = n_4 = n_5 = 10$. Therefore the Fundamental Principle gives the number of license plates as 26 x 10 x 10 x 10 x 10, the same result we obtained above.

EXAMPLE 2

How many "trains" each consisting of four different "cars" can be formed from Red (R), Green (G), Yellow (Y), Purple (P), Blue (B), and Orange (O) Cuisenaire rods? A train of four cars is simply a set of four rods laid end to end with the front car identified. If we refer to the Cuisenaire rods by their colors, and if we agree that the train is specified by naming the cars in order, front car first, then examples of trains are: RYBG, RBYG, RBYO and BGOY. Notice that the first two trains are different even though they contain the same cars since the cars are in different orders. Also, the train RYBG is different from GBYR for the same reason--remember that the front car is named first.

For each train, there are six choices for the first car. After the first car is chosen, there are five choices for the second car. After the first two cars are selected there are four choices for the third car, and after the first three cars are chosen there are three choices for the fourth car. Thus, there are

$$6 \times 5 \times 4 \times 3$$

different trains which can be constructed from the six different Cuisenaire rods. It is important to note that this technique determines the number of ordered selections of four elements from a set of six distinct elements. A statement which corresponds to that of Example 1 is the following:

> The number of ordered arrangements of r elements selected from a set with n distinct elements is
>
> $$n \times (n - 1) \times (n - 2) \times \ldots \times ([n - r] + 1)$$

A special case which occurs with sufficient frequency to merit special comment is

> The number of ordered arrangements of the elements of a set with n distinct elements is
>
> $$n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1.$$

Since any of the various changes in the order of the elements of an ordered set is known as a permutation, the principle stated above is known as the Principle of Permutations.

In this example there are six rods so $n = 6$, and we are interested in the number of arrangements of four rods; that is, $r = 4$. The first form of the Principle then gives the number of arrangements as $6 \times 5 \times 4 \times 3 = 360$.

EXAMPLE 3

Suppose that in Example 2 we were simply interested in the number of collections of four different rods which can be selected from the set of six rods, rather than in the number of trains. We know that the number of trains is $6 \times 5 \times 4 \times 3 = 360$. However, each collection of four rods can be used to construct several trains. In fact, using the special case of the Principle of Permutations, each collection of four rods can be used to construct $4 \times 3 \times 2 \times 1 = 24$ trains. Viewed in another way, there is a set of 24 trains each of which consists of the same four rods but in different orders. Continuing to reason in this way, the entire collection of 360 trains can be partitioned into groups of 24 trains, each group consisting of trains of the same four colored rods. Therefore, the number of collections of four rods is the number of groups of 24 trains in 360. That is, $360 \div 24$ or 15. The general principle used in this example is the Principle of Combinations:

> The number of ways that a collection of r elements can be selected from a set of n distinct elements is
>
> $$\frac{n \times (n - 1) \times (n - 2) \times \ldots \times ([n - r] + 1)}{r \times (r - 1) \times \ldots \times 2 \times 1}$$

In this example, n = 6 and r = 4. The number of collections of four different rods is

$$\frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} = \frac{360}{24} = 15.$$

DIRECTIONS:

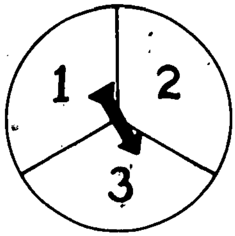
Work the problems designated by your instructor. In each case identify the principle(s) which are applicable to the situation and use them to solve the problem.

1. A class consists of 5 boys and 5 girls. In how many ways can a team consisting of 2 boys and 3 girls be selected?

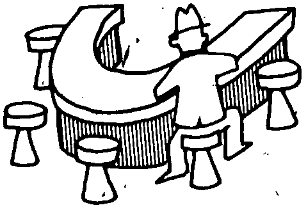
2. There are 11 girls in a class. How many volleyball teams of 9 girls can be formed?

3. A club with forty members is to elect a set of officers consisting of a president, a vice president, a secretary, and a treasurer. How many different sets of officers could the club elect?

4. An experiment consists of spinning the spinner shown at the right five times. How many possible outcomes are there?

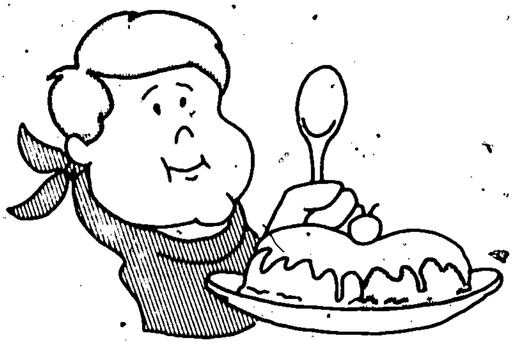


5. How many different ways are there to seat five people at a semicircular counter with five seats?



6. A student plans on registering for one art course, two social science courses and two education courses. If she is interested in three art courses, four social science courses and three education courses, from how many different sets of five courses must she make her choice?

7. An ice cream store offers 31 flavors of ice cream and four types of topping. If a sundae consists of two scoops of ice cream and a topping, how many different sundaes can be made? How many different triple-scoop ice cream cones can be made? What do you mean by "different" in each case?



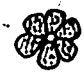
8. There are four airlines flying the Chicago-Los Angeles route and seven flying from Los Angeles to Honolulu. How many choices of carriers do you have for a Chicago-Los Angeles-Honolulu trip?

TEACHER TEASER

The Proliferous Poppies

A red poppy has two genes R and R; a white poppy has two genes W and W; and a pink poppy has two genes R and W. When one poppy is crossbred with another, each plant contributes one of its color genes to each of its offspring. The gene contributed by each parent is selected at random from the two genes of that parent.

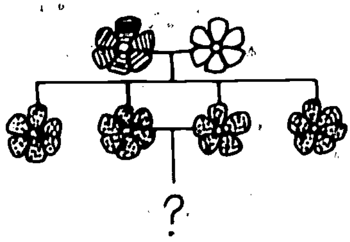
When a red poppy is crossbred with a white poppy, all offspring are pink. What is the percentage composition of the next generation if the pink poppies are crossbred with each other?

Colors of Flowers	Genes
	RR
	RW
	WW

Parental generation

First filial generation

Second filial generation



H. R. Jacobs, Mathematics, A Human Endeavor (San Francisco: W. H. Freeman & Co., 1970), p. 362. (Adapted).

ACTIVITY 16

INDEPENDENCE

FOCUS:

In this activity the ideas of independent and mutually exclusive events and conditional probability are introduced and exemplified. A knowledge of these concepts enables one to avoid many of the common errors in probabilistic reasoning.

DISCUSSION:

Suppose that you flip a fair coin. If you use the symmetry method and assign probability $\frac{1}{2}$ to the outcome of "heads" landing uppermost, then this probability is the same for every flip of the coin, regardless of the outcomes of previous flips. Of course, if you flip 100 straight heads you may begin to doubt your symmetry assumption; but if you retain this assumption, then the probability of heads is the same on every trial. The coin has no mechanism for remembering past outcomes or for adjusting future ones. It is therefore reasonable to say that the outcomes of any two flips are independent.

In more complicated experiments it may be very difficult to tell whether or not two events deserve to be called independent in the same sense as in the above example. For this reason we will give a precise definition and then several applications of the idea.

Let A and B be events (subsets of the sample space). Then $A \cap B$, the intersection of A and B--the set of all outcomes which are in both A and B, is another event. If A is the event that a black card is drawn from a bridge deck and B is the event that a queen is drawn, then $A \cap B$ is the event that a black queen is drawn.

> Let A and B be events with $Pr[B] \neq 0$. The <u>conditional probability</u> of A given B, written $Pr[A|B]$, is defined to be
>
> $$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} .$$
>
> The terminology "A given B" reflects the fact that $Pr[A|B]$ is the probability that A occurs if it is known that B occurs.

In the card-drawing example, if we suppose that the deck is completely shuffled and the draws are made at random, then

$$Pr[A] = \frac{26}{52} = \frac{1}{2}, \quad Pr[B] = \frac{4}{52} = \frac{1}{13}, \quad Pr[A \cap B] = \frac{2}{52} = \frac{1}{26}$$

and consequently,

$$Pr[A|B] = \frac{\frac{1}{26}}{\frac{1}{13}} = \frac{1}{2} .$$

Our conclusion is that the probability of drawing a black card given that we know a queen is drawn is $\frac{1}{2}$.

What is the probability of drawing a queen given that we know a black card is drawn?

$$Pr[\text{queen}|\text{black card}] = \frac{Pr[\text{queen and black card}]}{Pr[\text{black card}]} = \frac{\frac{2}{52}}{\frac{26}{52}} = \frac{1}{13}$$

Notice that in order to answer this question we used the definition of conditional probability with the event A equal to the draw of a queen and the event B equal to the draw of a black card.

In the special case that $Pr[A|B] = Pr[A]$, that is, the probability of A given B is the same as the probability of A, then we say that A and B are <u>independent events</u>. The terminology is very appropriate: A and B are independent if the occurrence of B does not affect the probability that A occurs. For example, suppose that a fair coin is tossed twice and the outcomes noted. The sample space for this experiment is

$$\{(H,H),\ (H,T),\ (T,T),\ (T,H)\}.$$

Using the method of symmetry we assign probability $\frac{1}{4}$ to each of these outcomes. Now let us determine the probability that the second toss results in heads given that the first results in heads. Let

$$A = \{\text{outcomes for which second toss is H}\}$$

$$B = \{\text{outcomes for which first toss is H}\}.$$

Then $A = \{(H,H),\ (T,H)\}$ and $B = \{(H,H),\ (H,T)\}$.

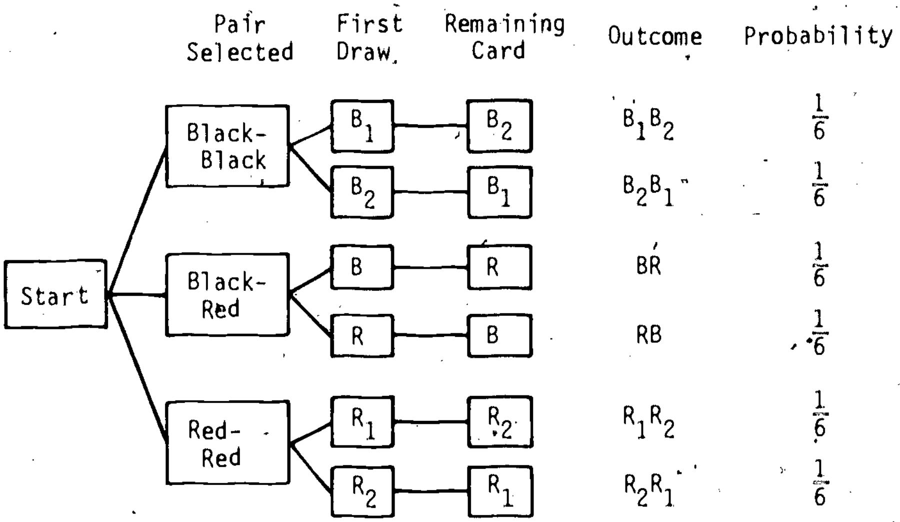
Also $A \cap B = \{(H,H)\}$. Using the definition we find that

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}.$$

Since Pr[heads on second toss] = $\frac{1}{2}$, we conclude that the outcomes on successive tosses of a fair coin are independent events.

To conclude this discussion we return to a point raised in Activity 3 in the discussion of the method of symmetry. The problem posed is the following. Suppose there are three pairs of cards turned face down on a table. One pair is known to consist of two black cards; one pair consists of two red cards; and one pair consists of one red and one black. Suppose a pair is selected at random and one card turned up. Suppose the card is black. What is the probability that the remaining card in the pair is black?

It is helpful to construct a tree diagram for this experiment. Since there are two black cards in one pair, we will distinguish between them by denoting them B_1 and B_2. Similarly, R_1 and R_2 will denote the two red cards in the pair containing only red cards. The probabilities are assigned by symmetry as shown on the following page.



We can compute Pr[remaining card black|card drawn is black] by using the definition of conditional probability. We have the events

$$X = \{\text{remaining card black}\} = \{B_1B_2,\ B_2B_1,\ RB\}$$

$$Y = \{\text{card drawn is black}\} = \{B_1B_2,\ B_2B_1,\ BR\},\ Pr[Y] = \frac{3}{6}$$

$$X \cap Y = \{B_1B_2,\ B_2B_1\},\ Pr[X \cap Y] = \frac{2}{6}.$$

Therefore

$$Pr[\text{remaining card black}|\text{card drawn is black}] = \frac{Pr[X \cap Y]}{Pr[Y]} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}.$$

That is, if the first card drawn is black, then the remaining card is black with probability $\frac{2}{3}$. In terms of frequencies, if you performed the experiment many times, you would expect that $\frac{2}{3}$ of the times you drew a black card it would be from the black-black pair. From this point of view, the answer seems entirely natural. You might find it interesting to look back at the discussion in Activity 3, p. 30.

DIRECTIONS:

Work the problems on the following page designated by your instructor.

1. A card is drawn at random from a completely shuffled bridge deck. What is the probability that it is the jack of diamonds given that it is a face card? Given that it is a red card?

2. A fair coin is flipped twice. What is the probability that the outcome is two heads? What is the probability that the outcome is two heads given that heads comes up at least once?

3. A card is drawn at random from a completely shuffled bridge deck. The event A is that it is a red card and the event B is that it is a face card. Are these events independent? Why?

4. A card is drawn at random from a completely shuffled bridge deck. The event X is that it is a queen, and the event Y is that it is a face card. Are these two events independent? Why?

5. Two events A and B are said to be <u>mutually exclusive</u> if $A \cap B$ is the empty set. Which of the following events are mutually exclusive? Find $A \cap B$ in each case.

   a) Two dice are rolled and the numbers of dots on the uppermost faces are noted. A is the event that one die shows 3; B is the event that the sum of the numbers is even.

   b) A coin s flipped three times and the face which lands uppermost on each flip is noted. A is the event of three heads; B is the event of at most one heads.

   c) A coin is flipped three times and the face which lands uppermost on each flip is recorded. A is the event that the second flip is heads; B is the event that the third flip is heads.

6. Are the events A and B of 5 c) independent?

7. Suppose that A and B are mutually exclusive events and $Pr[B] \neq 0$. Under what conditions are A and B independent?

8. A standard slot machine has three dials. Each dial has 20 symbols; each different symbol occurring the number of times shown on the following page.

	Dial 1	Dial 2	Dial 3
Bar	1	3	1
Bell	1	3	3
Plum	5	1	5
Orange	3	6	7
Cherry	7	7	0
Lemon	3	0	4
	20	20	20

The biggest payoff is for three bars. If you try playing the slot machine, what is the probability that you will win the biggest payoff? Assume that the slot machine is fair--that is, each outcome on each dial is equally likely to occur, and the outcomes on the three dials are independent.

ACTIVITY 17

EXPECTED VALUE

FOCUS:

The outcome of an experiment involving randomness is usually not determinable in advance. Therefore, if each outcome is associated with a number (think of a payoff), then this number is not determinable. However, if one knows the probabilities of the various outcomes, then the average or expected value of the payoff can be computed.

DISCUSSION:

Read the examples below and discuss in your groups.

EXAMPLE 1

Suppose that a fair coin is flipped and that you are paid 3¢ if heads turns up and 1¢ if tails turns up. What is your average gain per throw? Since the coin is assumed fair the probability of heads turning up is $\frac{1}{2}$ and the probability of tails turning up is $\frac{1}{2}$. It follows that you receive 3¢ with probability $\frac{1}{2}$ and 1¢ with probability $\frac{1}{2}$. Your expected gain in cents is therefore

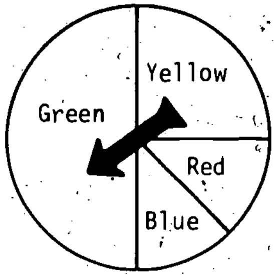
$$(3 \times \tfrac{1}{2}) + (1 \times \tfrac{1}{2}) = 4 \times \tfrac{1}{2} = 2.$$

Notice that the expected gain is different from either one of the payoffs.

EXAMPLE 2

Consider the spinner shown on the following page. Suppose that the payoffs associated with the various outcomes (stopping position of the pointer) are given as follows:

Outcome	Payoff	
blue	4¢	
red	3¢	
yellow	-2¢	(you must contribute 2¢!)
green	1¢	



If the pointer is equally likely to stop in any position, an argument based on symmetry (give the details!) yields

$$Pr[\text{blue}] = \frac{1}{8}, \qquad Pr[\text{yellow}] = \frac{1}{4},$$
$$Pr[\text{red}] = \frac{1}{8}, \qquad Pr[\text{green}] = \frac{1}{2}.$$

The expected value of the payoff in cents is

$$(4 \times \frac{1}{8}) + (3 \times \frac{1}{8}) + ((-2) \times \frac{1}{4}) + (1 \times \frac{1}{2}) = \frac{7}{8}$$

> In general, if an experiment has n outcomes $O_1, \ldots, O_n$, and if the payoffs associated with these outcomes are $a_1, \ldots, a_n$ respectively, then the expected payoff or the expected value of the payoff is
>
> $$a_1 \times Pr[O_1] + a_2 \times Pr[O_2] + \ldots + a_n \times Pr[O_n].$$

EXAMPLE 3

Suppose that you roll a fair die and receive a number of dollars equal to the number of dots on the face that turns up. What is your expected payoff on each roll of the die?

The outcomes (in numbers of dots) are 1, 2, 3, 4, 5, and 6; and since the die is assumed fair, each occurs with the probability $\frac{1}{6}$. The payoff for outcome 1 is \$1, for outcome 2 is \$2, etc. Using the general formula for expected value given above we have

$$\text{Expected value in dollars} = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$
$$= \frac{21}{6}.$$

Notice that the payoffs introduced here are always numbers. This is important, for the expected value is defined only in the case of numerical payoffs and is itself a number.

DIRECTIONS:

Work the exercises designated by your instructor.

1. Suppose that in Example 3 the payoffs were given as follows.

   a) If the outcome is odd, then the payoff (in dollars) is twice the number of dots. If the outcome is even, then the payoff is zero. What is the expected payoff?

   b) If the outcome is odd, then the payoff is twice the number of dots. If the outcome is even, then the payoff is negative and equal in magnitude to the number of dots (the player pays the house!). What is the expected payoff to the player?

2. The number of accidents on the Chicago Skyway on a Monday morning varies from 0 to 5. If the probabilities of the various numbers of accidents are given in the table, what is the expected number of accidents on a randomly selected Monday morning?

number of accidents	0	1	2	3	4	5
probability	.62	.15	.10	.08	.02	.03

3. A student estimates that she will earn a 4.0 grade-point average with probability $\frac{1}{8}$, a 3.5 with probability $\frac{1}{4}$, a 3.0 with probability $\frac{1}{2}$, and a 2.5 with probability $\frac{1}{8}$. What is her expected grade-point average for the semester?

4. A businessman in Milwaukee plans to open either a Brathaus or a Biergarten. The winters in Milwaukee can be quite cold, and on a cold day the Brathaus would do much more business than the Biergarten. On a warm day the situation would be reversed. In fact, he estimates that his profit on cold and warm winter days would be that shown in the table.

Profit in dollars	Brathaus	Biergarten
Cold winter day	300	100
Warm winter day	200	400

If the probability of a cold winter day is $\frac{2}{3}$ and the probability of a warm winter day is $\frac{1}{3}$:

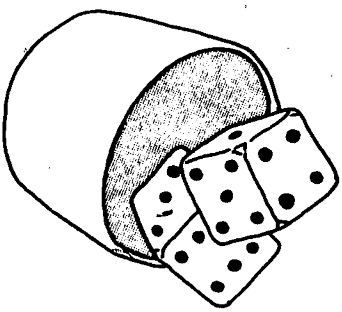
a) What is the expected profit from a Brathaus?

b) What is the expected profit from a Biergarten?

c) Which would tend to make the most money over the winter season?

d) Formulate and solve an analogous problem for the summer season.

TEACHER TEASER



In how many different ways may the numbers on a single die be marked, with the only condition that the 1 and 6, the 2 and 5, and the 3 and 4 must be on opposite sides?

REFERENCES

Discussions of the pedagogical issues which arise in teaching probability and statistics in the elementary school and examples of appropriate instructional activities are contained in the following references:

Fehr, Howard F., and Phillips, Jo McKeeby. Teaching Modern Mathematics in the Elementary School. 2d ed., Reading, Mass.: Addison-Wesley Publishing Co., 1972.

Page, D. A. "Probability," in The Growth of Mathematical Ideas, Grades K-12, Twenty-Fourth Yearbook of the National Council of Teachers of Mathematics. Washington, D.C., 1959.

Smith, R. R. "Probability in the Elementary School," in Enrichment Mathematics for the Grades, pp. 217-133. Twenty-seventh Yearbook of the National Council of Teachers of Mathematics. Washington, D. C., 1963.

The Teaching of Probability and Statistics. (Proceedings of the first C.S.M.P. International Conference Co-sponsored by Southern Illinois University and Central Midwestern Regional Educational Laboratory). Edited by Lennart Råde. New York: John Wiley & Sons, 1970.

The perspective which children bring to activities involving randomness is considered in the reference:

Piaget, Jean. The Child's Conception of Physical Causality, tr. by Marjorie Grabain. Totowa, N. J.: Littlefield, Adams and Co., 1969.

Probability materials written for school students, elementary and secondary, which expand upon the content of the probability sections of this unit include the following:

CEMREL. Elements of Mathematics. Book 0: Intuitive Background, Chapter 8: "An Introduction to Probability," St. Ann, Missouri: Central Midwestern Regional Education Laboratory, Inc., 1971.

Nuffield Mathematics Project, Probability and Statistics. New York: John Wiley and Sons, Inc., 1969.

School Mathematics Study Group. Introduction to Probability. Part 1: Basic Concepts, Part 2: Special Topics. Palo Alto, Calif.: Leland Stanford Junior University, 1966, 1967.

------------------------------. Probability for Intermediate Grades. Palo Alto, Calif.: Leland Stanford Junior University, 1966.

------------------------------. Probability for Primary Grades. Palo Alto, Calif.: Leland Stanford Junior University, 1966.

------------------------------. Secondary School Mathematics. Unit 6, Chapter 11, "Probability." Palo Alto, Calif.: Leland Stanford Junior University, 1966.

------------------------------. Secondary School Mathematics. Special edition, Chapter 7, "Probability." Palo Alto, Calif.: Leland Stanford Junior University, 1970.

Other sources of content-oriented material on probability or probability and statistics at approximately the same level as this unit include the references:

Brumfiel, C. F., and Krause, E. F. Elementary Mathematics for Teachers, Chapter A, Reading, Mass.: Addison-Wesley Publishing Co., 1969.

Jacobs, Harold R.. Mathematics, A Human Endeavor, Chapters 7, 8, 9. San Francisco: W. H. Freeman and Co., 1970.

Meserve, Bruce, and Sobel, Max. Contemporary Mathematics, Chapters 13, 14. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1972.

NCTM. More Topics in Mathematics for Elementary School Teachers, in Thirtieth Yearbook of NCTM, Booklet 16. Washington, D. C.: National Council of Teachers of Mathematics, 1969.

There are many books on the use and misuse of statistics written for the layman. The book Statistics: A Guide to the Unknown is used in Activity 13.

Campbell, Stephen K. Flaws and Fallacies in Statistical Thinking. Englewood Cliffs, New Jersey: Prentice-Hall, Inc. 1974.

Huff, Darrell. How to Lie with Statistics. New York: W. W. Norton & Co., Inc., 1954.

Tanur, Judith M., et al. Statistics: A Guide to the Unknown. San Francisco: Holden-Day, 1972.

The Joint Committee on the Curriculum in Statistics and Probability of the American Statistical Association and the National Council of Teachers of Mathematics has prepared a series of booklets for use by secondary students. The series is entitled Statistics by Example.

Mosteller, Frederick, ed. Statistics by Example. Part 1: Exploring Data, Part 2: Weighing Chances, Part 3: Detecting Patterns, Part 4: Finding Models. Reading, Mass.: Addison-Wesley, 1973.

Other examples of material prepared for the nonspecialist, this time on probability and games, are the books:

Huff, Darrell. How to Take a Chance. New York: W. W. Norton & Co., Inc., 1959.

McDonald, John. Strategy in Poker, Business and War. New York: W. W. Norton & Co., Inc., 1950.

Activities in probability and statistics suitable for use beginning in grade 7 are contained in:

Johnson, Donovan, et al. Activities in Mathematics. First course, Probability and Second course, Statistics. Glenview, Ill.: Scott, Foresman and Co., 1971.

REQUIRED MATERIALS

ACTIVITY	AUDIO-VISUAL	MANIPULATIVE AIDS	READINGS
Overview	Slide-tape: "Overview of Probability and Statistics," cassette recorder and projector. (Optional)		
1		Several circular cylinders with different cross section-to length ratios, a bag of 25 chips of 4 different colors (red, white, blue, green), a spinner with 3 equal colored sectors (blue, green, red), a fair coin.	
3		Two dice, a bridge deck.	
9		One spinner per group, paper and colored pencils.	
10		Answers to questions 1 and 3 of the assignment given on page of the Overview.	

ACTIVITY	AUDIO-VISUAL	MANIPULATIVE AIDS	READINGS
13 OPTIONAL			Mosteller, Frederick, and David L. Wallace, "Deciding Authorship," *Statistics: A Guide to the Unknown*, edited by Judith M. Tanur, *et al*. San Francisco: Holden-Day, 1972, pp. 164-174.

Continued from inside front cover

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Addition and Subtraction, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Transformational Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Number Theory,* and *Experiences in Problem Solving.*

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