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ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to mathematical topics for the elementary teacher. In addition to an introduction to the unit and an overview of number theory, the student text has sections on divisibility, prime numbers, and factorization; problems and problem solving; applications, connections, and generalizations; and an appendix that presents a detailed example of problem solving. The instructor's manual parallels the pupils text and provides comments and suggested procedure, major questions, rules for materials preparation, and answers to problems in the student document. (MP)

MATHEMATICS EDUCATION DEVELOPMENT CENTER

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NUMBER THEORY

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

Numeration
Addition and Subtraction
Multiplication and Division
Rational Numbers with Integers and Reals
Awareness Geometry
Transformational Geometry
Analysis of Shapes
Measurement
Number Theory
Probability and Statistics
Graphs: the Picturing of Information
Experiences in Problem Solving

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program,
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY,
CHICO

CALIFORNIA STATE UNIVERSITY,
NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT
DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY
EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-
GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugene, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

CONTENTS

INTRODUCTION TO THE NUMBER THEORY UNIT

Questions involving the counting numbers 1, 2, 3, ..., are as old as mathematics itself. Some natural questions, when viewed in the right light, are easily answered. Others, which appear equally natural and answerable, have required the efforts of some of the world's best mathematicians to resolve. Indeed, some have defied all attempts at solution and are of current research interest. Number theory, perhaps more than any other branch of mathematics, has profited from the efforts of amateurs. In this context we use the term <u>amateur</u> to refer to a nonprofessional; it carries no connotation of incapability. In fact, some of the most interesting and profound questions in the subject have been raised and studied by amateurs. The many contributions by amateurs have been encouraged by the fact that the problems are frequently quite near the surface; i.e., they occur in the normal course of a thoughtful study of the counting numbers, and they can be attacked by methods that do not depend on the development of an elaborate mathematical theory. The fact that number theory abounds with such questions is one cogent reason for including some number-theory work in the elementary school.

In addition to being a source of questions that can provide children with problem-solving experiences, number theory has a direct relationship to the arithmetic curriculum. Teachers find that number-theory activities can help to strengthen skills with basic facts in an interesting, nonroutine setting. Number-theory ideas are involved in computing common denominators for fractions. Besides, children just seem to enjoy looking for the patterns that can be found in the counting numbers.

The unit begins with an overview that focuses on the historical development of number theory and the place of number theory in the elementary school curriculum. There is a short list of definitions immediately following this Introduction so that the terminology and notation used in the overview and throughout the unit will be clear.

Section I includes the basic concepts of divisibility, primes, and factorization. Most of this material appears explicitly in elementary school programs.

Section II is concerned with problem solving. Elementary number theory is an extraordinarily fruitful source of easily understood but challenging problems, and this section presents an organizational scheme for attacking such problems. Several problems of varying degrees of difficulty are collected at the end of the section. You are invited to test your skill on them. Another problem like the ones of Section II is posed and solved in detail in the Appendix.

Section III illustrates how some of the ideas introduced earlier in the unit can be extended and applied in other situations.

Throughout the unit there are projects that are more substantial than exercises and pursue ideas not followed up in the activities. These projects, which may serve as the basis for reports to the class, round out this presentation of elementary number theory.

The unit closes with a selected bibliography for those who wish to delve further into some aspect of number theory. The references cited include some on history, content, and pedagogy.

DEFINITIONS OF TERMS USED IN NUMBER THEORY

The terms defined on this page are used throughout the unit. Many of them are familiar to you, but they are included so that you will be certain of their meanings. It is important that we have a precise vocabulary, in which the terms have the same meanings for all of us. In the definitions that follow, the symbols a, b, c, d, ... denote counting numbers, i.e., numbers selected from the set {1, 2, 3, ...}.

1. a is a <u>multiple</u> of b if there is a c such that a = bc. In this case, b and c are <u>factors</u> of a.
   Thus, 20 is a multiple of 5, and 4 and 5 are factors of 20.

2. The <u>set of factors</u> of a, or the <u>factor set</u> of a, is the set of <u>all</u> factors of a.
   The set of factors of 60 is
   {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}.

3. The <u>set of multiples</u> of a is the set of <u>all</u> multiples of a.
   For example, the set of multiples of 3 is
   {3, 6, 9, 12, 15, ...}.
   Note that the set of multiples of 3 is an infinite set.

4. a is <u>prime</u> if the factor set of a contains exactly two elements.
   The first eight prime numbers are 2, 3, 5, 7, 11, 13, 17, 19.
   The factor set of 5, for example, is {1, 5}.

5. a is <u>composite</u> if the factor set of a contains more than two elements.
   The first ten composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, 18. The factor set of 9, for example, is {1, 3, 9}.
   <u>Note</u>: The factor set of 1 is {1}. So 1 is neither prime nor composite. All other counting numbers are either prime or composite.

6. a <u>divides</u> b or a is a <u>divisor</u> of b if a is a factor of b.
   If a divides b, we write $a|b$. For example,
   5 divides 20 since $20 = 5\cdot 4$.

7. a and b are <u>relatively prime</u> if there is no c, $c \neq 1$, such that $c|a$ and $c|b$; i.e., a and b do not share a factor besides 1.
   The numbers 15 and 22 are relatively prime, even though neither one happens to be prime.

OVERVIEW

FOCUS:

You will be asked to read or view an overview of the contents of this unit. The first part of the overview will present a few highlights of the history of number theory. The remaining part will focus on the roles of number theory in the elementary mathematics curriculum. Two roles are illustrated: the application of number-theory ideas to the basic processes of arithmetic and the use of number theory as a medium for problem-solving experiences.

MATERIALS:

(Optional) The Mathematics-Methods Program slide-tape presentation entitled, "Overview of Number Theory."

DIRECTIONS:

Read the essay entitled "Overview of Number Theory" which starts on page 7 or view the slide-tape overview of the same title, and then engage in a brief classroom discussion of some of the points raised in it. The questions which follow can serve as a basis for discussion. These questions should be read before reading or viewing the overview.

1. Why is the strand "number theory" included in the elementary mathematics curriculum?
2. Some educators have suggested that the number-theory strand is ideally suited to a more child-oriented and less teacher-oriented instructional mode. Discuss this statement and provide arguments to support your position.
3. How do you know that Fermat's conjecture holds for $n = 2$?

4. Construct another magic square using the numbers 1 through 9.

A list of definitions and notations is given on pages 3/4. You may wish to look over this list before reading or viewing the overview. The terms defined there will be used throughout the unit.

AN OVERVIEW OF NUMBER THEORY

Number theory, as a branch of mathematics, has an especially rich history and, as a source of interesting and sometimes surprising problems, it is unsurpassed. There are topics from number theory that occur explicitly in the elementary school curriculum, e.g., odd and even numbers, prime and composite numbers, factor trees, and least common multiple. Some of these ideas--for instance, odd and even numbers, and the connection between the concept of prime numbers and the representation of sets of objects in arrays--can be presented at the primary level. Other ideas--for instance, factor trees and least common multiple--occur more naturally in the upper elementary grades. At all levels, one can pose interesting challenge-type problems with number-theoretic content. This unit contains a very brief introduction to some of the most basic ideas of number theory, a glimpse into the history of the subject and a few of its many famous problems, and a sample of the ways in which these ideas occur in the elementary curriculum. Throughout the unit you are urged to "participate" in the mathematics. Read with a pencil in hand, check the computations, and create your own examples. Some of the greatest number theorists of history have been amateurs.

Soon after early humans learned to count and to perform the arithmetic operations, they began to speculate on the properties of the counting numbers. Among the earliest indications of interest in matters that have number-theory content are the myths and superstitions of numerology. Many cultures had numbers to which they attached special significance. Examples

are the ancient Hindus and 10, Old Testament Jews and 7, and certain American Indians and 4. Since the Greeks expressed numbers by means of letters in their alphabet, each word and, in particular, each name was associated with a number. This association fostered interest in the mysticism of numbers. Relics of ancient numerology remain in our society. For instance, very few hotels or apartment buildings have a floor numbered 13.

One of the earliest problems that would today be classified as belonging to number theory is the problem of determining "Pythagorean triples." This designation, incidentally, is not entirely appropriate since the problem had been studied by the Babylonians several centuries before Pythagoras. The problem is that of finding number triples such as 3, 4, 5 for which $3^2 + 4^2 = 5^2$. The Babylonians had discovered some Pythagorean triples consisting of relatively large numbers, for example, 12,709, 13,500, 18,541, and evidently knew something of a general method for constructing them.

The prime numbers, that is numbers such as 2, 3, 5, 7, ..., which have no divisors other than 1 and themselves, were the subject of systematic study by the Greeks. Three of the thirteen books of Euclid's Elements (VII through IX) were devoted to number theory, and here we find Euclid's algorithm for determining the greatest common factor of a pair of counting numbers (see Activity 13) and a proof of the fact that there are infinitely many prime numbers (see Part B of Activity 5). About 100 years after Euclid, Eratosthenes developed a "sieving" procedure for identifying those counting numbers that are prime (see Part A of Activity 5). As a result of their interest in numerology, the Greeks investigated the properties of special types of numbers, for example, the <u>perfect numbers</u>, which are the sum of all their divisors (including 1 but excluding the number itself, as $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$).

In the seventeenth century, the earlier work on number theory was organized and significantly extended, and the outlines of what is now known as the mathematical area of number theory began to take shape. The French mathematician Pierre de Fermat (1601-1665) is considered by many to be the founder of the theory of numbers as an independent mathematical area. To convey the flavor of his research, it is easiest to give two of his results, which can be understood without any specialized knowledge. The two we have selected are:

<u>Every counting number can be expressed as the sum of the squares of four whole numbers.</u>

<u>If p is a prime and of the form $4n + 1$ where n is a counting number, then p can be expressed as the sum of the squares of two counting numbers. This expression is unique up to the order of the terms in the sum. No prime of the form $4n + 3$ can be so expressed.</u>

In addition to the many theorems for which he provided proofs, Fermat is famous for a theorem, or actually a conjecture, for which he did not leave a proof, at least so far as we know. It is easy to make conjectures in mathematics; it is difficult to make conjectures that influence the development of an area or stimulate a great deal of significant work. Fermat's conjecture, also known as Fermat's last theorem,

is of the latter sort. It is:

> The equation $m^k + n^k = p^k$ has solutions m, n, p in counting numbers <u>only</u> for $k = 1$ and $k = 2$ (examples for $k = 1$ are $3 + 2 = 5$ and $20 + 21 = 41$, and for $k = 2$ are $3^2 + 4^2 = 5^2$ and $12^2 + 5^2 = 13^2$).

Conjectures abound in number theory. Several others are mentioned in the exercises of the unit.

Like most areas in mathematics, number theory does not exist in isolation, but has significant connections with other areas. As an illustration we mention the number-theoretic aspects of certain geometric constructions.

A Euclidean construction is a geometric construction that can be performed using only straightedge and compass. The German mathematician C. F. Gauss (1777-1855) proved that there is a Euclidean construction of a regular polygon with N sides if and only if N is the product of distinct primes of a certain type and 2 raised to some power. The Greeks knew the Euclidean constructions for polygons with 2, 4, 8, 16, ... sides and with 3 and 5 sides. Combining these they could construct polygons with $3 \cdot 2^c$ and $5 \cdot 2^c$ sides, where c is any whole number.

Magic squares such as this one whose rows, columns and diagonals sum to 15, have a long history. Some date back to ancient China, while others are much more recent. Benjamin Franklin constructed some remarkable magic squares and also magic circles (see the reference <u>Invitation to Number Theory</u> by O. Ore).

2	9	4
7	5	3
6	1	8

There remain many unsolved problems in number theory that are the subject of current mathematical research. It is interesting that the modern digital computer is a useful tool for this research. The ability of the computer to carry out millions of arithmetic calculations each second permits the use of techniques that are completely unfeasible by hand or with desk calculators.

Certain topics from number theory, for example, odd and even numbers, multiples and factors, have gradually become a part of the elementary curriculum. Recently, other topics, such as primes and factor trees, have been added. One reason for the inclusion of selected topics from number theory is that they can be applied to other topics in the elementary mathematics curriculum. For example, odd and even numbers, multiples, and factors are closely related to the basic ideas of multiplication and division of whole numbers. Other examples are the use of the greatest common factor in simplifying a common fraction:

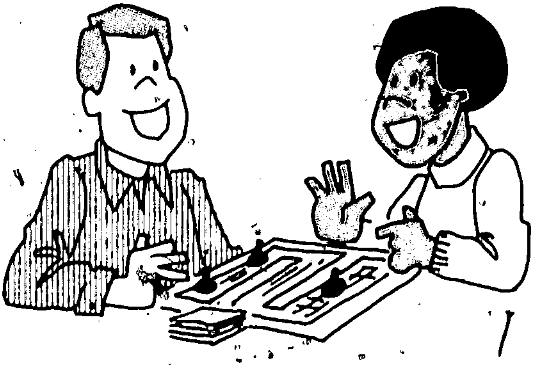
$$\frac{8}{12} = \frac{2\cdot 4}{3\cdot 4} = \frac{2}{3},$$

(here 4 is the greatest common factor of 8 and 12), and the use of the least common multiple in adding two fractions with different denominators:

$$\frac{5}{8} + \frac{1}{6} = \frac{15}{24} + \frac{4}{24} = \frac{19}{24}.$$

(Here 24 is the least common multiple of 8 and 6.) In this example, the least common multiple of 8 and 6 was used as a denominator in appropriate equivalent forms of $\frac{5}{8}$ and $\frac{1}{6}$.

Another reason for including topics from number theory in the elementary school is that there are many interesting and challenging problem-solving activities with a number-theory flavor. For example, we mention activities with figurate numbers (see Activity 9 and the Appendix) and the games of <u>Multo</u>* and <u>Prime Drag</u>**.



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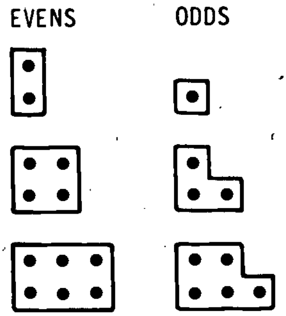
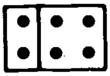
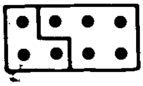
*<u>Multo</u>. Urbana, Illinois: University of Illinois Curriculum Laboratory/Booker T. Washington School, 1969.

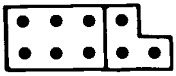
**<u>Prime Drag</u>. Palo Alto, Cal.: Creative Publications, Inc., 1969.

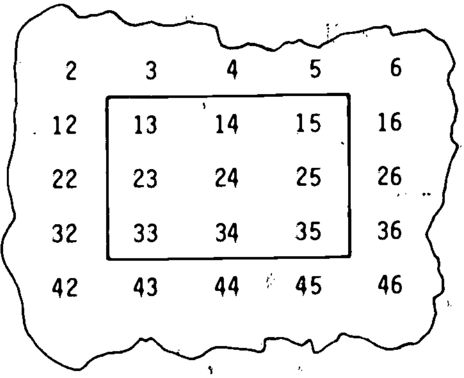


Although there are obviously more alternatives at the upper elementary grades, there are nevertheless many number-theoretic topics and situations that can be explored with primary-level children. For example, the properties of even and odd numbers can be developed with the use of rectangular arrays. The even numbers such as 2, 4, 6, can be represented by a rectangular array with two complete rows; but the odd numbers, such as 1, 3, 5, cannot be represented by a rectangular array with two complete rows. Using rectangular arrays can aid in establishing the arithmetic properties of the even and odd numbers. For example, an even number added to an even number results in an even number, an odd added to an odd results in an even, and an even added to an odd results in an odd. Also, there are interesting number patterns on hundred's charts and calendars. For example, have you observed that on a 10 x 10 hundred's chart none of the numbers surrounding 24 is divisible by the factors of 24, 4 and 6, while on a calendar there are numbers surrounding 24 which <u>are</u> divisible by 6 (18) and 4 (16)? Notice also that the numbers on one diagonal through the 24 on a calendar are divisible by 8, whereas none of the numbers on the other diagonal is divisible by 8, but they are all


even + even = even


odd + odd = even

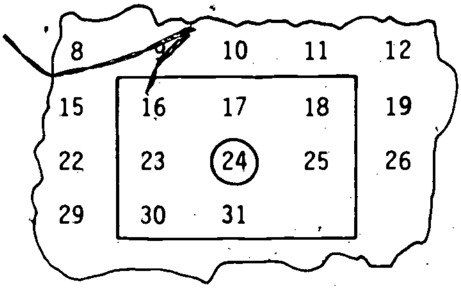

even + odd = odd



2	3	4	5	6
12	13	14	15	16
22	23	24	25	26
32	33	34	35	36
42	43	44	45	46

HUNDRED'S CHART

divisible by 3. Is there an explanation for these facts?



CALENDAR

Finally, there are problems in elementary mathematics that can be solved directly by number theory. For example, how many different rectangles can be constructed with sides of integral measures and an area of 24 square units? Are there more rectangles with different shapes with area 27 square units or 24 square units? It is clear that the answers to these questions are related to the <u>number of divisors</u> of 24 and 27. In fact, since 24 has eight divisors, the number of different-shaped rectangles with area 24 is four. Why? Similarly, the number of different-shaped rectangles of area 27 is two.

There are activities in the unit designed to expand your view of number theory and to provide you with ideas and techniques that may help you appreciate the number-theoretic content of the mathematics you will meet in and out of the classroom. There are also activities whose primary goal is to develop material that is frequently found in elementary textbooks. Finally, there are activities whose purpose is to alert you to some aspects of the problem-solving process.

DIVISIBILITY, PRIME NUMBERS AND FACTORIZATION

The prime numbers are the multiplicative building blocks of the counting numbers: if a counting number is factored into a product of smaller numbers each of which can be factored no further, then the smaller numbers are prime numbers. In addition to this basic property, there are many other fascinating relationships and patterns involving prime and counting numbers that are interesting in their own right.

The purpose of Activities 1 through 4 is the development of the concept of prime number, the prime numbers as "building blocks" in the counting numbers, and the related idea of divisibility. The questions "How many prime numbers are there?" and "How are the prime numbers distributed in the sequence of counting numbers?" are considered in Activity 5. The topics of Activity 6 are the notions of least common multiple and greatest common divisor and their relationship to prime numbers and prime factorization. The final activity, Activity 7, is a seminar, which reviews the section and asks questions related to the study of number theory as an intellectual activity and the role of number theory in the elementary school. Distributed through the section there are three projects that you will be asked to complete outside of class. They are more substantial than exercises and should serve as an introduction to some of the kinds of questions and mathematical problems lying near the surface in number theory.

MAJOR QUESTIONS

1. Discuss in your own words the statement "The prime numbers act as building blocks for the counting numbers." Give examples to illustrate your discussion. Are there other sets of numbers that serve as building blocks for the counting numbers in a different sense?

2. Discuss the advantages of using trains or tiles to introduce the concepts of prime and composite numbers with children. In your discussion, compare the method of trains or tiles with a method that proceeds by definition and examples.

3. Discuss how a multiplication table could be used to introduce the concepts of prime and composite numbers to children. Compare this method with others outlined in the section.

4. Is it worthwhile to introduce the concept of prime number in the elementary school? Support your answer.

5. Identify those number-theoretic concepts that are suitable for inclusion in the primary grades. For example, you might cite even and odd numbers.

6. How could Cuisenaire rods be used to introduce the idea of the least common multiple of two numbers?

7. From your general knowledge of mathematics, find another instance (there are several) in which a set of mathematical objects or concepts can be "constructed" in some sense from a proper subset of that set. [The counting numbers (the set), primes (proper subset), and the operation of multiplication (method of constructing) provide one such instance.] Describe how the whole set is to be constructed from the subset. Illustrate your explanation with examples.

ACTIVITY 1
DIVISIBILITY

FOCUS:

Many of the facts, problems, and results of elementary number theory involve the idea and properties of divisibility. In this activity we collect and organize some information that will be used throughout the unit.

DISCUSSION:

When one divides one counting number by another, the special case of a zero remainder is sufficiently interesting to merit further study. For example, 15 divided by 3 has remainder zero, as does 28 divided by 7. In such circumstances we say that 3 divides 15 evenly, or as we prefer here, simply 3 divides 15. Likewise we say 7 divides 28. We introduce the notation $7|28$, which should be read "7 divides 28," to express this divisibility property symbolically. Since $28 = 7 \times 4$, we also have $4|28$. In general, if a and b are counting numbers, then $a|b$ means that a divides b, or equivalently, b is a multiple of a.

EXAMPLES

$35 = 5\cdot7$, and consequently $5|35$ and $7|35$.

$20 = 5\cdot4$, and consequently $5|20$ and $4|20$.

Since $20 = 5\cdot2\cdot2$, we also have $2|20$.

Since $21 = 3\cdot7$ and $6 = 2\cdot3$, it follows that $3|21$ and $3|6$. Is it also true that $3|(21 + 6)$ and $3|(21 - 6)$? Using the fact that $21 + 6 = 27 = 3\cdot9$ and $21 - 6 = 15 = 3\cdot5$, we see that $3|(21 + 6)$ and $3|(21 - 6)$. Does this hold for numbers different from 3, 21 and 6? The answer is yes, and a general statement of the property exemplified above with

3, 21, and 6 is as follows:

> Let a, b, c be counting numbers, $a \neq 0$. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ and $a \mid (b - c)$.

To help us understand why this is the case, we will rewrite the argument given above for the special case $a = 3$, $b = 21$, $c = 6$, in one column and the argument for arbitrary a, b, c in an adjacent column.

	Special Case	General Case
i.	Since $3 \mid 21$, 21 is a multiple of 3; $21 = 3 \cdot 7$	Since $a \mid b$, b is a multiple of a; suppose $b = m \cdot a$, where m is a counting number
ii.	Since $3 \mid 6$, 6 is a multiple of 3; $6 = 2 \cdot 3$	Since $a \mid c$, c is a multiple of a; suppose $c = n \cdot a$, where n is a counting number
iii.	$21 + 6 = 7 \cdot 3 + 2 \cdot 3 = 9 \cdot 3$, so $3 \mid (21 + 6)$, by the definition of divisibility	$b + c = m \cdot a + n \cdot a = (m + n) \cdot a$, so $a \mid (b + c)$ by the definition of divisibility
iv.	$21 - 6 = 7 \cdot 3 - 2 \cdot 3 = 5 \cdot 3$, so $3 \mid (21 - 6)$	$b - c = m \cdot a - n \cdot a = (m - n) \cdot a$, so $a \mid (b - c)$

DIRECTIONS:

1. Write out an argument as in the Special Case above for $a = 6$, $b = 42$, $c = 18$.
2. Try to write out an argument, as in the Special Case above, for $a = 6$, $b = 42$, $c = 15$. What is wrong?
3. If $a \mid b$ and $a \mid c$, does it follow that $a \mid b \cdot c$? Why?
4. Give three examples that help you to conjecture an answer to the question: If $a \mid b$ and $a \mid (b + c)$, does $a \mid c$?

5. If $a|b$ and $b|c$, does $a|c$? Give three examples; then answer the question.

6. If $a|b$ and $a|c$, does $a|(2b + 3c)$? Give an example; then write out an argument in two columns, as above, which provides an answer to the general assertion.

ACTIVITY 2
PRIME AND COMPOSITE NUMBERS

FOCUS:

Prime numbers can be viewed as the basic building blocks or "indecomposable" counting numbers. In this activity, we introduce prime and composite numbers by working with either a set of Cuisenaire rods or a set of tiles. These two ways of introducing the ideas, both of which utilize concrete materials, are easily adapted to the elementary classroom.

MATERIALS:

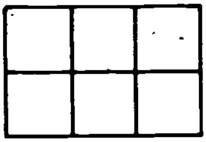
Set of Cuisenaire rods and at least 20 tiles.

DIRECTIONS:

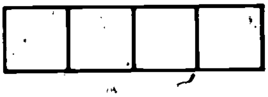
You (or your group) will be assigned either Part A or Part B by your instructor. After completing the assigned part, discuss and compare your work with someone who has completed the other part.

PART A: Primes and Tiles

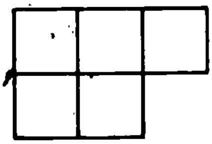
Tiles can be arranged to form <u>rectangular arrays</u>. The arrays shown in Figures 1 and 2 below are rectangular arrays, while the array shown in Figure 3 is not. (Rectangular arrays can be presented to children as chocolate bars--a candy bar divided into sections.) We will always think of rectangular arrays as those in which the sides of the tiles are parallel to the edges of the table on which you are working.



2 x 3 array
Figure 1



1 x 4 array
Figure 2



not a rectangular array
Figure 3

1. Take two of the tiles and determine how many rectangular arrays there are if a 1 x 2 array (two tiles across and one tile up) is considered different from a 2 x 1 array (one tile across and two tiles up). Deduce the set of divisors of 2 from the sizes of the arrays. This information has been recorded in the line labeled 2 in the leftmost column of Table A (next page).

2. Take three tiles and determine the number of rectangular arrays which can be formed from three tiles. Record this information in the line labeled 3 in Table A. Continue for four, five, six, ..., twelve tiles, and complete Table A.

3. Use Table A to identify those numbers between 2 and 12 (inclusive) that have only two divisors: 1 and the number. List them. All counting numbers with this property are called <u>prime</u> numbers.

4. Use Table A to identify those numbers between 2 and 12 (inclusive) that have <u>more</u> than two divisors. All counting numbers with this property are called <u>composite</u> numbers. Find those composite numbers that have an odd number of divisors. What is another way of describing these numbers?

5. There is one counting number that has less than two divisors. What is it? This number is placed in a class by itself since it is neither prime nor composite according to the definitions.

TABLE A

Number of Tiles	Number of Rectangular Arrays	Dimensions of Each Array	Number of Divisors	Divisors
2	2	2 x 1 1 x 2	2	1, 2
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				

It is interesting to note that activities like this one actually appear in elementary text series, cf., Heath Elementary Mathematics IV (1975), p. 232.

PART B: Primes and Trains

In this part an arrangement of equivalent Cuisenaire rods (rods of the same length) placed end to end will be called a <u>train</u>. Three trains of the same total length are shown in Figure 4.

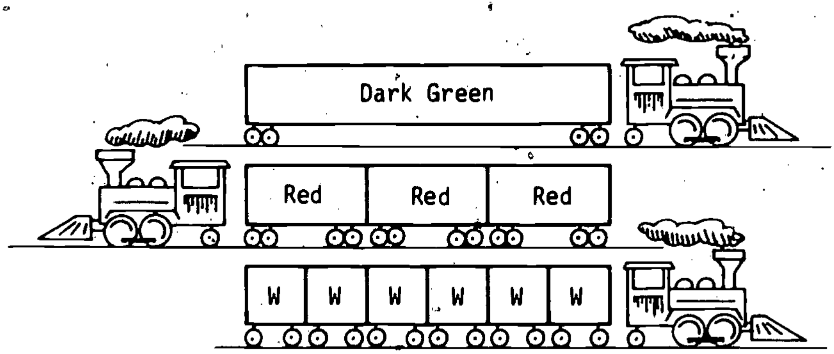


Figure 4

There is one more train of length 6. Can you find it?

Trains can be described mathematically by multiplication. For example, in Figure 4 we can describe the top train by

$$1 \times 6 \ (= 6),$$

the middle train by

$$3 \times 2 \ (= 6),$$

and the bottom train by

$$6 \times 1 \ (= 6).$$

In each case the first figure on the left refers to the number of rods and the second figure refers to the rod value. The product in parentheses gives the total length of the train. That is,

$$\underset{\substack{\uparrow \\ \text{Number of} \\ \text{rods in} \\ \text{the train}}}{3} \qquad \times \qquad \underset{\substack{\uparrow \\ \text{Rod} \\ \text{value}}}{2} \qquad \underset{\substack{\uparrow \\ \text{Length} \\ \text{of train}}}{(= 6)}$$

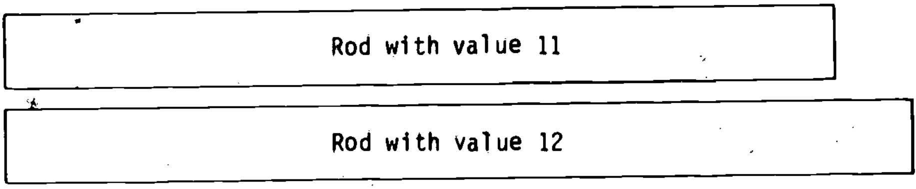
Number-Color Code

White is one unit

White (W)	1	Dark Green	6
Red	2	Black	7
Light Green	3	Brown	8
Purple	4	Blue	9
Yellow	5	Orange	10

1. How many trains have the same total length as a red rod? Deduce the divisors of 2 and record this information in the line labeled 2 in the leftmost column of Table B.

2. Repeat (1) for the light green rod. Continue for the rods with values 4, 5, 6, ..., 10. Rods of lengths 11 and 12 are pictured below. Use them to complete Table B.



3. Use Table B to determine which numbers between 2 and 12 (inclusive) can be the length of only two trains. Such numbers are called <u>prime</u>. How many divisors does each prime number have? (Be sure to count 1 and the number itself.)

4. Use Table B to determine which numbers between 2 and 12 (inclusive) can be the length of at least three trains. Such numbers are called <u>composite</u>. Some composite numbers can be the length of an odd number of trains. What is another way of describing these numbers?

5. A prime number has exactly two divisors and a composite number has more than two divisors. There is one counting number that has exactly one divisor. What is it? It is placed in a class by itself since it is neither prime nor composite, according to the definitions.

TABLE B

Color of Rods	Rod Number	Number of Trains of Equivalent Rods	Number of Divisors	Divisors
Red	2			
	3			
	4			
	5			
	6			
	7			
	8			
	9			
	10			
	11			
	12			

It is interesting to note that activities like this one actually appear in elementary text series, cf., Heath Elementary Mathematics IV (1975), p. 232.

ACTIVITY 3
FACTOR TREES AND FACTORIZATION

FOCUS:

The idea of the primes as "building blocks" is pursued in this activity, and the representation of any counting number as a product of primes is introduced. Factor trees and their relation to factorization will be considered. This activity consists of three parts:

Part A: Factor Trees
Part B: Factorization Into Primes
Part C: Exponential Notation and the Prime Factorization Theorem

PART A: Factor Trees

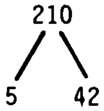
In constructing factor trees, we shall not admit factoring into factors one of which is 1. For example, we shall not admit $7 = 1 \cdot 7$ or $12 = 1 \cdot 12$.

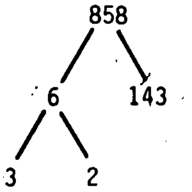
DISCUSSION:

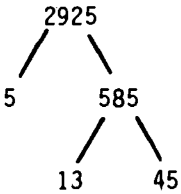
The construction of factor trees such as those shown on the accompanying textbook page (page 28) can be an interesting and enjoyable child activity in itself. Moreover, constructing a tree is a quick and easy method of finding the prime factors of a number. In this activity, the emphasis is on the construction of factor trees and on the relation between the factor trees of a number and the factorization of that number into prime factors. The use of the prime factorization of numbers in adding, multiplying, and "reducing" fractions will be the topic of Activity 6.

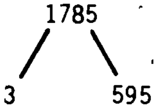
DIRECTIONS:

1. Examine the textbook page reproduced on page 28 of this unit. For each of the four numbers below, extend its factor tree until you have obtained <u>all</u> the prime factors of the numbers.

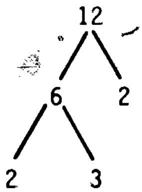




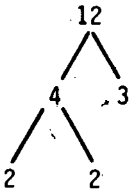




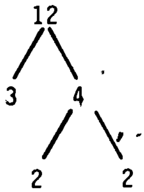
2. Let us agree that two factor trees are <u>different</u> if there is at least one number that is factored into one pair of factors in one tree and into a different pair of factors in the other tree. For example, consider the three factor trees of 12 shown below.



(i)



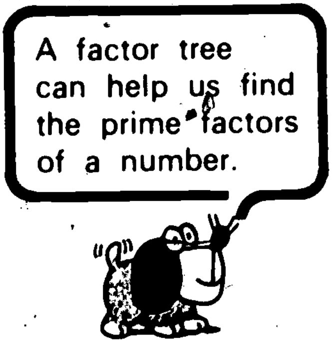
(ii)

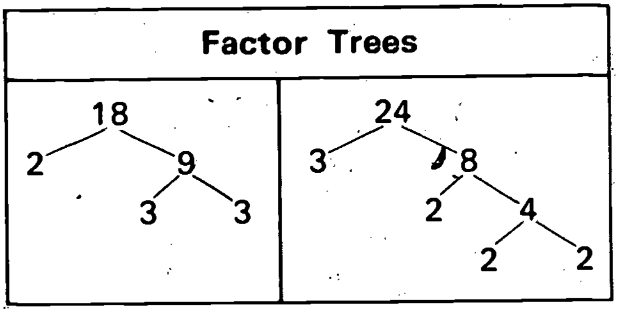


(iii)

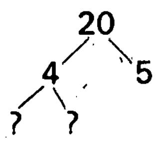
Factor trees (i) and (ii) are different, while the trees (ii) and (iii) are not different. The above example shows that a composite number may have more than one factor tree; i.e., it may have at least two different factor trees.

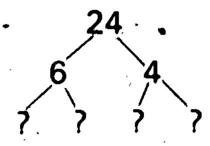
Performance Objective: To use a factor tree

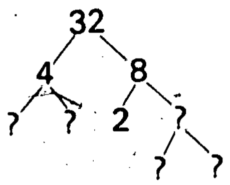




Copy and complete each factor tree.

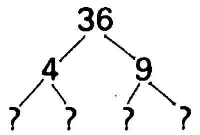
1.


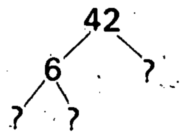
2.


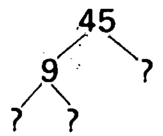
3.


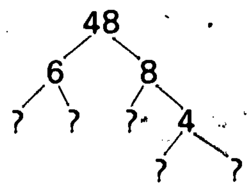
EXERCISES

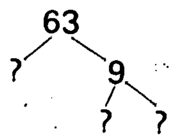
Copy and complete.

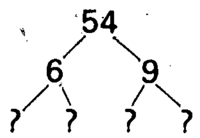
1.


2.


3.


4.


5.


6.


Make a factor tree for each.

1. 81 2. 124 3. 144 4. 108

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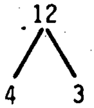
Silver Burdett Mathematics; Grade 4, p. 299. Reproduced with permission.

A factor tree will be called complete if the numbers at the ends of the "twigs" are all prime numbers. For example (i), (ii), and (iii) above are all complete factor trees while



is not complete, since 4 is composite.

a) Find all the different complete factor trees of 30. (Hint: There are three.)

b) Find all the different complete factor trees of 36. (Hint: There are six.)

c) Find all the different complete factor trees of 72.

3. Find three composite numbers that do not have at least two different factor trees. What is their common property?

4. Is each prime number already a complete factor tree? Given any whole number except 1, do you think that it is possible to construct a complete factor tree for that number? Explain your reasoning.

5. Write each of the following numbers as a product of primes.

30 =

36 =

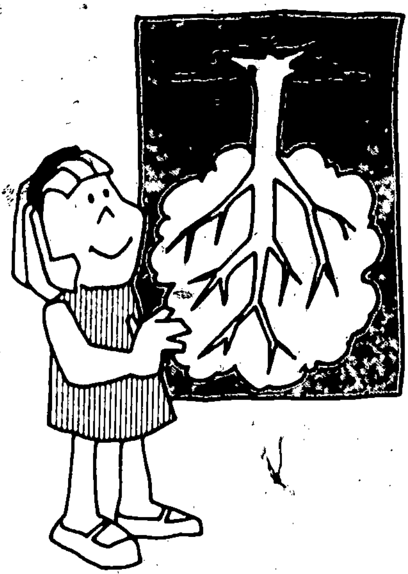
39 =

60 =

72 =

Given any composite whole number, is it possible to write it as a product of primes? Explain your reasoning and compare your answer to this question to the answer you gave to the second question in (4) above.

6. In considering factor "trees" did you ever ask yourself whether the arboreal metaphor really stands up? That is, you may have asked yourself why factor trees do not grow upward if they are supposed to be trees. In some text-books factor trees do grow upward. Consider the advantages and disadvantages of each notation. When trees are used extensively in class discussions the various parts are given names such as branch, fork, and twig.



PART B: Factorization into Primes

DISCUSSION:

After your experience in constructing factor trees in Part A, you probably came to feel that you can construct a complete factor tree for any counting number greater than 1. Although the terminology of factor trees is more appropriate for use with children, there is an economy of space and a conceptual ease associated with the factorization point of view. Indeed, another way of representing the factors of a counting number is to write the number as a <u>product of primes</u>. For example, 30 = 2·3·5 and 84 = 2·2·3·7 are representations of 30 and 84 as products of primes. Each counting number can be so represented, and a formal proof of this fact is not difficult. (Intuitively, one can argue that if a number is prime, then the representation is complete. If not, then it is composite and consequently is the product of two strictly smaller numbers. Each of these is either prime or composite. If either factor is composite, then that factor must be factorable into smaller numbers, and so on. Eventually a stage must be reached at which all the factors are primes.) This is the content of the Prime Factorization Theorem, which we now state.

> If N is a counting number greater than 1, then N can be written as a product of primes; that is, there are prime numbers $p_1, p_2, \ldots, p_k$ such that $N = p_1 p_2 \cdots p_k$. Note that k may be 1; that is, if N is prime then the factorization is complete.

For example, using the factor tree (i) of Part A we can write 12 = 2·2·3.

DIRECTIONS:

1. Write out a prime factorization of 100.

   100 =

……………………………………………………………………

Comment:

You may have observed in Part A, especially in exercise 2, that no matter how the factor tree for a number is constructed, one always ends up with the same prime factors and the same number of each factor. For example, the three factor trees for 30 (Exercise 2(b)) all involve the prime factors 2, 3, 5. A similar result holds in general.

> If N is a counting number greater than 1, then every prime factorization of N is the same except possibly for the order of the factors.

……………………………………………………………………

2. 36 = 4·9 = 6·6. Hence a number can have two composite factorizations that are not the same (even if the factors were reordered). Find another number with at least two different composite factorizations.

3. Why are the prime numbers called prime and the composite numbers called composite? Alternatively, do the names "prime" and "composite" provide reasonable descriptions of the characteristics of the corresponding numbers?

41

4. If one interprets a prime to be its own prime factorization, can he state that "every counting number greater than 1 has a prime factorization"? Can he state that "every counting number greater than 1 has a unique prime factorization (if the order of the prime factors is disregarded)"?

5. In what respects would the prime factorization results be less satisfying if the term "prime" had been defined so that the number 1 was a prime?

PART C: Exponential Notation and the Prime Factorization Theorem

DISCUSSION:

A more compact representation of composite numbers is possible if one uses exponents. For example, in Part B we wrote $12 = 2 \cdot 2 \cdot 3$ and, using exponential notation, this can be written as $12 = 2^2 \cdot 3$. Here the product $2 \cdot 2$ has been written in exponential form as 2^2. In general, if n is a counting number and $n \cdot n \cdots n$ is a product of m factors each of which is the number n, then, using exponential notation, this product can be written as n^m. Two examples are:

$$250 = 2 \cdot 5 \cdot 5 \cdot 5 = 2 \cdot 5^3 \quad \text{and} \quad 16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$$

DIRECTIONS:

1. Factor each of the following composite numbers and express the result using exponential notation.

   39 =

   60 =

   512 =

   27 =

2. State the prime factorization theorem (Part B, p. 31) using exponential notation.

3. Let p and q be primes and b a counting number. If $p|b$ and $q|b$, then $p \cdot q|b$.

   i) Verify this assertion in three special cases.

   ii) Show by example that the assertion is false (in general) if p or q is composite.

   iii) Give an argument that justifies the assertion as stated in the general case (i.e., do not consider more special cases).

PROJECT 1
E-PRIMES

COMMENTS

In the previous activities of this unit, the notions of prime number and factorization with respect to the set of counting numbers have been studied. Here the concern is with the factorization of numbers in the set E of even counting numbers: E = {2, 4, 6, 8, ...}. In this set there are numbers that cannot be written as a product of two other elements of the set. For example, 6 cannot be written as a product of two other elements of the set. (Of course, 6 = 2 x 3, but the number 3 is not in the set E.) An even number n will be called E-prime if n cannot be expressed as a product of elements of E. For example, 2 and 6 are E-primes while 4 is not since 4 = 2 x 2. An even number is said to be E-composite if it is not E-prime. Hence, 4 is E-composite. The object of this project is to explore the analogy between E-prime numbers and ordinary prime numbers. Some specific exercises and questions are posed below. As you answer these questions, compare your answers concerning E-primes with what you already know about ordinary primes.

EXERCISES AND QUESTIONS

1. Determine the first ten E-primes. (The first two E-prime numbers are 2 and 6.)
2. Can every E-composite number be factored into a product of E-primes?
3. List several even numbers that have only one factorization into E-primes. (Disregard the order of the factors.)
4. Find an even number whose E-prime factorization is not unique, that is, an even number that can be factored into products of E-primes in at least two different ways. (Disregard the order of the factors.)
5. Find a simple test to determine whether an even number is an E-prime. (Hint: Use the Prime Factorization Theorem.)

ACTIVITY 4

TESTING FOR DIVISORS

FOCUS:

This activity is organized as a group discovery session to determine divisibility tests for several small counting numbers. The goal is to develop a method of determining quickly and easily whether a small counting number divides a larger one.

DISCUSSION:

One method of showing that a number is composite is to find a divisor (different from 1 and the number). Also, even if a number is known to be composite, one may be interested in determining its factors. In this activity you will develop some divisibility tests (i.e., means of determining rapidly and easily whether one counting number divides another). For example, does 2 divide 720? Does 5? It is not necessary to actually perform the division to answer the questions. Indeed, the divisibility of a number by 2 and 5 can be determined simply by inspecting its digits. Several such tests will be developed in this activity.

DIRECTIONS:

After giving a brief introduction to the topic, your instructor will divide the class into several groups and assign small counting numbers to each group for investigation. For example, suppose that 2 is one of the numbers assigned to your group. Your problem is to determine a divisibility test for 2, that is, a method for determining whether a counting number is divisible by 2 by simply examining its digits.

Appoint one member of your group as a discussion leader and another as recorder. It is likely that most of your time will be spent in making guesses and working examples. Working examples and studying special cases, either independently or as a group, is a profitable activity. Do your work with pencil and paper so that you can share your ideas with others.

If you find yourself temporarily stumped, you might ask your instructor to point out some worthwhile questions to ask and/or helpful directions in which to proceed. Do not expect your instructor to provide you with tests!

If your group discovers a workable test, report it to your instructor. A summary of the results will be prepared, and after the group activity is completed the entire class will discuss the results. The question of justifying the tests will be considered. You should keep a record of the tests in the table "Summary of Divisibility Tests" on page 37.

TEACHER TEASER

Is $n^2 - n$ always divisible by 2?
Is $n^4 - n$ always divisible by 4?
Is $n^3 - n$ always divisible by 3?
Is $n^5 - n$ always divisible by 5?
Is $n^1 - n$ always divisible by 1?
Is there a pattern?

TABLE
SUMMARY OF DIVISIBILITY TESTS

DIVISOR	TEST
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

Some applications of the divisibility tests to factoring problems follow below:

1. Use the divisibility tests developed above to determine the divisors of each of the following counting numbers and factor each into a product of primes.

   a) 78

   b) 693

   c) 12,760

   d) 342,540

48

2. If two counting numbers have the same digits but in the reverse order (for example, 254 and 452), then their difference (the larger minus the smaller) is divisible by 9.

   a) Verify this assertion for the number 563.

   b) Verify this assertion for the number 378,501.

   c) Try to justify the assertion for a general three-digit number $N = 100a_2 + 10a_1 + a_0$, where $a_2 > a_0$.

PROJECT 2
HOW MANY NUMBERS TO TEST

COMMENTS

When one is faced with the problem of finding all of the divisors of a counting number N, there is an obvious advantage in using a systematic procedure. If one simply tries potential divisors of N at random, there is a high likelihood that some divisors will be missed and, especially if N is large, some effort will be duplicated. There are several different systematic procedures, but one of the most natural is to begin with 2 and try each counting number in order to determine whether or not it is a divisor of N. It is a useful fact that one need not try all the counting numbers smaller than N to determine all the divisors. In fact, one can stop well short of N. We can now formulate a precise question.

QUESTION

In order to determine all of the factors of N by testing each counting number in order, beginning with 2, what is the largest counting number that must be tested?

Denote this largest number that must be tested by L(N).

A useful aid in answering the question is contained in the following observation. If one knows that a counting number p is a factor of

N, then one can determine a counting number q such that $p \cdot q = N$. Indeed one can take $q = N/p$. It follows that if one knows the set of all factors p such that $p \cdot q = N$ and $p \leq q$, then one knows the set of all factors of N. Stated somewhat differently, every pair of factors of N contains a "smaller one" (or two equal ones). A knowledge of this "smaller one" determines them both.

EXERCISES AND QUESTIONS

1. Complete the table below.

Number N	Pairs of Factors of N	Smaller of the Pair	L(N)
24	1, 24 2, 12 3, 8 4, 6	1 2 3 4	4
12			
36			
60			

For each N the entry L(N) in the last column can be obtained by inspection from the third column. According to its definition, L(N) is the largest of the entries in column three for that N.

2. For a counting number N, let n be the largest counting number such that $n \cdot n \leq N$.

If N = 24, what is n?

If N = 12, what is n?

If N = 36, what is n?

If N = 60, what is n?

In each case how does n compare with L(N)?

3. Justify the following statement:

If $p \cdot q = N$, then either p or q must be less than or equal to n.

4. Answer the Question posed on page 40. (Hint: Can you conclude that $L(N) \leq n$?)

5. As an application of the above, determine the largest counting numbers that must be tested to find all the factors of each of the following counting numbers:

100 64 1008 80 230

ACTIVITY 5
DISTRIBUTION OF THE PRIMES

FOCUS:

Once you have absorbed the idea of a prime number, an immediate question is "Is there an easy method to determine which numbers are primes?" This turns out to be a question for which there is no entirely satisfactory answer. Unlike the even numbers (multiples of 2) or the multiples of 3 or 5, the prime numbers are not uniformly distributed in the sequence of counting numbers. The focus of this activity is to identify some properties of the set of prime numbers.

The activity consists of three parts: the use of a "sieving" method with a table of counting numbers to identify the primes included in the table; an investigation of the number of primes; and an investigation of the occurrence of long strings of consecutive composite numbers.

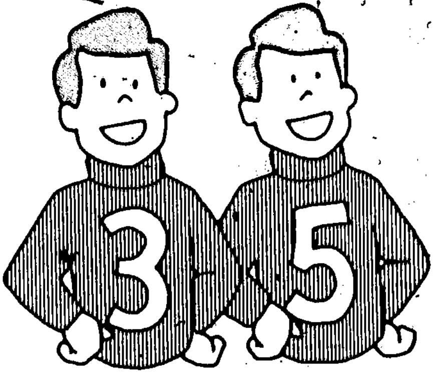
MATERIALS:

Set of Cuisenaire rods.

PART A: Identification of the Prime Numbers in a Table of Counting Numbers

DIRECTIONS:

1. The goal is to obtain all of the prime numbers between 2 and 103 by crossing out all the composite numbers in the table on page 45. This table of the first 100 (actually 2 through 103) counting numbers is arranged in six columns instead of the more common ten columns, in order to better display some of the features of the distribution of the primes in the sequence of counting numbers. Find a systematic method to identify the primes in the table and write out a careful description of the method and why it works.
2. Why might your method of identifying prime numbers be called a "sieving" technique?
3. Find several patterns in the table and list them. For example, where are the even numbers? Where are the multiples of 3? of 5? Where are the prime numbers?
4. Using this table as a guide, complete the following statement: every prime number greater than 3 is either one more than or one less than a multiple of ____.
5. How many primes are there between 2 and 100 (inclusive)?
6. Every prime number other than 2 is odd. Why? Two consecutive odd numbers that are primes are called twin primes. For example, 3 and 5 are twin primes. List all pairs of twin primes between 2 and 100.



7. A famous conjecture due to a Russian mathematician named Goldbach is that every even number greater than 2 is the sum of two

2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25
26	27	28	29	30	31
32	33	34	35	36	37
38	39	40	41	42	43
44	45	46	47	48	49
50	51	52	53	54	55
56	57	58	59	60	61
62	63	64	65	66	67
68	69	70	71	72	73
74	75	76	77	78	79
80	81	82	83	84	85
86	87	88	89	90	91
92	93	94	95	96	97
98	99	100	101	102	103

primes. For example, 12 = 5 + 7 and 20 = 17 + 3. Write each even number between 30 and 50 (inclusive) as the sum of two primes. Goldbach's conjecture remains unsettled. That is, no even number has been found that cannot be expressed as the sum of two primes; and no proof has been found that every even number can be so expressed.

8. With the exception of the pair 3, 5, the sum of any pair of twin primes is divisible by 12.
   a) Verify this assertion for three pairs of twin primes.
   b) Give an argument that justifies it in general.

9. Another conjecture by Goldbach (also unsettled) is that every odd number greater than 7 is the sum of three odd primes. For example, 9 = 3 + 3 + 5 and 15 = 3 + 5 + 7. Write each odd number between 31 and 51 (inclusive) as the sum of three odd primes.

10. Do the challenge problem at the end of the activity (page 51).

PART B: The Unlimited Supply of Primes

DISCUSSION:

When children or adults reflect on the idea of prime numbers they often pose a question like one of the following:

To begin, take a moment and verify that these questions are simply different forms of the same question, and that if one can be answered then all can be answered. Suppose you wish to answer the question, "Given any prime number, is there always a bigger one?" Depending on whether one believes the answer to be yes or no, one attacks the problem differently. If you believe the answer to be yes, then you might try taking various prime numbers and showing that in each case there is a bigger one. At some point you must give a general proof or justification of your conjecture. On the other hand, if you believe the answer is no, then you might try to find the largest prime and show conclusively that there are no larger primes.

You may be aware from your earlier work in mathematics (or you may have guessed from the title of this part of the activity) that the answer to the question is <u>yes</u>. That is, given any prime, there is always a larger one. To verify this, we will give a method that shows that for any prime p, there is a number which is

i) larger than p, and

ii) a prime

<u>Proposed Method</u>: Given p, find the product of all primes less than or equal to p and add 1.

Using the proposed method, we can determine the missing entries in the following table.

Given Prime	New Number Generated by Proposed Method	
2	2 + 1	= <u>3</u>
3	$(2\cdot3)+1$	= <u>7</u>
5	$(2\cdot3\cdot5)+1$	= ____
7	$(2\cdot3\cdot5\cdot7)+1$	= ____
11	$(\quad\quad\quad)+1$	= ____
13		= <u>30031 = 59·509</u>
17		= ____
⋮		
p	$(2\cdot3\cdot5\cdots p)+1$	= ____

DIRECTIONS:

1. Fill in the missing information in the table as far as (and including) the line corresponding to the prime 17.

   Note that the numbers generated from (i.e., resulting from applying the method to) the primes 5, 7, and 11 are all prime, but that the number generated from 13 is not a prime. However, this number, 30031, contains a prime factor, 59, which is larger than 13. Thus, although the new number generated by the method may not be prime, it leads to a larger prime number than we started with (at least in the cases we have examined).

2. Is there an infinite number of E-primes? (See Project 1.)

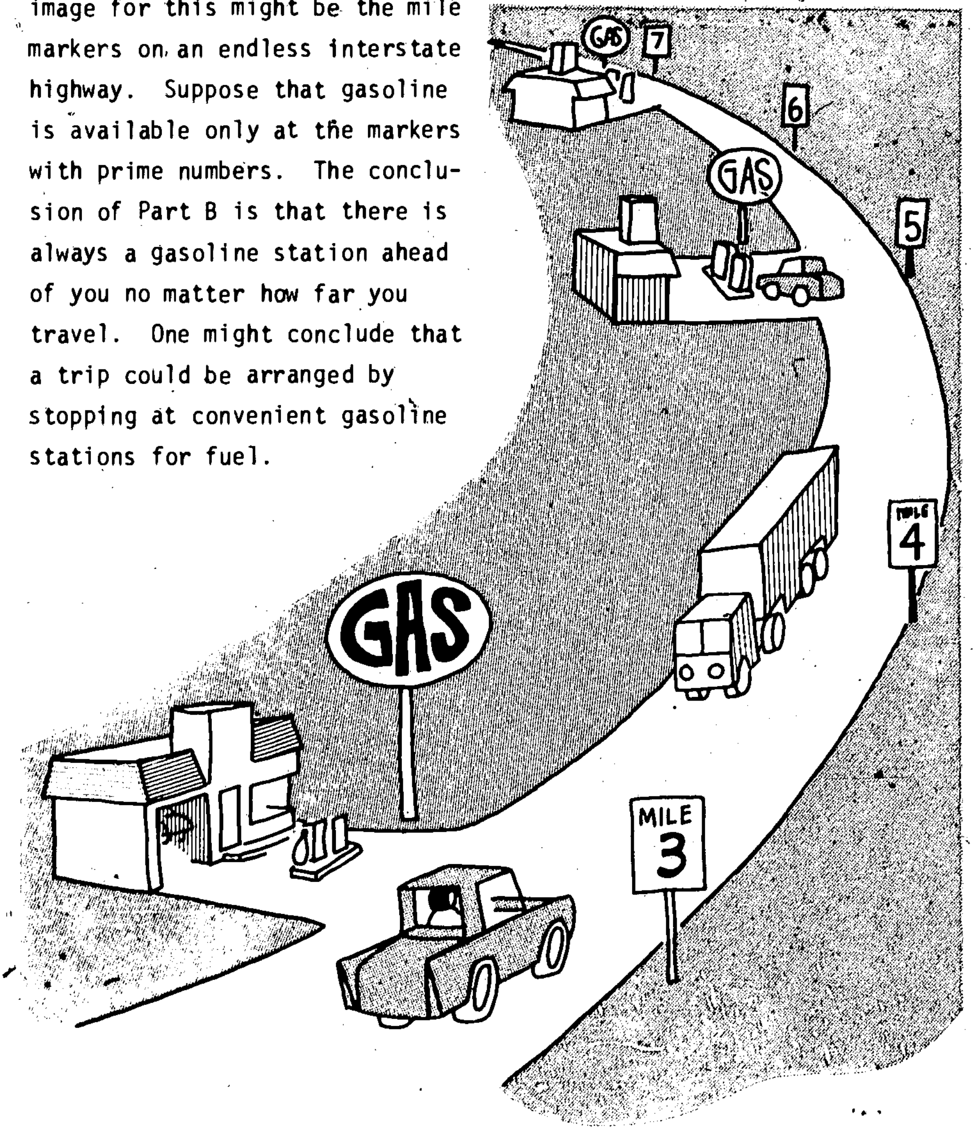
3. Additional insight into the proposed method can be obtained by thinking of the table in terms of Cuisenaire rods or other concrete terms. Use Cuisenaire rods to answer the questions and verify the statements that follow. Consider the prime number 5 and the newly generated number $(2 \cdot 3 \cdot 5) + 1$. Can 2 be a factor of this newly generated number? Can 3? Can 5? Consider 3, for instance. Since 3 is a factor of $2 \cdot 3 \cdot 5$, one can build a train of $(2 \cdot 5)$ rods of length 3 that has length $2 \cdot 3 \cdot 5$. From this it should be clear that it is impossible to obtain a train of rods of length 3 that has length $(2 \cdot 3 \cdot 5) + 1$. Thus, in checking whether $(2 \cdot 3 \cdot 5) + 1 = 31$ is a prime, we see that 2, 3 and 5 cannot be factors of 31. Finally, since 5 is the largest prime such that $5^2 < 31$, these are the only primes that need to be checked for factors of 31 (see Project 2).

4. Give an argument for a general prime p analogous to that given above for the prime 5.

PART C: Strings of Counting Numbers Containing No Primes

DISCUSSION:

From Part B we conclude that there are arbitrarily large prime numbers; i.e., there are prime numbers larger than any given number. An image for this might be the mile markers on an endless interstate highway. Suppose that gasoline is available only at the markers with prime numbers. The conclusion of Part B is that there is always a gasoline station ahead of you no matter how far you travel. One might conclude that a trip could be arranged by stopping at convenient gasoline stations for fuel.



However, for such a trip to be feasible, one would have to know the answer to the following kind of question:

> Could we pass a string of 1000 consecutive mile markers (counting numbers) without coming to a gasoline station (prime number)?

Before dealing with this question, answer the following questions in order to make sure you understand the "gasoline station" metaphor.

- How many gasoline stations are there in the first 100 miles?
- What is the maximum distance between gasoline stations (in the first 100 miles)?

To ask if there is a string of 1000 consecutive mile markers without a gasoline station is to ask if there is a string of 1000 consecutive composite numbers.

There are extensive tables of prime numbers, which, upon close examination, will reveal long strings of composite numbers. However, one does not need to rely on tables; there is an easy way to construct a string of composite numbers of any desired length. We illustrate the method by constructing a string of four consecutive composite numbers: 122, 123, 124, and 125 are composite. (Note that these are not the <u>first</u> four consecutive composite numbers in the sequence of counting numbers; in fact, 24, 25, 26, 27, and 28 are five consecutive composites.) It turns out to be easy to show that 122, 123, 124, 125 are composite by a method that can be generalized. Indeed,

$$\begin{aligned}
122 &= 120 + 2 = (2\cdot3\cdot4\cdot5) + 2 = 2[(3\cdot4\cdot5) + 1], \text{ so } 2|122,\\
123 &= 120 + 3 = (2\cdot3\cdot4\cdot5) + 3 = 3[(2\cdot4\cdot5) + 1], \text{ so } 3|123,\\
124 &= 120 + 4 = (2\cdot3\cdot4\cdot5) + 4 = 4[(2\cdot3\cdot5) + 1], \text{ so } 4|124,\\
125 &= 120 + 5 = (2\cdot3\cdot4\cdot5) + 5 = 5[(2\cdot3\cdot4) + 1], \text{ so } 5|125.
\end{aligned}$$

For notational convenience we often abbreviate $1\cdot2\cdot3\cdot4\cdot5$ by $5!$, and likewise for other whole numbers: $1\cdot2\cdot3\cdots p = p!$ (read "p factorial"). Using this notation, the above argument shows that $5! + 2$, $5! + 3$, $5! + 4$, $5! + 5$ are a set of consecutive composite numbers.

DIRECTIONS:

1. Use the method introduced above to produce a string of six consecutive composite numbers. Find a divisor of each and thereby show that your numbers are composite.
2. Tell how you would find a string of 1000 consecutive composite numbers.

CHALLENGE PROBLEM

Sequences of counting numbers that are uniformly distributed in some sense can usually be described by a mathematical formula. For example:

Evens	$2k$,	$k = 1, 2, 3, \ldots$
Squares	k^2,	$k = 1, 2, 3, \ldots$
Differences of Consecutive Cubes	$(k + 1)^3 - k^3$,	$k = 1, 2, 3, \ldots$

Mathematicians have studied the problem of finding a function whose value for each counting number is a prime. One can think of such a function as taking the input 1 and providing as output a prime, say p_1, taking input 2 and providing as output a prime p_2, and so on. Consider the function g defined by

$$g(k) = k^2 + k + 41 \quad \text{for } k \text{ a counting number.}$$

a) Show that $g(k)$ is prime for $k = 1, 2, 3, 4, 5, 6,$ and 7. (Hint: Use the table of Part A of this activity.) In fact $g(k)$ is prime for $k = 1, 2, 3, \ldots, 39$.

b) Show that $g(40)$ is composite, and factor it.

TEACHER TEASER

Prime triples are primes of the form $p - 2$, p, $p + 2$. There is exactly one prime triple. Find it. Provide an argument showing that there can be no others.

ACTIVITY 6

AN APPLICATION: GCF AND LCM

FOCUS:

The purpose of this activity is to introduce the concepts of greatest common factor and least common multiple and to investigate the relation between them and factorization into primes.

The activity is organized into two parts. Parts A and B develop (or review, for those students who remember the ideas from their school mathematics) the ideas of greatest common factor and least common multiple. Part C relates these concepts to prime factorization.

PART A: Least Common Multiple (LCM)

DISCUSSION:

Examine the textbook page on page 53 of this unit. Amy computed $\frac{1}{9} + \frac{5}{6}$ by expressing both $\frac{1}{9}$ and $\frac{5}{6}$ as the correct number of eighteenths. That is, she expressed them both as fractions with the common denominator 18. In identifying 18 as an appropriate denominator she might have argued as follows: The fraction $\frac{1}{9}$ is equivalent to $\frac{2}{18}$, $\frac{3}{27}$, $\frac{4}{36}$, $\frac{5}{45}$, ... and the fraction $\frac{5}{6}$ is equivalent to $\frac{10}{12}$, $\frac{15}{18}$, $\frac{20}{24}$, $\frac{25}{30}$ ... . In each case the denominators of the equivalent fractions are the multiples of 9 and 6 respectively. As shown on the textbook page,

{6, 12, 18, 24, 30, 36, 42, 48, 54, ...} is the set of multiples of 6

and

{9, 18, 27, 36, 45, 54, ...} is the set of multiples of 9.

It is evident that 18, 36, and 54 are (the first three) multiples that are common to both sets. Hence 18, 36, and 54 are called common multiples of 6 and 9. Any one of these could have been used as a denominator in the addition-of-fractions problem. The choice that

USING THE LEAST COMMON MULTIPLE

A. Suppose you want to add $\frac{1}{9}$ and $\frac{5}{6}$. Can you rename $\frac{1}{9}$ as some number of sixths? Can you rename $\frac{5}{6}$ as some number of ninths? Why must you rename both fractions?

- Copy and complete the following examples.

BETH

$$\begin{array}{rl} \frac{1}{9} & = \frac{6}{54} \\ +\frac{5}{6} & = \frac{45}{54} \\ \hline & \frac{51}{54}, \text{ or } \frac{??}{18} \end{array}$$

SALLY

$$\begin{array}{rl} \frac{1}{9} & = \frac{4}{36} \\ +\frac{5}{6} & = \frac{30}{36} \\ \hline & \frac{??}{36}, \text{ or } \frac{??}{18} \end{array}$$

AMY

$$\begin{array}{rl} \frac{1}{9} & = \frac{2}{18} \\ +\frac{5}{6} & = \frac{15}{18} \\ \hline & \frac{??}{18} \end{array}$$

- What did Beth use as a common denominator? Sally? Amy?
- How should their answers compare?

B. The number that Amy chose as a common denominator, 18, is the **least common denominator**.

$$\{6, 12, \enclose{circle}{18}, 24, 30, 36, 42, 48, 54, \cdots\}$$
$$\{9, \enclose{circle}{18}, 27, 36, 45, 54, \cdots\}$$

- *The least common denominator is the least common multiple of the denominators.*
- List the multiples of 2; of 3; 4; 5; 6; 8; 10.
- What is the least common denominator for $\frac{1}{2}$ and $\frac{1}{3}$? for $\frac{1}{8}$ and $\frac{2}{6}$? for $\frac{3}{5}$ and $\frac{1}{4}$? for $\frac{1}{2}$, $\frac{1}{10}$, and $\frac{1}{3}$?
- Copy and complete the following.

$\frac{1}{2}-\frac{1}{3}=$? $\quad$ $\frac{1}{8}+\frac{2}{6}=$? $\quad$ $\frac{3}{5}-\frac{1}{4}=$? $\quad$ $\frac{1}{2}+\frac{1}{10}+\frac{1}{3}=$?

C. Do 3 and 4 have any common factors? Do 11 and 2? 8 and 5?

- Compare the least common denominators with the denominators shown in each pair of fractions. How are they related?

	LEAST COMMON DENOMINATOR
$\frac{1}{4}, \frac{2}{3}$	12
$\frac{3}{11}, \frac{1}{2}$	22
$\frac{3}{8}, \frac{2}{5}$	40

- Copy and complete the following.

$\frac{1}{4}+\frac{2}{3}=$? $\quad$ $\frac{1}{2}-\frac{3}{11}=$? $\quad$ $\frac{3}{8}+\frac{2}{5}=$? $\quad$ $\frac{1}{6}+\frac{4}{7}=$?

gives the smallest denominator is 18. This is the smallest or least common multiple of 6 and 9.

DIRECTIONS:

1. Find the least common multiple of each pair of numbers given below:

   a) 9, 39

   b) 100, 225

   c) 12, 40

2. Until now we have been considering the set of counting numbers. If we look at the set of whole numbers {0, 1, 2, 3, ...}, then

   {0, 6, 12, 18, ...} is the set of (whole number) multiples of 6;

   {0, 9, 18, 27, ...} is the set of (whole number) multiples of 9.

   Hence a child would be completely right in saying that 0 is the least common multiple of 6 and 9 (with respect to the set of whole numbers). However, in adding fractions as on the textbook page, it is necessary to use counting-number multiples instead of whole-number multiples. Why?

3. As a group, formulate a definition of the term least common multiple of a pair of counting numbers. Later, each group will present its proposed definition, and the class will discuss which definitions are acceptable. Your instructor will then ditto or record on the blackboard the best definition(s). (By attempting to work out a definition of LCM instead of memorizing a printed definition, you should obtain a better feel for the LCM concept and also for the considerations that go into creating good definitions. The process of arriving at definitions is a very basic and important aspect of mathematical thinking.)

PART B: Greatest Common Factor (GCF)

DISCUSSION:

A knowledge of a systematic method for finding common factors or greatest common factors can be useful in multiplying or simplifying fractions. For example, to simplify $\frac{30}{45}$, the following method could be used:

$\{1, 2, 3, 5, 6, 10, 15, 30\}$ is the set of factors of 30.

$\{1, 3, 5, 9, 15, 45\}$ is the set of factors of 45.

The largest factor that is common to both [illegible]ts, the <u>greatest common factor</u> of 30 and 45, is 15. Therefore, 15 is the largest number that is a factor of both 30 and 45. This information can be used to simplify $\frac{30}{45}$, as follows:

$$\frac{30}{45} = \frac{2 \cdot 15}{3 \cdot 15} = \frac{2}{3}$$

DIRECTIONS:

1. Find the greatest common factor of each of the pairs of numbers given below:

   a) 126, 35

   b) 60, 75

   c) 143, 21

2. Working in small groups, formulate a definition of the term <u>greatest common factor</u>. Each group's proposed definition will be discussed by the class as a whole.

3. Give an example of a multiplication problem involving fractions in which finding a GCF would simplify the computation of the product. Complete the calculation of the product with the aid of the GCF.

4. Take a pair of numbers, find their GCF and LCM, and compute the product GCF x LCM. Repeat the process with a few other pairs. What do you observe?

5. Is the LCM of two numbers a multiple of the GCF of those two numbers? Try some examples.

6. Give a definition for the greatest common factor of <u>three</u> counting numbers. Use your definition and find the GCF of the following triples.

   a) 30, 21, 6

   b) 143, 21, 56

   c) 60, 210, 75

7. Let a and b be counting numbers and define a★b to be the greatest common factor of a and b. Does this define a <u>binary operation</u> on the set of counting numbers? (A binary operation on a set is a function which assigns a number from the set to each ordered pair of numbers from the set.) What properties does the operation ★ have?

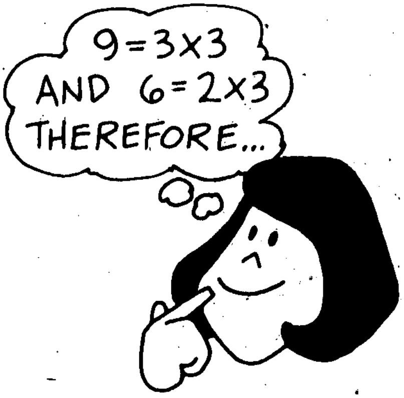
8. Suppose that the least common multiple of a and b is ab. What can you say about the common factors of a and b?

PART C: LCM, GCF, and Prime Factorization

DISCUSSION:

Sally attempted to find the least common multiple of 9 and 6 by reasoning as follows:

"9 = 3 x 3 and 6 = 2 x 3. Therefore, if a number is to be a multiple of 9, it must have two 3's in its prime factorization. If it is to be a multiple of 6, it must have a 2 and a 3 in its prime factorization. Consequently, the least common multiple of 9 and 6 must have one 2 and two 3's in its prime factorization: the LCM of 9 and 6 is 2 x 3 x 3 = 18."



DIRECTIONS:

1. Try Sally's method to find the LCM of each of the following pairs of numbers:

    a) 12, 40

    b) 54, 72

    c) 9, 39

2. Try to describe Sally's method succinctly and explain why it works.

……

Sally found her prime factorization method so successful for finding LCM's that she worked out a similar method for GCF's. Here's how she found the GCF of 30 and 45:

$$30 = 2 \times 3 \times 5$$
$$45 = 3 \times 3 \times 5$$

The GCF of 30 and 45 must be a factor of both 30 and 45. Since 2 is not a factor of 45, 2 cannot be in the prime factorization of the GCF. 3 can be; but 3 x 3 cannot be, since 3 x 3 is not a factor of 30. 5 is a factor of both 30 and 45. Putting this all together, the GCF of 30 and 45 must be 3 x 5 = 15.

……

3. Use Sally's method to find the GCF of each of the following pairs of numbers:

    a) 54, 72

    b) 60, 75

    c) 198, 162

4. Try to describe Sally's method succinctly and explain why it works.

5. Use Sally's methods for finding the LCM and GCF of a pair of numbers to explain what you observed in exercises 4 and 5 of Part B.

6. Explain how you can use the prime factorization of a number to find all of its factors. Also explain how you can use the prime factorization of a number to find immediately how many factors the number has. As examples to get you started, we list the following:

6 = 2 x 3, and 6 has four factors;
30 = 2 x 3 x 5, and 30 has eight factors;
210 = 2 x 3 x 5 x 7, and 210 has sixteen factors.

PROJECT 3

A PARLOR TRICK BASED ON NUMBER THEORY

An old chestnut on the parlor circuit can be described as follows: Ask a guest to write any three-digit number on a slip of paper, and to pass the paper to the next guest. This guest is to form a six-digit number by writing the original three digits in order twice. For example, if the original digits were 234, then the six-digit number is 234234. The paper is passed to the next guest, who is asked to divide the six-digit number by 7. You can add, "You need not be concerned about the remainder; there will be none." The paper is passed to the next guest, who is asked to divide the resulting quotient by 11. Again you may comment that there will be no remainder. The paper is passed to the next guest, who is asked to divide the resulting quotient by 13; again there will be no remainder. Finally, the paper is returned to the original guest, who observes that the last quotient is his original number!

EXAMPLE

a) original number	234
b) six-digit number	234234
c) quotient upon division of (b) by 7	33462
d) quotient upon division of (c) by 11	3042
e) quotient upon division of (d) by 13	234

What is the number-theoretic basis for the trick?

Can you construct a similar trick?

ACTIVITY 7

SEMINAR

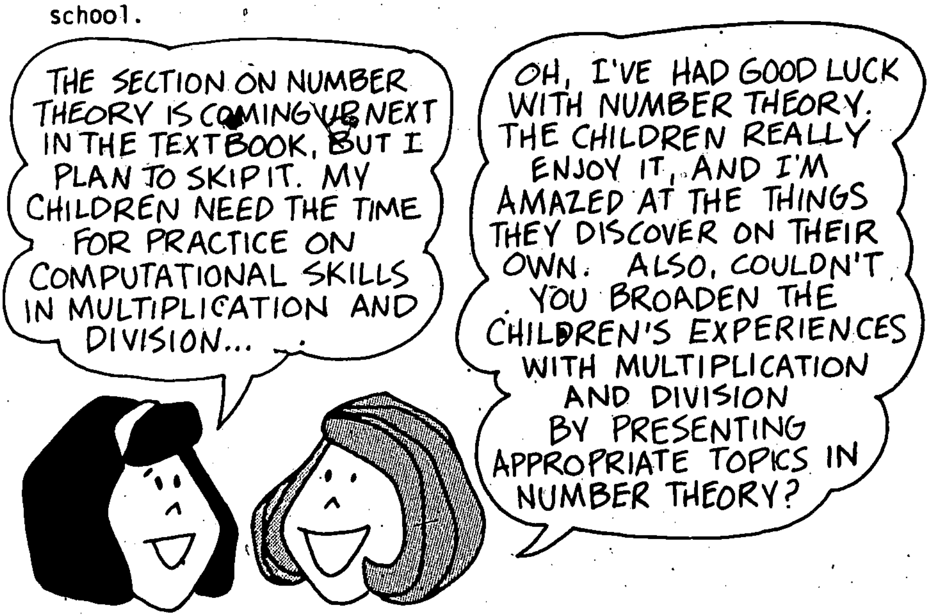
FOCUS:

This seminar considers the general question of the role of interesting number-theory problems in teaching and learning mathematics. Also, more specialized questions are raised concerning the relation of number theory to other topics in the elementary curriculum and concerning methods of introducing number theory to elementary children.

DIRECTIONS:

The questions that follow will serve as a basis for a class discussion.

1. Why is it that so many problems that occur in newspapers, magazines, and collections of mathematical puzzles have a number-theoretic base?
2. What are some instances of number theory as numerology, i.e., the mystique of numbers? (You may want to do a little library work on this one.)
3. Two teachers were talking in the faculty lounge of an elementary school.



In light of this dialogue, respond to the following questions.

a) What number-theory topics related to multiplication and division might the second teacher have in mind? Take at least one of the topics and discuss how you might teach it in order to clarify and extend the children's basic understanding of multiplication and division. (In particular, suggest some probing questions that might be used.)

b) Give some examples of number-theory problems that provide children with the opportunity to discover patterns. Suggest some settings, materials, and questions that might aid the children in the discovery process.

c) What additional arguments might be used to convince the first teacher that her line of action is unjustified?

Section II

PROBLEMS AND PROBLEM SOLVING

One of the goals of this unit is to provide you, the reader, with an opportunity to play the role of amateur mathematician. A part of the work of a mathematician is to construct theories; another part is to solve problems. Difficult problems (and this includes some that appear on the surface to be quite straightforward) may require sophisticated mathematical techniques for their solution. However, much less difficult problems may challenge your curiosity and inventiveness and still be solvable with the use of elementary ideas and techniques. The way in which problem solving can contribute to learning mathematics has been succinctly expressed by one of mathematics' greatest problem-solvers, George Polya:

> "Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations, he kills their interest, hampers their intellectual development and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking."

A great many problems that occur in the classroom, in textbooks, or in supplementary material--or that occur in everyday life, in daily activities, in newspapers, magazines or puzzle books--have a number-theoretic flavor. This section is devoted to the study of a few such problems, and to mathematical problem solving in somewhat more generality. The broad theme of the section is pursued in greater depth in the Experiences in Problem Solving unit of the Mathematics-Methods Program.

The purpose of Activity 8 is to get you started. It consists of three short problems and related exercises. The problems are worked out in varying degrees of detail and in a conceptual framework that should be useful to you in solving the problems in Activity 9 and in other mathematical problem-solving situations as well. Activity 9 contains a selection of problems; some are quite easy and others are fairly difficult. A completely worked out problem similar to those of Activity 9 is contained in the Appendix: An Example of Problem Solving.

MAJOR QUESTIONS

1. What are some advantages to organizing your approach to solving a problem?
2. In what ways might problem solving reinforce the study of standard topics in the elementary mathematics curriculum?
3. Write a paragraph describing how one standard topic might be presented in a problem-solving mode.
4. A common teacher reaction to a student's wrong approach to a problem is, "Read the problem again." What is the teacher trying to communicate to the student? What question could the teacher ask in place of this one that might more effectively achieve the objective?

ACTIVITY 8

ORGANIZING THE PROBLEM-SOLVING PROCESS

FOCUS:

This activity offers some guided experiences in solving mathematical problems. It provides a basic organizational scheme in which much of mathematical problem solving can be profitably viewed. (Read the discussion of Parts A, B, and C as a homework assignment, in preparation for class. In class you will discuss the examples and do the problems under "Directions.")

PART A

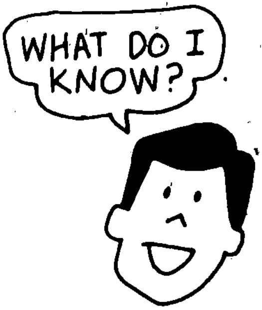
DISCUSSION:

From time to time in your study of mathematics, either as a regular part of the curriculum or as an extra-credit assignment, you have been asked to solve problems. Here we will focus more directly on the problem-solving <u>process</u> than on the task of getting answers to specific problems in the most efficient manner. We proceed in a discussion in which the main ideas are illustrated with examples. In keeping with the topic of the unit, we have selected relatively simple number-theoretic problems for our examples. You will have an opportunity to try your hand at similar problems, as well as some tougher ones, later on.

The place to begin solving any problem is to ask:

An attempt to state the problem in precise terms frequently suggests lines of attack. In asking for a precise statement of the problem, one often asks for a translation from a verbal form of the problem to a mathematical one.

Next, one asks,



It is important to decide what tools and techniques can be used in studying the situation. Again, sorting through your store of mathematical knowledge looking for something useful may suggest how to approach the problem.

Responding to these questions should be viewed as steps that one always goes through in solving a problem. From here on, the way in which one proceeds will vary from one problem to another. The approach used profitably on one problem--it might be termed a strategy--might be totally ineffective on another problem. We will mention and illustrate briefly a few ways in which problems can be approached. We turn now to our first example.

EXAMPLE 1

Are there any perfect squares in the sequence
11, 111, 1111, 11111, ...?

If we follow the approach suggested above, then the first step is to decide exactly what the problem asks. In this case, the answer is the following: Are there whole numbers n such that $n^2 = 11$, or $n^2 = 111$, or $n^2 = 1111, \ldots$?

Next we take inventory of what we know. We recall from Activity 2 that 11 is prime, and consequently there is no counting number n such that $n^2 = 11$. Also $10^2 = 100$ and $11^2 = 121$. Therefore, there is no n with $n^2 = 111$. Next, $33^2 = 1089$ and $34^2 = 1156$; so there is no n with $n^2 = 1111$.

So far we have considered a number of <u>special cases</u> of the problem. Frequently the solution of special cases provides insight into the general situation. In this case, the analysis of some special cases provides evidence that it is impossible to have a counting number n with $n^2 = 111...11$ for any number of 1's. We <u>guess</u> a solution: There are no perfect squares in the sequence 11, 111, ...

Let us see whether our guess can be supported or verified. We ask again, what information can we use to convince ourselves that our guess is correct? Remarkably, it is enough to know the simple fact that every counting number is either even or odd. In particular, if n is such that $n^2 = 111...11$, then n is either even or odd. We separate the problem into two subproblems and consider them separately.

First, if n is even, that is, if $n = 2k$, then

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Thus, if n is even, then n^2 is also even. But 111...11 is odd, so n cannot be even.

Next, suppose that $n^2 = 111...11$ and n is odd; i.e., $n = 2k + 1$. Then

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 111...11,$$

and consequently

$$4k^2 + 4k = 111...10.$$

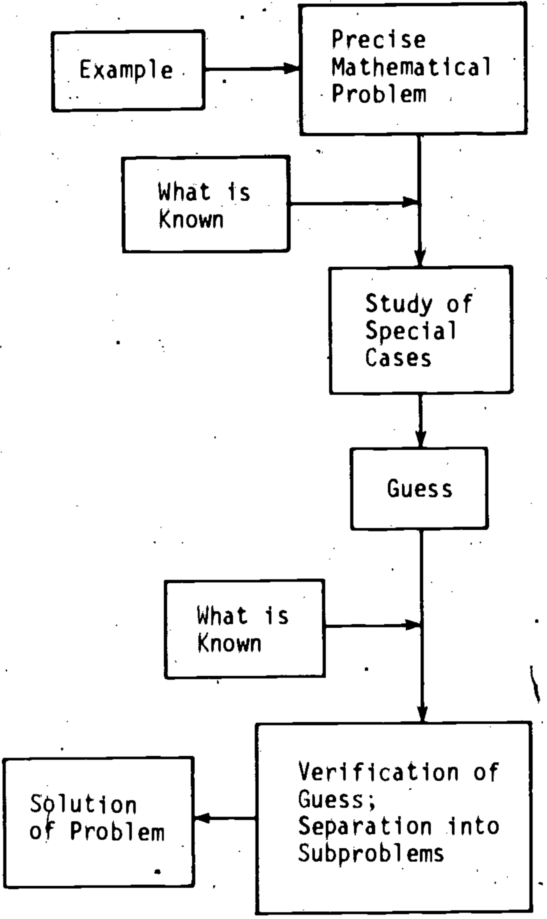
The last equality can be rewritten as $4(k^2 + k) = 111...10$. Since 4 divides every multiple of 100, it follows that if $4|111...10$, then 4 must divide 10. That is,

$$\begin{aligned} 111...1110 - 10 &= 111...1100 \\ &= 100 \times 111...11, \end{aligned}$$

and 4 divides 100 x 111...11, so if 4 divides 111...1110, 4 must also divide 10 (recall Activity 1). But since 4 does not divide 10, we conclude that 4 does not divide 111...10. This completes the argument that if n is odd, then n^2 cannot equal 111...11.

It is helpful to review the analysis of this problem, and the following diagram summarizes the main steps.

We introduced here the use of <u>special cases</u>, <u>guessing</u>, and <u>sub-problems</u>. The guessing noted in the argument is worthy of special comment. It is an example of an extremely important activity known as <u>hypothesis formulation</u>. In somewhat oversimplified terms, it is simply the generation of a guess based on the evidence. Although the generation of a hypothesis does not ordinarily settle a problem (it usually requires verification), it is an important first step. It sometimes happens that coming up with the right hypothesis is much more difficult than showing that the hypothesis is correct.



DIRECTIONS:

1. Determine whether there are any perfect squares in the sequence 22, 222, 2222,... How about the sequence 33, 333, 3333, ...? After making these determinations, think back and ask yourself, "Have I used the steps suggested above?"

PART B

DISCUSSION:

Another approach that is frequently useful in solving problems is to look for patterns. Patterns may be obvious from the formulation of the problem, or it may require some insight to recognize them. Since we are concerned with number-theoretic problems in this unit, the patterns we will look for are number patterns. The term number pattern means exactly that: an arrangement of numbers in which the numbers are positioned according to some discernible rule. The way in which one takes advantage of a number pattern to help solve a problem is best illustrated by an example. A final caveat is in order, however: Patterns can be deceiving things. They may lead you in fruitful directions, but they may also lead you astray. The directions pursued here are fruitful ones. There is an example of a pattern that leads to a dead end in the discussion of the conjecture on page 116 of the Appendix.

EXAMPLE 2

> Find the remainder when 3^{245} is divided by 5. Remember that 3^{245} means the number 3 multiplied by itself 245 times.

Clearly, this number is so large that it is impossible to write it out, actually perform the division, and check the remainder. Taking the first of our three suggestions, let us begin by asking, "What are we to do?" By the division algorithm (see the Multiplication and Division unit of the Mathematics-Methods Program), there are whole numbers q and r such that $0 \leq r < 5$ and $3^{245} = 5q + r$. Our problem is to determine the number r.

What is known? It may be useful to begin by looking at some simpler problems. We can certainly find the numbers q and r for small powers of 3. This is done in the table on the following page.

Exponent	Associated Number		Quotient q, remainder r when the associated number is expressed as 5q + r	
	Exponential Form (power of 3)	Decimal Form	q	r
0	1	1	0	1
1	3	3	0	3
2	3^2	9	1	4
3	3^3	27	5	2
4	3^4	81	16	1
5	3^5	243	48	3
6	3^6	729	145	4
7	3^7	2187	437	2
8	3^8	6561	1312	1

Notice the pattern: The remainders 1, 3, 4, and 2 occur in a regular repetition. Let us see if we can use this pattern to answer our original question; i.e., what remainder corresponds to the exponent 245? We observe that in every case illustrated in the table, an exponent which is a multiple of 4 corresponds to a remainder of 1:

Exponent		r
0	→	1
4	→	1
8	→	1

We guess that this continues to be true for higher exponents. If this is so, then the exponent 244(= 61 x 4) corresponds to remainder 1. Thus the next exponent, 245, ought to correspond to remainder 3. This is in fact the case. This assertion can be justified, but justification is not essential to make our point, namely, that a careful study of patterns is often helpful in problem solving.

In addition to the use of patterns, the main point of the example, we utilized guessing to formulate a hypothesis. Also, in the creation of the table from which we deduced a pattern we studied a number of simpler problems. As with special cases the study of simpler problems may aid problem solving by providing more information about the situation and possibly ideas which may work in the original problem.

DIRECTIONS:

1. Determine the remainder when 2^{89} is divided by 3.
2. Determine the remainder when 3^{197} is divided by 7.
3. Draw a diagram for the problem-solving process used in Example 2 similar to that drawn for Example 1 on page 68.

PART C

DISCUSSION:

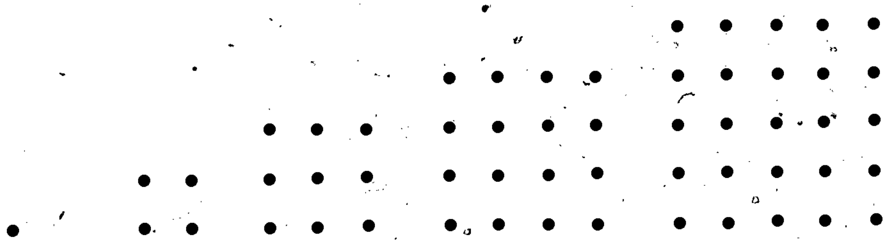
Parts A and B contained problems that you could understand at once. They involved only mathematical ideas with which you were already familiar. In the next example we will consider another way, and perhaps a more typical one, in which problems arise. We begin not with a specific problem but rather with a situation to be studied. In the course of our study we will formulate and solve some specific problems.

In addition to giving an instance in which problems arise in the course of a discussion, this example will illustrate the way in which appropriate mathematical notation can simplify expressions and thereby facilitate problem solving. One can view mathematical notation as a concise language and a systematic means of keeping track of information. We all appreciate how basic an understanding of the language is to communication.

The content selected for this example is "figurate" or "polygonal" numbers. It exemplifies the sort of material that can be presented at one level in the elementary school (middle grades) and that offers worthwhile food for thought for more advanced students.

EXAMPLE 3

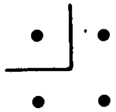
One can view the squares of the counting numbers, i.e., 1, 4, 9, 16, 25, ..., as those numbers that give the number of dots in a square array. It will be very convenient to have a notation that associates with each counting number n the n^{th} square number. We accomplish this by letting S_n (read "Ess sub n") denote the n^{th} square number. The diagram below illustrates S_1 through S_5.



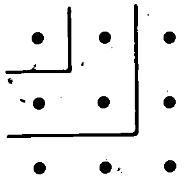
$S_1 = 1$ $S_2 = 4$ $S_3 = 9$ $S_4 = 16$ $S_5 = 25$

What is S_6? S_7? Draw a diagram for each.

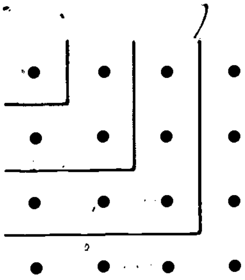
The square arrays can be partitioned to suggest interesting number-theoretic relations. One possible partitioning is illustrated below.



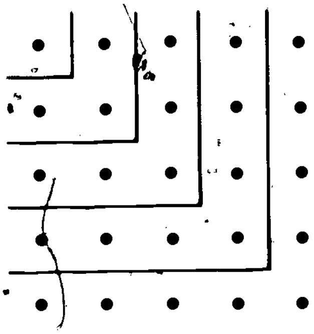
4 = 1 + 3



9 = 1 + 3 + 5



16 = 1 + 3 + 5 + 7



25 = 1 + 3 + 5 + 7 + 9

If this information is collected in a table, we have:

Number of Dots on a Side n	Number of Dots in Array S_n	Number of Subarrays Resulting from Partition	Number of Terms in Sum $1 + 3 + 5 + \ldots + \square = S_n$	$\square$
2	4	2	2	3
3	9	3	3	5
4	16	4	4	7
5	25	5	5	9

If one examines this table carefully and looks for patterns, one notes that in each line the entry under $\square$ is one less than twice n. That is, $\square = 2n - 1$. Since $\square$ is defined as the last term in the sum, we conclude that

$$S_n = n^2 = 1 + 3 + 5 + \ldots + (2n - 1).$$

Thus, a study of square numbers has led to an answer to the problem:

Find the sum of the first n odd numbers.

This question was not posed initially, but instead arose quite naturally in our discussion of square numbers.

DIRECTIONS:

1. Evaluate the sums $1 + 3 + 5 + 7 + 9$

   $1 + 3 + 5 + 7 + \ldots + 19$

   $1 + 3 + 5 + 7 + \ldots + 99$

2. Use the results of this example to evaluate

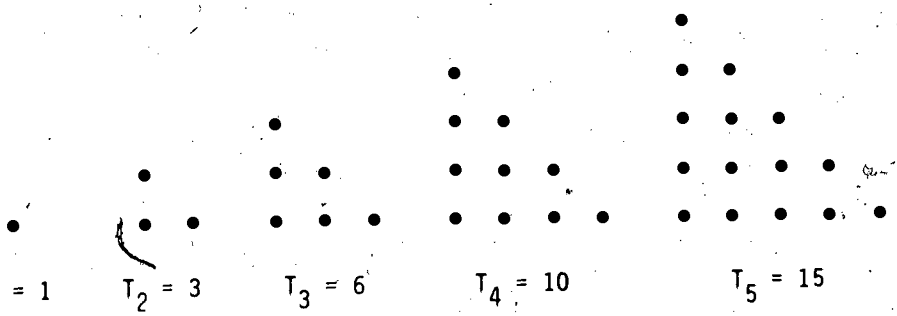
   $2 + 4 + 6 + \ldots + 222.$

PART D

DISCUSSION:

In Example 3 of Part C, we considered square numbers. There are other interesting "polygonal" numbers, and in this part we continue the discussion a bit further. Adopting a notation similar to that above, we let T_n be the number of dots in a (regular) triangular array with n dots on each "leg." We refer to T_n as the n^{th} triangular number. The triangular numbers T_1 through T_5 are illustrated.

EXAMPLE 4



You should determine T_6 and T_7 and illustrate them.

Since the number of dots in the top row of each triangle is 1, the number in the next row is 2, the number in the third row is 3, and so on, we have

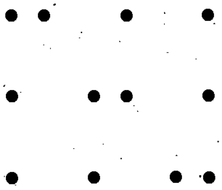
$$T_1 = 1$$
$$T_2 = 1 + 2$$
$$T_3 = 1 + 2 + 3.$$

and in general

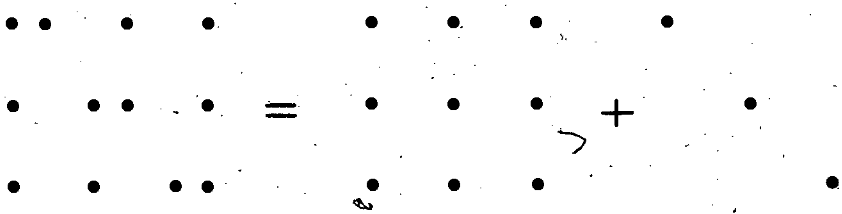
$$T_n = 1 + 2 + 3 + \ldots + n.$$

Given n, the number of dots in the n^{th} triangular array, that is T_n, can be computed in a variety of ways. We will describe one of them and leave another for you to do [see (2) below].

First, we use the above results on square numbers. We begin by observing that the third triangular array is related to the third square array as shown below.



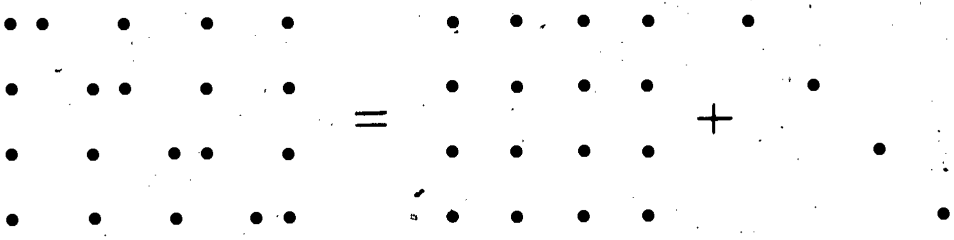
That is, the number of dots in the above array is twice the number in the third triangular array. But the number of dots in this array is also equal to the number of dots in the third square array plus 3. The 3 arises from the fact that the above figure has two dots in each diagonal spot instead of one as in the square array. Using a diagram we could express this as



or, in symbols, as

$$2T_3 = S_3 + 3.$$

The corresponding figure for the fourth array is



or $\qquad 2T_4 \qquad = \qquad S_4 \qquad + \qquad 4$

If the pattern holds true in general, then $2T_n = S_n + n$. But $S_n = n^2$ so we have $2T_n = n^2 + n = n(n + 1)$ or $T_n = \frac{1}{2}n(n + 1)$.

We have solved the problem:

Find the sum of the first n counting numbers.

DIRECTIONS:

1. Use the information deduced above about triangular numbers to help in evaluating the sums:

$$1 + 2 + 3 + \dots + 9$$
$$1 + 2 + 3 + \dots + 19$$
$$1 + 2 + 3 + \dots + 99.$$

2. Consider the following table

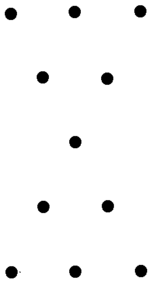
n	1	2	3	4	5	6
T_n	1	3	6	10	15	21
$8T_n$	8	24	48	80	120	168

Find another relationship between T_n and S_n. What does it mean geometrically? (<u>Hint</u>: Consider $2n + 1$.)

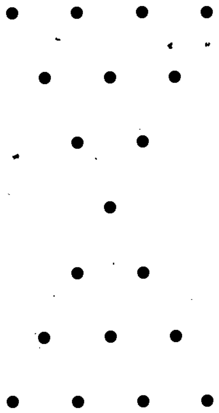
3. One can define hourglass numbers as in the following figures.

$H_1 = 1$

$H_2 = 5$



$H_3 = 11$



$H_4 = 19$

Find an expression for the n^{th} hourglass number, H_n. How is H_n related to T_n?

4. Identify the uses of patterns in the discussion of square numbers and triangular numbers.

ACTIVITY 9
PROBLEMS

FOCUS:

In this activity, you will have an opportunity to try to utilize the organizational scheme presented in the preceding activity to solve some problems on your own.

DISCUSSION:

The problems presented in this activity are not tied directly to specific subject matter, and their solutions are not to be found explicitly anywhere in the unit. However, the topics you have studied in the unit will be helpful in directing your thinking along productive lines. It is to be emphasized that you will be operating as an amateur mathematician and it is for you to decide which mathematical tools are to be used in each instance. It may well be that your approach to a problem will differ from the approach chosen by another member of the class. You may find it useful to review Activity 8 as you proceed.

DIRECTIONS:

A selected set of these problems will be assigned by your instructor. Work on them at home or during free moments in the scheduled class. Your assignment should be finished by the time the unit is completed. The problems of Part B are intended to be more challenging than those of Part A.

PART A

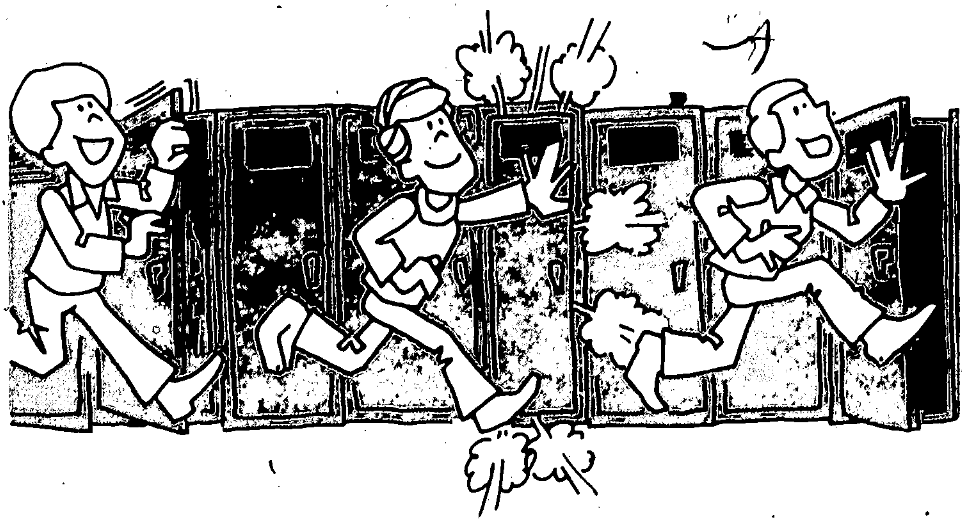
1. Is a number plus its square always even?
2. Is $n(n + 1)(n + 2)$ divisible by 2? By 3? By 6?
3. Can the square of each odd number be written in the form $8m + 1$ for some whole number m?

4. Ignoring the initial primes 2 and 3, is the product of each pair of twin primes divisible by 12?

5. Find the sum of the squares of the first n odd numbers.

PART B

1. There is a row of 1,000 lockers, some open and some closed. A boy runs down the row and opens every locker. A second boy runs down the row and beginning with the second locker he shuts every other locker. A third boy runs down the row and beginning with the third locker he changes the state of every third locker. That is, he opens those that are closed and closes those that are open. A fourth boy runs down the row and beginning with locker number 4 he changes the state of every fourth locker. The process continues until 1,000 boys have run down the row. Is the 1,000th locker open or closed? the 764th? [Hint: Do not work out the entire process by hand. Consider the results after the first few boys (six or so) have run down the row, and then look for a pattern.]



2. The Pythagorean theorem says that if z is the hypoteneuse of a right triangle with legs x and y, then $x^2 + y^2 = z^2$. Triples of whole numbers m, n, p such that $m^2 + n^2 = p^2$

are known as Pythagorean triples. Certain Pythagorean triples have the property that $p = n + 1$. For example

3, 4, 5 $(3^2 + 4^2 = 5^2)$ and 5, 12, 13 $(5^2 + 12^2 = 13^2)$.

Find five more such special Pythagorean triples.

3. A group of children decide to "play store," and they have items for sale at 1¢, 2¢, 3¢, ..., i.e., at all prices. Their money, however, consists of only two coins, a gleep worth 7¢ and a glop worth 23¢. A customer can purchase an item worth 5¢ (for example) by giving 4 gleeps and receiving 1 glop in change. What are the prices in cents of all of the items that can be purchased with gleeps and glops? What if a gleep were worth 6¢ and a glop 21¢?

4. Each letter stands for one of the digits 0, 1, 2, ..., 9. Find values of the letters that make the following true:

   a)
$$\begin{array}{r} \text{HOCUS} \\ +\ \text{POCUS} \\ \hline \text{PRESTO} \end{array}$$

   b) FORTY + TEN + TEN = SIXTY [Solve independently of (a).]

5. Three pirates have a chest full of gold pieces, which are to be divided between them. Before the division takes place, one of the pirates secretly counts the number of pieces and finds that if he forms three equal piles, then one piece is left over. Not being a generous man he adds the extra coin to one pile, takes the pile and leaves. Later the second pirate goes to the chest,

divides the gold into three piles, and again finds a piece left over. He adds the extra piece to one pile, takes it, and leaves quietly. The third pirate does likewise. Still later when the pirates meet to divide the treasure they find that the number of coins remaining is evenly divisible into three piles. How many coins were originally in the chest?

PROJECT 4
PASCAL'S TRIANGLE

Pascal's triangle is a triangular array of numbers that arises in several different ways in mathematics. Although it is named after the French mathematician Blaise Pascal (1623-1662), the table was known to the Moslem philosopher and mathematician Omar Khayyam over five centuries earlier, and there are references to still earlier occurrences. The table can be extended indefinitely, and the first ten rows are reproduced below.

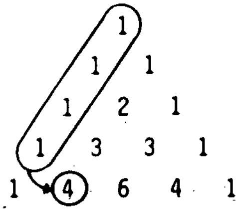
1
1 1
1 2 1
1 3 3 1
1 4 [illegible] 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
⋮

This triangular array contains a great many number patterns. As examples, we point out that the entries on the borders of the triangle are all 1's (remember that you should imagine the triangle continuing indefinitely downward) and each entry not on a border is the sum of the two entries in the line just above it to its right and left:

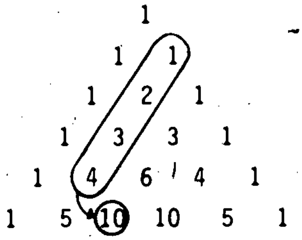
1
1 1
1 ② ①
1 3 ③ 1

1 + 2 = 3

Also, each entry is the sum of the entries on a diagonal segment. For example:



1 + 1 + 1 + 1 = 4



1 + 2 + 3 + 4 = 10

As a final example, note the pattern formed by the sum of the entries in each row:

1	1	= 1 = 2^0
1 1	1 + 1	= 2 = 2^1
1 2 1	1 + 2 + 1	= 4 = 2^2
1 3 3 1	1 + 3 + 3 + 1	= 8 = ?
1 4 6 4 1	1 + 4 + 6 + 4 + 1	= ? = ?

To check your understanding of these three patterns, it is suggested that you give three more examples of each of them.

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Much of the interest in Pascal's triangle is due to its ubiquitousness. We illustrate its occurrence in an unexpected setting with the following question:

Given the set {A, B, C, D}, how many different subsets containing one element are there?

How many different subsets containing two elements?

How many different subsets containing three elements?

How many different subsets containing four elements?

Can you connect this pattern of numbers with Pascal's triangle? How?

Be wise, Generalize!

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Section III

APPLICATIONS, CONNECTIONS AND GENERALIZATIONS

This section contains a collection of activities whose common ingredient is that they use the number theory developed in Section I. There are additional examples, an indication of some of the connections between ideas, and some mathematical generalizations. Although the topics discussed here do have implications for elementary school mathematics, e.g., clock arithmetic, we do not concentrate on these aspects of the subject. Instead, we seek to provide a natural mathematical setting in which several of the topics discussed earlier appear as special cases of more general situations. The reader who finds his appetite whetted by this brief glimpse into a vast and important area of mathematics is encouraged to pursue his interest. The references in the bibliography that are identified as oriented toward content or extensions of the mathematics would be an appropriate starting point.

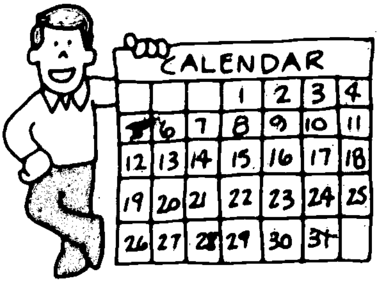
MAJOR QUESTIONS

1. Identify and discuss a real-world situation different from those of Activity 10, in which the concept of a <u>remainder class</u> arises in a natural way.
2. In what ways is the identification of a symbol with a remainder class described in Activity 11 similar to the identification of the numeral 3 with sets of 3 blocks, 3 balls, 3 pictures, etc.?

3. Do you agree with the statement "Congruence is a generalization of ordinary equality"? Why?

4. Identify two instances outside of number theory in which the partitioning feature of an equivalence relation is useful.

TEACHER TEASER



Billy was practicing addition by adding the numbers along each full week on the calendar. After a while Billy saw the following pattern for finding the sum of the numbers in a week: "Take the first day. Add 3. Multiply by 7." Try it. Why does it work?

ACTIVITY 10
REMAINDER CLASSES

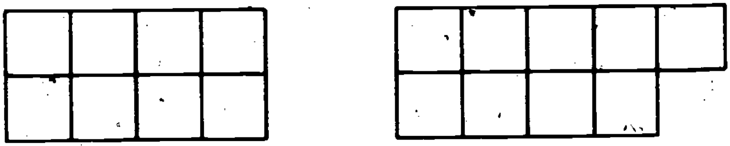
FOCUS:

One of the ways of viewing prime and composite numbers introduced in Activity 2 is considered further in this activity. The concept of a remainder class is introduced and real-world occurrences of remainder classes are discussed.

DIRECTIONS:

Review the definitions of prime and composite numbers and Part A of Activity 2. Do the following and be prepared to participate in a class discussion of number (4).

1. Certain rectangular and near-rectangular arrays can be formed from tiles arranged in horizontal rows of 2:



   What are the similar arrays that can be formed from tiles arranged in horizontal rows of 3? Of 4? Of 5?

2. The arrangements of (1) can be viewed from the standpoint of the division algorithm. (You recall that the division algorithm says that for whole numbers a and $b \neq 0$, there exist whole numbers q and r so that $a = bq + r$ and $0 \leq r < b$.)
   a) Describe the connection between these arrangements and the division algorithm. (You may choose to do parts (b) and (c) before answering this part.)

b) What are the possible remainders when any whole number is divided by 2? By 3? By 4? By 5? Relate your answers to rectangular and near-rectangular arrays.

c) What are the possible remainders when any whole number is divided by n?

3. Let us now think of selecting some whole number, say 3, and classifying or partitioning the set of whole numbers into disjoint subsets according to the remainder after division by 3. For example, we have

Number	0	1	2	3	4	5	6	7	8	9	10	11
Remainder after division by 3	0	1	2	0	1	2	0	1	2	0	1	2

Thus we assign 1, 4, 7, 10, ... to the same set, or remainder class, since they have the same remainder after division by 3. We can proceed similarly with remainders 0 and 2.

Remainder	Remainder Class
0	$\{0, 3, 6, 9, \ldots\}$
1	$\{1, 4, 7, 10, \ldots\}$
2	$\{2, 5, 8, 11, \ldots\}$

Notice that the remainder classes are disjoint (i.e., no number belongs to more than one remainder class), and that they exhaust the whole numbers (i.e., every whole number belongs to some remainder class). The importance of this comment is that partitioning the whole numbers into remainder classes is in fact an honest partitioning (see (d) below).

a) How many remainder classes are associated with division by 3?

b) How many remainder classes are associated with division by 2? What are they? List them in the same way the remainder classes for 3 are listed above.

c) How many remainder classes are associated with division by 5? What are they? List them as above.

d) Why is it that we can be sure that every whole number belongs to some remainder class and no whole number belongs to more than one remainder class?

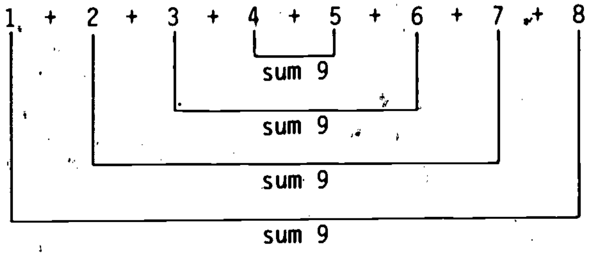
4. Discuss how the use of a 12-hour clock can be viewed as a use of remainder classes.

PROJECT 5

THE SUM OF THE FIRST N COUNTING NUMBERS

The problem of determining the sum $1 + 2 + 3 + \ldots + n$ admits of a variety of solutions. We shall interpret the problem as asking for a formula involving n that gives the sum $1 + 2 + 3 + \ldots + n$ for each choice of n, and we shall develop a means of <u>guessing</u> a formula that works. We shall not prove that our guess does in fact always work; but if one is to have absolute faith in one's results, such a proof should be given. This problem is approached differently in Activity 8.

We begin by studying the sum of an even number of counting numbers. For example $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$. Observe that by pairing the first and last, the second and second from last, and so on, one has a set of pairs each of which has the same sum.



In each case the sum is $9 = 8 + 1$. There are $4 = \frac{8}{2}$ such sums so by multiplication $1 + 2 + \ldots + 8 = 9 \cdot 4 = (8 + 1) \cdot \frac{8}{2}$.

Construct a diagram similar to the one above and find the sums:

$$1 + 2 + \ldots + 14$$
$$1 + 2 + \ldots + 20$$

Observe that in each case, for an appropriate choice of n (i.e., $n = 14$ and $n = 20$), the following equality holds.

$$(*) \qquad 1 + 2 + \ldots + n = (n + 1)\frac{n}{2} .$$

Check this formula for $n = 2, 4, 6, 8$, using Pascal's triangle (Project 4) or a straightforward computation.

We now have a conjecture for a formula that works when n is even. What if n is odd? If n is odd, then $n - 1$ is even, and we can use the formula for the sum of an even number of numbers. Replacing n in the formula (*) by $n - 1$ (which is legal since $n - 1$ is even), we obtain

$$\begin{aligned} 1 + 2 + \ldots + (n - 1) &= [(n - 1) + 1](\frac{n - 1}{2}) \\ &= n\frac{(n - 1)}{2}, \end{aligned}$$

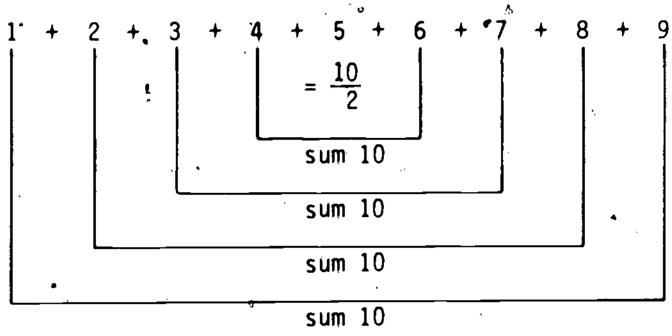
and therefore

$$\begin{aligned} 1 + 2 + \ldots + (n - 1) + n &= n\frac{(n - 1)}{2} + n \\ &= n(\frac{n - 1}{2} + 1) \\ &= n(\frac{(n - 1) + 2}{2}) \\ &= n(\frac{n + 1}{2}) , \end{aligned}$$

That is, the same formula holds for odd numbers n.

Check this formula for $n = 3, 5, 7, 9$, using Pascal's triangle or a straightforward computation.

Try to deduce the formula for odd n using the grouping idea exemplified in the diagram.



ACTIVITY 11

MODULAR ARITHMETIC I

FOCUS:

In this activity, definitions are made, notation is adopted, and the set of remainder classes associated with a counting number is given a mathematical structure. Selected properties of the resulting mathematical system are investigated.

DISCUSSION:

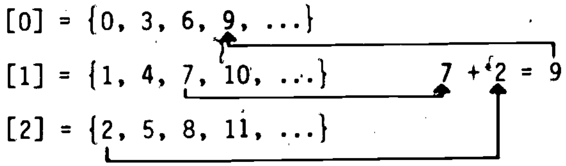
Consider again the remainder classes associated with division by 3.

Remainder	Remainder Class
0	$\{0, 3, 6, 9, \ldots\}$
1	$\{1, 4, 7, 10, \ldots\}$
2	$\{2, 5, 8, 11, \ldots\}$

We propose an abstract setting in which these remainder classes are viewed as entities in their own right. Each remainder class is to be thought of as a sort of number. Thus we introduce the symbol [0] to denote the remainder class consisting of those whole numbers which have a remainder 0 when divided by 3. Likewise we introduce symbols [1] and [2]. We can write, suggestively: $[0] = \{0, 3, 6, 9, \ldots\}$, $[1] = \{1, 4, 7, 10, \ldots\}$, $[2] = \{2, 5, 8, 11, \ldots\}$. To be precise we should write $[0]_3$, $[1]_3$, and $[2]_3$, or some similar notation, to indicate that these are the remainder classes for division by 3. We will not do so unless we wish to distinguish the remainder classes associated with division by different counting numbers.

Comment: This might be a good point to give some thought to Major Question 2. It is possible to introduce an arithmetic structure into this new system, and we proceed to give a definition of addition and multiplication. We begin with addition, and we first consider an example. The symbols [0], [1], and [2] denote the remainder classes associated with division by 3.

To add [1] and [2] we take any element of the associated remainder class for each; for instance, we might select 7 for [1] and 2 for [2], and perform ordinary addition on these numbers.



The result is 9, and we define the sum of [1] and [2] to be the remainder class to which 9 belongs. Since 9 belongs to [0], we define the sum of [1] and [2] to be [0].

[illegible] would be convenient to have a symbol to denote addition of rema[illegible]er classes as + denotes addition of ordinary numbers. To keep the notation to a minimum, we will continue to use + even though we are not adding ordinary numbers. The multiple use of + is common in mathematics and one has to interpret the meaning of the symbol from the context. Thus, if we write 1 + 2 we mean ordinary addition; and if we write [1] + [2] then we mean addition of remainder classes. With this convention regarding the use of the symbol +, we can write [1] + [2] = [0].

Continuing as above, one can construct an addition table for the remainder classes associated with division by 3.

+	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

We leave it for the reader to check that the sum of [1] and [2] is well defined in the sense that this sum is independent of the particular representatives selected (7 for [1] and 2 for [2] above). Such a check might consist of trying several different examples. It is true in general that the sums are independent of the representatives selected.

An appropriate definition of multiplication can be given along similar lines. Thus, to define the product of [0] and [2] we select any pair of representatives, one from each remainder class, multiply them together and note the remainder class of the product. For instance, we might select 9 from [0] and 8 from [2], which leads to 9 x 8 = 72. Since 72 is contained in [0] we define the product of [0] and [2] to be [0]. Extending the use of the symbol x as we did for + above, we will write [0] x [2] = [0]. The multiplication table for the remainder classes associated with division by 3 is given below.

x	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

DIRECTIONS:

1. Complete the following addition and multiplication tables.

+	$[0]_4$	$[1]_4$	$[2]_4$	$[3]_4$
$[0]_4$				
$[1]_4$				
$[2]_4$				
$[3]_4$				

$+$	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[0]_5$					
$[1]_5$					
$[2]_5$					
$[3]_5$					
$[4]_5$					

$\times$	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[0]_6$						
$[1]_6$						
$[2]_6$						
$[3]_6$						
$[4]_6$						
$[5]_6$						

2. The abstract systems constructed in this activity have some, but not all, of the properties of the system of ordinary whole numbers. In this exercise we shall briefly explore this comparison.

   a) Is there a number in the system $[0]_4$, $[1]_4$, $[2]_4$, $[3]_4$ that behaves as does 0 under addition in the ordinary whole numbers?

   b) Is there a number in the system of part (a) that behaves as does 1 under multiplication in the ordinary whole numbers?

c) In the ordinary whole numbers, if a product of two numbers is zero, then at least one of the numbers must be zero. Is this assertion true in the system of part (a)?

d) Answer each of the above questions for the system $[0]_5$, $[1]_5$, $[2]_5$, $[3]_5$, $[4]_5$.

3. It is important that addition of remainder classes is well defined in the sense that the sum is independent of the particular representatives selected. For example, in the discussion of adding $[1]_3$ and $[2]_3$ given above, we selected 7 for $[1]_3$ and 2 for $[2]_3$. We would have obtained the same result had we selected 4 for $[1]_3$ and 8 for $[2]_3$.

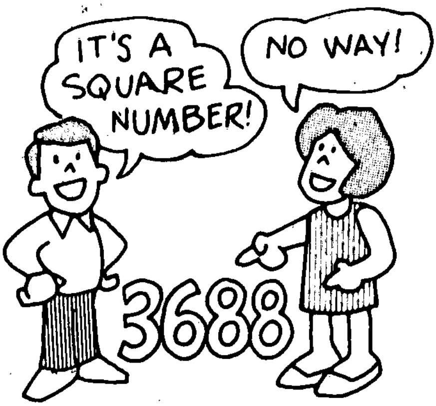
a) Select two different sets of representatives for each of the pairs of remainder classes in the following sums and check that the sums are well defined.

$[0]_3 + [2]_3$ $\qquad$ $[1]_4 + [3]_4$ $\qquad$ $[2]_5 + [4]_5$

b) Proceed as in (a) to check that the following products are well defined.

$[1]_3 \times [2]_3$ $\qquad$ $[2]_4 \times [0]_4$ $\qquad$ $[1]_5 \times [4]_5$

TEACHER TEASER



The product of a number multiplied by itself is called a square number. Billy says that 3688 is a square number. Sally says it can't be because its one's digit is an 8. Is either person right?

PROJECT 6

CASTING OUT NINES

Although you may think of modular arithmetic as a "modern" subject, in fact it forms the basis for a method of (partially) checking arithmetic computations that goes back at least to the sixteenth century. This method is known as "casting out nines" and is sometimes included in elementary textbooks.

The casting-out-nines technique rests on the fact that a counting number is congruent mod 9 to the sum of its digits. This assertion, which is not justified here, can be proved using an argument similar to the one you used in Activity 4 to show that a counting number is divisible by 9 if and only if the sum of its digits is divisible by 9.

We illustrate the technique with examples:

$$\begin{array}{r}373\\ \underline{+\ 486}\\ 859\end{array}\qquad \begin{array}{l}\left.\begin{array}{l}3+7+3\equiv 4 \bmod 9\\ 4+8+6\equiv 0 \bmod 9\end{array}\right\}\quad 4+0\equiv 4 \bmod 9\\ 8+5+9\equiv 4 \bmod 9\end{array}$$

Since the remainder class of 373 mod 9 (i.e., 4) plus the remainder class of 486 mod 9 (i.e., 0) is equal to the remainder class of 859 mod 9 (i.e., 4), the test shows that the addition could be correct. (See the note at the end of the discussion.)

$$\begin{array}{r}187\\ \underline{\times\ \ 53}\\ 561\\ \underline{935\ }\\ 9911\end{array}\qquad \begin{array}{l}\left.\begin{array}{r}1+8+7\equiv 7 \bmod 9\\ 5+3\equiv 8 \bmod 9\end{array}\right\}\quad 8\times 7=56\equiv 2 \bmod 9\\ \\ \\ 9+9+1+1\equiv 2 \bmod 9\end{array}$$

The product of the remainder classes of the factors is equal to the remainder class of the product, and consequently the computation is not shown to be false.

On the other hand, the erroneous addition

$$\begin{array}{r} 32 \\ 87 \\ \underline{+\ 21} \\ 142 \end{array} \qquad \begin{array}{l} \left.\begin{array}{r} 3 + 2 \equiv 5 \bmod 9 \\ 8 + 7 \equiv 6 \bmod 9 \\ 2 + 1 \equiv 3 \bmod 9 \end{array}\right\} \; 5 + 6 + 3 \equiv 5 \bmod 9 \\ 1 + 4 + 2 \equiv 7 \bmod 9 \end{array}$$

is shown to be false by the observation that the sum of the remainder classes (mod 9) of the addends is not equal to the remainder class of the (purported) sum.

Notice that the test is a <u>negative test</u>; i.e., the test can be used to <u>prove</u> that a computation is false. However, it can never be used to prove that a computation is correct. If casting out nines leads to consistent results, then we have more confidence in our calculations, but we are not certain that they are true. If casting out nines leads to inconsistent results, then we know that the computation is false.

<u>Note</u>: Each of these tests uses the mathematical fact that:

$$\text{if} \quad a \equiv b \bmod 9 \quad \text{and} \quad c \equiv d \bmod 9, \text{ then}$$
$$a + c \equiv b + d \bmod 9$$
$$\text{and} \qquad a \cdot c \equiv b \cdot d \bmod 9$$

1. Check the following computations using casting out nines:

$$481 + 653 + 98 + 124 = 1356$$
$$25 + 36 + 86 = 157$$
$$37 \times 255 = 9535$$
$$17 \times 41 = 697$$
$$58 \times 74 = 4382$$

   Is each of the answers correct?

2. Make an example of an addition where the answer is incorrect and yet casting out nines does not detect the error.

CHALLENGE PROBLEM

Activities 10 and 11 provide the mathematical background necessary to justify the assertion of the Note. Do so.

ACTIVITY 12

MODULAR ARITHMETIC II: CONGRUENCES, EQUIVALENCE RELATIONS, AND APPLICATIONS

FOCUS:

In this activity, the concepts presented in Activities 10 and 11 are considered further and the notion of congruence is introduced. The important idea of an equivalence relation is discussed, and congruence is shown to be an equivalence relation on the whole numbers. This activity consists of three parts:

PART A: Notation
PART B: Congruence as an Equivalence Relation
PART C: Applications

PART A: Notation

DISCUSSION:

The notions of remainder class and modular arithmetic that were introduced in the preceding activities in this section are interesting mathematical ideas in their own right. However, they also provide an appropriate setting in which to view a number of facts and problems. Here we will introduce some notation and define congruence, discuss congruence from the standpoint of equivalence relations, and give some applications of the ideas of this section. We start with notation,

In Activity 11 the symbols $[0]_3$, $[1]_3$, and $[2]_3$ were introduced to denote the remainder classes associated with division by 3. Let us now consider how one could determine whether two numbers are members of the same remainder class. The result is the following:

> Two whole numbers a and b belong to the same remainder class if and only if 3 divides (b - a).

Observe that there are two parts to this assertion. First, if a and b belong to the same remainder class, then $3|(b - a)$; and, second, if $3|(b - a)$, then a and b belong to the same remainder class. This result can be verified by using the fact from the division algorithm that each whole number can be represented as $3q + r$, where $r = 0, 1$, or 2.

Using this result we conclude that 4 and 28 belong to the same remainder class. Indeed, $28 - 4 = 24$, and 3 divides 24. Since 4 belongs to $[1]_3$ it follows that 28 does also. Likewise, 2 and 14 belong to $[2]_3$, and 3 and 15 belong to $[0]_3$.

DEFINITION

Let p be a counting number. Two whole numbers a and b are said to be congruent modulo p if p divides $b - a$, that is, if a and b are in the same remainder class associated with division by p. If a is congruent to b modulo p, we write $a \equiv b(\text{mod } p)$.

EXAMPLES

$$3 \equiv 8(\text{mod } 5),\ 8 \equiv 14(\text{mod } 3),\ 8 \equiv 14(\text{mod } 6)$$

DIRECTIONS:

1. Let $S = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. Determine which elements of S are congruent to 0 modulo 2. To 1 modulo 3? To 0 modulo 5? To 1 modulo 4? To 2 modulo 6?

2. The last two congruences in the examples immediately preceding the directions illustrate the following fact: If p and q are counting numbers and p divides q, then $a \equiv b(\text{mod } q)$ implies that $a \equiv b(\text{mod } p)$. Give three more examples of this fact.

3. Give as precise an argument as you can to justify the fact asserted in exercise 2.

4. If $a \equiv b(\text{mod } p)$ and $b \equiv c(\text{mod } p)$, then $a \equiv c(\text{mod } p)$. This assertion can be justified in detail using some results of Activity 1. Discuss why it should be true and provide a precise argument if you can.

5. Does there exist a whole number x such that $2x \equiv 3 \pmod 6$? Justify your answer.

6. If n is a counting number, determine if $3n - 1$ can ever be the square of a counting number. Hint: Use congruences.

7. Is there a counting number n such that 11 divides $4(n^2 + 1)$?

8. Discuss the 24-hour day and the 12-hour clock from the standpoint of congruence modulo 12.

PART B: Congruence as an Equivalence Relation

DISCUSSION:

If we consider the statement $5 \equiv 17 \pmod 6$, it is clear that the implication is that 5 is somehow related to 17. The precise statement is, of course, that 6 divides 17 - 5 or that 17 equals 5 plus some multiple of 6. In this very intuitive sense we will refer to congruence modulo p as a relation. The concept can be made much more precise.

Since 0 is contained in the remainder class $[0]_p$ for every counting number p, it follows that $a \equiv a \pmod p$ for every whole number a. Likewise, if $a \equiv b \pmod p$, then $b \equiv a \pmod p$. The first of these is known as the reflexive property of the relation $\equiv$ and the second is known as the symmetric property of that relation. The transitive property is described in the assertion of exercise 4 above. A relation that is reflexive, symmetric, and transitive is called an equivalence relation. Thus, congruence is an example of an equivalence relation.

The relation concept is a very general one and congruence is only one example. A somewhat more general framework in which to view congruence is the following: We have a set U and a correspondence R which relates each element of U with other elements of U. Suppose that the correspondence R is such that for every pair u and v of elements of U, either u is related to v by R or it is not. In the example of congruence modulo p, the set U is the set of whole numbers, and the relation is that of congruence; for every pair u and v of whole numbers, either $u \equiv v(\text{mod } p)$ or $u \not\equiv v(\text{mod } p)$. (Relations are discussed in more detail in Section III of the Graphs unit of the Mathematics-Methods Program.)

DIRECTIONS:

Consider each of the following relations and determine how it fits into the framework described above. Identify those that are equivalence relations, i.e., are reflexive, symmetric, and transitive.

1. Ordinary equality on the set of whole numbers.
2. The relation "less than" defined on the set of counting numbers.
3. The relation "divides" on the set of counting numbers (a is related to b if a divides b).
4. The relation "is the brother of" defined on a set of children.
5. The relation "is the son of" defined on the audience at a concert.
6. The relation $\sim$ defined on ordered pairs of counting numbers where the relation $\sim$ is defined by $(a,b) \sim (c,d)$ if $ad = bc$.
7. The relation "has the same prime factors as" defined on the set of counting numbers.

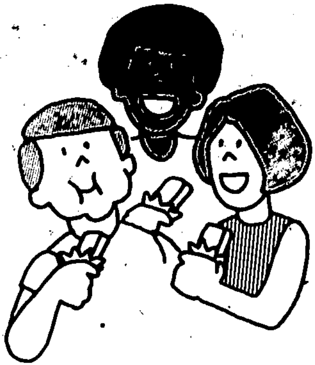
PART C: Applications

DISCUSSION:

It is well known that equations of the form $ax = b$ occur frequently in the applications of elementary mathematics. Likewise, there are problems arising outside of mathematics that lead to congruences of the form $ax \equiv b(\text{mod } p)$. Here a, b, and p are assumed known and the problem is to find a value of x for which the congruence is true. Unfortunately, even these simple congruences need not have any solutions. For example, the congruence $2x \equiv 1(\text{mod } 4)$ has no whole-number solutions. Indeed, for every whole number x the number $2x$ is even and, consequently, is either evenly divisible by 4 [i.e., $2x \equiv 0 (\text{mod } 4)$], or else it has remainder 2 when divided by 4 [$2x \equiv 2 (\text{mod } 4)$]. (See exercise 5 of Part B for another example.)

The following example illustrates how congruences may arise.

EXAMPLE



A box of candy bars is such that when it is divided equally among three children there are two bars left over, and when it is divided equally among five children there are four bars left over. What is the least number of bars the box can contain?

We proceed by writing the problem in mathematical form. Let x denote the unknown number of bars in the box. Then the conditions described in the problem are $x \equiv 2 (\text{mod } 3)$ and $x \equiv 4 (\text{mod } 5)$. That is, there are whole numbers m and n such that $x - 2 = 3m$ and $x - 4 = 5n$ or, expressing these facts somewhat differently, $x = 3m + 2$ and $x = 5n + 4$. Consider the two sets

$$M = \{\text{whole numbers } k \text{ such that } k = 3m + 2, \ m = 0, 1, 2, 3, \ldots\}$$
$$= \{2, 5, 8, 11, 14, \ldots\},$$
$$N = \{\text{whole numbers } k \text{ such that } k = 5n + 4, \ n = 0, 1, 2, 3, \ldots\}$$
$$= \{4, 9, 14, 19, \ldots\}.$$

The problem asks for the smallest number that is both in M and in N. This number is 14. Thus, the smallest number of bars that the box could contain is 14.

There are many applications of congruences to checking computations (see Project 6), to calendar and chronology problems (see exercise 4, Activity 10), to the scheduling of tournaments, and so on. Several of the references cited in the bibliography discuss these applications. In particular, the book, Invitation to Number Theory, by Oystein Ore contains a chapter on applications of congruences.

DIRECTIONS:

1. A recipe for a large batch of cookies calls for 5 eggs. Before baking several batches of cookies, there are a number of cartons of a dozen eggs and 3 additional eggs. After baking there is one egg left over. How many eggs were there to begin with?

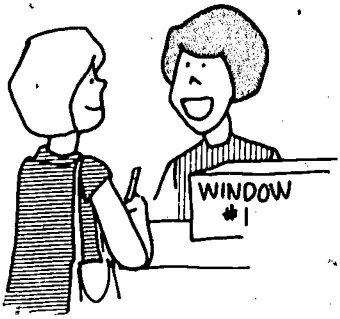
2. In the set of ordinary real numbers the number $\frac{1}{2}$ is that number which when multiplied by 2 gives 1. The number $\frac{1}{2}$ in mod 5 arithmetic is defined similarly.

   a) Compute $\frac{1}{2}\cdot\frac{1}{3}$, $\frac{1}{3}\cdot\frac{1}{4}$, $\frac{1}{2}\cdot 3$ in mod 5 arithmetic.

   b) A student says "$\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$ in mod 5 arithmetic just as in regular arithmetic." What did he mean, and was he correct?

3. Find a counting number x which satisfies both of the congruences $x \equiv 2 (\text{mod } 5)$ and $3x \equiv 1 (\text{mod } 8)$.

4. A woman cashed a check at a bank and the teller mistook the number of cents for the number of dollars and vice versa. After purchasing an item for 68 cents the woman discovered the error, and at that time she had exactly twice as much money as the value of the original check. Find one possible value for the check.



ACTIVITY 13

THE EUCLIDEAN ALGORITHM AND OTHER SELECTED TOPICS

FOCUS:

In this activity, the Euclidean algorithm and some of its consequences and applications are presented.

PART A

DISCUSSION:

The multiplication of 357 by 231 can be considered as an operation on the pair 357, 231, and there is a well-known method for computing the product 357 x 231. This method is sometimes referred to as the multiplication algorithm. An algorithm is simply a well-defined procedure to solve a problem. Frequently an algorithm to perform an operation (such as the multiplication of two three-digit numbers) consists of the step-by-step application of a number of simpler operations (in this case, a number of multiplications by one-digit numbers followed by an addition). There is an algorithm, known as the Euclidean algorithm, to determine the greatest common factor of two numbers. An algorithm to compute the GCF of two counting numbers is useful since the factors of either number may not be at all clear from inspection. For example, it might be quite laborious to determine by ad hoc techniques that the GCF of 867 and 1802 is 17.

The Euclidean algorithm for determining the GCF of two numbers is based on the repeated application of the familiar division algorithm. Recall that if a and b are counting numbers, then there are whole numbers q and r (usually known as the quotient and remainder), $0 \leq r < b$, such that

$$a = q \cdot b + r.$$

We illustrate the algorithm by an example, the problem of determining the GCF of 867 and 1802 posed above. We begin by using the division algorithm to write $1802 = 2\cdot 867 + 68$. Since the GCF of 867 and 1802 (hereafter referred to in this example simply as GCF) must divide 1802, it divides both sides of the equation $1802 = 2\cdot 867 + 68$; and since it divides 867, it must divide 68 (review Activity 1 if you are uncertain of the reasons for this). Consequently, the GCF must divide 1802, 867, <u>and</u> <u>68</u>.

Again, using the division algorithm, we can write $867 = 12\cdot 68 + 51$. Reasoning as above, since the GCF divides 867, it must divide $12\cdot 68 + 51$ and since it divides 68, it must divide 51. Therefore, the GCF divides 1802, 867, 68, <u>and</u> <u>51</u>.

Continuing, we can write $68 = 51 + 17$, and since the GCF divides 68 and 51, it must divide 17. At this point we know that the GCF divides 1802, 867, 68, 51, and 17.

Next we write $51 = 3\cdot 17$ and we note that in this application of the division algorithm there is a <u>zero</u> <u>remainder</u>. This is the signal that our work is finished and that the GCF of 1802 and 867 is 17. Indeed,

$$\begin{aligned} & 51 = 3\cdot 17 \\ \text{so}\quad & 68 = 3\cdot 17 + 17 = 4\cdot 17 \\ & 867 = 12\cdot 68 + 51 = 12\cdot 4\cdot 17 + 3\cdot 17 = 51\cdot 17. \\ \text{and}\quad & 1802 = 2\cdot 867 + 68 = 2\cdot 51\cdot 17 + 4\cdot 17 = 106\cdot 17. \end{aligned}$$

Thus $17|1802$ and $17|867$. Our work also shows that no larger counting number divides both 1802 and 867. We conclude that 17 is the GCF of 1802 and 867.

DIRECTIONS:

1. Find the GCF of the following pairs of numbers:

   a) 222, 98
   b) 748, 132
   c) 1536, 244

2. Find the LCM of 5436 and 2618. (<u>Hint</u>: Recall Exercise 4, Part B, Activity 6.)

3. A student determined the GCF of 1802 and 867 by writing the sequence of divisions shown below beginning at the right

$$\begin{array}{rrrr}
3 & 1 & 12 & 2 \\
17\,\overline{)51} & \overline{)68} & \overline{)867} & \overline{)1802} \\
\underline{51} & \underline{51} & \underline{816} & \underline{1734} \\
0 & 17 & 51 & 68
\end{array}$$

a) Explain how the algorithm works.

Work exercises (1a) and (1c) using this algorithm.

PART B

DISCUSSION:

Consider the problem of finding the GCF of 264 and 150. Using the Euclidean algorithm we have

i) $264 = 150 + 114$,

ii) $150 = 114 + 36$,

iii) $114 = 3\cdot 36 + 6$,

$36 = 6\cdot 6$,

and consequently the GCF of 264 and 150 is 6. Beginning with line (iii) of this set of equations, we can write

$$6 = 114 - 3\cdot 36.$$

Next, using line (ii) to write $36 = 150 - 114$, we can write

$$\begin{aligned} 6 &= 114 - 3(150 - 114) \\ &= 4\cdot 114 - 3\cdot 150. \end{aligned}$$

Finally, using line (i) to write $114 = 264 - 150$, we have

$$6 = 4(264 - 150) - 3\cdot 150$$

or $$6 = 4\cdot 264 - 7\cdot 150.$$

This shows that the GGF of 264 and 150, namely 6, can be written as

$$6 = x \cdot 264 + y \cdot 150,$$

where x and y are integers. This is a special case of the following fact.

> Let a and b be any counting numbers. Then there are integers x and y such that
>
> $$\text{GCF of } a \text{ and } b = x \cdot a + y \cdot b.$$
>
> Moreover, the GCF of a and b is the smallest counting number that can be expressed in this form, that is, the smallest counting number that can be written in the form $x \cdot a + y \cdot b$, where x and y are integers.

Counting numbers a and b are said to be relatively prime if the GCF of a and b is 1. Thus, 3 and 16 are relatively prime, but 6 and 16 are not. The GCF of 6 and 16 is 2.

DIRECTIONS:

1. Write the GCF of each of the following pairs of counting numbers in the form $x \cdot a + y \cdot b$.

   a) $a = 9, \quad b = 30$
   c) $a = 9, \quad b = 25$
   b) $a = 8, \quad b = 28$
   d) $a = 18, b = 42$

2. Justify the following statement. If a and b are relatively prime counting numbers, then there are integers x and y such that $x \cdot a + y \cdot b = 1$.

3. Decide which of the following pairs of counting numbers are relatively prime. For each pair that is relatively prime, find integers x and y such that $x \cdot a + y \cdot b = 1$.

   a) $a = 9, \quad b = 20$
   c) $a = 6, \quad b = 40$
   b) $a = 9, \quad b = 60$
   d) $a = 6, \quad b = 35$

4. Is the relation "is relatively prime to" an equivalence relation? If so, justify your claim; if not, find an example that illustrates that one of the properties of an equivalence relation fails to hold.

Appendix

AN EXAMPLE OF PROBLEM SOLVING

This appendix presents the solution of a problem comparable in difficulty to some of those in Part B of Activity 9, using the organizational scheme introduced in Activity 8. This example might be read profitably before working the problems of Activity 9. Remember that mathematics should be read with paper and pencil handy. You will need to pause to check calculations and to convince yourself that assertions made in the text are valid.

DISCUSSION:

Here we consider a somewhat more difficult problem of the same general type as those of Part B of Activity 9. The approach outlined here again involves looking at special cases, searching for patterns, and carefully examining the situation in terms of what we know about numbers. This example was suggested by a problem in Mathematical Discovery, Vol. II, by Polya (problem 15.48, page 166).

BACKGROUND FOR THE PROBLEM

We have considered in Activity 8 several examples of sets of numbers that can be represented by arrays of dots of a particular geometric form. We continue this idea and introduce the concept of a trapezoidal number. As you would expect, a trapezoidal number is one that can be associated with a trapezoidal array of dots. The arrays of interest to us are regular ones, that is, arrays in which the number of dots in any row is one more (or one fewer) than the number in adjacent

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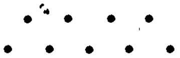
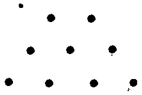
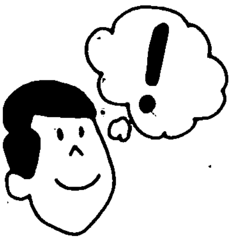
rows. For example, 9 is a trapezoidal number since there are 9 dots in the array  We notice that there are also 9 dots in the array  and in the (somewhat degenerate) array of 9 dots in a line  In fact, if we admit arrays of this last type, then it is clear that every number is trapezoidal. However, it is also clear that some numbers have several representations using trapezoidal arrays while others, 4 for example, have only one representation. This leads us to the basic problem to be studied here.

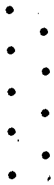
THE PROBLEM

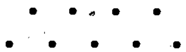
How many different trapezoidal arrays does each counting number have?

A SOLUTION OF THE PROBLEM

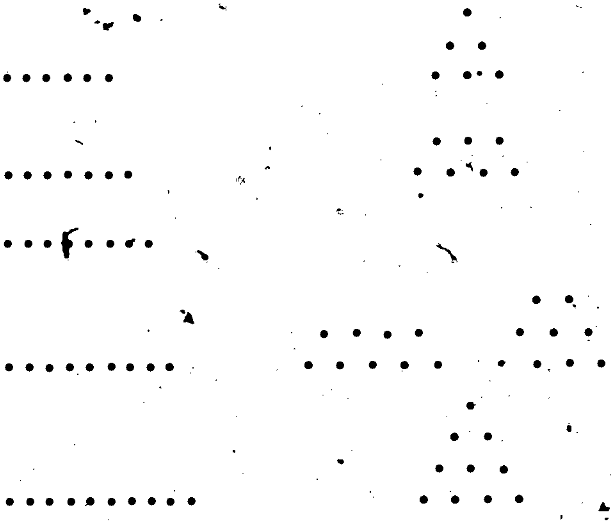
We begin by trying to understand the question more precisely. First, what is meant by "different" arrays? Taking, for example, arrays for the number 9, should the array



be considered as different from  ?

Both arrays are trapezoidal. If we view these two arrays as different, then, since there are infinitely many orientations for the basic array  we would conclude that there are infinitely many different arrays for each number. This does not appear to be an interesting answer to the question and we drop the idea of distinguishing between arrays on the basis of orientation. Instead we look for a different way of viewing arrays. A common characteristic of all the arrays discussed just above is that there are 5 dots in one row and 4 in another row, 5 + 4 = 9. This observation provides a viewpoint from which it is meaningful to consider all arrays with 5 dots in one row and 4 in another as <u>equivalent</u> arrays. It also generalizes to the other arrays for the number 9, namely 9 = 2 + 3 + 4 (all arrays with 2 dots, 3 dots and 4 dots are considered equivalent) and 9 = 9 (all arrays with nine dots in a line are considered equivalent). It makes sense to identify a trapezoidal array for the number n with a sequence of consecutive numbers whose sum is n. The reader should consider the geometric aspects of this identification.

Let n be a counting number and define Z(n) to be the number of different ways in which n can be written as a sum of consecutive counting numbers. For the first 10 counting numbers, we have the following diagram, which displays the trapezoidal arrays and the values Z(n).



n = 6 Z(6) = 2

n = 7 Z(7) = 2

n = 8 Z(8) = 1

n = 9 Z(9) = 3

n = 10 Z(10) = 2

Data for n = 1, 2, ... 40 are summarized in Table A.

TABLE A

n	Z(n)		n	Z(n)		n	Z(n)		n	Z(n)
1	1		11	2		21	4		31	2
2	1		12	2		22	2		32	1
3	2		13	2		23	2		33	4
4	1		14	2		24	2		34	2
5	2		15	4		25	3		35	4
6	2		16	1		26	2		36	3
7	2		17	2		27	4		37	2
8	1		18	3		28	2		38	2
9	3		19	2		29	2		39	4
10	2		20	2		30	4		40	2

The reader is invited to check several of these entries.

Our goal is to find a means for determining $Z(n)$ for an arbitrary counting number n. We begin by examining the above data in detail; i.e., we begin by <u>using what we know</u>. Although our goal is to determine $Z(n)$ for a given n, it is useful to look at the values $Z(n)$ and see if we can deduce a relation between these values and the associated values of n. This technique, i.e., the comparison of input n and output $Z(n)$, for specific values of n, frequently provides useful information. First of all, consider the n's for which $Z(n) = 1$. We have $n = 1, 2, 4, 8, 16, 32$. That is, for the range of n studied here, $Z(n) = 1$ in exactly those cases in which n is a power of 2. (Remember $2^0 = 1$.) At this point we would be justified in making our first conjecture (or guess, or hypothesis):

<u>Conjecture 1</u>:

> If n is a power of 2, then $Z(n) = 1$. That is, for every whole number k, $Z(2^k) = 1$.

To keep track of progress, it is suggested that the n's which are powers of 2 be crossed out on Table A.

There are many values of n for which $Z(n) = 2$, so let us bypass them for a moment and continue by looking at the values of n for which $Z(n) = 3$ and 4.

$$Z(n) = 3 \quad \text{for} \quad n = 9, 18, 25, 36$$

and

$$Z(n) = 4 \quad \text{for} \quad n = 15, 21, 27, 30, 33, 35, 39.$$

It is not easy to discern a pattern in either case. However, we do note that $Z(n) = 3$ for $n = 9$, for $n = 18 = 2\cdot 9$, and for $n = 36 = 2\cdot 18 = 2\cdot 2\cdot 9$. Also $Z(n) = 4$ for $n = 15$ and $n = 30 = 2\cdot 15$. With this clue we go back to the main table and we note that for every number n for which both n and $2n$ are included in the table, we have $Z(2n) = Z(n)$. We are ready to make another guess.

Conjecture 2:

For every value of n, $Z(2n) = Z(n)$.

The importance of this conjecture is that if we know how to determine $Z(3)$, then $Z(6)$, $Z(12)$, and $Z(24)$ can also be determined; if we can determine $Z(5)$ then $Z(10)$, $Z(20)$, and $Z(40)$ can also be determined; if $Z(7)$ can be determined, then so can $Z(14)$ and $Z(28)$, and so on. It is suggested that those values of n for which $Z(n)$ can be determined from values of $Z(n)$ for smaller n using the conjecture, i.e., $n = 6, 12, 24; 10, 20, 40; 14, 28; \ldots$, be crossed out on Table A.

Which counting numbers remain? Only those for which the associated value of Z cannot be determined by knowing the value $Z(n)$ for a smaller number n. Therefore, if one accepts Conjectures 1 and 2, then only the data in the much shortened Table B remain to be explained.

TABLE B

n	* 3	* 5	* 7	9	* 11	* 13	15	* 17	* 19	21	* 23	25	27	* 29	* 31	33	35	* 37	39
Z(n)	2	2	2	3	2	2	4	2	2	4	2	3	4	2	2	4	4	2	4

It is now time to ask how our knowledge of the counting numbers can be applied to aid in further understanding the situation. The sequence of n's remaining is a sequence of odd numbers, and it is natural to look for ways of subdividing the class. One natural way is to divide it into primes and composites. Each prime number is denoted by an asterisk in Table B. We observe that every prime number n has $Z(n) = 2$. On this basis we formulate another hypothesis.

Conjecture 3:

$$\boxed{\text{If } n \text{ is an odd prime, then } Z(n) = 2.}$$

Let us review again what we know in light of this conjecture. We recall from Activity 2 that one of the characteristics of a prime number is that it has exactly two factors. Therefore, another way of viewing Conjecture 3 is that if n is an odd prime, then $Z(n)$ gives the number of factors.

……

Let us digress for a moment and display the arrays for n if it is an odd prime. If n is odd, then $n = 2m + 1$ for some whole number m. Thus $n = m + (m + 1)$, and n has been written as the sum of two consecutive counting numbers. The other array is the trivial one with n dots.

$$\underbrace{\bullet\ \bullet\ \bullet \cdots \bullet}_{n \text{ dots}} \qquad\qquad \begin{array}{c} \overbrace{\bullet\ \bullet \cdots \bullet}^{m} \\ \underbrace{\bullet\ \bullet\ \bullet \cdots \bullet}_{m + 1} \end{array}$$

$$m + (m + 1) = n$$

Notice that this remark applies to all odd numbers whether primes or not. We conclude that every odd number greater than 1 has at least two arrays, a conclusion which is consistent with Conjecture 1. Why?

……

We proceed by following up the idea which led to Conjecture 3. In that discussion our concern was with prime numbers. However, every counting number can be written as a product of prime numbers (the results of Activity 3 are included in "what we know"), and this appears to be a connection worth exploiting. Write the prime factorization of each n in the empty center row of Table B. The result, omitting primes, is reproduced on the next page in Table C.

TABLE C

n	9	15	21	25	27	33	35	39
Prime Factorization	$3\cdot3$	$3\cdot5$	$3\cdot7$	$5\cdot5$	$3\cdot3\cdot3$	$3\cdot11$	$5\cdot7$	$3\cdot13$
$Z(n)$	3	4	4	3	4	4	4	4

We observe that each n that can be written as a product of two distinct odd primes has $Z(n) = 4$. Taking this together with Conjecture 3 we might propose

Conjecture:

> If $n = p_1p_2\cdots p_k$ where $p_1, p_2, \ldots p_k$ are distinct odd primes, then $Z(n) = 2k$.

Using the data of Table A we see that this conjecture checks when $k = 1$ and $k = 2$ and $n \leq 40$, that is for numbers less than or equal to 40 that have either one or two distinct odd prime factors. In other words, it checks for all the data contained in Table A. One might claim that this is a sufficient check and stop. However, it pays to be skeptical and to try another case: Let $n = 105 = 3\cdot5\cdot7$. In this case $k = 3$ and our conjecture is $Z(105) = 6$.

However, we have

$$\begin{aligned}
105 &= 105 \\
&= 52 + 53 \\
&= 34 + 35 + 36 \\
&= 19 + 20 + 21 + 22 + 23 \\
&= 15 + 16 + 17 + 18 + 19 + 20 \\
&= 12 + 13 + 14 + 15 + 16 + 17 + 18 \\
&= 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 \\
&= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14.
\end{aligned}$$

That is, Z(105) = 8; consequently this last conjecture is actually false even though it held for all the data in Table A. This points out the need for mathematical justification or proof. Proposals that appear to be reasonable, and that may in fact give correct results in many special cases, may nevertheless be false. They do not give correct results in general, i.e. in all cases. No number of examples or special cases can ever prove that an assertion is true, while a single example (as above) may prove it to be false. Examples that prove an assertion to be false are known as counterexamples and play a very important role in mathematical problem solving.

The failure of this last conjecture leads us to believe that looking for a method of computing Z(n) directly in terms of the number of prime factors of n is not fruitful. Rather than give up completely what appears to be a promising idea, we modify it slightly and consider the number of divisors of n. If n is the product of distinct odd primes, then all divisors of n are the products of the prime factors. Data can be obtained from Table C. We include data for the number 105 which proved to be helpful as a test case just above.

n	15	21	33	35	39	105
Prime Factors	3, 5	3, 7	3, 11	5, 7	3, 13	3, 5, 7
Divisors	1, 3 5, 15	1, 3 7, 21	1, 3 11, 33	1, 5 7, 35	1, 3 13, 39	1, 3, 5, 7 15, 21, 35, 105
Number of Divisors	4	4	4	4	4	8
Z(n)	4	4	4	4	4	8

This data provides evidence for another conjecture.

Conjecture 4:

> If n is the product of distinct odd primes, then Z(n) equals the number of divisors of n (including 1 and n).

The only remaining cases from Table A to be considered are those in which n is the product of odd primes, not all distinct. Relevant data taken from Table A and two additional cases are given below.

n	9	25	27	45	75
Prime Factorization	$3\cdot3$	$5\cdot5$	$3\cdot3\cdot3$	$3\cdot3\cdot5$	$3\cdot5\cdot5$
Z(n)	3	3	4	6	6

We invite the reader to find the arrays that show that Z(45) = 6.

Taking a cue from the argument leading to Conjecture 4, we augment this table by noting the divisors and the number of divisors.

n	9	25	27	45	75
Divisors	1, 3, 9	1, 5, 25	1, 3 9, 27	1, 3, 5, 9, 15, 45	1, 3, 5, 15, 25, 75
Number of Divisors	3	3	4	6	6

Thus, the fact that n is a product of <u>distinct</u> odd primes does not seem to be important in Conjecture 4. That conjecture can be modified to take this observation into account.

Conjecture 5:

If n is a product of odd primes, then $Z(n)$ is equal to the number of divisors of n (including 1 and n).

Let us now summarize our work. Our problem is to determine $Z(n)$ for an arbitrary counting number n. We know that n can be written as a product of primes. If it is a power of 2, then the first of our conjectures applies and we propose that $Z(2^k) = 1$. Powers of the prime 2 in the factorization of n turn out to be unimportant in the determination of $Z(n)$. Remember $Z(2n) = Z(n)$ for every counting number n is one of our conjectures. If n is a product of odd primes, then Conjecture 5 tells how $Z(n)$ is to be determined. Let us collect all of this into one final conjecture.

Final Conjecture:

Given any whole number n, $Z(n)$ is equal to the number of odd divisors of n, including 1 (and n if n is odd).

This conjecture is actually true and can be shown to be so by a careful mathematical argument.

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Grosswald, Emil. Topics from the Theory of Numbers. New York: Macmillan Co., 1966.

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A very useful book that combines history, content, and many carefully selected problems in number theory is

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REQUIRED MATERIALS

ACTIVITY	AUDIO-VISUAL	MANIPULATIVE AIDS
Overview	Slide-tape: "Overview of Number Theory," cassette tape recorder and projector. (Optional)	
2		Set of Cuisenaire rods, 20 tiles.
5		Set of Cuisenaire rods.

Continued from inside front cover

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Addition and Subtraction, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Transformational Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Probability and Statistics,* and *Experiences in Problem Solving.*

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INSTRUCTOR'S MANUAL
to accompany

NUMBER THEORY

Mathematics-Methods Program
units written under the direction of

John F. LeBlanc or Donald R. Kerr, Jr. or
Maynard Thompson

Instructor's Manual to accompany

NUMBER THEORY

Prepared by

Thomas L. Schroeder

Under the direction of

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CONTENTS

NUMBER THEORY
INSTRUCTOR'S MANUAL

INTRODUCTION:

This unit, like other units of the Mathematics-Methods Program, involves one as an adult learner in activities which have implications for teaching children. By working with concepts that children might learn, by studying the problem-solving processes that children might use, and by doing activities that might be modified for use with children, one grows in understanding and enjoyment of mathematics. The objective is to increase both students' competence and their desire to teach mathematics to children.

The "Introduction to the Number Theory Unit" which appears on pp. 1-4 of the unit describes the content of the unit and explains the spirit in which the unit was written. It would be a good idea for the instructor to become acquainted with this introduction in the process of deciding whether and how to use the unit.

THE CONTENT OF THE UNIT:

As is noted in the Introduction, the unit contains six major parts. They are (1) an overview which focuses on the historical development of number theory and the role of number theory in the elementary classroom (pp. 7-13); (2) a list of terms, definitions, and notations used in the unit; (3) Section I which presents the basic concepts of divisibility, primes, composites, and factorization; (4) Section II, concerned with problem solving, which explores a number of easily understood but challenging problems, presents an organizational scheme for attacking problems, and provides opportunities for solving problems of various difficulties; (5) Section III, on applications, which illustrates how some of the ideas introduced earlier can be extended and applied; and (6) an Appendix which presents another example of problem solving. At the end of the unit, a substantial bibliography can be found.

TIMETABLE SUGGESTIONS:

The time spent on this unit will depend upon a number of factors, including the mathematical background of the students, the time available for the unit, and the relative emphasis to be given to mathematics content and to teaching methods. The chart below suggests three alternatives for scheduling the work of the unit, each predicated on a different set of values and priorities. We have characterized the alternatives as:

A. Mathematics & Methods, Leisurely--for an integrated content and methods course in which there is time to deal with this unit in some detail. About 20-25 single periods would be needed.

B. Mathematics & Methods, Rushed--for an integrated content and methods course in which this unit has low priority or in which time is at a premium. About 11-15 single periods would be needed.

C. Mathematics Emphasis--for a course which is concerned mainly with mathematics content for prospective teachers. About 15-20 periods would be needed.

These are just three of many possible alternatives; we hope they will be helpful in deciding how to use the unit. The numbers in the table below are estimates of the number of class periods needed for each activity. The symbol "HW" indicates that all or part of the activity could be done as homework. When "HW" precedes the number of periods, advance preparation by students is suggested; when "HW" follows the number of periods, homework to finish the activity is intended. "HW" alone indicates that the entire activity could be done outside of class.

	Alternative Timetables		
Activity (A) or Project (P)	A. Mathematics & Methods, Leisurely	B. Mathematics & Methods, Rushed	C. Mathematics Emphasis
Overview	HW,1	HW,.5	HW,.5
A1	.5,HW	.5,HW	.5,HW
A2	1	.5	.5,HW
A3	1-2,HW	1,HW	HW,.5,HW
P1	HW,.5	HW, or omit	HW,.5
A4	1,HW	.5,HW	.5-1,HW
P2	HW,.5	HW	HW,.5
A5	2,HW	1-2,HW	2,HW
A6	2,HW	1-2,HW	1-2,HW
P3	HW,.5	HW	HW,.5
A7	HW,1-2	HW,1	HW, or omit
A8	2-3,HW	1-2,HW	2-3,HW
A9	HW	HW	HW
P4	HW,.5	HW, or omit	HW,.5
A10	1,HW	.5,HW	1,HW
P5	HW,.5	HW, or omit	HW,.5
A11	1,HW	.5-1,HW	1,HW
P6	HW,.5	HW,.5	HW,.5
A12	1-2,HW	1,HW	1-2,HW
A13	1-2,HW	1,HW	1-2,HW
Appendix	HW,1,HW	HW,.5 or omit	HW,1
Total	$19\frac{1}{2}$ - $24\frac{1}{2}$	$10\frac{1}{2}$ - $14\frac{1}{2}$	$15\frac{1}{2}$ - 20

OVERVIEW OF NUMBER THEORY

MATERIALS PREPARATION:

(Optional) The Mathematics-Methods Program slide-tape presentation entitled "Overview of Number Theory."

COMMENTS AND SUGGESTED PROCEDURE:

The content of this activity may be either the slide-tape presentation or the essay on pages 7-13 of the unit. No matter which alternative is chosen, students will gain more from the experience if they read the questions (1-4, pp. 5-6) first. These questions can serve as advance organizers to enhance the efficiency of their viewing or reading. Discussion of the questions could be a fairly brief class activity.

ANSWERS:

The comments which follow are not given as recommended answers to the questions posed in the unit, but are offered as samples of ideas which may be mentioned in the discussion.

1. Two major reasons for including the number theory strand in elementary school mathematics are (1) to extend and clarify concepts in the study of the whole numbers and the rational numbers, e.g., factor, multiple, least common denominator, GCF, and (2) to provide problem-solving experiences, e.g., the discovery of number patterns and generalizations. In addition to these subject-centered reasons there are learner-centered reasons including the possibility of helping students to enjoy working on puzzles and problems and the benefits of replacing dry tiresome

drills by self-directed practice in the service of solving a problem or detecting a pattern.

2. Problem-solving experiences such as those mentioned in (1) above provide opportunities for both individual and group exploration. Children like looking for number patterns and properties, and many of these can be identified by children without a great deal of teacher direction, e.g., "odds" and "evens." Carefully chosen materials and activity cards often help to make number patterns more obvious. Consider, for example, children working with odds and evens, E + E = E, E + O = O, etc. (the materials used in slides 24 and 25). On the other hand, some of the topics associated with the application of number theory ideas to rational numbers, e.g., least common denominator, may need more careful teacher direction.

3. Each right triangle with integral sides (e.g., 3, 4, 5) corresponds to a solution of the equation $x^2 + y^2 = z^2$. For example, $3^2 + 4^2 = 5^2$.

4. There are eight magic squares using the numbers 1 through 9. Each can be obtained from the one in slide 10 by a series of rotations and reflections.

8	1	6
3	5	7
4	9	2

SECTION I

DIVISIBILITY, PRIME NUMBERS, AND FACTORIZATION

INTRODUCTION:

This section includes the basic concepts of prime and composite, factor, multiple, and divisibility. These topics appear explicitly in most elementary school mathematics programs, and are often presented through activities similar to those of this section.

MAJOR QUESTIONS:

These discussion or essay format questions attempt to capture the essence of the section. They may be assigned as homework, or modified for use as examination items, or discussed in class. The comments which follow are not given as definitive or even model answers, but we hope they may be useful in stimulating thought and discussion.

1. The prime numbers act as the building blocks for the counting numbers, when the method of construction is multiplication. Another way of constructing the counting numbers would be to build on one (unity) using addition (or a successor function) as the method of construction.

   (e.g., 1 = 1, 2 = 1 + 1, 3 = (1 + 1) + 1, 4 = ((1 + 1) + 1) + 1,
   or, 1 = 1, 2 = S(1), 3 = S(S(1)), 4 = S(S(S(1))), etc.)

2. The arguments advanced to support the use of trains and tiles are all the familiar ones concerning the advantages of using concrete embodiments to develop concepts.

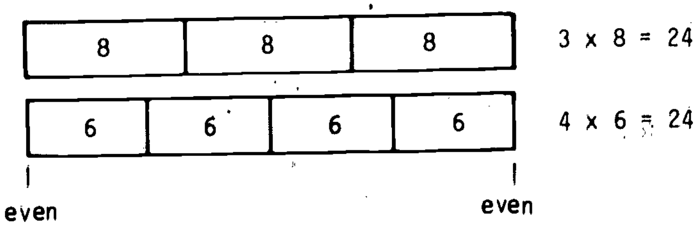
3. In the multiplication table all the products which are not 1 x n or n x 1 will be composites, but it does not follow that all the composites less than 100 will appear in the 10 x 10 multiplica-

tion table. For example, 39 = 3 x 13 does not appear in the table. Also the row and the column of multiples of 1 contain some primes and some composites.

4. Most students will probably argue in support of introducing prime numbers in the elementary school. Reasons may include the usefulness of prime factorizations (in LCM, GCF, etc.), the desirability of forming concepts and classifications of numbers, and the motivation and positive attitudes that can be developed through games and other activities.

5. Feasible topics for the elementary school might include just about any from the entire unit, but the topics from this section are most appropriate.

6. This question asks for a description of the method of finding the LCM of two numbers by "trains" (c.f., Activity 2). One lays out two trains or rods (of the respective lengths) starting "even," and searches for a place where they end "even." For example, the diagram below shows that the LCM of 6 and 8 is 24.



7. In addition to the method of constructing the counting numbers given in (1) above, one might mention the construction of all fractions from the unit fractions (fractions in the form $\frac{1}{n}$) by repeated addition, the construction of Euclidean geometry from points by set operations (the point-wise model of geometry), or all rigid transformations in the plane as a composition of flips (or slides, turns, and flips).

ACTIVITY 1
DIVISIBILITY

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

The objective of this activity is to explore the notion of one number "dividing" another. As is pointed out in the discussion, the term "divides" in this activity refers to division without a remainder or "divides evenly." After students have met the concept and its notation $(a|b)$, they investigate whether $a|b$ and $b|c$ implies $a|c$; $a|b$ and $a|c$ implies $a|b \pm c$; $a|b$ and $a|c$ implies $a|b \cdot c$; $a|b$ and $a|b + c$ implies $a|c$; and $a|b$ and $a|c$ implies $a|mb + nc$. The first part of the activity will probably require some exposition and discussion, but the conjectures are intended to be the type that students are able to work out on their own.

ANSWERS:

1. $6|42$ $\quad 42 = 6 \cdot n$ $\quad n = 7$
   $6|18$ $\quad 18 = 6 \cdot m$ $\quad m = 3$
   $42 + 18 = 6 \cdot 7 + 6 \cdot 3 = 6(7 + 3) = 6 \cdot 10$ $\quad$ so $6|(42 + 18)$
   $42 - 18 = 6 \cdot 7 + 6 \cdot 3 = 6(7 - 3) = 6 \cdot 4$ $\quad$ so $6|(42 - 18)$

2. $6|42$, but $6|15$ is false, i.e., $6\nmid 15$

3. Yes, $a|b \rightarrow b = ax$ and $a|c \rightarrow c = ay$ so $b \cdot c = ax \cdot ay$.
   $bc = a(xay)$ shows $a|bc$. It is even true that $a^2|bc$.

4. The conjecture is correct. The three examples given should be similar to the following:
   Let $a = 2$ $\quad b = 6$ $\quad b + c = 10$ $\quad a|b$ because $2|6$

$a|(b + c)$ because $2|10$. Now if $b + c = 10$ and $b = 6$, then $c = 4$ and $a|c$ since $2|4$.

5. The conjecture is correct. The three examples given should be similar to the following:
   Let $a = 2$ $b = 6$ $c = 24$.
   $a|b$ because $2|6$, $b|c$ because $6|24$.
   $a|c$ is also true since $2|24$.

6. The assertion is correct. An example of it is the following:
   Let $a = 2$ $b = 6$ $c = 8$.
   $a|b$ because $2|6$. $a|c$ because $2|8$. $(2b + 3c) = [(2\cdot6) + (3\cdot8)] = (12 + 24) = 36$. $2|36$ so $a|(2b + 3c)$. An argument in special case and general case supporting the assertion is the following.

SPECIAL CASE	GENERAL CASE
since $2|6$ 6 is a multiple of 2 $6 = 2 \times 3$	since $a|b$ b is a multiple of a $b = na$
since $2|8$ 8 is a multiple of 2 $8 = 2 \times 4$	since $a|c$ c is a multiple of a $c = ma$
$36 = [(2\cdot6) + (3\cdot8)] = [(2\cdot2\cdot3) + (3\cdot2\cdot4)] = 2[(2\cdot3) + (3\cdot4)]$ so $2|36$	$2b + 3c = 2na + 3ma = a(2n + 3m)$ so $a|(2b + 3c)$

ACTIVITY 2
PRIME AND COMPOSITE NUMBERS

MATERIALS PREPARATION:

Sets of about 20 tiles (square or rectangular), one set per student or group of 2-3 students in half the class; sets of Cuisenaire rods (about 5 of each color), one set per student or groups of 2-3 students in half the class.

COMMENTS AND SUGGESTED PROCEDURE:

It is intended that half the class will work individually or in small groups on Part A while the other half works individually or in small groups on Part B. As individuals and groups finish they should share and compare the experiences they have had working on the concepts of prime and composite numbers in these two different formats (arrays of tiles, trains of rods). The instructor may wish to check that the tables have been filled in and the questions answered completely and correctly before concluding the activity with some brief comments on the usefulness of these two embodiments in explaining prime and composite numbers to elementary school children.

ANSWERS:

Part A

1. There are two arrays and therefore two divisors of 2.

2. The completed table is

Number of Tiles	Number of Rectangular Arrays	Dimensions of Each Array	Number of Divisors	Divisors
3	2	1 x 3 3 x 1	2	1,3
4	3	1 x 4 2 x 2 4 x 1	3	1,2,4
5	2	1 x 5 5 x 1	2	1,5
6	4	1 x 6 2 x 3 3 x 2 6 x 1	4	1,2,3,6
7	2	1 x 7 7 x 1	2	1,7
8	4	1 x 8 2 x 4 4 x 2 8 x 1	4	1,2,4,8

TABLE (cont.)

Number of Tiles	Number of Rectangular Arrays	Dimensions of Each Array	Number of Divisors	Divisors
9	3	1 x 9 3 x 3 9 x 1	3	1,3,9
10	4	1 x 10 2 x 5 5 x 2 10 x 1	4	1,2,5,10
11	2	1 x 11 11 x 1	2	1,11
12	6	1 x 12 2 x 6 3 x 4 4 x 3 6 x 2 12 x 1	6	1,2,3,4, 6,12

3. 2,3,5,7,11

4. The composites between 2 and 12 (inclusive) are 4, 6, 8, 9, 10, 12; the composites with an odd number of divisors (4 and 9) are perfect squares.

5. 1

Part B

1. 2 trains

2. The completed table is

Color of Rod	Rod Number	Number of Trains of Equivalent Rods	Number of Divisors	Divisors
Red	2	2	2	1,2
Light Green	3	2	2	1,3
Purple	4	3	3	1,2,4

TABLE (cont.)

Color of Rod	Rod Number	Number of Trains of Equivalent Rods	Number of Divisors	Divisors
Yellow	5	2	2	1,5
Dark Green	6	4	4	1,2,3,6
Black	7	2	2	1,7
Brown	8	4	4	1,2,4,8
Blue	9	3	3	1,3,9
Orange	10	4	4	1,2,5,10
See p.24 of unit	11	2	2	1,11
See p.24 of unit	12	6	6	1,2,3,4, 6,12

3. 2,3,5,7,11; primes have exactly two divisors

4. The composite numbers with an odd number of trains (4 and 9) are perfect squares.

5. 1, the unit.

ACTIVITY 3
FACTOR TREES AND FACTORIZATION

MATERIALS PREPARATION:

None

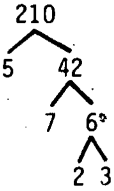
COMMENTS AND SUGGESTED PROCEDURE:

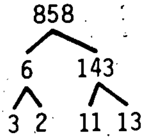
The content of this activity is divided into three parts: Part A: Factor Trees, Part B: Factorization into Primes, and Part C: Exponential Notation and the Prime Factorization Theorem. Most of this material should be easy for students to master by reading the discussions and following the directions. Some points worth discussing are the reasons for admitting factorization which has 1 for one of the factors, and questions A3, A6, B2, B5, and C3.

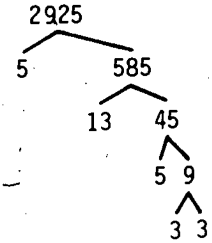
ANSWERS:

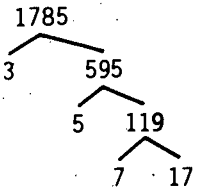
PART A: FACTOR TREES

1.

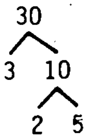


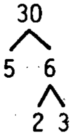


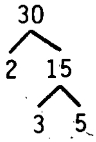




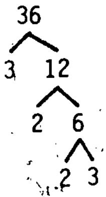
2. a)

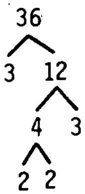


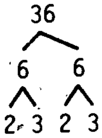


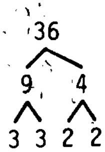


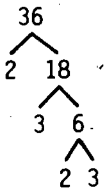
b)

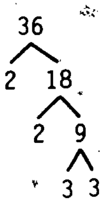




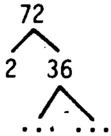


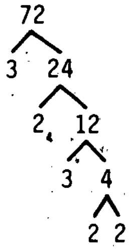


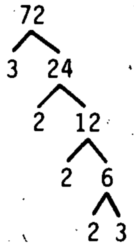


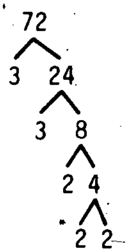


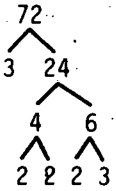
c) There are 15 different complete factor trees for 72. Six of them begin with and proceed from 36 in the six ways given in (b) above. The remaining nine are:

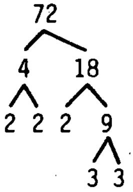


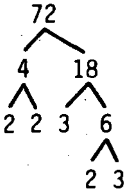


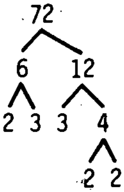


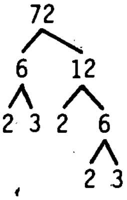


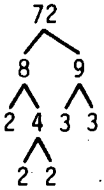






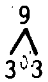




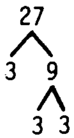


3. Three examples of composites with unique factor trees are

,



, and



The property common to such numbers is that they are either squares of primes, cubes of primes, or have exactly two prime factors.

4. Since each prime has only itself and one as factors, and since 1 has been "ruled out" of factor trees (because factoring out a 1 permits an unlimited number of branches to sprout at every fork), primes may be thought of as complete trees in themselves.

5. 30 = 2 x 3 x 5
   36 = 2 x 2 x 3 x 3
   39 = 3 x 13
   60 = 2 x 2 x 3 x 5
   72 = 2 x 2 x 2 x 3 x 3

   Yes, every composite number may be written as the product of primes. The primes that one must multiply together are the primes which appear at the "twigs" or "acorns" of the number's factor tree.

6. Having the tree grow upward has the advantage of more closely resembling a real tree, but there are two main disadvantages: First, as you write upward your hand covers the work you have already done, and may cause a smear especially if you are working in ink. Secondly, it may be difficult to estimate in advance how much space to allow to be sure that you will have enough space as the tree grows upward. Most elementary school children don't seem to mind that the "trees" are "upside down." The tree may even be considered as a root system--showing that the composites have their roots in the prime numbers.

PART B: FACTORIZATION INTO PRIMES

1. 100 = 2·2·5·5

2. Example: 72 = 9 x 8   72 = 12 x 6

3. The primes are the fundamental or primary multiplicative building blocks of the counting numbers greater than one. Composites can be thought of as being composed of primes. The composites are compound numbers.

4. Yes, yes. The primes must be thought of as having only one factor, since factors of 1 are not considered.

5. A number could have several prime factorizations if 1 were considered to be prime, for example: $6 = 2 \times 3$ $6 = 2 \times 3 \times 1$
$6 = 2 \times 3 \times 1 \times \ldots \times 1$

PART C: EXPONENTIAL NOTATION AND THE PRIME FACTORIZATION THEOREM

1. $39 = 3 \times 13$
$60 = 2^2 \times 3 \times 5$
$512 = 2^9$
$27 = 3^3$

2. If m is a composite counting number, then m is a product of prime powers; that is, there are counting numbers $e_1, e_2, \ldots, e_n$ and distinct prime numbers $p_1, p_2, \ldots, p_n$ such that
$m = p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$.

3. (i) The three special cases should be similar to the following: Let $p = 2$, $q = 3$, $b = 12$. Then $2|12$, $3|12$, $2 \cdot 3 = 6$ and $6|12$.

(ii) A counterexample in the case where p or q is composite might be similar to the following: Let $p = 3$, $q = 6$, $b = 12$. Then $3|12$, $6|12$, but $3 \cdot 6 = 18$ and $18 \nmid 12$ hence $p \cdot q|b$ is false.

(iii) $p|b$ and $q|b$ means that in the prime factorization of b, both p and q appear at least once each. So b may be written $b = p \times q \times$ (other factors) since the order of factors may be rearranged (multiplication is commutative). Hence, obviously $p \cdot q|b$.

PROJECT 1
E-PRIMES

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

The Projects in this unit provide brief excursions into mathematical ideas related to the main body of the unit, but not crucial to it. Each Project can be dealt with in a variety of ways; it may be given as optional or required homework assignments; it may be done in class individually or in small groups; it may be presented to the class by the instructor or by a student or group of students who have prepared it in advance; or it may be omitted. The problem-solving goals of this unit (i.e., the process goals as distinct from the content goals) are best served when each student has the opportunity to grapple with and solve at least some of these problems on his own. However, limitations on the time available or deficiencies in the students' background may prevent this.

ANSWERS:

1. {2, 6, 10, 14, 18, 22, 16, 30, 34, 38}. In fact, an even number is an E-composite if it is a multiple of 4.

2. Yes, the argument is analogous to the argument given for the existence of a factorization into ordinary primes.

3. 4 = 2 x 2
   8 = 2 x 2 x 2
   12 = 2 x 6
   16 = 2 x 2 x 2 x 2
   20 = 2 x 10
   24 = 2 x 2 x 6
   28 = 2 x 14 . . .

4. 36 = 2 x 18   36 = 6 x 6

5. An even number is an E-prime if it is <u>not</u> divisible by 4 (i.e., has a factor of 2 but not of 2^2).

ACTIVITY 4

TESTING FOR DIVISORS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

It is expected that students will know or can discover by working together in groups the tests for divisibility by 2, 3, ..., 13. After introducing the activity with some comments on the usefulness of being able to tell by inspection the factors of a whole number, you can set the class to work in groups of 2-3 finding the divisibility rules and stating them carefully. While the groups are working you should be available to encourage the hypothesis-making and testing process, to provide an occasional hint, but not to pass out the answers.

ANSWERS:

The table "Summary of Divisibility Tests" (p. 37) should be filled in as follows:

Divisor	Test
2	Is ones digit 0, 2, 4, 6, or 8?
3	Is sum of digits divisible by 3?
4	Is counting number defined by tens and ones digits divisible by 4?
5	Is ones digit 0 or 5?
7	From the right, group the digits by threes, and mark these groups alternately positive and negative; then total the signed groups. Is this sum divisible by 7?
9	Is sum of digits divisible by 9?
10	Is ones digit 0?
11	Mark digits alternately positive and negative from the right; then total the signed digits. Is this sum divisible by 11?

Divisor	Test
13	Compute the sum as in the test for 7. Is this sum divisible by 13?

1. a) 78 has a factor of 2 and a factor of 3. $78 \div 6 = 13$. Since 13 is prime, $78 = 2 \times 3 \times 13$ is the prime factorization.

   b) 693 is divisible by 3, 7, 9, and 11. The prime factorization is $693 = 3^2 \times 7 \times 11$.

   c) 12,760 is divisible by 2, 4, 5, 8, 10, and 11. The prime factorization is $12{,}760 = 2 \times 2 \times 2 \times 5 \times 11 \times 29$.

   d) $342{,}540 = 2^2 \times 3^2 \times 5 \times 11 \times 173$

2. a) $563 - 365 = 198 = 22 \times 9$

   b) $378{,}501 - 105{,}873 = 272{,}628 = 30{,}292 \times 9$

   c) Let $N = 100a_2 + 10a_1 + a_0$ so if N^* is N with its digits in reverse order, $N^* = 100a_0 + 10a_1 + a_2$. Assuming $a_2 > a_0$, $N - N^* = 100a_2 + 10a_1 + a_0 - (100a_0 + 10a_1 + a_2) = 99a_2 - 99a_0 = 99(a_2 - a_0)$.
   So $N - N^*$ is divisible by 9 (and also by 11).

PROJECT 2
HOW MANY NUMBERS TO TEST

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This Project is concerned with the fact that in testing a whole number n for divisors, one need not be concerned with divisors larger than $\sqrt{n}$. Indeed, if $d|n$ and if $d > \sqrt{n}$ then $n = dx$ where x is another divisor smaller than $\sqrt{n}$. Therefore, d would have been discovered as a divisor when x was tested. The alternative ways of dealing with the Project are mentioned in the instructor's notes to Project 1 on p. 17.

ANSWERS:

1. The completed table reads:

Number N	Pairs of factors of N	Smaller of the pair	L(N)
24	1,24 2,12 3,8 4,6	1 2 3 4	4
12	1,12 2,6 3,4	1 2 3	3
36	1,36 2,18 3,12 4,9 6,6	1 2 3 4 6	6
60	1,60 2,30 3,20 4,15 5,12 6,10	1 2 3 4 5 6	6

2. If $N = 24 \quad n = 4$
   $N = 12 \quad n = 3$
   $N = 36 \quad n = 6$
   $N = 60 \quad n = 7$
   In each case $L(n) \leq n$

3. To prove: If $p \cdot q = N$, then either p or q must be less than or equal to n. Here n is the largest counting number such that $n \cdot n \leq N$.

   Proof: Suppose that both $p > n$ and $q > n$, and suppose that p and q are labelled so that $p \geq q$. Then $q^2 > n^2$ and $q^2 \leq p \cdot q = N$. Consequently, n is not the largest counting number such that $n \cdot n \leq N$.

4. The largest number that must be tested is the largest counting number whose square is less than or equal to N.

5.

N	100	64	1008	80	230
L(N)	10	8	31	8	15
Reason	$10^2 = 100$ $11^2 > 100$	$8^2 = 64$ $9^2 > 64$	$31^2 < 1008$ $32^2 > 1008$	$8^2 < 80$ $9^2 > 80$	$15^2 < 230$ $16^2 > 230$

ACTIVITY 5
DISTRIBUTION OF THE PRIMES

MATERIALS PREPARATION:

Chart in unit (p. 45) prepared on chalkboard, chart, overhead projector transparency, etc.

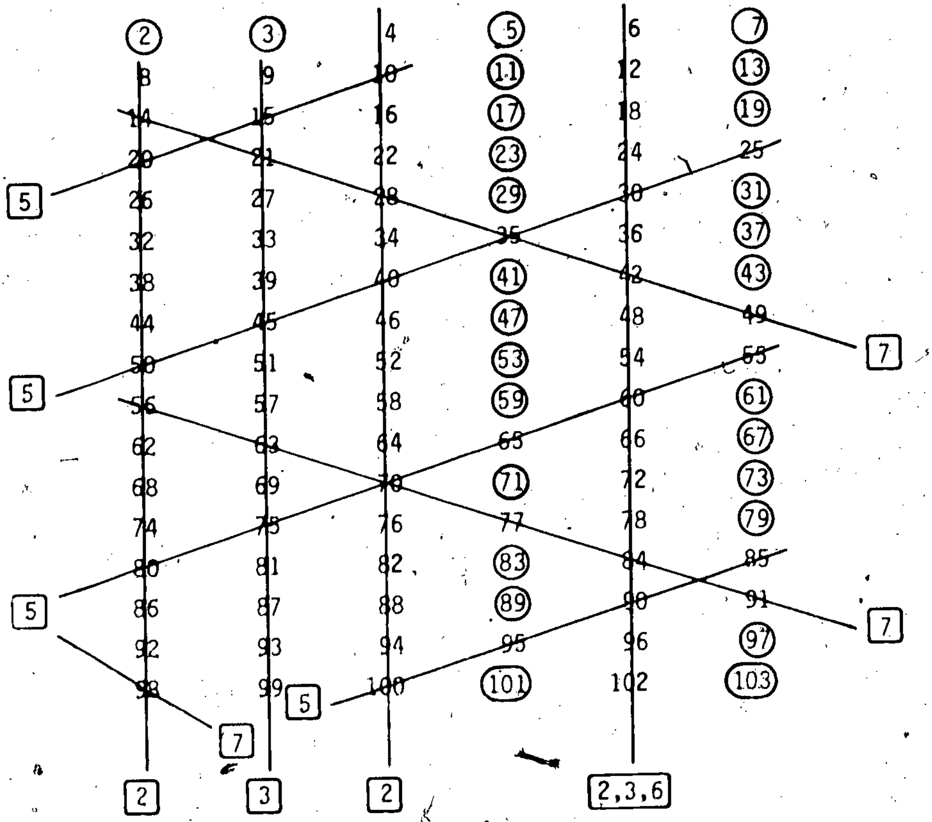
COMMENTS AND SUGGESTED PROCEDURE:

The content of this activity is divided into three parts: Part A: Identification of the Primes (Sieving Procedure); Part B: The Unlimited Supply of Primes (Constructing Primes); Part C: Strings Containing No Primes (Constructing Sequences of Consecutive Composites). The main ideas of the activity make interesting experiments worth doing and discussing in class. Also included are a number of exercises which can be done individually outside of class after the main ideas have been presented (e.g.; A: 4, 5, 6, 7, 8, 9, 10; B: 4; C: 1, 2; Challenge Problem.)

ANSWERS:

1. The arrangement of the numbers 2 through 103 is intended to suggest the means for crossing out all the composite numbers. All the numbers after 2 in the first column and all the numbers in the third and fifth columns are crossed out because they are even (i.e., multiples of 2). All the numbers after 3 in the

second column are crossed out because they are multiples of 3. (The numbers in the fifth column are also multiples of three, but they have already been crossed out because they are even. The even multiples of 3 are multiples of 6.) The multiples of 5 and of 7 are crossed out with the slanting lines shown in the diagram below. The numbers which remain have been circled. They are primes because we have crossed out all the multiples of all the numbers less than 10 and 10·10 = 100. (The multiples of 8 are also multiples of 2 and the multiples of 9 are also multiples of 3, and so have already been crossed out.)



2. The crossing lines are reminiscent of a sieve. The composites are held back and the primes sift through.

3. The patterns mentioned in this question are shown by the lines drawn in the diagram on page 22. All of the primes greater than 3 are in the fourth column or in the sixth column (i.e., they are either one more or one less than a multiple of 6).

4. six

5. 25

6. Every even number has a factor of two, so every even number greater than 2 (which is a prime) is composite. The twin primes less than 100 are 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31, 41 and 43, 59 and 61, and 71 and 73.

7. Below are possible (but not the only) solutions. As the conjecture is stated, the case of the two primes being equal (e.g., 34 = 17 + 17) is not excluded.

30 = 13 + 17 32 = 13 + 19 34 = 5 + 29 36 = 17 + 19
38 = 31 + 7 40 = 17 + 23 42 = 5 + 37 44 = 7 + 37
46 = 17 + 29 48 = 19 + 29 50 = 19 + 31

8. a) 5 + 7 = 12 12/12
41 + 43 = 84 12/84
71 + 73 = 144 12/144

b) In each pair of twin primes, the smaller one is one less than a multiple of 6 and the larger one is one more than a multiple of six. So let $P_1 = 6n - 1$, $P_2 = 6n + 1$. Thus, $P_1 + P_2 = 6n - 1 + 6n + 1 = 6n + 6n = 12n$. And $12n$ is obviously a multiple of 12.

9. Below are possible (but not the only) answers.

31 = 3 + 11 + 17
33 = 3 + 11 + 19
35 = 5 + 11 + 19
37 = 5 + 13 + 19
39 = 7 + 13 + 19
41 = 5 + 17 + 19
43 = 7 + 17 + 19
45 = 5 + 17 + 23
47 = 7 + 17 + 23
49 = 7 + 19 + 23
51 = 11 + 17 + 23

A special case of this conjecture is a conjecture made by the American mathematician Levy in 1964. Levy's conjecture is that every odd number greater than 7 is the sum of twice one prime plus another (i.e., $N = 2p + q$ where $N > 7$ is odd and p and q are distinct primes).

10. An answer to the challenge problem is given below.

a) $g(1) = 43$ $\quad g(5) = 71$
$g(2) = 47$ $\quad g(6) = 83$
$g(3) = 53$ $\quad g(7) = 97$
$g(4) = 61$

b)
$$\begin{aligned} g(40) &= (40)^2 + 40 + 41 \\ &= 40(40 + 1) + 41 \\ &= 40(41) + 41 = (41)^2 \end{aligned}$$

Part B

1. The table should be completed as follows.

Given Prime	New Numbers Generated by Proposed Method
2	$2 + 1 = 3$ (prime)
3	$(2\cdot3) + 1 = 7$ (prime)
5	$(2\cdot3\cdot5) + 1 = 31$ (prime)
7	$(2\cdot3\cdot5\cdot7) + 1 = 211$ (prime)
11	$(2\cdot3\cdot5\cdot7\cdot11) + 1 = 2311$ (prime)
13	$(2\cdot3\cdot5\cdot7\cdot11\cdot13) + 1 = 30031 = 59\cdot509$
17	$(2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17) + 1 = 510{,}511 = 19\cdot97\cdot277$

2. Yes, in fact in any set of an even number of consecutive even numbers there are as many E-primes as there are E-composites, because the E-primes are even numbers which are non-multiples of 4 and the E-composites are the multiples of 4.

3&4. Given a prime p, consider $(2\cdot3\cdot\ldots\cdot p) + 1$. The number one more than the product of all the primes less than or equal to p. This number has no prime factor less than or equal to p, because the remainder upon division by each prime less than or equal to p will be 1. Hence this number is either prime or has a prime factor larger than p.

Part C

1.

Number	5042	5043	5044	5055	5056	5057
Found by	7! + 2	7! + 3	7! + 4	7! + 5	7! + 6	7! + 7
Has a factor of	2	3	4	5	6	7

2. The 1000 consecutive composites could be found by evaluating [(1001)! + 2], [(1001)! + 3], ..., [(1001)! + 1000], [(1001)! + 1001]. Note, however, that each of these numbers is on the order of 4 x 10^{2570} (i.e., has 2,570 digits in its base ten numeral). We do not recommend that you ask students to give their answers to this question in expanded notation.

ACTIVITY 6
AN APPLICATION: GCF and LCM

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

The content of this activity is separated into three parts: Part A: LCM, Part B: GCF; and Part C: LCM, GCF, and Prime Factorization. In the first two parts the students work with the concepts and formulate definitions; in the third part, students explore alternate methods of finding LCM and GCF and some properties of the LCM and GCF of a number. A useful organizational sequence for this activity would be introduction, group work to formulate definitions, class discussion of the definitions, and exercise completion as homework.

ANSWERS:

Part A

1. a) 117

   b) 900

   c) 120

2. Zero cannot be the denominator of a fraction, because division by 0 is undefined.

3. Students' answers to this question should be complete sentences containing the words "multiple(s)," "common" (or "in both sets") and "least" (or "minimum").

Part B

1. a) 7

   b) 15

   c) 1

2. Students' answers to this question should be complete sentences containing the words "factor(s)," "common" (or "belonging to both sets"), and "largest" (or "greatest," etc.).

3. The example given should be similar to the following:

   $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ now GCF (6,12) = 6 so

   $\frac{6}{12} = \frac{6 \div 6}{12 \div 6} = \frac{1}{2}$

4. The product of their LCM and their GCF is the product of the two numbers (i.e., GCF (a,b) x LCM (a,b) = a·b).

5. Yes, any factor of either of the two numbers will be a factor of every common multiple of them.

6. The GCF of three counting numbers is the largest counting number which is a factor of all three of them.

   a) 3 b) 1 c) 15

7. As defined, $*$ is a binary operation. Some of its properties are:

   (1) $a*a = a$ (i.e., $*$ is idempotent)

   (2) $a*1 = 1$

   (3) $(a*b)*c = a*(b*c)$ (i.e., $*$ is associative)

   (4) $a*b = b*a$ (i.e., $*$ is commutative)

   (5) $a*b \leq a$ $\quad a*b \leq b$ hence

   $a*b*c \leq a$ $\quad a*b*c \leq b$ $\quad a*b*c \leq c$

   $a*b*c \leq a*b$ $\quad a*b*c \leq a*c$ $\quad a*b*c \leq b*c$

   (6) if p and q are primes $p*q = 1$

   (7) if $m|n$ $\quad m*n = m$

8. Using the result of exercise (4) above, the GCF of a and b must be 1. (i.e., they are relatively prime)

Part C

1. a) $12 = 2 \cdot 2 \cdot 3$ $\quad 40 = 2 \cdot 2 \cdot 2 \cdot 5$

   LCM $(12,40) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$

   b) $54 = 2 \cdot 3 \cdot 3 \cdot 3$ $\quad 72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$

   LCM $(54,72) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 216$

   c) $9 = 3 \cdot 3$ $\quad 39 = 3 \cdot 13$

   LCM $(9,39) = 3 \cdot 3 \cdot 13 = 117$

2. Sally's method is to write out the prime factorization of each of the two numbers. The LCM is the product of all the prime numbers which appear in either of the two prime factorizations, each prime taken as many times as the maximum number of times it appears in either prime factorization.

3. a) $54 = 2 \cdot 3 \cdot 3 \cdot 3$ $\quad 72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ $\quad$ GCF $(54,72) = 2 \cdot 3 \cdot 3 = 18$

   b) $60 = 2 \cdot 2 \cdot 3 \cdot 5$ $\quad 75 = 3 \cdot 5 \cdot 5$ $\quad$ GCF $(60,75) = 3 \cdot 5 = 15$

   c) $198 = 2 \cdot 3 \cdot 3 \cdot 11$ $\quad 162 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

   GCF $(198,162) = 2 \cdot 3 \cdot 3 = 18$

4. Sally's method is to write out the prime factorization of each of the two numbers. The GCF is the product of all the prime numbers which appear in both of the two prime factorizations,

each prime taken as many times as the minimum number of times it appears in either prime factorization.

5. The two facts to be explained are (1) that the product of the LCM and GCF of two numbers is equal to the product of the two numbers and (2) that the GCF of a pair of numbers is always a factor of their LCM. To see why these facts are true, we begin with an example. Let the two numbers be 420 and 90. We find their LCM and GCF by using the method of writing them as the products of primes.

$$420 = 2^2 \times 3 \times 5 \times 7$$
$$90 = 2 \times 3^2 \times 5$$

The LCM of 420 and 90 is the product of all the primes which appear in either of the two prime factorizations, each prime taken as many times as the maximum number of times it appears in either factorization. The GCF of 420 and 90 is the product of all the primes which appear in both of the two prime factorizations, each prime taken as many times as the minimum number of times it appears in either prime factorization. In the diagram below we have circled the factors taken in the LCM and boxed the factors taken in the GCF.

$$420 = \enclose{circle}{2^2} \times \boxed{3} \times \boxed{5} \times \enclose{circle}{7}$$
$$90 = \boxed{2} \times \enclose{circle}{3^2} \times \enclose{circle}{5}$$

$\enclose{circle}{\text{LCM}}(420,90) = 2^2\cdot 3^2\cdot 5\cdot 7 = 1260$

$\boxed{\text{GCF}}\ (420,90) = 2\cdot 3\cdot 5 = 30$

Since all of the factors of 420 and 90 are either boxed or circled, the product of the LCM and GCF is equal to the product of 420 and 90. Also note that the LCM contains all the factors which appear in the GCF (as well as others). This shows that the GCF is a factor of the LCM.

The steps used above are generalizable; that is, they apply to any two numbers a and b. First we write out the prime factorization of a and b. If a prime p appears in the factorization of a but not b, then it appears in the LCM(a,b) as many times as it appears in a. Likewise, if it appears in b but not in a, it appears in the LCM(a,b) as many times as it appears in b. If a prime appears in both factorizations, we must note how many times it appears in each. Suppose p appears s times in a and t times in b. Then it will appear in the LCM as many times as the larger of s and t and in the GCF as many times as the smaller of s and t. If it appears in both prime factorizations the same number of times, it appears in the LCM that number of times and in the GCF that number of times. Each and every prime factor in the two prime factorizations will be assigned to either the LCM or GCF. Thus the product of the LCM(a,b) and GCF(a,b) has exactly the same factors as the product of a and b, so they are equal.

6. If you know the prime factorization of a number, all the factors of that number may be found by taking all possible combinations of the prime factors (one at a time, two at a time, ..., all of them at once). The number of factors of any given counting number is the number of different subsets that can be formed from the set of its prime factors. If a counting number a has s different prime factors, then a has 2^s factors altogether. If a has a prime factorization of the form $a = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$ for $p_1 < p_2 < \ldots < p_n$ primes and each $k_i > 0$ for $i = 1, \ldots n$, then the number of factors of a is $(k_1 + 1)(k_2 + 1)\ldots(k_n + 1)$.

PROJECT 3

A PARLOR TRICK BASED ON NUMBER THEORY

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

If time allows, this trick should actually be performed in class in order for it to have the most impact. The solution of the trick could be presented by the instructor or by a class member; it could be assigned as classwork in groups; or it could be given as an optional or required homework assignment.

ANSWER:

When you make a six-digit number from a three-digit number by repeating the original digits you are in effect multiplying by 1001. The prime factorization of 1001 is 1001 = 7 x 11 x 13, so successive divisions of the six-digit number by 7, 11, and 13 will never leave a remainder, and the final result amounts to the original number first multiplied by 1001 then divided by 1001, or the original three-digit number.

Another trick similar to this one is the following. Give a guest a sheet of paper that has been divided into two columns, and invite him to choose any three-digit number and write it at the top of each of the two columns. In the example below, the guest has chosen 237. Invite another guest to choose another three-digit number and write it below the first number in the first column. In the example below, the second guest has chosen 695. You now announce that you will choose a number to write below the first guest's number in the second column. Ask another guest to multiply the two numbers in each of the two columns, then take the sum of the products. You can nonchalantly write down that his sum will be the six-digit number formed by repeating the digits of the first guest's number. The trick depends upon your choosing the second number in the second column so that its sum with the second guest's number is 1001. In the example below, the second guest chose 695, so you choose 1001 - 695 = 306.

237	237
695	306
1185	1422
21330	71100
142200	72522
164715	
72522	
237237	

ACTIVITY 7

SEMINAR

MATERIALS PREPARATION:

Some of the following elementary school textbooks may prove useful as a resource for this activity. They could be available for use in class or suggested for use by students in preparing for the class discussion.

Johnson, Donovan et al. Activities in Mathematics: First Course, Patterns. Glenview, Illinois: Scott Foresman and Co., 1971.

LeBlanc, John F. Experiences in Discovery: Enrichment Materials for Elementary Mathematics, Level D. Morristown, New Jersey: Silver Burdett Co., 1967.

Manks, John L. et al. Enlarging Mathematical Ideas. Teacher's ed., Boston: Ginn and Co., 1961.

Manks, John L. et al. Exploring Mathematical Ideas. Teacher's ed., Boston: Ginn and Co., 1961.

Manks, John L. et al. Extending Mathematical Ideas. Teacher's ed., Boston: Ginn and Co., 1961.

May, Lola J. Elementary Mathematics: Enrichment, 5. Teacher's ed., New York: Harcourt, Brace, and World, Inc., 1966.

May, Lola J. Elementary Mathematics: Enrichment, 6. Teacher's ed., New York: Harcourt, Brace, and World, Inc., 1966.

COMMENTS AND SUGGESTED PROCEDURE:

The essence of this activity is investigation of number theory work actually being done in elementary school classes, and discussion of it. Feasible organizational patterns for this activity include (a) having students write brief answers to the questions outside of class, (b) discussing the questions in small groups; or (c) having the instructor lead a class discussion of the questions. No matter how the class is handled, advance research by students will probably be necessary to insure that the discussion is based on classroom realities rather than on mere opinions of what ought to be.

ANSWERS:

The answers given below are by no means the only correct answers. They are, rather, observations that might be made (by the instructor, if need be) in the course of the discussion.

1. One reason that number theory is the basis of so many popular puzzles and tricks is that the layman has relatively more experience with the whole numbers, their properties, and the operations on them than with the objects and properties of other branches of mathematics. A second reason is that number theory is a rich branch of mathematics--rich in the sense that it contains a remarkable number of interesting results which arise from the well-known properties of the counting numbers.

2. There are many curious examples of numerology in primitive cultures; some of these are mentioned in the overview (page 5). The fact that modern man, too, is affected by number superstitions is well illustrated by the fact that very few hotels or apartment buildings have a thirteenth floor and by the fact that air travel volume is always noticeably lighter on a Friday the thirteenth than on comparable days.

3. a) Some possibilities are factors and multiples, primes and composites, GCF's and LCM's, odds and evens. In connection with odds and evens, a class could discuss what happens in each of the following cases.

Odd x Odd =
Odd x Even =
Even x Even =
Even x Odd =

The class could look at a few numerical examples and then look at rectangular arrays, etc.

Some probing questions might be:

A. Should odd x even = even x odd?

B. If the length of one side of a rectangle is an even number and the length of the other side is a counting number, can you tell whether the area is odd or even?

C. If the length of one side of a rectangle is an odd number and the length of the other side is a counting number, can you tell whether the area is odd or even?

D. What properties do 0 and 1 have with respect to multiplication? Do odd or even mimic 0 or 1?

b) A. One example would be to let the children complete or create multiplicative magic squares--square arrays of distinct counting numbers such that the product of the elements along each row, column, and main diagonal is the same.

	1	50
	10	
	100	

A partial array like the one at left could be given for the child to complete. The children could then try to construct their own squares. Once some children have constructed magic squares of their own, the teacher could ask whether it is possible to construct a magic square all of whose entries are primes or all of whose entries are squares, etc.

Activity 8 contains several more examples of pattern finding in solving number theory problems.

c) Teacher A claims she wishes to give the children more computational practice. If so, number theory itself can lead to a reasonable amount of such practice without resorting to tiresome drills. Often a child will perform many more arithmetic operations in the process of testing a number-theoretic hypothesis than he could ever be expected to do in a simple drill exercise. The substitution of intrinsic motivations for extrinsic ones makes all the difference.

To summarize, there appear to be two major reasons for including the number theory strand in elementary mathematics:

i) to extend and clarify concepts occurring in the study of the whole numbers and the rational numbers, e.g., factor, multiple and least common denominator;

ii) to provide problem-solving experiences, e.g., the discovery of number patterns and generalizations.

SECTION II

PROBLEMS AND PROBLEM SOLVING

INTRODUCTION:

This section is concerned with problems and problem-solving processes. Since number theory is such a fruitful source of easily posed, easily understood, yet challenging problems, the opportunity has been taken to present (in Activity 8) an organizational scheme for attacking problems. Activity 9 contains a number of problems of various levels of difficulties for students to work at and solve by the end of the unit. The section concludes with Project 4, on Pascal's Triangle.

MAJOR QUESTIONS:

1. Even in the simplest problems, ones in which the answer is apparent almost from the beginning, it is useful to guide one's thinking by asking, "What am I to do?, What do I know?, What can I conjecture?, Is it true?, and Have I solved the problem?" In problems which involve finding all the solutions to a given set of conditions (e.g., finding all the factors of some large numbers) it is necessary to have a systematic approach in order to know when all the solutions have been found. Problem solving always invokes reflective thinking, that is to say, not only thinking about the problem itself, but also thinking about the progress that one's thinking is making toward the solution.
2. Teachers often notice that their comprehension of a topic is enhanced in the process of preparing to teach that topic to students. This experience may be explained to a large extent as an instance of the organizing power of thought that is necessarily in the reflective mode. Presumably, problem-solving situations

that put the elementary school children's thinking into this mode would benefit them as well.

4. The teacher who gives this sort of correction to a student probably assumes that the student has incorrectly answered the question, "What exactly am I to do?" No matter how many times a student reads and rereads a problem, the teacher cannot be sure that the student has answered this question correctly. The teacher needs to ask questions to find out whether the student's idea of what he is to do or to find out matches the problem actually posed.

ACTIVITY 8

ORGANIZING THE PROBLEM-SOLVING PROCESS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This activity is clearly the longest and one of the two (along with Activity 9) most difficult ones of the entire unit. This activity also contains the best exemplars of the problem-solving process which the unit seeks to develop. It is suggested that the discussions and examples be dealt with thoroughly and that the students' exercises be checked carefully to see whether they have understood and correctly adopted the problem-solving techniques to which they are being exposed.

ANSWERS:

Part A

1. Both questions can be answered in the negative if one knows that the last digit of a perfect square can be 0, 1, 4, 5, 6, or 9 but never 2, 3, 7, or 8. Another proof that the sequence 2, 22, 222, ... contains no perfect squares is the following. Suppose $n^2 = 22 \ldots 2$, then since $22\ldots2$ is even, n^2 must be even which means n itself is even. So there is some k such that $n = 2k(2k)^2 = 22\ldots2$. This means $4k^2 = 2(11\ldots1)$ or that $2k^2 = 11\ldots1$. But this is clearly impossible because the left side $2k^2$ is even but the right side $11\ldots1$ is odd. Another proof that the sequence 3, 33, 333, ... contains no perfect squares is the following. Suppose $n^2 = 33\ldots3$, then $n^2 = 3(11\ldots1)$. This shows that n^2 has a factor of 3, so n itself

must have a factor of 3. Let n = 3k. Then $(3k)^2 = 3(11...1)$ or $9k^2 = 3(11...1)$ or $3k^2 = 11...1$. This shows that 11...1 has a factor of 3; let 3a = 11...1. For this to happen the last digit or a would have to be 7, but this is impossible because $a = k^2$ is a perfect square and there are no perfect squares that have a ones-digit of 7.

Part B

1. First construct the table.

N	2^N	$2^N = 3q + r$	
		3q	r
0	$2^0 = 1$	0	1
1	$2^1 = 2$	0	2
2	$2^2 = 4$	$1 \cdot 3$	1
3	$2^3 = 8$	$2 \cdot 3$	2
4	$2^4 = 16$	$5 \cdot 3$	1
5	$2^5 = 32$	$10 \cdot 3$	2
6	$2^6 = 64$	$21 \cdot 3$	1

We hypothesize that the remainder when 2^N is divided by 3 is 1 if N is even and 2 if N is odd. From this we predict that when 2^{89} is divided by 3 the remainder is 2.

2. First construct the table.

(SEE THE FOLLOWING PAGE)

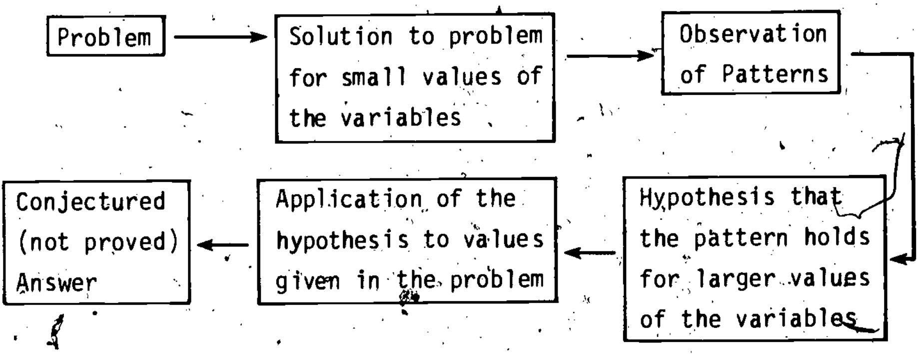
N	3^N	$3^N = 7q + r$	
		7q	r
0	$3^0 = 1$	$7\cdot 0$	1
1	$3^1 = 3$	$7\cdot 0$	3
2	$3^2 = 9$	$7\cdot 1$	2
3	$3^3 = 27$	$7\cdot 3$	6
4	$3^4 = 81$	$7\cdot 11$	4
5	$3^5 = 243$	$7\cdot 34$	5
6	$3^6 = 729$	$7\cdot 104$	1
7	$3^7 = 2187$	$7\cdot 312$	3
8	$3^8 = 6561$	$7\cdot 937$	2

We hypothesize that the remainder when dividing 3^N by 7 is determined by the remainder on dividing N by 6, and that if the remainder on dividing N by 6 is n then the remainder on dividing 3^N by 7 will be m where the relation between m and n is given by the following table.

n	0	1	2	3	4	5
m	1	3	2	6	4	5

Hence we predict that the remainder on dividing 3^{197} by 7 will be 5 because $197 \div 6 = 32$ R 5.

3. Students' answers should be similar to the following:



Part C

1. $1 + 3 + 5 + 7 + 9 = 5^2 = 25$

   $1 + 3 + 5 + 7 + \ldots + 19 = 10^2 = 100$

   $1 + 3 + 5 + 7 + \ldots + 99 = 100^2 = 10000$

2. $2 + 4 + 6 + \ldots + 222 =$

   $(1 + 1) + (3 + 1) + (5 + 1) + \ldots + (221 + 1) =$

   $(1 + 3 + 5 + \ldots + 221) + 1 \cdot 111 =$

   $(111)^2 + 111 = 12321 + 111 = 12432$

Part D

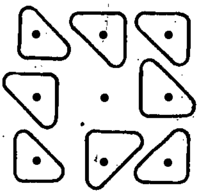
1. $1 + 2 + 3 + \ldots + 9 = \frac{1}{2} \cdot 9 \cdot 10 = 45$

   $1 + 2 + 3 + \ldots + 19 = \frac{1}{2} \cdot 19 \cdot 20 = 190$

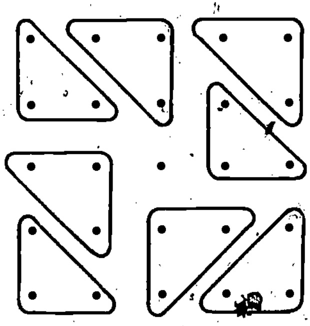
   $1 + 2 + 3 + \ldots + 99 = \frac{1}{2} \cdot 99 \cdot 100 = 4950$

2. The relation is $S_{2n+1} = 8T_n + 1$.
   This relation can be proved geometrically by observing the pattern shown below.

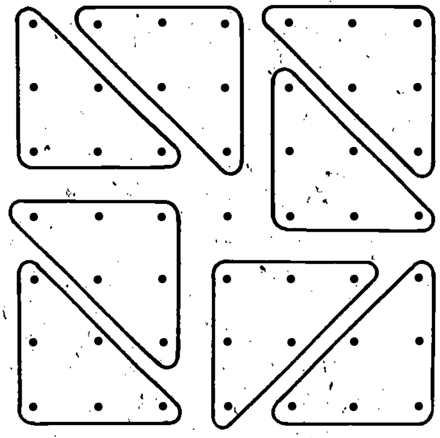
n = 1



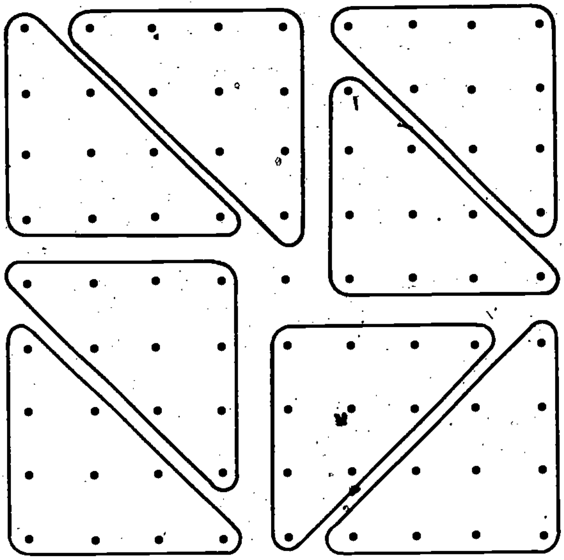
n = 2



n = 3



$n = 4$



The pattern for general n is now clear.

3. $H_n = 2T_n - 1$ $H_n = n^2 + n - 1 = n(n + 1) + 1$

4. Pattern finding has been an important element in each and every question of this activity. Some of the questions ask the student specifically to find a pattern (e.g., D2, D3); in others, the finding of a pattern has been a prerequisite or perhaps subconscious step in the process of finding a particular numerical answer.

ACTIVITY 9
PROBLEMS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

Unlike the problems in Activity 8 which were solved by copying and extending certain well-defined problem-solving strategies presented in the text, these problems are to be solved by individual students using their own idiosyncratic methods. The directions in the students' unit indicate that some or all of the problems will be assigned, that students will have a fair amount of time to work on them, and that the assignment is due by the end of the unit. The choice of which problems to assign, and of how much time to allow must be made on the basis of a careful analysis of students' abilities and backgrounds and the difficulties inherent in the problems. Note that the problems of Part B are considerably more difficult than those of Part A.

Part A

1. Yes. If n is even, n^2 is also even, and $n + n^2$ is the sum of two even numbers which is also even. If n is odd, n^2 is also odd and $n + n^2$ is the sum of two odd numbers which is even.

2. Yes, Yes, Yes. If n is even, both n and $n + 2$ are even so $n(n + 1)(n + 2)$ is divisible (twice) by 2. If n is odd then $(n + 1)$ is even hence $n(n + 1)(n + 2)$ is divisible (once) by 2.

3. Yes. If a number n is odd it can be written as $(2k + 1)$ for some $k = 1, 2, \ldots$ Thus $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. Now we know that $(k^2 + k)$ is even (Question 1 above) so $4(k^2 + k)$ has a factor of 8. Hence n may be written $n = 8m + 1$.

4. No, the product of two twin primes has no factors other than one, itself, the first prime and the second prime. However, the sum of two twin primes greater than 3 is divisible by 12, because the smaller may be written $6k - 1$ and the larger may be written $6k + 1$. Their sum $(6k + 1) + (6k - 1) = 12k$, which is clearly divisible by 12.

5. $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(4n^2 - 1)}{3}$ is very difficult to discover by means of pattern finding. However, one may write

$$\sum_{k=1}^{n} (2k - 1)^2 = \sum_{k=1}^{n} 4k^2 - 4k - 1 = 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k - n$$

to reduce the problem to one of finding a formula for the sum of the squares of the counting numbers and a formula for the sum of consecutive counting numbers. Knowing that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and that} \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

and using a little algebra yields the result above.

Part B

1. Closed, closed. Whether a locker is initially open or closed does not affect its (final) state after all boys have passed because the first boy opens all the lockers. Thus we may suppose that each locker is initially closed. This assumption allows us to make the following general statement: The state of the k^{th} locker is changed by the n^{th} boy if and only if k is a multiple of n. Therefore, the k^{th} locker changes state for each n that is a factor of k. If the total number of boys is no fewer than k, then the number of state changes that the k^{th} locker undergoes as the boys pass is just the number of factors of k. Since $1000 = 2^3 5^3$, 1000 has 4 x 4 or 16 factors. Since the 1000^{th} locker was initially closed and it underwent an even number of state changes, it is closed after the 1000^{th} boy has passed. The 764^{th} locker also ends up in a closed state because 764 equals 2^2 x 191 and therefore has 6 factors. As a matter of fact, all counting numbers have an even number of factors except those which are perfect squares. Thus all lockers except those whose numbers are perfect squares will be closed. Since 1000 and 764 are not perfect squares these lockers will be closed.

2. The next five Pythagorean triples generated by this algorithm are (7,24,25), (9,40,41), (11,60,61), (13,84,85), and (15,112,113).

3. a) Suppose a gleep is worth 7¢ and a glop is worth 23¢. Then giving the clerk 10 gleeps and receiving 3 glops in change

is equivalent to giving the clerk 1¢. Hence, the students can buy all items.

b) Suppose a gleep is worth 6¢ and a glop is worth 21¢. Then giving the clerk 1 glop and receiving 3 glops in change is equivalent to giving the clerk 3¢. Hence the students can buy any item (and only those items) whose cost is a multiple of 3¢.

(If s and t are the integral values in cents of two coins, then any item (and only those items) whose cost is a multiple of GCF(s,t) can be bought because

$$\{ms + nt > 0 : m, n \text{ are integers}\}$$

is just the set of counting number multiples of GCF(s,t).)

4. $\begin{array}{r} 92836 \\ +\ 12836 \\ \hline 105672 \end{array}$ $29786 + 850 + 850 = 31486$

5. The smallest solution to this problem is 25 coins, but $81n + 25$, where $n = 0, 1, 2, \ldots$, is an expression which gives all possible values for the number of coins originally in the chest. One method of solving the problem is to let n be the number of coins each pirate got in the final sharing and to generate expressions for the number of coins at each of the stage of the story.

$3n$	coins just before final sharing
$\frac{3}{2}(3n) + 1$	coins before 3rd pirate raids the chest
$\frac{3}{2}(\frac{9}{2}n + 1) + 1$	coins before 2nd pirate raids the chest
$\frac{3}{2}(\frac{27}{4}n + \frac{5}{2}) + 1$	coins before 1st pirate raids the chest

Thus, the original number of coins in the chest was $\frac{81}{8}n + \frac{19}{4}$, and we seek a value for n which makes this a counting number.

PROJECT 4

PASCAL'S TRIANGLE

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This project, like the other three which have preceded it can be presented in a variety of ways. See the notes to the other Projects for ideas.

ANSWER:

1. The set {A,B,C,D} contains 4 different subsets of one element, 6 different subsets of two elements, 4 different subsets of three elements, and 1 subset of four elements. If one notes that there is 1 subset containing no elements (the empty set), then one has a one-to-one correspondence between the number of subsets of different sizes and the numbers in the 4th row of Pascal's triangle. In general, the k^{th} entry of the n^{th} row of Pascal's triangle (counting the single 1 at the top as the 0^{th} row) gives the number of $k - 1$ element subsets of an n element set.

SECTION III

APPLICATIONS, CONNECTIONS, AND GENERALIZATIONS

INTRODUCTION:

This section contains activities which use the number theory ideas developed in Section I. As the title indicates, the section gives not only uses (applications) but also connections between ideas and some mathematical generalizations. For the most part, the ideas of this section do not have direct implications for the elementary school. Instead, the content is placed in a mathematical setting and pursued in such a way as to whet the student's appetite. For students who do wish to study more number theory, an extensive bibliography is included at the end of the unit.

MAJOR QUESTIONS:

1. The days of the week and months of the year may be thought of as remainder classes just as the 12-hour clock was in Activity 10, question 4. Activity 12, Part C contains other real-world instances of the concept of remainder classes.
2. The process of identifying a symbol with an equivalence class is essentially the same in both cases. In the association of numbers with numerals all sets of, say, 3 objects are equivalent in their threeness. In the association of symbols with remainder classes, it is numbers (themselves abstractions) which are abstractly related by the equivalence.
3. Yes. Equality in the ordinary sense has all the properties of congruence but not vice versa. Equality, then, is a stronger relation, but both congruence and equality are instances of the general concept of an equivalence relation.

4. In geometry there are many examples of partitioning a set by an equivalence relation. Both "is similar to" and "is congruent to" are equivalence relations defined on the set of geometric figures. The relation "has the same number of sides as" is a useful equivalence relation defined on the set of polygons. In algebra one can partition the set of linear equations in one variable into equivalence classes on the basis of slope (i.e., $y = m_1x + b_1$ is equivalent to $y = m_2x + b_2$ when $m_1 = m_2$). One notes, for example, that the sum and difference of two equations in the same equivalence class are also in the same class.

ACTIVITY 10
REMAINDER CLASSES

MATERIALS PREPARATION:

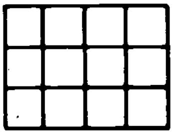
(Optional) sets of tiles.

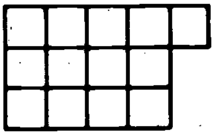
COMMENTS AND SUGGESTED PROCEDURE:

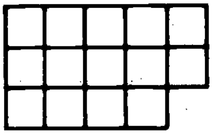
This activity introduces the concept of remainder classes through concrete (or semi-concrete) embodiments. The activity should not take long, nor is it difficult, but it should not be overlooked, because it provides the real-world roots to the concepts explored in Activities 11 and 12 (on modular arithmetic).

ANSWERS:

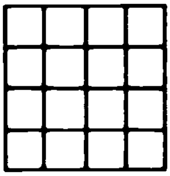
1. Rows of 3:

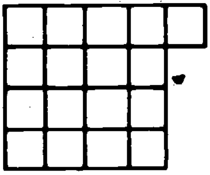


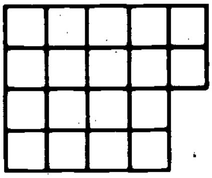


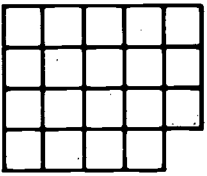


Rows of 4:

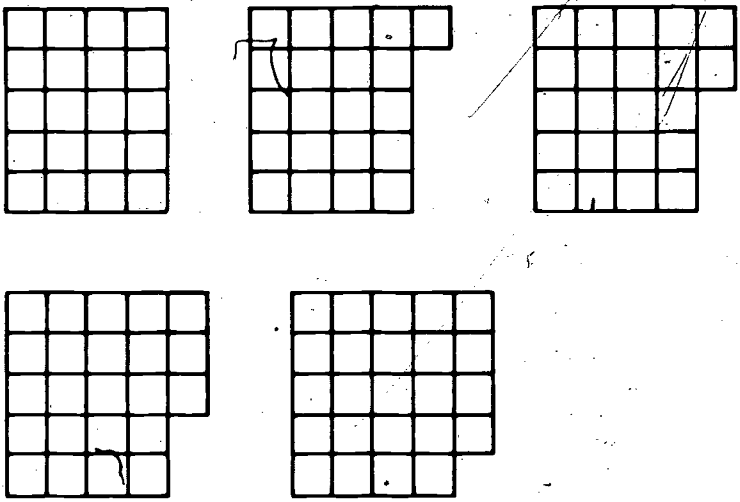








Rows of 5:



2. a) The different arrays show a number being partitioned into a rectangle of q rows each of length b (bq) and a part row (r) where the number in the part row r is less than the row length b.

   b) 0,1; 0,1,2; 0,1,2,3; 0,1,2,3,4

   c) 0,1,2,...,n - 1

3. a) Three: numbers with r = 0, with r = 1, and with r = 2

   b) Two: numbers with r = 0 (the even numbers) and those with r = 1 (the odd numbers)

   c) Five: those with r = 0, with r = 1, with r = 2, with r = 3, with r = 4.

   d) This is guaranteed by the division algorithm which asserts that given a and b there exist <u>unique</u> values for q and r such that $a = bq + r \quad 0 \leq r < b$.

4. The times on the 24-hour clock range from 0000 hours (0 hours 0 minutes after midnight) to 2359 hours (23 hours 59 minutes after

midnight). To convert from 24-hour clock to 12-hour clock we divide by 1200 and consider the quotient and the remainder. The quotient tells you whether the time is before or after noon, the remainder gives you the time in 12-hour clock. For example: 1415 hours = 1200 (1) + 215 so 1415 hours is 2:15 p.m.; 0720 = 1200 (0) + 720 so 0720 hours is 7:20 a.m.

PROJECT 5

THE SUM OF THE FIRST N COUNTING NUMBERS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This Project takes another look (from a different perspective) at a fact discovered in Activity 8. It may be dealt with in any of the several ways suggested in the instructor's notes to the other Projects.

ANSWERS:

$$1 + 2 + \dots + 14 = 1 + 2 + \dots + 13 + 14$$

sum 15

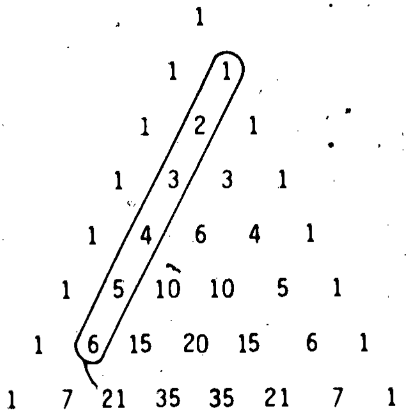
There are $\frac{14}{2}$ such sums, so $1 + 2 + \dots + 14 = 7 \cdot 15$.

$$1 + 2 + \dots + 20 = 1 + 2 + \dots + 19 + 20$$

sum 21

There are $\frac{20}{2}$ such sums, so $1 + 2 + \dots + 20 = 10 \cdot 21$.

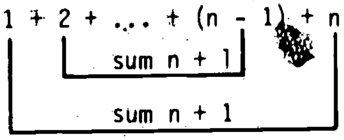
The use of Pascal's triangle to verify that $S_n = \frac{1}{2}n(n + 1)$ in the case of $n = 6$ is shown in the following diagram.



$\frac{1}{2}n(n + 1)$ for $n = 6$ is

$\frac{1}{2}\cdot 6\cdot 7 = 21$

The use of Pascal's triangle to show that the formula is true for n odd is straightforward. The grouping idea is more difficult, but can be carried out with an argument such as the following: In $1 + 2 + \ldots + n$ there are an odd number of terms when n is odd, but $\frac{(n - 1)}{2}$ pairs of them have the sum $n + 1$.



The middle term will be $\frac{n + 1}{2}$ so the whole sum will be

$$\frac{n - 1}{2}(n + 1) + \frac{n + 1}{2} = \frac{(n - 1)(n + 1)}{2} + \frac{n + 1}{2} =$$

$$\left(\frac{n + 1}{2}\right)((n - 1) + 1) = \left(\frac{n + 1}{2}\right)n = \frac{1}{2}n(n + 1).$$

ACTIVITY 11

MODULAR ARITHMETIC I

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This activity is concerned with supplying the notation to express formally the ideas developed in Activity 10. The presentation of the notation is followed by exercises in applying the symbols. An item that deserves special attention is the notion of "well-definedness" which appears in question 3. To a mathematician this is a natural concern; to the mathematically unsophisticated this notion may seem less natural and require some careful development. It might be useful to relate it to "vagueness" and to comment on the difficulties children have with a vague concept.

ANSWERS:

1.

+	$[0]_4$	$[1]_4$	$[2]_4$	$[3]_4$
$[0]_4$	[0]	[1]	[2]	[3]
$[1]_4$	[1]	[2]	[3]	[0]
$[2]_4$	[2]	[3]	[0]	[1]
$[3]_4$	[3]	[0]	[1]	[2]

+	$[0]_5$	$[1]_5$	$[2]_5$	$[3]_5$	$[4]_5$
$[0]_5$	[0]	[1]	[2]	[3]	[4]
$[1]_5$	[1]	[2]	[3]	[4]	[0]
$[2]_5$	[2]	[3]	[4]	[0]	[1]
$[3]_5$	[3]	[4]	[0]	[1]	[2]
$[4]_5$	[4]	[0]	[1]	[2]	[3]

x	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[0]_6$	[0]	[0]	[0]	[0]	[0]	[0]
$[1]_6$	[0]	[1]	[2]	[3]	[4]	[5]
$[2]_6$	[0]	[2]	[4]	[0]	[2]	[4]
$[3]_6$	[0]	[3]	[0]	[3]	[0]	[3]
$[4]_6$	[0]	[4]	[2]	[0]	[4]	[2]
$[5]_6$	[0]	[5]	[4]	[3]	[2]	[1]

2. a) yes, $[0]_4$

   b) yes, $[1]_4$

   c) no, $[2]_4 \times [2]_4 = [0]_4$ but neither factor $[2]_4$ is $[0]_4$.

   d) yes, $[0]_5$; yes $[1]_5$; yes $[a]_5 + [b]_5 = [0]$ requires either $[a]_5 = [0]_5$ or $[b]_5 = [0]_5$ or both. This can be verified by examining a table constructed for x on this system as in exercise 1.

3. a) Possible answers are

$[0]_3 + [2]_3 = [2]_3$ $[1]_4 + [3]_4 = [0]_4$ $[2]_5 + [4]_5 = [1]_5$

$6 + 14 = 20$ $5 + 7 = 12$ $2 + 14 = 16$

$18 + 8 = 26$ $13 + 15 = 28$ $7 + 19 = 26$

b) Possible answers are

$[1]_3 \times [2]_3 = [2]_3$ $[2]_4 \times [0]_4 = [0]_4$ $[1]_5 \times [4]_5 = [4]_5$

$4 \times 8 = 32$ $6 \times 4 = 24$ $6 \times 4 = 24$

$10 \times 11 = 110$ $10 \times 8 = 80$ $11 \times 9 = 99$

PROJECT 6
CASTING OUT NINES

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This Project deals with an interesting application of modular arithmetic which has been in use for several centuries. It is likely that in the past few people using this technique could explain its mathematical basis, and that their inability to do so was the result of never having investigated remainder classes and never having developed notation to describe them. Casting out nines is a good example of an interesting and useful result in number theory which is accessible to the layman.

ANSWERS:

1. $481 + 653 + 98 + 124 = 1356$ The sum is correct and casting out
   $[4] + [5] + [8] + [7] = [6]$ nines does not show it incorrect.

$25 + 36 + 86 \neq 157$ $[7] + [0] + [5] \neq [4]$	The sum is incorrect and casting out nines shows that it is incorrect.
$37 \times 255 \neq 9535$ $[1] \times [3] \neq [4]$	The product is incorrect and casting out nines shows that it is incorrect.
$17 \times 41 = 697$ $[8] \times [5] = [4]$	The product is correct and casting out nines does not show that it is incorrect.
$58 \times 74 \neq 4382$ $[4] \times [2] = [8]$	The product is incorrect, but casting out nines does not show that it is incorrect (i.e., casting out nines does not detect the error).
2. $73 + 42 \neq 124$ $[1] + [6] = [7]$	Any sum in which one digit is one too large and another digit is one too small will do.

Challenge Problem

Casting out nines depends upon two principles: (1) that if n is a counting number and $s(n)$ is the sum of its digits then $s(n) \equiv n(\text{mod } 9)$ and (2) if $a + b = c$ and $a \times b = d$, then in any modulus (and in particular in mod 9) $a(\text{mod } 9) + b(\text{mod } 9) = c(\text{mod } 9)$ and $a(\text{mod } 9) \times b(\text{mod } 9) = d(\text{mod } 9)$.

Principle (2) has been developed in question 3 above, and principle (1) can be justified as follows. In a has k digits and $a = a_0 + a_1 10 + a_2 10^2 + \ldots + a_{k-1}10^{k-1}$ where $0 \leq a_i \leq 9$ for $i = 0,1,\ldots,k - 1$, then $n = a_0 + a_1(9 + 1) + a_2(9 + 1)^2 + \ldots + a_{k-1}(9 + 1)^{k-1} = (a_0 + a_1 + a_2 + \ldots + a_n) + N$ where N is some number divisible by 9. Thus, $n \equiv (a_0 + a_1 + \ldots + a_{k-1})(\text{mod } 9)$.

ACTIVITY 12

MODULAR ARITHMETIC II:

CONGRUENCES, EQUIVALENCE RELATIONS, AND APPLICATIONS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This activity contains the most sophisticated mathematics of the entire unit. Part A deals with the notion and notation of an equivalence class. Part B deals with properties of equivalence classes in some generality. Part C contains problems which can be solved using modular arithmetic. Problems 1 and 4 in Part C are fairly difficult and may be used as challenge problems.

ANSWERS:

1. $\{x: x \in S, x \equiv 0 \pmod{2}\} = \{4, 6, 8, 10, 12, 14\}$

   $\{x: x \in S, x \equiv 1 \pmod{3}\} = \{4, 7, 10, 13\}$

   $\{x: x \in S, x \equiv 0 \pmod{5}\} = \{5, 10\}$

   $\{x: x \in S, x \equiv 1 \pmod{4}\} = \{5, 9, 13\}$

   $\{x: x \in S, x \equiv 2 \pmod{6}\} = \{8, 14\}$

2. Suitable examples are

   $9 \equiv 3 \pmod{6} \Rightarrow 9 \equiv 3 \pmod{3}$

   $12 \equiv 4 \pmod{8} \Rightarrow 12 \equiv 4 \pmod{2}$

3. If $p|q$ there is an integer x such that $px = q$. If $a \equiv b \pmod{q}$ then $q|(a - b)$, and therefore $px|(a - b)$. It follows that $p|(a - b)$ and therefore $a \equiv b \pmod{p}$.

4. The reason this "ought to be true" is that congruence mod p is an equivalence relation and equivalence relations are transitive. A proof of the assertion follows.

   If $a \equiv b \pmod{p}$ and $b \equiv c \pmod{p}$, then $p|(a - b)$ and $p|(b - c)$. If a number divides both of two numbers it divides their sum, so $p|[(a - b) + (b - c)]$ or $p|(a - c)$. Hence $a \equiv c \pmod{p}$.

5. Consider a number n such that $n \equiv 3(\text{mod } 6)$. Such a number must satisfy $6|(n - 3)$, so there is an integer b such that $n - 3 = 6k$ or equivalently $n = 6k - 3$. This shows that n must be odd. Thus any number congruent to 3 mod 6 is odd, and consequently $2x \equiv 3(\text{mod } 6)$ is impossible.

6. The key to this problem is to work in mod 3. In mod 3 any counting number m must be congruent to 0 or 1 or 2. If $m \equiv 0$ mod 3 then $m^2 \equiv 0$ mod 3. If $m \equiv 1(\text{mod } 3)$ then $m^2 \equiv 1(\text{mod } 3)$. If $m \equiv 2(\text{mod } 3)$ then $m^2 \equiv 1(\text{mod } 3)$. In no case is $m^2 \equiv 2(\text{mod } 3)$. But $(3n - 1) \equiv 2(\text{mod } 3)$, hence $(3n - 1)$ cannot be a perfect square.

7. No. If $4(n^2 + 1)$ is divisible by 11, then $n^2 + 1$ must be divisible by 11. Another way of saying this is that $n^2 + 1 \equiv 0(\text{mod } 11)$, or $n^2 \equiv 10(\text{mod } 11)$. We may check whether this is possible by constructing the following table.

$n(\text{mod } 11)$	0	1	2	3	4	5	6	7	8	9	10
$n^2(\text{mod } 11)$	0	1	4	9	5	3	3	5	9	4	1

The table shows that there is no n such that $n^2 \equiv 10(\text{mod } 11)$.

8. See the answer given in Activity 10, question 4.

Part B

1. An equivalence
2. Not an equivalence; not reflexive; not symmetric
3. Not an equivalence; not symmetric
4. Not an equivalence; not reflexive; not symmetric
5. Not an equivalence; not reflexive; not symmetric; not transitive
6. An equivalence; if one associates (a,b) with the fraction $\frac{a}{b}$ then the equivalence $\sim$ is the ordinary equivalence of two fractions.
7. An equivalence

Part C

1. 51 eggs. If n is the number of cartons of eggs originally and x is the number of batches of cookies, then $12n + 3 - 5x = 1$. Since $12n \equiv 0 \pmod{12}$ we have $3 - 5x \equiv 1 \pmod{12}$. Then $3 \equiv 5x + 1 \pmod{12}$ and consequently $5x \equiv 2 \pmod{12}$. Thus $5x$ belongs to $\{14, 26, 38, 50, \ldots\}$. The smallest possible value of $5x$ is 50 so $x = 10$. Substituting this in the original equation we get $12n + 3 - 50 = 1$ or $12n + 3 = 51$. Thus there were originally 51 eggs.

2. In mod 5 arithmetic $3 \cdot 2 \equiv 1 \pmod 5$, $4 \cdot 4 \equiv 1 \pmod 5$.

   a) $\frac{1}{2} \cdot \frac{1}{3}$ is 1 in mod 5 arithmetic

   $\frac{1}{3} \cdot \frac{1}{4}$ is $\frac{1}{2}$ in mod 5 arithmetic

   $\frac{1}{2} \cdot 3$ is 4 in mod 5 arithmetic

   b) Yes. He meant that the inverse of 2 in mod 5 arithmetic times the inverse of 2 is the inverse of 4 in mod 5 arithmetic. Since 3 is the inverse of 2 and 4 is its own inverse he claims that $3 \cdot 3 \equiv 4$ in mod 5 arithmetic; and he is right.

3. $x \equiv 2 \pmod 5$ implies that $x \in \{2, 7, 12, 17, 22, 27, 32, 37, \ldots\}$. $3x \equiv 1 \pmod 8$ implies that x must be odd, and consequently we need to consider only the odd numbers in the above set, i.e., $\{7, 17, 27, \ldots\}$. Since $3 \cdot 7 = 21 \equiv 5 \pmod 8$, $3 \cdot 17 = 51 \equiv 3 \pmod 8$, and $3 \cdot 27 = 81 \equiv 1 \pmod 8$, 27 is the smallest whole number solution to the pair of congruences.

4. The check could have been for \$10.21. To solve the problem, let A be the number of cents in the check and B be the number of dollars. The story tells us that $100A + B - 68 = 200B + 2A$, from which we may deduce that $B - 68 \equiv 2A \pmod{100}$. Now since there was only one purchase and only one possible regrouping of dollars to cents, one of the following must hold: either $2A = B - 68$ (Since $A \geq 0$, we conclude $B \geq 68$ and since B is two digits, $A \leq 16$. With such A and B, $100A + B - 68 = 200B + 2A$ is

impossible.); or 102A = B - 68 (Since B is a two-digit number, this is impossible.); or 2A = B + 32. By trial and error one has determined that it is the last of these which yields a whole number solution by substituting $A = \frac{B + 32}{2}$ into the original equation 100A + B - 68 = 200B + 2A. The details follow.

$$100\left(\frac{B + 32}{2}\right) + B - 68 = 200B + \frac{2(B + 32)}{2}$$

$$50B + 1600 + B - 68 = 200B + B + 32$$

$$1600 - 68 - 32 = 150B$$

$$1500 = 150B, \quad B = 10$$

$$A = \frac{B + 32}{2} = \frac{10 + 32}{2} = 21$$

ACTIVITY 13

THE EUCLIDEAN ALGORITHM AND OTHER SELECTED TOPICS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This final activity in the unit concerns the Euclidean Algorithm (for finding the GCF of two numbers) and the fact that given integers a and b there exist integers m and n such that am + bn = GCF(a,b). The verbal argument on pages 105-106 justifying the algorithm should be presented with discussion in class. Many students may lack the mathematical experience and maturity to follow such an argument presented only in print.

ANSWERS:

Part A

1. a) 2
   b) 44
   c) 4

2. The hint is a reminder that for any counting numbers a and b $LCM(a,b) \times GCF(a,b) = ab$. Thus $LCM(a,b) = \frac{ab}{GCF(a,b)}$. The GCF may be found by the following steps.

$$\begin{array}{r} 2 \text{ r } 200 \\ 2618\overline{)5436} \\ \underline{5236} \\ 200 \end{array} \qquad \begin{array}{r} 13 \text{ r } 18 \\ 200\overline{)2618} \\ \underline{2600} \\ 18 \end{array} \qquad \begin{array}{r} 11 \text{ r } 2 \\ 18\overline{)200} \\ \underline{198} \\ 2 \end{array} \qquad \begin{array}{r} 9 \text{ r } 0 \\ 2\overline{)18} \end{array}$$

Therefore, $LCM = \frac{5436 \times 2618}{2} = 7115724$

3. a) This is just exactly the Euclidean algorithm in a format which does not require rewriting the remainder as the next divisor.

b)

$$\begin{array}{r|rrrrr} & 3 & 3 & 1 & 3 & 2 \\ \hline 2 & 6) & 20) & 26) & 98) & 222 \\ & \underline{6} & \underline{18} & \underline{20} & \underline{78} & \underline{196} \\ & 0 & 2 & 6 & 20 & 26 \end{array}$$

$$\begin{array}{r|rrrrrr} & 3 & 1 & 1 & 2 & 3 & 6 \\ \hline 4 & 12) & 16) & 28) & 72) & 244) & 1536 \\ & \underline{12} & \underline{12} & \underline{16} & \underline{56} & \underline{216} & \underline{1464} \\ & 0 & 4 & 12 & 16 & 28 & 72 \end{array}$$

Part B

1. a) $GCF = 3 = (-3)\cdot 9 + 1\cdot 30$

   b) $GCF = 4 = (-3)\cdot 8 + 1\cdot 28$

   c) $GCF = 1 = (-11)\cdot 9 + 4\cdot 25$

   d) $GCF = 6 = (-2)\cdot 18 + 1\cdot 42$

2. The GCF of two relatively prime numbers is 1.

3. a) Relatively prime $1 = (-9)\cdot 9 + 4\cdot 20$

   b) $GCF = 3 \neq 1$

   c) $GCF = 2 \neq 1$

   d) Relatively prime $1 = 6\cdot 6 + (-1)(35)$

4. No, the relation is neither transitive nor reflexive. For example, 4 and 9 are relatively prime, and 9 and 8 are relatively prime, but 4 and 8 are <u>not</u> relatively prime. Also the GCF of a number and itself is itself, so a number cannot be relatively prime to itself.

TEACHER TEASER, page 36

$n^1 - n = 0$ is always divisible by 1

$n^2 - n = n(n - 1)$ is always divisible by 2 (one of two consecutive counting numbers is even)

$n^3 - n = (n - 1)(n)(n + 1)$ is always divisible by 3 (one of three consecutive numbers is a multiple of 3)

$n^4 - n = (n - 1)(n)(n^2 + n + 1)$ is <u>not</u> always divisible by 4 (if $n = 2$, $2^4 - 2 = 14$ which is not divisible by 4)

$n^5 - n = (n - 1)(n)(n + 1)(n^2 + 1)$ is always divisible by 5 (one of the factors is always a multiple of 5)

In general, if k is prime then $n^k - n$ is always divisible by k, no matter what whole number n is. If the student tests the case $n^6 - n$ or especially if he uses a computer to test several more cases, he will probably find this pattern.

TEACHER TEASER, page 51

The only prime triple is 3, 5, 7. To prove this suppose $p - 2$, p, $p + 2$ are primes and $p > 5$. If $p > 5$ then $p - 2$ and p are a pair of twin primes, which are one less and one more respectively than some multiple of 6. Let $p - 2 = 6n - 1$ and $p = 6n + 1$. Then $p + 2 = 6n + 1 + 2 = 6n + 3 = 3(2n + 1)$, which shows that $p + 2$ is not prime because it has a factor of 3. So it is impossible to have prime triple $p - 2$, p, $p + 2$ with $p > 5$.

TEACHER TEASER, page 86

Let n be the number of the first day of some week. Then the seven days are numbered n, n + 1, n + 2, n + 3, n + 4, n + 5, n + 6. Their sum is 7n + 21 = 7(n + 3). Another way of looking at this is to number the days [(n + 3) - 3], [(n + 3) - 2], [(n + 3) - 1], [n + 3], [(n + 3) + 1], [(n + 3) + 2], [(n + 3) + 3] and to notice that there are seven days each of which has an (n + 3) term and another term which is the additive inverse of that term for some other day. The total is 7(n + 3).

TEACHER TEASER, page 96

Sally is correct; the ones digit of a square can never be an 8. The ones digit of a square can be 0, 1, 4, 5, 6, or 9, but never 2, 3, 7, or 8.