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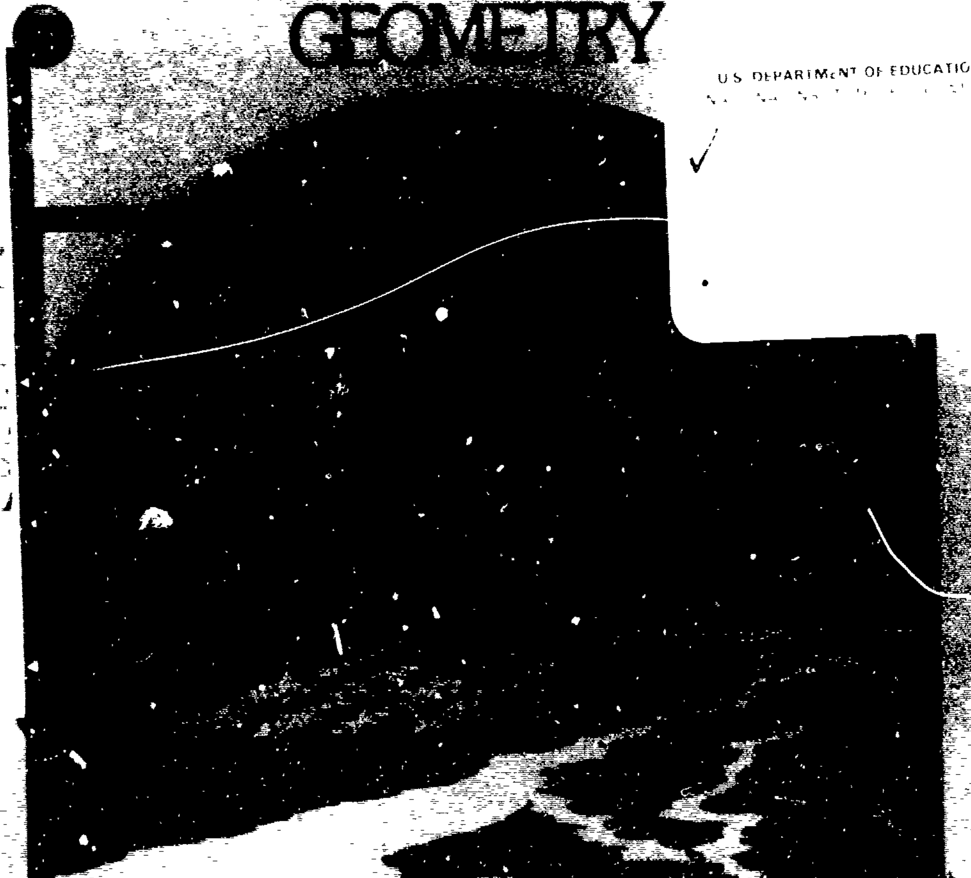
ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to geometry in the elementary school. In addition to an introduction to the unit and an overview, the text has sections on rigid projective and topological transformations. Transformations. (MP)

TRANSFORMATIONAL GEOMETRY



Mathematics-Methods Program
units written under the direction of

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

<u>Numeration</u>
<u>Addition and Subtraction</u>
<u>Multipl..ation and Division</u>
<u>Rational Numbers with Integers and Reals</u>
<u>Awareness Geometry</u>
<u>Transformational Geometry</u>
<u>Analysis of Shapes</u>
<u>Measurement</u>
<u>Number Theory</u>
<u>Probability and Statistics</u>
<u>Graphs: the Picturing of Information</u>
<u>Experiences in Problem Solving</u>

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugène, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

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CONTENTS

INTRODUCTION TO THE GEOMETRY UNITS OF THE MATHEMATICS-METHODS PROGRAM

Geometry to most people just means proving theorems about angles opposite equal sides, squares of hypotenuses, and such. This s natural since most people have their only exposure to geometry in high sch·.l where the traditional course has been built around such proofs. Geometry has been gradually working its way into the elementary school. Today's new textbooks contain a considerable amount of geometry.* Much of this material is being ignored or badly taught since many teachers see little relevance of this geometry to their own lives, to other aspects of the elementary school curriculum, or to the lives of their pupils. Moreover, some of the topics that are currently contained in textbooks were not taught when the teacher went to school and, therefore, are not fully understood by the teacher.

The geometry units of the Mathematics-Methods Program attempt to present geometry from a point of view that will bring out the potential for geometry with children. Geometry is presented as the <u>study of space experiences</u>. This point of view is not only consistent with the historical development of geometry, but it also keeps the focus on the relationship between geometry and the objects and shapes in our environment.

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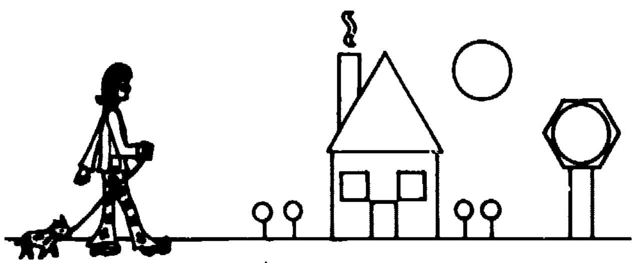
*Paul R. Trafton and John F. LeBlanc, "Informal Geometry in Grades K-6," in <u>The 36th Yearbook of the National Council of Teachers of Mathematics, 1973: Geometry in the Mathematics Curriculum</u>.

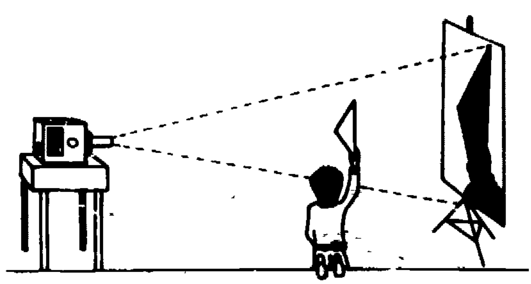
The study of space experiences addresses itself mainly to shapes. Shapes are abstractions from the environment. They can be informally investigated and analyzed. One can also study the changes (or transformations) that shapes undergo.

To effect this study of space experiences, four units have been developed.

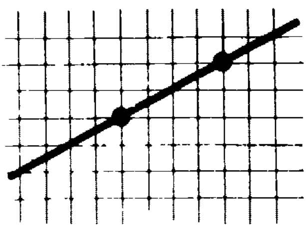
- The Awareness Geometry unit is designed to orient the prospective teacher to the informal study of geometry. In this unit one looks carefully at the environment, experiments with shapes that are observed there, and informally analyzes certain shapes. At the end of the unit, one is given experience with planning for geometry lessons with children.



- The Transformational Geometry unit studies changes that shapes can undergo. The unit is organized into the study of rigid transformations, projective transformations, and topological transformations. The presentation is informal and the focus is on concrete real-world examples of the concepts.

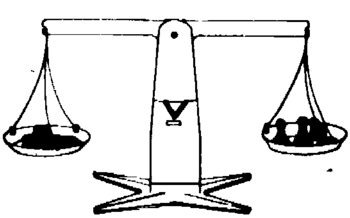


- The Analysis of Shapes unit studies straight lines, triangles and circles. The real-world occurrences and importance of each shape are investigated; each shape is informally analyzed to determine some of its important properties; and then the fruits of these analyses are applied to real-world problems. Many of the traditional topics of Euclidean geometry, including coordinate geometry, are considered here from a nontraditional point of view. There is also a section which deals with problems of verification and places into perspective the informal methods of elementary school geometry and the formal approach to high school geometry.



$$\frac{\text{up}}{\text{out}} = \frac{2}{4} = \frac{1}{2}$$

- The Measurement unit provides experiences with identifying attributes, choosing unit quantities of attributes, and determining numbers through comparisons. The emphasis is on informal, concrete, conceptual activities. There is a separate section which is devoted to child readiness and the planning of measurement activities for children. Metric units are used throughout. While measurement could have been included in the Analysis of Shapes unit, it has been placed in a separate unit because of its importance in the elementary school curriculum and in order to provide flexibility in the use of the units.



The four geometry units of the Mathematics-Methods Program are independent of one another. Any number of them can be used in any order. They can be used in a separate geometry course; they can be interspersed among other units of the Mathematics-Methods Program; or they can be used in conjunction with other materials.

These geometry units, like the other units of the Mathematics-Methods Program, involve one as an adult learner in activities which have implications for teaching children. One works with concepts that children might learn, with materials that children might use, and on activities that might be modified for use with children. The objective is to provide growth in understanding and enjoyment of geometry along with increased ability and desire to teach geometry to children.

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INTRODUCTION TO THE TRANSFORMATIONAL GEOMETRY UNIT

Geometry can be thought of as the study of space experiences. One very common category of space experience has to do with the changes that occur in objects. In this unit you will study changes or transformations.

Transformations can have many forms. Objects can be relocated without any change in size or shape. The appearance of an object can be altered by the interaction of the object and available light sources with the viewer's eye. Or an object can be radically changed in size and shape.

The Transformational Geometry unit begins with an overview of the content of the unit. Section I analyzes rigid transformations which, as you might guess, involve changes in location without changes in size or shape. In Section II you will be casting shadows to model projective transformations. And in Section III you will analyze the extreme but not totally unrestricted changes that result from topological transformations.

There are quite a few ideas here that will probably be new to you. Many of the activities presented could be modified for use with children. The hope is that you will become involved in the activities as an adult learner, without losing sight of their potential for use in the elementary school.

A WORKING OVERVIEW OF TRANSFORMATIONAL GEOMETRY

FOCUS:

Geometry is the study of space experiences, and transformational geometry is the study of those space experiences that involve change. In this overview you will be introduced to transformations in a real-world setting, and you will be introduced to three kinds of transformations. This is called a "working" overview since you will be asked to do things and and answer questions as you proceed.

MATERIALS:

Slide-tape presentation entitled "Overview of Transformational Geometry" (optional); balloons, construction paper, scissors, tracing paper; projector or penlight.

DIRECTIONS:

Read the following Working Overview, and respond to the numbered directions and questions as you come to them.

WORKING OVERVIEW

Historically, geometry has reflected the attempts of humans to analyze and control their experience on earth and in the space around them. Ancient Egyptians used measurements to delineate the lands that were inundated by frequent Nile floods. Ancient astronomers and mapmakers inspired much of the work in geometry. An interesting sidelight is that the geometry and the art of a civilization have often been reflected in each other. (William M. Ivins, Jr., Art and Geometry: A Study in Space Intuitions. New York: Dover Publications, Inc., 1964.)

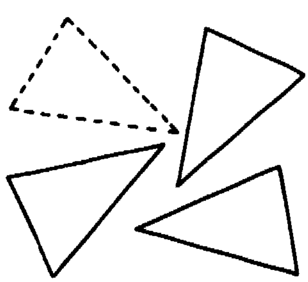
You may well associate the word "proof" with geometry, since you were probably introduced to deductive proofs in your high school geometry course. Actually, any area of mathematics involves proofs. It happens that, back around 400 B.C., Euclid organized the geometry that was known at that time into a deductive system in which each statement or theorem was proved by using a few basic assumptions. The availability of this development of geometry, together with the concrete nature of geometry, gave rise to a tradition of introducing students to deductive proofs in high school geometry. We mention this here to point out that geometry need not involve proofs. In particular, there is general agreement that proofs should not be prominent in elementary school geometry.

School geometry can be thought of as the study of space experiences. Transformational geometry studies those space experiences that involve change. For example, in your everyday life, you move yourself and other things; you represent things; you change the shape of things.

In this unit you will study three kinds of changes or <u>transformations</u>, namely:

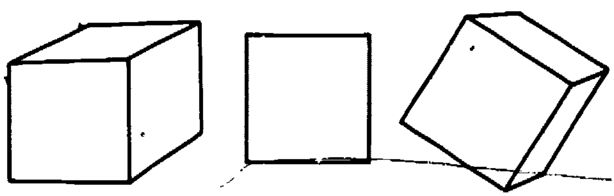
rigid transformations,
projective transformations,
and topological transformations.

A <u>rigid transformation</u> of an object changes its location but does not change its size or shape. For example, each of the solid-line triangles on the right is a rigid transformation of the dotted triangle. So you are performing a rigid transformation anytime that you relocate an object without changing its size or shape.

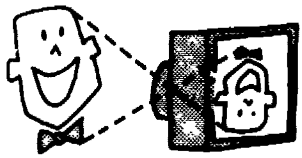


Projective transformations are a little harder to describe. Examples of projective transformations are those that result from visual phenomena--such changes as the change in shape which occurs from an object to its shadow:

or which result from viewing an object from different perspectives:

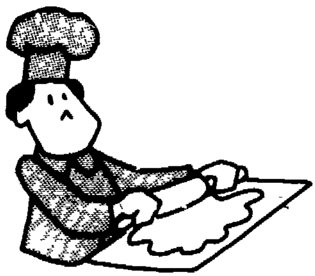


Actually, projecting slides and taking pictures involve projective transformations.

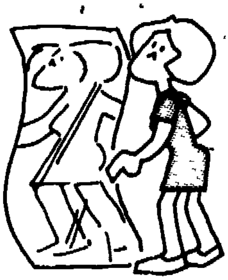


Notice that these projective transformations can involve change in size and some changes in shape, as well as changes in location.

Topological transformations can involve changes in size or shape as well as location. The changes in shape can be much more extreme than those for projective transformation. Any transformation is topological as long as it does not involve tearing (breaking) or pushing separate points together. For example, a baker with a rolling pin effects a topological change on dough:



An irregular mirror can result in an image of you that is a topological transformation of your actual appearance.



In this Transformational Geometry unit you will analyze rigid, projective, and topological transformations. Before going on to that analysis, do the following introductory activities, which should give you a better feeling for the three kinds of transformations. We do them in the reverse of the order mentioned above, so that you can proceed directly from the discussion of rigid transformations to the section of the unit that analyzes rigid transformations.

1. Topological Transformations

   a) Take a balloon and draw a face on it like this one.

b) Using both hands, you should change the balloon by stretching it, to help you answer the following questions. You may want to blow it up, too. (Note that any change that does not involve tearing the balloon or pressing separate parts together is a topological transformation.)

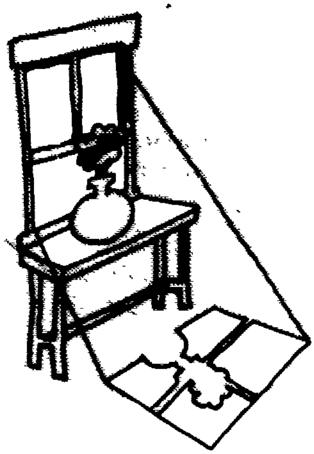
- Can a topological transformation make an object larger? (Try it with your balloon.)
- Can a topological transformation make a straight line crooked or a crooked line straight? (You can mark a straight line on the balloon.)
- Can a topological transformation move an object outside of a closed curve in the plane? That is, could this:  be transformed into this:  by a topological transformation? (Experiment with an eye on your balloon.)
- Can a topological transformation transform this:  into this:  ? (Try the other eye.)
- Can a topological transformation change a circle into an ellipse?
- Can a topological transformation disconnect things? That is, can this:  be changed into this:  ?

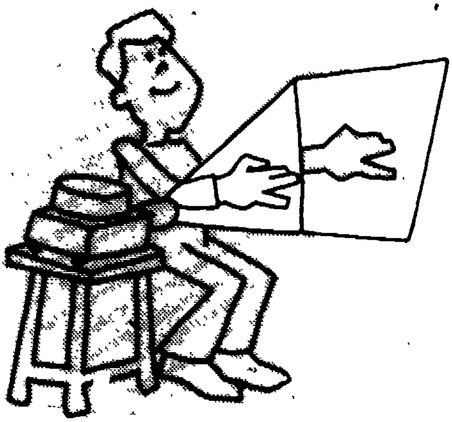
c) From your experiences in (b), summarize those things that topological transformations can and cannot change.

d) Two objects are called topologically equivalent if one can be changed into the other by using a topological transformation. Which of the letters in the alphabet are topologically equivalent to the letters C and D?

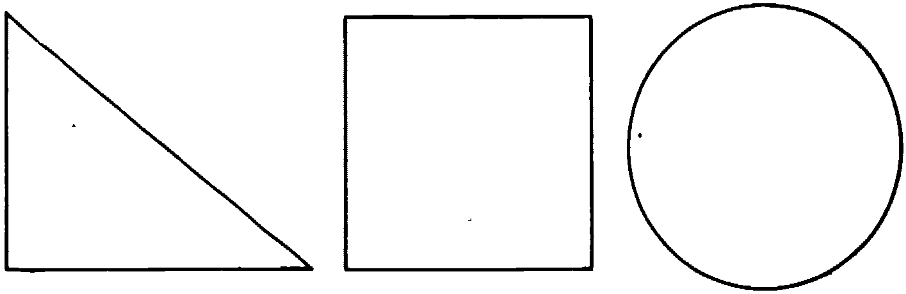
2. Projective Transformations

a) Arrange for a light source with which you can cast shadows. You can use either sunlight or an intense light source such as a projector or a pen light.*





b) Cut out shapes like these three.



Experiment with the shadows cast by them and by two pencils (or two rulers), to answer the following questions about projective transformations. You can assume that the changes from an object to its shadow are representative of the changes caused by projective transformations.

*Different kinds of projective transformations result from different light sources; but any light source that casts clear shadows will be adequate for this activity.

- Can a projective transformation change a straight line into a curved line? (Try with your pencils, and other shapes.)
- Can a projective transformation change the shape of an object?
- Can a projective transformation change a square shape into a nonsquare shape? (Try with your square.)
- Can a projective transformation change a triangular shape into a square shape?
- What can a projective transformation change a triangular shape into? What can change and what cannot? (Experiment by changing the position of the triangle.)
- What can a projective transformation change a circular shape into?

c) On the basis of your experience in (b), summarize what changes can and cannot be effected by projective transformations.

3. Rigid Transformation

Rigid transformations are simpler and easier to study. This makes their introduction easier than that of topological and projective transformations, and it will also enable us to analyze them in more depth (this analysis will be carried out in Section I). As the name implies, rigid transformations cannot change the size or shape of an object. They can, however, change its location. You should do the following.

a) Place a triangular shape on a large piece of paper and trace its location with a dotted line. Now pick up the shape and toss it to any other location on the paper and trace its new location with a solid line. The change from the dotted triangular shape to the solid one is a rigid transformation.

b) Briefly make a list of the changes that you make in objects in your life that are rigid transformations and those that are not. Which of the transformations that you listed would be good for helping a child to understand rigid transformations?

4. Summary

Below is a list of statements that might refer to a transformation. Put T on the line beside those statements that could refer to a topological transformation. Put P on the line beside those statements that could refer to a projective transformation. Put R on the line beside those statements that could refer to a rigid transformation. Leave the line blank if the statement refers to none of the three kinds of transformations. You will find that some of the statements can have several interpretations. You may also find that one statement may apply to more than one kind of transformation. Put all letters that are appropriate next to such a statement.

You may want to look back through certain of the preceding pages. You may even want to try certain transformations with pencil and paper or with objects that you have.

__________ Transforms a ball into a plate.

__________ Transforms a ball into a doughnut.

__________ Transforms a circle into an ellipse.

__________ Transforms an ellipse into a closed bottle shape.

__________ Turns a bottle upside down.

__________ Has an effect similar to the wind gently blowing the leaves of a tree.

__________ Transforms a tree (nonevergreen) and its leaves in the summer into the tree and its leaves in the winter.

__________ Transforms a standing person into a sitting person.

__________ Has the same effect as turning a yardstick into a meter stick.

__________ Transforms a ruler into its image in a convex mirror.

__________ Transforms a ruler into its image in a regular mirror.

__________ Transforms a ruler into its image in a magnifying mirror.

__________ Closes a window.

__________ Moves eyelashes, as when an eye is closing.

__________ Transforms you into your image in a calm pond.

__________ Transforms you into your image in a ripply pond.

Note: You have been introduced to each of three types of geometrical transformations. Each one will now be treated in more detail. We will start with rigid transformations, since there seems to be more interest in them in today's elementary school mathematics curriculum and since they have many interesting applications.

Section I

RIGID TRANSFORMATIONS

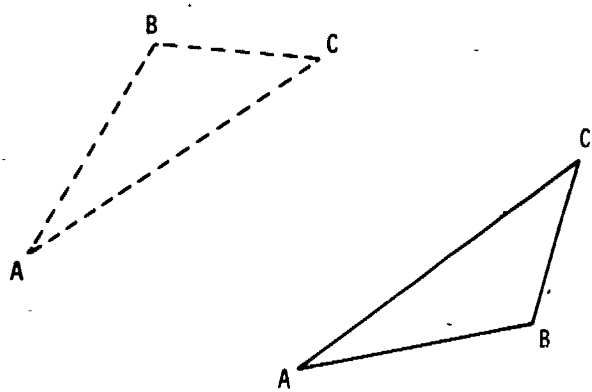
You have been introduced to the three kinds of transformations that will be studied in this unit. Sections I, II, and III will cover rigid, projective, and topological transformations, respectively. Rigid transformations are covered first, for several reasons. First of all, they are the most easily characterized and studied of the three kinds. The simple embodiment of a rigid transformation as the movement of objects without changing their size or shape is easily grasped by adults and children. Furthermore, rigid transformations relate nicely to the real world and to this unit's theme of "studying space experiences." Rigid transformations lead naturally to topics such as symmetry and tessellations, that have roles in the elementary curriculum. There is also the fact that rigid transformations are starting to appear explicitly in elementary school mathematics texts. The future of the study of rigid transformations in the elementary schools is still quite unclear. Much more investigation is needed into both the feasibility and the desirability of such study. But there is a great deal of current interest.

Activity 1 leads from an intuitive introduction to slides, flips, and turns, to accurate descriptions of these three kinds of simple rigid transformations. Then in Activity 2 you will discover the nice result that every rigid transformation can be expressed in terms of slides, flips, and turns. Activities 3, 4, and 5 investigate symmetry and tessellations, which provide applications of rigid transformations to the real world and interesting activities to do with children. Activity 6 introduces you to many of the available materials and references related to rigid transformations, and gives you an opportunity to develop activities for children and to try them out. Finally, Activity 7 involves a textbook analysis and seminar on the role of rigid transformations in the elementary school.

Throughout the section you should keep in mind the fact that an overall objective for studying geometry is to enrich your space experiences by organizing and analyzing them.

MAJOR QUESTIONS

1.



The triangle above started in the dotted-line position and was moved to the solid-line position. Describe, as accurately as you can, a sequence of slides, flips, and turns that could result in that movement.

2. Discuss the status of rigid transformations in the elementary school. Give attention to the following questions:

   a) Should rigid transformations be explicit or implicit in the study of symmetry?

   b) Do rigid transformations make geometry easier or harder to teach?

c) What is being done with rigid transformations in current texts?

3. Choose a topic that involves rigid transformations and list some objectives for that topic at a specific grade level. Outline four or five activities on that topic that are designed to achieve the objectives with children at the specified grade level.

ACTIVITY 1

SLIDES, FLIPS, AND TURNS

FOCUS:

In this activity you will analyze slides, flips and turns. Each of these three types of rigid transformations is introduced intuitively and then presented with more precision and detail. You should learn how to describe accurately any slide, flip or turn and how to perform any slide, flip or turn if you are given an accurate description.

MATERIALS:

Ruler, graph paper, tracing paper, protractor, compass; MIRA (optional).

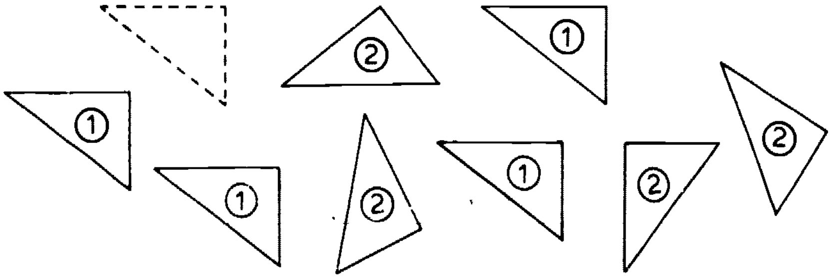
DISCUSSION:

Slides, flips, and turns are basic rigid transformations in that: (1) they are relatively easy to describe precisely; and (2) all rigid transformations can be analyzed in terms of them. Mathematicians call them translations, reflections, and rotations, respectively; but some people feel that the names "slides," "flips," and "turns" are more descriptive for children, so we use this terminology here.

DIRECTIONS:

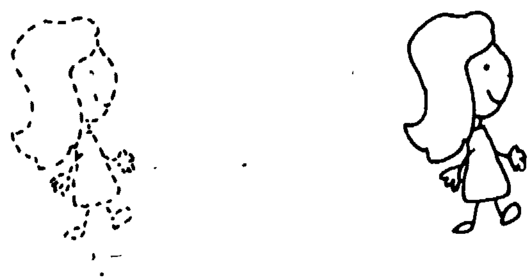
PART A: SLIDES

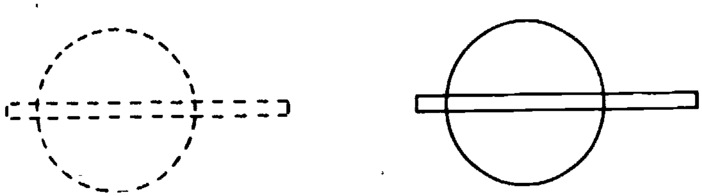
1. The dotted triangular shape below has been changed into each of the different solid triangular shapes by means of a different rigid transformation.

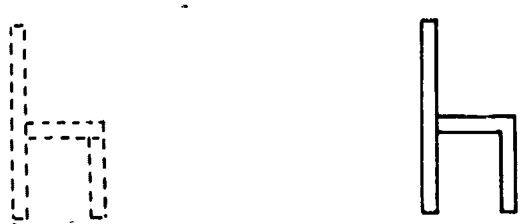


All of the triangles labeled ① are the result of <u>slide</u> transformation. None of the triangles labeled ② is the result of a slide transformation.

In each of the illustrations below, the solid line figure is the result of applying a slide transformation to the dotted line figure. Look at each and then try to describe what makes a slide a slide.



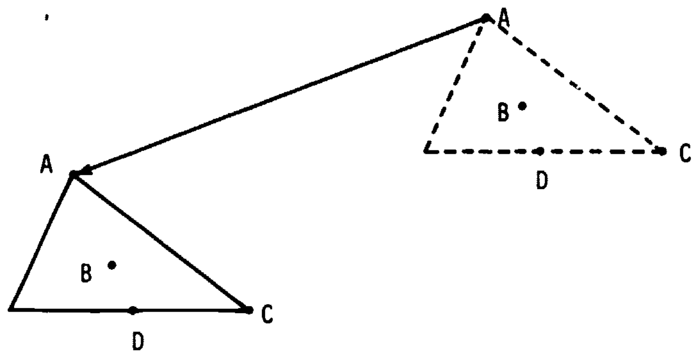




Do you see now what makes a slide a slide?

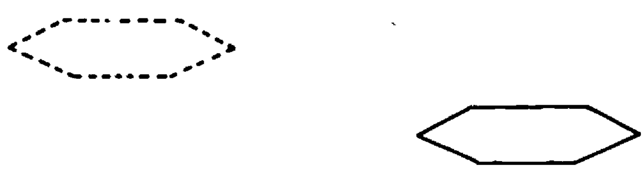
2. It is possible to very accurately describe a slide by means of a single arrow.

   a) A slide transformed the dotted triangular shape below into the solid one. We have labeled four points in the dotted triangular shape and the transformed points in the solid triangular shape, and we have connected one pair of points with an arrow. <u>You</u> connect the other pairs.



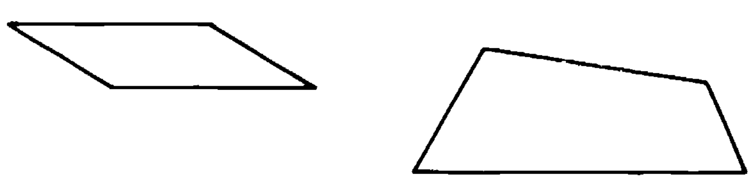
   What do you observe about the length and direction of the four arrows? What do you predict would happen if you drew arrows between other pairs of corresponding points?

   b)



   Choose four pairs of corresponding points in the above hexagonal (6-sided) shape and draw arrows from the original (dotted) shape to the transformed (solid) shape. Do the arrows have the same relationship as in (a)?

   c) Another rigid transformation is pictured here.



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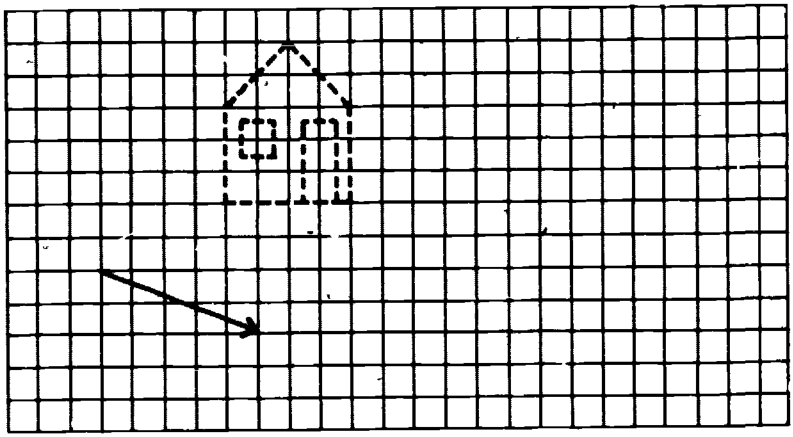
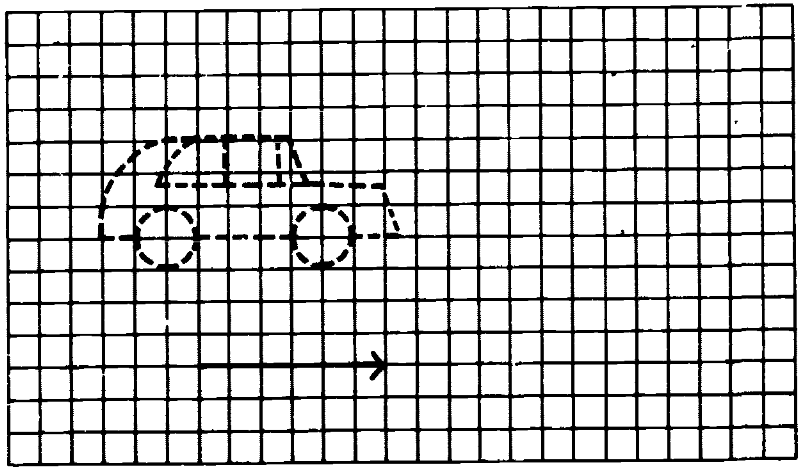
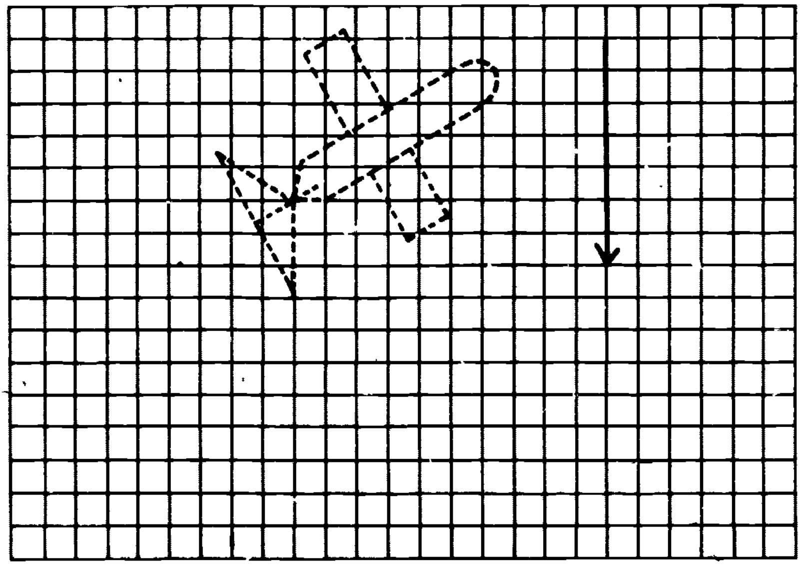
Join two corresponding pairs of points with arrows. What happened? Why?

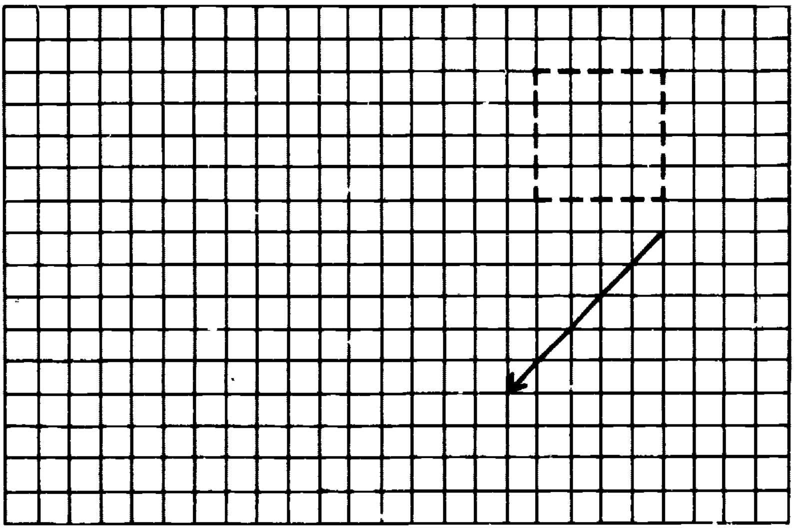
d) As you have doubtless noticed, when a transformation is a slide, all of the arrows between corresponding points have the same length, and are in the same direction. We call any arrow of the same length and direction as one of those a slide arrow for the transformation.



The arrow above is a slide arrow of a slide transformation. Apply the slide to the dotted house and accurately draw the result with solid lines. Do the same thing for each of the figures below.







e) (You may want to use graph paper or a grid for this problem.) Read (i) and (ii) carefully, so that you will be aware of the difference between them. Then do what they say to do.

i) Draw an object and a slide image of that object. Then draw an arrow that would completely and accurately describe the slide involved.

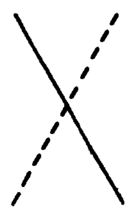
ii) Draw an object and accurately describe a slide by means of an arrow. Then draw the image of the object that would result from applying the slide.

If you can do (i) and (ii) with ease and understanding, you have mastered the accurate description of slides.

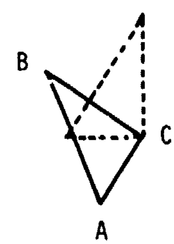
From now on when you are asked to accurately describe a slide transformation, you will be expected to draw a slide arrow for that transformation.

PART B: TURNS

3. We now direct our attention to turns. Read the captions of the following illustrations and do as you are instructed. (You will need tracing paper.)

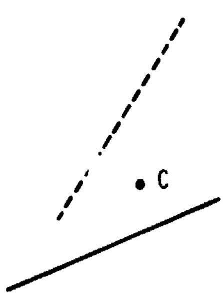


This change could result from a turn.

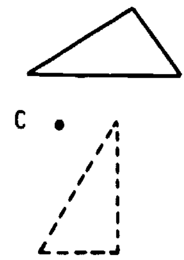


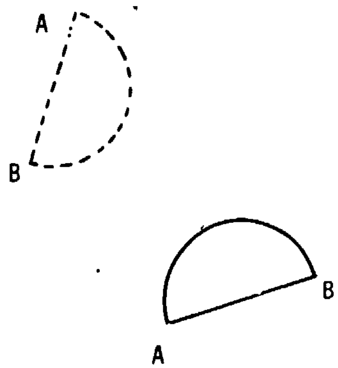
This change could result from a turn.

...and so could this one if you locate the center of the turn at the point marked C. (To test this statement, trace the dotted line on a sheet of tracing paper, pivot the paper at the point C, and see if you can rotate the paper so that the dotted line coincides with the solid one.)

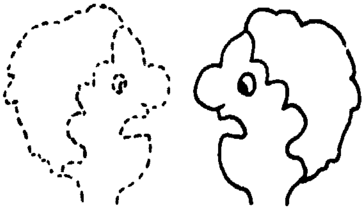


...and so could this one. (Locate the center of the turn at the point marked C.) To test this statement, trace the dotted triangle on a sheet of tracing paper, pivot the paper at the point C, and see if you can rotate the paper so that the dotted triangle coincides with the solid one.





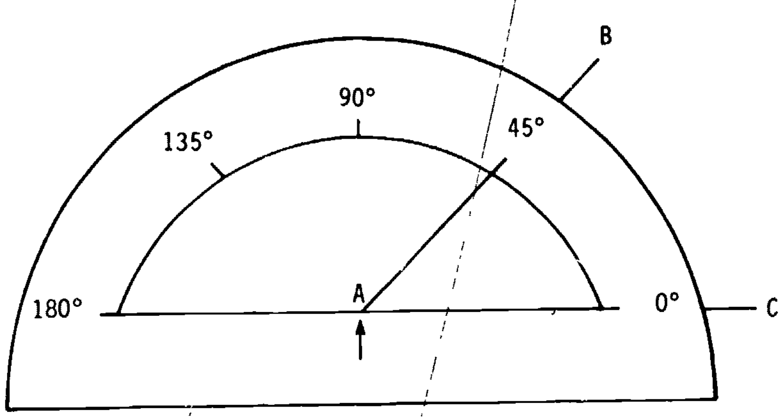
Could this change result from a turn? Try it. See if you can find the center of turn, using trial and error.



This change could not be the result of a turn. (Try it with a piece of tracing paper, as you did in the previous example.) This happens to be an example of a flip, which is the next type of transformation to be studied.

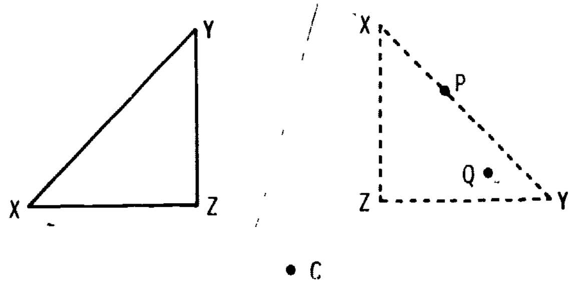
4. You have learned that a turn transformation involves a center and a rotation or turn about the center. Here you will have further experience with turns. (You will need a protractor and compass.)

You will be left a little more on your own in learning how to accurately describe turns. Instead of being told exactly how to describe a turn, you will be asked to perform an experiment and to arrive at a means of description as a result of your observations. It is hoped that you and your classmates will attempt to do this without the help of your instructor. However, if you get "bogged down," seek help.



In case you have forgotten how to use a protractor, here is one that is measuring the 45° angle BAC.

a)



In the figure above, a turn transformed the dotted triangular shape into the solid one. The center of the turn is at C. Use tracing paper, pinned down at C, in order to follow the paths along which the points X, P, and Q of the dotted triangular shape are transformed. Consider the following:

i) What is the shape of the paths along which the points traveled?

ii) What does C have to do with the paths?

iii) How do the angles of rotation of X, P, and Q compare?

iv) Through what angle did each point rotate? (Use your protractor.)

After you feel that you have experimented sufficiently, write down an accurate description of the turn that carried the dotted triangular shape into the solid one. Check your description with those of other individuals and groups.

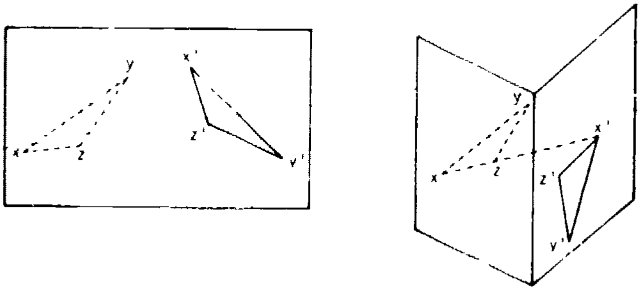
b) Draw a "before-and-after" picture of a turn and then precisely describe the turn.

c) One detail has been sloughed over here, even though you may have run into it in Exercise 4. How does one find the center of a turn if one is given a "before-and-after" picture of the turn? We challenge you to develop a procedure for finding centers of turns. You will be given a hint.

CHALLENGE

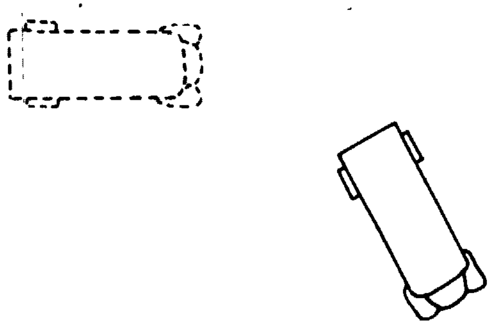
Write down a description of how to find the center of turn when you have been given a "before-and-after" diagram for the turn.

Hint: If a turn transforms the point x into the point x', the center of rotation lies on the crease that results when paper is folded so that the points x and x' touch each other. (See the figure below.)



So your task is to find out where on the crease the center C falls. (Don't forget that there are other points and can be other creases. You could also use the concept of perpendicular bisectors directly instead of folding as described above.)

d) Use the method that you have just discovered to find the center for the turn depicted below, which transforms the dotted figure into the solid one.



e) Pair up with a classmate, and each of you make up a problem like that in (d) for the other. Each help the other solve the problem if necessary.

f) Let T be the turn that rotates every point 60° counterclockwise about point C on the following page. Draw pictures of two objects before and after T has been applied to them.

g) Devise an activity for children that would make them aware of turns in their lives and that would help then analyze the paths along which points travel. (You may want to relate this to playground games and equipment. Could the body motions of dancing be used in this activity? Puppets, string, or rope may also be helpful aids.)

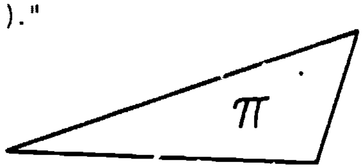
•C

From now on when you are asked to accurately describe a turn you will be expected to locate its center and to give the number of degrees of rotation about the center (and the direction of the rotation).

TEACHER TEASER

CAIN*

There was once a couple, Adam and Eve. They had a son, Cain. To develop Cain's intellect Adam liked to pose problems. One day he showed Cain a triangular pie π (see below) and he said: "You may choose any point O in the plane of π and rotate π through 180° about O. Call the resulting image π'. You get to eat $\pi \cap \pi'$ (i.e., the portion of the pie that is common to both π and π')."



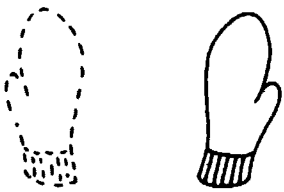
Answer the following questions. (Experimentation may prove to be your best approach.)

1. If Cain went hungry, where did he choose O?
2. If Cain got a piece of pie in the shape of a parallelogram, where did he choose O?
3. If Cain got a piece of pie in the shape of a hexagon, where did he choose O?
4. Can Cain get a triangular piece of pie?
5. Where should Cain choose O to get as much pie as possible?

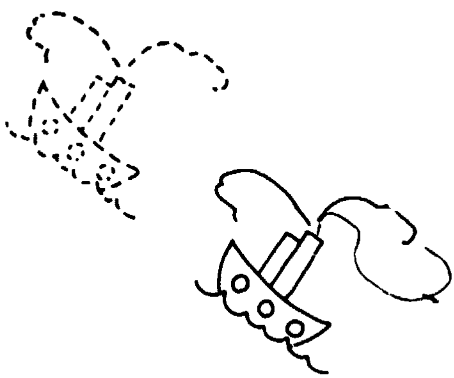
*This problem was taken from a paper presented by Professor Arthur Engel to the CSMP International Conference on the Teaching of Geometry at the Pre-College Level.

PART C: FLIPS

5. The third and final type of rigid transformation that we will study is the flip. (The MIRA* mirror seems to be a particularly effective device for investigating flips, so additional instructions have been included in parentheses for those who have a MIRA available.) Read the captions on the following illustrations and do as you are instructed.

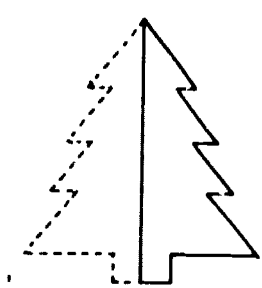


This change could result from a flip.



This change could not result from a single flip. Why? (MIRA: Place your mirror between the two boats and try to make the image of the dotted boat coincide with the solid one.)

This is a result of a flip. Some people say that an object after a flip is a mirror image of the object before the flip. Can you see why? Indicate where the mirror should be. (MIRA: Verify where the mirror should be.)



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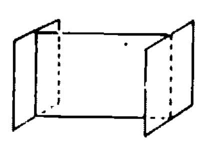
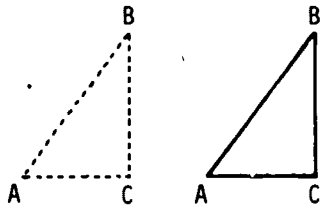
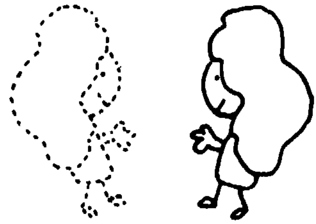
*MIRA is the trade name for a commercially produced product that is transparent and yet reflects. It allows one to easily draw the flip image of objects.

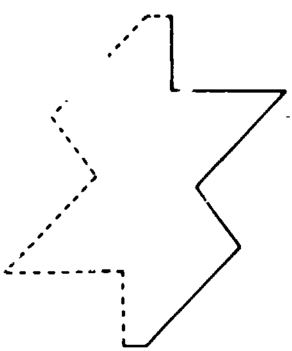






This change could not be the result of a single flip. (MIRA: Try it.)

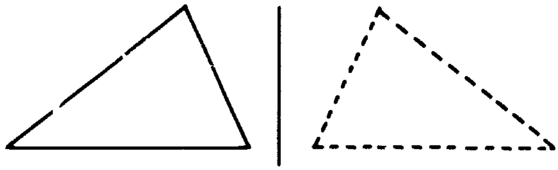
This change could result from a flip. (MIRA: Do it. Where's the line of reflection?)



But this is not the result of a flip. What rigid transformation could it result from?

6. A flip turns out to be fairly easy to describe accurately. Each flip is completely determined by a single line (called the line of reflection). In this part, then, you will want to learn to apply flips to objects, given the line of reflection; and you will want to learn to find the line of reflection, given a picture of what a flip does to an object. Again, an experiment seems in order.

a)

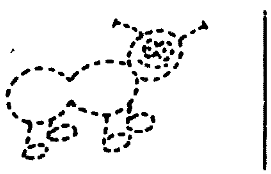


Analyze what the flip depicted in the "before-and-after" drawing above does to several points. Then do (i) and (ii) below.

i) Write a sentence that describes what the flip does to each point.

ii) Write a description of how to find the line of reflection if you have been given a "before-and-after" representation of the flip. (Would folding be helpful?) (MIRA: Do (i) and (ii) without your mirror. Then write a description of how you can find the line of reflection using your mirror.)

b) Given the object and the line of reflection below, draw the image of the object after the implied flip. (MIRA: This is much easier with your mirror.)



c) Pair up with a classmate to pose problems like the one in (b). Check solutions and provide help where needed.

d) Describe the relationship between the line joining a point with its flip image point and the line of reflection of the flip?

e) Suppose that you are given a "before-and-after" description of a transformation. How could you use folding to decide whether the transformation is a rotation or a reflection?

From now on, when you are asked to accurately describe a flip, you will be expected to draw its flip line.

PART D: SUMMARY

7. This is a summary of properties of slides, flips and turns.

Let S stand for "slide."

Let F stand for "flip."

Let T stand for "turn."

Put any or all of S, F, and T in the box next to each sentence, depending on whether the sentence applies to a slide, flip, or turn, respectively, when applied either to a specific figure or to the whole plane. If a sentence does not apply to any of the rigid transformations, leave its box empty.

- [ ] Moves every point of the plane (i.e., has no fixed points).
- [ ] Moves every point of the plane except one (i.e., has one fixed point).
- [ ] Moves every point of the plane except the points of one line (i.e., has a line of fixed points).
- [ ] Moves points along concentric arcs through an equal angle.
- [ ] Moves points an equal distance along parallel lines.
- [ ] Makes some objects smaller.
- [ ] Does not change the shapes of objects.
- [ ] Moves points along parallel lines but not necessarily the same distance.
- [ ] Changes right-left orientation.
- [ ] Transforms an object into a congruent object.

8. There are materials developed for the MIRA mirror that you may want to investigate further. (See reference 19 on page 83.)

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ACTIVITY 2

DECOMPOSITION OF RIGID TRANSFORMATIONS INTO SLIDES, FLIPS, AND TURNS

FOCUS:

You will be introduced to the idea of composition of rigid transformations. You should gain an intuitive grasp of this idea, and you should learn how to apply it, its notation, and its terminology.

You will then see the importance of slides, flips, and turns by convincing yourself that every rigid transformation can be expressed as a composition of these transformations, each of which is easy to describe precisely.

This activity consists of three parts:

Part A: The Challenges
Part B: Resources for the Challenges
Part C: Applications

MATERIALS:

Graph paper, string, scissors, construction paper, protractor, compass.

DISCUSSION:

Part A of this activity consists of two challenges. Part B consists of several subactivities, which are designed to help you meet the challenges, and Part C consists of some exercises that relate rigid transformations to the real world. You can proceed in one of three ways:

Alternative 1: Tackle the challenges in Part A, taking advantage of only those subactivites in Part B that you need to help you meet them.

Alternative 2: Work your way through Part B systematically, with an eye toward meeting the challenges in Part A as a wrap-up for the activity.

Alternative 3: Do only Subactivity for Challenge 1 on page 42 and omit the rest of the activity.

Alternative 1 places you more in a problem-solving mode and relieves the student with a strong background of the burden of covering "old territory."

Alternative 2 provides a more systematic route through the activity and will probably help many students gain some interesting insights.

Alternative 3 provides a way of getting to the "punch line" of the activity in the shortest possible time. Those who are on a particularly tight schedule could do Alternative 3 alone.

PART A: THE CHALLENGES

Challenge 1: Verify that every rigid transformation can be decomposed into slides, flips, and turns.

Challenge 2: Determine and verify which of the following statements are true and which are false.

a) Every rigid transformation is a composition of slides.

b) Every slide is a composition of flips.

c) Every flip is a composition of turns.

d) Every turn is a composition of flips.

e) Every rigid transformation is a composition of flips.

DIRECTIONS:

To verify that a statement is true, you should describe how to precisely determine the slides, flips, and turns involved. For the slides you must describe how to determine the appropriate arrow; for the flips, the appropriate line of reflection; and for the turns, the appropriate center, angle, and direction.

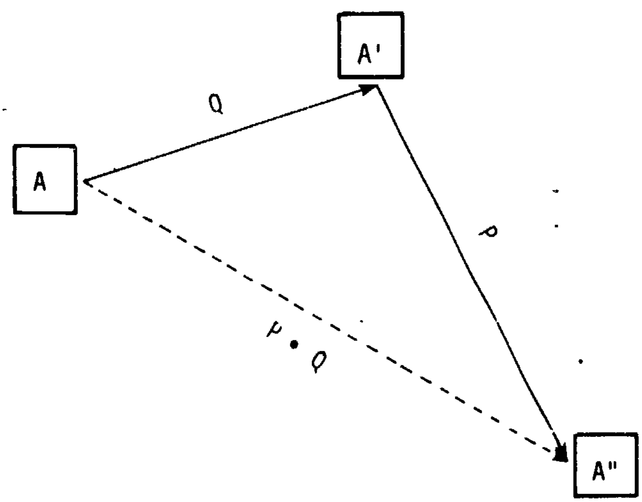
To verify that a statement is false, you will probably find it easiest to give an example that shows that the statement does not always hold (i.e., a counterexample).

PART B: RESOURCES FOR THE CHALLENGES

1. Subactivity: Composition of Transformations: (This subactivity relates to both challenges.) The composition of two transformations is another transformation. It is the one transformation that has the same effect on objects as the two would if they were applied one after the other.

Definition: Let P and Q be the names of two transformations. Then P • Q (read "P composed with Q") is the single transformation that results from applying the transformation Q to an object and then the transformation P. (Q • P is the result of P first and then Q.)

EXAMPLE 1



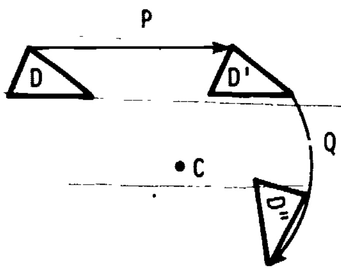
Q takes the box from position A to position A'.
P takes the box from position A' to position A".
P • Q takes the box from position A to position A".

In this example Q, P, and P • Q are all slides, so that if the transformation Q is determined by the arrow labeled "Q" and P is determined by the arrow labeled "P," then the transformation P • Q is determined by the arrow labeled "P • Q."

EXAMPLE 2

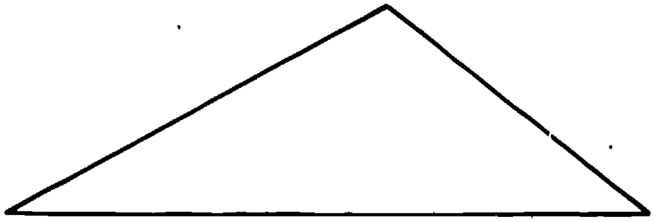
P is a slide of one inch to the right.

Q is a quarter-turn clockwise about point C.

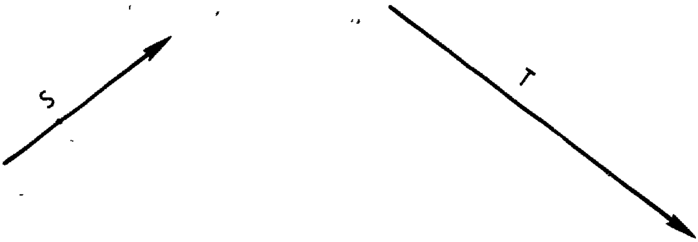


Q • P takes D into D". Now you try P • Q on some graph paper.

Note: Below are some activities to sharpen your comprehension and skill concerning compositions. Cut out a triangle like the one below to use in modeling the transformations that are described.



a) The slides S and T are described by the arrows below. Give an accurate description of the transformation S • T. Is S • T a slide?



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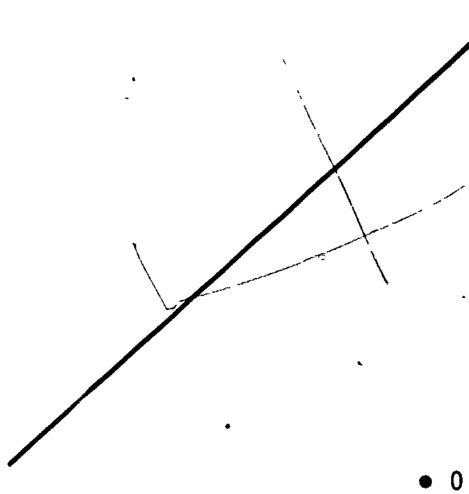
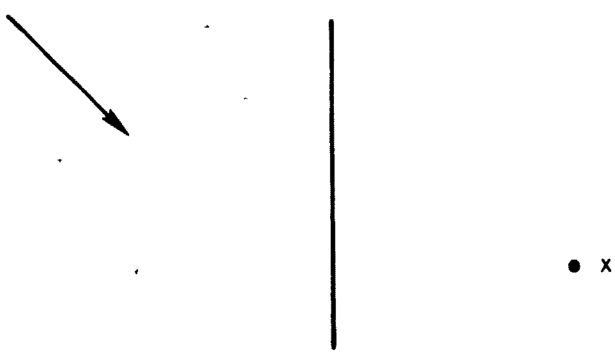
b)



Above are a line and a point. Call the flip about the line "F," and let the turn (about the point) of one quarter-circle counterclockwise be called "T." Draw an object and then draw the image of the object after T • F has been applied to it. (Use your cutout triangle if you like, and draw it on a piece of graph paper if you like. Make it clear in your drawing which is the object and which is its image.)

<u>Note</u>: If x is a point and P is a transformation, we use the symbol P(x) to stand for the image of x after P is applied to it.

c) Let P be determined by the arrow below and Q by the line. Find and label P(x), Q(x), P • Q(x).



d) If you have the time, use the tiles on the floor of your classroom (or graph paper, or squares on a chessboard) for reference, and make up and physically act out transformations P and Q and then P • Q and Q • P.

e) For a change of pace, get up, take a walking trip, and describe your trip as a composition of slides, flips, and turns (you may not need flips). Can you dream up games for children that would make them conscious of the components of their movements?

2. Subactivity for Challenge 1. Place your cut-out triangle on the table or floor and mark its position. Then pick it up and toss it randomly to another position. Mark the new position. Now use the cut-out triangle to analyze its change of position. (A slide, then a turn, and then a flip, if necessary, will work.) Repeat this experiment several times until you feel that you can describe a general procedure for decomposing any rigid transformation into slides, flips, and turns.

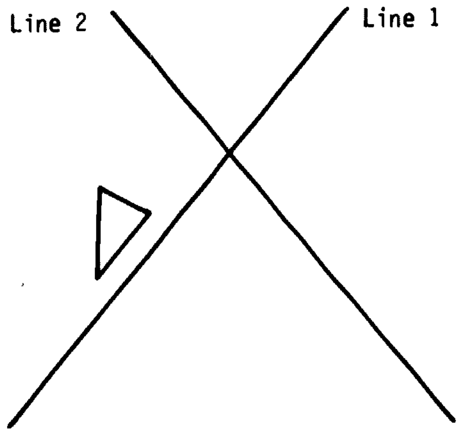
3. Subactivity for Challenge 2:

a) If P and Q are both slides (i.e., described by arrows), what about P • Q? Can you indicate how you would describe the composition of any two slides? (First experiment with your cut-out triangle; then use arrows to describe your examples.) How would you convince someone that no flip is a slide?

b) Again your cut-out triangle will help. Slide it, marking its position before and after the slide; try to figure out if there are two flips whose composition would effect the same change. If you think you can always find such flips, try to describe how you would find the appropriate two lines of reflection for any slide; i.e., if you are given a slide arrow, how would you find two flip lines the composition of whose flips would be the slide? If you don't think every slide is a composition of two flips, try to find a slide that you can show is not the composition of flips.

c) Experiment with your cut-out triangle. See what happens when you flip it. See what happens when you use the composition of rotations (with the same or with different centers). See what generalizations you can make.

d) Here it might be good to look at compositions of flips first.



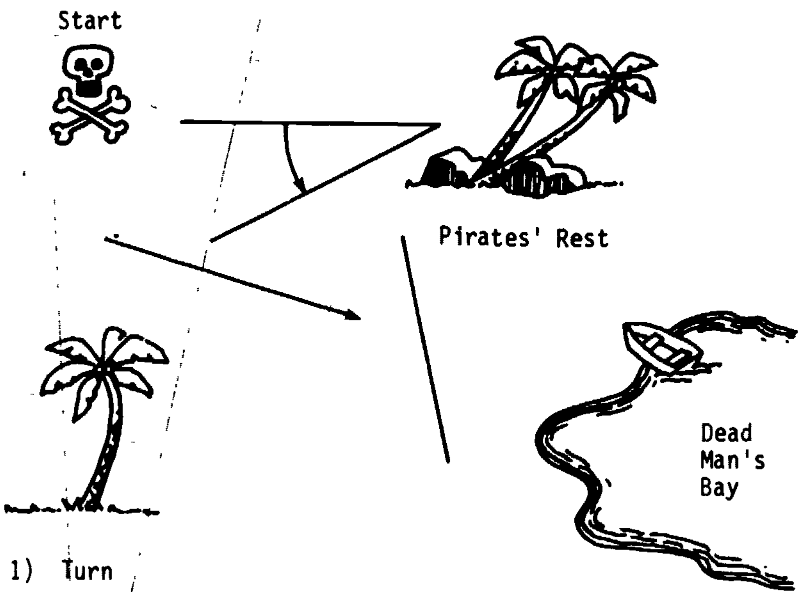
Flip the triangle above about line 1 and then about line 2. Can you see a rotation that might do the same job? Describe it. Try the above with your cut-out triangle and string lines. Now try to reverse it; i.e., try taking a turn and finding the flip lines that will work for it. Give a general description of how to do it for any turn.

e) There are two ways to handle this one. The direct way is to toss your triangle as before and try to find the appropriate flip lines. Another interesting way is to logically combine Challenge 1 with Challenge 2 (b) and (d).

PART C: APPLICATIONS

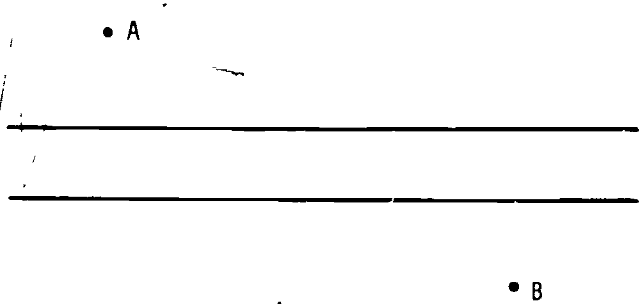
Part of the reason for studying rigid transformations is to gain greater insight into, and therefore a richer interacti with, c.iange in the real world. Rigid transformations can also be used directly to solve problems. Here are a couple.* (<u>Hint</u>: In problems 2 through 5 you may find flips to be very useful.)

1. Can you find the buried treasure?



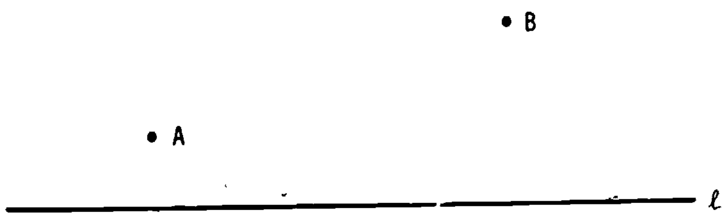
   1) Turn
   2) Slide
   3) Flip

2. Two towns, A and B are separated by a river with parallel banks, as shown below. If a bridge is to be built over the river perpendicular to its banks, where should the bridge be located so that the connecting road is the shortest distance?

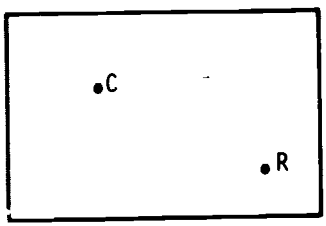


*These problems were communicated to the author by Professors James Riley and Christian Hirsch.

3. Two farms, A and B, and a power line ℓ are situated as shown below. If there is only one transformer with separate wires running to each farm, where should the transformer be located so that the minimum length of wire will be used?



4. The figure below shows the plan of a billiard table and the positons of the cue ball C and a red ball R.

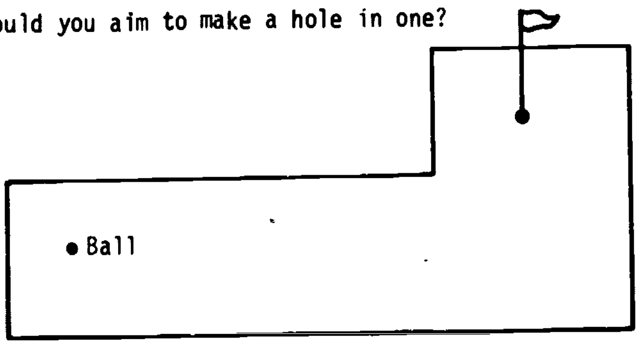


a) Draw a one-cushion path from the cue ball to the red ball. How many such paths are there?

b) Draw a two-cushion path from C to R. How many are there?

c) Can you draw a three-cushion path from C to R?

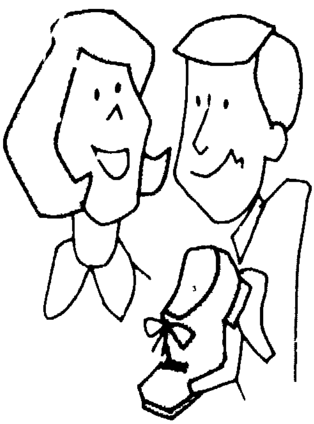
5. The figure below shows a par 2 hole on a miniature golf course. Where would you aim to make a hole in one?



6. Imagine that you are in the command module and that you have to give instructions to the pilot of the lunar landing module in order for that module to dock with your module. What kinds of terminology would you use in your instructions? Note: This exercise is an example of three-dimensional rigid transformations. They are easy to work with intuitively but hard to work with precisely. The decision not to emphasize three-dimensional transformations here is made reluctantly.

7. Experiment with three-dimensional rigid transformations. What would you replace two-dimensional slides, flips, and turns by? That is, which kinds of easily described three-dimensional rigid transformations can be used to describe all three-dimensional rigid transformations?

TEACHER TEASER



Sam said to Sally: "Since 2 x 3 = 3 x 2, P • Q = Q • P." Sally said to Sam: "That's like saying that putting your shoes on and then tying them is the same as tying your shoes and then putting them on."

Who is right? That is, is the composition of rigid transformations commutative?

ACTIVITY 3

COORDINATE ANALYSIS OF RIGID TRANSFORMATIONS

FOCUS:

One of the important mathematical advances of the last 2000 years was the tie that was established between geometry and algebra by Descartes. In this activity you will take advantage of this tie to generate formulas, in terms of Cartesian coordinate systems, for certain kinds of slides, flips, and turns.

MATERIALS:

Graph paper and a ruler.

DISCUSSION:

You have already solved the problem of describing slides, flips, turns, and general rigid transformations precisely in terms of arrows, lines, angles, directions, points, and compositions. You will now be confronted with the problem of writing down a formula in terms of a coordinate system for any rigid transformation.

Unfortunately, the complete solution to this problem is too intricate to be included here. So we have provided two challenges, each of which deals with a portion or special case of the problem. Then we have provided two subactivities, each of which consists of exercises designed to help you meet and go beyond the challenges. If you can meet the challenges you may skip the subactivities.

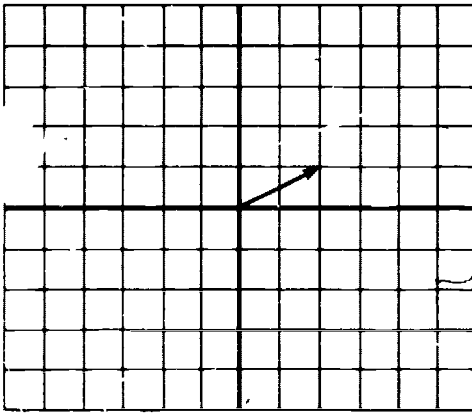
Challenge 1: A slide S moves the point (2,3) to the point (5,1). (The points are expressed in rectangular Cartesian coordinates.) Describe in terms of a formula what S will do to any point (x,y).

Challenge 2: A flip F moves the point (1,2) to the point (5,2). Find a formula which describes what F will do to any point (x,y).

Note: In this activity we will use the notation M(x,y) to stand for the image of the point (x,y) under the transformation M. So, for example, in Challenge 1 you are told that S(2,3) = (5,1) and you are asked to find S(x,y).

1. Subactivity for Challenge 1:

Exercise 1: Let M be the slide determined by the arrow in the following diagram.



Note: The arrow goes from (0,0) to (2,1); i.e., M(0,0) = (2,1).

Find M(1,2) =

M(-3,4) =

M(4,0) =

Can you distinguish a pattern in the change of coordinates? What is it? How is the first coordinate affected each time? The second?

Can you find M(x,y) for any point (x,y)?

Exercise 2: If you know that a slide N moves the point (1,3) to the point (7,4), can you determine the horizontal shift and the vertical shift? What does N(x,y) equal?

Exercise 3: Let P be the slide that takes (0,0) into (a,b). Find P(x,y) for each point (x,y).

Exericse 4: Let Q be the slide that takes (c,d) into (a,b). Find Q(x,y) for each point (x,y).

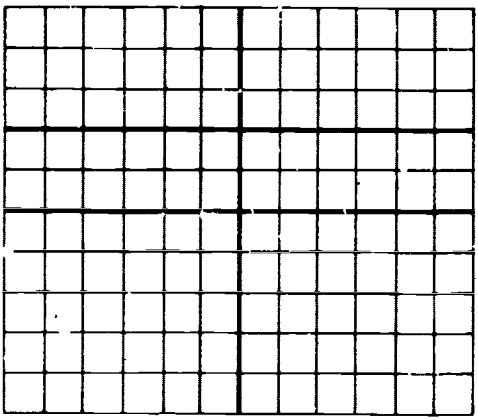
In solving Exercise 4 you have solved the general problem of representing a slide in terms of coordinates.

Exercise 5: Can you see that knowing what Q does to one point is the same as knowing an arrow? Explain.

2. Subactivity for Challenge 2

The challenge is to find F(x,y) if you know that F is a flip and that F(1,2) = (5,2). Recall from Activity 2 that a flip is completely determined by its line of reflection.

Exercise 6: Let L be a flip about the line y = 2. What is the image of the point (3,4)? (That is, find L(3,4).) (Use the grid on the next page.)



Does L change the x-coordinate?

Does L change the y-coordinate?

Find L(4,2) =

L(4,3) =

L(4,4) =

L(4,5) =

Do you see a pattern? What does L(x,y) equal?

Exercise 7: Let L be the flip about the line y = a. Find L(x,y) for any point (x,y).

Exercise 8: Let L be the flip about the line x = b. Find L(x,y) for any point (x,y).

5?

You have now solved the problem of writing a formula for any flip about a line parallel to the x-axis or the y-axis. Unfortunately, the general problem of representing a flip is too intricate to be solved here.

SUMMARY

You have been studying some of the fundamental examples of rigid transformations considered as point functions, functions tnat associate a point in the plane with another (perhaps the same) point in the plane. You can now precisely describe certain transformations to any anyone who understands your notation and terminology. It is possible to go on and write formulas for any rigid transformation, but we will not do it here. If you are interested in studying coordinate representations of rigid transformations further, you may want to look over one of the following references.

Bouwsma, Ward D. Geometry for Teachers. New York: Macmillan Co., 1972.

Choquet, Gustave. Geometry in a Modern Setting. Boston: Houghton Mifflin Co., 1969.

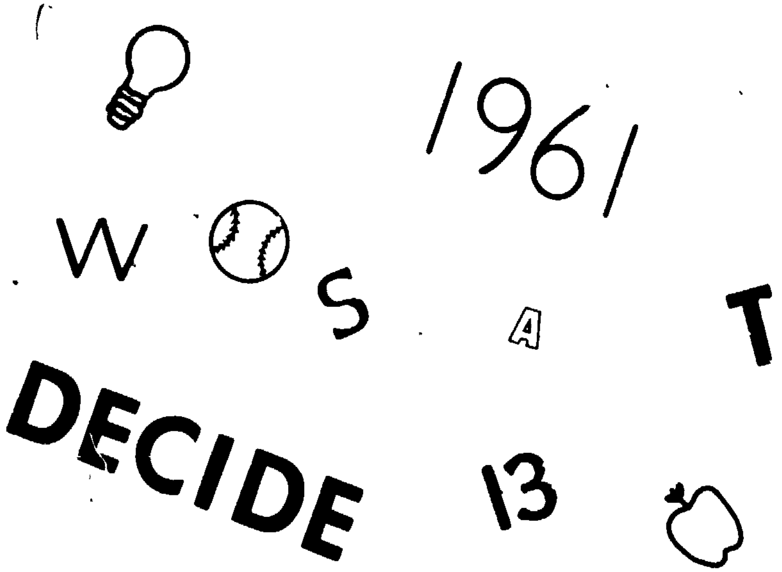
Gans, David. Transformations and Geometry. New York: Appleton-Century-Crofts, 1969. (Best reference for general description.)

Haag, Hardgrove, and Hill. Elementary Geometry. Reading, Mass.: Addison-Wesley, 1970. (Intuitive approach.)

ACTIVITY 4

SYMMETRY

Each of these objects is symmetric.



Have all of the symmetric letters been listed above? Can you see any difference between the kind of symmetry exemplified by "A" and that possessed by "S"? Can you identify symmetric objects in your classroom?

FOCUS:

In this activity you will gain experience with recognizing symmetric objects, with using turns and flips to analyze the kinds of symmetries possessed by an object, and with folding and cutting to create designs that have flip symmetries. You will also take advantage of these experiences in developing a strategy for introducing symmetry to children.

MATERIALS:

Scissors, ruler, construction paper and tracing paper, elementary texts, references (see page 57) (optional), and MIRA mirrors, if available.

DISCUSSION:

Symmetry is so common in our lives that it would be easy to overlook it. Our bodies have symmetries; most physical objects that we make have symmetries; artists either seek symmetry or avoid it consciously; and in a conceptual rather than a physical sense, symmetry is a tool of the musician and the physicist.

Part of the appeal of symmetry seems to be aesthetic. Symmetry seems to appeal to a sense of order that is shared by man and by nature. Symmetry also has practical values. Symmetry in architectural design provides offsetting forces and hence structural stability. Symmetry in product design makes it possible to easily reproduce several parts of a product from a single part.

Symmetry has a definite place in the elementary curriculum. Not only is it an important concept that should become part of a child's awareness, but it is a concept that has many embodiments in the child's environment and that can be introduced intuitively by involving the child in doing and making. Further, it provides a nice application for the concepts of rigid transformational geometry.

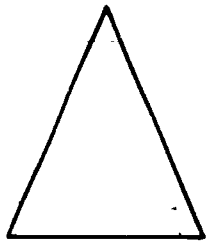
In this activity symmetry will be introduced intuitively. Turns and flips will be applied to analyze symmetries. Folding will be used to effect flip symmetries, and you will have an opportunity to plan a strategy for introducing symmetries to children.

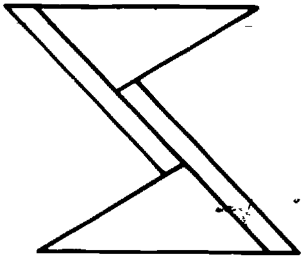
As in Activity 1, there will be instructions in parentheses for those who have MIRA mirrors.

DIRECTIONS:

1. The objective here is to familiarize you with both flip symmetries and turn symmetries.

a) Trace and cut out the two shapes below, and experiment to determine which rigid transformations make each shape coincide with itself.

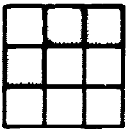


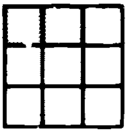


It is, of course, true for any figure that a turn of 360° about any point will make it coincide with itself. The interesting thing to find is what <u>other</u> rigid transformations will "work."

b) An object like that in the lefthand figure above is said to have a flip symmetry. Some objects have one symmetry, some have none, some have many. Describe the kinds of symmetries of <u>each</u> of the objects on the first page of this activity. Make a model of and experiment with any one that gives you trouble. (MIRA: Your mirror will make the flip symmetries easy to detect.)

c) Fill in the appropriate squares so that each of the grids below has exactly one flip symmetry. Draw in the line of symmetry, i.e., the line about which it can be flipped without changing appearance. (MIRA: Again, use your mirror.)



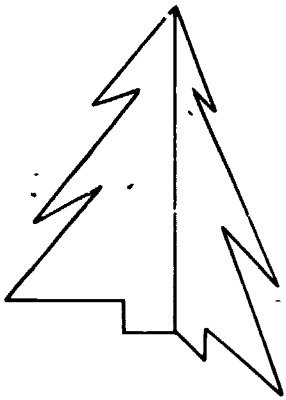


Notice that there are several ways to solve this problem.

d) Make two 4 x 4 grids (instead of 3 x 3) and fill in boxes so that the resulting pattern has more than one line of symmetry.

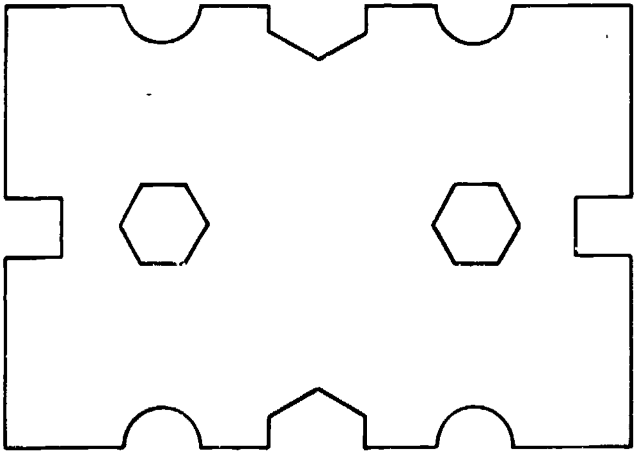
e) Do as in (c) and (d) for turn symmetries, i.e., turns besides 360° that make the figure coincide with itself.

2. Most people can remember making cut-out Christmas trees as a child by folding a piece of construction paper and cutting.

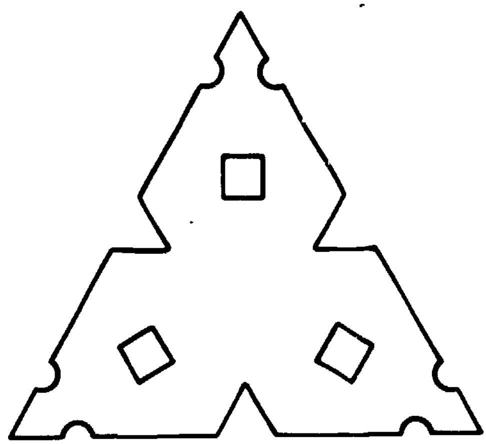


Such a process guarantees a design with at least one line of symmetry. Now you will gain some experience in constructing with paper and scissors designs that have various lines of symmetry.

a) Use a folding and cutting technique to produce an object with two lines of symmetry. (MIRA: Check the lines of symmetry on the example below.)



b) Similarly, make one with three lines of symmetry as below.



c) If you are enjoying doing this, by all means try making shapes with other symmetries.

3. As was indicated in the introduction, symmetry is a part of a child's environment and, as you have seen in (1) and (2), there are some "making and doing" activities that illustrate and apply symmetry. Here you are to:

a) Choose a grade level.

b) Identify some objectives for symmetry at that grade level. (You may want to look at some elementary texts, or you may want to work out your own objectives.)

c) Outline four or five lessons* on symmetry designed to achieve a few selected objectives at the chosen grade level.

d) Present your outline to the rest of the class.

In preparing yourself for this activity, you may want to become familiar with some of the following references if they are available to you, and you should pay particular attention to the following considerations.

---

*MIRA: You may want to build a part of your lessons around MIRA.

- Relate symmetry to the child's world (the child's body, possessions, home, classroom, etc.) and to nature.
- Have the child doing and making.
- Apply symmetry to other aspects of the child's school work, e.g., letters, numerals, words, art class.

If it is appropriate, you may want to prepare some transparencies and present your unit to the rest of the class.

For references, see the following entries in the bibliography on pages 82-83. (1), (5), (6), (16), (17), (18--pp. 25-33).

Also consider Materials for Mirror Cards which are available through Elementary Science Study of the McGraw-Hill Book Co. (1967) and consider any current elementary school mathematics text series.

ACTIVITY 5

USING SYMMETRY TO ANALYZE SHAPES

FOCUS:

Here the role of symmetry in analyzing shapes will be explored as part of the Transformational Geometry unit's theme of organizing and analyzing space experiences. The concept of order of symmetry will be the principal tool in the analysis.

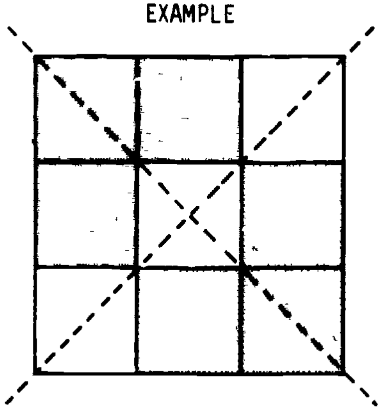
MATERIALS:

Rectangular prism, e.g., a brick, or a wooden block.

DIRECTIONS:

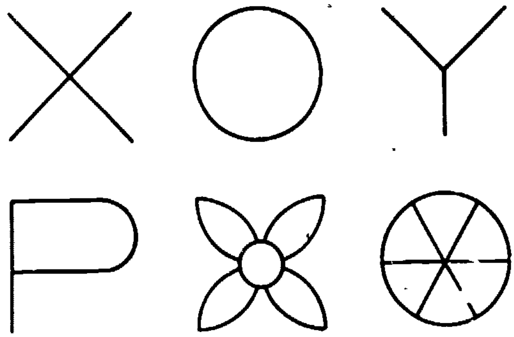
1. The order of a kind of symmetry for an object is the number of different symmetry transformations of that kind for the object.

EXAMPLE



The order of flip symmetry of this design is 2, since the reflection about each of two lines (the two diagonals) is a symmetry transformation. The order of turn symmetry is 2, since a turn of 360° or of 180° about its center makes it coincide with itself. Since a turn of 360° is a symmetry transformation for any shape, every shape has an order of turn symmetry of at least 1.

a) Classify each of the following shapes in terms of the orders of its symmetries.



b) Create a shape with each of the following sets of symmetry orders.

	TURN	FLIP
i)	1	0
ii)	2	0
iii)	1	1
iv)	2	2
v)	3	3

c) Can you conjecture, in terms of your experience with compositions of flips, why it might be impossible to have a shape with order of flip symmetry 2 and order of turn symmetry 1?

d) OPTIONAL: *Is the composition of two symmetry transformations necessarily a symmetry transformation?*

2. One important geometric activity is analyzing and classifying shapes. Here you will be asked to use orders o symmetries to partially analyze the family of triangles and the family of quadrilaterals.

   a) Fill in the table on the following page concerning triangles.

Types of Triangle	Order of Turn Symmetry	Order of Flip Symmetry
Scalene		
Isosceles (nonequilateral)		
Equilateral		

If you are given the order for a triangle, can you tell the type of triangle? Would there have been any problem if "right triangle" had been included as a type?

b) Below is an order chart for quadrilaterals, to be filled in as above.

Type of Quadrilateral	Order of Turn Symmetry	Order of Flip Symmetry
No parallel sides		
Trapezoid (nonparallelogram)		
Parallelogram (nonrhombus, nonrectangle)		
Rhombus (nonrectangle)		
Rectangle (nonsquare)		
Square		

Have you considered kite-shaped objects as part of the first category?

If you are given the orders of a quadrilateral, can you tell the type of quadrilateral? (Pay particular attention to the first two categories.)

The analysis and classification that you have been doing here is a kind of formalization of the common everyday act of becoming familiar with a shape by flipping it and turning it in your hand. Do you agree?

3. The concept of symmetry is certainly meaningful for solid shapes as well as the plane (two-dimensional) shapes studied above. Instead of a line of reflection, one needs a plane and instead of a point of rotation one needs a line.

   a) Investigate the orders of symmetries of a rectangular prism (e.g., a brick). (For line symmetries, ask how many ways it can be put into a rectangular box.)

   b) The body has symmetries and so do many other physical objects (if you are not too literal or precise). Outline a sequence of questions that you might ask a third-grade child to sensitize him or her to symmetries.

TEACHER TEASER

Sam said to Sally, "I have designed a better mousetrap." Sally said to Sam, "Fabulous, what does it look like?" Sam said to Sally, "I can't tell you since my patent is pending, but I will say that the design has two flip symmet and no turn symmetry besides 360°." Sally sighed with disgust and contempt saying, "You obviously have lied or err'd."

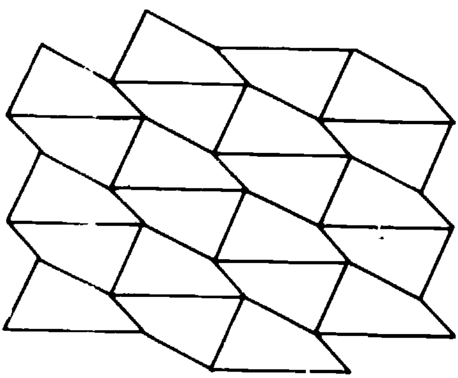
Can you tell why Sally was so disgusted with Sam's description of his mousetrap? It may help to think back to your experience with decomposing rigid transformations.

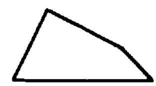
ACTIVITY 6

TESSELLATIONS

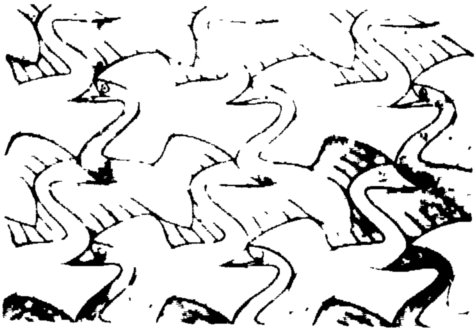
INTRODUCTION:

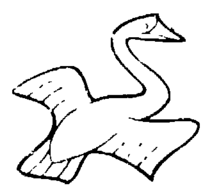
The concept of tessellation is a very rich one that lends itself to the study of several mathematical concepts as well as to the art of graphic design.





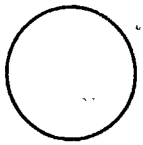
This shape tessellates. That is, an infinite number of copies of it can cover the whole plane without overlapping or leaving gaps.



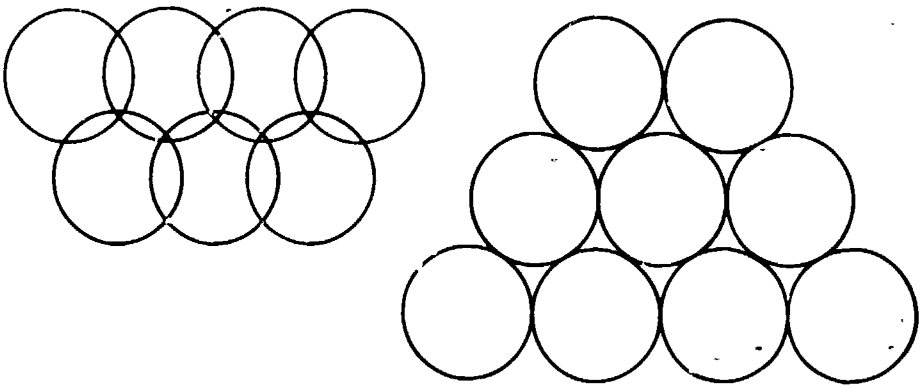


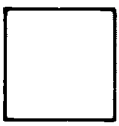
This shape also tessellates.*

*M.C. Escher. Swans. Reproduced by permission of the Escher Foundation, Haags Gemeentemuseum, The Hague.

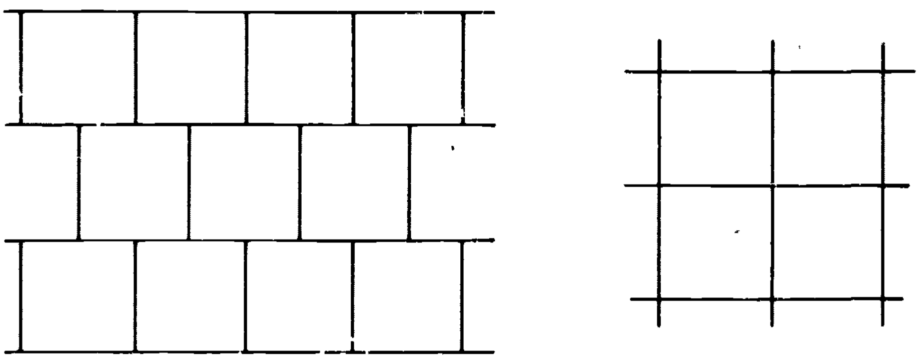


This shape does not tessellate. No matter how you arrange copies of it, they will either leave gaps or overlap.





This shape can be the fundamental unit for many different tessellation patterns.



The patterns produced by a square shape are called regular tessellations, since the square is a regular polygon (equal straight sides and equal angles).

Look around you. The floor in your classroom is probably tessellated. The chances are that someone in the room has a tessellated fabric as part of his or her clothing. Many works of art (especially op-art) involve tessellations; and, in particular, the graphic artist, M. C. Escher, created many tessellations out of such unusual fundamental units as birds, fish, and men on horseback (see pages 62 and 66). Solid shapes can tessellate as well, but solid shapes will not be focused on here. Many of the tessellations around you use square, rectangular, and triangular shapes as fundamental units. You may want to collect instances of more interesting tessellations. They could be used to catch the imagination of children when you are teaching.

FOCUS:

You will be introduced to the concept of tessellations of the plane and its relationships to: Measurement, Analysis of Shapes, Number Factorization, Rigid Transformation, Symmetry, Number Patterns, and Nature.

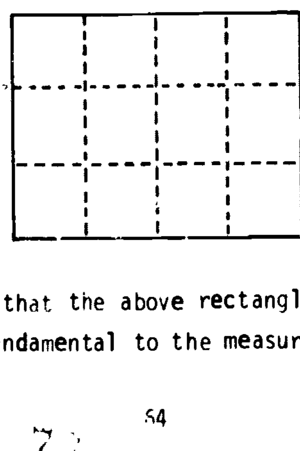
In this activity there are five smaller activities designed to bring to light the relationship of tessellations to the concepts listed above. Each of these subactivities is independent of the others so that you may do some or all of them.

MATERIALS:

Scissors, construction paper, ruler.

SUBACTIVITY 1: TESSELLATION AND MEASUREMENT OF AREA

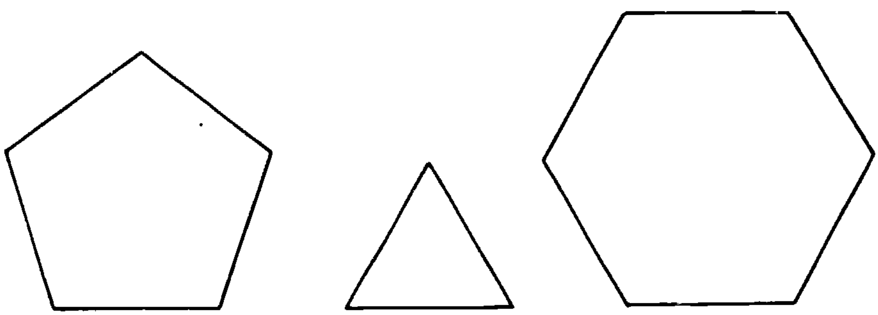
DISCUSSION:



You will quickly agree that the above rectangle has an area of 12 square centimeters. Fundamental to the measurement of this area is the choosing of a unit. (In this case the unit is a square shape, whose sides have lengths of one centimeter, and which is called a square centimeter.) Also fundamental is the covering of the rectangle with copies of the unit without leaving gaps and without overlapping. The square centimeter is obviously not the only unit that can be used for measurement. You will see, in this subactivity, that units of measurement in the plane need not even be square shapes. In fact you will see that many different shapes can be fundamental units for tessellations (i.e., they will cover the plane without gaps or overlaps), and this tessellation property is all that is required for a shape to be a unit of measurement in the plane. This subactivity helps you to explore shapes that tessellate, and hence that can be units for measurement, by putting you in a problem-solving situation. The problem is that of choosing the most economical tile for tiling a bathroom floor.

DIRECTIONS:

1. Cut out four copies of each of these three shapes.



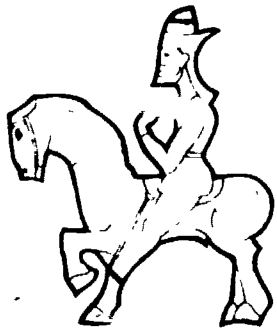
Shape #1 Shape #2 Shape #3

Experiment with each shape to determine which tessellate and which don't.

2. If you were going to tile a bathroom floor and you could buy tiles like shape #2 for 30¢ each and tiles like shape #3 for $1.50 each, which shape would be more economical? (If you want

to, you can cut out several copies of each shape and experiment with a 1-foot-by-2-foot model of a bathroom floor.)

3. *OPTIONAL: M. C. Escher designed this man on horseback* as the fundamental unit in a tessellation. How much could you pay for tiles of this shape and still have it be the most economical shape?*



4. Children enjoy tessellations and could learn a great deal by finding different fundamental units in the world around them, and by solving tiling problems. Outline a series of four or five lessons on tessellations for a specific grade level. (This is an open sort of topic for children, in which they can concentrate on problem solving and investigation rather than on content learning.)

SUBACTIVITY 2: REGULAR POLYGONAL SHAPES THAT TESSELLATE

DISCUSSION:

Below are three questions and some hints to help you answer them. The questions are designed to help you completely solve the problem of "Which regular polygonal shapes tessellate?"

DIRECTIONS:

Answer, using the hints where necessary, each of the following three questions:

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*From M. C. Escher, Horseman. Reproduced by permission of the Escher Foundation, Haags Gemeentemuseum, The Hague.

1. What are the measurements of the vertex angles of each of the following regular polygons?

   a) Regular triangle

   b) Square

   c) Regular pentagon

   d) Regular hexagon

   e) Regular octagon

   You way find the following facts to be useful:

   - When a shape tessellates the sum of the angles where vertices meet is 360°.
   - The sum of the angles of a triangle is 180°.
   - Each polygonal shape can be cut up into triangular shapes.
   - Each regular polygon has equal sides and equal angles.

2. Some of the regular polygonal shapes above tessellate and some do not. Explain why, in terms of their angles. Remember the comment about the sum of the angles where copies of tessellatable shapes meet.

3. How many regular polygonal shapes tessellate? Explain this in terms of the whole-number factors of the number 360.

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SUBACTIVITY 3: TESSELLATIONS, RIGID TRANSFORMATIONS AND SYMMETRIES

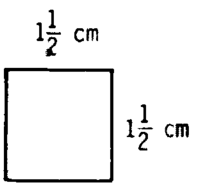
DISCUSSION:

The focus of this subactivity is on the question, "Why are tessellations studied in a section on rigid transformations?" Part A shows that rigid transformations provide the link between a fundamental unit and a tessellation of the plane with that fundamental unit. That is, given a fundamental unit, there is a describable (infinite) set of rigid transformations that will move the fundamental unit so as to cover the entire plane. Part B explores some relationships between tessellations and symmetry.

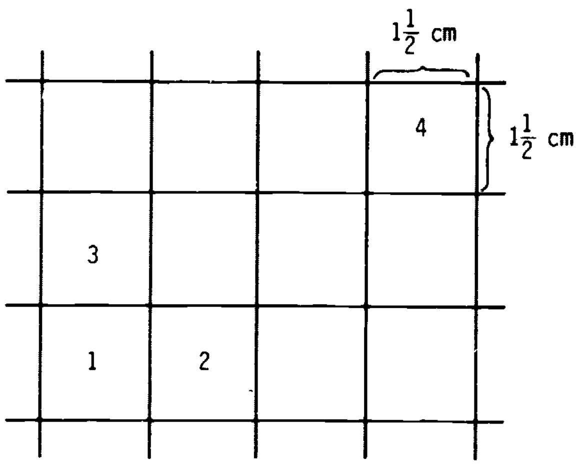
PART A

EXAMPLE

If one wants to describe a set of rigid transformations that will cover the entire plane with a $1\frac{1}{2}$ cm square shape, one can proceed as follows:



Fundamental Unit



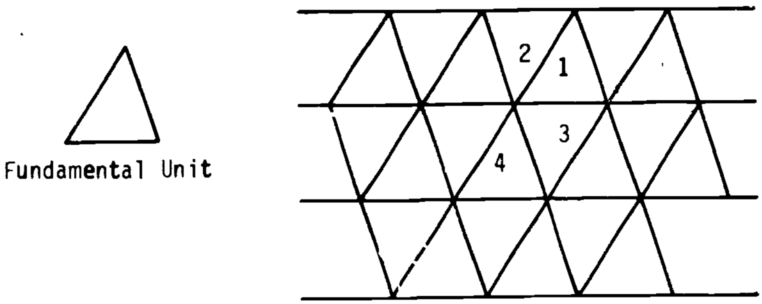
In the above tessellation, one needs a slide of $1\frac{1}{2}$ cm to the right to move the fundamental unit from position 1 to position 2, and a slide of $\frac{1}{2}$ cm up, to move the fundamental unit from position 1 to position 3. To get from position 1 to position 4, one needs the composition of three $1\frac{1}{2}$ cm slides to the right and two $1\frac{1}{2}$ cm slides up.

One can move the fundamental unit from position 1 to other locations in the plane by various compositions of other vertical and horizontal slides. One can, therefore, describe a set of transformations that will cover the entire plane with that fundamental unit as $1\frac{1}{2}$ cm slides to the right, to the left, up, and down, and all possible compositions of such slides.

DIRECTIONS:

Answer the questions that accompany the following two tessellations.

1.



a) Describe a single rigid transformation that will move the fundamental unit from:

Position 1 to position 2

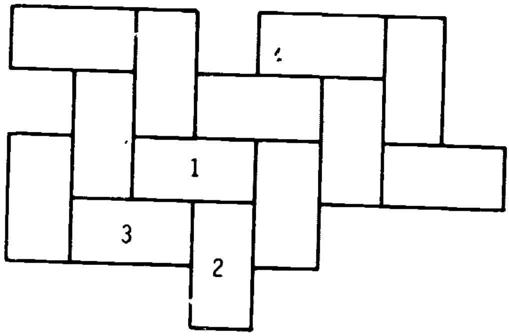
Position 1 to position 3

Position 1 to position 4

b) Describe a set of rigid transformations that will cover the entire plane with the fundamental unit.

2.

Fundamental Unit



a) Describe a single rigid transformation that will move the fundamental unit from:

Position 1 to position 2

Position 1 to position 3

Position 2 to position 3

b) Describe a set of rigid transformations that will cover the entire plane with the fundamental unit.

PART B

In Activity 5 you studied turn and flip symmetries. None of the (finite) shapes or patterns studied there had slide symmetries. Here we will discover that (infinite) tessellation patterns can have slide symmetries as well as turn symmetries and flip symmetries.

A slide of $1\frac{1}{2}$ cm to the right will make the tessellation pattern in the example in Part A coincide with itself. So this slide is a symmetry transformation for that patter . There are many others.

1. Determine a slide symmetry transformation for the other two patterns in Part A.

2. Determine a symmetry transformation that is not a slide, for one of the patterns in Part A. You may wish to trace the pattern and experiment with the traced pattern on top of the original pattern.

3. Can you create an argument to convince one of your classmates that no finite shape or pattern can have a slide symmetry?

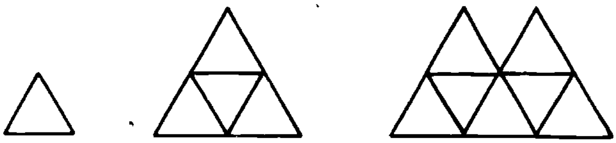
SUBACTIVITY 4: NUMBER PATTERNS AND TESSELLATIONS

DISCUSSION:

This subactivity models a kind of number-pattern investigation that children might enjoy and profit from.

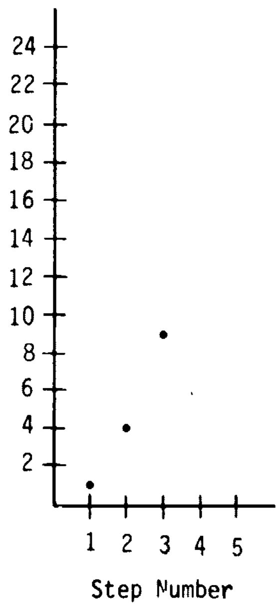
DIRECTIONS:

1. Start with an equilateral triangular shape and build successively larger ones as illustrated below.



Complete this one.

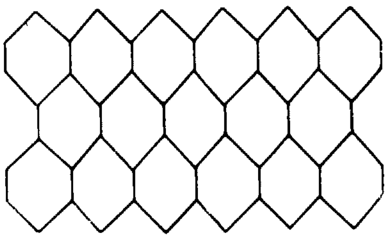
Complete the graph below, which records the number of smaller equilateral triangular shapes required to make each larger one.

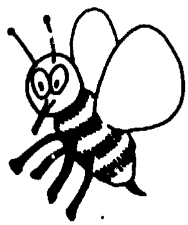


2. Repeat (1) with two more tessellatable shapes, e.g., square shape, parallelogram shape, hexagon shape. Are there any characteristics that are common to the three graphs? What number patterns have you observed?

3. Outline an activity for children that will investigate other number patterns in other tessellations. (You will need to play with some shapes in order to discover other patterns.)

SUBACTIVITY 5: TESSELLATIONS IN NATURE

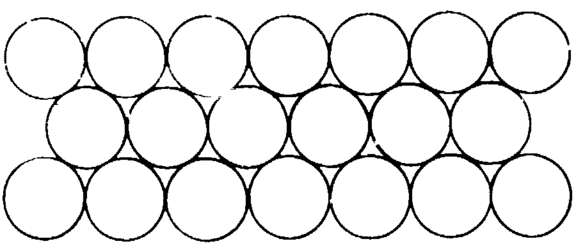




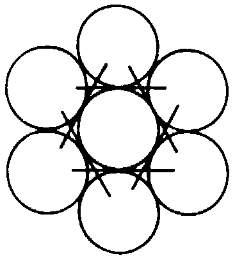
DISCUSSION:

Why is the honeycomb a hexagonal tessellation? We hope that, if we give you a few facts and an activity to do at home, you will be able to figure out the "mathematical abilities" of a bee.

Fact 1: Circles minimize perimeter per given area. That is to say, if you want to enclose a region with a given area by the shortest possible curve, your region had better have a circular boundary. Liquid naturally seeks this minimal-perimeter-for-area state, so that when a bee puddles liquid on a honeycomb, the liquid takes a circular shape. (It is actually spherical, but this whole problem is more easily viewed in two dimensions.)



Fact 2: If you pack congruent circles as compactly as possible so that the circles just touch, you will see that each circle is touched by exactly six other circles; and if you draw the tangents of those circles at the points where they touch, you will get a regular hexagon that encloses the middle circle.



DIRECTIONS:

1. With the above facts in mind, the next time you bake cookies with a circular shape, make sure that, when you put them on the cookie sheet, at least one cookie is surrounded by cookies that just touch it and touch each other as the figure shows. Find out what the shape of the surrounded cookie in the middle will be after being baked. Does this shed any light on how the honeycomb takes the shape it does?

2. Name some other hexagonal tessellations that you find in nature. (Look at the fruit counter in the grocery store.)

3. Outline an appropriate activity for children on tessellations in nature. For references you may refer to the bibliography in Activity 7.

ACTIVITY 7

EXPERIENCES WITH GEOMETRY MATERIALS

FOCUS:

There are many clever, interesting, and valuable activities related to rigid transformations that one can do with children. This activity will give you some familiarity with related materials and references that are available; you will have an opportunity to use these materials and references in developing activities for children; and you will present your activities to children or to classmates.

MATERIALS:

Geoblocks, geoboards, various kinds of graph paper, MIRA materials, and mirror cards, as well as those of the references listed on pages 82-83 that are available.

DIRECTIONS: (summarized on page 79)

In this assignment you will find a list of topics that would be appropriate for you to teach to different elementary grades. Along with each topic you will find reading references. Each group of three or four of you is expected to choose one topic and to take two or three days to prepare an activity directed to a certain grade level on the topic chosen. On "Activity Day" you will have a chance to try your activity on your classmates and/or children.

I. In class today you are expected to:

   A. Look at the list of topics on pages 80-81.

   B. Form groups of three or four and choose the topic you want to work on.

   C. Report to your instructor the names in your group and the topic chosen. Choice of topics follows the rule of "first come first served."

D. Go to the desk where some material aids are displayed, and familiarize yourself with them, particularly with the geoblocks, geoboards, MIRA materials, and mirror cards. It is important that you get a well-formed idea about these aid materials so that you can use them knowledgeably and purposefully in your activities.

E. Carefully look over the chart that follows.

F. Read through the rest of this activity.

II. A. What to do with the aid materials

The aid materials displayed include: geoboards, geoblocks, MIRA materials, and mirror cards, and different kinds of graph paper. These are some of the commercially produced materials that we know of, and think are appropriate to use. Many of you might want to plan activities using other materials such as potatoes or ink blots or folding paper. And some of you might want students to act themselves rather than use material. You are free to use any material aids that you want.

B. Points to take into consideration while writing your activity

1. The reading references do not give you a ready-made activity that suits your purposes; instead they:

   a) Reinforce your understanding of the content involved;

   b) Suggest to you different approaches;

   c) Present you with activities which, if you decide to use them, you must adjust to your topic as well as to the level of difficulty needed at the grade you are writing for.

2. The activity should not take more than ten minutes when your peers are performing it. To adjust the time, try it yourself. It is expected to take longer with children.

3. You should write the objective of the activity and the materials needed for it before you proceed with the activity.

4. Keep in min that the activity is directed toward children in grades K-6 and not to your peer students. The level of difficulty in the textbook series might give you an idea as to the appropriate level of complexity or difficulty to aim at in your activity.

5. While writing your activity you may want to write some notes to the teacher who might use it.

6. You should put emphasis on play and action that will lead the child to learn a concept rather than on telling the concept to the child.

7. Make sure the directions in your activity are clear and to the point.

C. How to prepare your station for activity day

Prepare a sufficient number of copies of your activity. You will probably be using aid materials in your activity. If the aid materials can be provided by your instructor, make sure to ask him/her for the materials at least one day before Activity Day. If they cannot be so provided make sure <u>you</u> bring them to your station.

III. Activity Day

A. How to prepare your station

1. Hand in one copy of your activity to your instructor.

2. Choose a table as your station and mark it with the number of your activity (as numbered on pages 80-81).

3. Place the copies of your activity on the table and set up all of the materials needed in performing the activity.

4. Provide some sheets of paper for groups performing your activity to write their evaluations, criticisms, and suggestions.

B. What to do then

1. Choose a station set up by another group.

2. Read and then perform the activity you find at that station.

3. Discuss the activity and write down your evaluation, criticism, and suggestions to the writers of the activity, paying particular attention to:

a) Clarity of instructions

b) Appropriateness for the grade level designated

c) The motivation it provides

d) The math content it provides

4. When you are finished move to another station.

Note: You need not cover all stations, but cover thoroughly those that you do.

IV. Activity day (with children)

A. Preparation

1. Work out details as to time and place of class.

2. Prepare materials for class.

B. What to do

1. Facilitate the experience of a small group of children with your activity.

2. Observe how the children interact with your materials and instruction.

V. Follow-up work with your activity

Your experiences with children or with other activities, as well as the criticisms of other groups, should provide you with some ideas for revision of your activity.

A. Make any revisions that you feel will improve your activity.

B. Hand in the revised activity to your instructor, along with some comments concerning the reasons for your changes.

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PROCEDURAL CHART

I. Today Includes:	II. Next Three Days at Home	III. Activity Day in University Class	IV. Activity Day in Elementary Classroom	V. Next Two Days at Home
Each group of three choose topic from list on pages 80-81.	Read guidelines for activity preparation.	Each group will have prepared an activity.	Each group will have prepared an activity.	Each group will revise their own activity.
Submit topic title to the instructor.	Read references.	Each group will hand in one copy of activity and prepare a station where other groups can do activity.	Each group will make arrangements to teach activity to small group of children.	Each group will hand in revised activity.
Study aids related to topics.	Plan activities.			
	Plan and discuss materials needed with instructor.	Each group will do several activities prepared by other groups.	Each group will teach activity to small group of children.	
		Each group will record an evaluation of each activity done.	Each group will evaluate and modify activity	

This chart provides five possible steps that can be taken related to Activity Day. Some classes may do all five. Other classes may just do I, II, III or I, II, IV.

TOPICS AND REFERENCES

(Number refers to references in bibliography on pp. 82-83.

I. Symmetry in Nature, 1st or 2nd grade level

1 (pp. 58-107); 4; 9 (Book 1, pp. T252-T254); 16; 17; 18; 19; 22; 23; 25; 26

II. Symmetry of Shapes, 3rd or 4th grade level

1 (pp. 58-107); 4; 5; 6; 8 (pp. 30-32, 59-85); 9 (Book 3, p. T362 and Book 4, pp. T146-T147); 10 (pp. 57-59, 77-79, and 103-106); 13 (pp. 57-64); 16; 17; 18; 19; 20 (pp. 32-36); 26

III. Introducing the Three Rigid Motions, 3rd grade level

3; 4; 8 (pp. 27-33, 34-49, and 59-66); 9 (Book 3, pp. T139-T140 and T144-T148, Book 4, pp. T140 and T145-T147, and Book 6, pp. T156-T157); 10 (pp. 100-103 and 207-217); 15 (pp. 102-103 and 105-107); 19; 26

IV. How to Make Drawings of Motions Easy, 6th grade level

1 (pp. 73-83); 12 (pp. 61-64 and 67-69); 15 (pp. 105-107); 20 (pp. 34-44)

V. Rigid Motions and Symmetries Leading to the Concept of Congruence, 4th grade level

1 (pp. 58-107); 10 (pp. 89-94); 11 (Book 6, pp. 249-258); 12 (pp. 25-29, 38-39 and 46-49), 14 (pp. 115-120); 19; 20; 26

VI. Rotations Leading to a Study of Angles, 5th grade level

3; 8 (pp. 10, 27-28, and 34-52); 9 (Book 3, pp. T40 and T148-T151, Book 4, p. T360, and Book 6, pp. T152-T153); 15 (pp. 105-106); 21 (Book 2, pp. 77-89 and Book 3, pp. 29-31)

VII. Symmetries Leading to the Introduction of Fractions, 2nd grade level

7; 9 (Book 2, pp. T270-T309); 19

VIII. Symmetries in 3-D Objects, 3rd or 6th grade level

1 (pp. 58-107); 9 (Book 4, pp. T366-T371); 13 (pp. 57-64); 20 (pp. 136-137); 21 (Book 2, pp. 25-33); 24

IX. Tessellations, 5th grade level

10 (pp. 46-48 and 147-161); 13 (pp. 17-20); 15 (pp. 102-103 and 105-107); 20 (pp. 136-149); 21 (Book 2, pp. 36-45 and 55-56, and Book 3, pp. 27-28 and 61-63)

X. Tessellations Leading to Area, 3rd grade level

2; 9 (Book 3, pp. T166-T167 and Book 4, pp. T164-T174 and T362-T363); 10 (pp. 46-48 and 147-161); 13 (pp. 17-20); 21 (Book 3, p. 61)

XI. Symmetries Leading to Group Structure, 5th grade level

8 (pp. 10-13, 46-55 and 69-86); 15 (pp. 108-109); 21 (Book 3, pp. 48-54)

XII. Symmetry in Music and Poetry, 5th and 6th grade levels

18 (pp. 98-103).

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ACTIVITY 8

RIGID TRANSFORMATIONS IN THE ELEMENTARY SCHOOL

FOCUS:

In this activity you will analyze the transformational geometry material that appears in elementary mathematics text series, and you will engage in a seminar on the role of transformational geometry in the elementary school.

MATERIALS:

Several current and some older elementary mathematics text series. These could be located in a library or resource center if the textbook analysis is to be done outside of class.

DISCUSSION:

There is not general agreement among mathematics educators as to what role geometry should play in the elementary school mathematics curriculum. In particular, the role of rigid transformations in the elementary school is presently undetermined. Rigid transformations have been studied to some extent at the junior high level and to a greater extent at the senior high level, but their potential for and value in the elementary school curriculum are little tested.

Some would agree that a rigid transformation approach to geometry is particularly appropriate at the elementary level since it lends itself to activities that involve physical objects and movement. Others contend that rigid transformations serve to emphasize the relationship between geometry and the real world. Some also point out that a transformation is a function and hence that the study of rigid transformations tends to reinforce the function concept, which is generally agreed to be of great mathematical importance.

On the other side of the debate are arguments that the explicit introduction of rigid transformations into the elementary school confuses teachers and pupils, and, moreover, that the rationale for this introduction is based on untested claims. Some also argue that the unifying effects of such concepts as that of function are lost on elementary school children. The argument goes on that the traditional geometry curriculum implicitly contains most of the concepts of the transformational approach, is carefully developed, well tested, familiar, and has a higher probability of being well taught.

Elementary textbooks reflect the above arguments. Some contain no rigid transformations; several contain some rigid transformations; few, if any, have thoroughly integrated rigid transformations into their geometry development.

In Part A of this activity you will be asked to analyze two elementary text series, one that contains some rigid transformations and one that contains none.

In Part B you will be asked to consider the role of rigid transformations in teaching symmetry.

In Part C you will be asked to take part in a seminar on the role of rigid transformations in the elementary school.

PART A

Pick an elementary text series that includes rigid transformations and one that does not. Then read the questions below and use them to direct your analysis of the geometry content in the two series.

1. From a series that contains rigid transformations answer the following questions:
   a) How do the sections on rigid transformations fit in with the development of the geometry content as a whole?
   b) Cite one instance from the textbook series where rigid transformations are utilized to facilitate the introduction of other concepts.
2. From a series that does not include rigid transformations, answer the following questions:

a) How do the geometric goals as a whole differ from those of the series you considered in (1) above? Do any important concepts seem to be overlooked?

b) Do you feel that the geometry in this series would be easier, harder, or of about the same difficulty to teach as the geometry in the series with rigid transformations?

PART B

1. Investigate a text series that explicitly uses rigid transformations to introduce symmetry and one that does not. Consider the following questions in your investigation.

   a) Which approach is richer, clearer, more teachable, more likely to appeal to children?

   b) Which approach relates symmetry more directly to other mathematical concepts and to the real world?

2. In the seminar discuss the role of rigid transformations in teaching symmetry. Consider the following questions in your discussion.

   a) What advantages do you see in the explicit use of rigid transformations to introduce symmetry? Is it more advantageous for point symmetry than line symmetry?

   b) What disadvantages do you see?

   c) List the various kinds of symmetry that you know of (e.g., point, line, rotational, reflectional, translational), and discuss various ways of presenting each (e.g., MIRA mirrors, tracing paper, cutting, folding, ink blots).

PART C

Engage in a seminar on the role of rigid transformations in the elementary school. Below are some questions that the class can address itself to.

1. Did the text series that contained rigid transformations seem to be enhanced in any way by their presence?

   a) Was it more or less teachable?

   b) Did it seem more or less interesting?

   c) Did it tie its geometry more or less closely to the real world?

2. Were there sections of the series not containing rigid transformations that could be enhanced by the introduction of rigid transformations? If so, how?

3. If you were assigned to teach from either of the text series, which of the following would you do?

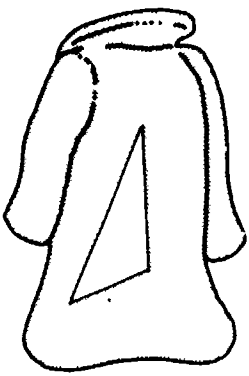
   a) Skip the rigid transformations in the series that includes it. If so, why?

   b) Introduce rigid transformations into the series that does not include it. If so, how, where, and why?

   c) Teach either series as it is written.

TEACHER TEASER

The Tragic Mistake of the Poor Tailor of Sikinia*



In Sikinia people are poor, but everyone owns a ferocious dog. These dogs tear triangular holes in the clothes of passersby. There was a poor tailor who was making a living by patching up the holes which resulted from such mishaps.

The secret dream of our poor tailor was to become rich by mending mink coats. One day he had his big chance. A lady came in with a mink coat that had a huge triangular hole in the back (see picture above). Our poor tailor had never mended furs before, but only regular cloth. And he made a tragic mistake. On mink, hair grows on one side only. The other side is clean-shaven. You cannot turn it over like plain cloth which looks the same on both sides. But our poor tailor had to learn this the hard way. He cut a mink patch to fit the hole, but it fit only on the wrong side (see figure above). What to do now? How can we help our poor tailor?

*This problem was taken from a paper presented by Professor Arthur Engel to the CSMP International Conference on the Teaching of Geometry at the Pre-College Level.

Section II

PROJECTIVE TRANSFORMATIONS

In some ways projective transformations are a little harder to deal with than rigid transformations. One can easily model a rigid transformation by repositioning objects. But, as you will see, modeling projective transformations involves casting shadows of objects. So representing a projection on paper involves picturing an object, a light source, and a shadow. Also, one can describe the invariants of rigid transformations with such everyday terms as "size" and "shape" or "distance." Projective invariants, on the other hand, prove to be more complex and involve the comparison of ratios. However, casting shadows is fun and can lead to surprising results. You should enjoy this brief beginning at organizing space experiences that involve visual change.

Activity 9 provides an opportunity for exploration with shadow casting, both with parallel light rays (e.g., from the sun) and with point-source light rays (e.g., from a small flashlight). Activity 10 begins to systematize your observations by helping you to single out some of the invariants of projective transformations. Finally, in Activity 11 you will find that the familiar topic of similarity can be investigated using projective transformations that involve a point light source and an object, which is placed parallel to the screen on which you are casting its shadow. By the end of your experiences with this section, you should be able to answer the following questions.

MAJOR QUESTIONS

1. List some invariants of projective transformations and give everyday examples of the implications of these invariants; e.g., the shadow of a flagpole will be straight.

2. Outline a sequence of activities that you might do with children to build on and systematize knowledge gained from their everyday experiences with shadows. Make your goals and the nature of your activities clear.

3. Describe clearly the relationship between the concept of similarity, proportions involving ratios of corresponding sides, and shadows of objects cast by a point light source.

ACTIVITY 9

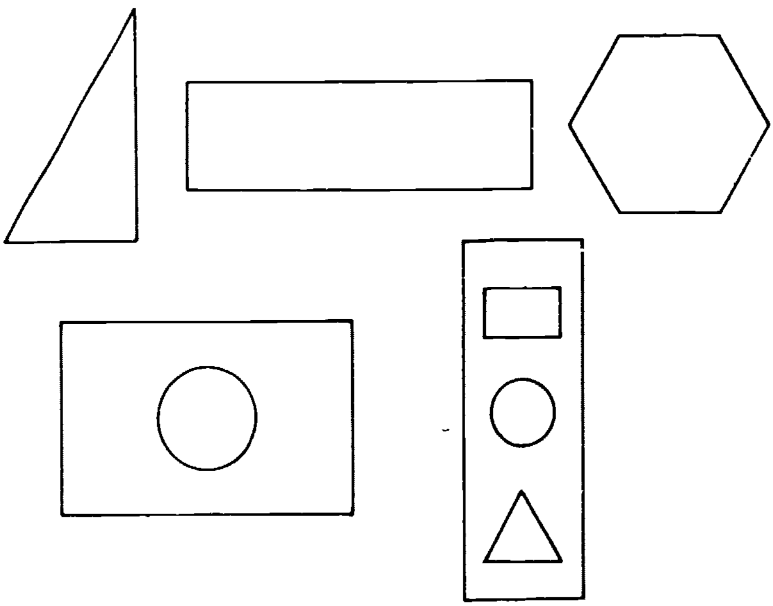
CASTING SHADOWS

FOCUS:

Projective transformations can be modeled by casting shadows of objects. This activity will provide an opportunity to experiment with shadow casting.

MATERIALS:

Point source of light (best approximated by a pen light). Parallel source of light (best approximated by sunlight); if sunlight is not available a slide projector used at a distance of 6 feet or more can provide a fair approximation to parallel light. The following cut-out shapes:



DIRECTIONS:

1. Experiment freely with point-source and parallel light shadows of the objects that you are given and any others that you might

have. (If time is short, you could experiment with one rather than both kinds of light source.)

2. Below are five sets of shapes. In each of Figures 1 through 5 below, shape (a) shows the the basic shape. Circle those shapes among the remaining shapes for the respective figure, that could be shadows cast by the basic shape. Both point-source and parallel light can be considered. Experiment with your shapes.

Figure 1

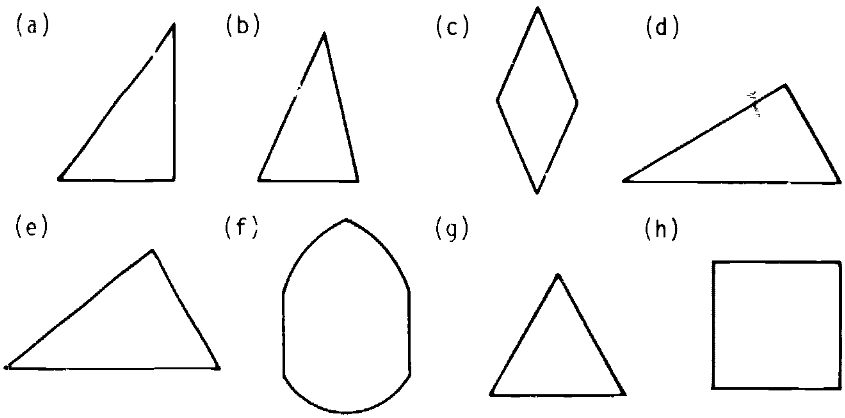


Figure 2

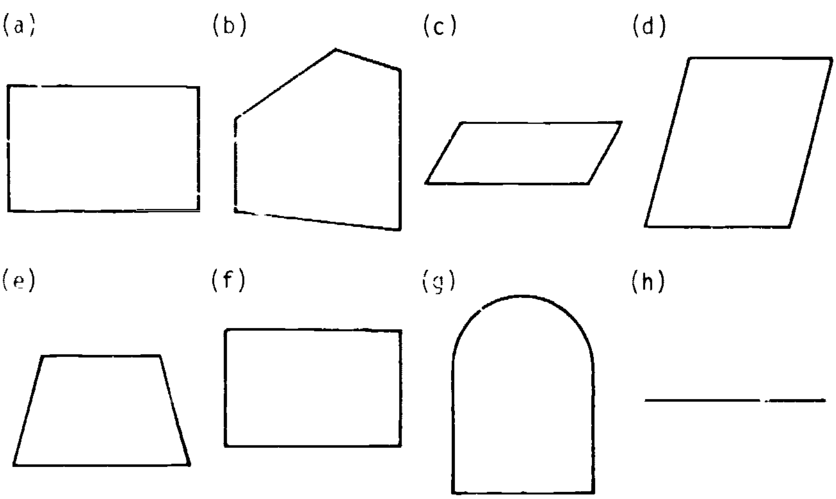


Figure 3

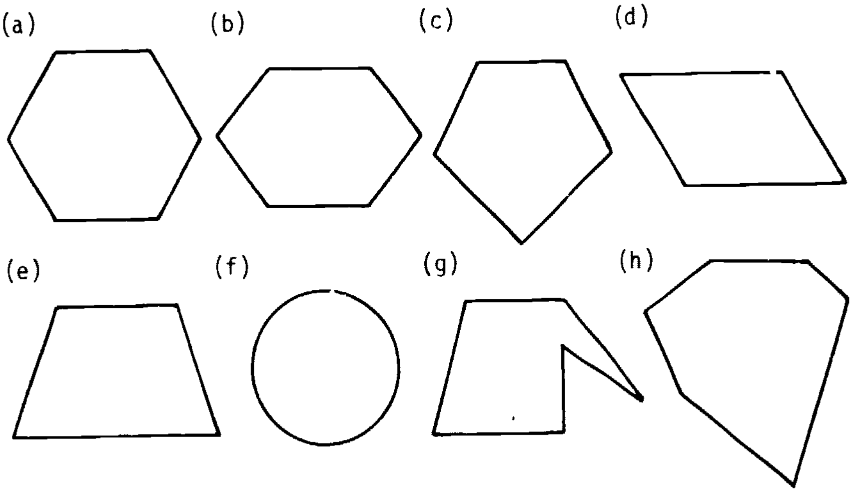


Figure 4

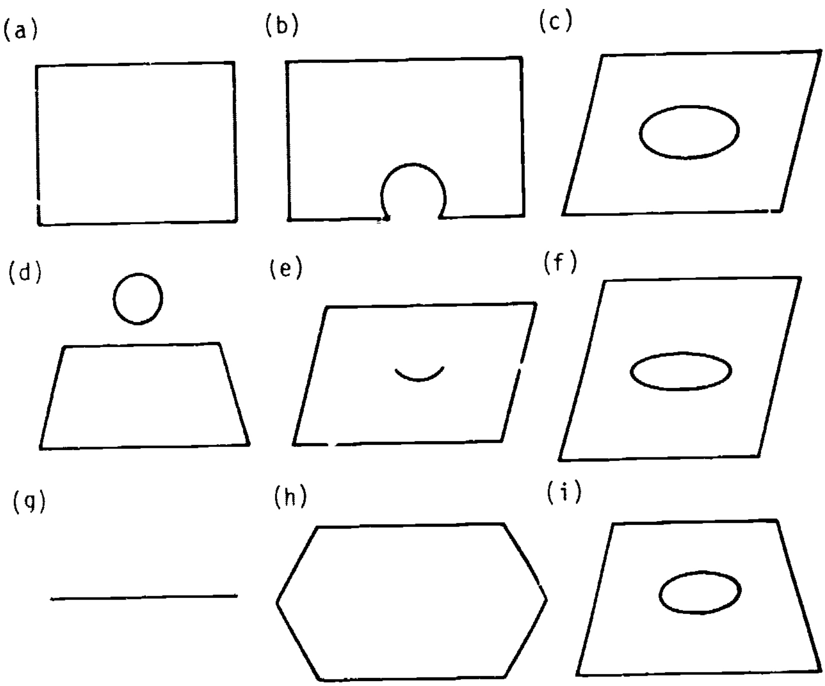
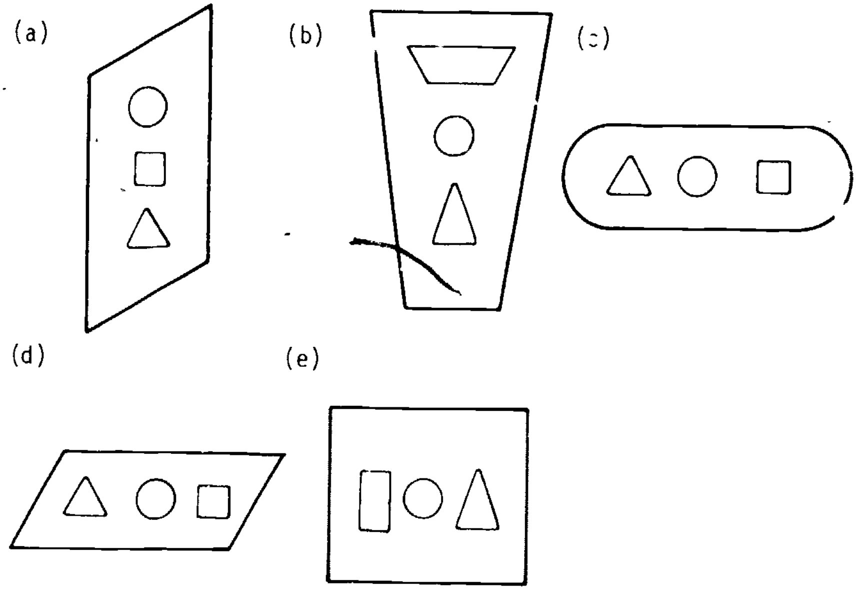


Figure 5



HISTORICAL HIGHLIGHT

Projective geometry was invented by Gérard Desargues (1593-1662). His objective was to make the diverse facts known about perspective understandable to his fellow architects and engineers. His work was ignored by mathematicians and forgotten until a manuscript copy was discovered two centuries later.

ACTIVITY 10

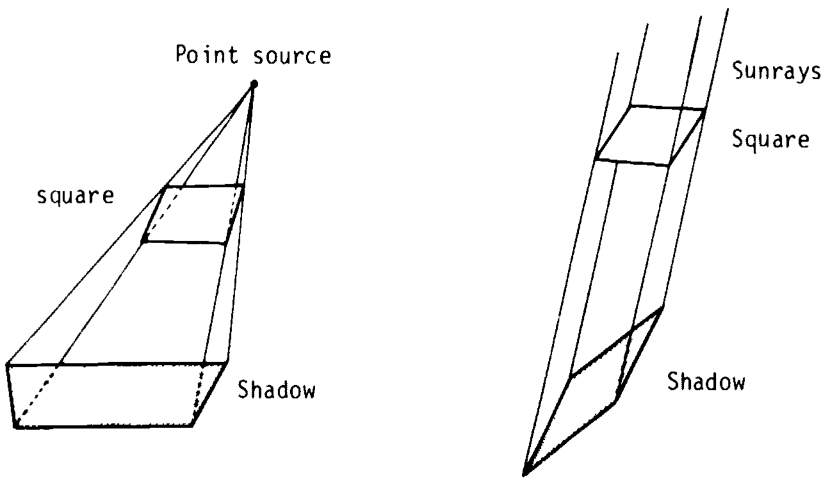
INVARIANTS UNDER PROJECTIVE TRANSFORMATIONS

FOCUS:

In this activity you will have a closer look at projective transformations as embodied in shadow casting, and you will determine the properties that a shape and its shadow share and the properties that they do not share.

DISCUSSION:

In studying the relationships between objects and their shadows, we have to keep in mind that the shape of the shadow depends not only on the object but also on the kind of light source, i.e., whether it is a parallel light source or a point light source. It also depends on the position of the object relative to the light source and relative to the plane where the shadow is cast. The following diagrams may help to clarify these ideas.



A shadow of a square from a point source may be quite different from a shadow of the same square cast by the sun's rays.

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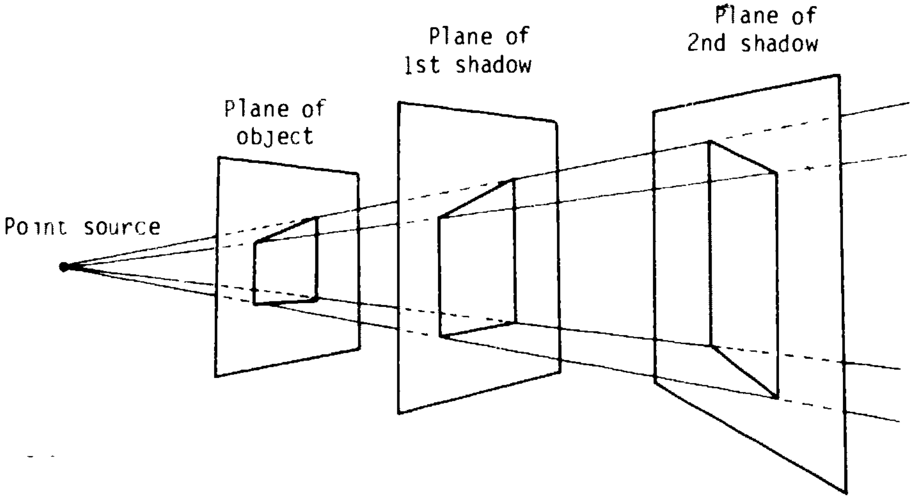


Varying the angle between the object and the screen causes the shadow to vary.

This activity is designed to help you find the projective invariants (i.e., the common properties between the shape of an object and the shapes of its shadows). In Part I you will first be asked to cast point-source shadows for two or more objects; and, while using each object, you will be asked to produce several shadows of that object by changing the relative positions of the object and the plane where the shadow will be cast. Then you will be asked to record certain observations in a chart provided for you. You will be asked to repeat the same process with a parallel light source.*

Part II will deal with certain ratios that are shared by an object and its shadows in sunlight projections. These ratios will enable you to determine when a shadow is a sunlight shadow, when it is not, and why.

*One of the light sources can be omitted if time is short. If the parallel light source is omitted, you may want to skip part II of this activity.

PART I

MATERIALS:

Pen light, sunlight (or slide projector), construction paper, scissors, brown wrapping paper or newsprint ruler, protractor.

DIRECTIONS: (To be done in groups of two or more)

A. You are going to cast shadows using a point light source.

1. Cut out a polygonal shape of your choice and cast its shadow on a large piece of wrapping paper laid on the floor. Have a classmate outline the shadow on the paper as you hold the shape and the light source still. Record your observations for each of the following on the chart on page 98.

   a) Measure the sides of your polygonal shape and the corresponding sides of the shadow. Record your measurement.

   b) Measure the angles of your polygonal shape and the corresponding angles of the shadow and record your measurement.

   c) Count the number of sides of your polygonal shape and the number of sides of its shadow.

   d) Observe whether straight edges cast straight shadows or curved shadows. What can you say about shadows of curved edges?

   e) Observe whether parallel edges make parallel-edged shadows.

2. Make such measurements and observations for two more shadows of the same shape, keeping in mind the different positions of the shape relative to the wrapping paper that gave you the different shadows.

3. Punch a hole in the center of the polygonal shape and observe the position of the hole in the shadow.

POINT-SOURCE PROJECTIONS

1st shape: 2nd shape:

	1st shape	1st shadow	2nd shadow	3rd shadow	2nd shape	1st shadow	2nd shadow	3rd shadow
Lengths of sides								
Measures of angles								
Number of sides								
Position of shape								
Are any two sides parallel?								
Are all sides straight?								
Does the hole in the center of the shape cast a shadow that is in the center?								
Is the order of the notches preserved?								

SUNLIGHT PROJECTIONS

1st shape: ______ 2nd shape: ______

	1st shape	1st shadow	2nd shadow	3rd shadow	2nd shape	1st shadow	2nd shadow	3rd shadow
Lengths of sides								
Measures of angles								
Number of sides								
Position of shape								
Are any two sides parallel?								
Are all sides straight?								
Does the hole in the center of the shape cast a shadow that is in the center?								
Is the order of the notches preserved?								

4. Cut three different-sized notches on one of the sides of your polygonal shape. Observe the shadow that they make. What can you say in comparing the order in which the notches appear on the shape, and the order on the shadow? If you find it difficult to answer this question, recall your findings on Figure 5 in Activity 9.

5. Repeat (1) through (4) using another polygonal shape.

B. Repeat section A, using a parallel light source. Be sure to fill in your data on the chart on page 99.

C. Make a summary of your observations as to what properties are shared by an object and its shadows and what properties are not, under

a) Point-source transformations,

b) Parallel light transformatons.

If you have any other observations, by all means record them.

PART II

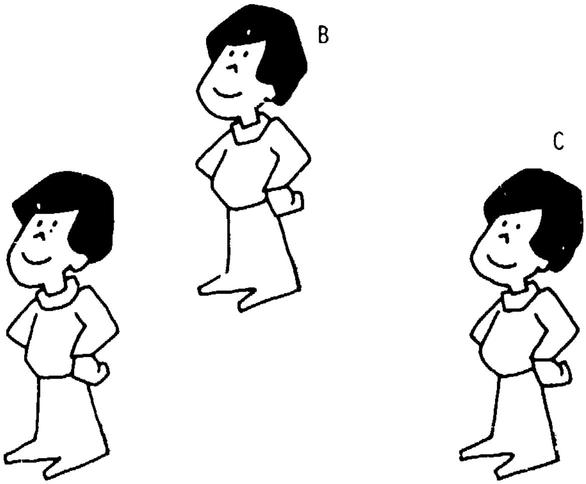
DISCUSSION.

Part I of this activity did not help you to find all of the invariants of sunlight projections. One invariant happens to be less obvious, though quite important, and we will be studying it here.

It is obvious by now that length is not preserved under projective transformations. But even though length changes, it does not change haphazardly; a certain proportionality is preserved.

Remember that in Part I you punched a hole in a shape, and you found out that in sunlight projections the hole in the shadow is also in the center; it may have been bigger or smaller but its position in the shape remained the same. This maintenance of relative position can be more precisely expressed in terms of the equality of ratios (proportionality).

Proportionality or constancy of ratio is a familiar idea. You can probably look at a shadow and be able to tell whether it is a sunlight shadow or not. For example, which of the shadows in the illustration below is a sunlight shadow?



Some of these shadows cannot possibly be shadows cast by the sun. For example, in the shadow in (a), the size of the head and the size of the body do not have the same ratio as they do for the actual person; the head is too large relative to the body. Similarly, in (c), the body is too long in comparison to the size of the head. Everyday experience suggests that the shadow in (b) is the shadow that is cast by sunlight. The aim of Part II is to formalize your intuitive knowledge in terms of proportions.

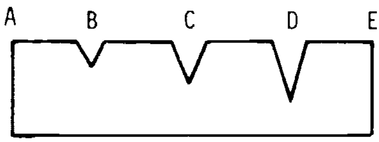
MATERIALS:

A strip of construction paper, scissors, sunlight, and ruler.

DIRECTIONS:

1. Cut three different-sized notches on one of the edges of a strip of construction paper, leaving equal distances between them, as shown in the figure below. You will notice that notch C is halfway along the strip, and notches B and D are $\frac{1}{4}$ and $\frac{3}{4}$ of the way respectively.

11.



2. Using sunlight (or an approximation of it), cast a shadow of the strip. Mark the location of the endpoints of the shadow of the strip and the location of the notches.

3. Call the points on the shadow A', B', C', D', E'.

4. Measure the following distances and record them.

AE	EC	ED	AD	DC
A'E'	E'C'	E'D'	A'D'	D'C'

5. Compare the following ratios:

$\frac{AE}{ED}$	$\frac{A'E'}{E'D'}$
$\frac{AE}{EC}$	$\frac{A'E'}{E'C'}$
$\frac{AD}{DC}$	$\frac{A'D'}{D'C'}$

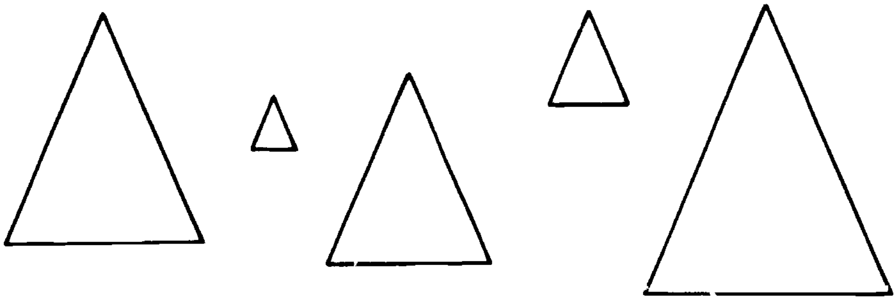
6. Cast another shadow of the same strip. Make the same measurements as before and compare the ratios you get to those you already have.

7. Make another strip with two or three notches that are not necessarily equidistant from each other. Repeat the experiment and record your findings.

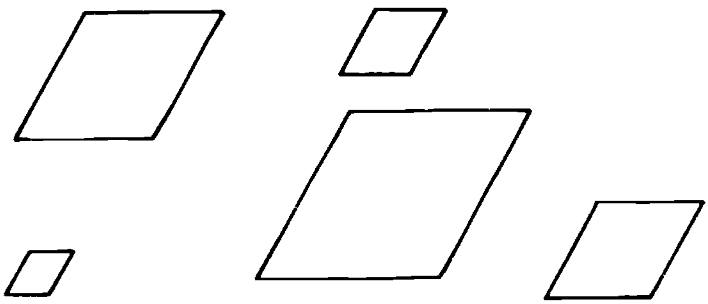
8 Explain what we mean by saying that ratio is an invariant under sunlight transformations.

9. Outline a drawing activity for children that would increase their awareness of proportionality between physical objects and their shadows.

ACTIVITY 11
SIMILARITY



All of these are <u>similar</u> triangles. They all have the <u>same shape</u>.



All of these are <u>similar</u> parallelograms. They all have the same shape.

FOCUS:

The concept of similar shapes can be studied in the setting of <u>point-source projective transformations with object and screen parallel</u>. You will have an opportunity to investigate similarity using a physical model of point-source light rays, and then to apply the equivalent-ratios property of similarity to real-world problems.

MATERIALS:

Ten 5-foot pieces of string for each group, cardboard, scissors, yardstick, large sheets of wrapping paper, scotch tape, protractor.

DISCUSSION:

The concept of similarity is not completely foreign to you. You might make any one of the following synonymous statements.

The two objects are <u>similar</u>.

The two objects have the <u>same shape</u>.

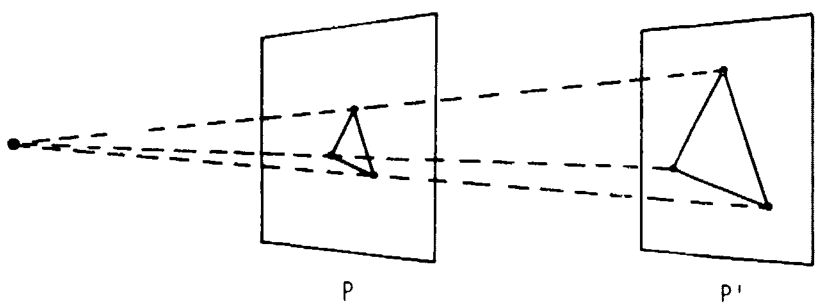
One of the two objects is an <u>enlargement</u> of the other.

In this activity you will investigate two more such statements.

The <u>ratios</u> of corresponding lines in the two objects are <u>equivalent</u>.

One of the two objects is a <u>point-source projection</u> of the other on a parallel screen.

This last statement can be interpreted by a drawing.

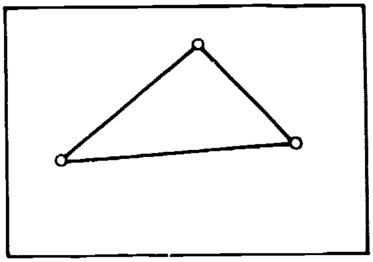


The requirement is that the plane P, where the object is, and the plane P', where its shadow is, be parallel.

The principal objective of this activity is for you to convince yourself experimentally of the fact that the equal-ratios characterization and the point-source, parallel-object characterization both reflect the concept of similarity as you have known it.

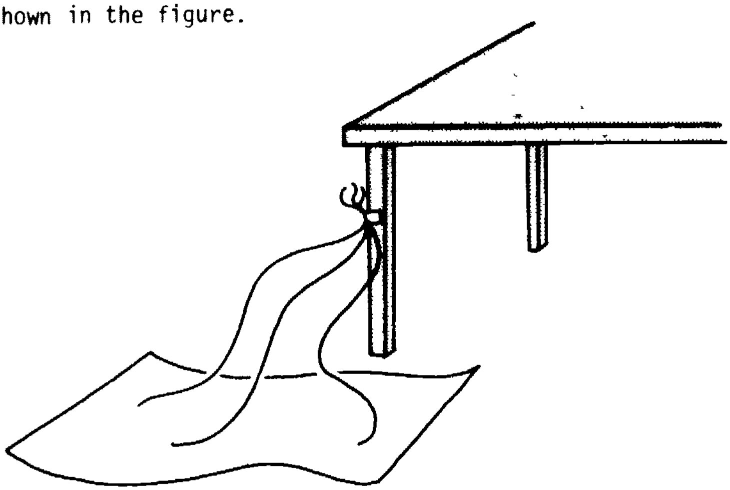
DIRECTIONS:*

1. Take a piece of cardboard and punch three holes in it, forming a triangle.



Draw straight lines between the holes to emphasize the triangle.

2. Tie three strings together and tape them to the wall, leg of the chair, leg of table, or any other place that is suitable, as shown in the figure.

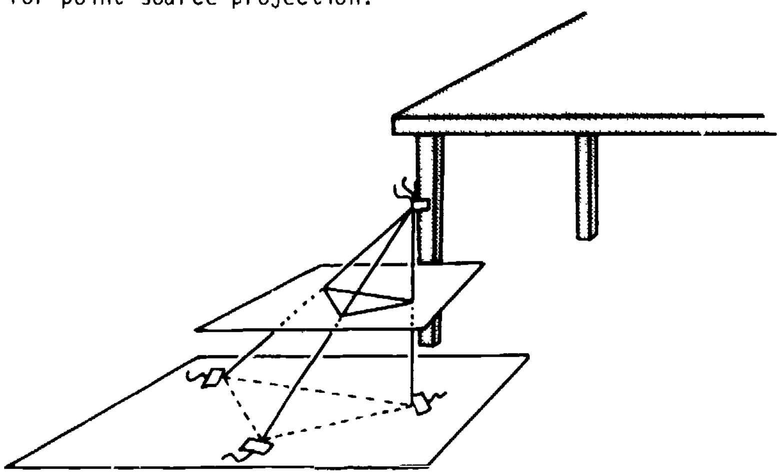


3. Spread the big sheet of wrapping paper on the floor beneath the strings.

*We are indebted to Dienes, Z. P., and E. W. Golding, Geometry Through Transformations, I: Geometry of Distortion. New York: Herder and Herder, 1967, pp. 73-74, for the idea of this physical model.

4. Thread the strings through the holes in the cardboard. Locate the cardboard parallel to the floor between the strings and stretch the strings taut. Tape or paper-clip the cardboard to the strings so that it stays parallel to the floor. Make a mark where the string touches the wrapping paper, and tape the string there

5. After stretching all three strings and taping them on the wrapping paper, look at your model and convince yourself that the stretched strings can represent light rays while the knot represents the point source and that what you really have is a model for point-source projection.

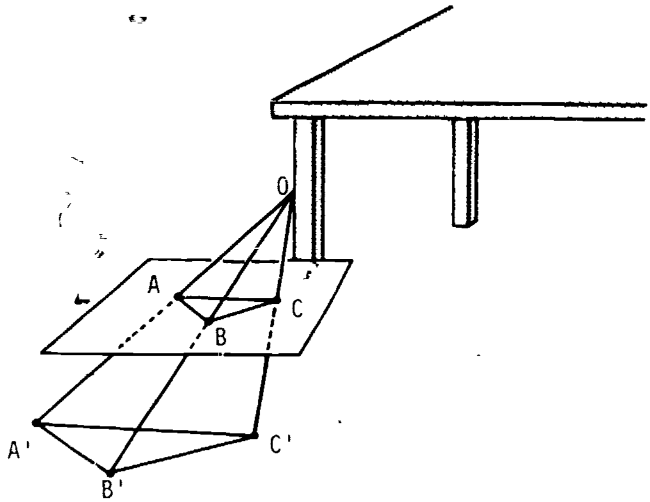


6. You have built your model, and now what you have to do is to start using your model to study similarity.

   a) First of all, do you agree that your object triangular shape and your shadow triangular shape are similar? If not, repeat the taping, making sure that the object shape is parallel to the floor.

   b) Referring to the drawing on the following page, measure and record the following lengths.

   AB, A'B'
   BC, B'C'
   AC, A'C'



c) Compare $\frac{AB}{A'B'}$ to $\frac{BC}{B'C'}$ to $\frac{AC}{A'C'}$.

d) As an optional point of interest, measure OA, OA', OB, OB', OC, and OC', and compare corresponding ratios.

e) Measure and compare the angles of the triangles ABC and A'B'C'.

7. Now repeat steps 4 through 6, using a different triangular shape. What do your measurements tell you?

8. Construct another string model for some polygonal shape besides a triangular one. Repeat the experiment using the polygonal shape, and relate your conclusions about it and its shadows to those you reached about the triangular shapes and their shadows.

9. Discuss the following questions with your classmates.

a) What general conclusions concerning similarity can you make from your experiences in (1) through (8)?

b) Were the corresponding angles of your similar triangular shapes equal? If the angles of two triangles are equal, must the triangles be similar? What about the same questions for two quadrilaterals?

c) When will a shape and its <u>sunlight</u> shadow be similar?

10. The following statements involve everyday applications of similarity. You will be asked to explain each statement. You may first want to look over the following list of review facts about proportions. Some of them will be helpful in explaining the statements.

if $\frac{a}{b} = \frac{c}{d}$, then $ad = cb$.

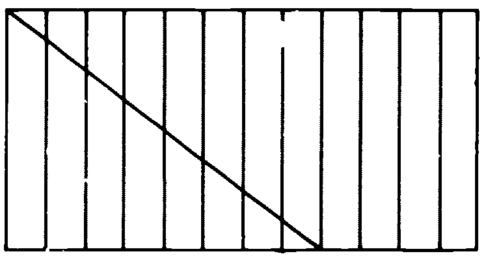
if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

if $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a + b}{b} = \frac{c + d}{d}$.

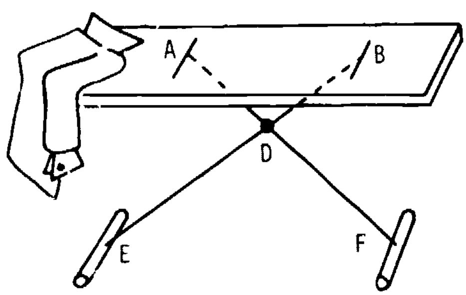
Explain why:

a) If you lay a rod across window bars, the bars subdivide the rod into equal distances. Why?

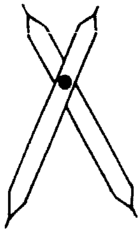


b) How could you use a piece of ruled notebook paper to mark off 11 equally spaced marks on the edge of a 3 x 5 card?

c) What makes your ironing board, when adjusted, always parallel to the floor.

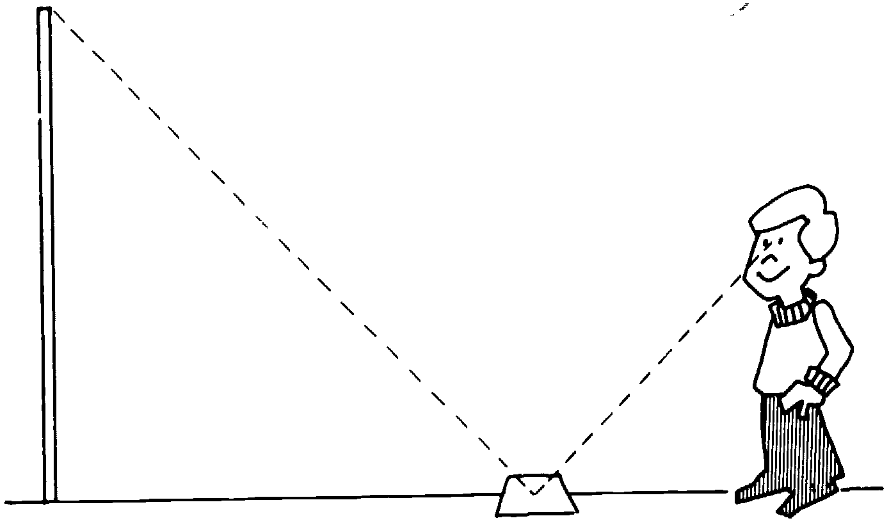


d) A proportional divider is an instrument used in drafting, that reduces or enlarges the scale from one drawing to another. How does it work? What geometric concepts are used?



e) The shadow of a 20-foot-high tree in front of a house is 30 feet long. The shadow of the house is 60 feet long. How high is the house?

f) To determine the height of a pole you can place a mirror on the floor and keep moving back until you see the tip of the pole in the mirror. How does this procedure work? Explain.



(10 (a), (c), (d), and (f) are adapted from Keedy, Marvin L., et al. Exploring Geometry, New York; Holt, Rinehart, and Winston, Inc., 1967, pages 309-314.)

Section III

TOPOLOGICAL TRANSFORMATIONS

The terms closed and inside are among the most basic descriptive terms that one can apply to objects. This is true in the sense that the most "radical" changes that are usually studied in transformational geometry do not change the properties of a curve being closed or of an object being inside a closed curve. In this section we will study topological transformations, which are those space experiences which involve any changes in size or shape of objects except those that involve tearing (breaking) or compressing separate points into the same point.

In Section I you used the physical movement of objects to explore the effects of rigid transformations. Such transformations preserved the size and shape of objects and allowed changes only in position and orientation. Projective transformations, on the other hand, were embodied in the casting of shadows of objects. In Section II you found that while size and sometimes even shape could change under projection, they could do so only in a very orderly way, so that, for example, straightness and certain important ratios were preserved. In Section III, you will learn that topological transformations change size and shape. Moreover, no such shapelike properties as straightness, number of sides, or ratio are preserved. In Activity 12 you will experiment with such embodiments as the stretching of balloons or rubber sheets and the molding of clay, to discover what properties are left unchanged (invariant) by topological transformations. Activity 13 investigates those invariants that are discussed most in the elementary school. In Activity 14 the Euler relationship for graphs is studied, as being an invariant of classical mathematical significance. Finally, Activity 15 will be a seminar on transformational invariants. (Note: You may want to review the portion of the overview on pages 10-11 that relate to topological transformations before starting this section.)

MAJOR QUESTIONS

1. "There are more topological transformations than there are projective or rigid transformations, and there are fewer rigid transformations than there are projective or topological." Explain why this statement is true.
2. Outline four or five activities that you might do with first- or second-graders to help them grasp the important topological invariants: closed, simple and inside.
3. List as many topological invariants as you can think of and try to give an everyday implication of each; e.g., inside--the egg must break in order for the chick to be born.

ACTIVITY 12

EXPLORING TOPOLOGICAL TRANSFORMATIONS

FOCUS:

Topological transformations can effect radical changes in objects. Here you will have an opportunity to explore those properties of an object that are changed, as well as those that are not changed, by a topological transformation.

MATERIALS:

Balloons, rubber sheet, or "stretchy" material; also clay, dough, or some material that can be molded.

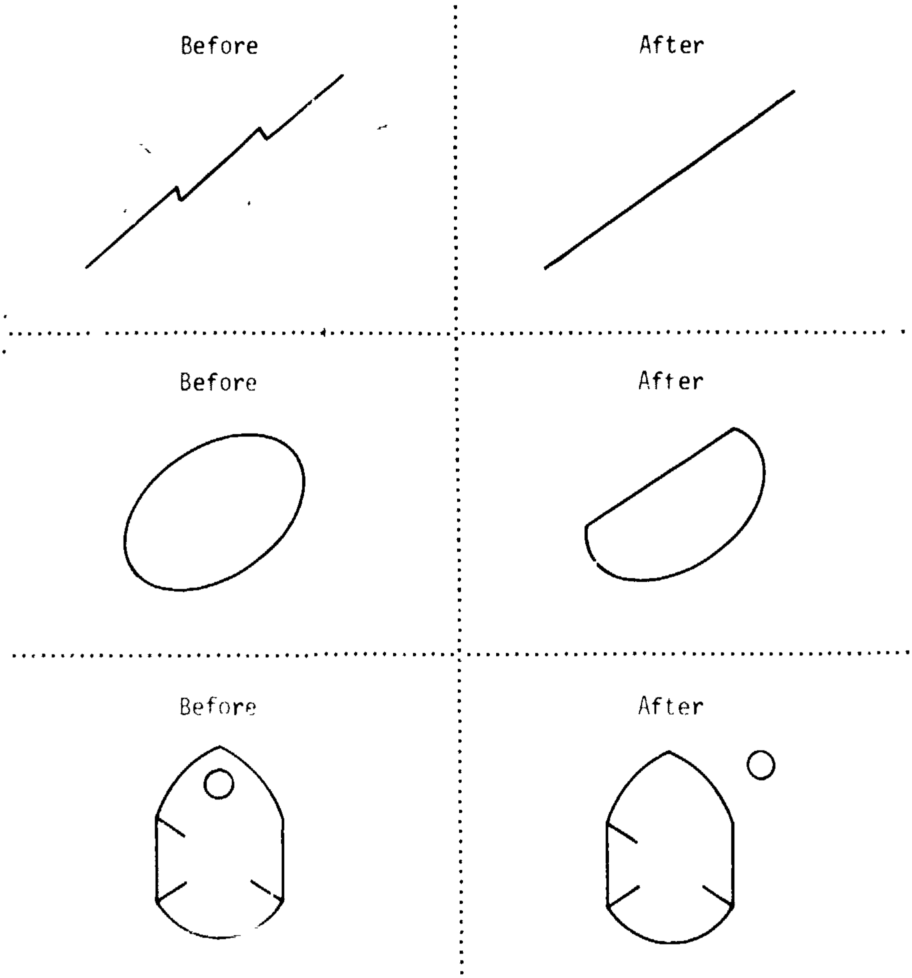
DISCUSSION

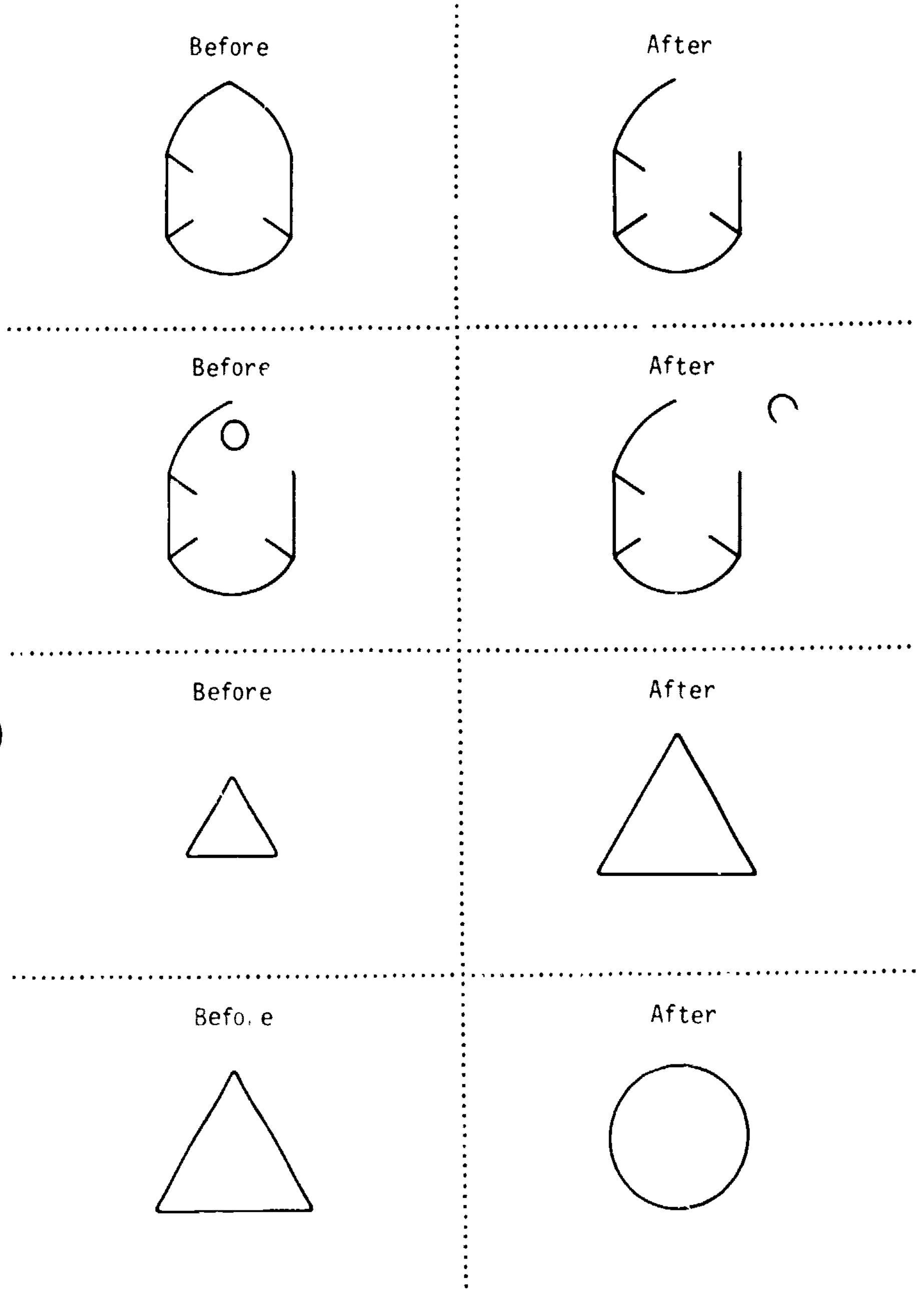
Topological transformations can be modeled by stretching or twisting without tearing, and by molding. In this activity you can explore two-dimensional topological transformations by stretching and twisting shapes drawn on a balloon, rubber sheet, or other piece of "stretchy" material. Three-dimensional topological transformations can be explored by molding clay, dough, or some similar material.

DIRECTIONS:

1. Explore two-dimensional topological transformations. Do this by drawing shapes on an elastic two-dimensional surface and seeing how the shape can be transformed by stretching and twisting. Several people's hands stretching at once can produce interesting effects. One can also model some topological transformations by drawing on a sheet of paper, crumpling the paper, and observing the effects on the drawing.
2. Similarly explore three-dimensional topological transformations using some moldable substance.

3. Each of the pictures on this and the following pages represents an object before and after a transformation has been applied to it. Try to reconstruct the transformation with your two-dimensional or three-dimensional material and indicate which of the pictured transformations are <u>topological</u>. In the case of the nontopological transformations, indicate whether a <u>tearing</u> or a <u>compressing</u> together of separate points causes the transformation to be nontopological.

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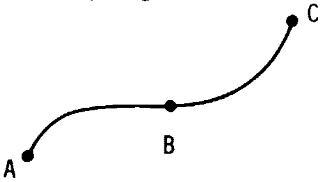

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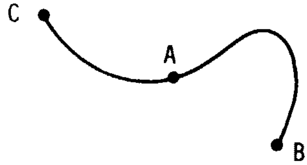
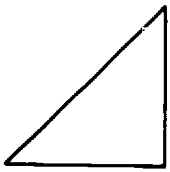


Before



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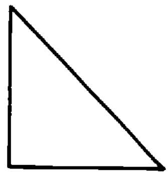

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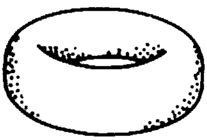

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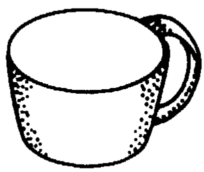

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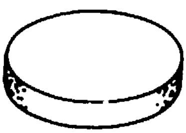

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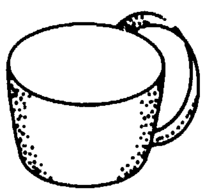


4. If one object can be changed into another by a topological transformation, the objects are called <u>topologically equivalent</u>. List all of the letters of the alphabet that are topologically equivalent to the letter "C."

   Do the same for "O."

5. Which of the following are topologically equivalent to a solid ball?

sausage	flower vase	cheddar cheese
doughnut	loaf of bread	pipe
supper plate	Swiss cheese	shirt

6. Below is a list of properties of objects. Circle those that are not changed by topological transformations (i.e., are invariant).

straightness	consisting of 10 points
closedness	having one hole
ratio of corresponding sides	being close together
being a polygon	being two feet long
being inside a closed shapc	being in a certain order on a curve
being round	

ACTIVITY 13

CERTAIN IMPORTANT TOPOLOGICAL INVARIANTS

FOCUS:

The property of being a simple closed curve and the property of being inside or outside such a curve are topological invariants. These properties are explored here; and, in particular, you will devise a scheme for determining whether a point is inside or outside a simple closed curve.

MATERIALS:

Current elementary mathematics textbook series, scissors, construction paper, ruler (optional).

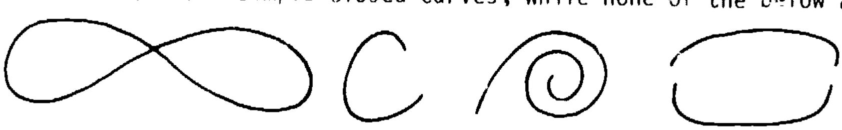
DISCUSSION:

Recall that two objects are <u>topologically equivalent</u> if one can be transformed into the other by a topological transformation.

Definition: Any curve that is topologically equivalent to a circle is called a <u>simple closed curve</u>.

EXAMPLES

The above are all simple closed curves, while none of the below are:

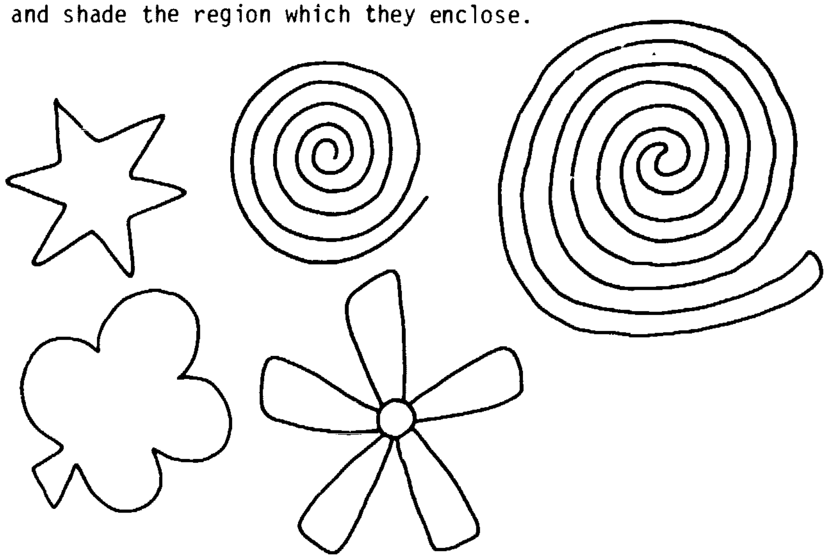


The first is closed, but not simple (it crosses itself). The rest fail to be closed, but are simple. Can you see why none of these is topologically equivalent to a circle?

To transform a circle into any of the latter shapes, one would either have to break the circle or bring separate points on the circle together. Note that simple closed curves are called simple because the boundaries do not intersect themselves, and closed because when you trace the boundary you end up where you started without tracing any path twice.

DIRECTIONS:

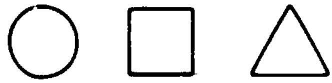
1. Indicate which of the following curves are simple closed curves, and shade the region which they enclose.



2. The concept of topological equivalence might be hard for a child to grasp (even to pronounce!). One could introduce children to simple closed curves just by giving them examples. Or one could give them a piece of string tied in a loop and use that as a part of an activity.

   a) Discuss with your classmates how you would present the concept of simple closed curves to children. Can you think of an easy description of a simple closed curve that children could understand? Can you think of ways to relate simple closed curves to their everyday experiences? How about a story involving simple closed curves?

b) Discuss in what ways a child could consider these three shapes as being the same.



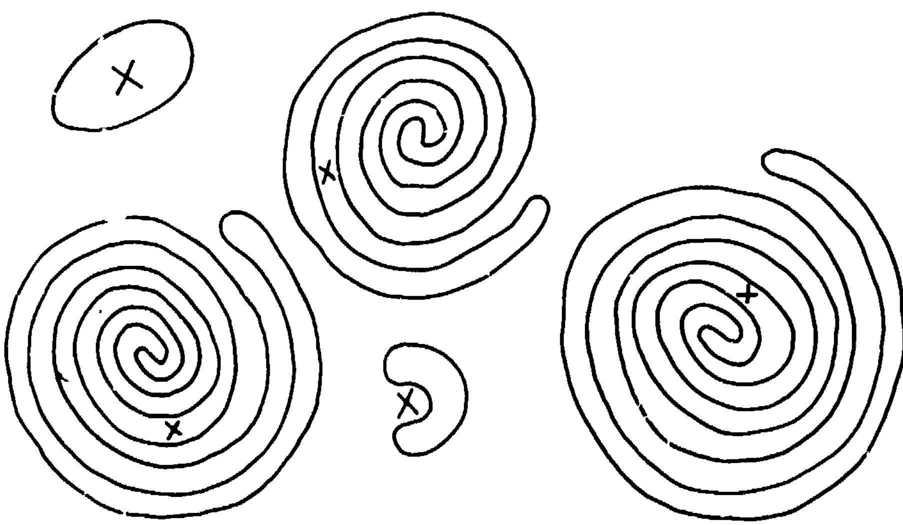
3. Look through a current elementary mathematics text series and find where "open," "closed," "inside," and "outside" are introduced.

   a) Is there any way that you could take advantage of what you have learned in this unit to do a better job of presenting the concepts behind these terms?

   b) Did you encounter other instances of topological concepts being introduced in the texts?

> <u>Jordan Curve Theorem</u>: A simple closed curve separates the plane into two regions, one that is bounded (the inside) and one that is unbounded (the outside).

4. In each of the following cases determine whether the x is inside or outside the curve.



5. Determine a simple way to tell whether a point is inside or outside a simple closed curve.

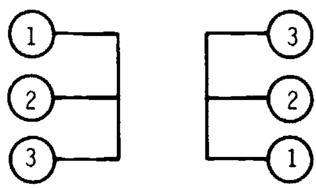
   With several simple closed curves and several points inside and outside the simple closed curves, follow the following steps:

   a) Draw a straight line from the point to the edge of the page.

   b) Look for a relationship between the number of intersections of the straight line with the simple closed curve and whether the point is inside or outside the curve.

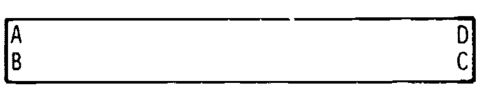
6. <u>The Caliph and His Daughter's Boy Friends</u>. To select a husband for his daughter the caliph declared that anyone who will solve the following puzzle will be able to marry his daughter. (Heaven knows what would have happened if a woman solved the puzzle!)

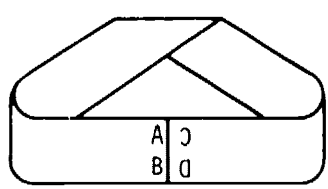
   The puzzle was to connect like numbers in the figure below with lines that do not intersect each other or any other line.



   It is said that the caliph's daughter died an old maid. What do you think? Can you use the Jordan Curve Theorem and its consequences to explain your conclusion?

7. *OPTIONAL: Can the Möbius strip solve the caliph's puzzle? Cut a piece of paper about 1 inch wide and 12 inches long. Label it as shown on the left below. Give the strip half a twist, and tape the ends together as in the righthand figure.*





a) If the caliph's puzzle were drawn on this strip instead of on plane paper, could the puzzle be solved? Use colored pencils to help you find out.

b) Fun with the Möbius strip.

(1) A cylinder has two surfaces, an inside surface and an outside one, so if you want to paint each surface of a cylinder with a different color you will need two colors.

How many do you need for the Möbius strip?

(2) Cut the Möbius strip lengthwise along a line in the center of the strip. What happens?

(3) Cut the Möbius strip lengthwise along a line one third of the way in from the edge. What do you get?

ACTIVITY 14

GAMES, GRAPHS, AND EULER

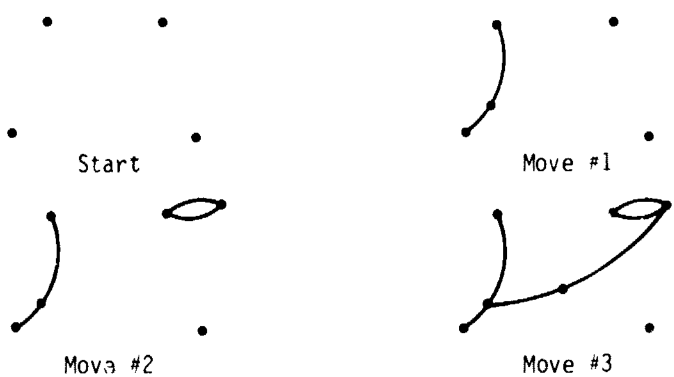
FOCUS:

A game that children might enjoy will be used to generate some graphs. An analysis of these graphs should give rise to the Euler (pronounced "oiler") formula.

DIRECTIONS:

1. Play the following game (two or three times) in pairs.
   a) Put four dots on a sheet of paper.
   b) Take turns drawing arcs (curves) according to these rules.
      i) Each arc must join two dots or join one dot to itself.
      ii) No arc may cross itself, cross another arc, or pass through a dot that is already drawn.
      iii) When an arc is drawn, a new dot must be placed somewhere on it.
      iv) Each dot must have no more than three arcs joining it.
   c) The winner is the player who draws the last possible "legal" arc.

EXAMPLE



2. Answer the following questions about the game.

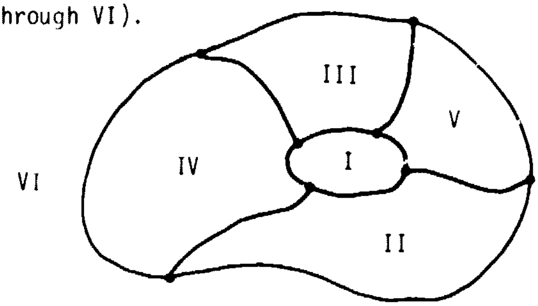
   a) When the game starts, each of the four dots may have three arcs meet there; i.e., there are 12 "arc meetings" possible. What happens to the number of arc meetings possible after each move?

   b) Does there always have to be a winner? Why or why not? Is the outcome of the game affected by who starts?

   c) Try the game with three dots instead of four. What difference does this change make?

   d) Try the game with direction (b) iv replaced by "Each dot must have at least three arcs joining it." What difference does this change make?

3. Graphs are composed of dots and arcs, and they divide the plane into regions. For example, the following graph has 12 arcs and 8 dots, and divides the plane into six regions (numbered I through VI).



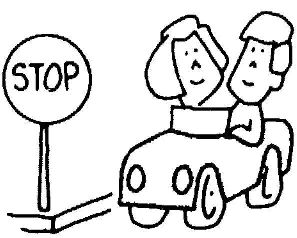
Using the graphs from your completed games in (1), fill in the following table.

Graph #	A # of Arcs	D # of Dots	R # of Regions	(D + R) - A
1				
2				
3				
4				
⋮				

Write down the formula suggested by this table.

4. Are each of D, R, and A topological invariants of a graph?

TEACHER TEASER



Sam and Sally were traveling in Geometryland. They had just passed through Rigidburg when they came to another town. Sam said, "I wonder whether this is Topologytown or Projectionville." Sally said, "I don't know. Let's see if we can figure it out." Just then Sam ran a stop sign. Sally said, "Watch out. Didn't you see the sign?" Sam said, "Yes, but it's round. I'm used to regular octagons. What's more, I was preoccupied by all of the windows in the town. They are different shapes Some are circles, some are triangles, and some are rectangles." "Aha," said Sally, "I know what town this is." What town was it? How did Sally know?

ACTIVITY 15

SEMINAR ON TRANSFORMATIONAL INVARIANTS

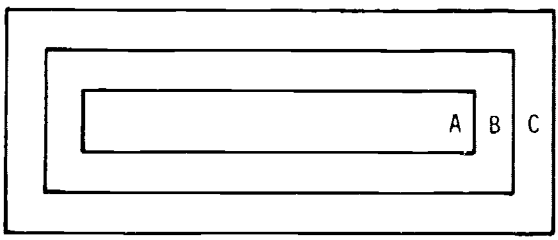
FOCUS:

In Sections I, II, and III of this unit you have investigated rigid, projective, and topological transformations. One feature of the investigation of each type of transformation has been a listing of the invariants (properties left unchanged) of that transformation. This seminar will review these invariants and will investigate applications of the concept of invariants to the elementary classroom.

DIRECTIONS:

1. As a class, list the invariants for the three different kinds of transformations. (Be careful with projective transformations.)

2.



If each of the rectangles A, B, and C contains all of the invariants of transformations of a particular kind, i.e., projective, topological, and rigid, write the appropriate type of transformation next to each of the letters below to indicate which kind of invariants would be in which rectangle.

A ______________________________

B ______________________________

C ______________________________

3. "It is important for kids to know that some things happen all of the time, some never happen, and some happen in very special circumstances." Use this statement as a point of departure for a discussion of the role of transformational invariants in the elementary school.

   - Would you have kids learn them by heart?
   - Would you ignore them altogether?
   - Would you play "What could happen if..." games?

4. Describe the sequence of transformations that steel undergoes from the time that it is molten in the steel mill until the time that it drives away as a new car.

   - Do problems like this hold any promise for the elementary classroom?
   - Can you think of other such problems for children?

REQUIRED MATERIALS

ACTIVITY	MANIPULATIVE AIDS	SUPPLIES	AUDIO-VISUAL AND OTHER
Overview	Balloon, projector or penlight.	Construction paper, scissors, tracing paper.	Slide-tape, "Overview of Transformational Geometry," cassette recorder, projector. (Optional)
1	MIRA. (Optional)	Ruler, graph paper, tracing paper, protractor, compass.	
2		Graph paper, string, scissors, construction paper, protractor, compass.	
3		Graph paper, ruler	
4	MIRA. (Optional)	Scissors, ruler, construction paper, tracing paper.	Elementary mathematics textbook series; references on page 57 (optional).
5	Rectangular prism.		
6		Scissors, construction paper, ruler.	
7	Geoblocks, geoboards, MIRA materials, mirror cards.	Various kinds of graph paper.	References on pages 82 and 83 that are available.

ACTIVITY	MANIPULATIVE AIDS	SUPPLIES	AUDIO-VISUAL AND OTHER
8			Several current and some older elementary math text series.
9	Penlight, sunlight or slide projector, cut-cut shapes on page 91.		
10	Penlight, sunli[illegible] slide projector	Construction paper, scissors, brown wrapping paper or newsprint, ruler.	
11		Ten 5-foot pieces of string for each group, cardboard, scissors, yardstick, wrapping paper, Scotch tape, pro-[illegible]tor.	
1[illegible]	Balloons, rubber sheet or "stretchy" material: [illegible], dough, or some material that can be molded		
1[illegible]		Scissors, construction paper, ruler (Optional)	Current elementary math textbook series.

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Continued from inside front cover

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* * * * * * * * * *

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Addition and Subtraction, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Number Theory, Probability and Statistics,* and *Experiences in Problem Solving.*

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