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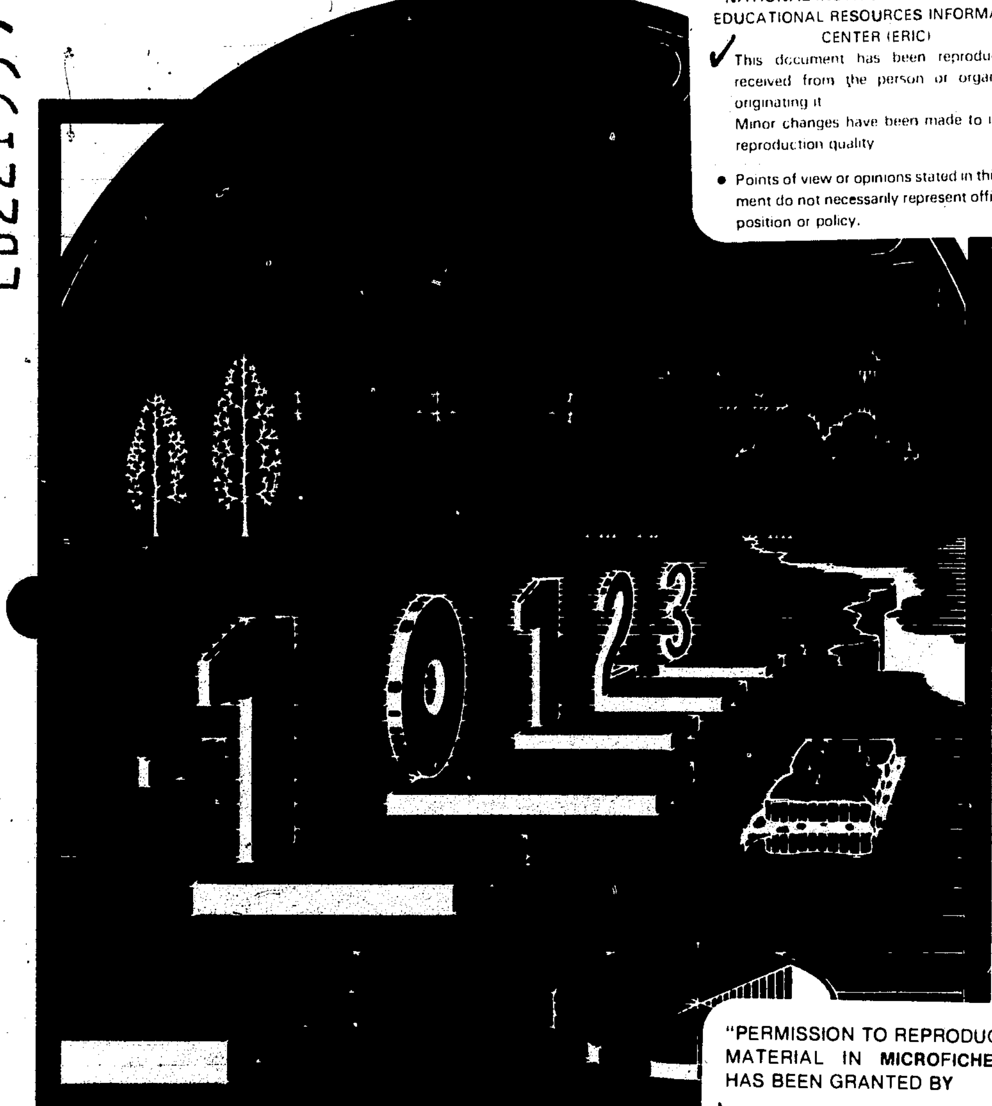
ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to the basic number work in the elementary school. In addition to an introduction to the unit, the text has sections on integers in the elementary school, rational numbers in the elementary school, mathematics of the rational numbers, rational numbers as decimals, the real number system, and appendices that cover self-test answers, the properties of number systems, and skill builder exercises. (MP)

RATIONAL NUMBERS
with Integers and Reals



Mathematics-Methods Program
units written under the direction of

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MATHEMATICS
EDUCATION DEVELOPMENT CENTER

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RATIONAL NUMBERS WITH INTEGERS AND REALS

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

<u>Numeration</u>
<u>Addition and Subtraction</u>
<u>Multiplication and Division</u>
<u>Rational Numbers with Integers and Reals</u>
<u>Awareness Geometry</u>
<u>Transformational Geometry</u>
<u>Analysis of Shapes</u>
<u>Measurement</u>
<u>Number Theory</u>
<u>Probability and Statistics</u>
<u>Graphs: the Picturing of Information</u>
<u>Experiences in Problem Solving</u>

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE
UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugene, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER
FALLS

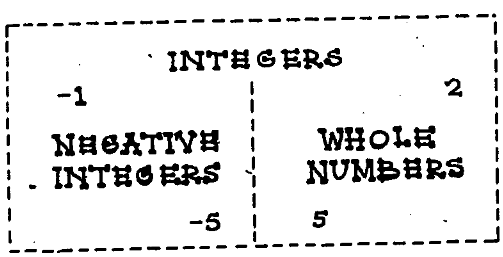
UNIVERSITY OF WISCONSIN/STEVENS
POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

CONTENTS

INTRODUCTION TO RATIONAL NUMBERS WITH INTEGERS AND REALS

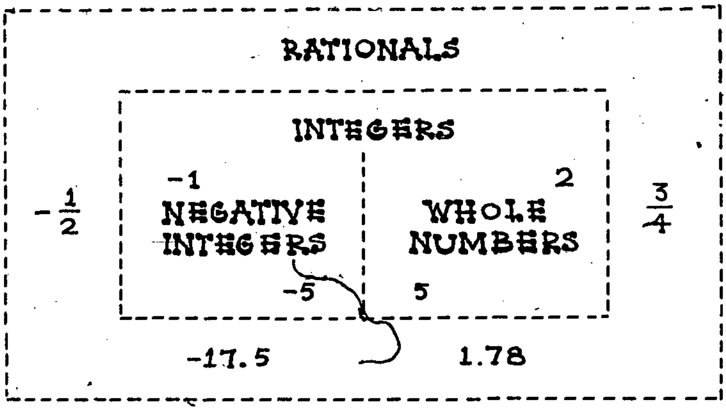
In this unit you will study three sets of numbers. You are familiar with these numbers either from your everyday experiences or from your study of mathematics in elementary school and high school. In Section I you will study the set of numbers called the <u>integers</u>. The set of integers is comprised of ..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ... You should recognize the nonnegative integers as the whole numbers.



Sections II, III and IV focus on the study of <u>rational numbers</u>. The set of rational numbers includes numbers such as the following:

$$-3\frac{1}{2},\ -\frac{2}{1},\ -0.5,\ 0,\ \frac{1}{3},\ 1\frac{1}{2},\ \frac{8}{3},\ 3.72$$

As you can see in the diagram on the next page, the set of rational numbers includes the set of integers, which in turn includes the set of whole numbers. Rational numbers can be expressed in the form of fractions ($\frac{1}{2}$) or decimals (0.5). In Section II you will study rational numbers expressed as fractions.



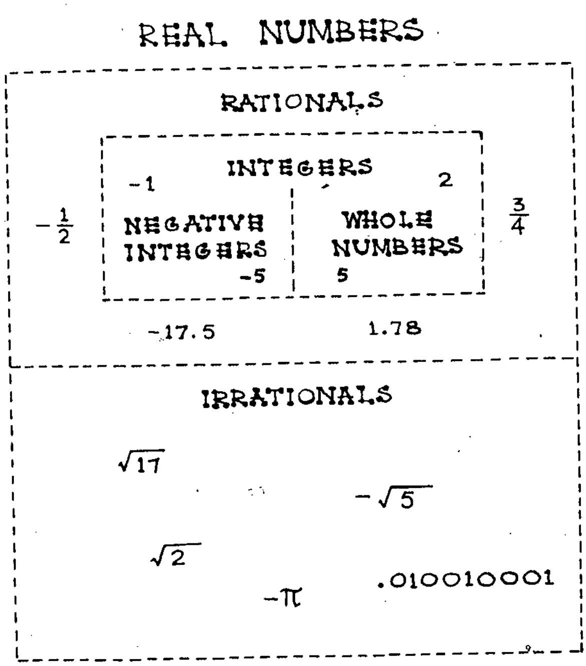
Section II will also introduce some pedagogical procedures for developing rational numbers with elementary school children. Section III develops some of the interesting and important mathematical ideas associated with rational numbers, and Section IV presents rational numbers expressed as decimals.

You are probably least familiar with the set of real numbers. Sections V presents a brief introduction to the set of real numbers. This set of numbers includes such numbers as $-\sqrt{5}$, -2, 0, $\sqrt{2}$, π, $6\frac{1}{4}$. Note that the set of real numbers includes the set of rationals, which in turn includes the set of integers, which in turn includes the set of whole numbers. The real numbers also include the irrational numbers. Irrational numbers are numbers that cannot be expressed in the form of $\frac{a}{b}$ (where a and b are integers and $b \neq 0$). For example, the number $\sqrt{2}$ is an irrational number.

At the end of the unit is an Appendix entitled "Properties of Number Systems." This appendix reviews and summarizes the formal mathematical properties of the different number systems presented in the unit.

Children are informally aware of numbers others than whole numbers at a very early age. In a physical sense, they recognize and use such rational numbers as one-half and one-third. They also have experiences with integers in playing games in which they go "in the hole," or through experiences with temperatures below zero.

The elementary school curriculum formalizes these experiences with numbers other than whole numbers. The order in which these sets of numbers are studied in the school is different from what would seem to be a mathematically logical order. A logical order based on the mathematical structure would be whole numbers, integers, rational numbers, and then real numbers. The order followed in the standard school curriculum is whole numbers, positive rational numbers, integers, negative rational numbers, and real numbers. (Real numbers are not usually introduced until grades 7 - 8.) Later you will be asked to give reasons why the instructional or pedagogical sequence differs from the logical order.



As with other units of the Mathematics-Methods Program you are encouraged to collect activities from various sources to add to your notes for this unit. These activities can serve as a resource when, as a teacher, you will need to motivate, reinforce, or provide enrichment opportunities for children.

Section I

INTEGERS IN THE ELEMENTARY SCHOOL

In this section you will have a chance to review your understanding and skills with integers. At the same time, some suggestions will be provided for introducing the set of integers and the operations with integers to elementary school children.

Although children have had informal experiences with integers outside of school, there is some difficulty in developing the operations from physical embodiments. This difficulty is more apparent in the cases of developing multiplication and division. For example, in motivating the operations with whole numbers, several embodiments or representations are readily available. 2 x 3 can be embodied by two sets of three objects each, or can be represented pictorially by the array $\begin{smallmatrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\end{smallmatrix}$ with two rows, each containing three objects. How does one motivate -2 x -3? Or -12 ÷ 4? We must rely more on patterns or other mathematical relationships to develop these skills with children.

This section begins with an "Overview and Summary of Rational Numbers with Integers and Reals." Following the overview, the activities focus on the nature of integers and operations with integers.

MAJOR QUESTIONS

1. a) a + b = c (a, b and c integers)
   For what values of a and b will c be negative? positive?

b) a x b = c (a, b and c integers)

For what values of a and b will c be negative? positive?

2. The operations with integers are more difficult to teach to children than are operations with whole numbers. Why are they more difficult? What special steps can a teacher take to help a child understand the operations with integers?

3. Give examples of concepts or techniques that you think would be well presented to children by means of a physical embodiment, a pictorial representation, and a mathematical relationship. For each concept or technique, indicate how you would present it to children.

4. What prrperties hold for the set of integers and not for the set of whole numbers? Give examples.

ACTIVITY 1
OVERVIEW AND SUMMARY OF RATIONAL NUMBERS WITH INTEGERS AND REALS

FOCUS:

This activity will provide an overview and summary of the concepts to be discussed in this unit. Specifically, each of the sets of numbers to be discussed in this unit (the integers, the rational numbers, and the real numbers) will be described. Further, the role of these numbers in the elementary school mathematics program will be outlined. You should not expect to learn new concepts. You should, instead, expect to gain a perspective on the content of this unit and on how it appears in the elementary curriculum.

MATERIALS:

(Optional) The Mathematics Methods Program slide-tape, "Overview of Rational Numbers with Integers and Reals."

DIRECTIONS:

1. Consider the following questions in preparation for doing (2) or (3).

   a) What real-world situations can be used to motivate the integers, rational numbers, and real numbers?

   b) What are the interrelationships between the set of rational numbers, the set of integers, and the set of real numbers?

   c) Which form, fraction or decimal, is more efficient for each fundamental operation with the set of rational numbers?

   d) How do you convert a rational number in fraction form to one in decimal form?

   e) How do you know that irrational numbers exist?

2. View the slide-tape "Overview of Rational Numbers with Integers and Reals," and then briefly discuss each of the questions in (1).

3. Read the following "Rational Numbers with Integers and Reals--An Overview of Their Occurrence in the Elementary School," and then briefly discuss each of the questions in (1).

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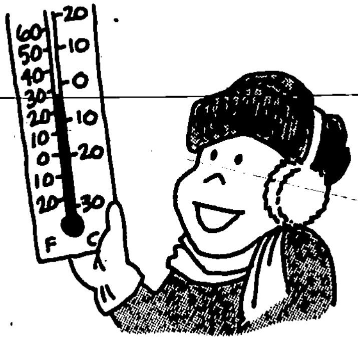
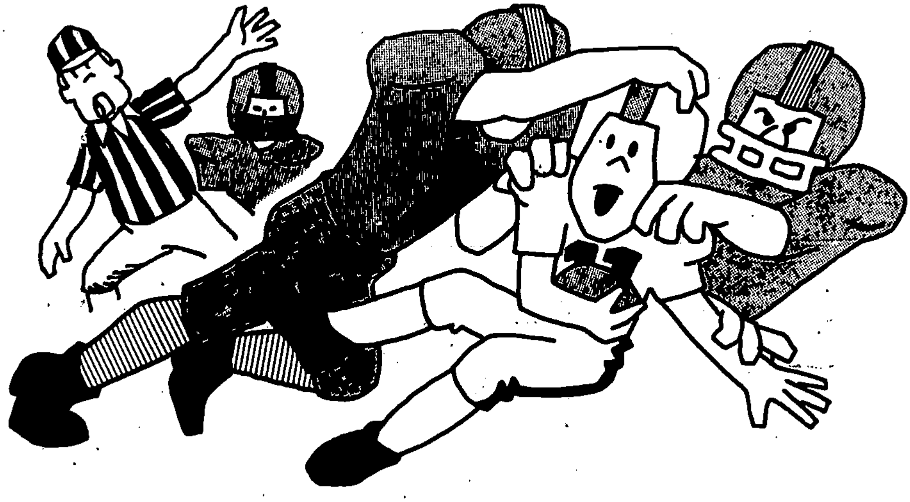
RATIONAL NUMBERS WITH INTEGERS AND REALS--
AN OVERVIEW OF THEIR OCCURRENCE IN THE ELEMENTARY SCHOOL

Basic Terminology

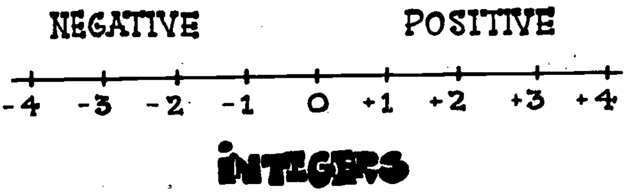


In the elementary school mathematics program, much time and attention are spent developing whole numbers. Children learn to tell "how many," to add, to subtract, to multiply, and to divide, using whole numbers. Whole numbers are used to help children interpret their world. There are, however, real-world situations where the child needs other kinds of numbers to explain a physical situation. One of these is the measurement of temperature on a cold day.

Another situation, a four-yard loss in football, might be represented by the number, called an integer, "negative four" (-4). A gain of yardage might be represented by an integer like "positive three" (+3). Some situations suggest negative integers, others positive integers.



Children in the elementary school learn about integers through informal experiences. Often the integers are introduced using a pictorial representation such as the number line.



The whole numbers, together with the negatives of whole numbers, are called the integers. The study of the integers is more formally treated in junior high school.

Another set of numbers that is introduced in the elementary school is the set of rational numbers. Before continuing with the overview, it may be helpful to quickly review some terminology that is used with the set of rational numbers. The term "rational number" is used to refer to an abstraction, e.g., "halfness," as opposed to the symbols or numerals ($\frac{1}{2}$, $\frac{2}{4}$, 0.5, $\frac{50}{100}$, etc.) used to represent the rational numbers.

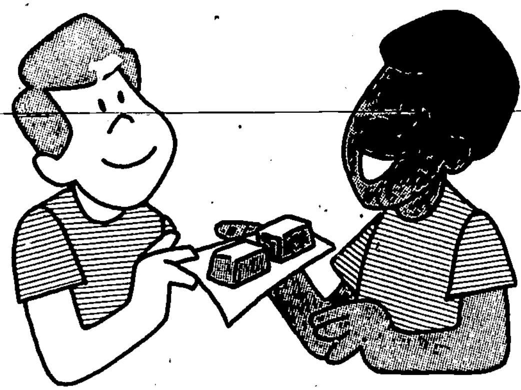
Rational numbers may be expressed in two forms. One form is the fraction form. Thus, $-5\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and $2\frac{1}{3}$ are all fractions that represent rational numbers. Another form of representing rational numbers is the decimal form. Thus, the same rational numbers given above can be expressed as -5.25, 0.5, 0.75, $2.3\overline{3}$. As you study the overview, these various terms, "rational numbers," "fractions," and "decimals," will be used with the meanings just described.

Introducing the Rational Numbers--The Situations

Children's study of rational numbers begins with informal experiences based on real-world situations. The development of rational numbers receives considerable time and attention in the elementary school. This development includes the following phases:

1. Introducing rational numbers by using appropriate models,

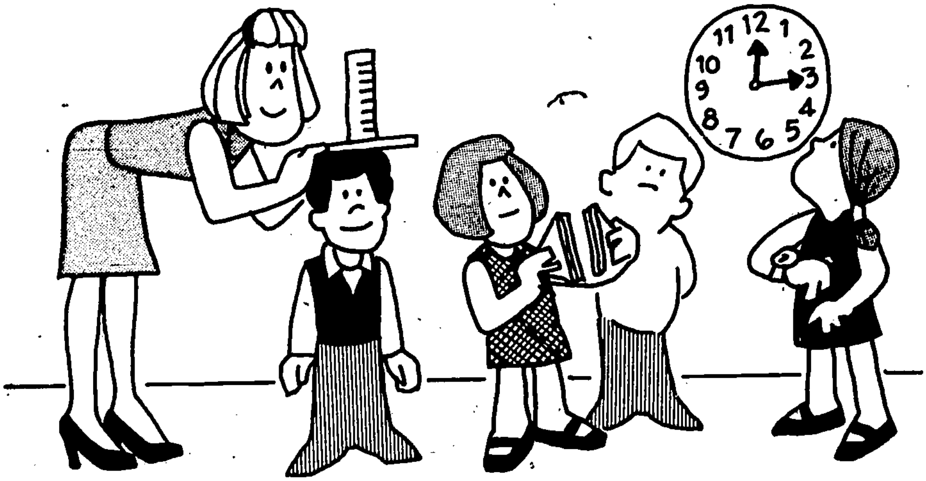
2. Equivalent fractions,
3. Operations with fractions,
4. Operations with decimals.



The introduction of rational numbers is based on the child's experience with fractional parts. For example, suppose a friend invites David to have a piece of his candy bar. The language used to describe his portion should develop naturally from the child's language to mathematical language. In this case, it is possible to use a pair of whole numbers to express the amount of candy David is taking: "one piece of two equal pieces." Another way of expressing the amount that David has taken is to say "He has taken <u>one-half</u> of the candy bar." Later, David will express this amount, one-half, as the fraction $\frac{1}{2}$.

ONE-HALF $\quad \frac{1}{2} \quad$ ONE PIECE OUT OF TWO PIECES

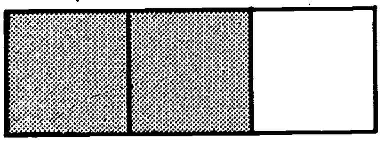
This illustration suggests some other examples of situations in which the elementary school child may have encountered fractions. As with other mathematical ideas, children's study of rational numbers should begin with and develop from experiences that he or she brings to the classroom.



Introducing Rational Numbers--The Models

The introduction of rational numbers should include several models or pictorial representations to assist the child in forming fundamental concepts. One model commonly used is the region. In this example, the child might be asked how much of the total region has been shaded. Although the answer could be given as two out of three equal parts, the standard terminology, two-thirds, can be developed easily.

REGION MODEL

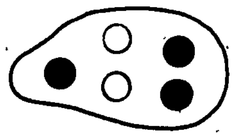


TWO-THIRDS IS SHADED

$\frac{2}{3}$

A second model is the set model. Here the question might be, "What part of the total number of objects is black?" "What fractional part is black?"

SET MODEL

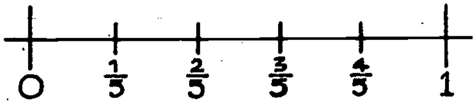


THREE-FIFTHS IS BLACK

$\frac{3}{5}$

A third model used to represent rational numbers is the number line. The number line shown here is commonly used to introduce the child to equivalent fractions, as well as to the operations of addition and subtraction.

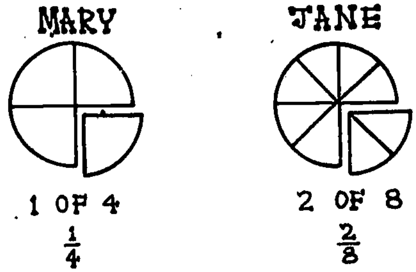
NUMBER LINE



The Concept of Equivalent Fractions

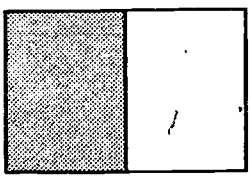
After the child has had some experiences recognizing and finding rational numbers associated with the region, set, or number line models, the development of equivalent fractions is begun.

WHO TOOK MORE?



The concept of equivalent fractions is fundamental to the rest of the development of rational numbers, and this concept should be presented with care. Through careful use of models, the child can see that $\frac{1}{4}$ of the cake is equivalent to $\frac{2}{8}$. Some models have been presented for introducing rational numbers. These same models are used in developing the concept of equivalent fractions--fractions that name the same rational number. A presentation of some methods used with children in developing equivalent fractions follows.

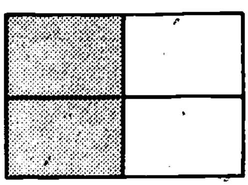
One way of introducing equivalent fractions is to begin by using the region model. To find fractions equivalent to one-half, the child begins by shading one of the two parts shown here.



1 OF 2

$\frac{1}{2}$

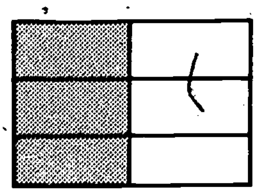
Through paper-folding or dividing the region into four equal parts, the child can become convinced that two-fourths is the same as one-half.



2 OF 4

$\frac{2}{4} = \frac{1}{2}$

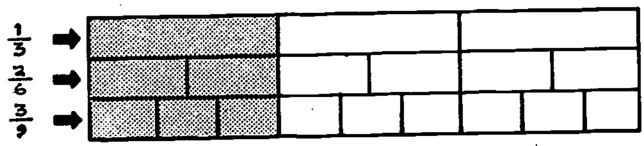
Similarly, the original region could be folded or divided into six equal parts. This model will encourage the child to see that three-sixths is the same as two-fourths, which is the same as one-half. In this way, a child can generate an endless set of equivalent fractions.



3 OF 6

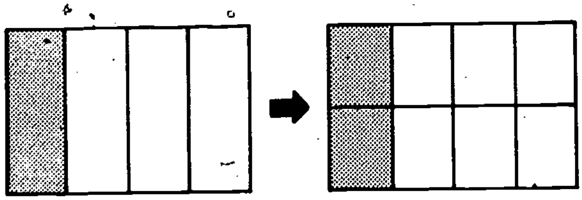
$\frac{3}{6} = \frac{2}{4} = \frac{1}{2}$

Fraction bars, a variation on the number line model, are also used for determining equivalent fractions. To find those fractions equivalent to $\frac{1}{3}$, the child can compare other fractional parts of a whole which are equivalent to $\frac{1}{3}$. Here we can see that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$ and $\frac{3}{9}$.



$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$$

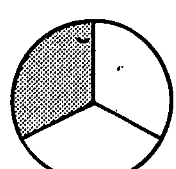
After the informal development, a more formal approach to developing equivalent fractions, using multiplication of whole numbers, can be introduced.



$$\frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$$

The child is encouraged to discover that the way to find an equivalent fraction is to multiply the numerator and denominator by the same number.

A whole set of equivalent fractions can be generated, each of which represents the shaded portion of the circle shown here.

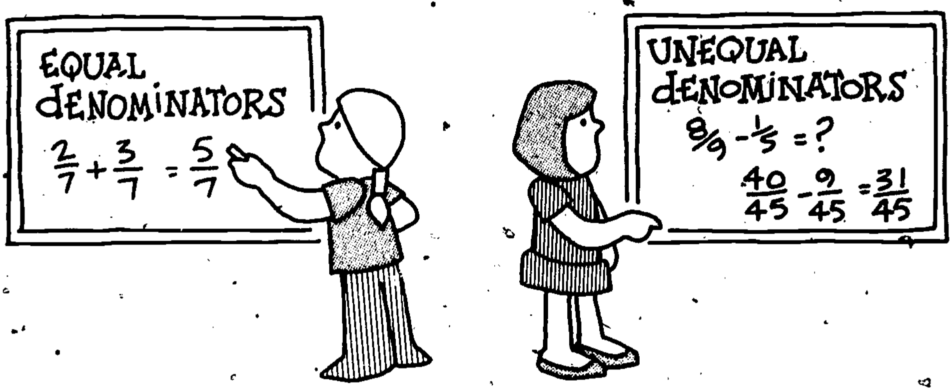


$$\left\{ \frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12} \ldots \right\}$$

Developing the Operations with Fractions

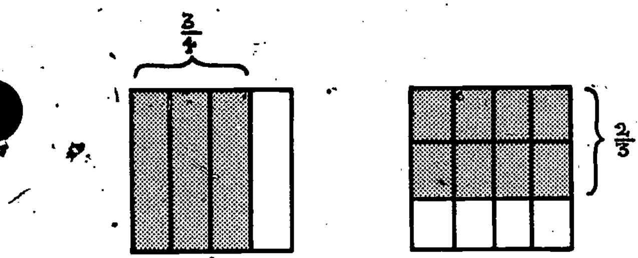
The development of the operations with fractions is based on the models presented in the child's introduction to rational numbers and the study of equivalent fractions. The operations with fractions are presented using physical embodiments or pictorial representations when appropriate.

The development of addition and subtraction of rational numbers begins with examples in which the denominators are equal (lefthand picture). The representation used to aid in the solution of this problem might be the number line, the region, or the set.

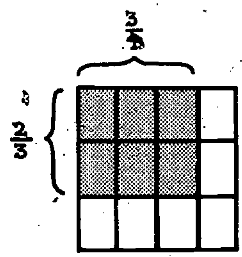


Later, rational numbers with unequal denominators are introduced. In the subtraction problem in the righthand picture, the child found fractions equivalent to $\frac{8}{9}$ and $\frac{1}{5}$ with the same denominator, and then subtracted.

The region model is often used in the development of multiplication of fractions. The example $\frac{2}{3} \times \frac{3}{4}$ is illustrated by first considering the region associated with $\frac{3}{4}$ of the total rectangle. Often the child is encouraged to think of $\frac{2}{3} \times \frac{3}{4}$ as meaning "$\frac{2}{3}$ of $\frac{3}{4}$." Using this interpretation, $\frac{2}{3}$ of the rectangle is shaded. The resulting divisions of the rectangle into thirds and fourths form 12 equal units.

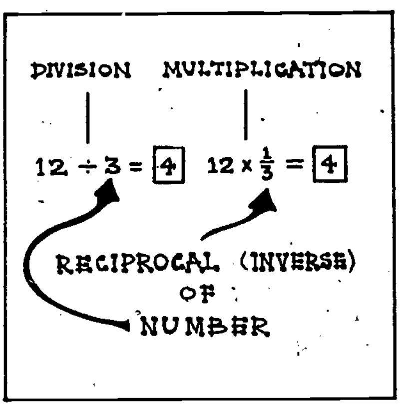


This pictorial representation shows that $\frac{2}{3} \times \frac{3}{4}$ results in having six of the 12 equal units shaded. This representation gives insight into the multiplication example of:



$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$$

Division with fractions presents some difficulty, in that it is not easy to identify a physical embodiment or pictorial representation that seems helpful. Although several approaches can be used, many teachers find that it is best to rely on the inverse relationship between multiplication and division. In writing a division sentence as a multiplication sentence, one uses the reciprocal (or inverse) of the divisor. Thus, since $\frac{1}{3}$ is the reciprocal of 3, the division sentence $12 \div 3 = 4$ can be written as $12 \times \frac{1}{3} = 4$.



• OBSERVE THE PATTERN

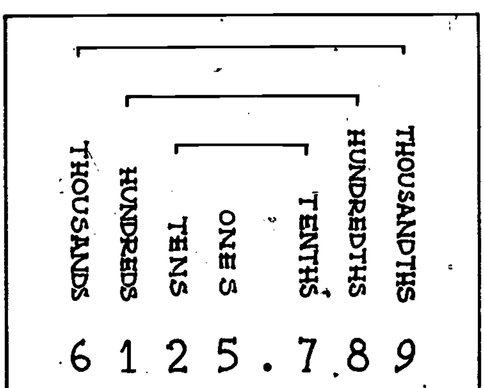
DIVISION		MULTIPLICATION
$12 \div 4 = \square$	$\longleftrightarrow$	$12 \times \frac{1}{4} = \square$
$12 \div 3 = \square$	$\longleftrightarrow$	$12 \times \frac{1}{3} = \square$
$12 \div 2 = \square$	$\longleftrightarrow$	$12 \times \frac{1}{2} = \square$

Children initially become confident of this procedure by using simple examples and observing patterns. Later, when they have gained greater sophistication with arithmetic manipulations, they can verify this procedure using more complicated mathematical arguments.

Rational Numbers Expressed as Decimals

Children's study of rational numbers expressed as decimals follows their study of the operations with fractions. With the wider usage of the metric system, which uses decimal notation, the emphasis on decimal study may increase.

Decimals are used to represent rational numbers. Decimal notation is also consistent with the child's earlier study of place value with whole numbers. The symmetry of the decimal numeration system is studied as the children learn to read and to write decimals.



Skill in writing decimals as fractions, and vice versa, is developed. Some interesting insights into rational numbers can be acquired in this conversion process.

Operations with decimals are based upon understanding of the numeration system and upon operations with whole numbers and with fractions. The operations with decimals are informally begun in the primary grades with situations involving money, but are not formally developed until the intermediate and middle grades.

In the intermediate or middle grades, children discover that there are three types of decimals:

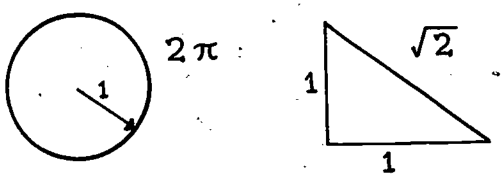
1. Terminating decimals, that is, decimals which have an ending, such as 0.5, 0.75, 0.375, and so on;
2. Repeating decimals, that is, decimals which go on forever, but in which a single digit or sequence of digits keeps repeating. Examples are $0.3\overline{3}$, $0.31\overline{31}$, $0.173\overline{173}$. The bar over the digits indicates the digits that are repeated indefinitely.
3. Nonrepeating decimals, that is, decimals which go on forever, but which are not formed by repeating a single sequence of digits. For example, the decimal 0.1010010001... is not formed by repeating a single sequence of digits.

<u>A Glimpse at the Irrational Numbers</u>

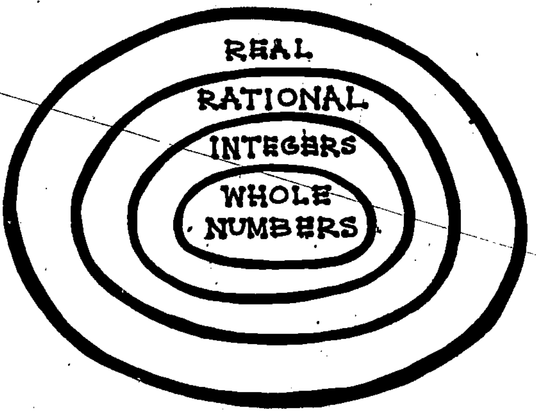
While both terminating and repeating decimals represent rational numbers, decimals which are nonrepeating do <u>not</u> represent rational numbers. These decimals represent a different kind of number called an <u>irrational number</u>.

Children in the elementary school do not formally study the irrational numbers, but they do encounter such irrational numbers as pi (π) and $\sqrt{2}$.

SOME IRRATIONAL NUMBERS



The set of irrational numbers forms part of the set of real numbers. In the diagram below, we see that the real numbers consist of the set of rational numbers and the set of irrational numbers. Further, we are reminded that the integers are contained in the rationals and contain the whole numbers.



This overview has presented the development of numbers as found in the elementary school. In addition to the whole numbers, children study integers and rational numbers. The foundations for the real numbers are also carefully laid for the children's later mathematical study.

ACTIVITY 2

INTRODUCING THREE RESOURCES FOR TEACHERS

FOCUS:

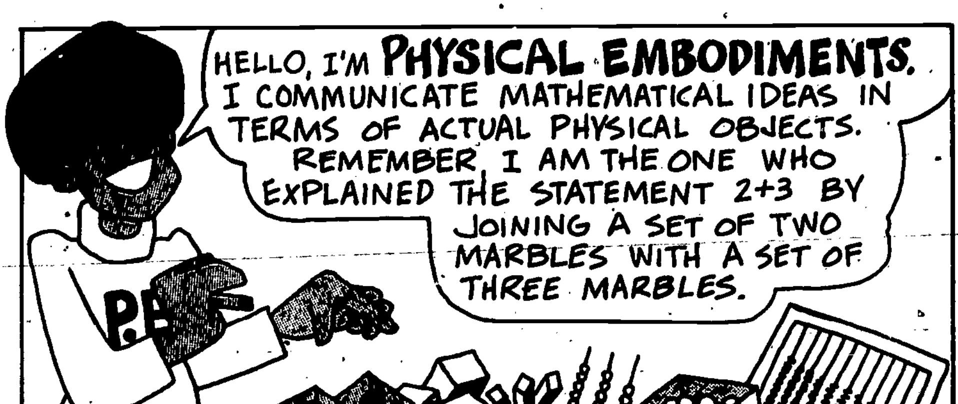
In this activity you will be introduced to three resources that can be helpful in explaining mathematics to children:

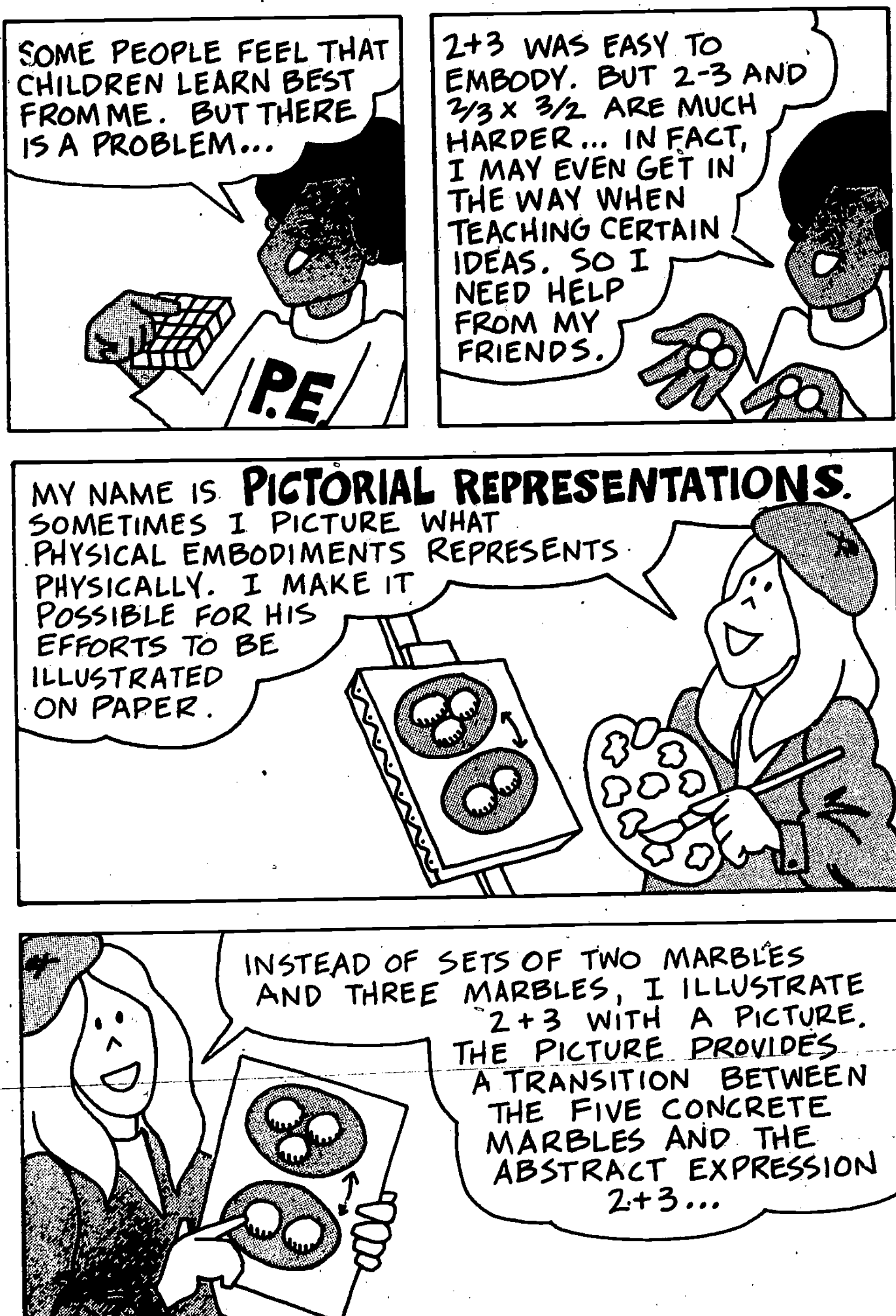
Physical Embodiments,
Pictorial Representations, and
Mathematical Relationships.

Then you will have an opportunity to attempt to explain certain mathematical statements involving integers and rational numbers, in order to gain some insights into the problems that you will face as a teacher. The remainder of the Rational Numbers with Integers and Reals unit will focus on helping you to learn to apply the three resources to these problems.

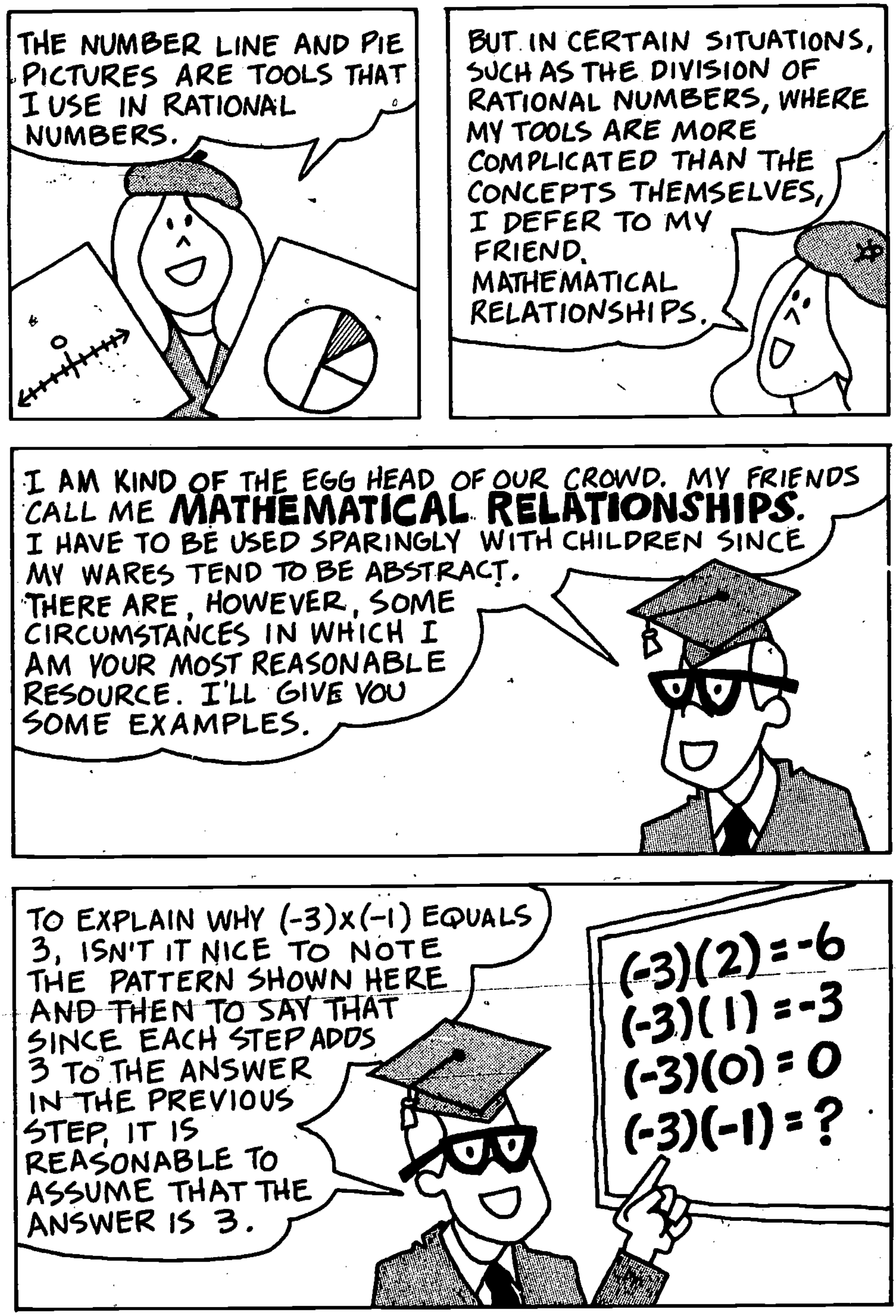
DISCUSSION:

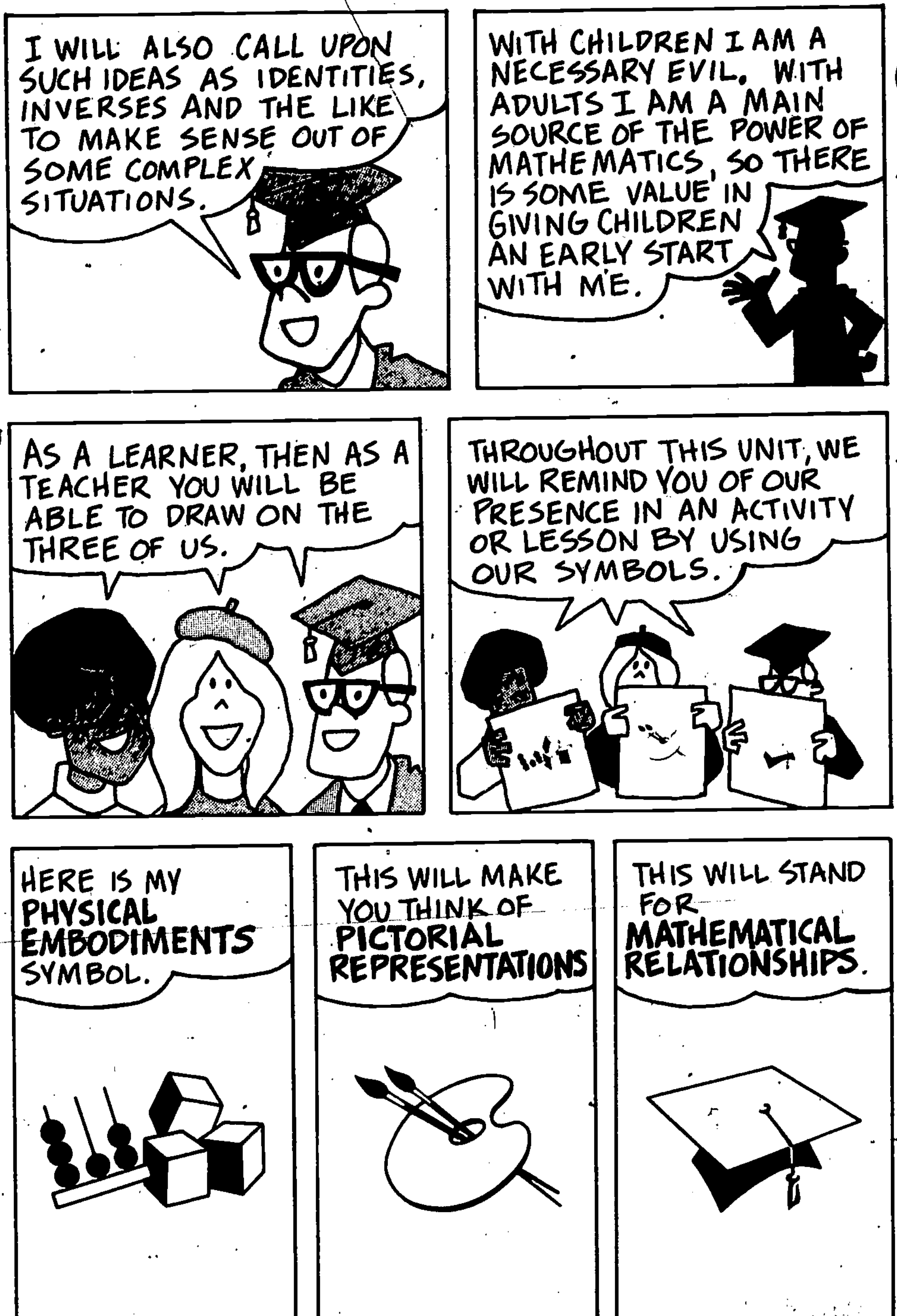
As a teacher, you will need all of the help that you can get in explaining integers and rational numbers to children. Let us introduce you to three resources who will come to your aid in times of need.





SOME PEOPLE FEEL THAT CHILDREN LEARN BEST FROM ME. BUT THERE IS A PROBLEM...
P.E.
2+3 WAS EASY TO EMBODY. BUT 2-3 AND 2/3 x 3/2 ARE MUCH HARDER... IN FACT, I MAY EVEN GET IN THE WAY WHEN TEACHING CERTAIN IDEAS. SO I NEED HELP FROM MY FRIENDS.
MY NAME IS PICTORIAL REPRESENTATIONS. SOMETIMES I PICTURE WHAT PHYSICAL EMBODIMENTS REPRESENTS PHYSICALLY. I MAKE IT POSSIBLE FOR HIS EFFORTS TO BE ILLUSTRATED ON PAPER.
INSTEAD OF SETS OF TWO MARBLES AND THREE MARBLES, I ILLUSTRATE 2+3 WITH A PICTURE. THE PICTURE PROVIDES A TRANSITION BETWEEN THE FIVE CONCRETE MARBLES AND THE ABSTRACT EXPRESSION 2+3...


THE NUMBER LINE AND PIE PICTURES ARE TOOLS THAT I USE IN RATIONAL NUMBERS.
0
BUT IN CERTAIN SITUATIONS, SUCH AS THE DIVISION OF RATIONAL NUMBERS, WHERE MY TOOLS ARE MORE COMPLICATED THAN THE CONCEPTS THEMSELVES, I DEFER TO MY FRIEND, MATHEMATICAL RELATIONSHIPS.
I AM KIND OF THE EGG HEAD OF OUR CROWD. MY FRIENDS CALL ME **MATHEMATICAL RELATIONSHIPS.** I HAVE TO BE USED SPARINGLY WITH CHILDREN SINCE MY WARES TEND TO BE ABSTRACT. THERE ARE, HOWEVER, SOME CIRCUMSTANCES IN WHICH I AM YOUR MOST REASONABLE RESOURCE. I'LL GIVE YOU SOME EXAMPLES.
TO EXPLAIN WHY (-3)x(-1) EQUALS 3, ISN'T IT NICE TO NOTE THE PATTERN SHOWN HERE AND THEN TO SAY THAT SINCE EACH STEP ADDS 3 TO THE ANSWER IN THE PREVIOUS STEP, IT IS REASONABLE TO ASSUME THAT THE ANSWER IS 3.
(-3)(2) = -6
(-3)(1) = -3
(-3)(0) = 0
(-3)(-1) = ?


I WILL ALSO CALL UPON SUCH IDEAS AS IDENTITIES, INVERSES AND THE LIKE TO MAKE SENSE OUT OF SOME COMPLEX SITUATIONS.
WITH CHILDREN I AM A NECESSARY EVIL. WITH ADULTS I AM A MAIN SOURCE OF THE POWER OF MATHEMATICS, SO THERE IS SOME VALUE IN GIVING CHILDREN AN EARLY START WITH ME.
AS A LEARNER, THEN AS A TEACHER YOU WILL BE ABLE TO DRAW ON THE THREE OF US.
THROUGHOUT THIS UNIT, WE WILL REMIND YOU OF OUR PRESENCE IN AN ACTIVITY OR LESSON BY USING OUR SYMBOLS.
HERE IS MY **PHYSICAL EMBODIMENTS** SYMBOL.
THIS WILL MAKE YOU THINK OF **PICTORIAL REPRESENTATIONS**
THIS WILL STAND FOR **MATHEMATICAL RELATIONSHIPS**.

DIRECTIONS:

In the tasks that follow, you may experience some frustration. You should, however, gain insights into some of the problems that our three resource friends face as they attempt to help you teach children.

1. Consider the statement $8 - 5 = 3$. What physical or real-world embodiments of the statement could you give to a young child?
2. Now consider $5 - 8 = -3$. What embodiments or representations could you give a child for this statement? (You may want to work with the number line, or with credits and debits.)
3. Shifting your attention to noninteger rational numbers, consider the following question. How would you relate the symbol $\frac{3}{4}$ to a child's real world?

<u>Comment</u>: We could raise further questions concerning $\frac{1}{2} \div \frac{3}{4}$ or even $-\frac{1}{2} \div \frac{3}{4}$, but our point should already be made.

Throughout the remainder of the unit we will call upon the three resources (Physical Embodiments, Pictorial Representations, and Mathematical Relationships). It may sometimes appear that you are jumping from one mode of explaining to another. This is because we are attempting to provide you with the full power that each of the resources can bring to you to prepare you for your role as a teacher.

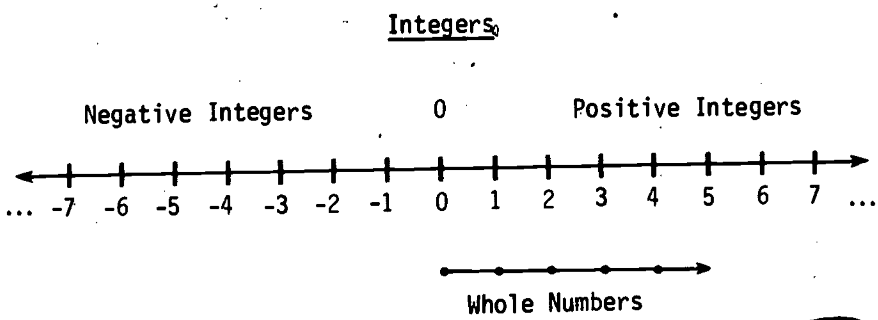
ACTIVITY 3
INTEGERS IN LIFE AND IN SCHOOL

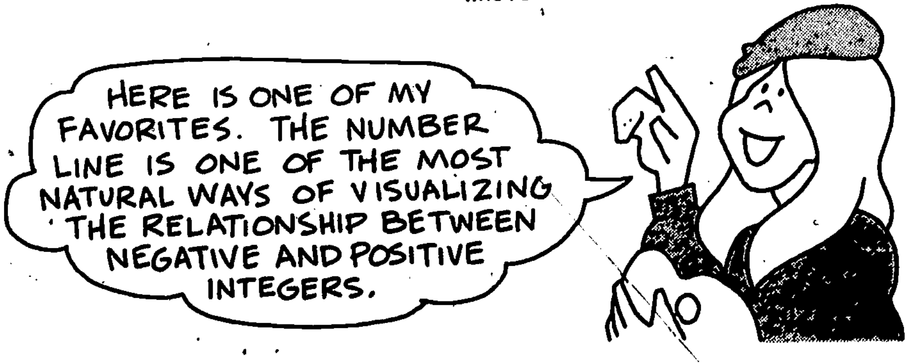
FOCUS:

This activity focuses on instances of integers in the real world, on how integers are related to the whole numbers, and how and when they might be introduced to elementary school children. You should come away with no doubt in your mind as to which numbers are integers.

DISCUSSION:

Even though the initial number work in the elementary school is focused on the whole numbers, children do use negative integers informally in real-life situations. In games, children may develop the feeling that negative integers are the "opposite" of whole numbers. While their informal ideas may not match the illustrations below, very little effort is needed to formalize their experiences gradually.





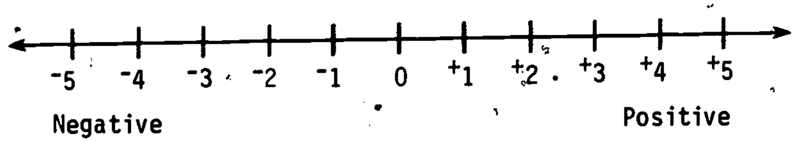
DIRECTIONS:

1. Discuss instances in the English language in which the word "negative" is used and what meaning it has in those instances, e.g., negative attitude.

2. Make a list of examples in which negative integers are used in real-life situations, e.g., temperature.

3. The integers are often introduced in the elementary school by using a raised sign to indicate whether the number is positive or negative, for example, $^{-}7$, $^{+}8$. Even though the positive integers are the same as the whole numbers, on which we did not use a + sign, some mathematics educators feel that the use of raised signs helps children focus on the fact that integers include both positive and negative integers (and zero), and not just the negative integers. The illustration below shows how such an introduction might be pictured.

Integers

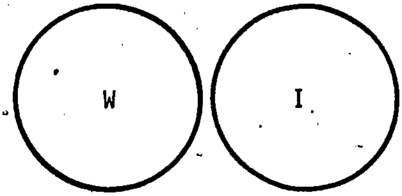


Discuss this approach and cite the possible advantages and disadvantages that placing a raised sign on the nonzero integers might have.

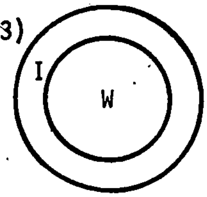
4. A technique commonly employed in the elementary classroom is to illustrate relationships among sets by using pictures called Venn diagrams.

   a) Venn diagrams are used on the following page to illustrate the relationship between the set of integers (I) and the set of whole numbers (W). Choose the Venn diagram that correctly illustrates that relationship.

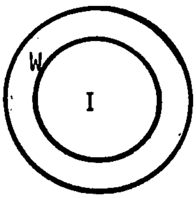
(1)



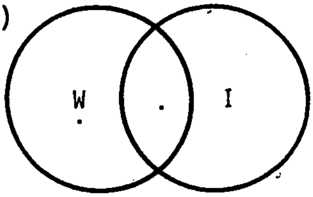
(3)



(2)



(4)



b) Use Venn diagrams to show the relationship between:

(1) The set of even whole numbers (W_e) and the set of whole numbers (W);

(2) The positive integers (I^+) and the nonpositive integers (N);

(3) The multiples of 3, the multiples of 2, and the whole numbers.

5. At what grade level do children begin their study of integers? Before looking in elementary texts, make a reasonable guess. Then refer to some elementary texts to check your guess.

Topic	Guess (Grade)	Textbook (Grade)
Introduction to integers		
Addition of integers		
Subtraction of integers		
Multiplication of integers		
Division of integers		

Why do you think that the study of integers is delayed until later in the elementary school?

6. A sequence is a set of numbers arranged in some order. Find the pattern in each of the following sequences and write the next five terms. Such patterns can be fun, and they help to strengthen familiarity with integers.

a) 2, -2, 0, 0, -2, 2, -4, 4, __, __, __, __, __,

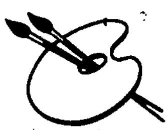
b) 2, 1, -1, -4, -8, -13, __, __, __, __, __,

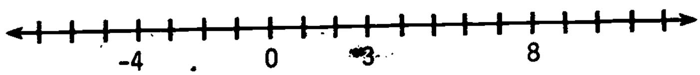
c) -7, -5, -1, 5, 13, 23, 35, __, __, __, __, __,

d) 0, -1, -1, -2, -3, -5, -8, __, __, __, __, __,

Make a sequence of your own and see whether a classmate can find the pattern.

7. There is one very natural order that is imposed on the integers. It extends the "greater than" order on the set of whole numbers. Pictorially, one can use the number line and say that a is greater than b if a is to the right of b on the number line.





a) Locate the following pairs of integers on the number line, and for each pair circle the greater integer.

i) -5 and 2	ii) -8 and 6
iii) 8 and 3	iv) -1 and 1
v) 0 and -6	vi) 0 and 18
vii) -27 and -34	viii) -346 and -340

b) Note that there is a mathematical characterization of the natural order on the whole numbers. That is, we say a is greater than b if a - b is a whole number. Would this characterization work for integers? Try it on the pairs in part (a). If you can't handle the subtraction of negative integers, hang on. Relief will come in Activity 5.

8. The negative integers give rise to a different but interesting ordering. This ordering is often useful in situations where distance is the main concern. To define this order, we need to define the absolute value of an integer. The absolute value of the integer a is defined to be the distance of a from 0. We use the symbol $|a|$. So, for example, $|3| = 3$ and $|-3| = 3$, since both 3 and -3 are at a distance of 3 from 0.

a) Find the absolute value of each of the following integers.

i) -5	(ii) 39	(iii) 422
iv) -23	(v) 422 - 63	(vi) 63 - 422
vii) 0	(viii) 1	(ix) -1

b) Define the "absolutely greater" ordering by saying that a is absolutely greater than b if $|a|$ is greater than $|b|$. So, for example, -3 is absolutely greater than 2 since $|-3|$ is greater than $|2|$. Circle the absolutely greater member of each of the following pairs.

i) 0 and -3	ii) 5 and 2
iii) -5 and -2	iv) -1 and -2
v) -739 and 740	vi) -2 and 2

c) How would you define the concept of "absolutely equal"?

d) For which integers x is it true that $|x - 3|$ is greater than 4?

e) Which of the following are always true (i.e., true for every value of p or q)?

i) $|p| = p$;

ii) $|p| = p$ or $-p$;

iii) $|p - q| = |q - p|$;

iv) $|p| + |q|$ is greater than or equal to $|p + q|$.

ACTIVITY 4

ADDITION AND SUBTRACTION OF INTEGERS

FOCUS:

This activity will provide some techniques for presenting addition and subtraction of integers to children. It can also serve as an opportunity for you to enhance your own understanding and skill with integers. Each of your three friends (Physical Embodiments, Pictorial Representations, and Mathematical Relationships) will come into play.

MATERIALS:

Paper and scissors, spinners and counters or chips.

DISCUSSION:

Although children bring to school many informal real-life experiences with integers, the formal development of the operations, particularly subtraction, is often difficult to model with Physical Embodiments. Consequently, you will find that Pictorial Representations and Mathematical Relationships tend to dominate the instruction in this activity.

Generally, an appropriate instructional plan for teaching addition and subtraction of integers to children might proceed as follows.

A. Provide many informal readiness activities involving addition of integers.

B. Develop the operation of addition.

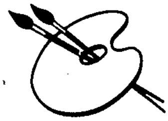
C. Develop the operation of subtraction as the <u>inverse</u> of addition.

The organization of this activity follows the instructional procedure suggested above. A Part D has also been included, which suggests some games that can be helpful in developing integer concepts with children. To gain insight into these games, <u>you</u> should play them.

DIRECTIONS:

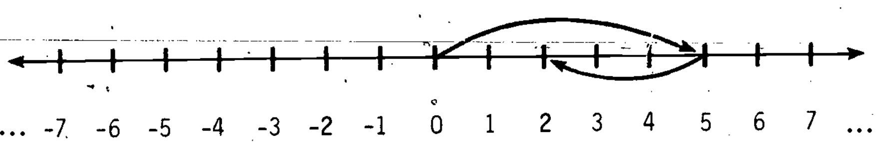
PART A: INFORMAL READINESS ACTIVITIES FOR ADDITION OF INTEGERS

1. The number line provides a good model for informal work with the integers. A positive integer can be represented as a movement to the right, a negative integer as a movement to the left, and addition (+) is accomplished by combining these movements.



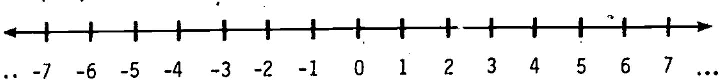
EXAMPLE

$5 + (-3) = 2$ 5 right, 3 left

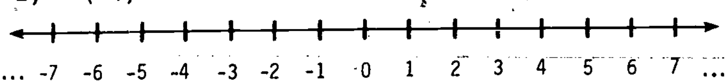


Illustrate the following addition expressions on the number lines provided.

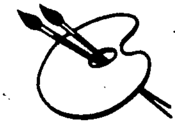
a) $3 + (-4)$



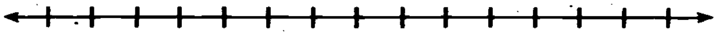
b) $(-2) + (-4)$



2. An unusual rabbit's "hops" could also be used. Starting at home (0), he could hop forward and backward on a number line. Draw the following "hops" and indicate what addition expression is represented by your picture.

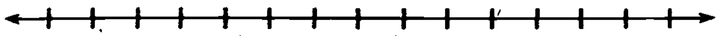


5 forward, 2 backward



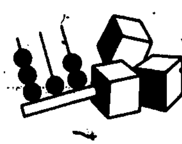
Addition expression:

3 forward, 3 backward



Addition expression:

Children using a number line can act out the "hops"; or, for a less strenuous activity, they could step forward and backward.



3. Some other informal preparatory activities for addition of integers include posing such situations as:

"You have 15 points and lose 20. How many points do you have?"

"The temperature is -5°. It warms up 20°. What is the temperature?"

4. a) Why isn't the physical embodiment of combining sets of objects as useful for addition of integers as it was for whole numbers?

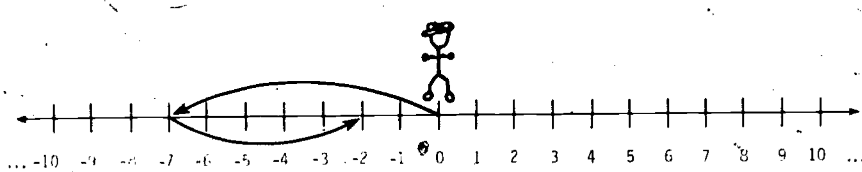
b) Brainstorm some other activities that might provide readiness for addition of integers.

PART B: DEVELOPING ADDITION OF INTEGERS

1. Building on their ability to interpret such statements as 5 + (-3) in terms of movement or hops to the left and right, one can ask children to determine a single integer or answer that is equal to 5 + (-3); for example, 5 + (-3) = 2. Complete the sentence below.

EXAMPLE

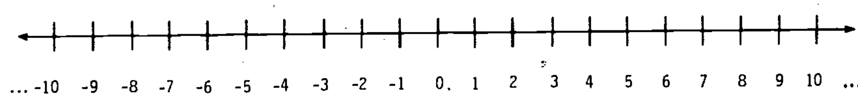
(-7) + 5 =



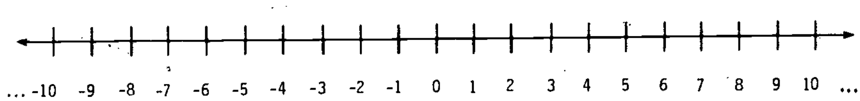
Note: Make sure children are counting units (i.e., hops)--not points--on the number line. Some children tend to count 0 as 1 or -1.

Draw arrows to illustrate the moves of Mr. Stick on the following number lines.

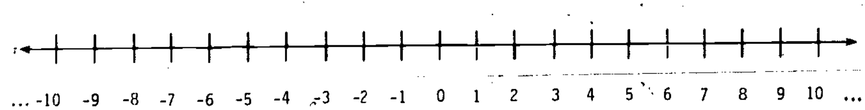
a) 7 + (-5) =



b) (-5) + (-3) =



c) (-6) + 4 =



d) 3 + (-2) =

... -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 ...

e) 4 + 3 =

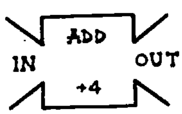
... -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 ...

f) (-5) + 4 =

... -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 ...

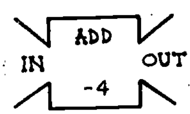
2. Another way in which children can practice addition of integers is using the "function machine." Supply the input and output pairs for the following four function machines.

a) 2 positives



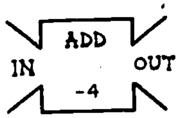
in	out
3	
6	10
8	
10	

b) 2 negatives



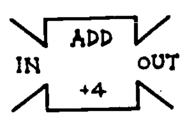
in	out
-3	
-6	
-8	
-10	

c) 1 positive
1 negative

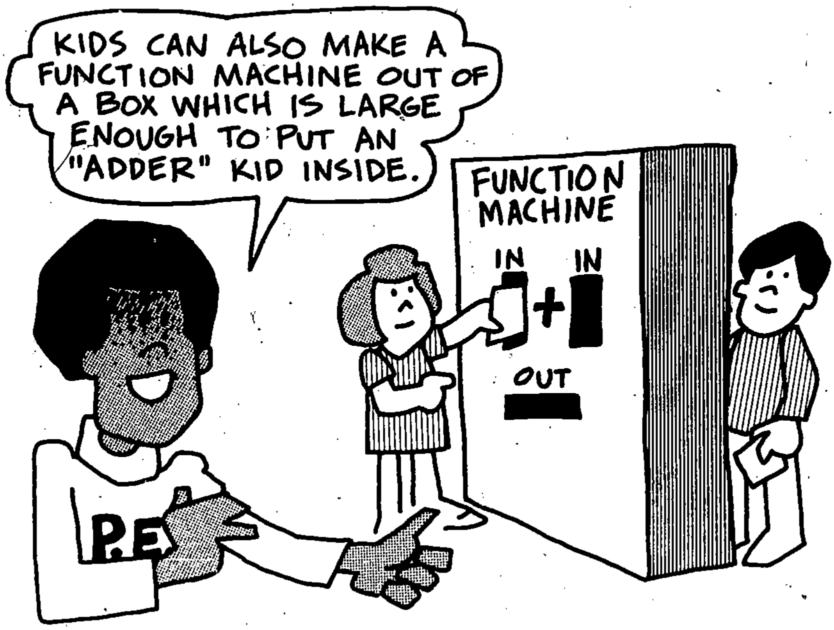


in	out
3	
6	
8	4
10	

d) 1 negative
1 positive



in	out
-3	
-6	
-8	
-10	



PART C: DEVELOPING SUBTRACTION OF INTEGERS

It is important that new learnings for children be built on what they have already learned. In this part, subtraction of integers will be built on the addition of integers and the subtraction of whole numbers. We will rely most heavily on Mathematical Relationships, especially in using pattern finding.

1. The key to our approach is that <u>subtracting an integer</u> is the same as adding its additive inverse*; that is, $a - b = a + (-b)$.

---

*For a definition of <u>additive inverse</u>, see Appendix B.

We justify this fact by observing that it works in certain familiar cases. In the lefthand column below are subtraction problems, which children will have learned to solve in their whole-number work. In the righthand column are addition problems, which have just been developed in work with integers. Solve each pair of problems, noting that the answers are the same. You may want to use hops on the number line to solve the problems on the right.

	Subtraction (Whole numbers)	Addition (Integers)
a)	$13 - 8 =$	$13 + (-8) =$
b)	$15 - 7 =$	$15 + (-7) =$
c)	$16 - 5 =$	$16 + (-5) =$
d)	$23 - 13 =$	$23 + (-13) =$

2. Using the fact that $a - b = a + (-b)$, express each of the following as an addition sentence.

a) $19 - 4 = 15$

b) $21 - 7 = 14$

c) $17 - 6 = 11$

3. We have not yet dealt with problems like $5 - (-2)$. In (1) and (2) you have seen that:

$$13 - 8 = 13 + (-8)$$

$$15 - 7 = 15 + (-7)$$

$$19 - 4 = 19 + (-4)$$

or, in general,

$$a - b = a + (-b);$$

so it makes sense to write

$$5 - (-2) = 5 + (-(-2)).$$

Complete each of the following in the same way.

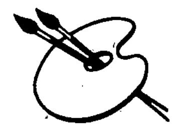
a) 7 - (-8) = 7 +

b) 5 - (-1) = 5 +

c) 13 - (-5) = 13 +

The trouble is, what does -(-2) equal? Well, -(-2) = 2. This can be made to seem reasonable in two ways.

<u>Way 1</u>: -2 is the opposite of 2, so
-(-2) is the opposite of -2, which is 2.
Visualize this on the number line.



<u>Way 2</u>: 2 + (-2) = 0, that is, -2 is the additive inverse of 2. So -(-2) should be the additive inverse of -2. But 2 + (-2) = 0, so 2 is the additive inverse of -2; that is, -(-2) = 2.

Fill in the following.

-(-8) =

-(-1) =

-(-5) =

7 - (-8) = 7 + =

5 - (-1) = 5 + =

13 - (-5) = 13 + =

8 - (-2) = 8 + =

4. Using the fact that -(-b) = b, write each of the following addition sentences as a subtraction sentence. (This is being done to help cement the relationship between addition and subtraction.)

a) 5 + 3 = 8 b) -4 + (-7) = -11

c) 8 + 5 = 13 d) -1 + 1 = 0

5. To build your skill with the ideas presented here, solve the following problems.

a) 4 - (-4) =	b) 8 - (-8) =
c) 13 - (-13) =	d) 17 - (-18) =
e) 7 + 9 =	f) 3 + 4 =
g) 6 + 13 =	h) 12 + 15 =

6. State a general rule for writing a subtraction sentence as an addition sentence.

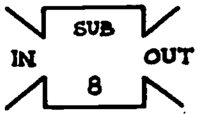
7. Another way of helping children gain some insight into subtraction of integers is by presenting a carefully sequenced set of examples in which a pattern can be observed.

For each of (a), (b), and (c), complete the table by observing the emerging pattern.

a) Start at the top.	b) Start at the top or bottom.	c) Start here *. Work up and down.
7 - 3 =	7 - 3 =	3 - (-1) =
7 - 4 =	7 - 2 =	3 - 0 =
7 - 5 =	7 - 1 =	3 - 1 =
7 - 6 =	7 - 0 =	3 - 2 =
7 - 7 =	7 - (-1) =	*3 - 3 =
7 - 8 =	7 - (-2) =	3 - 4 =
7 - 9 =	7 - (-3) =	3 - 5 =
7 - 10 =	7 - (-4) =	3 - 6 =

8. The "function machine" can be used as another aid in developing subtraction of integers. Again, the subtraction rule should be changed to an addition rule, as we have been doing.

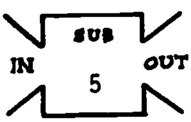
Rule: Sub 8 or
Add $\boxed{-8}$



In	Out
2	-6
4	-4
6	-2
8	0

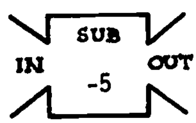
Complete the following:

a) Rule: Sub 5 or
Add $\boxed{}$



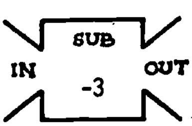
in	out
2	
4	
6	1
8	

b) Rule: Sub -5 or
Add $\boxed{}$



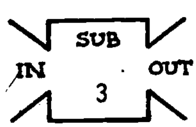
in	out
-2	
-4	
-6	
-8	

c) Rule: Sub -3 or
Add $\boxed{}$



in	out
1	
3	
5	
7	

d) Rule: Sub 3 or
Add $\boxed{}$



in	out
-1	
-3	
-5	
-7	

PART D: SOME GAMES RELATED TO ADDITION AND SUBTRACTION OF INTEGERS

Games can provide a nice change of pace in the elementary classroom. They are especially appropriate when they serve to reinforce or extend concepts that are being studied. These games are intended for you. Play them and discuss how the game should be modified for children.

1. A gain-loss game is representative of another model for illustrating operations with integers. The game is based on the following rules, and is appropriate for playing in the elementary school classroom.

   i) The number indicates which chips, L or G, are taken.

   ii) The operation tells how to interpret the next move.

   Addition means "take the chips indicated."

   Subtraction means "take chips <u>opposite</u> of those indicated."

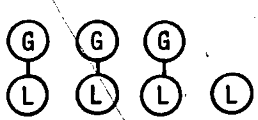
   iii) The answer is obtained by comparing the gain and loss chips and recording how many remain after making one-to-one pairings of G and L.

EXAMPLE A

3 + (-4)

3: take 3 gain chips
+: take the next chips as indicated
(-4): take 4 loss chips

After making pairings among the 3 gain and 4 loss chips, only one loss chip remains. So the answer is 1 loss or -1.

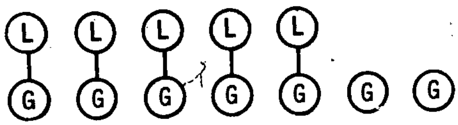


EXAMPLE B

(-5) - (-7)

(-5): take 5 loss chips
-: take chips <u>opposite</u> of those indicated next
(-7): take 7 gain chips

The result is represented as

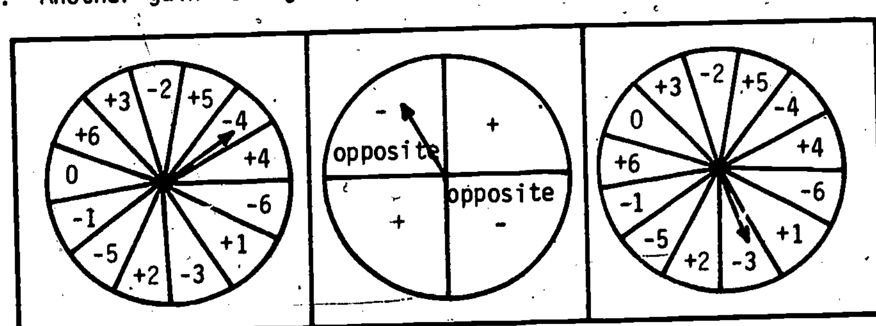


So the answer is 2 gain, or +2.

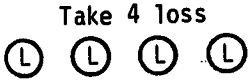
Cut out 10 loss and 10 gain discs and solve the following examples using the discs.

a) 3 + (-5)
b) (-6) - (-2)
c) 4 + 4
d) 6 - 4
e) (-5) - 4
f) (-6) + 6
g) 5 - (-4)
h) (-4) + (-5)

2. Another gain-loss game consists of three spinners.

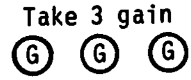


Take 4 loss



Take opposite

Take 3 gain



(-4) - (-3) = -1

Spinner 1 tells how many gain or loss chips to take. Spinner 2 tells whether one should take the chips indicated on Spinner 3 or the opposite chips. Spinner 3 tells how many gain or loss chips to take, depending on the results of 2. Children could make up equations as a result of using the three spinners.

3. Try making up another model for teaching addition and subtraction of integers. For example, can you think of a model or game using the following?

   "An elevator travels up and down going to the floors above and below ground level."

4. Discuss: What are advantages and disadvantages of presenting models through games, as illustrated above? Is there a danger that sometimes the model is more difficult to understand than the process itself?

5. For each of the following real-world problems, write a mathematical sentence that will help you solve the problem. Then find the solution.

   a) Bill had a score of 50 points in Canasta. In the next game he lost 20 points, and in the third game he lost 60 points. What was his score after three games?

   b) Kathy earned $14 babysitting and $17 working as a checkout person in the local grocery store. She then spent $9 for a pair of jeans and $7 for a shirt. How much money did she have left?

   c) One winter day the temperature at International Falls, Minnesota, dropped suddenly. At noon, the recorded temperature was 28°F above zero. By midnight, the temperature had dropped to -17°F. What was the difference between the noon and midnight temperatures?

6. Write a real-world problem that could be represented by each of the following mathematical sentences.

   a) 17 + (-12) + 24

   b) 27 - (-5) - (-13)

   c) (-26) + (-57) + (-43)

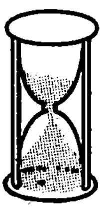
ACTIVITY 5

MULTIPLICATION AND DIVISION OF INTEGERS

FOCUS:

Multiplication and division of integers are usually delayed until the child has gained sufficient maturity to deal easily with Mathematical Relationships. This activity will present some techniques for justifying the operations of multiplication and division of integers, using some pictorial representations and, most often, mathematical relationships.

HISTORICAL HIGHLIGHTS



Fibonacci Numbers*

The Fibonacci sequence of numbers has interested mathematicians and nonmathematicians for ages. It is named after the medieval mathematician, Leonardo (Fibonacci) de Pisa. The sequence begins with 1 and is described as the number obtained by adding the two previous numbers in the sequence. Thus we obtain

1, 1, 2, 3, 5, 8, 13, 21, ...

Find the next five numbers in the sequence

_____ _____ _____ _____ _____

Interestingly, this sequence occurs often in nature. For example, many plants appear to form pairs of spirals in their flowers or fruit. These spirals in the daisy occur regularly with 21 one way and 34 the other. For pine cones, it is 5 one one way and 8 the other, while for pineapples it is 8 and 13. What relationship do these observations have with the Fibonacci sequence?

*Bergamini, David, Mathematics (New York: Time-Life, 1970).

DISCUSSION:

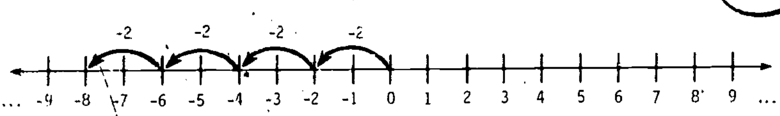
You already know how to solve the problems 2 x 3 or 8 ÷ 4, but what about 2 x (-3), (-2) x (-3), (-8) ÷ 4, and (-8) ÷ (-4)? The answers -6, 6, -2, and 2 are difficult to rationalize using techniques similar to those developed for addition and subtraction. There is one approach to multiplication of a positive integer times a negative one, that uses pictorial representations. Otherwise, we will have to rely on number patterns and on those number properties that are true for the integers. Remember, the goal is to develop a rationale for answering any multiplication or division problem involving integers. The examples below will summarize the results obtained by multiplying or dividing any combination of integers.

EXAMPLES

3 x 4 = <u>12</u>	10 ÷ 5 = <u>2</u>
5 x (-2) = <u>-10</u>	9 ÷ (-3) = <u>-3</u>
(-3) x 2 = <u>-6</u>	(-8) ÷ 4 = <u>-2</u>
(-2) x (-4) = <u>8</u>	(-6) ÷ (-2) = <u>3</u>

DIRECTIONS:

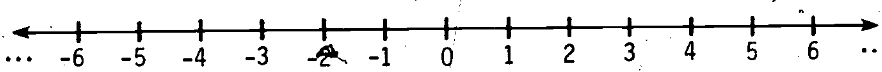
1. The one opportunity for Pictorial Representations to get into the act is by interpreting multiplications of the form 4 x (-2) as <u>repeated addition</u> (-2) + (-2) + (-2) + (-2), and representing it on the number line in terms of backward hops.



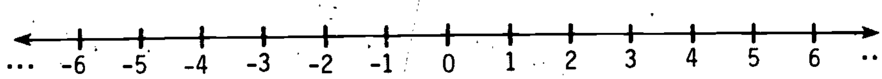
This approach has the advantage of building on one of the whole number models for multiplication and on the addition of integers. A problem comes up in interpreting (-2) x 4 unless one just accepts the fact that multiplication is <u>commutative</u>, so that (-2) x 4 = 4 x (-2).*

Solve each of the following multiplication problems by representing it on the number line.

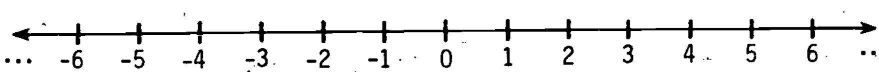
2 x (-3) =



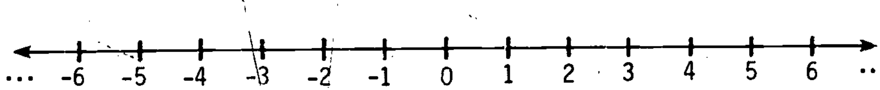
5 x (-1) =



(-3) x 2 =



3 x (-2) =



*See Appendix B for discussion of commutativity.

2. Pattern finding is a Mathematical Relationships approach to multiplying integers. Answer the multiplication problems in (A) and (B), and summarize the pattern(s) you observe.

A	B
4 x 4 = _____	8 x 3 = _____
4 x 3 = _____	8 x 2 = _____
4 x 2 = _____	8 x 1 = _____
4 x 1 = _____	8 x 0 = _____
4 x 0 = _____	8 x (-1) = _____
4 x (-1) = _____	8 x (-2) = _____
4 x (-2) = _____	8 x (-3) = _____

3. In a similar manner, answer each of the multiplication problems in (A) and (B) below, and summarize the patterns you observe. Use the fact suggested in (1) and (2), that the <u>product</u> of a positive integer and a negative integer is a <u>negative</u> integer.

A	B
(-2) x 4 = _____	(-5) x 3 = _____
(-2) x 3 = _____	(-5) x 2 = _____
(-2) x 2 = _____	(-5) x 1 = _____
(-2) x 1 = _____	(-5) x 0 = _____
(-2) x 0 = _____	(-5) x (-1) = _____
(-2) x (-1) = _____	(-5) x (-2) = _____
(-2) x (-2) = _____	(-5) x (-3) = _____
(-2) x (-3) = _____	(-5) x (-4) = _____

4. Using the results of (1), (2), and (3), summarize multiplication of integers by answering the following:

a) A positive integer times a positive integer is a ________.

b) A positive integer times a negative integer is a ________.

c) A negative integer times a positive integer is a ________.

d) A negative integer times a negative integer is a ________.

5. A second technique for establishing the rules for multiplying integers depends on the mathematical properties that are true for the set of integers.

Developing Multiplication and Division Using Number Properties*

For each of the following properties, a and b represent integers.

$a \times b = b \times a$	Commutative
$a \times (b \times c) = (a \times b) \times c$	Associative
$a \times (b + c) = (a \times b) + (a \times c)$	Distributive
$a \times 0 = 0$	Zero property for multiplication
$a \times 1 = 1$	Multiplicative identity
$a + (-a) = 0$ and if $a + b = 0$, then $b = -a$.	Additive inverse

Study each of the properties above. Identify the property that makes each of the following statements true, and give another example of the same property.

a) Statement: $6 \times [4 + (-2)] = (6 \times 4) + [6 \times (-2)]$
Property: ________________ Example:

b) Statement: $(-6) \times 4 = 4 \times (-6)$
Property: ________________ Example:

*For a more complete discussion of these properties, see Appendix B.

c) Statement: $(-4) \times 0 = 0$
Property: ________________ Example:

d) Statement: $[(-5) \times 3] \times (-2) = (-5) \times [3 \times (-2)]$
Property: ________________ Example:

e) Statement: $1 \times (-3) = -3$
Property: ________________ Example:

f) Statement: $(5 \times 7) + [(-5) \times 7] = 0$
Property: ________________ Example:

6. Multiplication of integers can be explained by using the six properties listed above plus a knowledge of the basic facts. The four cases that must be considered are illustrated below. The first is simply multiplication of two whole numbers. The second, a positive integer times a negative integer, is done for you. The third is much like the second, but you should provide the reasons. The final case will allow you to try your hand at writing the steps and reasons. If you have difficulty, ask classmates or your instructor.

a) <u>a positive times a positive</u>: $3 \times 2 =$
(answered in the same manner as the product of two whole numbers)

b) <u>a positive times a negative</u>: $3 \times (-2) =$

$3 \times 0 = 0$	Zero property of multiplication $a \times 0 = 0$
$3 \times [2 + (-2)] = 0$	$2 + -2 = 0$ since -2 is the additive inverse of 2. $a + (-a) = 0$
$(3 \times 2) + [3 \times (-2)] = 0$	Distributive property $3 \times [2 + (-2)] =$ $(3 \times 2) + [3 \times (-2)]$ $a \times (b + c) + a \times b + a \times c$

$6 + [3 \times (-2)] = 0$ — Basic fact: $3 \times 2 = 6$

$3 \times (-2) = -6$ — Additive inverse; when $3 \times (-2)$ is added to 6, you get 0. $a + (-a) = 0$

c) <u>a negative times a positive</u>: $(-3) \times 2 =$

$0 \times 2 = 0$

$[3 + (-3)] \times 2 = 0$

$(3 \times 2) + [(-3) \times 2] = 0$

$6 + [(-3) \times 2] = 0$

$(-3) \times 2 = -6$

d) <u>a negative times a negative</u>: $(-3) \times (-2) =$

$(-3) \times 0 = 0$

7. Once multiplication of integers has been firmly established, division can be dealt with by relying on a whole number relationship between multiplication and division. For example,

$\frac{8}{4} = 2$ is equivalent to $8 = 4 \times 2$; or, generally,

$$\frac{a}{b} = c \quad \text{is equivalent to} \quad a = b \times c.$$

So, to solve the problem

$$-24 \div 8 = \square$$

it seems reasonable to replace it with the problem $-24 = 8 \times \square$ the solution to which is -3. Using this strategy, rewrite each of the following division examples as a multiplication sentence, and solve.

Division Example	Multiplication Sentence
a) $12 \div (-3) = \square$	$(-3) \times \square = 12$
b) $(-15) \div (-5) =$	
c) $(-81) \div 9 =$	
d) $(72) \div (-12) =$	
e) $(-144) \div (-36) =$	
f) $96 \div 16 =$	

8. For each of the following real-world problems, write a mathematical sentence that will help you solve the problem. Then find the solution.

   a) The lowest temperatures reported for a five-day period in Chicago during the month of January were: 1°, -7°, -11°, -8°, 6°. What was the average low temperature for the five-day period?

   b) Every week for seven weeks, Mr. Young's gas station recorded a net loss of \$73 (a profit of -\$73). What was his net profit for the total seven weeks?

c) At the end of a year, three partners in a business venture determined that their new enterprise had lost a total of $7,323. If the three partners were to share in the profits of the business, what was each partner's share of the profit?

9. Write a real-world problem that could be represented by each of the following mathematical sentences.

a) (-3) x (-16)

b) (-324) ÷ 9

c) (-6110) ÷ (-5)

d) 4 x (-112)

10. One of the reasons for expanding the set of whole numbers to the set of integers was to be able to describe a wider class of real-world situations. Another reason is to give wider and richer use to the four operations +, -, x, ÷. In the questions below, you will have an opportunity to compare the whole numbers and the integers with respect to certain properties. After the next section, you will make this comparison between the integers and the rational numbers.

a) A set is closed* under an operation if, when the operation is applied to two members of the set, a <u>third</u> member of the set results. For example, the whole numbers are closed under addition, since a + b is a whole number whenever a and b are whole numbers. Under which of the following operations is the set of whole numbers closed?

i) subtraction

ii) multiplication

iii) division

*This term is discussed further in Appendix B.

b) Under which of the following operations is the set of integers closed?

i) addition

ii) subtraction

iii) multiplication

iv) division

c) In which of the following sets of numbers is the equation $2 + x = 1$ solvable for x?

i) whole numbers

ii) integers

iii) rational numbers (positive and negative fractions)

d) In which of the following sets of numbers is the equation $4x = 2$ solvable?

i) whole numbers

ii) integers

iii) rational numbers

Section II

RATIONAL NUMBERS IN THE ELEMENTARY SCHOOL

In Section I, the set of integers was studied. There are many instances in which children have informally used integers (including negative integers) in their lives. The equation 5 - 12 = ☐ reflects a potentially realistic situation; e.g., the temperature is 5 degrees and drops 12 degrees. The study of integers provides a good foundation for helping the child understand and interpret real-world situations involving integers.

In a similar manner, most children have had several experiences involving rational numbers. The example of sharing objects so that fractional parts result is not uncommon. "You four children may split this bag of M & M's." "You may take one-half of the last piece of cake." Other real-world instances often occur in measurement of length, weight, volume, etc. Dividing a bottle of Coke among three children, or measuring a piece of paper and finding it is a little more than 11 inches long, are further instances of experiences with rational numbers that children bring with them to school. The formalization of these experiences begins in the first year of school and continues throughout the child's total school experience. The concepts associated with rational numbers seem to be more difficult to teach and learn than those associated with whole numbers (or even integers). Although the rational-number concepts may indeed be more complex, a contributing part of the difficulty may be the teacher's own lack of confidence and understanding of these concepts.

This section will focus on several important rational-number concepts that are taught in the schools. The first activity provides a self-test to help you measure your own proficiency and skill in working examples associated with rational numbers.* You will have the responsibility of making sure that you possess the basic skills with rational numbers before completing this unit. The remaining activities in this section have a dual focus: (1) to develop basic rational-number concepts so that you have a firm understanding of them; and (2) to develop these concepts using a pedagogical approach that is appropriate for use in the schools.

Before beginning your work in this section it would be helpful to:

1. Review and discuss the "Overview of Rational Numbers with Integers and Reals" presented in Activity 1, Section I.
2. Read carefully the titles of the activities in this section to gain some perspective as to where they lead. You should observe that the study of rational numbers, like that of whole numbers and integers, proceeds from the real world. Real-world referents and examples using physical embodiments and pictorial representations are identified, and the mathematical terminology and symbolism are developed in as natural a setting as possible. You will also note that this development from real-world examples to mathematical symbolism is not as easily done with rational numbers as it was with whole numbers.

A few comments about terminology would be helpful here. The overview briefly pointed out the possible confusion between the terms "rational numbers," "fractions," and "decimals." At times we talk about the <u>number</u>; at other times we talk about the <u>numeral</u> (or symbol) that represents the number.

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*This test includes skills with fractions as well as with decimal representations of rational numbers.

A. When we talk about the number, we are talking about the abstraction, such as "halfness," rather than about the symbol or numeral that is used to describe the number.

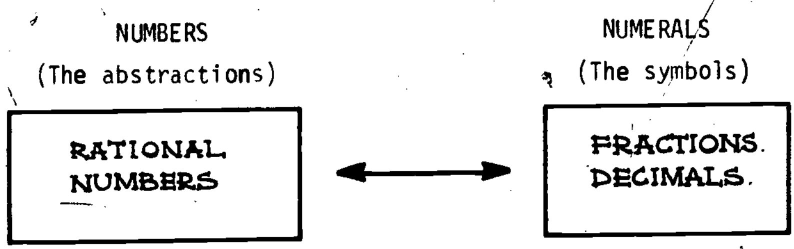
Rational numbers refers to the total set of numbers, positive, zero, and negative.*

B. When we talk about the numerals, we are talking about the symbols (including words) with which we represent numbers.

Fractions refers to symbols that represent rational numbers in the form $\frac{a}{b}$ ($\frac{1}{2}$, $\frac{2}{3}$, etc.), where a, b are integer representations ($b \neq 0$).

Decimals refers to symbols that represent rational numbers in the form of a sequence of digits with a decimal point, for example, 0.5, 0.666...

A study of this diagram may help you organize these terms.



MAJOR QUESTIONS

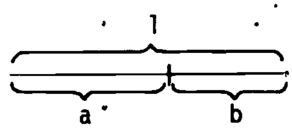
1. Why do you think that the learning of rational-number concepts presents more difficulty than the learning of whole-number concepts? Include some specific examples to support your discussion.

*The term fractional number is also used in the elementary school to describe the nonnegative rational numbers, that is, zero and the positive rational numbers. The elementary school texts probably use the term "fractional number" since it is derived from fraction, a word which the children are likely to hear and use in their environment.

2. Why is the concept of equivalent fractions so important in the development of rational-number concepts? (One possible way of answering this question is to identify all of the concepts and skills that are dependent on the concept of equivalent fractions.)

The Golden Section

Early Greek planners were fascinated by the division of a distance in such a way that the ratio of the whole length to the larger part corresponded geometrically to the ratio of the larger part to the smaller.



If the entire length is 1, then the diagram above shows that $\frac{b}{a} = \frac{a}{a + b}$ is the proportion described above.

This division of a distance, called the <u>Golden Section</u>, was used by Greek architects and artists, and continues to intrigue contemporary painters and musicians. Several other parts of this unit will explore the Golden Section and its application; but first, try to find a value for a (the Golden Ratio). One method is by trial and error; another is using algebraic manipulation. However, it is possible to arrive at an estimate as close as you wish using the Fibonacci sequence mentioned in Section I, page 44. That is, by taking ratios of successive Fibonacci numbers, the resulting sequence provides a closer and closer approximation to the Golden Ratio. Thus, the first approximation is $\frac{1}{1}$, or 1; the second is $\frac{1}{2}$, and the third is $\frac{2}{3}$.

If you continue this process, what observations can you make about the value of the Golden Ratio, using the Fibonacci sequence?

ACTIVITY 6

SELF-TEST OF SKILLS WITH RATIONAL NUMBERS

FOCUS:

As future elementary school teachers, you should realize the importance of developing your skills with rational numbers. This self-test of these skills will indicate which areas require improvement before the completion of this unit.

DISCUSSION:

Since you are going to have to explain rational-number computations to others, there is little sense in rote memorization of procedures to gain the necessary skills. You should take the time to develop an understanding of the basis for each of these skills. If you need to review some computational procedures, the developments in many elementary school mathematics texts are carefully done, and you will find them helpful in your own learning.

In taking this test, you will note that the items are mixed and are often presented in sentence form rather than in simple example form. The reason for this is that such questions as "What is $\frac{2}{3}$ of 8?" occur in real-life situations more often than simply $\frac{2}{3} \times 8$.

DIRECTIONS:

1. Take the test, displaying as much work as possible to make the error analysis easier.
2. Correct the test, using the answers in Appendix A, pp. 217-218.
3. Classify the errors that you made, e.g., addition of fractions, multiplication of decimals, etc.
4. Determine the probable cause of the errors.
5. Identify a procedure for relearning those fundamental skills that you do not possess, and complete the relearning before the

end of this unit. This step is your responsibility. The Skill Builder Exercises in Appendix C have been constructed to aid in the development of the specific skills identified in (3) above.

Self-Test

Part A: Common Fractions

1. $\frac{3}{7} + \frac{2}{7} =$

2. $4 \times \frac{1}{2} =$

3. $\frac{9}{12} - \frac{7}{12} =$

4. $\frac{3}{4} \div \frac{2}{3} =$

5. $\frac{6}{7} + \frac{3}{4} =$

6. $\frac{7}{16} - \frac{1}{3} =$

7. $\frac{7}{8} \times \frac{4}{5} =$

8. $5 \div 7 =$

9. How many sevenths are there in one?

10. Write $1\frac{5}{8}$ in terms of eighths (as a number of eighths).

11. What is $\frac{2}{3}$ of $\frac{3}{4}$?

12. $\frac{3}{4}$ is what part of $\frac{7}{8}$?

13. Find the whole numbers m and n if $\frac{2}{7} = \frac{m}{14} = \frac{6}{n}$.

14. Find the whole number p if $\frac{6}{14} = \frac{p}{35}$.

15. Place the following in order from least to greatest:

a) $\frac{3}{8}$ b) $\frac{1}{2}$ c) $\frac{3}{4}$ d) $\frac{3}{16}$ e) $\frac{9}{15}$

16. Express each of the following in simplest form.

a) $\frac{32}{48} =$ b) $\frac{52}{40} =$ c) $\frac{48}{3} =$

17. What is the least common multiple (LCM) of:

a) 3 and 7 b) 6 and 8 c) 3 and 4 and 6

18. What is the greatest common factor (GCF) of:

a) 6 and 8 b) 28 and 35 c) 72 and 144

19. $2\frac{3}{4} + 1\frac{5}{9} =$

20. $1\frac{4}{5} - \frac{7}{9} =$

21. $4\frac{2}{9} - 2\frac{5}{6} =$

22. $3\frac{3}{4} \times 8\frac{3}{7} =$

23. $1\frac{7}{8} \times 6\frac{3}{4} \times 2\frac{2}{3} =$

24. $1\frac{6}{7} \div 3\frac{1}{2} =$

25. $1\frac{1}{2} + (\frac{-3}{8}) + 7\frac{2}{5} =$

26. $4\frac{7}{8} - (-3\frac{1}{3}) =$

27. $(\frac{-7}{8}) \times (-2\frac{1}{5}) =$

28. $(\frac{-5}{6}) \times (2\frac{3}{10}) =$

29. $(\frac{-1}{2}) \div (-3) =$

30. $3\frac{2}{5} \div (-2\frac{1}{6}) =$

Part B: Decimals

1. 1.5 + 0.0617 =
2. 32.45 x 0.02 =
3. 67.85 - 15.317 =
4. 4.033 ÷ 0.037 =
5. How many thousandths are there in one-tenth?
6. Place the following in order from least to greatest:

   a) 0.29 b) 0.74 c) 0.271 d) 2.065 e) 0.09874

7. 4.01 ÷ (-0.04) =
8. 7.31 + (-35.678) =
9. 6.31 x (-2.003) =
10. 16.8 - (-33.07) =
11. (-7.32) + (-10.8) =
12. (-43.6) - (+7.25) =
13. (-1.03) x (-0.023) =
14. (-73.25) ÷ (-0.05) =
15. Express $\frac{3}{8}$ as a decimal.
16. Express $\frac{7}{3}$ as a decimal to the nearest thousandth.
17. Express $\frac{2}{11}$ as a decimal.
18. Express 0.7312 as a fraction.
19. Express 0.6666... as a fraction.
20. Express 0.323232... as a fraction.

ACTIVITY 7

PHYSICAL EMBODIMENTS FOR RATIONAL NUMBERS

FOCUS:

In this activity you will have an opportunity to discuss different physical embodiments that might be used for introducing rational numbers to young children.

DISCUSSION:

The whole number <u>three</u> is an abstraction that is concretely or physically embodied in sets containing three objects, and that is represented by the numerals "three," "3," "2 + 1," etc. The integer <u>negative three</u> is also an abstract concept, which can be physically (or at least pictorially) embodied by three backward hops on the number line. It is, of course, represented by the numerals "negative three," "-3," "5 - 8," etc. In the same way, the rational number <u>one-half</u> is an abstract concept that can be physically embodied in at least the three different ways described in this activity. It can be represented by the numerals "one-half," "$\frac{1}{2}$," "1 ÷ 2," "0.5" etc. In fact, one-half has an <u>infinite number</u> of representations. This fact will be discussed later.

In the work on the next page, three kinds of embodiments are given that can prove helpful in developing the concept of rational numbers with children. These embodiments lend themselves particularly well to fractional representations.

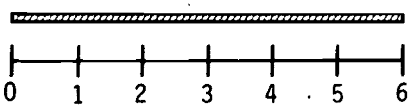
DIRECTIONS:

Three physical embodiments for introducing rational numbers expressed as fractions are given below. Study each of them and then answer the questions that follow.

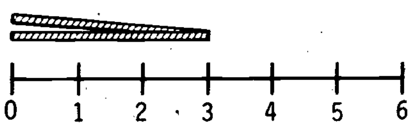
A. <u>Using a Measurement Model</u>

Hold up a piece of yarn against a number line displayed in front of the room. The first time hold up a piece that is 6 units long.

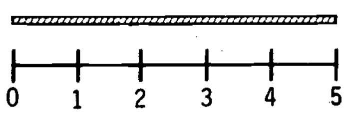
Ask: How long is the yarn?



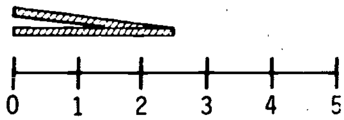
Then ask: If I folded it in half, how long would it be? (Hold up the folded yarn for the children to see that it is 3 units long. Next, take a piece of yarn 5 units long.)



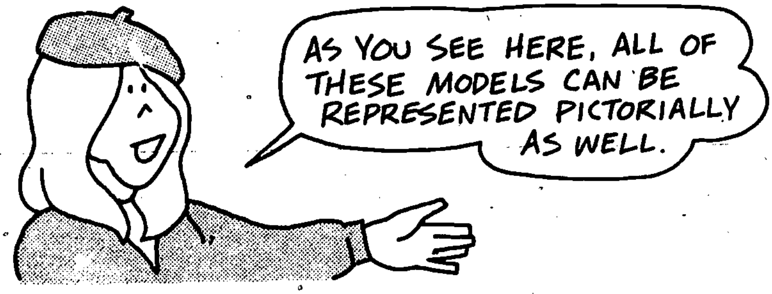
Ask: How long is the yarn?

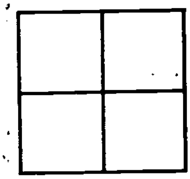


Then ask: If I folded it in half, how long would it be? (Hold up the folded yarn for the children to see that it is $2\frac{1}{2}$ units long.)

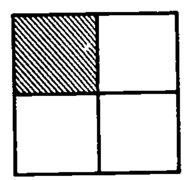


B. Using a Region Model (Part of a whole)

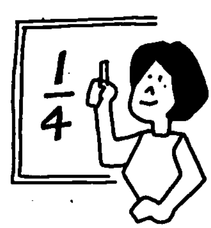




Take a large piece of light-colored construction paper. Fold it into four parts. Ask the children to count the number of parts into which the paper has been folded. Ask if each part is equal to the others. This can be demonstrated by folding the parts onto each other.



Shade one part of the paper using a dark-colored marker. Ask the pupils how many parts are shaded. Write "1" on the chalkboard. Ask how many parts there are in all. Write 4 below the 1.



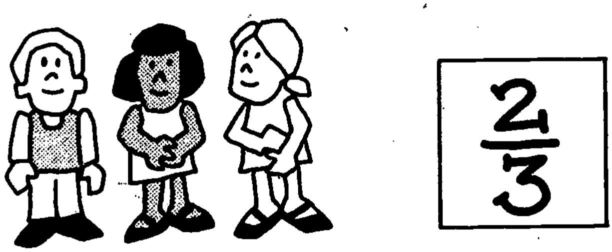
Now draw the line between the numerals so that the fraction $\frac{1}{4}$ is seen on the board. Tell the children that the numeral on the board is called the fraction <u>one-fourth</u>. Ask them again what the 4 means.

Repeat the procedure for the unshaded $\frac{3}{4}$ portion. (Similar developments could be done using a circle or a rectangle. Particular care must be taken that equal (or congruent) subdivisions can be found easily.)

C. Using a Set Model

Bring three children to the front of the room, choosing two girls and one boy. Ask the children to think of the three children as a set.

Ask how many boys, how many girls there are in all. Now ask what part of the set is girls. Develop this by asking again how many girls. Write 2 on the board. Ask how many in the set in all. Write 3 on the board below the 2. Now draw the line between the numerals so that the fraction $\frac{2}{3}$ is seen on the board. Tell the children that the numeral on the board is called the fraction two-thirds. Ask them again what the 2 means, what the 3 means.



Repeat the procedure to develop the fraction of the set that contains boys.

Discuss the following questions. Summarize your discussions.

1. Discuss how you might extend Example A above (the measurement model) to develop the writing of fractions.

2. The three examples illustrate the three types of physical embodiments usually associated with rational numbers: the number line (ruler) or measure approach; the part of a whole (region) approach; the part of a set approach. Which of the three do you feel would be the easiest for children to understand? Which would be the most difficult?

3. Order the following activities for children in the sequence that you feel would be most appropriate.

a) Ask children to tell what the top numeral (numerator) in a fraction means, what the bottom numeral (denominator) in a fraction means.

b) Ask children to partition objects into halves, thirds, quarters.

c) Ask children for examples of the use of fractions in their lives.

d) Ask children to match shaded regions with a fraction.

e) Ask children to write the fractions for one-half, one-third, one-fourth, etc.

f) Ask children to match part of a set with a fraction.

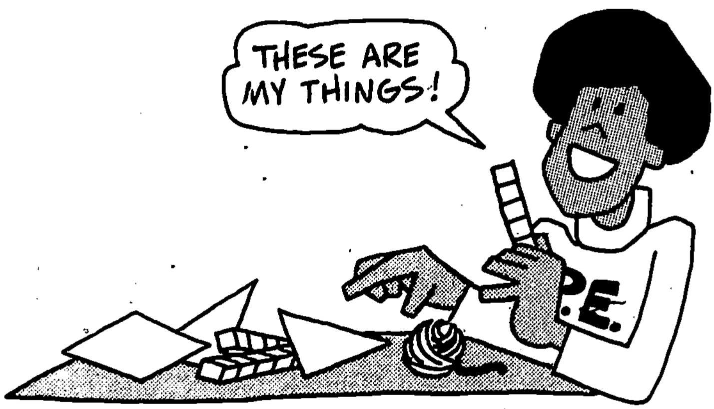
ACTIVITY 8

INTRODUCING RATIONAL NUMBERS

FOCUS:

In this activity you will have an opportunity to take part in three introductory lessons, which are aimed at a more precise understanding of rational numbers.

MATERIALS:



You will find it helpful to have available the materials suggested for the children. These materials are:

Section A Cuisenaire rods or strips of graph paper of the same width, ranging in length from 1 unit to 10 units.

Section B Paper squares, rectangles, isosceles triangles, equilateral triangles, and scalene triangles.

Section C Twelve objects, six of which are alike and six of which are different; yarn or string for making a closed loop and for partitioning.

DISCUSSION:

The child's introduction to rational numbers is through the notion of fractions and fractional parts. Fractions are represented using pairs of integers--usually in the form $\frac{a}{b}$, sometimes in the form (a,b). The ratio of two lengths (a 4-inch length and a 6-inch length), when used in introducing rational numbers, is seen as $\frac{4}{6}$ or $\frac{2}{3}$ (assuming the shorter is compared to the longer). Sets and parts of sets are compared in a similar fashion. This activity will focus on using pairs of numbers as a natural way to teach children about rational numbers.

DIRECTIONS:

These activities are intended for children. However, you should do the activities in groups as if you were children. A different member of the group can play the role of teacher for each of Sections A, B, and C, giving the instructions for the section to the rest of the group.

As a group, discuss and answer the questions that follow each section. When acting as the teacher, phrase each of the directions and questions as if you were talking with children.

Section A
(Measurement Model)

1. Ask the child to take a brown rod or an 8-unit strip of paper. His or her task is to find two--but only two--rods (or strips) of the same size which together give the same length as the original rod or strip. That is, he or she must find half the original.
2. Repeat (1), using a different original rod or strip. (The dark green rod or 6-unit strip is suggested.)
3. Ask the child whether half the brown rod is the same length as half the dark green rod.
4. Give the child a rod (e.g., the red rod), and ask him to find the rod of which it is half.
5. Then ask the child to find all the rods in the box for which there are other rods half their length. (It is important that

the child not be left with the misconception that an odd-length bar cannot be divided into two or four equal lengths.)

Section A--Questions

1. Analyze the child's activities to see whether they are repeated examples of a single experience or whether they provide fundamentally different experiences.

2. How would you assure the child that the odd-length bars can also be divided in half?

3. Write four questions you can ask children, which follow the same sequence as above, but which make the meaning of "quarter" more precise. State which rods or strips you propose to use.

Section B
(Region Model)

1. Ask the child to take a square and fold it so that one side of the square fits exactly on the other side.

2. Extend (1) by asking the child in how many <u>different</u> ways he can fold a square in half.

3. Ask the child to fold equilateral, isosceles, and scalene triangles in half, if he can.

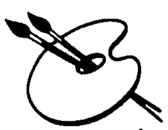
4. Ask similar questions for folding halves of a circle and rectangle.

Section B--Questions

1. Draw rough diagrams of the answers you would expect from the child's activities in folding.

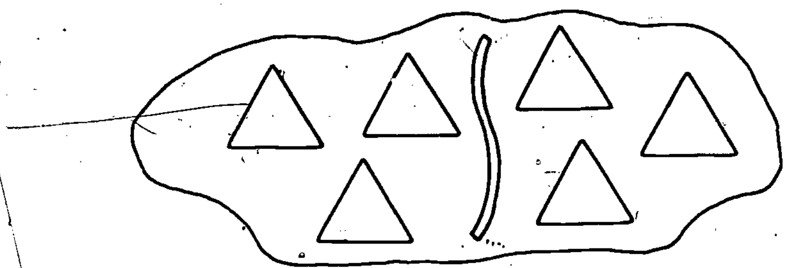
2. Suppose you are allowed to draw a line to form halves rather than just folding. How many ways can you draw a line to form halves?

3. How might you use Pictorial Representations for the example in (2) above? What would the advantages and disadvantages be?



Section C
(Set Model)

1. Ask the child to put six objects within a loop of string and use a ribbon to partition this set into two subsets so that there are an equal number of objects in each. Then ask him how many is half of 6.



2. Ask questions to extend the activity to experiences for finding half of four objects, eight objects, etc.

Section C--Questions

1. What instructions for partitioning would you give in order to demonstrate quarters?
2. Does it make any difference to your approach if the objects are not alike?
3. Suppose a child looks at the set above and says, "I don't see $\frac{1}{2}$. I see 3." How would you handle this observation?
4. The three different approaches (A, B, and C) were applied to the same number (one-half) so that you can make a comparison. Which approach do you feel most comfortable with? Why? Do you think that the same would be true for children?

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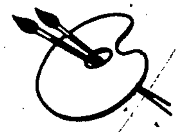
Assignment

1. The introductory lessons suggested above used models based on physical objects. A variation of the measurement model is the number line.



Write three or four questions you might ask a child in developing the concept of "half," "quarter," and "third," using a number line.

2. For each of the following problems, write a mathematical expression that describes the situation.

   a) The bakery prepared 42 glazed donuts, 36 plain donuts, and 24 sugar donuts for one day's sales. What fractional part of all the donuts baked was represented by the plain donuts?

   b) On a trip from Chicago to Denver, John drove 5 hours, Mary drove 4 hours, Susan drove 5 hours, and James drove 6 hours. What fraction of the time did James drive?

3. Write a real-world problem that could be represented by each of the following rational numbers.

   a) $\frac{2}{7}$ b) $1\frac{7}{8}$

ACTIVITY 9

$\frac{a}{b} = a \div b$

FOCUS:

Most readers will know that $\frac{a}{b}$ and $a \div b$ represent the same rational number. In this activity, you will consider the problem of making this seem reasonable to a child.

DISCUSSION:

You probably remember converting from a fractional representation of a rational number to a decimal representation by means of division, e.g.,

$$\frac{1}{4} = 0.25 \quad \text{since} \quad \begin{array}{r} 0.25 \\ 4\overline{)1.00} \\ \underline{8} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

When you do this you are using the fact that $\frac{1}{4}$ and $1 \div 4$ represent the same rational number. This may seem reasonable to you, but how would you explain it to a child who:

i) learned about division in the context of whole numbers?

ii) learned later about fractional representation of rational numbers in a context like that of the previous activity?

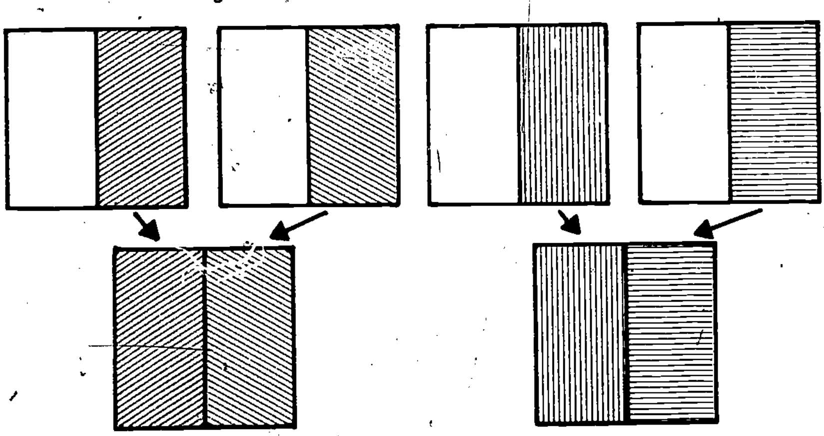
DIRECTIONS:

1. Here are two approaches to making $\frac{a}{b} = a \div b$ seem reasonable. Read them.

   - "Mary has two pies and wants to divide them equally among five children. How much pie does each person get?"

     This story can be interpreted with either of the symbols $2 \div 5$ or $\frac{2}{5}$.

- $\frac{4}{2}$ is a fraction that represents a rational number, and $4 \div 2 = 2$ is a whole number. It seems reasonable to agree that $\frac{4}{2} = 4 \div 2$ since, if we take four regions, separate them into halves, and take 4 of these halves, we end up with two regions.



a) One of these explanations relates to the child's experience with division, and one relates to the child's understanding of fractions. Explain which is which. Do you see any problems with either?

b) Make up a rationale for the equality of $\frac{1}{4}$ and $1 \div 4$.

c) Make up a rationale ~~for~~ the equality of $\frac{4}{1}$ and 4.

d) Use the fact that $\frac{a}{b} = a \div b$ to explain why it seems reasonable that every integer is a rational number.

2. Division of whole numbers can be related to multiplication by means of the statement that

$$a \div b = x \text{ is equivalent to } a = bx.$$

a) Use the above statement and the fact that $a \div b = \frac{a}{b}$ to show that

$$\frac{a}{b} = \frac{c}{d} \text{ is equivalent to } ad = bc.$$

b) Use what you have learned to date to show that

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}.$$

ACTIVITY 10

INTRODUCING EQUIVALENT FRACTIONS

FOCUS:

In this activity you will have an opportunity to take part in some experiences, suggested for children, to develop the concept of equivalent fractions.

MATERIALS:

Pieces of paper (any size; $8\frac{1}{2}$" x 11", for example); twelve strips of paper of equal length (1" x 12"--or use centimeters); compass.

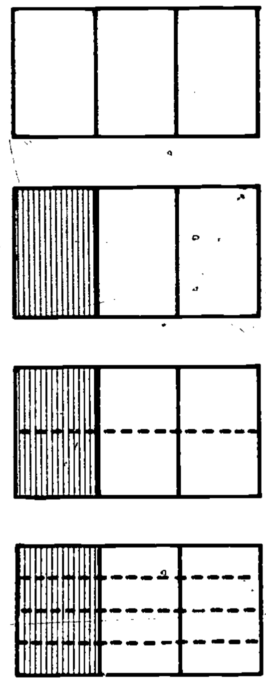
DISCUSSION:

The concept of equivalence is one of the important ideas that run throughout the mathematics curriculum. Equivalence of fractions is a particularly important idea as well as a practical one. Children will be expected to use this idea freely when adding or subtracting rational numbers. The concept of equivalent fractions is also important in determining order (which is greater) among fractions. At the simplest level, it is important for children to be able to recognize equivalent fractions. As you have seen, fractions are names for rational numbers. Two fractions are called equivalent if they are names for the same rational number.

DIRECTIONS:

In small groups, work through these activities as if you were children. Take turns acting as the teacher. Be sure to answer the questions that follow each section.

Section A
(Region Model)



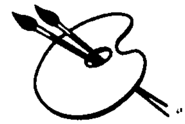
1. Take a piece of paper which is divided into three parts. (Do this by folding.)
2. Shade one portion of the paper. "How many parts of the paper are there in all? How many parts are shaded? What fractional part is shaded?"
3. Fold this paper in half. "How many parts in all? How many parts are shaded? What fraction tells how much is shaded? Does $\frac{1}{3} = \frac{2}{6}$?"
4. Fold the paper in half, then fold it in half again. "How many parts in all? How many are shaded? What fraction tells how much is shaded? Does $\frac{1}{3} = \frac{2}{6} = \frac{4}{12}$?"

<u>Section A--Questions</u>

1. Would it be advisable, when working with children, to continue folding the paper to get three other fractions equivalent to $\frac{1}{3}$? Why or why not?
2. In the above embodiments of $\frac{1}{3}$ and $\frac{2}{6}$, how do you know that they represent the same rational number?
3. Develop a set of questions using paper-folding to establish the equivalence of $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$.

   How would you use paper-folding to show that $1 = \frac{2}{2} = \frac{4}{4}$?

Section B
(Measurement Model)



1. Take one of the twelve strips of paper. Place a large "1" in the center. Take another, mark it into halves and write $\frac{1}{2}$ on each part. Do the same for thirds, fourths, etc., up to twelfths. (For children you could do this on a single piece of paper with the divisions already given, as shown in Figure 1 on page 78.)

1											

$\frac{1}{2}$	$\frac{1}{2}$

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

2. Take the strip divided into halves. Match it with the other strips, to see if any other fraction matches the $\frac{1}{2}$ mark. Does $\frac{1}{2} = \frac{1}{3}$? $\frac{1}{2} = \frac{2}{3}$? How many $\frac{1}{4}$'s does $\frac{1}{2}$ equal?

$\frac{1}{2}$	$\frac{1}{2}$

$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$\frac{1}{2}$	$\frac{1}{2}$

$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

3. Line up all the fraction strips equivalent to $\frac{1}{2}$; to $\frac{1}{3}$; to $\frac{1}{4}$.

4. How many twelfths are there in $\frac{3}{4}$? How many in $\frac{2}{3}$? Which is the larger, $\frac{3}{4}$ or $\frac{2}{3}$?

5. Place in order of size (least to greatest): $\frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{5}{6}$.

6. How many sixteenths are there in a whole strip? How many twentieths?

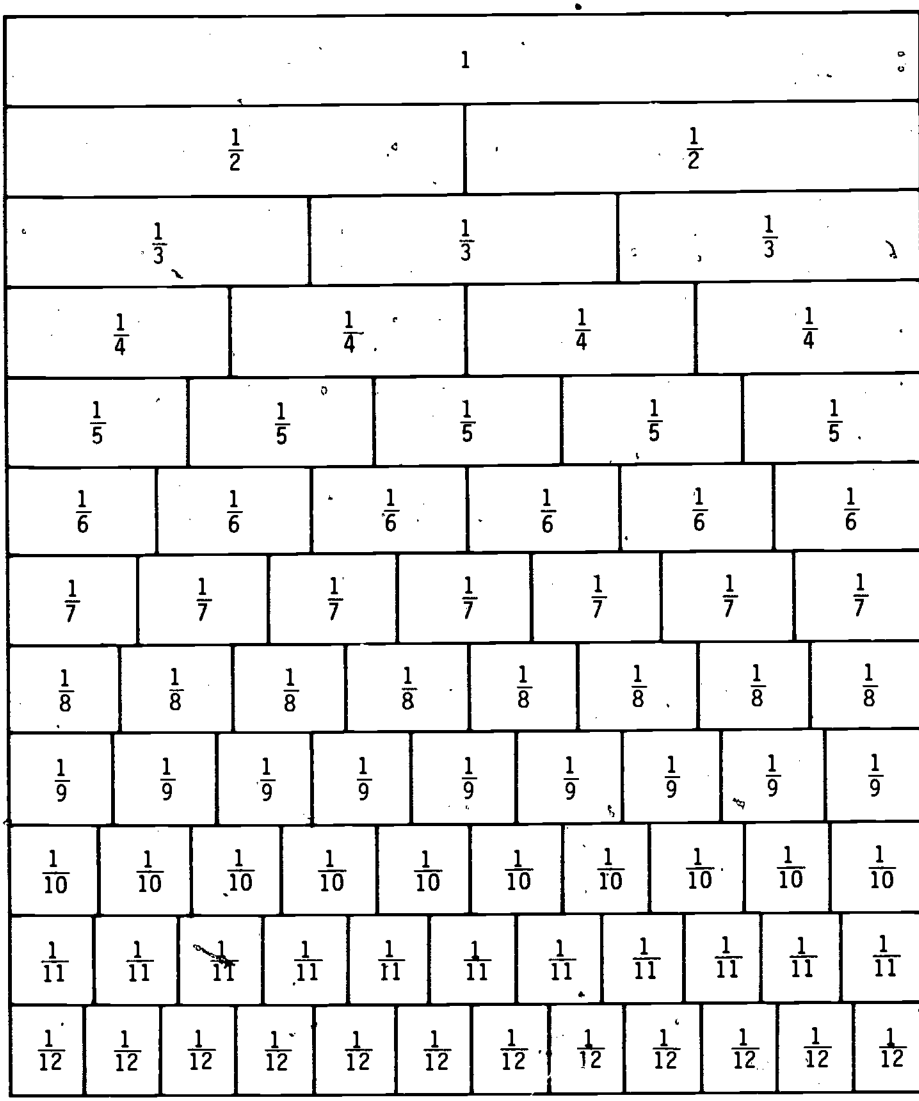

1
1/2 1/2
1/3 1/3 1/3
1/4 1/4 1/4 1/4
1/5 1/5 1/5 1/5 1/5
1/6 1/6 1/6 1/6 1/6 1/6
1/7 1/7 1/7 1/7 1/7 1/7 1/7
1/8 1/8 1/8 1/8 1/8 1/8 1/8 1/8
1/9 1/9 1/9 1/9 1/9 1/9 1/9 1/9 1/9
1/10 1/10 1/10 1/10 1/10 1/10 1/10 1/10 1/10 1/10
1/11 1/11 1/11 1/11 1/11 1/11 1/11 1/11 1/11 1/11 1/11
1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12

Figure 1

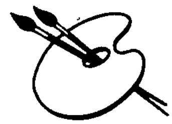
Section B--Questions

1. Describe how you might use these strips to show equivalence by shading in portions--rather than by cutting and comparing.

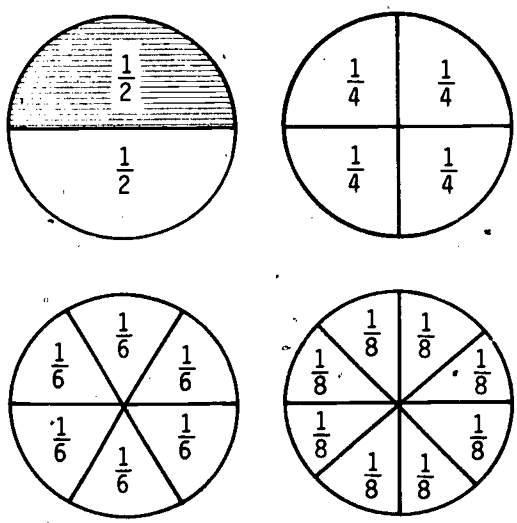
2. Write a few questions for children showing how you might ask them to use the strips to determine which of two fractions is greater. For example, which is greater, $\frac{1}{3}$ or $\frac{1}{4}$? $\frac{2}{5}$ or $\frac{3}{7}$?

3. How would you use the strips to show that $1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} \ldots$?

Section C
(Region Model)



1. Use the compass to draw four circles. Partition one into halves, one into fourths, one into sixths, and one into eighths. Shade one-half of the circle partitioned into halves.



2. How many sections of the circle partitioned into fourths would one shade to be equivalent to $\frac{1}{2}$? How many sixths? How many eighths?

Section C--Questions

1. Write a few questions developing the use of circular regions for fractions equivalent to $\frac{2}{3}$. How many circles would you use? How would you partition them?
2. Would you ask children to use a compass to construct their own circles? Why or why not? If not, how might you prepare or have children prepare the circular regions?

Sections A through C have suggested experiences to develop an intuitive understanding of equivalent fractions. Section D outlines a lesson that would begin to formalize this understanding, as well as to develop a procedure for generating equivalent fractions.

Section D

1. Study the sets of equivalent fractions at the right. Can you give the next five fractions for each set?

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \_\_, \_\_, \_\_, \_\_, \_\_\right\}$$

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \_\_, \_\_, \_\_, \_\_, \_\_\right\}$$

$$\left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \_\_, \_\_, \_\_, \_\_, \_\_\right\}$$

2. What pattern or rule can you find to give an equivalent fraction?

$$\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \ldots\right\}$$

3. (After pupils discuss possible patterns, you can develop the multiplication pattern.) Try multiplying both numbers by 2, by 3, by 4 and so forth.

$$\left\{\frac{1}{2} \rightarrow \frac{1 \times \boxed{2}}{2 \times \boxed{2}} = \frac{2}{4}\right\}$$

$$\left\{\frac{1}{2} \rightarrow \frac{1 \times \boxed{3}}{2 \times \boxed{3}} = \frac{3}{6}\right\}$$

$$\left\{\frac{1}{2} \rightarrow \frac{1 \times \boxed{4}}{2 \times \boxed{4}} = \frac{4}{8}\right\}$$

Section D--Questions

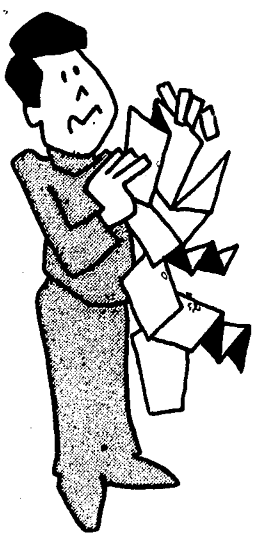
1. What patterns do you note in the set of equivalent fractions shown in (1) above?

2. Which rational numbers would you use to introduce equivalent fractions to children? In what order would you introduce them?

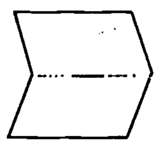
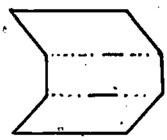
3. Which of the approaches presented in Sections A, B, and C lends itself best to illustrating comparisons of fractions?

4. When you introduce equivalent fractions to children they will not yet have studied multiplication of fractions. The multiplication procedure for generating equivalent fractions that was evolved above can be explained in terms of multiplying fractions by fractions equivalent to 1. How would you explain this procedure to children who know how to multiply fractions?

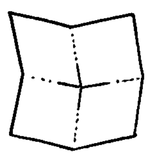
TEACHER TEASER



You can partition a piece of paper into

halves,  thirds, 

and fourths by folding.



Can you partition it into fifths by folding only?

ACTIVITY 11

USING EQUIVALENT FRACTIONS

FOCUS:

This activity extends the concept of equivalent fractions and develops some useful techniques for finding equivalent fractions.

DISCUSSION:

Equivalent fractions name the same rational number. Sometimes it is useful to rename fractions so that they have the same (a common) denominator. For example, it is often easier to use $\frac{1}{4}$ and $\frac{2}{3}$ in the form $\frac{3}{12}$ and $\frac{8}{12}$. Some techniques for finding common denominators will be developed in this activity.

Another application of the concept of equivalent fractions is the reverse process of "reducing" fractions to a simpler, or to the simplest, form. The fraction $\frac{18}{24}$ can be rewritten as $\frac{9}{12}$ or $\frac{3}{4}$. Techniques for "reducing" fractions will also be presented in this activity.

DIRECTIONS:

1. Finding common denominators

Three techniques for finding a common denominator are presented here. Study and discuss each method, working the examples.

a) Find a common denominator by generating a set of equivalent fractions.

i) List in order the next five equivalent fractions.

$\frac{3}{4}, \frac{6}{8}$, __, __, __, __, __

$\frac{1}{3}, \frac{2}{6}$, __, __, __, __, __

ii) Which fractions equivalent to $\frac{3}{4}$ and $\frac{1}{3}$ have a common denominator?

b) Find a common denominator by using the product of the denominators. To find a common denominator, one may multiply the denominators. For example, a common denominator for $\frac{5}{6}$ and $\frac{3}{8}$ is 6 x 8 or 48.

i) Discuss how one might find fractions equivalent to $\frac{5}{6}$ and $\frac{3}{8}$ with a denominator of 48.

<u>Hint</u>: Complete the following:

$$\frac{5}{6} = \frac{5 \times \square}{6 \times 8} = \frac{5 \times \square}{48} = \frac{\triangle}{48}$$

$$\frac{3}{8} = \frac{3 \times \square}{8 \times 6} = \frac{3 \times \square}{48} = \frac{\triangle}{48}$$

The common denominator found by this method is not always the <u>least</u> common denominator (LCD). In teaching children, there is some danger in overemphasizing the usefulness of finding the least (smallest) common denominator.

ii) Find a common denominator using the product-of-the-denominators method <u>and</u> write equivalent fractions for the following.

$\frac{2}{5}, \frac{3}{7}$ $\qquad$ $\frac{3}{10}, \frac{4}{5}$ $\qquad$ $\frac{8}{9}, \frac{2}{3}$

c) Find a common denominator using the <u>prime factorization</u> method. This method results in finding the LCD and is generally taught in the middle school. The method assumes skill in renaming numbers as products of primes. Study and discuss the example that follows.

<u>Find the LCD of $\frac{1}{6}$ and $\frac{3}{8}$.</u>

The prime factorization of 6 is 2 x 3.

The prime factorization of 8 is 2 x 2 x 2.

The LCD is the product which includes the factors of each number:

2 x 2 x 2 x 3 or 24

i) Why must the LCD include the factors of each number? Why does this method provide the <u>least</u> common denominator?

ii) For each of the pairs of fractions below use this method to find the LCD and use the LCD to write equivalent fractions.

$\frac{13}{16}, \frac{5}{12}$ $\qquad$ $\frac{7}{9}, \frac{5}{6}$ $\qquad$ $\frac{7}{10}, \frac{3}{15}$

iii) <u>Discuss</u>: What are some advantages and disadvantages of the prime factorization method? What other method could you use to find the LCD?

2. <u>Finding the simplest form of fractions (lowest terms)</u>

In comparing the equivalent fractions $\frac{6}{8}$ and $\frac{12}{16}$, we say that $\frac{6}{8}$ is in simpler form than $\frac{12}{16}$ because the "terms" (numerator and denominator) are smaller in $\frac{6}{8}$ than in $\frac{12}{16}$. But even $\frac{6}{8}$ can be "reduced" to $\frac{3}{4}$. The procedure for generating equivalent fractions is to multiply numerator and denominator by the same number.

$$\frac{a \times n}{b \times n}$$

For example,

$\frac{3 \times 2}{4 \times 2}$ and $\frac{3 \times 4}{4 \times 4}$ are equivalent to $\frac{3}{4}$.

Conversely, the procedure for finding fractions in simpler form is to divide each term by the same number. The procedure for finding the simplest form is to divide by the <u>greatest</u> common factor, e.g.,

<u>Simpler form</u>	<u>Simplest form</u>
$\frac{12}{16} = \frac{12 \div 2}{16 \div 2} = \frac{6}{8}$	$\frac{12}{16} = \frac{12 \div 4}{16 \div 4} = \frac{3}{4}$

a) Find the simplest form for the following fractions:

$\frac{8}{18}$ $\qquad$ $\frac{24}{52}$ $\qquad$ $\frac{36}{54}$ $\qquad$ $\frac{72}{104}$ $\qquad$ $\frac{96}{144}$

b) Analyze the procedure you used in reducing these fractions, and outline the steps you would advise ~~a child~~ to use (including the justification for each) in reducing fractions to simplest form.

c) The most efficient way of finding a fraction in simplest form is to identify the greatest common factor (GCF) of the numerator and denominator. To find the simplest form of $\frac{18}{24}$, we must find the GCF of 18 and 24. To see that the GCF of 18 and 24 is 6, note that the factors of 18 are 2, 3, 6, 9, and the factors of 24 are 2, 3, 4, 6, 8, 12, 24. The common factors are 2, 3, 6. So the greatest common factor is 6. By dividing the numerator and denominator by 6, the simplest form can immediately be found: $\frac{18 \div 6}{24 \div 6} = \frac{3}{4}$. For each fraction below, find the GCF and the simplest form.

$\frac{16}{36}$ $\frac{28}{42}$ $\frac{36}{54}$ $\frac{24}{56}$

d) Two numbers are relatively prime if they have no common factors (1 always excepted). Give examples of five pairs of numbers that are relatively prime.

e) "A fraction is in simplest form if the numerator and denominator are relatively prime." Do you agree or disagree with this statement? Give your reasons.

ACTIVITY 12

ORDERING THE RATIONAL NUMBERS

FOCUS:

It is sometimes important to be able to order rational numbers (i.e., determine which is largest). In this activity you will learn how to do this using equivalent fractions.

DISCUSSION:

Children have little difficulty determining which of two whole numbers is greater. The symmetry of the place value system and counting procedures provide insight and an intuitive "feel" for order. The task of determining which of two rational numbers is greater, for example, $\frac{4}{13}$ or $\frac{5}{14}$, is not so easy. Some techniques are needed. A pictorial technique was developed in Activity 10. The use of equivalent fractions will provide the basic technique for this activity. We will consider comparing fractions which have:

1. equal denominators,
2. unequal numerators and unequal denominators,
3. equal numerators.

DIRECTIONS:

1. <u>Equal Denominators</u>

   To order rational numbers represented by fractions with equal denominators, it is necessary only to <u>compare the numerators</u>. Determine which number in each of the following pairs is greater.

   $\frac{2}{3}, \frac{1}{3}$ $\qquad$ $\frac{3}{12}, \frac{7}{12}$ $\qquad$ $\frac{6}{7}, \frac{12}{7}$ $\qquad$ $3\frac{4}{9}, 3\frac{1}{9}$

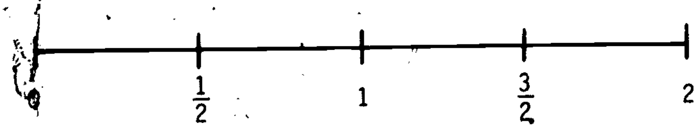
2. Unequal Denominators--Unequal Numerators

a) Although many rational numbers that are compared have unequal denominators, it is not difficult to write them with equal denominators, using your knowledge of equivalent fractions. Determine which number in each of the following pairs is greater by finding equivalent fractions with equal denominators.

$\frac{2}{7}, \frac{1}{3}$ $\qquad$ $\frac{6}{5}, 1\frac{2}{9}$ $\qquad$ $\frac{5}{8}, \frac{7}{12}$ $\qquad$ $\frac{9}{10}, \frac{8}{9}$

b) Locate each of the following numbers on the number line below.

$\frac{2}{3}, \frac{5}{6}, \frac{7}{5}, \frac{3}{7}, \frac{8}{10}, \frac{6}{4}, \frac{1}{5}$



3. Equal Numerators

a) Fractions are easily compared if their numerators are equal. Determine which number in each of the following pairs is greater. (Do not convert the numbers so that their denominators are equal.)

$\frac{1}{8}, \frac{1}{9}$ $\qquad$ $\frac{3}{7}, \frac{3}{6}$ $\qquad$ $\frac{8}{9}, \frac{8}{10}$ $\qquad$ $3\frac{2}{5}, 3\frac{2}{3}$

b) In your own words, describe how you determine which of two numbers is greater when the numerators are equal.

4. Cross-Product Method

If the denominators of the two fractions being compared are not equal, one technique for determining which is greater is to multiply both fractions by the product of their denominators. The following steps will illustrate the "cross-product method" for fractions $\frac{3}{5}$ and $\frac{2}{3}$.

a) Which is greater? $\frac{3}{5}$ or $\frac{2}{3}$?

b) Multiply both by 5 x 3. $(5 \times 3) \times \frac{3}{5} = \frac{5 \times 3 \times 3}{5}$,

$(5 \times 3) \times \frac{2}{3} = \frac{5 \times 3 \times 2}{3}$.

c) Simplify each fraction. $\frac{5 \times 3 \times 3}{5} = 3 \times 3 = 9$,

$\frac{5 \times 3 \times 2}{3} = 5 \times 2 = 10$.

d) Compare the whole numbers. 10 is greater.

e) Which is greater? $\frac{3}{5}$ or $\frac{2}{3}$? (Remember 10 was the result of multiplying $\frac{2}{3}$ by 5 x 3. We are assuming that the greater one is still greater after multiplying by 5 x 3.)

Using the "cross-product method," determine which of the numbers in each of the following pairs is greater.

$\frac{3}{4}, \frac{2}{3}$ $\qquad$ $\frac{5}{7}, \frac{2}{3}$ $\qquad$ $\frac{7}{5}, \frac{4}{3}$ $\qquad$ $\frac{8}{9}, \frac{7}{8}$

5. Decimal Method (Decimals developed later)

If decimals have been discussed, another way of comparing rational numbers is to express them in decimal form. For example, the fractions $\frac{1}{3}$ and $\frac{2}{5}$, written as decimals, are $0.33\overline{3}$ and 0.4. Here it is easy to determine that 0.4 is greater than $0.33\overline{3}$, and hence that $\frac{2}{5}$ is greater than $\frac{1}{3}$. Use the decimal method to determine which number in each of the following pairs is greater.

$\frac{1}{3}, \frac{7}{20}$ $\qquad$ $\frac{4}{25}, \frac{1}{8}$ $\qquad$ $\frac{4}{5}, \frac{3}{4}$

6. For each of the following problems, write a mathematical expression that describes the situation.

a) One weekend Matt was successful in picking the winners in 8 out of 15 basketball games. The following weekend he picked 10 winners out of 19 games. On which weekend did Matt pick the greater fraction of the winners?

b) The Blue Jay scout troop had 5 girls and 7 boys among its members. The Bear scout troop had 7 girls and 8 boys. Which troop had the greater fraction of boys?

7. Write a problem that might be represented by each of the following expressions.

a) $\frac{2}{7} < \frac{3}{5}$ b) $\frac{6}{11} < \frac{5}{6}$

TEACHER TEASER

Little is known about the life of Diophantus, the Greek father of algebra, except his age at death, which has been preserved in the famous riddle shown here.

Diophantus' youth lasted $\frac{1}{6}$ of his life.

He grew a beard after $\frac{1}{12}$ more.

After $\frac{1}{7}$ more of his life he married.

Five years later he had a son.

His son lived exactly $\frac{1}{2}$ as long as his father.

Diophantus died just 4 years afterwards.

How old was Diophantus when he died? (If you need the equation you can find it on page 100.)

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ACTIVITY 13

ADDITION OF RATIONAL NUMBERS

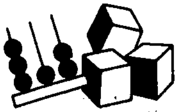
FOCUS:

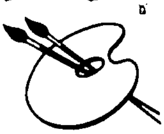
In developing an operation like addition on rational numbers, one must be concerned with the concept of sum as well as with a process for computing sums. In this activity, you will be studying lessons that attend to both the concept and the process of finding the sum of rational numbers that are expressed as fractions.

MATERIALS:

Cuisenaire rods or strips of graph paper; elementary school mathematics textbook series.

DISCUSSION:





Addition of rational numbers should be introduced using Physical Embodiments or Pictorial Representations.

As you recall from previous activities, the usual embodiments and representations for rational numbers are part of a set, part of a region, and the number line. In this activity, you will find that the part-of-a-set embodiment does not work very well for operations on rational numbers. The development of addition of rational numbers in this activity has the following steps.

Lesson 1: Physical Embodiments and Pictorial Representations of sums of rational numbers expressed as fractions with equal and unequal denominators--no computations.

Lesson 2: Embodiments and Representations of sums of fractions with equal denominators--sums computed.

Lesson 3: Computation of sums of fractions with equal denominators--no embodiments.

Lesson 4: Computation of sums of fractions with unequal denominators, by expressing the fractions as equivalent fractions with equal denominators and then adding, as in Lesson 3--no embodiments.

For example, to compute

$$\begin{array}{r} \frac{1}{3} \\ +\ \frac{1}{2} \\ \hline \end{array},$$

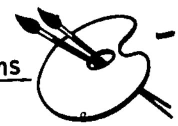
we find $\frac{1}{3} = \frac{2}{6}$ and $\frac{1}{2} = \frac{3}{6}$. So we can write:

$$\begin{array}{rcl} \frac{1}{3} & = & \frac{2}{6} \\ +\ \frac{1}{2} & = & \frac{3}{6} \\ \hline & & \frac{5}{6} \end{array}.$$

DIRECTIONS:

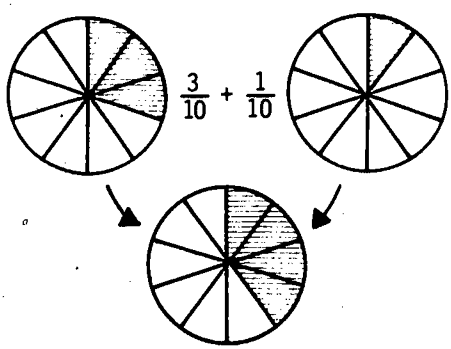
Work through these lessons as if you were learning the material for the first time. You may want to work in small groups--taking turns playing the role of teacher.

1. Lesson 1

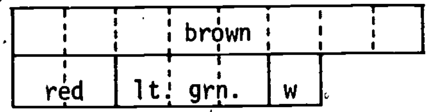


Embodiments and Representations of Sums of Fractions

Two pies were prepared for Thanksgiving dinner; $\frac{3}{10}$ of one pie was left, and $\frac{1}{10}$ of the other. Represent the remaining pie.



Use the brown Cuisenaire rod as 1, the white as $\frac{1}{8}$, the red as $\frac{1}{4}$, etc. Make a chart on the side, if you wish, with the appropriate values. (You may use the strips of paper prepared for Activity 10 in place of Cuisenaire rods.)

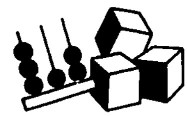


Embody $\frac{1}{8} + \frac{3}{8}$, using the rods or strips.

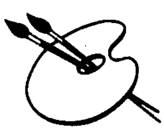
Embody $\frac{1}{8} + \frac{1}{4}$, using the rods or strips.

Embody $\frac{1}{4} + \frac{3}{8}$, using the rods or strips.

Embody $\frac{3}{4} + \frac{1}{8}$, using the rods or strips.



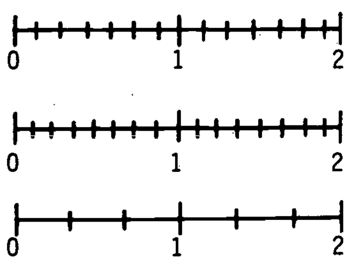
Use a number line to represent the following sums.



$\frac{3}{7} + \frac{3}{7}$

$\frac{1}{4} + \frac{5}{8}$

$\frac{2}{3} + \frac{8}{6}$



Lesson 1--Questions

a) By the time children encounter addition of fractions, they have already had considerable experience with addition of whole numbers and with the representation of rational numbers as fractions. Explain how each part of Lesson 1 builds on this previous experience.

b) Are the commutative and associative properties for addition true for rational numbers? Give examples to illustrate your answer. These properties are discussed in Appendix B.

Determine the appropriate teaching order for the following addition problems. Give reasons for the order you have selected.

a) $1\frac{1}{2} + 1\frac{1}{3}$

b) $\frac{1}{6} + \frac{5}{12}$

c) $1\frac{1}{2} + \frac{1}{2}$

d) $\frac{1}{3} + \frac{2}{3}$

e) $\frac{3}{4} + \frac{1}{7}$

f) $\frac{1}{4} + \frac{1}{4}$

2. Lesson 2

Determination of Sum of Fractions with Equal Denominators from Embodiments and Representations

Return to Lesson 1 and determine each of the following sums by analyzing its embodiment or representation.

$\frac{3}{10} + \frac{1}{10} =$

$\frac{1}{8} + \frac{3}{8} =$

$\frac{3}{7} + \frac{3}{7} =$

$\frac{2}{3} + \frac{4}{3} =$

Look back at these examples. Do you see a way to compute the sum without referring to the embodiment or representation? Describe the way.

Lesson 2--Questions

a) Use one of the three embodiments and representations (circles, strips or rods, number line) to determine the sum $\frac{1}{2} + \frac{1}{3}$. How does this example differ from those in Lessons 1 and 2? What special problems does this difference cause?

b) Use a part-of-a-set model to embody $\frac{1}{3} + \frac{2}{3}$. What problems do you face here?

3. Lesson 3

Adding Fractions with Equal Denominators without Embodiments or Representations

$\frac{1}{5} + \frac{2}{5}$ can be thought of as "one of something plus two more of that thing." Thinking of it in that way, it is natural to write

$$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}.$$

Verbalize each of the following as "so many of something plus so many more equal __________" to determine what each sum equals.

$\frac{1}{3} + \frac{2}{3} =$ $\quad\quad$ $\frac{4}{5} + \frac{6}{5} =$ $\quad\quad$ $\frac{12}{731} + \frac{21}{731} =$

State a rule or algorithm for adding fractions with equal denominators. Then use your algorithm to solve the following.

$\frac{1}{7} + \frac{4}{7} + \frac{9}{7} =$ $\quad\quad$ $\frac{18}{315} + \frac{1}{315} =$

Lesson 3--Questions

Where does the "so many of something plus so many more ..." language break down on a problem like $\frac{2}{5} + \frac{3}{7}$?

4. Lesson 4

Adding Fractions with Unequal Denominators

Now the children know what addition of rational numbers means, and they know how to add fractions that have equal denominators. It is time to learn to add <u>any</u> fractions using the following facts:

Equivalent fractions represent the same rational number. If two fractions have unequal denominators, two additional fractions can be found that have equal denominators and are equivalent to the first fractions. So, presented with the problem of adding a pair of fractions with unequal denominators, you should find equivalent fractions with equal denominators, and then add them, using the skills that you have already developed.

For example, to find $\frac{1}{4} + \frac{5}{6}$, we proceed as follows.

i) Find equivalent fractions with equal (common) denominators; e.g.,

$\frac{1}{4} = \frac{6}{24}$ $\frac{5}{6} = \frac{20}{24}$

ii) Add the new fractions.

$\frac{6}{24} + \frac{20}{24} = \frac{26}{24}$

iii) Represent the sum as the sum of the original fractions.

$\frac{1}{4} + \frac{5}{6} = \frac{26}{24}$.

Go through the above three steps to find $\frac{1}{4} + \frac{2}{5}$.

i) $\frac{1}{4} =$

$\frac{2}{5} =$

ii)

iii) $\frac{1}{4} + \frac{2}{5} =$

Apply the above steps to the following problems.

$\frac{1}{2} + \frac{1}{3} =$ $\frac{1}{4} + \frac{1}{8} =$ $\frac{1}{32} + \frac{16}{40} =$

In several of the above examples, the common denominators may have been larger than they needed to be. Activity 11 provides several appropriate approaches to developing techniques for finding the least common denominator.

5. Several elementary school textbooks use the number line to develop addition of rational numbers. Study such a development and give two advantages and two disadvantages (rising from potential ambiguities) for such an approach. (For example, does the number line model fit all situations? Do children have experience with measurement?)

6. What difficulties do you feel children will most frequently encounter in addition of rational numbers?

7. Devise one activity for children to promote practice in addition of rational numbers. (You may find some suggestions in the teacher's editions of elementary school textbooks.)

8. Write mathematical sentences that represent the following problems. Then solve the problems.

   a) The recipes for cookies, a pie, and bread call for $2\frac{1}{2}$, $\frac{2}{3}$, and $2\frac{1}{4}$ cups of flour, respectively. How much flour is needed to make all three items?

   b) At the celebrated Calaveras County Frog-Jumping contest, each entrant is allowed three jumps. One ambitious amphibian was able to accomplish $6\frac{3}{8}$ feet, $7\frac{2}{3}$ feet, and $5\frac{1}{4}$ feet. What was the total length jumped?

9. Write a real-world problem that might be represented by each of the following mathematical sentences.

   a) $\frac{7}{8} + \frac{5}{8} + 3\frac{1}{4}$ b) $4\frac{3}{8} + 2\frac{1}{4}$

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ACTIVITY 14

SUBTRACTION OF RATIONAL NUMBERS

FOCUS:

In this activity, you will have an opportunity to compare subtraction and addition of rational numbers and to note areas of difficulty peculiar to subtraction.

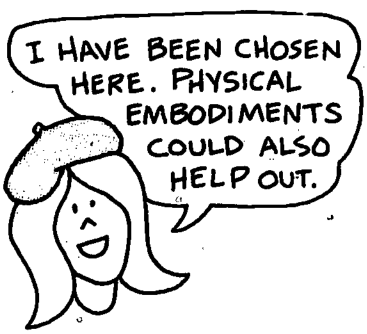
DISCUSSION:

As with addition, the subtraction of rational numbers is easiest to embody or represent by using models with the same number of subdivisions. In these "equal denominator" situations, the representation is analogous to the "backward hops" on the number line used with integers and to the "take away" model used with whole numbers.

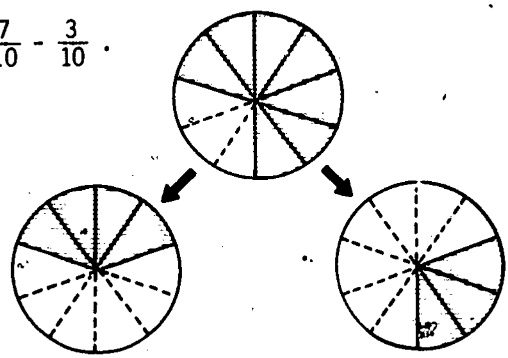
So we will proceed, as we did with addition, to present the concept of subtraction using the "equal denominator" situations. Then we will learn to compute subtraction of fractions with equal denominators, and finally we will convert unequal denominator problems into equal denominator ones.

Region Model

To illustrate subtraction using the region model, consider the problem:



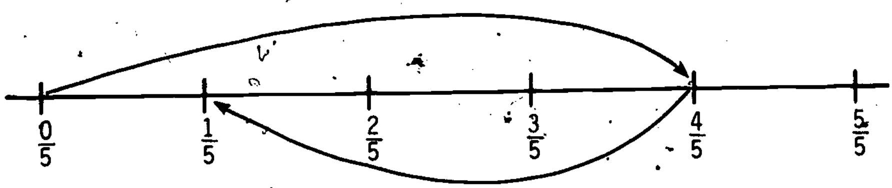
$\frac{7}{10} - \frac{3}{10}$.



Here $\frac{3}{10}$ is removed, to leave $\frac{4}{10}$.

Measurement Model

The typical pictorial representation used in a measurement model is the number line. The jumps on the number line below illustrate the solution to the problem, $\frac{4}{5} - \frac{3}{5}$.



DIRECTIONS:

1. a) What previously learned representations could you refer to in a child's experience as a basis for developing each of the above representations?

   b) State a rule for subtracting fractions with equal denominators.

   c) Compute each of the following.

   $\frac{2}{3} - \frac{1}{3}$ $\qquad$ $\frac{53}{271} - \frac{27}{271}$

   $\frac{3}{4} - \frac{2}{4}$ $\qquad$ $\frac{16}{17} - \frac{11}{17}$

   $\frac{5}{3} - \frac{4}{3}$ $\qquad$ $\frac{296}{513} - \frac{171}{513}$

2. Referring to the Discussion above and to the models developed in Activity 13, outline an introductory lesson using either:

   a) a circular region model, or

   b) a number line model,

   to develop the concept of subtraction of rational numbers.

3. As we said, to deal with unequal denominators, just find equivalent fractions with equal denominators and proceed as in (1) and (2). Compute each of the following.

$\frac{2}{3} - \frac{1}{2}$ $\frac{3}{13} - \frac{1}{11}$ $2 - \frac{4}{3}$

$\frac{1}{7} - \frac{1}{6}$ $\frac{29}{120} - \frac{4}{40}$ $\frac{5}{2} - 1$

$\frac{21}{32} - \frac{9}{16}$ $\frac{1}{10} - \frac{1}{100}$ $2\frac{1}{2} - 1\frac{1}{3}$

4. Study the examples below. Put them in an appropriate sequential instructional order. Justify your order.

   a) $3\frac{2}{3} - 1\frac{1}{2}$ b) $1\frac{1}{3} - \frac{2}{3}$

   c) $\frac{3}{4} - \frac{1}{4}$ d) $4\frac{3}{5} - 1\frac{1}{5}$

   e) $2 - \frac{1}{2}$ f) $\frac{3}{4} - \frac{1}{2}$

   g) $\frac{5}{6} - \frac{2}{5}$ h) $3\frac{1}{5} - 1\frac{2}{3}$

5. For two of the examples in (4) above, write an appropriate objective. For example, for $\frac{3}{4} - \frac{1}{4}$ one might say: "to subtract simple fractions having a common denominator without 'borrowing.'"

6. In the example at the right, children must rename a whole number as a fraction before subtracting; that is, $1 = \frac{4}{4}$.

$$\begin{array}{r} 3\frac{1}{4} = 2\frac{5}{4} \\ -\ 1\frac{3}{4} = 1\frac{3}{4} \\ \hline \end{array}$$

   a) In what way is this procedure similar to a procedure used in the subtraction of whole numbers?--for example, 43 - 17?

   b) What concept is involved in the rewriting of 1 as $\frac{4}{4}$? Give an illustration of the type of exercises you might give to help children acquire that concept. (You might find it helpful to refer to the region model of $\frac{1}{4}$.)

7. Introspection, or analyzing your own thinking, can often be helpful in gaining some insights into an appropriate teaching sequence. The analysis of an algorithm is particularly helpful when trying to assess causes of pupil errors. As you work the

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subtraction example given below, analyze each step you take. Briefly record these steps. The analysis might begin this way:

i) "I notice that the example is a subtraction example."

ii) "The denominators are not equal," etc.

$$7\frac{1}{8} - 3\frac{4}{5}$$

8. Identify the step(s) in the subtraction example that you think would require new learning skills or present difficulties for children. Explain your choice.

9. Throughout this activity, the examples have included only positive numbers. This has been done because only the positive rationals are taught in the elementary school and because it helps to emphasize methods for explaining the operations. The set of rational numbers, however, includes both positive and negative numbers. In performing the operations on negative rational numbers, one uses the same algorithms. Further, the techniques for determining the sign associated with the answer are the same as those you discovered in your work with integers in Section I. Compute each of the following, using these rules.

a) $\frac{3}{4} + (\frac{-8}{9})$ b) $\frac{4}{7} - \frac{9}{12}$

c) $(\frac{-15}{3}) - (\frac{-3}{8})$ d) $(-4\frac{2}{7}) + 2\frac{5}{6}$

10. For each of the following real-world problems, write a mathematical sentence that will help you solve the problem. Then find the solution.

a) The opening price of a certain stock was listed at $68\frac{3}{8}$ on Monday morning. By the end of the week, it was listed at $61\frac{3}{4}$. How much had it dropped in five days?

> The equation for Teacher Teaser on page 89 is:
> $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x$, where x is his age.

b) In 1913, the record time for the mile run was 4 minutes, $14\frac{6}{10}$ seconds. In 1975, the record was 3 minutes, $49\frac{4}{10}$ seconds. How much less time was needed to complete the race in 1975?

11. Write a real-world problem that could be solved by solving each of the following sentences.

a) $17\frac{1}{2} - 2\frac{1}{4}$ b) $\frac{3}{4} - \frac{2}{3}$

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ACTIVITY 15

MULTIPLICATION OF RATIONAL NUMBERS

FOCUS:

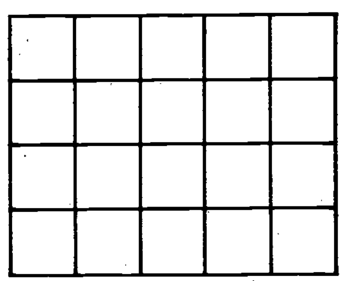
In this activity you will have an opportunity to study a model that is commonly used for introductory work in multiplication of rational numbers. Then you will work on the algorithm for multiplying fractions.

MATERIALS:

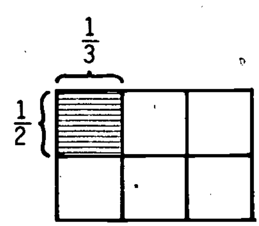
Elementary school math textbook series

DISCUSSION:

In introducing multiplication of whole numbers, one of the models used was the array. For the multiplication problem 4 x 5, the region was partitioned into 4 units by 5 units.



In a somewhat similar fashion, the concept of a region and subregions is used in multiplication of rational numbers. The main difference is that the total region used is considered as representing 1 unit. The factors involved in the multiplication subdivide the unit. Thus, $\frac{1}{2} \times \frac{1}{3}$ can be seen as 1 subregion out of the total of 6 subregions. The typical language used in the elementary school is "$\frac{1}{2}$ of $\frac{1}{3}$." The mathematical interpretation of "of" in multiplication with fractions is "times." You may wish to discuss in class the relationship between "of" and "times" in multiplying fractions.

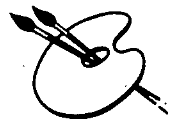


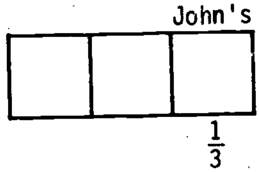
DIRECTIONS:

Study the model presented below, and then discuss and answer the questions that follow it.

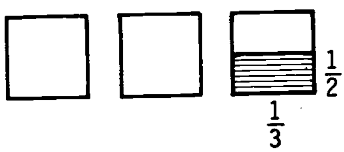
MODEL: THE REGION

1. *Suppose three boys had a very large candy bar to share. John got $\frac{1}{3}$. Show $\frac{1}{3}$.

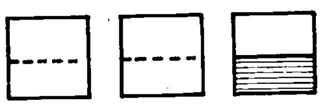




2. John decided he would not eat all of his share at once. Instead, he ate $\frac{1}{2}$ of his share. Show $\frac{1}{2}$ of John's share.



3. What portion of the entire candy bar did John eat?

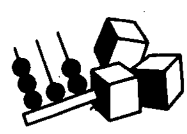


4. $\frac{1}{2} \times \frac{1}{3}$ means the same as $\frac{1}{2}$ of $\frac{1}{3}$. What is $\frac{1}{2} \times \frac{1}{3}$?

$\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$,

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

*This example could be carried out using a paper-folding technique. Fold a piece of paper into thirds. Keeping the paper folded in thirds, fold it in half. Now shade the resulting folded shape with a marker. Unfold the paper and show that $\frac{1}{6}$ of the total area has been shaded.

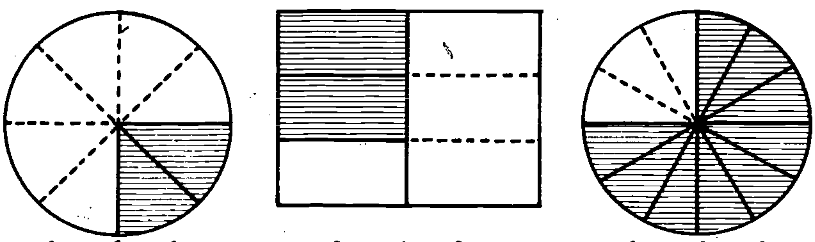


1. Make a drawing using the region model for each of the following examples.

   a) $\frac{2}{3} \times \frac{3}{4}$ b) $\frac{1}{2} \times \frac{2}{5}$ c) $\frac{3}{5} \times \frac{2}{3}$

2. In multiplication with whole numbers (other than 0), the product is always greater than or equal to any of the factors. How would you explain to a child that the product is <u>less</u> than the factors when using fractions less than 1? For example, $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

3. Study the models and the associated number sentences. Once the concept of the product of rational numbers has been developed, could a pattern for the multiplication algorithm be discovered from examples of this sort? Do you think the rectangular or the circular region model would be easier for a child to grasp? Or does it make any difference?



$\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$ $\frac{2}{3}$ of $\frac{1}{2} = \frac{2}{6}$ $\frac{1}{3}$ of $\frac{3}{4} = \frac{3}{12}$

4. Place the following examples in order of difficulty.

   ______ a) $3\frac{2}{3} \times 6\frac{1}{2} \times 9\frac{4}{7}$ ______ b) $3\frac{3}{4} \times \frac{2}{3}$

   ______ c) $\frac{1}{3} \times 2$ ______ d) $1\frac{2}{3} \times 2\frac{4}{5}$

   ______ e) $\frac{1}{2} \times \frac{1}{3}$

5. Mark the following examples as true or false. Correct the false examples.

_____ a) $2\frac{1}{2} \times 6\frac{1}{4} = 6\frac{1}{4} \times 2\frac{1}{2}$

_____ b) $(3\frac{3}{5} \times 2\frac{1}{8}) \times 6\frac{1}{2} = 3\frac{3}{5} \times (2\frac{1}{8} \times 6\frac{1}{2})$

_____ c) $3\frac{4}{5} \times 2\frac{1}{2} = (3 \times 2) + (\frac{4}{5} \times \frac{1}{2})$

_____ d) $3\frac{4}{5} \times 2\frac{1}{2} = (3 + \frac{4}{5}) \times 2\frac{1}{2}$

_____ e) $3\frac{4}{5} \times 2\frac{1}{2} = (3 + \frac{4}{5}) + (2 + \frac{1}{2})$

6. a) State a rule or algorithm for multiplying fractions.

   b) Apply your algorithm to compute each of the following.

$\frac{1}{8} \times \frac{3}{4}$ $\qquad$ $\frac{1}{8} \times \frac{1}{7} \times \frac{1}{6}$

$\frac{5}{4} \times \frac{4}{5}$ $\qquad$ $\frac{9}{10} \times \frac{8}{3} \times \frac{11}{2}$

$\frac{1}{2} \times \frac{1}{3}$ $\qquad$ $\frac{121}{732} \times \frac{15}{69}$

7. Which of the following properties* hold for multiplication of rational numbers? Give an example (or counterexample) for each.

   a) Associative property

   b) Commutative property

   c) Distributive property (over addition)

   d) Closure

8. Study the development of multiplication of rational numbers in an elementary textbook series. Record for discussion any similarities or dissimilarities to the presentation in this activity.

*These properties are discussed in Appendix B.

9. The analysis of an algorithm is particularly helpful when trying to assess causes of pupil errors. As you work the multiplication example given below, analyze each step you take. Briefly record these steps. The exercise might begin this way:

$$3\frac{2}{3} \times 2\frac{1}{4} =$$

a) "I notice that the example is a multiplication problem."

b) "The fractions are mixed."

.
.
.

10. Identify the step(s) in the multiplication example that you think would require new learning skills or present difficulties for children. Why?

11. a) A fast turtle can go $\frac{1}{10}$ mile in an hour. At this rate, how far can he go in 24 hours?

b) The United States government has calculated that the average household spends $\frac{2}{5}$ of its income on food. If the gross weekly pay for a family is $175, what should be the amount that is spent for food?

12. Write a real-world problem that could be represented by each of the following sentences.

a) $1\frac{3}{4} \times 2\frac{1}{2}$ b) $2\frac{3}{8} \times (3\frac{1}{4})$

ACTIVITY 16

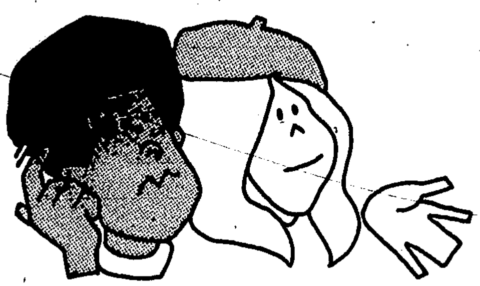
DIVISION OF RATIONAL NUMBERS

FOCUS:

In this activity you will have an opportunity to study and discuss developmental activities and problems associated with division of rational numbers.

DISCUSSION:

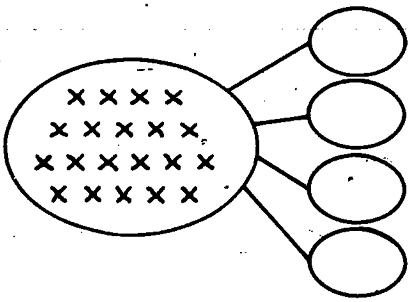
Whenever possible it seems important to use Physical Embodiments or Pictorial Representations in introducing a concept to children. In particular, we have made an effort to present embodiments and representations of rational numbers, the operations on them, and the corresponding computations with fractions. Division of rational numbers poses special problems. Most embodiments and representations that have been proposed have proved to be harder than the concept of division itself. Perhaps a Nobel Prize in mathematics education should be given to the person who discovers a good physical embodiment or pictorial representation of division of rational numbers. In this activity, we will present a "concrete" embodiment which introduces the concept of division of rational numbers; and then we will proceed to develop computational procedures for dividing fractions, with Mathematical Relationships leading the way.



DIRECTIONS:

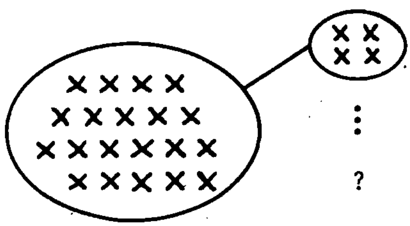
1. You recall that there are two standard situations that call for division of whole numbers.

There are 20 objects to be divided into 4 equal sets; how many in each set?



How many in each set?

There are 20 objects to be separated into sets with 4 objects in each; how many sets?



How many sets?

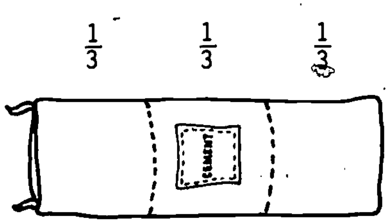
Both of these situations call for computing $20 \div 4$.

There follows a division situation for rational numbers that is analogous to one of the above. Study it and then answer the questions which follow it.

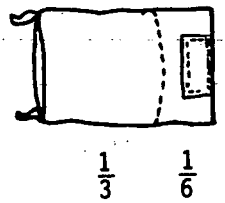
EXAMPLE

Mr. Jones uses $\frac{1}{3}$ bag of cement for each section of sidewalk he prepares.

a) If he has 1 bag, how many sections can he prepare? $1 \div \frac{1}{3} = 3$.



b) If he has $\frac{1}{2}$ bag, how many sections can he prepare? $\frac{1}{2} \div \frac{1}{3} = 1\frac{1}{2}$.



Since $\frac{1}{6}$ is $\frac{1}{2}$ of $\frac{1}{3}$, he can prepare 1 section plus $\frac{1}{2}$ of a section or $1\frac{1}{2}$ sections.

a) Devise a similar story to portray $\frac{2}{3} \div \frac{1}{4}$.

b) Discuss how helpful you feel such stories are.

c) Discuss the following story problem as a possible introductory situation for division with fractions.

Ms. Hall has $4\frac{1}{2}$ cups of sugar. It takes $1\frac{1}{2}$ cups for each batch of cookies. How many batches can she bake?

d) Which of the two standard division situations mentioned above is embodied in the cement example, the cookie example, and the example that you made up?

2. Three methods of division with fractions will be presented here. Each method is presented using portions of child lessons which are boxed in. In addition, the child lesson is amplified and discussed at an adult level. Be sure to work through the child-level presentation as well, making sure you understand the adult-level explanations, and answer any questions that are asked. Be attentive to identifying possible sources of confusion for the child. Discuss each procedure to be sure you understand and could explain it.

a) <u>Inverse Relationship Method</u>

The child is already familiar with whole number division and with multiplication of fractions. Here, through a series of examples, the child is encouraged to see that:

i) A division sentence can be written as a multiplication sentence; for example,

$$12 \div 4 = \frac{12}{4} \text{ and } \frac{12}{4} = 12 \times \frac{1}{4}$$

$$\text{or } 12 \div 4 = 12 \times \frac{1}{4}$$

In general: $a \div b = \frac{a}{b} = a \times \frac{1}{b}$

ii) There is a pattern between two of the numbers of the division and multiplication sentences; for example, 4 and $\frac{1}{4}$, 3 and $\frac{1}{3}$, etc. In general, writing a multiplication sentence for a division sentence one writes $\frac{1}{b}$ in place of b; that is,

$$a \div b = a \times \frac{1}{b}.$$

Portion of Child's Lesson

Solve each of the division and multiplication sentences.

Division Sentence	Multiplication Sentence
$12 \div 4 =$	$12 \times \frac{1}{4} =$
$9 \div 3 =$	$9 \times \frac{1}{3} =$
$4 \div 2 =$	$4 \times \frac{1}{2} =$
$5 \div 5 =$	$5 \times \frac{1}{5} =$

What numbers are different in the two sentences? How are they related? Write multiplication sentences for the following:

$15 \div 3 =$ ______________________

$18 \div 6 =$ ______________________

Now the child is introduced to the notion of the multiplicative inverse of a fraction. Two numbers are inverses if their product is 1. Thus, $\frac{1}{4}$ and 4 are inverses, as are $\frac{2}{3}$ and $\frac{3}{2}$, as well as b and $\frac{1}{b}$, since

$$\frac{1}{4} \times 4 = 1; \quad \frac{2}{3} \times \frac{3}{2} = 1; \quad b \times \frac{1}{b} = 1.$$

Portion of Child's Lesson

Find an inverse for each number. Check by finding the product.

Number	Inverse	Product
4	$\frac{1}{4}$	$4 \times \frac{1}{4} = 1$
3		
2		
5		
$\frac{2}{3}$	$\frac{3}{2}$	$\frac{2}{3} \times \frac{3}{2} = 1$
$\frac{3}{2}$		
$\frac{1}{4}$		
$\frac{12}{4}$		
$\frac{2}{7}$		

Finally the child is asked to write a multiplication sentence for any division sentence, replacing the divisor with its inverse. It is not inappropriate to allow the child to discover that this is equivalent to "inverting the divisor" and multiplying. Note that an appropriate order of development is to solve

i) a whole number divided by a whole number;

ii) a whole number divided by a fraction;

iii) a fraction divided by a fraction.

Portion of Child's Lesson

Write a multiplication sentence for each of the following division sentences. Solve.

Division Sentence	Multiplication Sentence
$18 \div 9$	$18 \times \frac{1}{9} = \frac{18}{9}$
$15 \div 4$	
$7 \div \frac{1}{3}$	
$4 \div \frac{2}{3}$	
$8 \div \frac{3}{2}$	
$\frac{2}{8} \div \frac{3}{4}$	
$\frac{5}{9} \div \frac{6}{5}$	
$\frac{2}{7} \div 8$	

b) <u>Common Denominator Method</u>

Another method presented in current textbooks for dividing with fractions is the common denominator method. Children have had practice in finding common denominators for adding and subtracting with fractions, so that this prerequisite skill has been established.

Portion of Child's Lesson

Study the example below which uses the common denominator method and then solve the examples below using this method.

$$\frac{5}{8} \div \frac{1}{6} = \frac{15}{24} \div \frac{4}{24}$$
$$= \frac{15 \div 4}{24 \div 24}$$
$$= \frac{15 \div 4}{1}$$
$$= 15 \div 4$$
$$= \frac{15}{4}$$

The first step calls for the child to find a common denominator. The second step seems reasonable to children even though it is hard to explain directly. You can see (with some concentration) that it is true, as follows:

$$\frac{15}{24} \div \frac{4}{24} = \frac{15}{24} \times \frac{24}{4} = \frac{15 \times 24}{24 \times 4} = \frac{15 \times 24}{4 \times 24} = \frac{15}{4} \times \frac{24}{24} = \frac{15}{4} \div \frac{24}{24}$$

$$= \frac{\frac{15}{4}}{\frac{24}{24}} = \frac{15 \div 4}{24 \div 24}$$

i) Try to see if you can justify each equality in the above chain.

ii) Use the common denominator method to solve each of the following.

a) $\frac{3}{4} \div \frac{2}{3}$ b) $\frac{4}{5} \div \frac{1}{6}$

c) $\frac{3}{4} \div \frac{5}{6}$ d) $1\frac{1}{3} \div \frac{4}{5}$

e) $2\frac{3}{8} \div 1\frac{2}{3}$ f) $2\frac{1}{6} \div 3\frac{1}{7}$

c) Complex Fraction Method

In presenting this method to a child, you should keep the following facts in mind. You know that any indicated division can be written as a fraction. For example, 8 ÷ 11 can be written as $\frac{8}{11}$. In a similar way, $\frac{1}{3} \div \frac{1}{2}$ can be written as

$$\frac{\frac{1}{3}}{\frac{1}{2}}.$$

The complex fraction method is based upon the following concepts.

i) $\frac{7}{8} \div \frac{3}{5} = \frac{\frac{7}{8}}{\frac{3}{5}}$

ii) a x 1 = a for any number a.

iii) 1 can be written in an infinite number of ways: as $\frac{3}{3}$, $\frac{\frac{1}{2}}{\frac{1}{2}}$, etc.

Portion of Child's Lesson

Study the example below and discuss each step before working the examples that follow.

$$\frac{7}{8} \div \frac{3}{5} = \frac{\frac{7}{8}}{\frac{3}{5}}$$

$$= \frac{\frac{7}{8}}{\frac{3}{5}} \times \frac{\frac{5}{3}}{\frac{5}{3}}$$

$$= \frac{\frac{7}{8} \times \frac{5}{3}}{1}$$

$$= \frac{7}{8} \times \frac{5}{3} = \frac{35}{24}$$

Use the complex fraction method to solve the following examples.

a) $\frac{3}{5} \div \frac{2}{4}$ b) $\frac{3}{4} \div \frac{8}{9}$ c) $\frac{2}{5} \div \frac{3}{7}$

d) $\frac{4}{5} \div \frac{3}{8}$ e) $1\frac{2}{3} \div \frac{4}{5}$ f) $2\frac{3}{4} \div 1\frac{7}{9}$

How is the complex fraction method similar to the inverse relationship method? Which of the above methods do you feel would be easiest to explain to a child?

3. Put the following examples in an appropriate teaching sequence. Justify your sequence.

_____ a) $6\frac{7}{8} \div 4\frac{3}{4}$

_____ b) $\frac{1}{3} \div \frac{3}{4}$

_____ c) $4 \div \frac{1}{7}$

_____ d) $3\frac{1}{2} \div \frac{2}{5}$

4. Tell which of the following are true and which are false. Correct the false examples.

a) $3\frac{1}{3} \div 2\frac{1}{5} = \frac{10}{3} \times \frac{11}{5}$

b) $3\frac{1}{3} \div 2\frac{1}{5} = (3 \div 2) + (\frac{1}{3} \div \frac{1}{5})$

c) $3\frac{1}{3} \div 2\frac{1}{5} = 2\frac{1}{5} \div 3\frac{1}{3}$

d) $(1\frac{7}{8} \div 2\frac{1}{2}) \div 3\frac{1}{4} = 1\frac{7}{8} \div (2\frac{1}{2} \div 3\frac{1}{4})$

5. In division with whole numbers (except when dividing by 1), the quotient is less than the dividend. In division with fractions this is not always true. Discuss. When is the quotient greater than the dividend? Why does this occur?

6. Throughout this activity, the examples have included only positive numbers. This has been done because only the positive ra-

tionals are taught in the elementary school and because they help to emphasize methods for explaining the operations. The set of rational numbers, however, includes both positive and negative numbers. In performing the operations on negative rational numbers one uses the same algorithms. Further, the techniques for determining the sign associated with the answer are the same as those you discovered in your work with integers in Section I. Answer each of the following problems using these rules.

a) $(-\frac{3}{8}) \times \frac{4}{7}$

b) $(-\frac{6}{8}) \times (-1\frac{2}{3})$

c) $(3\frac{2}{5}) \div (-\frac{3}{5})$

d) $(-2\frac{1}{3}) \div (-1\frac{1}{4})$

7. For each of the following real world problems write a mathematical sentence that will help you solve the problem. Then find the solution.

   a) An elementary school class made scarves for a money-making project. They needed $\frac{1}{4}$ of a yard of material for 1 scarf. How many scarves could they get from $3\frac{1}{2}$ yards of material?

   b) Mr. Smith bought a bicycle for his 3 children to share. On one afternoon they rode the bicycle for $2\frac{3}{4}$ hours. If they actually did share the use of the bicycle equally, how much time did each one ride?

8. Write a real-world problem that could be represented by each of the following sentences.

   a) $8 \div 1\frac{1}{8}$ b) $\frac{3}{4} \div (-1\frac{1}{2})$

ACTIVITY 17

ANALYSIS OF ERROR PATTERNS FOR RATIONAL NUMBERS

FOCUS:

What happens when you get all through teaching some aspect of rational numbers to a child, and he still can't solve the related problems? An important part of an elementary school teacher's job is the continual analysis of the problems children encounter when completing mathematical exercises. This activity will provide several examples of common errors made by children when working with rational numbers. An opportunity will be provided for you to recommend possible causes for these errors and remedies to the causes.

DIRECTIONS:

1. Carefully read each worksheet that has been completed by an elementary school child. You should note the kind of error you observe for each child.
2. For each worksheet:
   a) Describe in words the error pattern that you have noted.
   b) Work the last two computations (the ones in the box) as the child might have.
   c) Suggest an activity that might have prevented this error pattern--or that you might use for remediation.

   Worksheet I has been completed for you, in order to suggest the kinds of analyses and responses that you might give.

WORKSHEET I

Name: *Jim*

A. $\frac{1}{2} + \frac{3}{4} = \frac{4}{6}$

B. $\frac{7}{9} + \frac{2}{3} = \frac{9}{12}$

C. $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

D. $\frac{4}{5} + \frac{5}{8} = \frac{9}{13}$

E. $\frac{1}{8} + \frac{6}{7} = \frac{7}{15}$

F. $\frac{2}{7} + \frac{4}{7} = \frac{6}{7}$

G. $\frac{3}{8} + \frac{2}{5} = \frac{5}{13}$

H. $\frac{9}{11} + \frac{2}{7} = \frac{11}{18}$

Error Pattern Noted:

In adding fractions with unequal denominators, Jim is adding numerators and denominators.

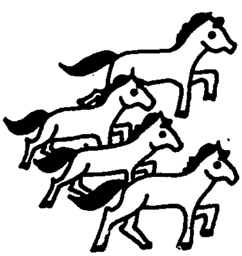
Diagnosis of Possible Causes:

- Jim may not have learned how to find common denominators, so he adds denominators, not knowing what else to do. The fact that he solves the common denominator problems accurately suggests this.
- Jim's problem may lie deeper. He may not have grasped any one of the three concepts: equivalent fractions, addition of rational numbers, and rational numbers.

Possible Remedial Steps:

You might start investigating by asking Jim why he adds denominators sometimes but not always. Then, depending on Jim's answer, you could work your way back through finding common denominators, equivalent fractions, addition of rational numbers, and rational numbers. In each case, provide Jim with introductory activities to see that he grasps the concepts. Listen to what Jim says for cues.

TEACHER TEASER



A man's will provided that his horses should be shared among his three sons as follows: $\frac{1}{2}$ to his oldest son, $\frac{1}{3}$ to his middle son, and $\frac{1}{9}$ to his youngest son. Unfortunately, the man had only 17 horses when he died, and the sons realized that $\frac{1}{2}$ of 17, $\frac{1}{3}$ of 17 and $\frac{1}{9}$ of 17 would not give whole numbers of horses. The lawyer administering the will agreed to lend the sons a horse so that they would have 18 which they could share as follows: 9 ($\frac{1}{2}$ of 18) to the oldest son, 6 ($\frac{1}{3}$ of 18) to the middle son, and 2 ($\frac{1}{9}$ of 18) to the youngest son. But this division of the 18 horses uses 9 + 6 + 2 = 17 horses so they returned the extra horse to the lawyer. The three sons were satisfied with this solution to the problem, but the fact remains that 17 horses cannot be divided evenly in halves, thirds, and ninths. What is wrong?

WORKSHEET II

Name: Susie

A. $\frac{3}{8} \times \frac{4}{8} = \frac{12}{8}$

B. $\frac{2}{7} \times \frac{3}{7} = \frac{6}{7}$

C. $\frac{1}{3} \times \frac{2}{5} = \frac{3}{15}$

D. $\frac{1}{5} \times \frac{2}{5} = \frac{2}{5}$

E. $\frac{7}{9} \times \frac{2}{9} = \frac{14}{9}$

F. $\frac{3}{4} \times \frac{2}{3} = \frac{6}{12}$

G. $\frac{4}{3} \times \frac{1}{3} =$

H. $\frac{6}{11} \times \frac{2}{11} =$

Error Pattern Noted:

Diagnosis of Possible Causes:

Possible Remedial Steps:

WORKSHEET III

Name: Jeffrey

A. $3\frac{3}{8} - 1\frac{1}{2} = 2\frac{1}{8}$

B. $4\frac{2}{6} - 2\frac{1}{2} = 2\frac{1}{6}$

C. $2\frac{5}{6} - 1\frac{2}{3} = 1\frac{1}{6}$

D. $6\frac{1}{3} - 2\frac{4}{6} = 4\frac{2}{3}$

E. $4\frac{7}{8} - 1\frac{1}{4} = 3\frac{5}{8}$

F. $5\frac{3}{4} - 2\frac{1}{4} = 3\frac{2}{4}$

G. $3\frac{2}{3} - 1\frac{5}{6} =$

H. $7\frac{2}{9} - 3\frac{1}{3} =$

Error Pattern Noted:

Diagnosis of Possible Causes:

Possible Remedial Steps:

WORKSHEET IV

Name: Jane

A. $\frac{7}{8} \div \frac{2}{5} = \frac{16}{35}$

B. $\frac{2}{4} \div \frac{3}{4} = \frac{8}{12}$

C. $\frac{3}{4} \div \frac{4}{5} = \frac{16}{15}$

D. $\frac{1}{8} \div \frac{2}{3} = \frac{16}{3}$

E. $\frac{3}{5} \div \frac{2}{5} = \frac{15}{10}$

F. $\frac{7}{9} \div \frac{2}{7} = \frac{18}{49}$

G. $\frac{5}{6} \div \frac{4}{7} =$

H. $\frac{8}{7} \div \frac{2}{5} =$

Error Pattern Noted:

Diagnosis of Possible Causes:

Possible Remedial Steps:

ACTIVITY 18

SEMINAR

FOCUS:

Rational numbers are a topic of major importance in the elementary school mathematics curriculum--and a particularly difficult topic to teach. Teaching rational numbers is difficult because there are several aspects that are more complicated than is true for whole numbers (e.g., order, equivalence). This activity will provide an opportunity to discuss some of these difficulties and to gain some perspective for your future teaching. Some possible questions are raised to instigate your thinking and the class discussion.

MATERIALS:

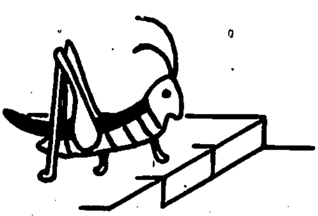
Elementary school mathematics textbook series.

DIRECTIONS:

1. Examine a textbook series and list the major ideas related to rational numbers developed in each grade. (Use the topics of Activities 6 through 17 in this section as a guide to your list of topics.)
2. What are the key ideas that must be carefully developed for each of the operations with fractions?
3. Why might children have some difficulty in determining which of two fractions is greater? (Do they have the same difficulty with ordering whole numbers?)
4. Children expect a "bigger" answer for multiplication and a "smaller" answer for division. (The bonds relating multiplication to "bigger" and division to "smaller" have been well established with whole numbers.) How can you help children see that the opposite is true with fractional numbers less than 1?

5. Do physical referents <u>always</u> help? Are there some instances in which the explanation of the model is more complicated than the process it purports to explain? Give some examples.

TEACHER TEASER



A grasshopper standing at one edge of a road wants to jump across to the other side of the road, but because there is a headwind he covers only $\frac{9}{10}$ of the distance across on his first try, so he jumps again, and although he aims to land just at the edge of the road, the headwind again reduces his jump by $\frac{1}{10}$. The grasshopper jumps again and again, each time covering $\frac{9}{10}$ as much as he intended. Show (1) that he will never get all the way across the road, and (2) that if he makes an infinite number of jumps, the total distance he covers is exactly the width of the road.

Section III

MATHEMATICS OF THE RATIONAL NUMBERS

The activities in Section II presented the mathematical content and pedagogy of rational numbers as they relate to the elementary school curriculum. You had an opportunity both to strengthen your own understanding of rational number concepts and to consider techniques for presenting these concepts to children.

Section III engages you in some more formal consideration of the material that was presented informally in Section II. For example, formal definitions of the addition, subtraction, multiplication, and division of rational numbers will be introduced. The formal properties satisfied by these four operations on the set of rational numbers will be developed and contrasted with those properties that are satisfied by the operations on the whole numbers. Observing these similarities and differences should help provide perspective on the whole numbers and rational numbers, which are the two primary objects of study in the elementary mathematics curriculum.

In addition to <u>formalizing</u> topics that were developed informally in Section II, Section III introduces the <u>density</u> of the rational numbers and provides some introductory work with the concept of a <u>group</u>, which is an important unifying concept in mathematics and its applications.

MAJOR QUESTIONS

1. Give generalized definitions of the four operations on the set of rational numbers. Illustrate each with an example.

2. List the properties of the system of rational numbers. Which of these properties hold for the system of rational numbers, but not for the system of whole numbers?

3. Several topics are discussed in this section. Briefly describe what each of the following means:

   a) The rational numbers form a dense set.

   b) The set of rational numbers, with the operation of addition, forms a group.

TEACHER TEASER

To motivate his son to do his homework carefully, a man agreed to pay his son a dime for each correct problem, and to fine him a nickel for each incorrect problem. There were 30 problems, and it turned out that the boy earned 90 cents. How many problems did he get right?

ACTIVITY 19

SUMMARIZING THE OPERATIONS AND RELATIONS FOR THE RATIONAL NUMBERS

FOCUS:

This activity will serve to summarize formally some of the results that were observed informally in Section II. You will develop a generalized definition of the operations of addition, subtraction, multiplication, and division, and the relations of equivalence and order.

DIRECTIONS:

1. Review the operations as they were described in Section II. Using the results that you observed there, write a general rule for addition, subtraction, multiplication, and division. That is, write a rule that tells how to operate on any two rational numbers that might be selected.

   Check your rule by applying it to a numerical example. The case for addition has been done for you.

   ADDITION:

   For a, b, c, and d, where $b \neq 0$ and $d \neq 0$,

   $$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

   $$\frac{2}{3} + \frac{4}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{3 \cdot 4}{3 \cdot 5} = \frac{2 \cdot 5 + 3 \cdot 4}{3 \cdot 5} = \frac{22}{15}$$

   SUBTRACTION:

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MULTIPLICATION:

DIVISION:

2. Some of the following problems have been incorrectly worked. Correct those that are wrong and give at least one guess as to why or how the error was made:

a) $\frac{3}{4} + \frac{7}{8} = \frac{10}{12}$

b) $\frac{5}{3} \times \frac{2}{3} = \frac{10}{3}$

c) $\frac{15}{10} - \frac{6}{5} = \frac{9}{10}$

d) $\frac{8}{12} - \frac{1}{12} - \frac{2}{12} = \frac{5}{12}$

e) $\frac{15}{8} \div \frac{3}{8} = \frac{5}{8}$

f) $\frac{8}{15} + \frac{8}{12} = \frac{8}{27}$

g) $\frac{6}{7} \times \frac{1}{2} - \frac{2}{5} = \frac{6}{70}$

h) $\frac{4}{9} + \frac{1}{3} \times \frac{5}{6} = \frac{35}{54}$

i) $\frac{8}{27} \div \frac{3}{4} = \frac{2}{9}$

j) $3 \times \frac{3}{4} = \frac{9}{12}$

3. a) Write a general rule for determining the order of two rational numbers; i.e., determine when it is true that $\frac{a}{b} < \frac{c}{d}$.

b) Indicate which of the following are true and why or why not.

$\frac{2}{3} < \frac{1}{2}$	$\frac{3}{4} < \frac{4}{5}$	$\frac{111}{213} < \frac{110}{214}$
$\frac{9}{8} < \frac{10}{9}$	$\frac{1}{3} < \frac{1}{2}$	$1 < \frac{5}{4}$

4. a) Write a rule stating when two fractions are equivalent.

b) Indicate which of the following are true and why or why not.

$\frac{1}{3}$ and $\frac{2}{9}$ are equivalent.

$\frac{3}{4}$ and $\frac{1\frac{1}{2}}{2}$ are equivalent.

16 and $\frac{80}{5}$ are equivalent.

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ACTIVITY 20

A GEOMETRIC LOOK AT EQUIVALENT FRACTIONS

FOCUS:

This activity provides an opportunity to explore a geometric representation of sets of equivalent fractions.

MATERIALS:

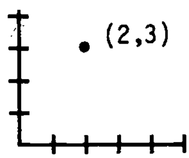
Ruler, graph paper.

DISCUSSION:

You may have seen that the rational number represented by the fraction $\frac{a}{b}$ is also represented by the ordered pair (a,b) and that the ordered pair (a,b) also represents a location with respect to coordinate axes. This relationship will enable us to geometrically visu-

$\frac{a}{b} \longrightarrow (a,b) \longrightarrow$ location of point

e.g., $\frac{2}{3} \longrightarrow (2,3) \longrightarrow$



alize relationships among fractions. In particular, we will attempt to visualize sets of equivalent fractions. There will also be some reverse associations attempted. That is, from geometric observations we will look back to see what might be true about equivalent fractions. This means that we will follow the arrows backwards:

$\frac{a}{b} \longleftarrow (a,b) \longleftarrow$ geometric location

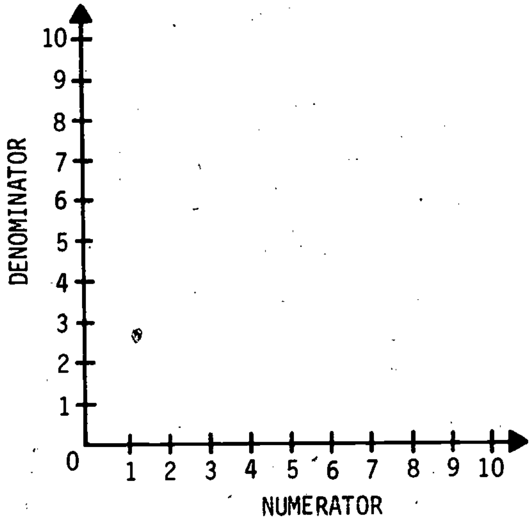
DIRECTIONS:

1. $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}\right\}$

   Write the set of ordered pairs associated with the above set of equivalent fractions.

$\{(1,2), (\quad),$

2. a) Plot the points represented by the ordered pairs in the set of exercise 1.



b) Place a ruler along the plotted points. What do you notice about the set of points you have plotted?

3. Write the set of ordered pairs associated with the following set of equivalent fractions.

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}\right\}$$

4. a) Plot the points represented by the ordered pairs in the set of exercise 3 on the graph on the following page.

b) What do you notice about the set of points?

5. Write ordered pairs for each of the following sets of equivalent fractions:

a) $\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}\right\}$

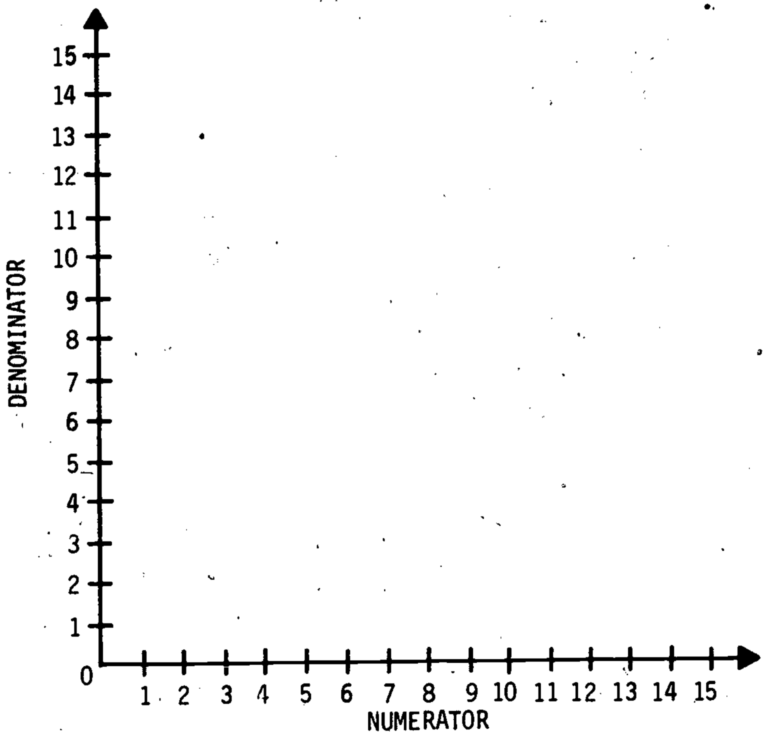
b) $\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}\right\}$

c) $\left\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \frac{15}{20}\right\}$

d) $\left\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}, \frac{5}{20}\right\}$

e) $\left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}\right\}$

Graph for (4)



6. Plot each set of points in (5) on the same grid on the next page. Connect the points associated with each set of equivalent fractions with a line. Label each line a, b, c, d, or e to represent each set.

7. a) Order the fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{4}, \frac{1}{3}$ from greatest to least.

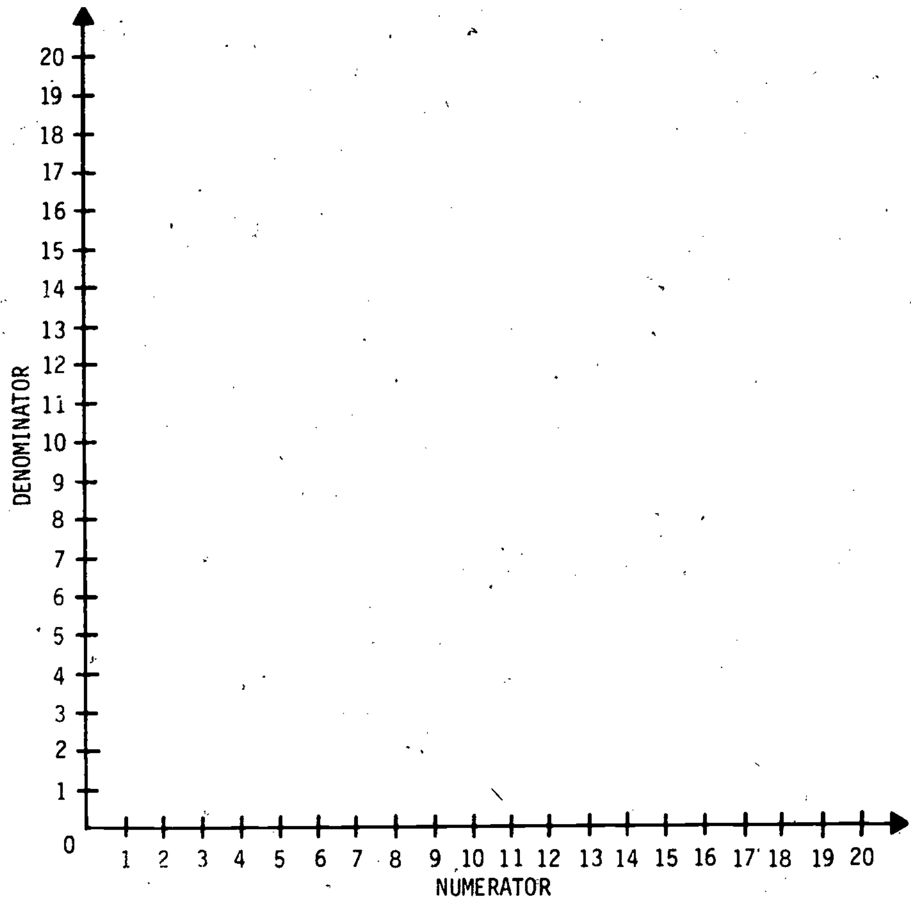
   b) Compare the order of the fractions with the plotted lines in (6). What pattern do you observe?

   c) What is the slope of each line?

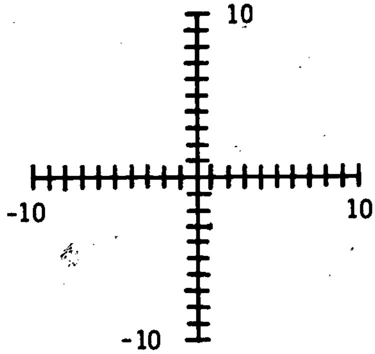
   d) If one extended the line through the origin of the graph, all lines would pass through the point (0,0). Discuss why division by 0 is undefined. What value would you assign to $\frac{0}{0}$? Which equivalence class of fractions should $\frac{0}{0}$ be in?

Now to see if we can make any inferences about equivalent fractions from the graphs.

Graph for (6)



8. Use a piece of graph paper and draw the "numerator" and "denominator" axes in such a way that all four quadrants can be used. Label the axis to 10 in all directions.



a) Plot the set of equivalent fractions $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}$. Draw a line through these points and extend it into the third quadrant. Does this line pass through (-1,-2)? Is the fraction $\frac{-1}{-2}$ equivalent to the other members of the set?

b) In a similar fashion (on the same graph) plot the set of equivalent fractions $-\frac{1}{2}, -\frac{2}{4}, -\frac{3}{6}, -\frac{4}{8}$. Draw and extend the line into the fourth quadrant. List four other points (in quadrant 4) and their associated fractions that are equivalent to the set.

c) How about the points on the straight lines in between the points you have plotted? For example, are $\frac{\frac{1}{78}}{\frac{2}{78}}$ and $\frac{1}{2}$ equivalent?

ACTIVITY 21

ORDER AND DENSITY OF RATIONAL NUMBERS

FOCUS:

Children learn early the meaning of "more than" and "less than." As elementary school teachers, you should be familiar with some of the important topics associated with order properties. This activitiy will consider the following topics:

- Determining Order
- The Law of the Trichotomy
- Density

DISCUSSION:

For any two numbers, you know that one of two conditions must hold. That is, either two numbers are equal to each other, or they are not. If the two numbers are $7 + 5$ and 3×4, they are equal. However, if the numbers are $\frac{7}{8}$ and $\frac{2}{3} - \frac{1}{7}$, they are not.

If two numbers are not equal, an additional alternative exists. Either the first is less than the second, or the first is greater than the second. Together, the conditions just described constitute the Law of the Trichotomy.

LAW OF THE TRICHOTOMY

For any two rational numbers (this includes the integers, which include the whole numbers), r and s, exactly one of the following conditions must hold:

i) $r < s$ (r is less than s)

ii) $r = s$ (r equals s)

iii) $r > s$ (r is greater than s)

With the introduction of the symbolism and the "less than--greater than" terminology, it is necessary to define these expressions. The following definition is only for the case when p is less than q. A similar definition can be made for p greater than q.

> The number p is less than the number q ($p < q$) if there is some positive number r such that $p + r = q$.

DIRECTIONS:

1. Indicate which one of the relationships (<, >, or =) is true for the following pairs of numbers. In the unequal cases, find the positive number r which one needs to add to the smaller number to get the greater.

   a) 4, 6 b) $\frac{3}{4}, \frac{2}{3}$

   c) $-4, \frac{-8}{3}$ d) $2\frac{3}{4}, \frac{11}{4}$

2. For the set of whole numbers, it is not difficult to determine whether $a < b$, $a = b$, or $a > b$. However, the task is not so easy for some other elements from the set of rational numbers. One approach suggested was that equivalent fractions should be found for both numbers being compared, so that the denominators will be equal. Using this technique, determine which one of the relationships (<, >, or =) is true for the following pairs of numbers. Again, find the positive number that must be added to the smaller number to get the larger number.

   a) $\frac{7}{8}, \frac{8}{9}$ b) $\frac{16}{13}, \frac{4}{3}$

   c) $-\frac{5}{16}, -\frac{1}{3}$ d) $4\frac{3}{7}, \frac{22}{5}$

3. Write a general rule that will tell you that $\frac{a}{b}$ is less than $\frac{c}{d}$ in each of the following three situations.

   a) b is equal to d.

b) a is equal to c.

c) Neither of the above conditions necessarily holds.

4. Determine whether each of the following is true or false. (An expression is true only if it is true in <u>every</u> case.) The letters p, q, r, and s represent rational numbers. When you have answered each expression, compare results with other members of your group. If there are too many here for the time you have, choose three or four to do.

a) If $p < q$, then $p + r < q + r$.

b) If $p < q$, then $p \times r < q \times r$.

c) If $p < q$ and $r \neq 0$, then $\frac{p}{r} < \frac{q}{r}$.

d) If $p \leq q$ and $q \leq p$, then $p = q$.

e) If $p + r < q + r$, then $p < q$.

f) If $pr < qr$, then $p < q$.

g) If $p < q$, then $q < p$.

h) If $p < q$ and $q < r$, then $p < r$.

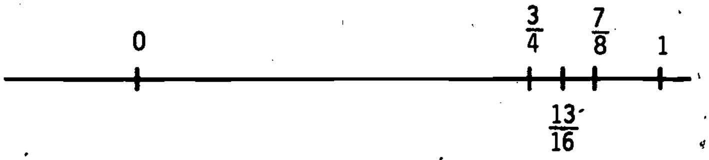
i) If $p < q$ and $r < s$, then $p + r < q + s$.

j) If $p < q$ and $r < s$, then $pr < qs$.

5. The concept of <u>density</u> is concerned with the existence of gaps in sets of numbers. A formal definition is as follows.

> A set of numbers is <u>dense</u> if between any two numbers of the set there is another number of the set.

The set of rational numbers is dense. To get some feeling for this, consider the numbers $\frac{3}{4}$ and $\frac{7}{8}$.



The number $\frac{13}{16}$ lies between them.

a) Find two other numbers that lie between $\frac{3}{4}$ and $\frac{7}{8}$.

b) Find a number between $-\frac{11}{7}$ and $-\frac{21}{14}$.

c) Can you see a way of finding a rational number between <u>any</u> two rational numbers? If so, describe it.

6. The following are three subsets of the rational numbers. Determine whether each is dense. (Remember that between <u>any</u> two numbers in the set there must be another <u>in the set</u>.)

a) the whole numbers;

b) the integers;

c) the rational numbers between 0 and 1.

d) the rational numbers whose fractional representations <u>in lowest terms</u> have even numerators (for example, $\frac{2}{7}$, $\frac{4}{5}$; the fraction $\frac{2}{6}$ does not count since it is $\frac{1}{3}$ in lowest terms).

7. Back to the density of the set of rational numbers: one way to show that the rationals are dense is to show that there is a rational number halfway between any two rational numbers. (The way to find the number halfway between two numbers is to find the average of the numbers; that is, $\frac{x + y}{2}$ is halfway between x and y.)

a) Find the rational number halfway between each of the following pairs of rational numbers. For one pair, actually show that the number is halfway.

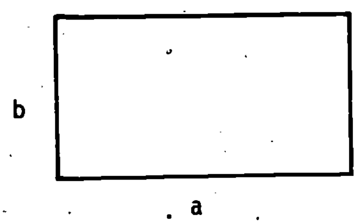
$\frac{3}{4}, \frac{2}{3}$ $\qquad$ $-\frac{3}{7}, \frac{2}{5}$ $\qquad$ $\frac{12}{5}, \frac{17}{8}$

b) Find a general formula for the number halfway between $\frac{a}{b}$ and $\frac{c}{d}$. Does it have to be a rational number? Explain. What does this say about the density of the rational numbers?

8. The concept of density is not difficult for children if it is presented using the notion of "between." Outline how you would introduce this topic in the elementary school classroom. Obtain a critique of your lesson from other members of your group.

THE GOLDEN SECTION REVISITED

Greek architects, painters, and sculptors used the ratio of the Golden Section to construct a Golden Rectangle. Because this rectangle was considered to be esthetically pleasing to the human eye, it was used extensively in the construction of buildings and works of art. Using the same property as the Golden Section, the Golden Rectangle was similar to the one pictured here, whose dimensions a and b have the property that $\frac{a}{a+b} = \frac{b}{a}$.



A painting by the twentieth-century artist Mondriaan, entitled "Black, White and Red," has at least nine examples of the Golden Section or Golden Rectangle.

ACTIVITY 22

REVIEWING NUMBER PROPERTIES

FOCUS:

This activity reviews the properties of the set of whole numbers for addition, subtraction, multiplication and division.

DISCUSSION:

- Closure
- Commutativity
- Associativity
- Distributivity
- Existence of identity
- Existence of inverses

are all properties related to operations on a number system. In early work with whole numbers, not much emphasis is put on these properties. The identities 0 and 1 are studied when working with addition and multiplication of whole numbers. The commutative property cuts in half the number of basic addition and multiplication facts that a child needs to learn. To some extent for children and to a greater extent for you, the properties become more important as you try to understand the systems of integers and rational numbers and computation with the operations on these systems. This activity will help you review the number properties of the system of whole numbers, in preparation for your work with properties of the rational numbers in the next activity. You may want to attempt the activity before you refer to Appendix B where the number properties are defined and explained.

DIRECTIONS:

1. Discuss the meaning of each property listed in the Discussion. (Examples are helpful.)

2. To test your understanding of the six properties listed above, consider each of the following arithmetical statements. Decide whether each is true or false; and for those statements that are true, state which property is being illustrated. The letters a, b, c, and d stand for whole numbers, and the properties apply only to operations within the set of whole numbers. (Note: A statement is false if there is one instance when it is false, and is true only if it is always true.)

T or F		Property (if true)
___	a) $(a \times b) + c = (b \times a) + c$	______
___	b) $(a \times b) + c = c + (a \times b)$	______
___	c) $(a \times b) + c = a \times (b + c)$	______
___	d) $(a \times b) + c = (a + c) \times (b + c)$	______
___	e) $(a \times b) \times c = (a \times c) \times (b \times c)$	______
___	f) $(a \times b) \times c = c \times (a \times b)$	______
___	g) $(a \times b) \times c = a \times (b \times c)$	______
___	h) $(a + b) \times c = (a \times c) + (b \times c)$	______
___	i) $(a + b) \times c = (b + a) \times c$	______
___	j) $(a + b) \times (c + d) = (b + a) \times (d + c)$	______
___	k) $a \times (b \times \frac{1}{b}) = a \times 1 \quad (b \neq 0)$	______
___	l) $(a \div b) \div c = a \div (b \div c)$ $(b \neq 0, c \neq 0)$	______
___	m) $(a - b) + c = a - (b + c)$	______
___	n) $a \times (b - b) = a \times 0$	______
___	o) $a + 0 = a$	______
___	p) $a + b$ is a whole number	______
___	q) $a - b$ is a whole number	______
___	r) $c \div d$ is a whole number $(d \neq 0)$	______

___ s) $(a \div b) \times c = (a \times c) \div (b \times c)$ ____________
$(b \neq 0)$

___ t) $(a + b) \div c = (a \div c) + (b \div c)$ ____________
$(c \neq 0)$

3. You have already observed results in this and other units concerning the properties that are satisifed within the set of whole numbers. Summarize these results by checking those properties in the following table that are satisfied for each operation. Remember: These apply only for the set of whole numbers.

Properties of the Set of Whole Numbers

	Addition	Subtraction	Multiplication	Division
Closure				
Commutativity				
Associativity				
Identity*				
Inverse**				

4. The distributive property was not included in the previous table because two operations are required in order to check that it is true. Using the four operations given in the table above, list those pairs of operations that satisfy the distributive property for elements from the set of whole numbers, and the order in which these operations must be distributed.

*There is an identity for the operation.
**Each element has an inverse for the operation.

ACTIVITY 23

EXTENDING THE PROPERTIES TO THE RATIONAL NUMBERS

FOCUS:

In the previous activity, you saw that for none of addition, subtraction, multiplication, and division does the set of whole numbers satisfy all the following properties: closure, commutativity, associativity, distributivity, and the existence of identity and inverses. This activity will explore those properties that are satisfied by the rational numbers for addition and multiplication. Similar consideration could be given to subtraction and division.

MATERIALS:

An elementary school mathematics textbook series.

DIRECTIONS:

1. For each of the following terms, write a definition. The statements are to be true for rational numbers, but the consistency of the subject implies that the statements will be similar to ones you have written earlier. Use mathematics books or other units to help you. Also, for each property, find an example of it as it is used in an elementary school textbook.

Closure for addition	Example
	Definition
Closure for multiplication	Example
	Definition

Commutative property for addition	Example	
	Definition	
Commutative property for multiplication	Example	
	Definition	
Associative property for addition	Example	
	Definition	
Associative property for multiplication	Example	
	Definition	
Identity element for addition	Example	
	Definition	

Identity element for multiplication	Example	
	Definition	
Inverse element for addition	Example	
	Definition	
Inverse element for multiplication	Example	
	Definition	

2. The inverse element in multiplication of rational numbers expressed as fractions is also called the <u>reciprocal</u> of the element. What is the inverse element for multiplication--or reciprocal--for each of the following?

$$\frac{5}{8}, \quad \frac{2}{7}, \quad \text{and} \quad -\frac{9}{4}$$

3. Check the properties in the following table that are satisfied by each operation for the set of rationals.

Properties of the Set of Rational Numbers

	Addition	Subtraction	Multiplication	Division
Closure				
Commutativity				
Associativity				
Identity				
Inverse				

4. Compare the completed table above with the table you constructed for the set of whole numbers in the last activity. What additional properties are satisfied by the set of rational numbers that are not satisfied by the whole numbers? Are there any properties that are true for the whole numbers that are not true for the rationals?

ACTIVITY 24
GROUPS

FOCUS:

The previous activities of this section have shown you some of the properties that are satisfied by the set of rational numbers. With the addition properties, the rational numbers form a structure to which mathematicians give the special name group. The purpose of this activity is to describe a group, and to explore some examples of sets and operations on those sets which form groups.

DISCUSSION:

From the previous work, you know that the set of rational numbers satisfies the following properties:

- Closure for addition
- Associativity of addition
- Existence of additive identity
- Existence of additive inverse elements for each element.

With these properties, mathematicians say that the set of rational numbers forms a group with respect to addition. Thus,

> For a given operation any set of numbers forms a group if the operation is closed and associative, if the set possesses an identity element for the operation, and if each element of the set has an inverse with respect to the identity element and the operation.

Each of these properties was verified for addition of the rational numbers in the previous activity.

DIRECTIONS:

1. Using the notion of a group described in the Discussion, decide whether each of the following satisfies the four properties necessary to form a group.

   a) The operation ⓧ defined on elements from the set {0, 1, 2, 3, 4} by the following table. (To find a ⓧ b, find a in the left column and b in the row across the top; then look at the intersection of the a row and b column; for example, 2 ⓧ 3 = 1.)

ⓧ	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

   b) the whole numbers with the operation of addition;

   c) the whole numbers with the operation of multiplication;

   d) the rational numbers with the operation of addition;

   e) the rational numbers with the operation of multiplication;

   f) the rational numbers (excluding 0) with the operation of multiplication;

   g) the rational numbers (excluding 0) with the operation of division.

2. The examples in (1) are all within the realm of mathematics. However, there are numerous examples of groups in the world

around you.* For each of the following examples, check to see whether it is a group.

a) <u>light switch</u>: Suppose that you consider a switch as having two possibilities, switching (S) or leaving alone (L). Thus, a <u>switching</u> followed by a <u>leaving alone</u> looks like a switching; a switching followed by a switching looks like a leaving alone, etc. Complete the table below and determine whether the set with the operation "$\curvearrowright$" is a group.

$\curvearrowright$	S	L
S		
L		

b) <u>clock</u>: Construct a table on the following page showing the sum of times on a regular twelve-hour clock. Think of the number on the left as the <u>starting time</u> and the number along the top as the <u>amount of time added</u> to the starting time. For example, from this table you see that $4 \oplus_{12} 9 = 1$. Does the operation of addition on the clock form a group? Which hour acts like the identity for $\oplus$?

3. Sometimes, the operation considered for each set may also be commutative. In the case where the operation on the set forms a group and is also commutative, the set with the operation is called a commutative group. Check the groups in exercises (1) and (2) to see whether they also form commutative groups.

*It may surprise you to know that the concept of "group" is one of the powerful unifying ideas in human knowledge. Scientists in the social as well as physical sciences have utilized groups for their descriptive and predictive power. Mathematicians have long found group theory to be a rich area of investigation.

$\oplus_{12}$	1	2	3	4	5	6	7	8	9	10	11	12
1												
2												
3												
4							11	12	1			
5												
6												
7												
8						2						
9												
10												
11												
12												

Section IV

RATIONAL NUMBERS AS DECIMALS

In Section II, the focus was on rational numbers expressed as fractions. It was pointed out that rational numbers could be expressed both as fractions and as decimals. This section describes and develops some of the key concepts related to the understanding and teaching of rational numbers expressed as decimals. You should keep in mind that these are the same old rational numbers but are just called by different names.

Decimal notation relates to several concepts that have already been presented in the elementary school. Among these relationships between decimals and earlier topics are:

- Extending the numeration system used to express whole numbers to a numeration system that expresses rational numbers;
- Reviewing rational numbers expressed as fractions by converting them to decimals using whole-number division techniques;
- Using the decimal place-value system as a means of explaining the structure of the metric system of measurement units.

The study of decimals also provides a natural way to lead from the rational numbers to the real numbers (to be developed in Section V).

The decimals, like other mathematical ideas, should be developed from real-world settings. Children come to school familiar with some

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instances of decimals. Dollars-and-cents notation, gas mileage, and sports statistics provide natural situations from which more formal instruction in decimals can follow.

On the other hand, as one begins to develop the operations with decimals, particularly multiplication and division, it is important that the students have a sufficiently clear understanding of the Mathematical Relationships on which to base these operations. It is difficult and impractical, if not impossible, to develop an understanding of these operations based on real-world referents in the form of Physical Embodiments or Pictorial Representations.

A consideration which should be kept in mind as you study the decimals is the probable role of decimals in the schools of the future. The use of the hand calculator in the schools is becoming widespread. The calculator will allow pupils to perform calculations in many ways different from those they presently use. For example, multiplication and division with decimals, which at best is tedious, becomes routine. (Helping the pupil understand what he is doing will still be important, however.) Presently, if one adds the fractions $\frac{3}{5} + \frac{1}{8}$ the display on the calculator will be the decimal sum, 0.725, rather than the fractional sum.* Finally, with the transition to the metric measurement system which uses decimal rather than fractional notation, there is no question that there will be an increased emphasis on decimals in the schools.

This section focuses on the decimal system in the following ways: as an extension of the whole-number numeration system, as a way of expressing rational numbers, as closely related to the structure of the metric system, and as a set of numerals with algorithms.

Activities 25, 26, and 27 present decimal concepts to you. In Activities 28 and 29 attention is given to developing decimal concepts with children.

*Calculators may be produced at a later date which would also display fractional notation.

MAJOR QUESTIONS

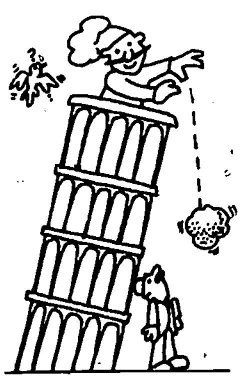
1. What general technique can be used for converting repeating decimals to fractional form?

2. Use instances to describe when it is easier to perform operations on rational numbers

   a) using decimals rather than fractions, and

   b) using fractions rather than decimals.

TEACHER TEASER



The Learning Tower of Pizza is exactly 100 spaghetti strands tall. A mad Italian chef drops a very springy ball of lasagna from the top of the tower. It rebounds to one-tenth of the height from which it falls, and keeps on bouncing indefinitely. Calculate how far (in spaghetti strands) it travels altogether.

Hint: Use decimal notation (Italy is a metric country) to express the distance traveled on each bounce, but give your final answer as a fraction.

ACTIVITY 25

EXTENDING THE NUMERATION SYSTEM TO DECIMALS

FOCUS:

The usual base 10 numeration system that you use to express whole numbers is also called the decimal numeration system. Rational numbers can be expressed either in fractional form or in decimal form. The representation of rational numbers as decimals arises quite naturally and consistently from the study of the numeration system for whole numbers. This activity will review this system and show you how the system can be extended to express rational numbers.

DISCUSSION:

In the development of the decimal numeration system for whole numbers the numeral is often "expanded" to illustrate the structure and symmetry of the system. For instance, for the numeral 3217 one can use several expanded forms for illustration.

$$3000 + 200 + 10 + 7$$

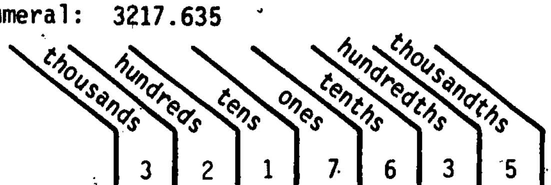
$$\text{or} \quad (3 \times 1000) + (2 \times 100) + (1 \times 10) + 7 \times 1$$

$$\text{or} \quad (3 \times 10^3) + (2 \times 10^2) + (1 \times 10^1) + (7 \times 10^0)$$

Similarly, when emphasizing the structure and symmetry of the numeration system for rational numbers, it is often useful to "expand" the numeral. The following diagram provides an aid for children when writing decimals in expanded form. It also aids children in reading decimals (which they find to be a bothersome task).

EXAMPLE

Standard Numeral: 3217.635



Expanded Forms:

a) $3000 + 200 + 10 + 7 + 0.6 + 0.03 + 0.005$

b) $(3 \times 1000) + (2 \times 100) + (1 \times 10) + (7 \times 1) + (6 \times 0.1) +$
$(3 \times 0.01) + (5 \times 0.001)$

c) $(3 \times 1000) + (2 \times 100) + (1 \times 10) + (7 \times 1) + (6 \times \frac{1}{10}) +$
$(3 \times \frac{1}{100}) + (5 \times \frac{1}{1000})$

d) $(3 \times 10^3) + (2 \times 10^2) + (1 \times 10^1) + (7 \times 10^0) + (6 \times 10^{-1}) +$
$(3 \times 10^{-2}) + (5 \times 10^{-3})$

Notice the symmetry about the "ones" place--tens on one side, tenths on the other; hundreds on one side, hundredths on the other, etc.

DIRECTIONS:

1. Write each of the following numbers in expanded form using the indicated notation.

   a) 2348.4 (use form (a) above);

   b) 70617 (use form (b));

   c) 3008.13 (use form (c));

   d) 27.006 (use form (d)).

2. For each of the following expansions, write the corresponding standard numeral.

   a) three hundred seven and six-tenths.

   b) $(2 \times 10^3) + (6 \times 10^2) + (7 \times 10^1) + (2 \times 10^{-1}) + (4 \times 10^{-2})$

   c) $(3 \times 10) + (6 \times 1) + (8 \times \frac{1}{100}) + (9 \times \frac{1}{1000})$

   d) $(4 \times 10^4) + (7 \times 10^1) + (2 \times 10^0) + (3 \times 10^{-2}) + (4 \times 10^{-4})$

3. How is each of the following changed when it is multiplied by 10? Write the standard numeral.

   a) 3267

   b) 32.67

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c) $(2 \times 10^1) + (6 \times 10^0) + (4 \times 10^{-1}) + (7 \times 10^{-2})$

d) $(7 \times 10^{-2}) + (3 \times 10^{-4})$

4. Some elementary textbooks simplify multiplication of whole numbers by 10 by instructing the student to simply annex a zero. Using the results of (3) above, what problems arise with rational numbers in decimal notation when this technique is used?

The Golden Section Continued

One example of the Golden Section lies in the area of music. The composer Bartók is particularly noted for the use of the ratio 0.618 in his compositions.

For example, the first movement of Sonata for Two Pianos and Percussion comprises 443 bars. Thus, its Golden Section, using the ratio defined earlier, is approximately 274. An examination of the work shows that the recapitulation starts precisely at the 274th bar.

This is not the only example of the use of the Golden Section in works by Bartók. In Movement I of Contrasts, there are 93 bars. Again the recapitulation begins at the bar marking its Golden Section. The same thing is observed for the Divertimento, and for "From the Diary of a Fly," in Vol. VI of Mikrokosmos.

ACTIVITY 26

APPLICATION OF DECIMALS: THE METRIC SYSTEM

FOCUS:

Increasing foreign trade and international travel emphasize the gap between the English system of measure we use and the metric system that most of the world uses. Although Congress declared the metric system a legal system of measure in 1866, we have not, as a country, made much progress in adopting the metric system. The next ten years are likely to bring considerable attention to the metric system of units in the schools and throughout the country. This activity will explore advantages of the metric system, which scientists and technicians have utilized for decades. In particular, a strong analogy between our decimal numeration system and the metric system is presented.

DISCUSSION:

The metric system is included here because it depends heavily on the structure of the decimal numeration system. That is, the relationships between the various units of the metric system rely on the fact that a change from one unit to the next is accomplished by multiplication or division by 10. For example, 3 meters can be changed to centimeters by multiplying by 100. To convert 5.1 milliliters to centiliters, one divides 5.1 by 10, which results in 0.51 centiliters. From these examples, one can see that a decimal representation allows for much greater facility in working with metric units than with the English units, in which you frequently find yourself dividing and multiplying by numbers like 3, 12, and 5280.

The relationships between units in the metric system are easy to see because the units' names are formed from a root word and a prefix. The root word for mass is <u>gram</u>, for length is <u>meter</u>, and for volume is <u>liter</u>. These root words are combined with the prefixes milli- for thousandth, centi- for hundredth, kilo- for thousand, and others less commonly used to form decimal multiples and submultiples of the units.

The chart below shows the relationship between the prefixes and the decimal place values. Note that the units place has no entry since the root word, meter, liter, or gram, is used alone there. The last three entries in each column indicate the abbreviations that are used for the metric units for length, mass and volume.

Place Value	thousands 10^3	hundreds 10^2	tens 10^1	units 10^0	tenths 10^{-1}	hundredths 10^{-2}	thousandths 10^{-3}
Metric Prefix	kilo-	hecto-	deca-		deci-	centi-	milli-
Length (meter)	km	hm	dam	m	dm	cm	mm
Mass (gram)	kg	hg	dag	g	dg	cg	mg
Volume (liter)	kl	hl	dal	l	dl	cl	ml

DIRECTIONS:

1. Fill in the cells of the table on the following page; the first one is done for you. Note that in the numerals in this table, spaces are used to group digits by threes to facilitate reading;

commas are not used. Thus you see 30 520 mm. An exception is made in cases where leaving a space would separate a single digit, e.g., 3052 cm. Furthermore, for measurements less than 1 unit, a zero precedes the decimal point, e.g., 0.13 cm.

kilometers	meters	centimeters	millimeters
0.030 52	30.52	3052	30 520
1.3624			
	311.67		
		7266.7	

2. Add the measures and write the sum in the indicated form.

   a) 2 km + 13.7 m = ________________ km

   b) 237.6 cm + 1778 mm = ________________ mm

   c) 3072 m + 471.6 cm = ________________ cm

3. Fill in the first blank with an operation and the second with the appropriate numeral.

   To change 3114.2 mm to meters, ________________ the millimeters by ____________.

4. Fill in the cells.

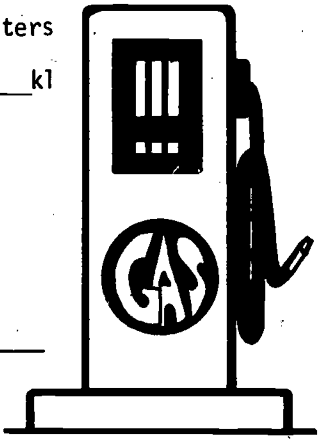
kiloliters	liters	centiliters	milliliters
3.0512			
	3.98		
		8314.5	
1			

5. Add the measures and write the sum in the indicated form.

3.625 kl + 0.03 kl + 4.203 kl = _____ liters

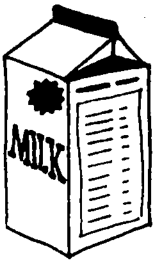
837 liters + 1426 ml + 732 531 ml = ______ kl

59 372 cl + 602.4 liters = _______ cl



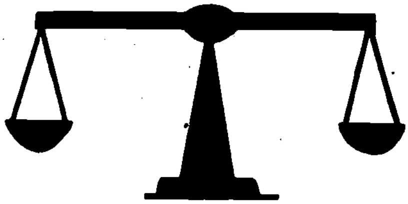
6. Fill in the first blank with an operation and the next with the appropriate numeral:

To change 31 114.2 kl to liters, __________ the kiloliters by _________.



7. Fill in the cells.

kilograms	grams	centigrams	milligrams
			375 124
	7734.		
		4440.36	

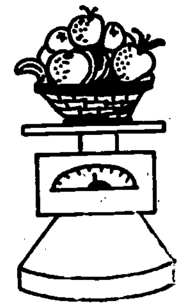


8. Add the measures and write the sum in the indicated form.

440 cg + 37.5 g = ______ g

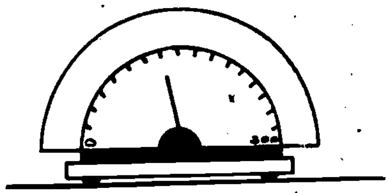
345.23 g + 3946.2 g + 0.07 kg = ______ kg

439.5 mg + 0.345 g = ______ mg



9. Fill in the first blank with an operation and the second with an appropriate numeral.

To change your weight in grams to kilograms, ________ your weight by ________.



10. Two hypothetical approaches to teaching the metric system are suggested. One advocates developing the ability to convert from the English system to the metric (e.g., 3'4" = ______ cm). The second would teach only the metric system. Discuss the advantages and disadvantages of each approach. Consider alternatives.

11. Investigate the use of metric measures in your community, including services and industries. Prepare a concurring or dissenting view to the claim that we must teach the metric system.

ACTIVITY 27

TERMINATING AND NONTERMINATING DECIMALS

FOCUS:

Since decimals and fractions both represent rational numbers, it is reasonable to expect some system for converting from the decimal representation of a number to a fractional representation and vice versa. Certain conversions you already know, e.g., $\frac{1}{2}$ to 0.5 and 0.001 to $\frac{1}{1000}$. In this activity we will deal with a general system for conversion.

DISCUSSION:

Before considering techniques for converting fractions or decimals it will be necessary to describe three terms associated with three forms of a decimal. The terms are <u>terminating</u>, <u>repeating</u>, and <u>nonrepeating</u>.

<u>Terminating decimal</u> designates such decimals as 0.5, 0.25, 0.37, 0.26118. In each case, the decimal expansion stops or terminates at some point.

A decimal is called <u>terminating</u> if its representation does not continue indefinitely but stops at some point.

The other two decimal forms are what you would expect. The second form does not terminate, but does repeat as it continues. For example: $0.313131\ldots = 0.3\overline{131}$ and $2.1231231231\ldots = 2.123\overline{123}$

A decimal is called <u>repeating</u> if it continues indefinitely, but at some point begins to repeat and continues to repeat. The portion of the decimal which repeats is indicated by a bar.

Some examples of this type of decimal include $0.3\overline{3}$, $0.6\overline{6}$, and $0.7\overline{7}$ which repeat immediately with one digit; $0.27\overline{27}$, $0.41\overline{41}$, and $0.93\overline{93}$, which repeat immediately with two digits; and $1.63\overline{3}$, $0.276\overline{76}$, and $0.23476\overline{6}$, which repeat after some point.

A decimal is called <u>nonrepeating</u> if it continues indefinitely, but does not at any point begin to repeat a sequence of digits.

Examples of this type of decimal can also be constructed, but care must be taken that they do not at any point begin to repeat. Thus some method for constructing them must be devised. The decimals listed below do not repeat even though they are formed with a discernible pattern.

a) 3.101001000100001...

b) 0.2113111121111112111111112...

c) 11.606116000611116...

Can you describe the patterns that were used to construct (a), (b), and (c)? Could you describe a nonrepeating decimal without describing such a pattern? Decimals which are nonrepeating represent <u>irrational</u> numbers. Irrational numbers will be discussed in Section V. We will not consider them further in this activity.

Decimals that terminate or that repeat represent rational numbers, which also have fractional representations. The following exercises will illustrate how to convert fractions to decimals and decimals to fractions. You are probably familiar with some of these techniques already.

DIRECTIONS:

1. In converting from a fractional representation of a rational number to a decimal representation we take advantage of the fact that $\frac{a}{b} = a \div b$. That is, we divide the denominator of the fraction into the numerator using the standard division algorithm and paying particular attention to the placement of decimal points. This process is carried out in detail for the fraction $\frac{2}{7}$.

$\frac{2}{7} = 2 \div 7 = 0.285714\overline{285714}.$

To see this we carry out the division

$$
\begin{array}{r}
0.285714 \\
7\overline{)\,2.000000} \\
\underline{1\;4} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
10 \\
\underline{7} \\
30 \\
\underline{28} \\
2
\end{array}
$$

At this point you see that you have a division problem that you had previously, namely $7\overline{)2}$. So you know what will follow is the same as what came before, i.e., that digits will repeat.

a) Carry out the steps in the divison of 7 into 2 far enough to convince yourself that the digits actually do repeat.

b) Convert each of the following fractions to decimal form, and decide whether the decimal is terminating, repeating, or nonrepeating.

Fractional form	Decimal form	Type of decimal
$\frac{2}{9}$		
$\frac{8}{11}$		
$\frac{7}{8}$		
$\frac{4}{18}$		
$\frac{9}{4}$		

2. OPTIONAL: *In an effort to further relate this division process to the division of whole numbers as you have done it before, we present the following explanation. You may or may not find it helpful. Most children would find it confusing.*

$$\begin{array}{r} 0.2 \\ 7\overline{)\,2.0} \\ \underline{1\ 4} \\ 6 \end{array}$$

Think of dividing up 2 or 20 tenths into 7 equal sets. Each set would contain 2 tenths and 6 tenths would remain.

$$\begin{array}{r} 0.28 \\ 7\overline{)\,2.00} \\ \underline{1\ 4\ \ } \\ 60 \\ \underline{56} \\ 4 \end{array}$$

Similarly, think of dividing up 60 hundredths into 7 equal sets. Each set would contain 8 hundredths and 4 hundredths would remain.

$$\begin{array}{r} 0.285 \\ 7\overline{)2.000} \\ \underline{1\;4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 5 \end{array}$$

Analyze the next step in this explanation of the division of 7 into 2 as shown at the left.

3. Analysis of the prime factors* of the denominators of fractions and of powers of ten will provide some insight into which fractions repeat and which terminate, when written as decimals.

   a) List the prime factors of 10, 100, 1000, 10000.

   b) Each of the rational numbers $\frac{3}{25}$, $\frac{3}{8}$, $\frac{7}{20}$, $\frac{3}{40}$, and $\frac{5}{16}$ can be expressed as a terminating decimal. List the prime factors for the denominator in each fraction.

   c) What are the only prime factors that you found in both (a) and (b)? What seems to be true concerning prime factors and terminating decimals?

   d) Can you see a relationship between the number of times a given prime is repeated in the prime factorization of the denominator and the number of places in the decimal equivalent of the fraction? What is it?

4. To convert a terminating decimal to a fraction, we use the numeration properties discussed in Activity 25. Use the following example as a model, and change each decimal below to a fraction.

Converting a Terminating Decimal to a Fraction

The decimal 0.625 can be written as $(6 \times \frac{1}{10}) + (2 \times \frac{1}{100}) + 5 \times \frac{1}{1000})$ or $\frac{6}{10} + \frac{2}{100} + \frac{5}{1000}$ or $\frac{600}{1000} + \frac{20}{1000} + \frac{5}{1000} = \frac{625}{1000}$. If desired, this fraction can be simplified to obtain $\frac{5}{8}$.

*The prime factorization of a number is the number expressed as a product of primes. For example, the prime factorization of 12 is 2 x 2 x 3. The prime factorization of 56 is 2 x 2 x 2 x 7; the prime factorization of 56 is not 8 x 7 since 8 is not prime.

a) 0.33

b) 2.167

c) 17.406

5. Check to see whether the results of (4) are consistent with those of (3).

6. To convert a repeating decimal to a fraction requires a little more algebraic sophistication. However, the technique can be summarized as multiplication of the original number once or twice to obtain equivalent decimal portions in the resulting products. This process, followed by a subtraction and then a division, results in the fraction form. The two examples below illustrate this process.

Converting a Nonterminating Repeating Decimal to a Fraction

Example 1

$x = 0.777\overline{7}$

Multiply

1. $10x = 7.77\overline{7}$

Subtract

2. $10x - x = 7.77\overline{7} - 0.777\overline{7}$

or

$9x = 7$

Divide

3. $x = \frac{7}{9}$

Example 2

$x = 0.2133\overline{3}$

Multiply

1. $100x = 21.33\overline{3}$

and

$1000x = 213.33\overline{3}$

Subtract

2. $1000x - 100x = 213.33\overline{3} - 21.33\overline{3}$

or

$900x = 192$

Divide

3. $x = \frac{192}{900} = \frac{16}{75}$

Note that in Step 1 (Multiply) of the procedure illustrated on the previous page one multiplies by powers of 10 so that after the subtraction process only an integer will remain.

Use this procedure to convert each of the following decimals to fractions. Show your work.

a) $0.66\overline{6}$

b) $3.1244\overline{4}$

c) $0.3737\overline{37}$

d) $0.999\overline{9}$

e) $0.499\overline{9}$

f) $0.8131\overline{313}$

7. After you have completed each of the four previous exercises compare the answers that you obtained with those of other members of your group. Resolve any differences.

8. Each rational number can be expressed as a decimal so that the decimal terminates or repeats. For example, $\frac{1}{4} = 0.25$ and $\frac{1}{8} = 0.125$ represent rational numbers which terminate, while $\frac{1}{3} = 0.33\overline{3}$ and $\frac{1}{6} = 0.166\overline{6}$ are repeating decimals. Certain rational numbers which form repeating decimals have interesting relationships.

Consider the rational numbers whose denominator is 11, i.e., $\frac{1}{11}, \frac{2}{11}, \ldots, \frac{10}{11}$. Complete the table on the following page by finding that portion of the rational number which repeats when written in decimal form.

$\frac{n}{11}$	Decimal form
$\frac{1}{11}$	$0.\overline{09}$
$\frac{2}{11}$	
$\frac{3}{11}$	
$\frac{4}{11}$	$0.\overline{36}$
$\frac{5}{11}$	
$\frac{6}{11}$	
$\frac{7}{11}$	
$\frac{8}{11}$	
$\frac{9}{11}$	$0.\overline{81}$
$\frac{10}{11}$	

a) What patterns do you observe?

b) If you know some of the expansions in the table above, how can you predict others?

c) Complete a similar table for rational numbers with 7 as the denominator.

d) Do you observe the same patterns? Are there additional observations you can make?

e) What other rational numbers would you expect to have those properties that were observed for 7 and 11?

ACTIVITY 28

INTRODUCING DECIMALS TO CHILDREN

FOCUS:

Now that your skill with decimal representation of rational numbers has been strengthened, it is time to consider presenting decimals to children. In this activity you will have a chance to consider some physical embodiments of decimals.

MATERIALS:

Dienes multibase arithmetic blocks in base ten; play money including cents, ~~dimes,~~ dollar bills, ten-dollar bills, and one-hundred-dollar bills (suggested for children).

DIRECTIONS:

1. Children will already have had (by the time you get to decimal representation of rational numbers) considerable experience with decimal representation of whole numbers. You should be sure that they understand, for example, that

   - $435 = 4 \times 100 + 3 \times 10 + 5 \times 1$.
   - 4 hundreds is also 40 tens.
   - 10×37 is 10 times 7 ones plus 10 times 3 tens.

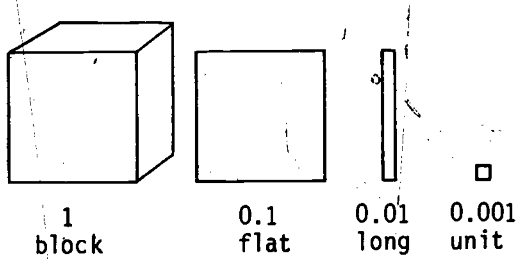
   Briefly outline some questions and activities that you might use to determine whether a child has grasped and remembers these concepts and to help teach or remind a child of these concepts. (You may want to bring in the use of such materials as multibase arithmetic blocks and the abacus.)

2. In their initial work with rational numbers, children learn that rational numbers can represent parts of a whole. Base 10 multibase arithmetic blocks (MAB) provide a good physical embodiment for this part-whole relationship that leads nicely to decimal

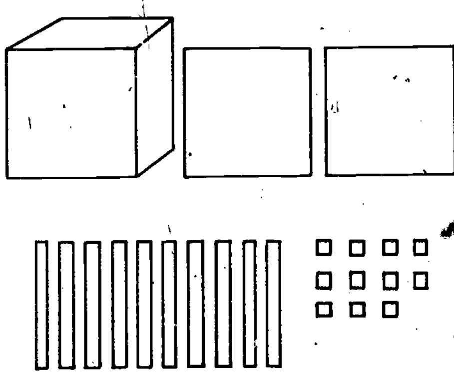
representations. The children will have several <u>blocks</u>, <u>flats</u>, <u>longs</u>, and <u>units</u> where

10 flats are equivalent to 1 block.
10 longs are equivalent to 1 flat.
10 units are equivalent to 1 long.

If the block is treated as 1, then the flat is 0.1, the long is 0.01, and the unit is 0.001.

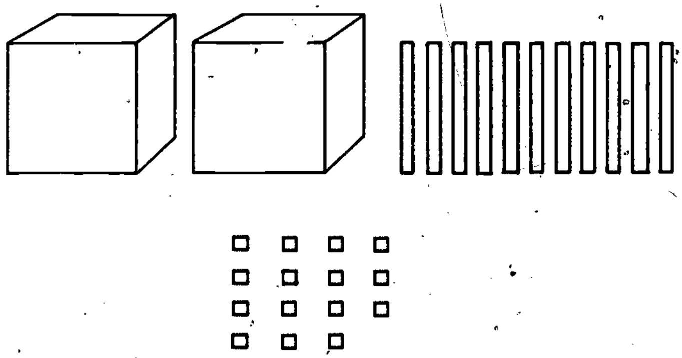


a) Treating the block as 1, give an equivalent representation which uses as few pieces as possible and then write the decimal that each of the following combinations of MAB represents.



Equivalent Representation:

Decimal Numeral:



Equivalent Representation:

Decimal Numeral:

b) For each of the following decimals, represent the decimal with your MAB and then record your representation in the space provided in the table on the following page.

	Blocks	Flats	Longs	Units
2.301				□
0.007				
0.123				
3.041				

c) Outline the steps you might take a child through to use MAB to represent rational numbers and then finally to represent the rational numbers as decimals.

3. All children who come to school have some experience with money. By the time that they are ready for the study of decimal representation of rational numbers, most children have had considerable experience with money, coins in particular. It would be a shame not to take advantage of that experience.

a) Outline how you might use money to introduce decimal representations of rational numbers.

b) In what ways (other than money) does the decimal system enter into the child's everyday life?

ACTIVITY 29

ADDITION AND SUBTRACTION WITH DECIMALS

FOCUS:

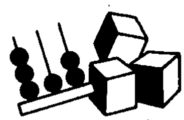
Once children have the concept of the decimal representation of a rational number in mind, they can be introduced to addition and subtraction with decimals. In this activity you will go through such an introduction in a way that parallels a possible child development.

MATERIALS:

Dienes multibase arithmetic blocks (MAB) in base ten.

DIRECTIONS:

1. Children will already have encountered the concept of addition of rational numbers in terms of combining embodiments of those numbers, and the concept of subtraction of rational numbers in terms of taking a portion of the embodiment of a number away. Use your MAB to embody each of the following addition statements. Then record your embodiment in the place provided. For the answer, record a second (equivalent) embodiment which uses the smallest possible number of pieces.



EXAMPLE

Statement	Embodiment			
	Blocks	Flats	Longs	Units
1.8	1	8		
+ 1.4	1	4		
Totals	2	12		
Answer (after exchange)	3	2		

Statement	Embodiment			
	Blocks	Flats	Longs	Units
2.301				
+ 1.189				
Totals				
Answer (after exchange)				

	Blocks	Flats	Longs	Units
3.713				
+ 0.405				
Totals				
Answer (after exchange)				

2. In the subtraction process one often has to rename a number in order to be able to subtract. The term "borrowing," which is often used to describe this process, is particularly descriptive when subtraction is being embodied using MAB. You are to embody the subtraction statements below as you did the addition ones in (1). Record the borrowing act when it is necessary.

Statement	Embodiment			
	Blocks	Flats	Longs	Units
2.8				
- 1.7				
Totals				
Answer (after exchange)				

184

Statement	Embodiment			
	Blocks	Flats	Longs	Units
2.301				
- 1.189				
Totals				
Answer (after exchange)				

	Blocks	Flats	Longs	Units
0.34				
- 0.24				
Totals				
Answer (after exchange)				

3. A step after embodying statements with MAB is to represent problems as follows

ones	tenths
1	3
2	6
3	9

represents

$$\begin{array}{r} 1.3 \\ +\ \underline{2.6} \\ 3.9 \end{array}$$

ones	tenths	hundredths
2	4	3
1	7	5
3	11	8

ones	tenths	hundredths
2	4	3
1	7	5
4	1	8

represents

$$\begin{array}{r} 2.43 \\ +\ \underline{1.75} \\ 4.18 \end{array}$$

a) Represent each of the statements in (1) above according to this scheme.

b) Represent each of the statements in (2) above according to this scheme.

4. One of the difficulties children have in adding or subtracting with decimals is in adding tens to tens, ones to ones, tenths to tenths, etc. What techniques could you suggest to help children avoid errors similar to that shown at the right?

$$\begin{array}{r} 1.3 \\ 2.84 \\ 10.69 \\ 37.5 \\ \hline 174.1 \end{array}$$

5. How would you explain to children that 0.35, 0.350, and 0.350000 have the same value?

6. List several real-world applications that you might use to help motivate addition and subtraction with decimals.

7. a) How are addition and subtraction with decimals like addition and subtraction of whole numbers?

   b) How are they different?

   c) What specific difficulties do you think children have in addition and subtraction with decimals?

8. For each of the following real-world problems, write a mathematical sentence that will help you solve the problem. Then find the solution.

   a) In 1904, Henry Ford set a land speed record of 147.01 kilometers per hour. In 1964, Art Arfons set a land speed record of 863.57 km/hr. How much faster was Arfons' speed than Ford's?

   b) The highest outdoor air temperature ever recorded at Indianapolis is 41.7° C; the lowest is -31.6° C. What is the difference between these two extreme temperatures?

c) The world's tallest manmade structure is the tower of television station KTHI in North Dakota. Its top is 0.629 km above sea level. The world's deepest oil well is in Beckham County, California; it is 9.16 km deep. What is the difference between these extremes?

9. Write a real-world problem that could be represented by each of the following examples.

   a) 12.116 + 125.4 + 6.005

   b) 173.45 - 12.266

   c) 2133.4 + 4.77 + (-63.411)

187

ACTIVITY 30

MULTIPLICATION WITH DECIMALS

FOCUS:

In this activity you will have an opportunity to study some procedures and algorithms for multiplication with decimals.

DISCUSSION:

The teaching of multiplication of rational numbers represented as decimals presents some problems because there is not a good concrete embodiment or pictorial representation on which to base introductory work. So we will rely on Mathematical Relations. As always, it is important to relate a new concept to appropriate earlier learnings of the child. In this case, the multiplication of decimals can be related to the multiplication of fractions with denominators equal to powers of 10, using the following steps:

a) writing decimals as fractions;

b) multiplying with fractions;

c) using the pattern for multiplying and dividing by 10 (or powers of 10) to convert back to a decimal.

It is important for you to analyze the steps involved in multiplving with decimals.

DIRECTIONS:

1. Discuss each of the indicated steps shown on the following page for the multiplication problem:

$$3.78 \times 0.21$$

a) Write the problem as the product of two fractions.

$$3.78 \times 0.21 = \frac{378}{100} \times \frac{21}{100}$$

b) Multiply, using the algorithm for multiplying with fractions.

$$= \frac{7938}{10000}$$

c) Write the resulting fraction as a decimal.

$$= 0.7938$$

2. Each of the above subskills (a), (b), and (c), should be practiced by children. The following will provide examples appropriate for such a development.

   a) Write the following in fractional form.

   2.5 x 0.36 3.02 x 0.5 6.27 x 0.317

   b) Multiply the fractions from (a), using the algorithm for multiplying with fractions.

   c) Write each resulting fraction as a decimal (use dividing by 10 and powers of 10).

3. Use the procedure described above to multiply the following pairs of decimals. Make sure each step that you use is clear.

   a) 15.23 x 3.6 b) 2.004 x 0.05

4. State a rule for dividing by 10 or powers of 10.

5. After the children have had experience with the three-step process described above, ask them to find a pattern and state a rule for the placement of the decimal point. Remember that learning a rule by rote and not with understanding may result in their inability to reconstruct the rule later if they forget it. Study the following examples and the answers. Find the pattern and state the rule.

	EXAMPLES	ANSWERS
a)	15.28 x 3.5	53.480
b)	1.204 x 0.23	0.27692
c)	7.6 x 52	395.2

"In multiplying two decimals one can locate the place for the decimal point in the product by

____

____."

6. Do the following multiplication problems by performing whole-number multiplications, and then using your rule in (5) to locate the decimal point.

23.9 x 1.03	0.2134 x 32.1	13.4 x 0.021

ACTIVITY 31
DIVISION WITH DECIMALS

FOCUS:

In this activity you will have an opportunity to study some procedures and algorithms for division with decimals.

DISCUSSION:

Division with decimals represents the same difficulty in teaching as multiplication does, because of the fact that there is not an appropriate physical or pictorial model to motivate this learning. One can, as with multiplication of decimals, provide a careful development to help the child gain insight and understanding of division with decimals by basing the procedure on previous learnings:

a) writing a decimal as a fraction,

b) dividing with fractions,

c) converting a fraction to a decimal by dividing.

Three approaches to explaining division with decimals are presented. Study all of them to determine which of them you would prefer to use. Perhaps you will not feel any is appropriate for you and will wish to explore other techniques or methods; you are encouraged to do so. It may be worth noting that children will already have been introduced to the concept of division of rational numbers, as well as to computations involving division of fractions.

DIRECTIONS:

1. Study and discuss each of the procedures below and complete the exercises that follow.

<u>Method A</u>

EXAMPLE $2.5\overline{)1.72}$

1) Write the division example as a fraction.

$$2.5\overline{)1.72} \qquad \frac{1.72}{2.5}$$

2) Multiply by 1 (= $\frac{100}{100}$) using an appropriate power of 10.

$$\frac{1.72}{2.5} \times \frac{100}{100} = \frac{172}{250}$$

3) Convert the fraction to a decimal by dividing.

<u>Note</u>: The pupils should be advised as to the required precision of the answer. In this case the answer "comes out even" at the thousandths place; i.e., the decimal is a terminating one.

$$\begin{array}{r} 0.688 \\ 250\overline{)172.000} \\ \underline{150\ 0} \\ 22\ 00 \\ \underline{20\ 00} \\ 2\ 000 \end{array}$$

Use Method A to solve the following examples (to the nearest thousandth).

$3.4\overline{)0.07}$ $\qquad$ $0.05\overline{)1.23}$ $\qquad$ $6.7\overline{)1.3}$

<u>Method B</u>

EXAMPLE $\quad 2.5\overline{)1.72}$

1) Write the example in sentence form.

$$2.5\overline{)1.72} \qquad 1.72 \div 2.5$$

2) Write each decimal as a fraction.

$$1.72 \div 2.5 = \frac{172}{100} \div \frac{25}{10}$$

3) Divide, using the fractions.

$$\frac{172}{100} \div \frac{25}{10} = \frac{172}{100} \times \frac{10}{25}$$

$$= \frac{1720}{2500}$$

4) Convert the fraction to a decimal by dividing.

$$
\begin{array}{r}
0.688 \\
2500\overline{)\,1720.000} \\
\underline{1500\ 0} \\
220\ 00 \\
\underline{200\ 00} \\
20\ 000
\end{array}
$$

Use Method B to solve the following examples (to the nearest thousandth).

$3.4\overline{)\,0.07}$ $\quad$ $0.05\overline{)\,1.23}$ $\quad$ $6.7\overline{)\,1.3}$

Method C (short method)

This method is similar to Method A except that the divisor and dividend are multiplied by a multiple of 10 determined by the number of decimal places in the divisor. For example

$$2.5\overline{)\,1.72} \text{ becomes } 25\overline{)\,17.2}$$

Each is multiplied (mentally) by 10 since 2.5 has only one decimal place. The usual technique for showing this method is:

$$2.5\overline{)\,1.72}$$

1) Place arrows in appropriate places to show the resulting division using Method C. Give the multiple of 10 by which you mentally multiplied the divisor and dividend.

Multiple of 10

a) $0.317\overline{)\,2.8679}$ ____________________

b) $13.5\overline{)\,678.5}$ ____________________

c) $0.017\overline{)\,61.8}$ ____________________

2) Solve the following using Method C.

a) $6.7\overline{)\,5.219}$

b) $7.84\overline{)\,61.72}$

c) $0.17\overline{)\,34.68}$

2. Study the exercises which you have completed. Suppose you did each division example as if there were no decimal point. Identify a pattern and a rule that would tell you where the decimal point should be placed in your answer.

3. Make a list of prerequisite skills that pupils must have to perform the computations above. What seem to be the advantages and disadvantages of Methods A, B, and C?

4. In the example 0.0256 ÷ 1.6, each of the steps shown below can be justified mathematically. For each of the steps provide a justification.

   a) $\frac{0.0256}{1.6} = \frac{0.0256}{1.6} \times \frac{10}{10}$

   b) $\frac{0.0256}{1.6} = \frac{0.256}{16}$

   c) $\frac{0.256}{16} = 0.016$

5. Discuss: Should estimation be used in the teaching of multiplication and division with decimals? Give examples to support your position.

6. For each of the following real-world problems, write a mathematical sentence that will help you solve the problem. Then find the solution.

   a) The circumference of the earth is about 2500 miles. Using 3.14 as an approximation for π find the diameter of the earth.

   b) The average annual rainfall at Miami is 59.76 inches. The average annual rainfall at Reno is 7.2 inches. How many times as much rain as Reno does Miami get?

   c) A stack of one million one-dollar bills is approximately 3750 inches thick. How tall is a stack of one thousand one-dollar bills? One hundred?

7. Write a real-world problem that could be represented by each of the following examples.

a) 12.45 ÷ 1.5

b) -3.445 ÷ 0.5

<u>A Decimal Approximation for the Golden Section</u>

Earlier in this unit, page 58, it was stated that the Golden Section (Golden Rectangle) can be approximated using ratios of successive numbers in the Fibonacci sequence. Now that you know that each rational number can be represented as a decimal, find a decimal approximation to the Golden Section which is correct to at least three decimal places.

Can you think of a physical object which most college students might sometime use and which is an approximation to the Golden Rectangle?

ACTIVITY 32

ANALYSIS OF ERROR PATTERNS FOR DECIMALS

FOCUS:

The analysis of pupil errors is one task that an elementary school teacher must continually perform. This activity will illustrate some of the common errors made by pupils when solving arithmetic problems involving decimals. An opportunity will also be provided for you to recommend some techniques for correcting these errors.

DIRECTIONS:

1. Read each worksheet included in this activity and briefly summarize the error pattern that you observed for each child.
2. Answer problems G) and H) making the same error that the child might make, as evidenced by the error pattern you observed.
3. List at least two possible causes for the error pattern noted on each worksheet.
4. Give a couple of questions that you might ask a child, to help you get to the source of the problem.
5. Recommend some immediate activities that might be used to assist the child in seeing that an error has been made or in preventing the error from recurring.

WORKSHEET I

NAME Chris

A) 2.1 + 3.06 = 3.27 B) 22.4 + 11.1 = 33.5

C) 6.41 + 2.37 = 8.78 [illegible] 19.3 + 4.115 = 4.308

E) 97.22 + 8.36 = 105.58 F) 6.002 + .34 = 6.036

G) 523.6 + 47.8 =	H) 71.28 + 2.311 =

Error Pattern Noted:

Diagnosis of Possible Causes:

Possible Remedial Steps:

WORKSHEET II

NAME Joyce

A) 3.42
-2.175
1.255

B) 1.074
- .82
.254

C) 34.28
- 25.7
8.58

D) 61.31
- 42.2174
19.1074

E) 3.716
- 2.178
1.538

F) 35.87
- 13.8254
22.0554

G) 317.81
- 28.476

H) 68.237
- 31.98

Error Pattern Noted:

Diagnosis of Possible Causes:

Possible Remedial Steps:

WORKSHEET III

NAME Billy

A)
```
  2.71
x  .03
------
  8.13
```

B)
```
  16.2
x  .12
------
   324
  162
------
 1,944
```

C)
```
  22.3
x  3.7
------
  1561
 669
------
 825.1
```

D)
```
   .233
x   .52
-------
   466
 1165
-------
 .12116
```

E)
```
   1.005
x   .214
--------
    4020
   1005
  2010
--------
 215.070
```

F)
```
  19.1
x  3.4
------
   764
  573
------
 649.4
```

G)
```
  15.2
x  6.5
------
```

H)
```
  2.35
x  2.3
------
```

Error Pattern Noted:

Diagnosis of Possible Causes:

Possible Remedial Steps:

WORKSHEET IV

NAME Roger

A)
```
       15.2
2.5 ⟌ 380
      25
      130
      125
        50
        50
```

B)
```
        26.3
.42 ⟌ 1104.6
      84
      264
      152
       126
       126
```

C)
```
          .0516
1.62 ⟌ 8.3592
       810
        259
        162
         972
         972
```

D)
```
        2.36
5.7 ⟌ 134.52
      114
       205
       171
        342
        342
```

E)
```
      .79
38 ⟌ 30.02
     266
      342
      342
```

F)
```
         34.
11.5 ⟌ 3910
       345
        460
        460
```

G) $22.3 \overline{)105.256}$

H) $.57 \overline{)7524}$

Error Pattern Noted:

Diagnosis of Possible Causes:

Possible Remedial Steps:

Section V

THE REAL NUMBER SYSTEM

In Section IV, rational numbers were expressed by using a decimal numeration system. For example, the fraction $\frac{3}{8}$ was written using the terminating decimal 0.375 and the fraction $\frac{2}{3}$ was written using the repeating decimal $0.66\overline{6}$ or 0.666... Using the strategy described there, we can write every rational number in one of those two forms; and every decimal of those forms represents a rational number.

Examining the different types of decimals suggests that there may exist some numbers that do not belong to the set of rational numbers. For example, 0.010010001..., which does not terminate and does not repeat, is one such decimal. The numbers which can be represented by nonterminating, nonrepeating decimals are called irrational numbers. The irrational numbers are not typically included in the elementary school curriculum, but appear in most junior high school textbooks, so you should at least be familiar with them.

$$\text{Rational} \rightarrow \text{Real} \leftarrow \text{Irrational}$$

Combining the set of irrational numbers with the set of rational numbers gives yet a new set of numbers, called the <u>real numbers</u>.

Thus, any real number must belong to one of two sets, the rationals or the irrationals. This section will provide a brief introduction to the set of real numbers.

Since we have already studied rational numbers, the focus of this section will be on irrational numbers. Remember that real numbers include the rationals as well as the irrationals.

Historically, the existence of irrational numbers has been known since the time of the Greeks. The Pythagorean school is credited with the discovery; but the irrationality of $\sqrt{2}$, for example, was difficult for many Greek mathematicians to accept--even the Pythagoreans themselves. They assumed that everything depended on the whole numbers, and had evolved a theory of proportion which was limited to magnitudes having a common unit of measure. The discovery of the irrationality of $\sqrt{2}$ posed a threat to this theoretical structure, so for some time the Pythagoreans attempted to keep the existence of irrational numbers secret. Members were threatened with death for disclosing information about these new numbers.

By 370 B.C., the "scandal" was resolved by a pupil of Plato, Eudoxus, who devised a new definition of proportion. His handling of the irrationals appears in the fifth book of Euclid's Elements and is similar to that of the mathematician Dedekind who, in 1872, gave the first rigorous and modern description of the set of real numbers.

At that time, mathematicians were seeking a way to provide a solid foundation for the new mathematics that was being developed. To accomplish this task, it was necessary to formally define the set of irrational numbers. It is interesting to note that this formal definition of the set of irrational numbers and, consequently, of the set of real numbers occurred only 100 years ago.

The description of irrational numbers given in this section will be informal and intuitive, not formal and rigorous.

MAJOR QUESTIONS

1. Give three different instances that would prevent us from thinking that all numbers are rational.

2. Find some instances where irrational numbers are introduced in the elementary school. Describe how you might develop one such instance.

3. Write a paragraph describing the relationships among the following sets of numbers: wholes, rationals, reals, integers, irrationals.

ACTIVITY 33

IRRATIONAL NUMBERS

FOCUS:

This activity introduces irrational numbers and different ways of representing them.

MATERIALS:

Ruler and geoboard or dot paper.

DISCUSSION:

Two ways that one can recognize the need for irrational numbers are:

- by looking at the decimal numeration system,
- by analyzing certain geometric objects.

Let's pursue these.

DIRECTIONS:

1. a) 0.121221222122221... is a nonrepeating decimal. Explain how you can tell.

   b) Write two more nonrepeating decimals.

   c) Can these nonrepeating decimals represent rational numbers? (What do you know about the decimal representation of a rational number?)

2. *OPTIONAL: Let the sequence $r_1, r_2, r_3, r_4, r_5, \ldots$ be the following.*

   $r_1 = 0.1$ $r_5 = 0.12122$

   $r_2 = 0.12$.

   $r_3 = 0.121$.

   $r_4 = 0.1212$.

a) *The numbers $r_1, r_2, r_3, r_4, r_5, \ldots$ are all rational numbers. How can you tell?*

b) *What is the relationship between the sequence $r_1, r_2, r_3, r_4, r_5, \ldots$ and the irrational number*

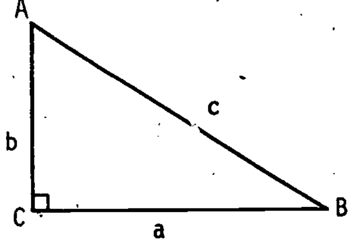
$$r = 0.121221222122221\ldots$$

defined in 1(a)?

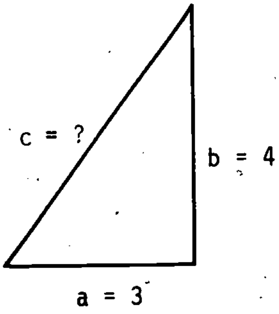
c) *Explain what one might mean by saying that $r_1, r_2, r_3, r_4, \ldots$ __converges__ to r. (Try roughly picturing $r_1, r_2, r_3, r_4, \ldots$ and r on a number line.) This is an example of a sequence of rational numbers that converges to an irrational number.*

3. Geometric objects like right triangles and circles were thoroughly studied by the Greek mathematicians who preceded Euclid. It was in analyzing these objects that they discovered the fact that <u>rational numbers were not enough</u>. You should recall that, using the Pythagorean Theorem, it is possible to find the length of the third side of a right triangle, given the lengths of the other two sides. For triangle ABC pictured below, with sides a, b, and c, the Pythagorean Theorem states that:

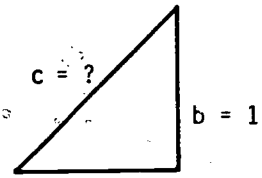
$$a^2 + b^2 = c^2.$$



a) Use the Pythagorean Theorem to find the length of the third side of the triangle on the right.



b) How about this one?



c) List three more pairs of integers for a and b that do not yield integer values for c.

> Fact: If a whole number is not a perfect square,* its square root is an irrational number. For example,
>
> $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, $\sqrt{15}$, $\sqrt{17}$
>
> are all irrational numbers. This can be proved. We won't do it here.

d) Briefly discuss the quandary that the results of questions (a), (b), and (c) presented to the Pythagoreans, who felt that all of geometry could be accomplished using rational numbers. (To refresh your memory about the Pythagoreans, you may want to reread the introduction to Section V.)

4. a) Two constructions of irrational numbers are shown below. Choose either Figure A or Figure B and briefly describe how the lengths $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ are constructed.

*The whole number n is a perfect square if there is a whole number m so that $m^2 = n$. Examples of perfect squares are 1, 4, 9, 16, 25,...

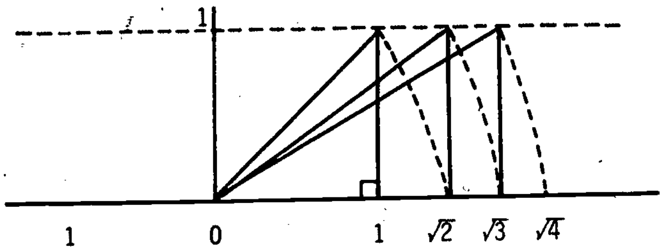


Figure A

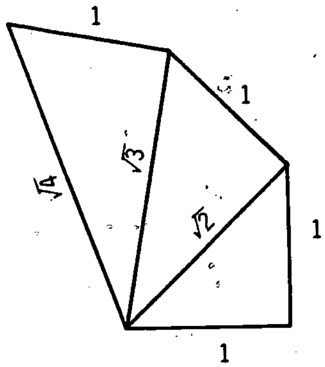


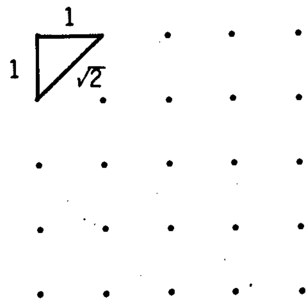
Figure B

b) Using the same technique, construct $\sqrt{5}$ and $\sqrt{6}$ on each diagram.

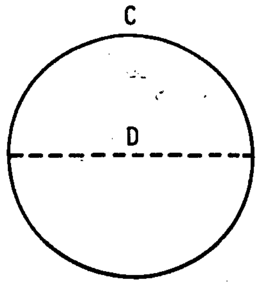
5. Rational and irrational lengths can be represented on dot paper (or on a geoboard). See how many of the following numbers can be represented using this technique.

$$\sqrt{3},\ \sqrt{4},\ \sqrt{5},\ \sqrt{6},\ \sqrt{7},\ \sqrt{8},\ \sqrt{9},\ \sqrt{10}$$

$\sqrt{2}$ has been constructed in the illustration below.



Fact:



The circumference (C) of a circle is the distance around it and the diameter (D) is the greatest distance across. Surprisingly, the ratio $\frac{C}{D}$ is the same for every circle. The constant value of that ratio is the irrational number which we call "pi" (π). You have seen this fact expressed as

$$\frac{C}{D} = \pi \quad \text{or} \quad C = \pi D.$$

It is possible, but difficult, to prove that π is irrational. The first six digits of the (nonrepeating) decimal representation of π are 3.14159... π has been computed to thousands of decimal places using electronic computers.

π is developed more thoroughly in the <u>Measurement</u> unit of the Mathematics-Methods Program.

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ACTIVITY 34

RATIONAL APPROXIMATIONS OF IRRATIONAL NUMBERS

FOCUS:

In this activity a procedure will be developed for finding rational numbers that are close to the irrational number $\sqrt{n}$ where the whole number n is not a perfect square.

DISCUSSION:

As a point of departure, recall that the square root of a number a is that number $\sqrt{a}$ which when multiplied by itself gives a. Thus the square root of 25 is 5 since 5 x 5 = 25. We write $\sqrt{25} = 5$. Similarly, $\sqrt{16} = 4$ and $\sqrt{121} = 11$. What is $\sqrt{7}$? It is not a rational number. But whatever it is, we know that $\sqrt{7} \times \sqrt{7} = 7$.

It is often inconvenient to use irrational numbers. For example, if one wished to know the sum or product of 5 and $\sqrt{7}$ the answer $5 + \sqrt{7}$ or $5\sqrt{7}$ does not lend much insight into the magnitude of the answer. It is often more convenient to use a rational approximation of the irrational number. For example, if one uses the approximation

$$\sqrt{7} \approx 2.449490,^*$$

$$\text{then} \quad 5 + \sqrt{7} \approx 7.449490$$

$$\text{and} \quad 5\sqrt{7} \approx 12.247450.$$

(Note that each of the decimals on the right is terminating, so that it represents a rational number.)

At one time, teaching children to find rational approximations to the square root of a number was considered an important process in school. Today it is not considered an important skill since one can use a table of such values (found in many books) or, better yet,

---

*The "wiggly" equals sign $\approx$ means "approximately equals."

use a hand calculator. The procedure below for finding square root values, however, is important in that it gives some insight into the technique of finding successively closer approximations to a number. The emphasis in this activity, then, is on the approximation process rather than on the skill involved in finding square roots.

DIRECTIONS:

The example below presents one strategy for finding the square root of a number. Study the procedure and use it to find the square roots in the exercises that follow. (Do not use a table of square roots or a hand calculator!)

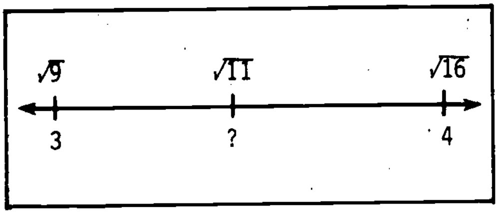
EXAMPLE

David wanted to find a decimal approximation of $\sqrt{11}$ to the nearest thousandth. He made an initial estimate of 3.5 since he reasoned that

$$\sqrt{9} < \sqrt{11} < \sqrt{16}$$

so

$$3 < \sqrt{11} < 4.$$

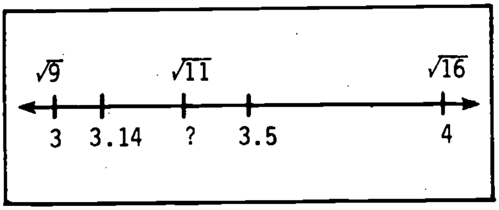


David divided

$$\frac{11}{3.5} \approx 3.14$$

So David concluded that

$$3.14 < \sqrt{11} < 3.5.$$



(This is reasonable because: $3.14 \approx \frac{11}{3.5}$ so $3.14 \times 3.5 \approx 11$. But $\sqrt{11} \times \sqrt{11} = 11$, so 3.14 must be too small and 3.5 too large.) David chooses an estimate halfway between 3.14 and 3.5:

$$\frac{3.14 + 3.5}{2} = 3.32$$

David divides 11 by his new estimate:

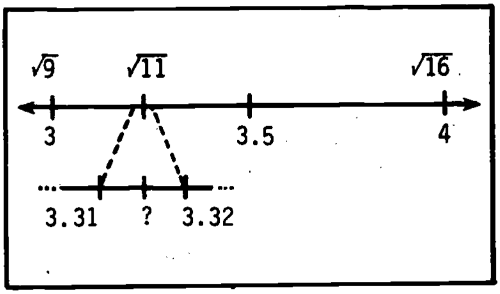
$$\frac{11}{3.32} \approx 3.31$$

Again he reasons that

$$3.31 < \sqrt{11} < 3.32$$

(You supply the argument this time.) David's new estimate is

3.315



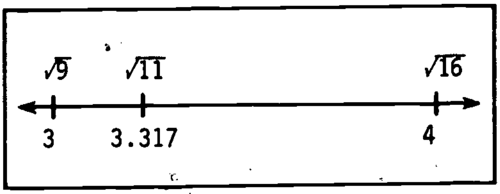
After dividing, he decides that

$$3.315 < \sqrt{11} < 3.318.$$

Another estimate (halfway between 3.315 and 3.318) is 3.3165. After dividing 11 by 3.3165, therefore, he finds:

$$3.3165 < \sqrt{11} < 3.3167$$

Since these estimates agree in the thousandths place, David decided that he could choose either one and round it off to the nearest thousandth. So David's estimate to the nearest thousandth was 3.317. He wrote $\sqrt{11} \approx 3.317$.



1. Give the argument that David would have used to conclude that $3.3165 < \sqrt{11} < 3.3167$.

2. You know that $\sqrt{9} = 3$ and $\sqrt{16} = 4$.

   a) Which is greater, $\sqrt{9}$ or $\sqrt{10}$?

   b) The $\sqrt{10}$ is between the integers _____ and _____.

3. For each irrational number below find the perfect squares and the two integers between which the irrational number lies.

	Irrational number	Perfect squares	Integer pairs
	$\sqrt{5}$	4 and 9	2 and 3
a)	$\sqrt{8}$		
b)	$\sqrt{23}$		

4. Use the averaging technique to find an approximate decimal equivalent (correct to the nearest hundredth) for the following irrational numbers.

a) $\sqrt{5}$ b) $\sqrt{23}$

ACTIVITY 35

THE REALS: THE COMPLETE NUMBER SYSTEM

FOCUS:

The integers have additive inverses while the whole numbers do not. The rational numbers contributed multiplicative inverses to our list of number properties. In this activity we will see that the real numbers do have a property called <u>completeness</u> which the rational numbers lack. You will find that the property is a little subtle.

DISCUSSION:

In Activity 33 we saw that the sequence

$r_1 = 0.1$
$r_2 = 0.12$
$r_3 = 0.121$
$r_4 = 0.1212$
$r_5 = 0.12122$
.
.
.

converges (gets nearer and nearer) to the irrational number $r = 0.121221222122221\ldots$ That is, the sequence r_1, r_2, r_3, r_4, r_5, ... of rational numbers zeroes in on (converges to) the target r which is not a rational number.

In Activity 34 we had two sequences of rational numbers

3.14	3.5
3.31	3.32
3.315	3.318
3.3165	3.3167

$\sqrt{11}$

which zeroed in on the irrational target number $\sqrt{11}$.

Can you see the shortcoming of the rational number system? It has these sequences that are zeroing in on a missing target. There are holes! The rational numbers lack <u>completeness</u>. The real numbers have completeness in the sense that every sequence of real numbers that is zeroing in on a target will find the target in the real number system.

DIRECTIONS:

1. For the sequence $r_1, r_2, r_3, r_4, r_5, \ldots$ and for the rational sequence converging to $\sqrt{11}$, draw a magnified number line that pictures the sequences zeroing in on (converging to) their respective target numbers r and $\sqrt{11}$.

2. Find two more sequences of rational numbers that are converging to a target which is not in the rational number system. What number does each converge to?

3. $S_1 = 0.921221222122221\ldots$
   $S_2 = 0.191221222122221\ldots$
   $S_3 = 0.129221222122221\ldots$
   $S_4 = 0.121921222122221\ldots$
   $S_5 = 0.121291222122221\ldots$
   $S_6 = 0.121229222122221\ldots$
   $S_7 = 0.121221922122221\ldots$
   $S_8 = 0.121221292122221\ldots$
   $S_9 = 0.121221229122221\ldots$
   $S_{10} = 0.121221222922221\ldots$
   .
   .
   .

   a) Compare this sequence with $r_1, r_2, r_3, r_4, r_5, \ldots$ and r. Describe how $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, \ldots$ was created.

   b) Are the numbers in the sequence rational or irrational?

c) Compute $S_1 - r_1$

$S_2 - r_2$

$S_3 - r_3$

.

.

.

d) What target is $S_1, S_2, S_3, \ldots$ zeroing in on?

e) Explain how all of this illustrates the statement that every sequence of real numbers which zeroes in on a target will find that target in the real numbers. How does this example differ from the examples in (1) and (2)?

4. *OPTIONAL: Here is a tricky one for you. Do you think that if a sequence of irrational numbers zeroed in on a target, that target would be irrational? The answer is "no." It's up to you to find a sequence of irrational numbers that zeroes in on a rational target.*

ACTIVITY 36

CARDINALITY OF THE RATIONAL NUMBERS

FOCUS:

The cardinality of a set refers to how large it is, i.e., to how many objects are in it. As new sets of numbers are introduced, it is interesting to investigate their relative size. In this activity you will have an opportunity to investigate the relative sizes of the set of whole numbers, the set of integers, the set of rational numbers, and the set of real numbers.

DISCUSSION:

The <u>cardinality</u> of a set refers to the number of objects in the set. For example, the cardinality of the set A

$$A = \{1, 3, 5, 7, 9, 11\}$$

is 6, because A contains 6 objects. We tell "how many" in a set by putting the objects in one-to-one correspondence with the counting numbers. That's what counting is!

$$\begin{array}{cccccccc} \{1, & 3, & 5, & 7, & 9, & 11\} & & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ \{1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots\} \end{array}$$

Similarly the cardinality of the set B is 10 since

$$\begin{array}{rcccccccccccc} B = & \{2, & 5, & 7, & 12, & 29, & 37, & 45, & 59, & 73, & 88\} & & \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ & \{1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & \ldots\} \end{array}$$

Although the examples given above represent finite sets, we are going to be interested here in sets which contain an infinite number of elements. Some examples of infinite sets include the counting numbers, the even numbers, the multiples of three, the integers, and the rational numbers. The rule for determining whether two infinite sets have the same cardinality is the same as for finite sets. That is, for all sets, we say:

> Two sets of objects have the same cardinality if there is a one-to-one correspondence between their objects.

Each of the finite sets below has different cardinality.

$$\{a, b, c\}$$
$$\{a, b, c, d\}$$
$$\{a, b, c, d, e\}$$

You might guess that the set N of counting numbers and the set N_E of even numbers have different cardinality, since the set of counting numbers properly contains the even numbers.

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$$
$$N_E = \{2, 4, 6, 8, 10, 12, 14, 16, \ldots\}$$

Yet these sets have the <u>same cardinality</u>! To show this, we must show that there is a one-to-one correspondence between the two sets. That is, we must show a relationship between N and N_E which relates each element of N to one element of N_E and each element of N_E to one element of N. The arrows below graphically illustrate a one-to-one correspondence between N and N_E.

$$\begin{array}{rccccccccl} N = \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ N_E = \{ & 2, & 4, & 6, & 8, & 10, & 12, & 14, & 16, & \ldots\} \end{array}$$

This illustration that N and N_E have the same cardinal number should tip you off that analysis of infinite sets may produce some weird and unexpected results.

DIRECTIONS:

1. Describe verbally a one-to-one correspondence between N and N_E. Try to describe a general rule for relating the numbers of one set to those in the other.

2. Describe graphically and verbally a one-to-one correspondence between each of the following sets and the set of counting numbers, to show that each has the same cardinality as the set of counting numbers.

   a) the odd numbers,

   b) the multiples of 3,

   c) the multiples of 10.

3. Rather than drawing or verbally describing a one-to-one correspondence we can usually write an algebraic equation which associates the elements between the two sets in question. For example, one function which describes the one-to-one correspondence between the set of counting numbers and the even numbers is

$$y = 2 \cdot x$$

   where x is a counting number and y is an even number. Thus, if $x = 1$, then y is 2; if x is 6, then y is 12; etc. For each of the sets in (2), write an equation that describes the one-to-one correspondence between those sets and the set of counting numbers.

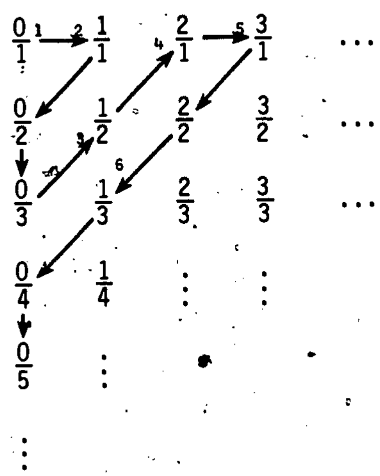
4. By now you should not be too surprised to learn that N and I (the integers) also have the same cardinality. See if you can come up with a one-to-one correspondence.

5. In order to compare the counting numbers and the set of rational numbers, it is first necessary to display the nonnegative rational numbers in some manner so that all of them are included. One way is pictured on the next page. Study the display to convince yourself that all rational numbers are represented. (Note that each rational number appears more than once.)

$$\begin{array}{ccccccccc}
\frac{0}{1} & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
\frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\
\frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\
\frac{0}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
\frac{0}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \cdots & \\
\frac{0}{6} & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}$$

Can you suggest a way to show a correspondence between the set of counting numbers and the set of rational numbers?

6. If you were not successful in finding a technique for describing the correspondence, consider the pattern shown below for a portion of the display.



That is,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & & \\ \frac{0}{1} & \frac{1}{1} & \frac{1}{2} & \cdots & \cdots \end{array}$$

a) Continue the list above by indicating which rational numbers are related to 4, 5, 6, 7, 8, and 9.

b) This gives a one-to-one correspondence between N and the nonnegative rational numbers. Can you figure out a correspondence between N and all of the rational numbers? (Look for an analogy with the correspondence between N and I.)

c) What does all of this say about the cardinality of N and I?

7. *OPTIONAL: By now you may suspect that all infinite sets have the same cardinality as N. This is* <u>*not*</u> *true. Here we will provide you with the steps necessary to show that there are infinite sets with different cardinality. It will be up to you to analyze what we have done.*

Let S stand for the set of all real numbers between 0 and 1 whose decimal expansions contain only 0's and 1's. For example,

.1111...

.0101...

.110011...

and many numbers both rational and irrational that are difficult to describe are all in S. <u>*Suppose*</u> *that the following is* <u>*any*</u> *one-to-one correspondence between S and N*

$$\begin{array}{c} 1 \longleftrightarrow s_1 \\ 2 \longleftrightarrow s_2 \\ 3 \longleftrightarrow s_3 \\ \vdots \\ n \longleftrightarrow s_n \\ \vdots \end{array}$$

where $s_1, s_2, s_3, \ldots, s_n, \ldots$ are all members of S. Define a real number x whose decimal contains 0's and 1's (i.e., a member of S) as follows:

The first digit in the decimal representation of x is

$$\begin{cases} 0 \text{ if the first digit in } s_1 \text{ is } 1 \\ 1 \text{ if the first digit in } s_1 \text{ is } 0. \end{cases}$$

The second digit in x is

$$\begin{cases} 0 \text{ if the second digit in } s_2 \text{ is } 1 \\ 1 \text{ if the second digit in } s_2 \text{ is } 0. \end{cases}$$

The third digit in x is

$$\begin{cases} 0 \text{ if the third digit in } s_3 \text{ is } 1 \\ 1 \text{ if the third digit in } s_3 \text{ is } 0. \end{cases}$$

a) Is x in S? Explain.

b) Is x equal to any of the numbers, s_1, s_2, s_3, ..., s_n ...? Explain.

c) What does all this say about the cardinality of N and S?

8. OPTIONAL: Use (7) to show that the real numbers do not have the same cardinality as the counting numbers.

ACTIVITY 37

COMPARING NUMBER SYSTEMS

FOCUS:

Throughout this unit, several number systems have been discussed. Each of these has grown out of a set of numbers already studied. This activity will give insight into the relationships among these sets of numbers.

DIRECTIONS:

1. For each of the numbers in the chart below, indicate whether it belongs to the set of whole numbers, integers, rational numbers, irrational numbers, or real numbers.

	Wholes	Integers	Rationals	Irrationals	Reals
-5					
$-\sqrt{4}$					
17					
$\frac{147}{3}$					
$-\sqrt{13}$					
-0.432					
0					
$\sqrt{6}$					

2. Determine whether each of the following is true or false. Remember that a statement is true only if it is always true.

a) Every whole number is a (an)

_____ integer.
_____ rational number.
_____ irrational number.
_____ real number.

b) Every integer is a (an)

_____ whole number.
_____ rational number.
_____ irrational number.
_____ real number.

c) Every rational number is a (an)

_____ whole number.
_____ integer.
_____ irrational number.
_____ real number.

d) Every irrational number is a (an)

_____ whole number
_____ integer.
_____ rational number.
_____ real number.

e) Every real number is a (an)

_____ whole number.
_____ integer.
_____ rational number.
_____ irrational number.

APPENDIX A

SELF-TEST ANSWERS (ACTIVITY 6)

Part A: Common Fractions

1. $\frac{5}{7}$
2. 2 (or $\frac{4}{2}$)
3. $\frac{2}{12}$ or $\frac{1}{6}$
4. $\frac{9}{8}$ or $1\frac{1}{8}$
5. $\frac{45}{28}$ or $1\frac{17}{28}$
6. $\frac{5}{48}$
7. $\frac{28}{40}$ or $\frac{7}{10}$
8. $\frac{5}{7}$
9. 7
10. $\frac{13}{8}$
11. $\frac{6}{12}$ or $\frac{1}{2}$
12. $\frac{24}{28}$ or $\frac{6}{7}$
13. m = 4; n = 21
14. p = 15
15. d, a, b, e, c
16. (a) $\frac{2}{3}$ (b) $1\frac{3}{10}$ (c) 16
17. (a) 21 (b) 24 (c) 12
18. (a) 2 (b) 7 (c) 72
19. $4\frac{11}{36}$ or $\frac{155}{36}$
20. $\frac{46}{45}$ or $1\frac{1}{45}$
21. $\frac{25}{18}$ or $1\frac{7}{18}$
22. $\frac{885}{28}$ or $31\frac{17}{28}$
23. $33\frac{3}{4}$ or $\frac{135}{4}$
24. $\frac{26}{49}$
25. $\frac{341}{40}$ or $8\frac{21}{40}$
26. $\frac{197}{24}$ or $8\frac{5}{24}$
27. $\frac{77}{40}$ or $1\frac{37}{40}$
28. $\frac{-115}{60}$ or $-1\frac{11}{12}$
29. $\frac{1}{6}$
30. $\frac{-102}{65}$ or $-1\frac{37}{65}$

Part B: Decimals

1. 1.5617
2. 0.6490
3. 52.533
4. 109
5. 100
6. e, c, a, b, d
7. -100.25
8. -28.368
9. -12.63893
10. 49.87
11. -18.12
12. -50.85

13. 0.02369

14. 1465

15. 0.375

16. 2.333

17. $0.18\overline{18}\ldots$

18. $\frac{457}{625}$

19. $\frac{2}{3}$

20. $\frac{32}{99}$

APPENDIX B

THE PROPERTIES OF NUMBER SYSTEMS

DISCUSSION:

The properties of number systems are associated with the words:

- closure
- commutative
- associative
- identity
- inverse
- distributive

The different ways that these properties are used in teaching, learning, and doing mathematics loosely correspond to different stages in their historical evolution.

The concepts of the counting numbers, such as 1, 2, 3 and so on, arose naturally in the life of primitive humans. One would even have to imagine that the primitive hunter knew that killing one beast and then two more would yield the same amount of meat as killing two and then one more. It seems most doubtful, however, that the concept of commutativity was formally understood. Wouldn't you also guess that the hunter knew that if he killed no more, he had no more? We do know on the other hand that the formal idea of zero (or additive identity) was a long time coming.

As number systems have become more developed and more sophisticated, more properties have been observed and more have been needed. Typically, a property is observed to be true for one set of numbers and then it is used as a part of the definition of a new (usually larger) set of numbers.

In learning basic facts such as 2 + 3 = 5 or 2 x 3 = 6, children can observe the commutativity of + and x. The facts that $a + b = b + a$ and $a \times b = b \times a$ cut the number of basic facts to be learned exactly in half. Whenever a child adds 2 + 3 + 4, he must use some form of the associative law. Usually the child will add 2 and 3 and then 4, or 3 and 4 and then 2. In this case, the child is using the fact that $(a + b) + c = a + (b + c)$, even though a teacher might feel that making a point of the associative property would only confuse the child.

Another way the basic properties are useful is in explaining why certain algorithms work the way they do. For example, the multiplicative identity is used to explain why one inverts and multiplies when dividing fractions. In more advanced number work and for adult understanding, many of the rules and formulas that are taught to children intuitively and by analogy can be explained precisely in terms of the basic number properties.

A more sophisticated and probably much more important fact about the basic number properties is that they are the roots from which much of the power of abstract mathematics grows. Abstract systems that are the objects of study in mathematics are usually defined using properties that are selections from, or modifications of, the basic number properties (see, e.g., Activity 24 on Groups). These systems help to uncover many similarities in apparently dissimilar situations, thus giving insight into the structure of phenomena in the universe.

We will now list each of the six basic number properties and will provide:

a) a general statement of the property,

b) examples to convey the meaning of the property

c) a statement of when it does and does not hold.

1. The Closure Property

a) This property has to be defined by a specific operation on a specific set of numbers. A set is closed under an operation if, whenever two numbers in the set are combined by the operation, a number in the set results.

b) The whole numbers are closed under + since the sum of any two whole numbers is a whole number. The whole numbers are not closed under ÷ since 2 ÷ 3 is not a whole number even though 2 and 3 are.

c)

	+	-	x	÷
Whole numbers	Yes	No	Yes	No
Integers	Yes	Yes	Yes	No
Rational numbers	Yes	Yes	Yes	Yes*
Real numbers	Yes	Yes	Yes	Yes

The above table indicates which sets of numbers are closed under which operations.

2. The Commutative Property**

a) $a + b = b + a$ (Commutative property for addition)

$a \times b = b \times a$ (Commutative property for multiplication)

b) $7 + 16 = 23 = 16 + 7$

$39 \times 751 = 29{,}289 = 751 \times 39$

$-\frac{2}{3} + \frac{3}{5} = \frac{-1}{15} = \frac{3}{5} + (\frac{-2}{3})$

$3 \times (-2) = -6 = (-2) \times 3$

*Actually, the rational numbers excluding 0 are closed under ÷.

**The properties from 2 on are generally discussed only for + and x, since - and ÷ can be defined in terms of + and x and their inverses.

c) The commutative property holds for addition and multiplication on all sets of real numbers. The commutative property does not hold for subtraction or division except in very special cases.

3. <u>The Associative Property</u>

a) $(a + b) + c = a + (b + c)$
$(a \times b) \times c = a \times (b \times c)$

b) $(21 + 3) + 7 = 24 + 7 = 31$
$21 + (3 + 7) = 21 + 10 = 31$
$(-3 \times 2) \times 5 = -6 \times 5 = -30$
$(-3) \times (2 \times 5) = -3 \times 10 = -30$

c) Addition and multiplication are associative on all sets of numbers. (Subtraction and division are not associative on any sets of numbers that are of interest to us.)

4. <u>The Identity Property</u>

a) $0 + a = a + 0 = a$ for each a
(0 is called the additive identity)

$1 \times a = a \times 1 = a$ for each a
(1 is called the multiplicative identity)

b) $0 + 2391 = 2391 + 0 = 2391$
$1 \times \frac{3}{4} = \frac{3}{4} \times 1 = \frac{3}{4}$

c) 0 acts as additive identity for any set of numbers that contains 0, and 1 acts as multiplicative identity for any set of numbers that contains 1.

5. <u>The Inverse Property</u>

a) This property must be discussed relative to a set. A set is said to have the additive inverse property if for each a in the set, there is a b in the set so that $a + b = b + a = 0$. The multiplicative inverse property requires that, for each a ($\neq 0$) in the set, there is a b in the set so that $a \times b = b \times a = 1$.

b) The integers have the additive inverse property, e.g., $3 + (-3) = (-3) + 3 = 0$, $(-71) + 71 = 71 + (-71) = 0$. The integers do not have the multiplicative inverse property since $3 \times b = 1$ cannot happen if b is an integer. The rational numbers do, however, have the multiplicative inverse property since, if b is a nonzero rational number, $\frac{1}{b}$ is a rational number and $b \times \frac{1}{b} = \frac{1}{b} \times b = 1$.

c)

	Additive Inverse Property	Multiplicative Inverse Property
Whole numbers	No	No
Integers	Yes	No
Rational numbers	Yes	Yes
Real numbers	Yes	Yes

6. <u>The Distributive Property</u>

a) $a \times (b + c) = (a \times b) + (a \times c)$

b) $7 \times (5 + -3) = (7 \times 5) + (7 \times (-3))$
$= 35 + (-21)$
$= 14$

c) The distributive property holds on any set that is closed under + and x.

7. <u>Summary Table</u>

The following table summarizes the applicability of the six basic number properties to the four number systems we have studied.

	closure +	closure x	commutativity +	commutativity x	associativity +	associativity x	identity +	identity x	inverse +	inverse x	distributivity of x over +
Whole numbers	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No	Yes
Integers	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes
Rational numbers	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Real numbers	Yes	Yes	Yes	Yes	Yes	Yes'	Yes	Yes	Yes	Yes	Yes

APPENDIX C

SKILL BUILDER EXERCISES

DIRECTIONS:

Do these computations. Give your answers in simplest form. Show your work.

Exercise 1: Mixed Practice with Fractions

1. $\frac{1}{12} + \frac{7}{12} =$
2. $\frac{3}{4} - \frac{3}{8} =$
3. $\frac{3}{4} \times \frac{3}{4} =$
4. $\frac{2}{3} \div \frac{4}{5} =$
5. $\frac{0}{3} \times \frac{5}{14} =$
6. $\frac{7}{10} - \frac{3}{10} =$
7. $\frac{1}{2} + \frac{2}{5} =$
8. $\frac{3}{4} + \frac{1}{6} =$
9. $\frac{2}{3} - \frac{3}{5} =$
10. $3 \times \frac{5}{6} =$
11. $\frac{5}{8} \div \frac{3}{8} =$
12. $4 \div \frac{1}{2} =$
13. $\frac{3}{8} \times 1\frac{1}{3} =$
14. $\frac{5}{6} - \frac{3}{8} =$
15. $\frac{3}{10} + \frac{1}{2} =$
16. $1\frac{1}{5} + 2\frac{3}{10} =$
17. $1\frac{1}{4} - \frac{3}{4} =$
18. $1\frac{2}{3} \times 2\frac{2}{5} =$
19. $\frac{3}{4} \div 6 =$
20. $1\frac{3}{4} \div \frac{2}{3} =$

Exercise 2: Addition of Fractions

1. $\frac{1}{4} + \frac{7}{12} =$
2. $\frac{2}{3} + \frac{1}{5} =$
3. $\frac{1}{6} + \frac{7}{9} =$
4. $\frac{5}{12} + \frac{7}{18} =$
5. $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} =$
6. $\frac{5}{6} + \frac{7}{8} =$
7. $1\frac{1}{3} + 2\frac{1}{2} + \frac{1}{6} =$
8. $7\frac{3}{4} + 1\frac{5}{6} =$
9. $\frac{3}{8} + \frac{1}{6} + \frac{3}{10} =$
10. $\frac{2}{3} + \frac{4}{5} + \frac{6}{7} =$

Exercise 3: Subtraction of Fractions

1. $\frac{5}{8} - \frac{3}{8} =$
2. $\frac{5}{6} - \frac{1}{3} =$
3. $\frac{5}{6} - \frac{3}{4} =$
4. $\frac{1}{7} - \frac{1}{9} =$
5. $\frac{3}{8} - \frac{2}{9} =$
6. $2 - 1\frac{3}{8} =$
7. $4\frac{1}{4} - 2\frac{2}{3} =$
8. $3\frac{3}{16} - 1\frac{5}{8} =$
9. $\frac{11}{36} - \frac{11}{64} =$
10. $\frac{10}{21} - \frac{13}{28} =$

Exercise 4: Multiplication of Fractions

1. $\frac{2}{3} \times \frac{4}{5} =$
2. $12 \times \frac{2}{3} =$
3. $\frac{3}{8} \times \frac{4}{9} =$
4. $\frac{15}{16} \times \frac{20}{21} =$
5. $\frac{2}{3} \times \frac{3}{4} \times \frac{5}{6} =$
6. $9\frac{1}{5} \times 1\frac{1}{2} =$
7. $2\frac{2}{9} \times 3\frac{3}{5} =$
8. $8\frac{1}{3} \times 7\frac{4}{5} =$
9. $11\frac{2}{3} \times 12\frac{3}{5} =$
10. $5\frac{1}{2} \times 6\frac{2}{5} \times 3\frac{2}{11} =$

Exercise 5: Division of Fractions

1. $\frac{5}{6} \div \frac{1}{8} =$
2. $\frac{5}{6} \div \frac{5}{12} =$
3. $\frac{7}{12} \div \frac{5}{6} =$
4. $\frac{1}{5} \div 3 =$
5. $6 \div \frac{3}{4} =$
6. $2\frac{5}{8} \div 3 =$
7. $\frac{2}{3} \div \frac{1}{2} =$
8. $7\frac{1}{3} \div 2\frac{1}{5} =$
9. $4\frac{1}{4} \div 1\frac{5}{12} =$
10. $1\frac{5}{16} \div 3\frac{1}{2} =$

Exercise 6: Addition and Subtraction of Decimals

1. $0.3 + 0.75 =$
2. $2.87 - 1.6 =$
3. $0.0048 + 0.0109 =$
4. $1 - 0.00005 =$
5. $18.45 - 9.672$
6. $7.238 + 0.09 =$

7. 2.3 - 0.008 =

8. 1.34 + 21.091 =

9. 2.9093 + 1.1907

10. 12.472 - 8.79

Exercise 7: Multiplication of Decimals

1. 0.3 x 0.2 =

2. 1.2 x 0.08

3. 20 x 0.15 =

4. 27.8 x 2.2 =

5. 39.5 x 8.64 =

6. 1.02 x 3.5 =

7. 120 x 0.09 =

8. 0.0017 x 0.058 =

9. 17.25 x 4.4 =

10. 800 x 0.0014 =

Exercise 8: Division of Decimals

1. 24.12 ÷ 4 =

2. 88.2 ÷ 4.2 =

3. 4.263 ÷ 0.21 =

4. 0.8 ÷ 0.16 =

5. 3.9 ÷ 0.03 =

6. 6 ÷ 0.15 =

7. 0.0012 ÷ 0.2 =

8. 0.009 ÷ 20 =

9. 199.823 ÷ 2.47 =

10. 449.55 ÷ 0.185 =

Exercise 9: Percentages

1. What fraction in simplest form (lowest terms) is equivalent to 44%?
2. Write $\frac{3}{20}$ as a percentage.
3. What percent of 50 is 40?
4. What percent of 40 is 50?
5. What is 45% of 125?
6. How much sales tax (at 4%) do you pay when buying a $3,500 car?
7. What is 0.1% of one million dollars?
8. How much do you pay for a $42.95 article that is on sale for 15% off?
9. The wholesale price of an item is $8.40. If the markup is 30%, what is the retail price?

10. What was the percentage of decrease when the speed limit was lowered from 70 mph to 55 mph? (Give your answer correct to the nearest tenth of a percent.)

Exercise 10: Mixed Practice with Rational Numbers

1. $(\frac{3}{8} + \frac{1}{4}) \div 5\frac{1}{3} =$

2. $(3\frac{1}{2} + \frac{2}{5}) - 3\frac{3}{4} =$

3. $\frac{7}{12} - (\frac{4}{9} \div \frac{4}{3}) =$

4. $(\frac{5}{8} + \frac{3}{4}) \div \frac{4}{5} =$

5. $\frac{7}{8} \div (\frac{2}{3} + \frac{1}{6}) =$

6. $\frac{5}{7} - (\frac{1}{2} \times \frac{2}{3}) =$

7. $\frac{1}{2} + (\frac{2}{3} \times \frac{4}{5}) =$

8. $(1\frac{1}{3} \times 3\frac{3}{4}) \div 2\frac{5}{8} =$

9. $(\frac{5}{6} \times 1\frac{5}{16}) + (\frac{11}{16} \times \frac{5}{6}) =$

10. $(\frac{1}{2} \times \frac{3}{8}) + (\frac{5}{6} \times \frac{1}{2}) =$

11. Find $2\frac{1}{2}\%$ of $2\frac{1}{2}$

12. (85% of 40) + $5\frac{1}{2} =$

13. Find 42% of 28.3

14. $(17.4 - 0.05) \div 0.347 =$

15. $(0.82 \times 24.1) \div 1.205 =$

16. $(93.7 + 2.38) \div 0.08 =$

17. $(0.036 \div 40) \div 90 =$

18. $(36 \div 0.009) \div 0.4 =$

19. (35% of 15) $\div \frac{7}{8} =$

20. $3\frac{1}{4} \times (3\frac{2}{3} \div 7\frac{4}{5}) \times 2\frac{2}{5} =$

REQUIRED MATERIALS

ACTIVITY	AUDIO-VISUAL AND OTHER RESOURCES	MANIPULATIVE AIDS
1	Slide-tape: "Overview of Rational Numbers with Integers and Reals," cassette recorder and projector. (Optional)	
3	Elementary school mathematics textbook series	
4		Paper and scissors, spinners, counters or chips
8		Cuisenaire rods or strips of graph paper, paper shapes, six alike and six unalike objects, yarn or string
10		Paper, strips of paper, compass, protractor
13	Elementary school mathematics textbook series	Cuisenaire rods or strips of graph paper
15	Elementary school mathematics textbook series	

ACTIVITY	AUDIO-VISUAL AND OTHER RESOURCES	MANIPULATIVE AIDS
20		Ruler, graph paper
23	Elementary school mathematics textbook series	
28		Dienes multibase arithmetic blocks in base ten
29		Dienes multibase arithmetic blocks in base ten
33		Ruler and geoboard or dot paper

Continued from inside front cover

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* * * * * * * * * *

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Addition and Subtraction, Multiplication and Division, Awareness Geometry, Transformational Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Number Theory, Probability and Statistics,* and *Experiences in Problem Solving.*

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