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ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to the basic number work in the elementary school. In addition to an introduction to the unit, the text has sections on developing initial concepts in addition and subtraction, basic mathematical content in addition and subtraction, developing the basic addition and subtraction facts, algorithms and problem solving, and appendices that present an overview of cuisenaire rods and the properties of number systems. (MP)

MATHEMATICS EDUCATION DEVELOPMENT CENTER

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ADDITION AND SUBTRACTION

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

- Numeration
- Addition and Subtraction
- Multiplication and Division
- Rational Numbers with Integers and Reals
- Awareness Geometry
- Transformational Geometry
- Analysis of Shapes
- Measurement
- Number Theory
- Probability and Statistics
- Graphs: the Picturing of Information
- Experiences in Problem Solving

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

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Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugene, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

CONTENTS

INTRODUCTION TO THE ADDITION AND SUBTRACTION UNIT

Every day of their lives people are confronted with problems whose solutions require the operations of addition and subtraction. Typically such problems occur in shopping, banking, driving, sports, planning vacations and countless other everyday pursuits. Even preschool children may have informal experiences in addition and subtraction when playing with money, toys, and games in their homes.

As a result addition and subtraction find an important place in the elementary school mathematics curriculum. Children are usually introduced to addition and subtraction early in the primary grades, and continue to expand their knowledge of these operations in later years. Since the child's world is so rich in examples embodying addition and subtraction, an abundance of real-world situations exists for introducing these operations to children. The development of addition and subtraction from the real world will be a major focus throughout this unit.

The <u>Addition and Subtraction</u> unit is divided into four major parts:

I. Developing Initial Concepts in Addition and Subtraction

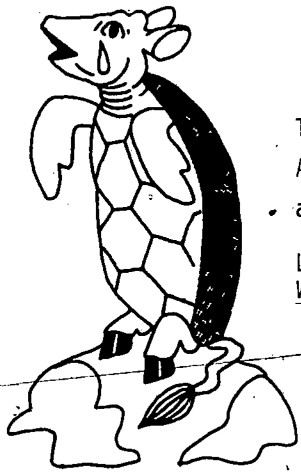
II. Basic Mathematical Content in Addition and Subtraction

III. Developing the Basic Addition and Subtraction Facts

IV. Developing Algorithms and Problem Solving in Addition and Subtraction

In the introduction to each section, an attempt has been made to give perspective to the activities included, and major questions have been posed to help you focus on important issues. Teacher Teasers (mathematical puzzles) are interspersed throughout the unit for your interest and enjoyment.

There are times when teachers need to call upon their resources to motivate, reinforce, or provide enrichment opportunities for children. Games, wisely selected and used, can be quite useful for such purposes. You are encouraged to keep a game file and add to it from your unit experiences and personal study.



The different branches of arithmetic--Ambition, Distraction, Uglification, and Derision.

Lewis Carroll, Alice's Adventures in Wonderland

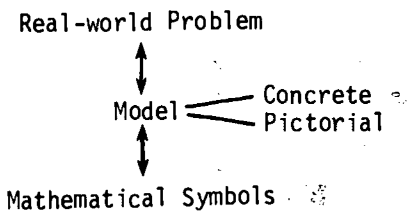
Section I

DEVELOPING INITIAL CONCEPTS IN ADDITION AND SUBTRACTION

Since addition and subtraction are the first operations introduced in the elementary school, it is extremely important that a teacher have a clear understanding of these concepts and how they might be presented to children. Not only is skill in addition and subtraction very functional in everyday life, but children's attitudes toward mathematics may be shaped at this early stage in their school experience. The clarity of the presentation of addition and subtraction ideas (and, relatedly, the strength of the child's grasp of them) may have a direct bearing on the formation of the child's attitude toward mathematics.

This section begins with an overview of addition and subtraction in the elementary school. Activities 2 and 3 present a description of some of the readiness work done by children in preparation for addition and subtraction. A thorough understanding of these prerequisites is essential since many children will have to be retaught these learnings. Activities 4 and 5 develop the three models used in early addition and subtraction: the set model, the number line model, and the function machine model. Activity 6 presents you with an opportunity to analyze some child errors in early subtraction. Activity 7 discusses the relationship between addition and subtraction and raises several questions about some potentially difficult lessons for children.

The point of view presented throughout this section is that mathematics should be developed from a real-world perspective. That is, mathematical learning for the young child should flow from real-world experiences to symbols. The diagram below will appear repeatedly in this section.



Major Questions

1. Describe a model which you could use to develop basic addition and subtraction. Include

   a) how the model can be related to a real-world problem

   b) how it can be used to process a solution

   c) how the solution reflects back to the original problem.

2. Explain the role of counting in helping children develop the concepts of:

   a) number

   b) addition

   c) subtraction.

3. What are some of the difficulties children experience with subtraction? Why do these occur? Suggest any emphasis in your teaching of subtraction which might help to avoid these difficulties.

ACTIVITY 1

OVERVIEW OF ADDITION AND SUBTRACTION IN THE ELEMENTARY SCHOOL

FOCUS:

While addition and subtraction seem easy to us, children do not usually come to school knowing these operations. This activity offers an opportunity for you to become acquainted with the general scope and sequence of the learning of addition and subtraction as it occurs in the elementary school. This overview should provide perspective on the content and motivation for your study of the Addition and Subtraction unit.

MATERIALS:

(Optional) The Mathematics-Methods Program slide-tape presentation, "Addition and Subtraction in the Elementary School."

DIRECTIONS:

1. Read the essay "Overview of Addition and Subtraction in the Elementary School" (or view the slide-tape presentation). Try to capture the general scope and sequence of the development of the two operations presented here.

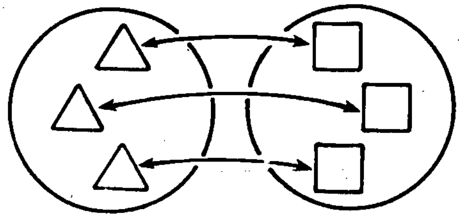
2. By what means is the real world related to mathematics in addition and subtraction?

3. There are three basic approaches to addition and subtraction. These are represented by sets, measure (number line), and function (function machine). Discuss each and make up some examples of problems (or activities) to illustrate each approach.

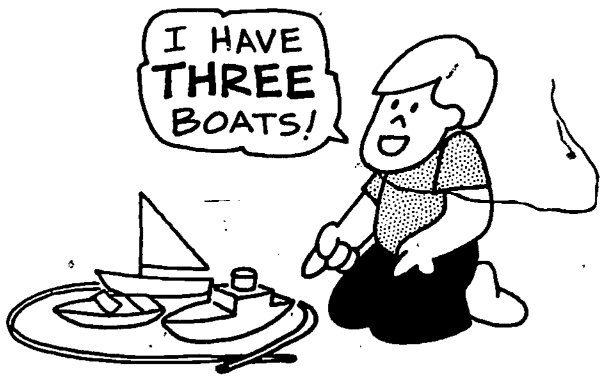
OVERVIEW OF

ADDITION AND SUBTRACTION IN THE ELEMENTARY SCHOOL

For any person--especially for a child--it is important to relate new concepts to concepts that are real, familiar, and concrete. So, when the concept of number is first introduced to children, it is done in the context of sets of concrete objects.



Children first learn to recognize one-to-one correspondences between sets of objects. Once they grasp the concept of equal numerosity of sets, they can then start to attach number names to sets in a meaningful way.



Similarly, in introducing the concepts of addition and subtraction to children, the teacher starts with a real problem and models the problem with concrete objects or with pictures of concrete objects. The child then manipulates the concrete objects to represent the solution of the real problem. When symbols are introduced to children, they are introduced as translations of the model of the real problem and its solution.

Real-world Problem

↕

Model → Concrete, Pictorial

↕

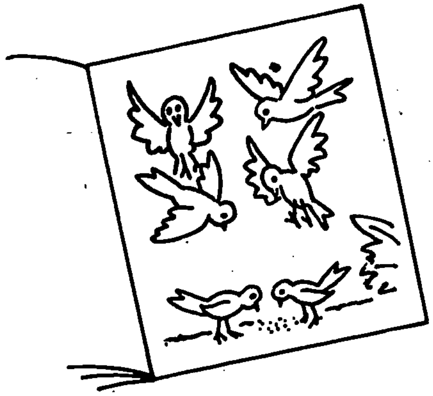
Mathematical Symbols

For example, if the objective is to introduce the basic concept of <u>addition</u>, a real-world problem is presented which suggests <u>joining</u> sets.

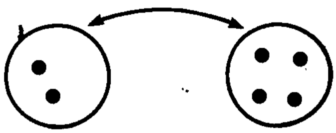
2 birds eating

4 more arrive.

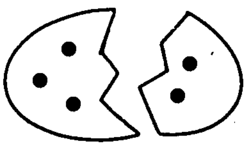
How many now?



This situation can be modeled by using a set of two marbles and a set of four marbles and by combining or joining those sets. The child subsequently learns to attach the symbols 2 + 4 to the situation.



Subtraction suggests separating part of a set. For example,



5 birds eating

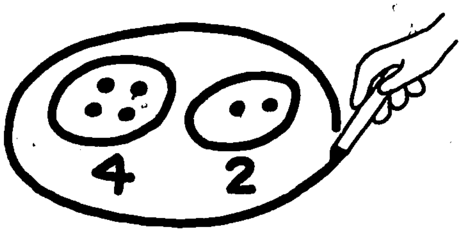
2 fly away.

How many are left?

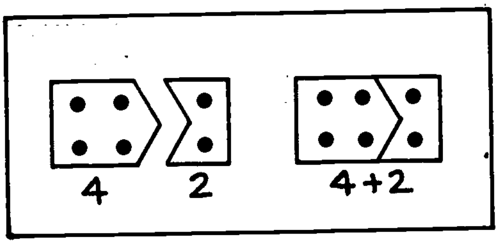
In every case, the child works with a concrete or physical model (sometimes pictorially represented). The model in turn gives meaning to the mathematical symbols used to represent it.

Now let's consider the instructional sequence for addition in a little more detail. Addition is introduced very early to children--at least by first grade. At this level, it is probably advisable to start with an oral presentation of a problem using actual objects.

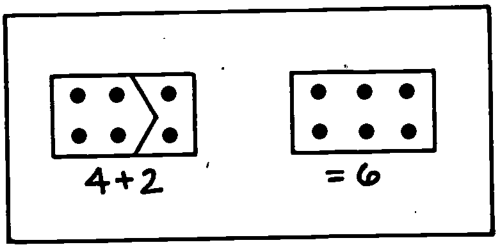
Carol can solve this problem by actually counting the crayons. A first level of abstraction can soon be introduced by having the child model problems using counters in place of the actual objects. The problem can also be modeled by having the child draw a picture.



In bridging the gap from these real situations involving joining to symbolic addition sentences, a diagram is sometimes helpful.

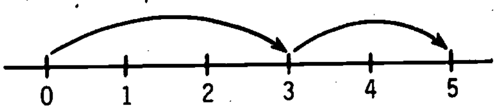


Finally, the child writes a number sentence in response to the question: "What is 4 + 2 the same as?"



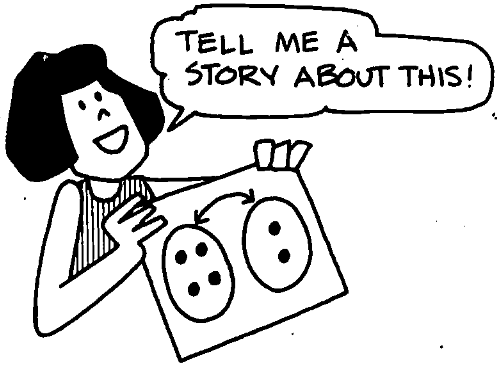
Addition situations can also be modeled using a <u>measurement</u> approach. For example, John may be faced with the problem:

A natural model for this problem is the number line.

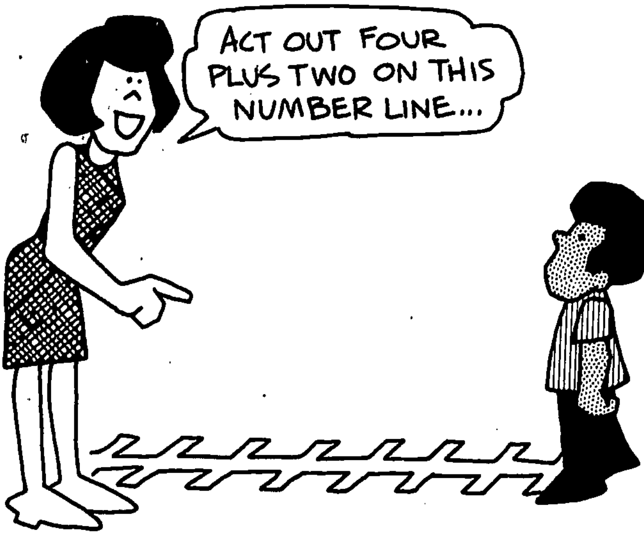


$$3 + 2 = 5$$

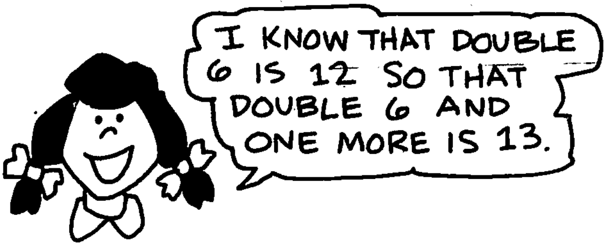
To further strengthen the child's grasp of the addition concept the above steps can be reversed. The child can be given the concrete situation and asked to tell a story for which it is a model.



Or the child can be given a number sentence and asked to represent it physically or pictorially and tell a story about it.



It is only when the teacher is confident that addition sentences have real meaning for a child, that work on the basic facts is begun. To facilitate the learning of basic facts, children are encouraged to observe patterns among numbers. For example, observing the commutative law, i.e., that a + b = b + a, reduces by half the number of facts that need to be memorized. Other patterns are also used. For example, to find 6 + 7 a child may use doubles.

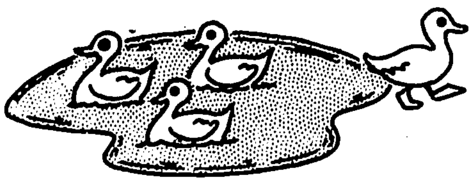


It is important that the child memorize certain basic addition facts. But every effort is made to give meaning to those facts and to minimize their number.

Subtraction is presented to children using a sequence of experiences parallel to those for teaching addition. The diagram below shows the three steps.

Real Problem:

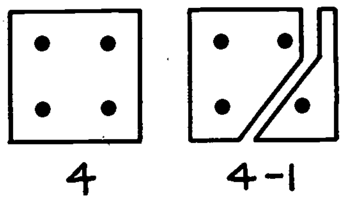
Concrete or Pictorial Model:

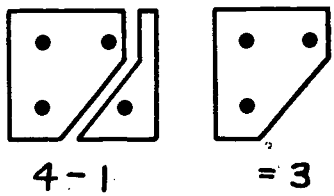


Symbolic Representation:

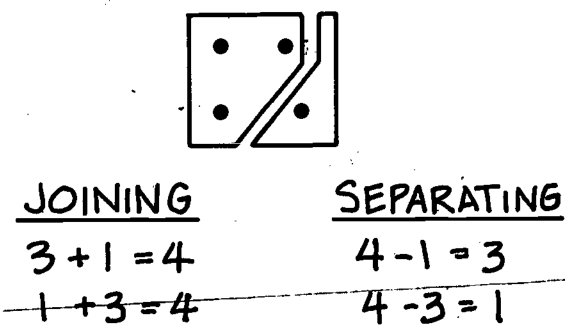
$$4 - 1 = 3$$

You may have noted in the pictorial model one of the difficulties which makes the subtraction concept more difficult to grasp--once the one duck leaves the four, the original set of four is no longer intact. There is a tendency for the child to focus on only the number remaining instead of on all three numbers involved. As with transitional sequences for addition, diagrams can be helpful in establishing the relationship between the separating situation and the subtraction sentence.





Having established the concept of subtraction, the teacher naturally turns to the basic facts. Since addition and subtraction are inverses of each other, it makes sense to establish this inverse relationship so that children can infer the basic facts of subtraction from those of addition. Children can come to see the relationship between addition and subtraction by viewing the same concrete model from the <u>joining</u> and the <u>separating</u> points of view.



Addition requires children to find the sum of two addends.

$$\underset{\text{addend}}{6} + \underset{\text{addend}}{2} = \underset{\text{sum}}{\boxed{8}}$$

Subtraction, on the other hand, requires them to find the missing addend.

$$\underset{\text{sum}}{8} - \underset{\text{addend}}{2} = \underset{\text{addend}}{\boxed{6}}$$

In the first case, they ask what the sum of 6 and 2 is; in the second case, they ask what number added to 2 makes 8.

The ultimate objective of instruction in addition and subtraction is that in any situation the child will be able to select the operation which is called for and will be able to carry out that operation with understanding and ease. In some situations it is not enough for the child to understand the concepts and to know the basic facts. The child must also know how to solve multiple-digit addition and subtraction problems such as

$$\begin{array}{r} 189 \\ +\ 138 \\ \hline \end{array} \qquad \begin{array}{r} 5078 \\ -\ 3996 \\ \hline \end{array}$$

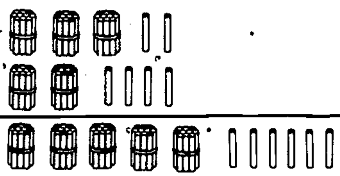
To solve these problems, systematic procedures called algorithms are needed. The development of these algorithms is like that of the initial concepts: a real-world problem is posed; the problem is modeled; and mathematical symbols are used to describe the model and the solution. At each step in the development understanding is stressed, but the ultimate goal is skill in performing the algorithms automatically.

To give meaning to the problem $\begin{array}{r} 32 \\ +\ 24 \\ \hline \end{array}$, the problem can be modeled with bundling sticks.

3 tens 2 ones

+ 2 tens 4 ones

5 tens 6 ones

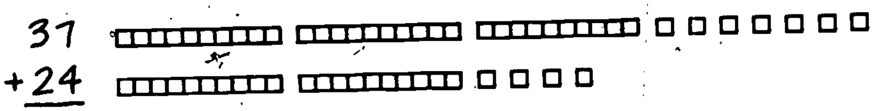


The child physically joins the bundling sticks to get 56. Then the physical act is recorded using tens and ones. One can see here the heavy reliance that is placed on the basic numeration concepts of grouping and place value.

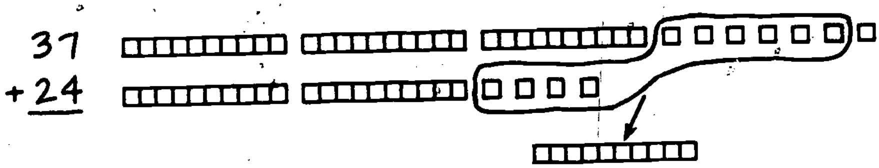
Concrete materials are particularly helpful in giving meaning to the concept of "carrying." For example, the problem

$$\begin{array}{r} 37 \\ +\ 24 \\ \hline \end{array}$$

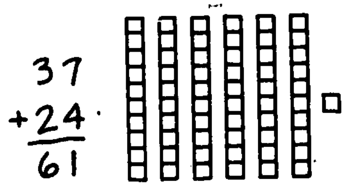
can be represented using Dienes blocks.



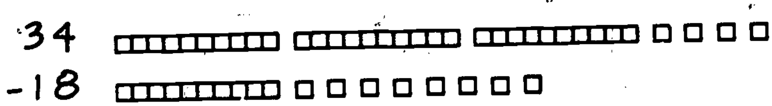
The blocks are joined, and ten units are traded for one long (10's block).



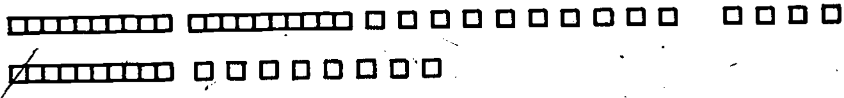
Finally, the sum is recorded.



Borrowing or renaming in subtraction also gains meaning when the problem is modeled concretely. For example, consider this sequence of steps.



$$\begin{array}{r} {}^{2}\cancel{3}{}^{14}\cancel{4} \\ -18 \\ \hline \end{array}$$



$$\begin{array}{r} 34 \\ -18 \\ \hline 16 \end{array}$$

As the child becomes comfortable with the algorithms, dependence on physical aids is reduced and eventually computations involving larger numbers are carried out with no aids at all.

$$\begin{array}{r} {}^{2}\cancel{3}\,{}^{11}\cancel{2}\,{}^{17}\cancel{7} \\ -\ 189 \\ \hline 138 \end{array} \qquad \begin{array}{r} 5078 \\ +\ 3996 \\ \hline 9074 \end{array}$$

Since it is very easy with large numbers to make a "small" error which makes a big difference, it is important for children to learn to estimate their answers by rounding off. For example, the subtraction and addition problems above can be estimated by computing:

$$\begin{array}{r} 300 \\ -200 \\ \hline 100 \end{array} \qquad \begin{array}{r} 5000 \\ +4000 \\ \hline 9000 \end{array}$$

Just as with earlier addition and subtraction work it is important for children to relate their computations to real-world situations. This is accomplished by posing many computational problems in real-world settings and also by asking the child to provide a real-world setting for a computational problem.

In this overview you have seen how addition and subtraction are developed in elementary school mathematics. The first notions of these operations arise in the real world and evolve through concrete and pictorial models to the symbolic form. Mastery of the basic combinations evolves from these initial ideas and from the recognition of number properties and patterns in addition and subtraction. Finally the child learns the algorithms which depend both on operation (addition and subtraction) concepts and place value concepts. This unit will explore each of these topics in more depth.

REAL-WORLD PROBLEM

↓

CONCRETE AND PICTORIAL MODELS

↓

SYMBOLS] ← PROPERTIES

↓

BASIC COMBINATIONS] ← PATTERNS

↓

(PLACE VALUE)

↓

ADDITION AND SUBTRACTION ALGORITHMS

ACTIVITY 2

READINESS FOR ADDITION AND SUBTRACTION

FOCUS:

In this activity you will have an opportunity to identify some of the learnings a child should have acquired before formal introductory work in addition and subtraction begins.

MATERIALS:

Several first-grade textbooks (one or two per group of four students).

DISCUSSION:

Children begin their number work with varying amounts of previous experience. For that reason first-grade text materials usually begin with several lessons that focus on classifying objects (by color, size, etc.), on ordering objects (smallest to largest, etc.), on matching sets in a one-to-one fashion, and so forth. All of these activities are done in preparation for the basic work of addition and subtraction of whole numbers. This activity will help you identify some of the learnings that children should have before the development of addition and subtraction is begun. This will be done by presenting a series of anecdotes followed by questions. At the end of the activity you will be asked to generate a list of prerequisite learnings for beginning work in addition and subtraction and compare it to the list you generated at the beginning of the activity.

DIRECTIONS:

1. Working in groups of four, list all of the prerequisites you can think of that a child should have before beginning work in addition and subtraction. Be as specific as you can; provide examples for each learning you identify. Do not proceed to 2) until you feel you have completed your list.

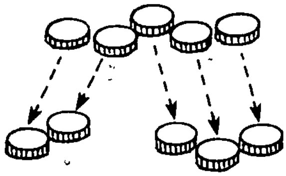
2. Some anecdotes follow which illustrate the lack of acquisition of some learning on the part of a child. Read and discuss each anecdote with your group. Then answer the questions which follow.

Anecdote A

Teacher: Judy, here are some checkers. Do you want to touch them?

Judy: Yes. I have some just like them.

Teacher: Judy, I am going to push the checkers apart like this. Are there as many checkers now as there were before?



Judy: No.

Teacher: Are there more checkers or are there fewer checkers?

Judy: There are more.

Teacher: Why do you think there are more?

Judy: Because there are!

Try to answer the following questions before the Discussion of Anecdote A.

a) Why do you think Judy thought there were more checkers after the teacher had moved them than there were before?

b) What is the relationship between this anecdote and addition and subtraction?

Discussion of Anecdote A

Judy exhibits what Piaget and others have termed a lack of conservation of numerousness. In other words, Judy thinks that the number of checkers changes when the checkers are moved. Perhaps she thinks there are more checkers because they are "spread out" or because there are two groups instead of one. Several researchers have identified this lack of conservation of numerousness to exist in many first-grade children.

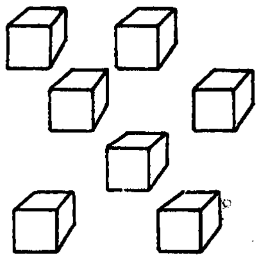
Conservation of numerousness is essential for addition (or subtraction) since addition reflects the joining of sets. Thus, if a child is asked to join a set of 2 and a set of 3 and he (she) thinks that the number in the set changes with movement, addition would be meaningless.

Anecdote B

Teacher: Kurt, can you count to 10 for me?

Kurt: Yes. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Teacher: Good. Now, Kurt, can you count and tell me how many blocks I have on the table?



Kurt: Yes. (Pointing with his finger) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.

(Kurt has pointed almost randomly to the blocks as he counted. He has counted some more than once.)

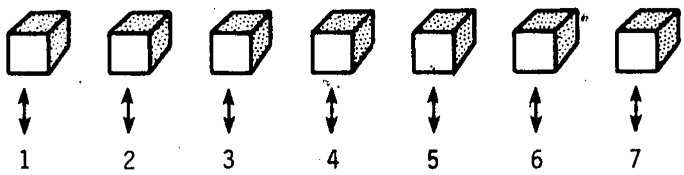
Try to answer the questions which follow before reading the Discussion of Anecdote B.

a) Why does Kurt count 11 blocks when there are only 7?

b) How is counting related to addition?

Discussion of Anecdote B

There is a difference between <u>rote</u> and <u>rational</u> counting. In rote counting a child merely recites the numbers in order much like reciting the alphabet. Rational counting implies a one-to-one matching between an object and a counting number. A one-to-one matching means a pairing of elements in two different sets so that each element of one set is matched with exactly one element of another set. In rational counting one of the sets is the set of counting numbers; the other set is a set of elements to be counted. This one-to-one matching can be illustrated as follows:



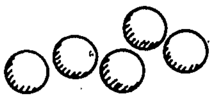
Thus, the number of blocks in the set is determined by the last counting number matched with the last block.

Obviously, Kurt does not possess the ability to count rationally. Work with one-to-one matching of sets is one of the activities in which children can engage to foster rational counting.

The relationship between rational counting and addition (or subtraction) should be evident. Counting is one of the most basic techniques in solving addition (and subtraction) examples. In adding two sets a child might count to find the sum. If he (she) is not able to count rationally, then the addition is meaningless.

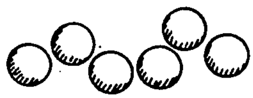
<u>Anecdote C</u>

Teacher: Mike, how many marbles are shown here?



Mike: Let's see. 1, 2, 3, 4, 5. There are five.

Teacher: Okay. How many are shown in this picture?



Mike: 1, 2, 3, 4, 5, 6. There are six.

Teacher: Good. Now, Mike, which picture shows more marbles, the picture with five marbles or the picture with six marbles?

Mike: Ummmm...I don't know.

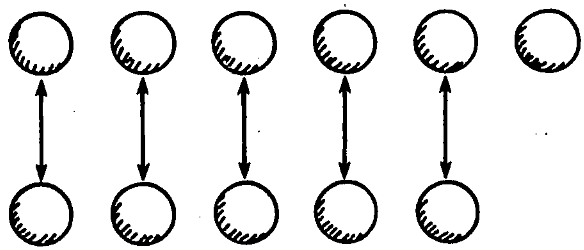
Try to answer the following questions before reading the Discussion of Anecdote C.

a) In spite of Mike's ability to count, why do you suppose he could not pick the set with more?

b) What relationship does comparison of numbers have with addition (subtraction)?

Discussion of Anecdote C

Although Mike can count rationally it is clear that he does not have a grasp of order. The concept of "more than" and "less than" can be developed by using the one-to-one matching technique described in the discussion of Anecdote B. Thus, in helping a child understand the meaning of "more than" and "less than," a teacher can present the child with two sets to be matched. The set having some left over has "more"; the set not having any left has "less."

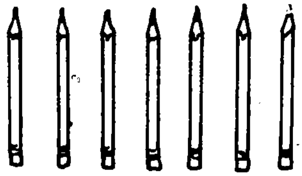


A child must have the order of numbers firmly established before addition and subtraction can have meaning for that child. Intuitively, a child should know that in addition (except with 0) the answer should be greater while in subtraction (except with 0) the answer should be less.

Anecdote D

Teacher: Linda, can you take six pencils out of this box (box has 40-50 pencils) and put them on the table?

Linda: Yes. (She then proceeds to place seven pencils on the table.)



Teacher: Do you have six pencils on the table?

Linda: Yes.

Teacher: Do you want to check to see if you have six pencils?

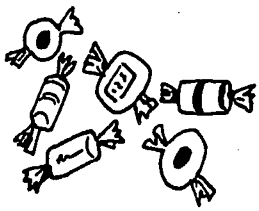
Linda: Okay. (She mumbles something as if counting while rearranging the pencils. Finally, she looks up and says) Yes, six.

Try to answer the following questions before proceeding to the Discussion of Anecdote D.

a) What do you think the cause of Linda's error might be?

b) How is the ability to select (or represent) a number of objects related to addition (or subtraction)?

Discussion of Anecdote D



The ability to count does not always imply that the child can select a given number of objects. Being able to count the number in the set shown at the right is a different ability from producing a set of six.

In adding (and subtracting) the child is often given a number sentence such as 4 + 3 = and asked to draw 4 objects and then 3 objects and to tell how many in all. If the child is not able to represent the sets accurately, then the addition example is, of course, meaningless.

3. Study a first-grade textbook to identify examples of activities for children that attend to the learnings described in Anecdotes B, C, and D. (Activities related to conservation of numerousness can be found in some kindergarten texts. Activities related to conservation must be done on an individual basis and do not lend themselves to paper-and-pencil activities. You may want to discuss conservation of numerousness with your instructor or he (she) may recommend some readings for you.)

4. Refer to the list of prerequisite learnings for addition and subtraction which you developed in 1) above. Discuss your list and use the anecdotes and the first-grade textbooks to help you revise your list.

ACTIVITY 3

WRITING A READINESS ACTIVITY FOR CHILDREN

FOCUS:

In this activity you will have an opportunity to write an activity for an objective chosen from one of three phases of early work in mathematics.

MATERIALS:

Kindergarten and first-grade textbooks; several blank transparencies and an overhead projector.

DISCUSSION:

In the previous activity you were asked to develop a list of learnings that a child should acquire before formal work in addition and subtraction begins. In this activity you will gain a broader perspective on children's early work in mathematics.

In many cases the child has had a year or so (preschool and kindergarten) of activities which do not focus on formal work in addition and subtraction. Instead these activities develop ideas related to number and mathematics in general. Establishing communication with the child, developing vocabulary, and organizing his already vast array of informal quantitative experiences are all part of early learning experiences related to mathematics. Clearly this activity cannot focus on all early mathematical experiences appropriate for young children. It does, however, bring attention to three phases of preaddition work.

1. Work occurring prior to work with numbers
2. Work associated with the development of numbers
3. Work specifically related to preaddition-subtraction.

DIRECTIONS:

Groups of three should be formed. Each group will be assigned three objectives, one from each of groups A, B, and C listed below. Each member of the group should take one objective and develop a brief outline of an activity that could be used with children.

The activity should be written on a transparency so that it can be projected and discussed in class. You should feel free to consult with the members of your group about your activity before a class presentation is made.

Reference

Payne, Joseph N., ed. Mathematics Learning in Early Childhood. NCTM Thirty-seventh Yearbook. Reston, Va.: NCTM, 1975.

SOME OBJECTIVES FOR EARLY WORK IN MATHEMATICS

Note: Additional objectives may be supplied by your instructor or you may suggest some from your work in Activity 2.

A. Prenumber

The child should be able to:

1. Describe and construct sets of objects in terms of a common property such as color, shape, and size.
2. Pair members of two sets in order to determine which set has more elements, which set has fewer elements.
3. Pair members of two sets to determine whether two sets have the same number of elements.

B. Number

The child should be able to:

4. Recognize sets of 0, 1, 2, ..., 10 elements.
5. Write the numeral that represents the number of elements in a set (0 - 10).
6. Write numerals for sets and identify the numerals that name the most (the least).

C. Preaddition and Subtraction

The child should be able to:

7. Join two disjoint sets, name the numbers for each of these sets and the number of the new set formed by joining.
8. Given sets of fewer than nine members, form sets of 9 by selecting the correct number of additional members.
9. Identify two sets which when joined form a set of a given number (e.g., two sets which form a set of 8).
10. Identify the numbers for two disjoint subsets of a given set.

ACTIVITY 4

USING AIDS TO INTRODUCE ADDITION AND SUBTRACTION

FOCUS:

In this activity you will use some of the aids used in early addition and subtraction lessons for children. Further, you will be asked to identify several lessons in textbooks which use aids.

MATERIALS:

Elementary school textbooks (grades 1 and 2 in particular), including teacher's editions; counters, bundling sticks, or other set aids.

DISCUSSION:

A basic procedure in the mathematical instruction of children is to move from some physical situation to a symbolic representation of it. This transition from the physical situation to the symbols of mathematics is shown in the diagram below. The physical situation is usually posed in the form of a problem.

Real-world Problem

↕

Model — Concrete / Pictorial

↕

Mathematical Symbols

For example, the following problem might be posed:

"John has 5 marbles.

He wins 2 from a friend.

How many marbles does John have now?"

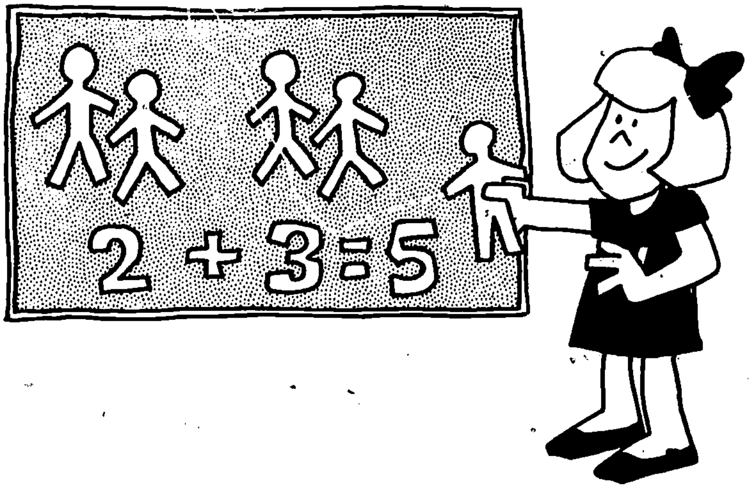
Such a problem should be modeled using concrete objects or a picture. In this problem actual marbles might be used or some popsicle sticks or blocks. In later lessons a picture of the marbles may suffice, but initially actual objects ought to be used. Since the textbook pages cannot present actual objects, teachers supplement the textbook pages by providing actual objects.

Children should have many experiences interpreting problem situations with a model--either <u>concrete</u> or <u>pictorial</u>--before using symbols. Even after the introduction of symbols, children should interpret problem situations with models. Conversely, children should make up problem situations from sentences like 5 + 2 = $\square$ and from pictures.

This activity will acquaint you with two basic models used for addition and subtraction. These are the set model and the number line model. A third model is the operator or function machine model. This third model is used less frequently in the primary grades.

DIRECTIONS:

1. <u>The Set Model</u>: In the set model individual concrete aids such as popsicle sticks, buttons, chips, paper clips, etc., are used so that each number is represented by a discrete number of objects. For example, in representing 5, five objects such as five chips would be used. Likewise, in a pictorial representation, five distinct objects would be pictured. Cutouts of such objects are often used in conjunction with the flannel board.



The set model is the most widely used of all three models in current textbooks. The set model has the advantage that most children's initial exposure to number has been through the set model. Children have often counted toy cars, marbles, and other discrete objects.

Textbooks try to use pictures of objects that are familiar to children and ask them to focus on the number of those objects. In early addition, for example, the illustrations might show two separate sets of familiar objects being joined.

3 2 3 + 2

In the teacher's edition, the authors of the textbooks often suggest that the teacher use real objects to supplement the lesson.

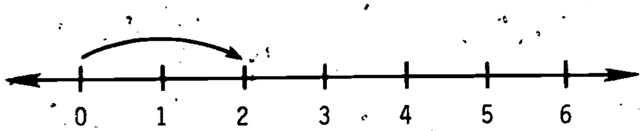
a) Look in some primary-grade textbooks and identify four or five lessons in which a set model is used to introduce addition or subtraction. Discuss these lessons in your group.

b) Using some set aids, take turns demonstrating how a child might do the following examples.

$6 + 3 = \square$	$2 + 5 = \square$
$7 - 4 = \square$	$8 - 4 = \square$

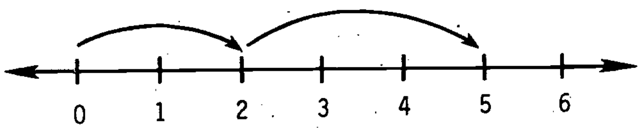
c) For each example in (b) above, make up a problem that might be represented by the number sentence.

2. Number Line: In the number line a correspondence is set up between points on the line and the set of whole numbers. At a later stage this correspondence can be extended to include the entire set of real numbers.



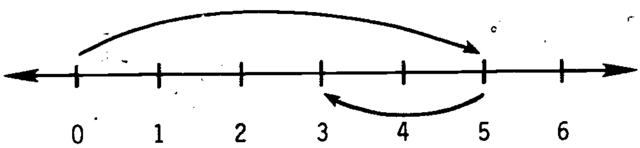
Numbers are represented by lengths on the number line. 2 is represented by a "move," "jump," "step," of two units from 0. Words such as "move," "jump," and "step," are useful in conveying meaning to children. The approach is dynamic; it involves action.

The procedure for adding 2 and 3 on the number line is illustrated below.



2 + 3 = 5

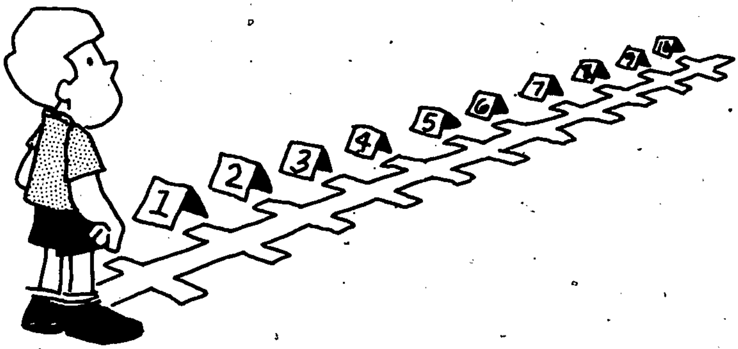
The child marks off a measure of 2, and then combines with this a measure of 3. The result of this combination is a measure of 5. Since this approach involves the combination of measures, the number line is said to provide a "measure approach" to addition. The procedure for subtracting 2 from 5 on the number line is illustrated as follows:



5 - 2 = 3

Using the number line children can readily find the solution to a number sentence such as 5 - 2 = $\boxed{}$. Once the child has moved forward 5 and then back 2, the result of the operation is immediately visible.

In practice a teacher will usually provide experiences on a "walk-on" number line before having children use individual number lines or number lines in their textbooks. A "walk-on" number line is pictured below. It can be constructed by placing cards in a line on the floor, by using chalk on the floor, or by using tape. "Start" (or 0) is written on the initial card (or point), 1 on the next, and so on. Children are then asked to step out various combinations on the number line.



For further reading you may wish to consult Minnemast (1968), Unit 13, "Interpretations of Addition and Subtraction."

a) Examine some primary-grade textbooks and identify four or five lessons in which a number line model is used. Discuss these lessons in your group.

b) Draw some number lines and represent each of the following examples on a number line.

$3 + 5 = \square$ $\qquad$ $6 + 2 = \square$

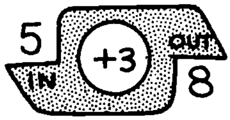
$7 - 4 = \square$ $\qquad$ $8 - 2 = \square$

c) For each example in (b) above make up a problem which might be represented by the number sentence.

<u>Note</u>: *(Optional) Appendix A contains a description of the development and use of a popular set of materials called Cuisenaire rods. These materials are based on the number line model and use colored sticks to represent numbers.*

3. <u>The Function Machine Model</u>: The function machine model is based on an operator approach. In an operator approach one can think of addition or subtraction as an operator which "acts" on a number. For example, suppose there is a "plus 2" operator. Then if the number 3 is operated on by "plus 2" the result is 5. Similarly, if the operator is "subtract 4" and 6 is operated on, the result is 2.

The function machine is usually pictured as a mechanical device that does something to a number put into the machine. The illustration at the right is suggestive of one used in a popular elementary textbook. The function machine is normally not used to introduce addition and subtraction. Rather it is used as a means of providing an alternate approach to the set model or as a means of drill. Some directions for making and using a function machine are given on page 33.



a) Examine some primary-grade textbooks and identify four or five lessons in which a function machine model is used. Discuss these lessons in your group.

b) Show how the following tables would be completed using the rules (operators) "add 2" and "subtract 3" respectively.

Rule: Add 2

Input	2	1	3	5	6	4	7
Output							

Rule: Subtract 3

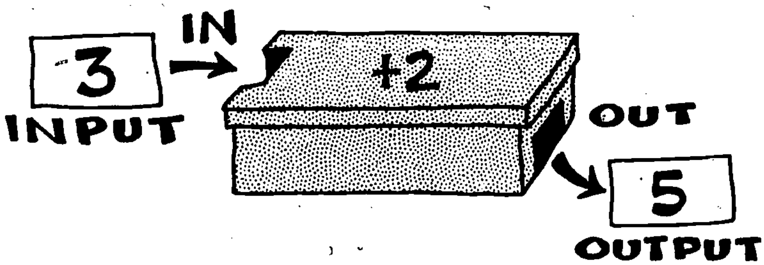
Input	4	9	10	11	5	7	8
Output							

4. Discuss whether you think it is better to introduce addition and subtraction using one model or several models.

FUNCTION MACHINE

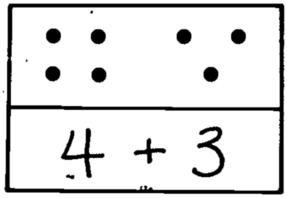
To make a function machine, cut holes in a small box and label them "IN" and "OUT." Make a set of function cards that can be taped to the box to show what function the box is to perform. The box shown below performs the "add 2" function.

The use of a function machine is illustrated for the example 3 + 2. Imagine that 2 marbles ha e been placed in the machine. If we put 3 marbles into the machine (IN), the machine will add 2 marbles. By pouring out the contents of the machine the result 5 is obtained. (In using the function machine for subtraction the machine will have to be changed slightly. How?) After some experience with the machine the marbles can be dispensed with and children can use cards for the input and output as shown in this illustration.



As the name suggests, the "function machine" illustrates the fact that a binary operation such as addition is a function. In the description above the "add 2" function was used. Naturally the children do not learn about functions here, but it is hoped that ideas like "function" will continue to grow out of experiences such as the function machine. The main use of the function machine in elementary mathematics is to provide practice in the basic facts of addition and subtraction.

5. "Counting on" is a procedure often used by children to find the sum in an addition example. For instance, when given the example at the right, some children say, pointing to the group of four and then to each dot of the three, "Four, five, six, seven." Other children, who do not count on, look at the example and start at the beginning, "One, two three, four, five, six, seven." Which model would be most useful in helping children acquire a "counting on" procedure?



ACTIVITY 5

THREE MODELS FOR SUBTRACTION

FOCUS:

In this activity you will be challenged to classify several subtraction problems as representing a take-away or missing-addend or comparison situation.

DISCUSSION:

Real-world problems that involve addition usually present little difficulty for students once they associate the notion of joining with addition. On the other hand, subtraction can be associated with three different situations.

1. Take-away
2. Missing addend
3. Comparison

Each of these situations will be exemplified in this activity.

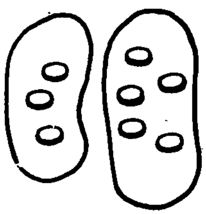
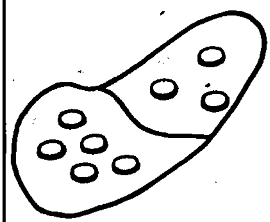
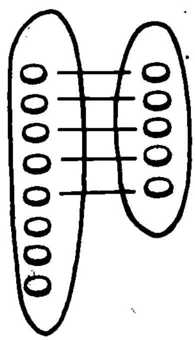
DIRECTIONS:

1. Study the three subtraction situations illustrated in the table on page 36. Then classify the problems which follow the illustrations as "take-away" or "missing addend" or "comparison."
2. Take part in a class discussion using the following questions as a focus:
   a) How are the models for the three problems alike? How are they different?
   b) Suggest an order of difficulty for the three kinds of problems. Justify your order.

<u>Or</u> (Instead of (2))

3. Several researchers* have found that most pupils find the comparison type of subtraction problem to be the most difficult. In your groups discuss why this might be true, and offer some suggestions to help overcome the difficulty.

Three Types of Subtraction Situations

	1. Take-Away	2. Missing Addend	3. Comparison
PROBLEM	Tom has 8¢; he spends 5¢. How much is left?	Tom has 5¢; a candy bar costs 8¢. How much more does he need?	Tom has 5¢; Jill has 8¢. How much more does Jill have than Tom?
MODEL			
SYMBOLS	8 - 5 = ☐	5 + ☐ = 8	8 - 5 = ☐ (5 + ☐ = 8)

PROBLEMS

1. In a ring toss game Patty has 60 points, Jim has 72. How many more points does Patty need before she has as many as Jim?

*Joseph N. Payne (ed.), <u>Mathematics Learning in Early Childhood. NCTM Thirty-seventh Yearbook</u>. (Reston, Va.: 1975), p. 55.

2. Sally has 53 pennies. Marcia has 45 pennies. How much more money does Sally have than Marcia?

3. Bill has 86 stamps in his collection. He gave 45 stamps to his brother Tom. How many stamps does Bill have left?

4. It takes Mary 40 minutes to walk to her friend's house. She has been walking for 15 minutes. How much longer does she still have to walk?

ACTIVITY 6

ERRORS IN EARLY SUBTRACTION

FOCUS:

Some pupil pages in early subtraction are presented to give you an opportunity to identify errors and the probable causes of those errors, and to suggest appropriate remedial activities. You should view this as a problem-solving opportunity. Much of your work as a teacher will focus on analyzing the kinds of errors children make so that your findings can be used to determine appropriate corrective measures.

DISCUSSION:

Children tend to experience more difficulties with subtraction than with addition. The purpose of this activity is to acquaint you with some of the errors children make in subtraction. An opportunity will be provided for you to analyze the children's errors and to attempt tentative remedies. The question, "Why is subtraction more difficult than addition?" should be kept in mind throughout the activity.

DIRECTIONS:

Three Pupil Pages A-C are presented with the objectives for each lesson provided at the beginning. The Pupil Pages show typical errors made by children when first learning subtraction.

1. Complete the Pupil Pages by answering the questions listed on each sheet. Keep in mind the important ideas contained in the following paradigm when suggesting remedial treatments for the pupils.

Real-world Problem

↕

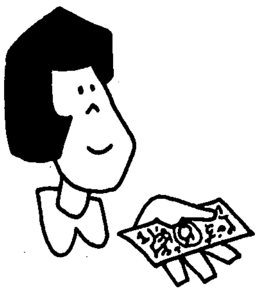
Model < Concrete / Pictorial

↕

Mathematical Symbols

2. Answer the questions at the end of this activity after discussing them in your group.

TEACHER TEASER



<u>No Change</u>

"Give me change for a dollar, please," said the customer.

"I'm sorry," said Miss Jones, the cashier, after searching through the cash register, "but I can't do it with the coins I have here."

"Can you change a half-dollar then?"

"No, in fact," she said, "I couldn't even make change for a quarter, dime or nickel!"

"Do you have any coins at all?" asked the customer.

"Oh, yes," said Miss Jones. "I have $1.15 in coins."

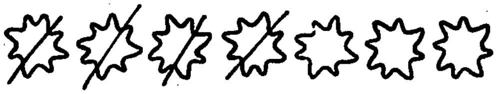
Assuming that each coin was less than a dollar, exactly what coins were in the cash register?

PUPIL PAGE A

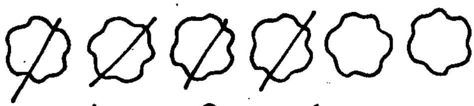
Name Lisa

Objective: To use a set model to introduce subtraction.

Directions: Write a number sentence which shows what has happened.

1. 

4 - 3 = 7

2. 

4 - 2 = 6

3. 

3 - 5 = 8

Questions:

1. Identify Lisa's errors and any pattern you may see in the errors.

2. What is a probable cause of the errors? Do you see anything inherently confusing in the model that might have caused Lisa's errors?

3. Outline an approach you might use to help Lisa overcome her errors.

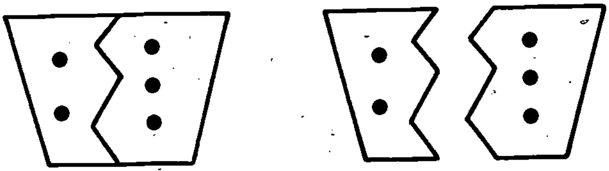
PUPIL PAGE B

Name AL

Objectives: To recognize or write a subtraction expression which represents a subset being removed from a set.

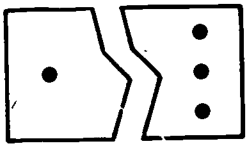
Directions:

1. Write an expression that the picture shows.



Answer: 5-3

2. Circle the correct number expression.

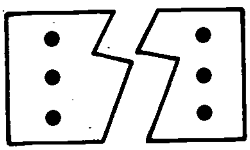


3 - 1

4 - 3

4 - 2

3. Write an expression that the picture shows.



Answer: 3-3

Questions:

1. Identify Al's errors and a probable cause for each error.

2. Is there any significance to the fact that Al answered example (1) correctly? How might the model be confusing to Al?

3. Outline a treatment you would provide to help Al overcome his errors.

53

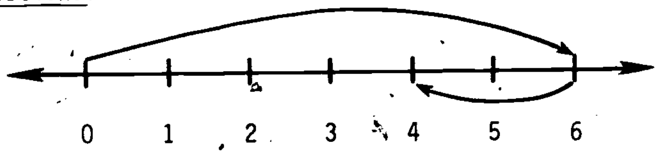
PUPIL PAGE C

Name: Jean

Objective: To write subtraction sentences for some "hops" on a number line.

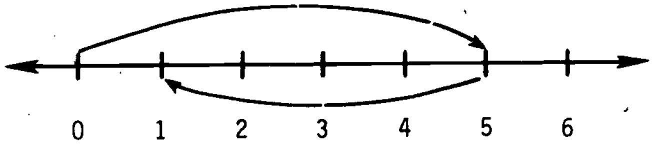
Directions:

1.



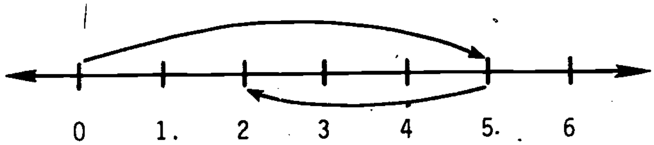
6 - 4 = 2

2.



5 - 1 = 4

3.



5 - 2 = 3

Questions:

1. Identify Jean's errors and indicate a probable cause for each error.

2. Is there anything about the model that might be confusing to Jean?

3. Outline a procedure which might help Jean overcome her errors.

Questions for group discussion

1. Why do you think that subtraction presents more difficulty for children than addition?

2. Are the models (set, number line) as helpful for subtraction as they are for addition?

3. Consider explaining $7 - 4 = \square$ to a child. Which is more help- ful (if either), using concrete aids or pictorial aids? What are the advantages and disadvantages of each?

4. What seems to be the best possible sequence of aids for helping children in early subtraction: using story problems, using a set model, using a number line model, using a function machine model? Justify the sequence you propose.

ACTIVITY 7

RELATING ADDITION AND SUBTRACTION

FOCUS:

In this activity you will have an opportunity to discuss some activities which may help children understand the relationship between addition and subtraction.

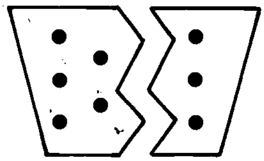
DISCUSSION:

Addition and subtraction are inverse operations. As children learn about each operation, they also develop an intuitive understanding of this inverse relationship. Later, when they are learning basic number facts, this relationship is studied explicitly because it has a definite practical value.

DIRECTIONS:

Discuss ~~and answer~~ the following questions.

1. The set model shown at right can suggest both addition and subtraction. The act of "joining" corresponds to addition; the act of "separating" corresponds to subtracting. Write two addition and two subtraction sentences which correspond to the model in this illustration.



a)

b)

c)

d)

2. Solving addition and subtraction sentences is important in elementary mathematics. Many different number sentences can be formed by using only addition and subtraction, e.g., $3 + 5 = \square$, $3 + \square = 8$, and $8 - 3 = \square$. Choose one basic fact and write down as many different forms of number sentences as you can, using only addition and subtraction. (<u>Hint</u>: Remember the symmetric property of equality.)

3. How could the inverse relationship between addition and subtraction (addend-addend-sum form) be employed in solving such sentences? Illustrate one case only.

4. Which sentence do you feel would be most difficult for children to solve?

5. *OPTIONAL: Read the article "Some Factors Associated with Pupils' Performance Level on Simple Open Addition and Subtraction Sentences" by J. Fred Weaver which appeared in the <u>Arithmetic Teacher</u>, November 1971. Compare your answer to question 6 above with Weaver's comments.*

6. During the late sixties and early seventies, sentences like $4 + \square = 7$ often appeared in first-grade work (they still do in some textbooks). Teachers universally had difficulty teaching such a form. Can you speculate why such difficulty occurred? (<u>Hint</u>: Children often wrote $4 + \boxed{11} = 7$.)

Section II

BASIC MATHEMATICAL CONTENT IN ADDITION AND SUBTRACTION

In Section I the focus of the activities was on the initial learnings of young children related to addition and subtraction. While the emphasis for that section was principally on the pedagogical aspects, there are fundamental mathematical concepts related to addition and subtraction that are essential for the teacher to know.

Section II focuses on these mathematical concepts. It includes a self-test (Activity 8) designed to measure your understanding of these basic concepts. Following this test are some activities related to the mathematical properties of addition and subtraction. Activity 9 provides an opportunity for you to review some basic concepts and terminology related to sets. In Activity 10 you are asked to interpret the properties as used in the elementary school. Activity 11 is an activity whose focus is on applying the properties to a mathematical structure.

MAJOR QUESTIONS

1. What are the basic properties of addition and subtraction and what role might they play in the development of initial concepts for young children?

2. What properties hold for addition and subtraction of whole numbers? Give examples and counterexamples to show properties that hold and properties that do not hold.

ACTIVITY 8

SELF-TEST

FOCUS:

This activity is designed to help you measure your understanding of certain mathematical concepts that are fundamental to the addition and subtraction of whole numbers.

DISCUSSION:

As is true with a class of elementary school children, you and the other members of your class have a variety of mathematical backgrounds. In particular, some of you will understand those concepts concerning sets and the relationships between whole numbers that are fundamental to addition and subtraction of whole numbers. Others of you will need additional work with some of those concepts.

The test that follows is a self-test and is designed to help you determine what your deficiencies are. It is your responsibility to take the test, check your answers, and take (with your instructor's help) whatever steps are necessary to remedy your deficiencies.

DIRECTIONS:

1. ~~Take the self-test that follows.~~
2. Check your work against the answer key supplied by your instructor.
3. Identify what your weak points are (you may wish to discuss them with your instructor).
4. Take steps to deal with your weaknesses. Among the resources that you can use are:
   a) Activity 9, which is concerned with basic concepts concerning sets.

b) Activity 10 and Appendix B, which discuss those basic properties of the whole numbers which are relevant to addition and subtraction.

c) Other mathematics references for adults and children which discuss basic ideas of sets and whole numbers.

d) Your classmates and your instructor. When you feel that you are through, it is up to you to decide if you have overcome any weaknesses.

SELF-TEST

ADDITION AND SUBTRACTION OF WHOLE NUMBERS

Part I

Circle the correct response to each item.

1. If $A = \{1, 4, 5\}$ and $B = \{3, 5, 8, 9\}$ then $A \cup B$ is the set:

   a) $\{5\}$

   b) $\{1, 3, 5, 8, 9\}$

   c) $\{1, 3, 4, 5, 8, 9\}$

   d) $\{1, 4, 5\} \{3, 5, 8, 9\}$

2. If $A = \{a, b, c, e, f\}$ and $B = \{c\}$, then the set difference $A - B$ is the set:

   a) $\{a, b, c, e, f\}$

   b) $\{a, b, e, f\}$

   c) $\{c\}$

   d) None of the above

3. If $A \cup B = \{a, b, c, d, e\}$ and $A = \{a, d\}$, then which of the following could not be B?

   a) $\{a, b, c, e\}$

   b) $\{c, d, e\}$

   c) $\{b, c, d, e\}$

   d) $\{b, c, e\}$

4. For which pair of sets A, B does $N(A \cup B) = N(A) + N(B)$?

   a) $A = \{1, 2, 5\}$ $B = \{2, 7, 8, 0, 4\}$

   b) $A = \{a, z, p\}$ $B = \{h, i, p, j, a\}$

   c) $A = \{r, 1, d\}$ $B = \{w, y, b, j, c\}$

   d) None of the above

5. For which pair of sets A, B does $N(A - B) = N(A) - N(B)$?

a) $A = \{1, 2, 5\}$ $B = \{3, 4\}$

b) $A = \{1, 2, 5\}$ $B = \{2, 4\}$

c) $A = \{1, 2, 5\}$ $B = \{2, 5\}$

d) $A = \{1, 2, 5\}$ $B = \{1, 2, 5, 7\}$

6. Which of the following is not an example of an operation with the closure property?

a) addition on the even counting numbers

b) subtraction on $\{0, 1, -1, 2, -2, \ldots\}$

c) addition on {multiples of 10}

d) subtraction on the counting numbers

7. The expression $(5 + 2) + 3 = 5 + (2 + 3)$ illustrates which of the following properties?

a) associativity

b) associativity and commutativity

c) commutativity

d) none of the above

8. Which of the following relationships is always true? (p, q, r are counting numbers.)

a) $p - (q - r) = (p - q) - r$

b) $p - q = q - r$

c) $p - 0 = 0 - p$

d) $(p + q) - q = p$

9. Which statement is an example of the identity property for addition of counting numbers?

   a) $p + 0 = 0 + p = p$ for all counting numbers p

   b) $p + q = q + p$ for all counting numbers p, q

   c) If p and q are counting numbers, $p + q$ is a counting number.

   d) $(p + q) + r = p + (q + r)$ for all counting numbers

10. If $a + b = b + a = a$, then:

    a) $a = b$

    b) $b = 0$

    c) $a = 1$

    d) $b = 1$

11. If each of 6 people has at least $1.50 and they want to chip in and buy a $7-tie for a friend, then:

    a) They won't have enough money to buy the tie.

    b) Not enough information is given to determine whether or not they could buy the tie.

    c) They have more than enough money to buy the tie.

12. If 5 people volunteer to bring Pepsi to a party for 30 people, and each of the volunteers brings no more than 7 Pepsis, then:

    a) Not enough information is given to determine whether or not each person at the party could have at least one Pepsi.

    b) There will be enough Pepsis so that each person at the party will be able to have at least one.

    c) There won't be enough Pepsis for everyone at the party to have at least one.

    d) There will be exactly enough Pepsis.

Part II

Each of the following exemplifies a property or principle in addition or subtraction of whole numbers. Identify the property or principle used. (Give a general symbolic statement wherever possible.)

1. In adding $\begin{array}{r} 24 \\ +\ 37 \\ \hline \end{array}$ one person may start at the top of each column and add down, and another may start at the bottom of each column and add up. Both people get the correct answer.

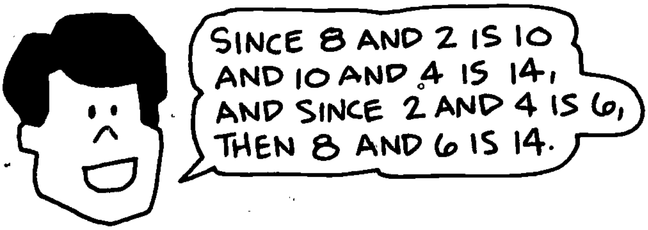
   Principle:

2. $8 + 0 =$ ______

   Principle:

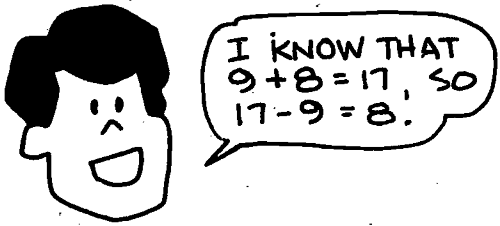
3. $8 + 6 =$ ______

   Principle:



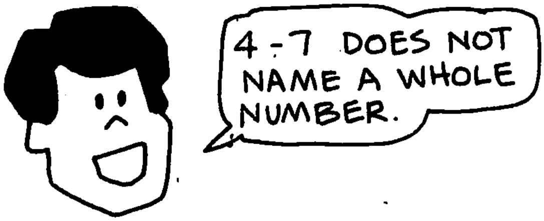
4. 17 - 9 = ______

Principle:



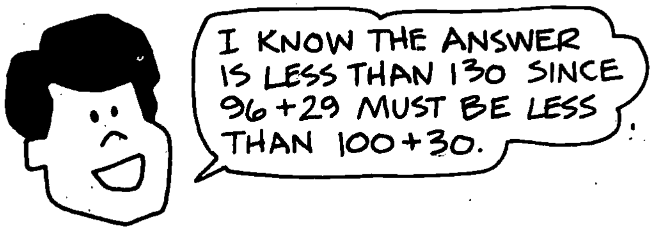
5. $\begin{array}{r} 64 \\ -\ 27 \\ \hline \end{array}$

Principle:



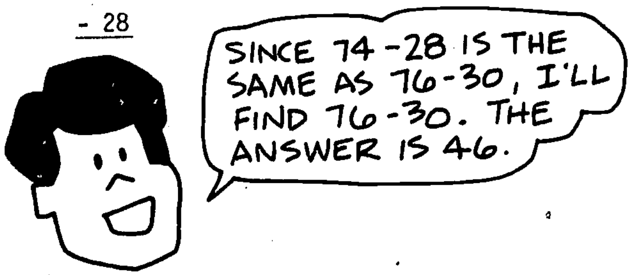
6. $\begin{array}{r} 96 \\ +\ 29 \\ \hline \end{array}$

Principle:



7. $\begin{array}{r} 74 \\ -\ 28 \\ \hline \end{array}$

Principle:



ACTIVITY 9

SETS FOR THE ELEMENTARY SCHOOL

FOCUS:

The word "set" has come to be used in most elementary textbooks. In this activity you will be introduced to

- the language and symbols of sets
- operations on sets
- counting sets
- sets in elementary textbooks.

DISCUSSION:

The so-called new math has been and still is a topic of controversy to parents and teachers. One source of the controversy which developed in the 1960's was some new and unfamiliar language which was used in the elementary school textbooks. The word set and some related words such as element, subset, union, and intersection were among the new terms. Mathematicians rightly claimed that the language of sets was not new; in fact, it had been around for 70 years. They also claimed that sets were simple and provided a powerful and general language which was compatible with good pedagogy. Parents and teachers, on the other hand, rightly claimed that the change in language was causing confusion or difficulty for adults which was being transmitted to the children.

In the last fifteen years the use of sets in textbooks has moderated, but it has not vanished altogether. This activity will introduce you to those ideas concerning sets which have importance in elementary school mathematics. One of the goals of the activity is to make you feel more comfortable with sets whenever you encounter them.

DIRECTIONS:

1. Language of sets: Any collection of objects can be referred to as a set. Each of the objects in a set is called an element of or a member of the set.

For example, you can speak of the set of students in your class. You are a member of or element of that set. Another example of a common set is the set of whole numbers. 0, 1, 2, etc., are elements of that set. The set can be described using the symbol {0, 1, 2, ...}. The three dots indicate that the list continues indefinitely.

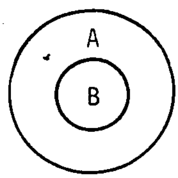
a) Name the set {1, 3, 5, ...} by describing its elements. Is 62 an element of the set? Is 131?

b) Name the set {..., -2, -1, 0, 1, 2, ...} by describing its elements. Are there any numbers which are not in this set? What are they?

c) Have a classmate tell you elements of a set as well as elements which are not in the set. Name the set by describing its elements.

2. Subsets: A subset of a set is a set which is completely contained in it. For example, the set of boys in your class is a subset of the set of students in the class. This is true even if all of the students are boys or none of the students are boys. In the latter case, the subset is called the empty set. The empty set is the set with no elements. Can you see why we say that the empty set is a subset of every set? The empty set is often designated in elementary mathematics texts by { }.

If the larger circle stands for a set called A, the smaller one for a set called B, then B is a subset of A. We use the symbol



$B \subset A$ to indicate that B is a subset of A. Note that this symbol $\subset$ is analogous to the symbol < which is used to designate "less than" for numbers. We will observe other (non-accidental) parallels between set language and number language.

a) Name four subsets of the set of students in your class. Name two sets which are not subsets of the set of students in your class.

b) Let $A = \{1, 2, 3, 4, 5, 6\}$

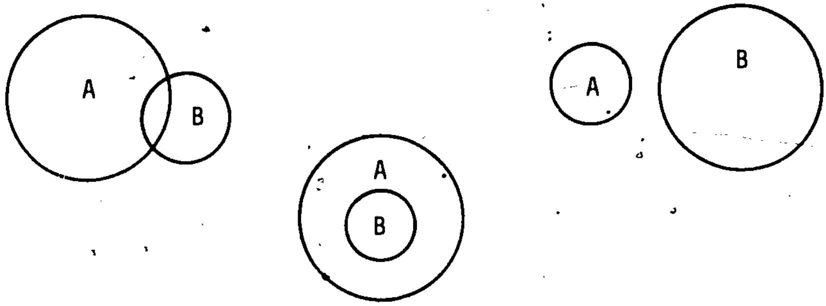
$B = \{2, 4, 6\}$

$C = \{1, 2\}$

Which of the following are true?

$A \subset B$, $B \subset A$, $C \subset A$, $A \subset C$, $B \subset C$, $C \subset B$

c)



Which of the above pictures $B \subset A$? Which $A \subset B$?

d) How would you explain the concept of the empty set to a child? Try by explaining it to a classmate.

3. Union, intersection and difference of sets: We say that +, -, x, and ÷ are operations on numbers since each operation combines pairs of numbers to give numbers. For example, + combines the numbers 2 and 3 to give the number 5. We write $2 + 3 = 5$. There are also operations on sets. The union operation combines two sets to give a set which contains all of the elements that were in either or both sets. For example, the union of $\{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\}$ is the set $\{1, 2, 3, 4, 5, 6\}$.

Symbolically we write

$$\{1, 2, 3, 4\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}.$$

The <u>intersection</u> operation combines two sets to give the set of elements common to the two sets. For example, the intersection of $\{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\}$ is the set $\{3, 4\}$. This time we write

$$\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}.$$

The <u>difference</u> operation applied to two sets gives those elements which are in the first set but which are not in the second set. For example, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6\}$, $A - B = \{1, 2\}$.

a) Verbally describe each of the following sets.

- The union of the set of boy students and the set of girl students in your class. The intersection of these two sets.
- The intersection of the set of integers and the set of rational numbers (fractions).
- The union of the set of tall women and the set of all women.
- The union of a set containing three marbles and a set containing two marbles. (You have to be careful here. The answer depends on whether the two sets overlap.)

b) Let $A = \{1, 2, 3, 4, 5, 6\}$ $B = \{1, 3, 5\}$ $C = \{2, 4, 6, 8\}$

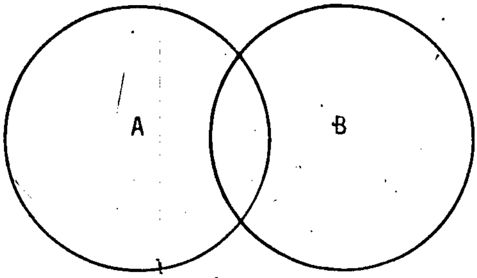
Then	$A \cup B =$	$A - B =$
	$B \cup C =$	$B - C =$
	$A \cap C =$	$A - C =$
	$B \cap C =$	

Is $B \cup C \subset A$? Is $A - B \subset A$?

Is $A \cap B \subset A$? Is $A - B \subset B$?

Is $B \cap C \subset B$?

c)



Shade $A \cup B$ with horizontal lines.
Shade $A \cap B$ with vertical lines.
Shade $B - A$ with slanted lines.

d) Which of the following are always true for any sets A, B, and C? (You may want to experiment with sets and with pictures.) If you can, give a verbal argument explaining why each is or is not true.

$A \subset B$
$A \cap C \subset B$
$A \cup B \subset B$
$B \subset A \cup B$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A - B = B - A$

4. Counting Sets: How many students are present in your class today? Answer this questions before you read on.

In answering the question you probably started counting by pointing to a student and saying "one," pointing to another and saying "two" and so on. The number that you said when you pointed to the last student in the class was the number of students in the class. What you were doing was setting up a one-to-one correspondence between the set of the first so many counting numbers and the set of students in your class. This is the general way that sets are counted. For example, we say that $\{a, b, c\}$ has three elements because we can set up a one-to-one correspondence between the elements of the set and the elements of the set of counting numbers beginning with 1 and ending with 3.

$$\begin{array}{ccc} \{1, & 2, & 3\} \\ \updownarrow & \updownarrow & \updownarrow \\ \{a, & b, & c\} \end{array}$$

Sometimes we will use the symbol $N(\{a, b, c\}) = 3$ to indicate that $\{a, b, c\}$ contains three elements.

a) How many coins do you have in your pocket or purse? Observe how you went about answering the question.

b) Let $A = \{1, 3, 5, \ldots, 21\}$

$B = \{2, 4, 6, 8\}$

$C = \{3, 6, 9, \ldots, 21\}$.

Find $N(A)$, $N(B)$, and $N(C)$.

Find $N(A \cap B)$, $N(B \cup C)$, $N(A \cap C)$, AND $N(A - C)$.

c) For which sets P and Q would it be true that

$N(P \cup Q) = N(P) + N(Q)$?

For which sets P and Q would it be true that

$N(P \cap Q) = 0$?

For which sets P and Q would it be true that

$N(P - Q) = N(P) - N(Q)$?

d) What does (c) have to say about a possible relationship between set union and difference and teaching addition and subtraction to children?

e) The idea of one-to-one correspondence is usually introduced to children before they start working formally with numbers. Can you explain why? Do you know any children who can say the numbers up to 10 in order, but who can not reliably report the number of objects in sets?

5. Sets in Elementary Texts:

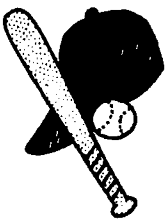
a) Look through an elementary text series, grades K-3 and fill in the following table.

Topic	Grade where the topic first appears	Appears explicitly or implicitly
one-to-one correspondence		
union		
intersection		
empty set		

If possible, it would be most interesting to compare a text series published in the middle to late 60's with one published in the middle 70's.

b) Write a brief outline of a development for children which starts with sets of objects and goes through the concept of whole number to the concept of addition of whole numbers.

TEACHER TEASER



Three sports writers picked the following teams to win next Sunday.

Writer #1: {Yankees, Twins, Orioles, Tigers}

Writer #2: {White Sox, Orioles, Angels, Yankees}

Writer #3: {Tigers, Yankees, Indians, Angels}

None of the writers picked the Royals to win. The teams mentioned are the only teams that will be playing on Sunday. Determine the teams that played each other. (You may find intersecting sets to be useful.)

ACTIVITY 10

DEFINITIONS AND EXAMPLES OF TERMS

FOCUS:

In this activity you are asked to define and illustrate properties and terms associated with addition and subtraction. This will provide a review of some of your mathematical vocabulary and an opportunity to look at elementary school texts for examples.

MATERIALS:

Elementary mathematics textbooks, grades K-6.

DIRECTIONS:

Complete the table below, providing illustrative examples from elementary school mathematics textbooks. Be sure to cite the grade level of the textbook in which you found the example. In providing the definition or statement, you may wish to consult Appendix B for further discussion on the properties of numbers.

Terms	Textbook Example	Grade Level	Definition or Statement
Identity Property of Addition	5 + 0 = 5	1	The sum of zero and any whole number is that whole number: a + 0 = a where a is a whole number.
Closure Property of Addition			
Commutative Property of Addition			

Terms	Textbook Example	Grade Level	Definition or Statement
Associative Property of Addition			
Addend			
Sum			
Inverse Property of Addition and Subtraction			

ACTIVITY 11
"ADDO AND SUBTRACTO"

FOCUS:

This activity provides an opportunity to use the properties and definitions you reviewed in Activity 10. However, instead of using the standard numerals, letters are used.

DIRECTIONS:

A = {a, b, c, d} is a finite set of numbers. A binary operation "addo" designated by $\oplus$ is defined on A by the following operation table:

$\oplus$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Note, for example, that: b $\oplus$ c = d.

1. Is the operation $\oplus$ closed for A? Why or Why not?

2. Is the operation $\oplus$ commutative on A? Explain your reasoning.

3. Partially verify the associative law by completing and checking the following cases:

   i) b $\oplus$ (c $\oplus$ d) =

   ii) (c $\oplus$ d) $\oplus$ c =

4. Find the identity element for (+).

5. The binary operation "subtracto" designated by (-) is the inverse operation of "addo." For any pair of numbers x, y in A, x (-) y is defined as that number z in A such that z (+) y = x. Using this definition for "subtracto" and the operation table for "addo" given previously, complete the "subtracto" table which follows:

(-)	a	b	c	d
a	a	d		
b				
c				
d				

Note: a (-) b = d. In finding a (-) b, think of a as the sum and b as one of the addends.

6. Were you able to see a pattern after writing down the entries for the first two rows? Explain the pattern.

7. Is the operation (-) closed on A? Why or why not?

8. Find b (-) c and c (-) b. Is the operation (-) commutative on A?

9. Find (b (-) c) (-) d and b (-) (c (-) d). Is the operation (-) associative on A?

10. Is there an identity element for (-) on A? Explain your reasoning.

11. List two properties for which (-) on A and subtraction on W (the set of whole numbers) behave in the same way.

12. List one property for which (-) on A and subtraction on W behave differently.

Section III

DEVELOPING THE BASIC ADDITION AND SUBTRACTION FACTS

The diagram pictured below illustrates the three stages in learning the basic concepts and number combinations for addition and subtraction.

Introduction →	Getting Ready to Memorize →	Memorization
1. Real-world problems	1. Use of number properties	1. Classroom drill
2. Using concrete or pictorial models	2. Relation of subtraction to addition	2. Independent drill--games, puzzles, flash cards, etc.
3. Writing mathematical sentences	3. Other number patterns and thinking strategies	
4. Solving mathematical sentences using the appropriate model		

Section III looks at the kinds of activities which can be used to help children "get ready" to memorize the basic addition and subtraction facts. These experiences should help children to organize their ideas about addition and subtraction and to develop thinking strategies designed to increase their understanding of the relationships among numbers. Activities 12, 13 and 14 focus on the "getting ready" stage. Activity 15 provides an opportunity to discuss the role of drill for children in their learning of basic facts. Throughout this section, and, in fact, throughout the entire course, you should be collecting games, puzzles and other activities that would motivate children to memorize the basic facts.

MAJOR QUESTIONS

The following questions focus on the main issues incorporated in this section:

1. Make a list of any addition and subtraction patterns you find during your work with this section which would be useful in the "getting ready to memorize" stage.
2. How would you use drill, short quizzes, games, etc., in helping the children in your class to master the basic combinations? List several games and activities useful for this "memorization" stage. (Make use of any discussion which occurs in Activity 15.)

ACTIVITY 12

THINKING PATTERNS IN ADDITION AND SUBTRACTION

FOCUS:

In this activity you will identify various thinking patterns that children might use in working with addition and subtraction combinations prior to memorization.

DISCUSSION:

Unfortunately, many teachers omit the most important stage in developing the basic number facts. They use many good techniques to introduce addition and subtraction. They help the children see the relationship between real-world instances and the mathematical representation of addition and subtraction. Later they provide many interesting activities for drill in memorizing the basic combinations. But the stage between the introduction and memorization is often omitted, perhaps because it is a difficult stage to develop and because it does not lend itself to paper-and-pencil activities. In fact, textbooks are unable to provide appropriate lessons for this stage.* The stage between the introduction and the memorization

*It is appropriate to point out here that textbooks can provide an excellent source of material for many lessons, but a teacher cannot rely on the textbook as the total program.

of basic number combinations is here called "getting ready to memorize." In this stage the children should be encouraged to explore a variety of ways of thinking about the number combinations. A second component of this stage is for the children to verbalize the ways in which they think about these combinations. In this activity you will have an opportunity to identify and list ways in which children might think about some number facts.

As a point of information, the terms "basic facts," "number facts," "basic combinations," etc. are used interchangeably and refer to those examples in which each number is represented by a single digit (0 to 9). Thus, 8 + 7 is considered a basic fact whereas 12 + 4 is not since 12 is not a single-digit numeral.

DIRECTIONS:

Read the entire set of directions before proceeding with (1).

1. Read and study the transcripts on pages 72 to 76 of three actual interviews with children. Identify, discuss and list the thinking strategy exhibited by each child in the interview. Use the "Summary of Patterns" chart on page 71 to list the strategies used by the children. Where possible, identify any mathematical property or concept used in the strategy.

2. Role-play by taking turns presenting a basic addition or subtraction number combination to the other members of your group (three or four students). Ask your fellow students to think of as many ways as possible of finding the answer to the combination. List these ways along with the list started in (1) above.

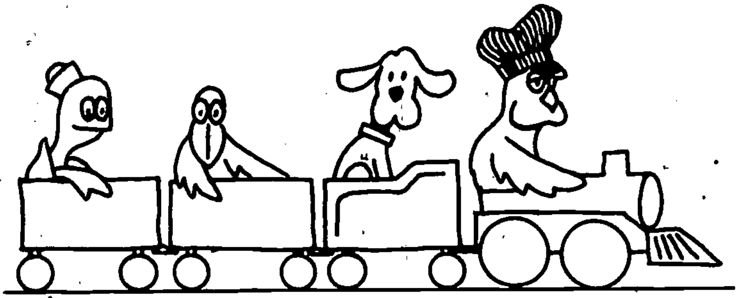
3. Discuss why this "getting ready to memorize" stage is so important in the child's growth in mathematics. Discuss the role of verbalization in the child's mathematical growth. Is there a fine line between "pushing children" to verbalize before they are ready and encouraging children to bring to the surface the strategies they are using to get the answers?

Write in expanded form.

Add ones, tens, and hundreds.

$$\begin{array}{r}345\\ +278\\ \hline\end{array} \qquad \begin{array}{r}300 + 40 + 5\\ 200 + 70 + 8\\ \hline\end{array}$$

$$\boxed{} + \boxed{} + \boxed{} = \boxed{}$$



A

$$\begin{array}{r}26\\ +47\\ \hline\end{array}$$

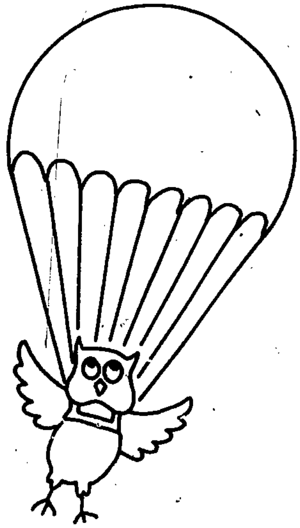
B

ADDITION
CARD SET

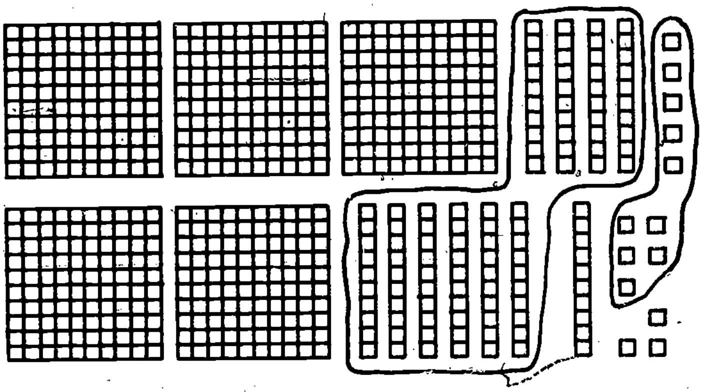
ADDITION
CARD SET

ADD:

$$\begin{array}{r} 43 \\ +35 \\ \hline \end{array}$$



C



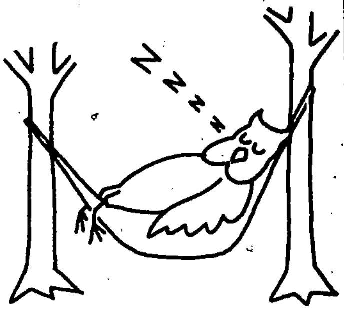
	3 hundreds	4 tens	5 ones
+	2 hundreds	7 tens	8 ones
	5 hundreds	11 tens	13 ones

6 hundreds 2 tens 3 ones

D

ADDITION
CARD SET

ADDITION
CARD SET



Tens	Ones
////	///
///	/////

$$
\begin{array}{r}
43 \\
+\ 35 \\
\hline
\end{array}
$$

E

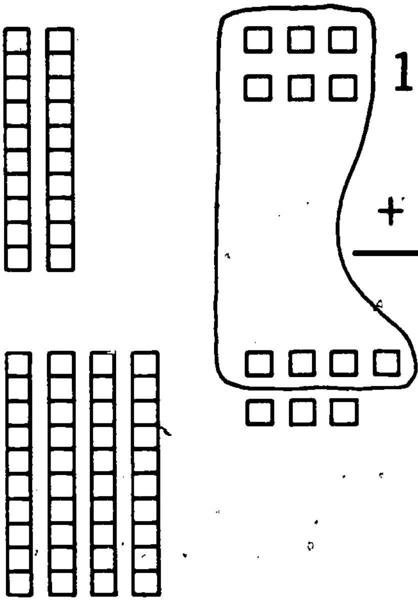
Hundreds	Tens	Ones
///	////	/////
// 0	//// /// 0	///// ///
6	2	3

$$
\begin{array}{r}
\overset{1}{3}45 \\
+\ 278 \\
\hline
623
\end{array}
$$

F

ADDITION
CARD SET

ADDITION
CARD SET



1 Ten

2 Tens 6 Ones

\+ 4 Tens 7 Ones

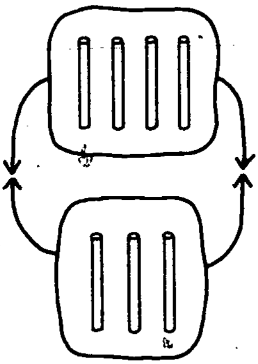
---

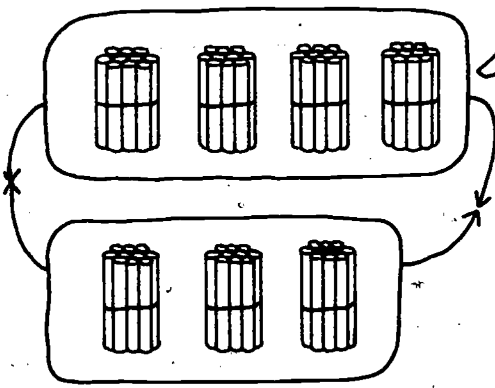
3 Ones

G

Adding Ones

Adding Tens





4	4 Tens	⇨	40
+ 3	+ 3 Tens		+ 30

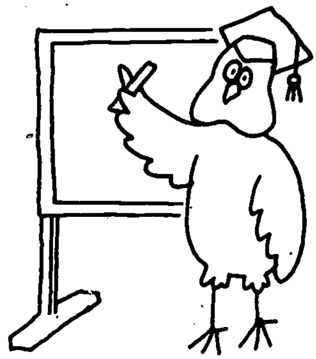
H

ADDITION
CARD SET

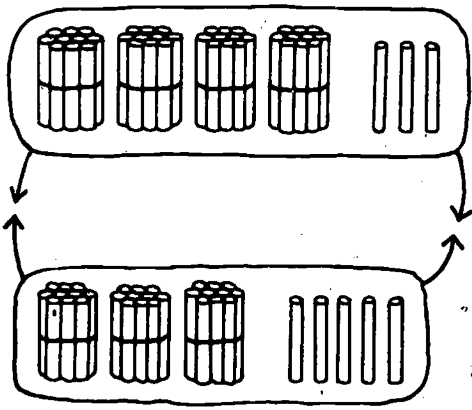
ADDITION
CARD SET

ADD:

$$
\begin{array}{r}
345 \\
+\ 278 \\
\hline
\end{array}
$$



I



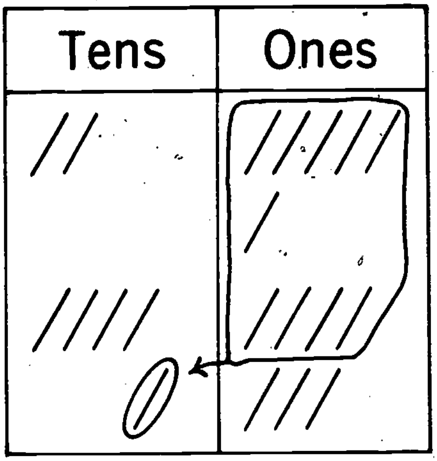
4 Tens 3 Ones

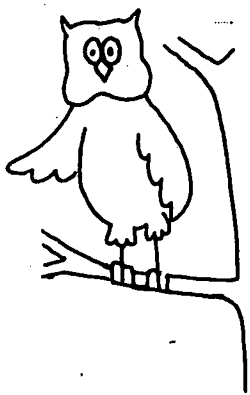
+ 3 Tens 5 Ones

J

ADDITION
CARD SET

ADDITION
CARD SET





$$\begin{array}{r} 26 \\ +\ 47 \\ \hline \end{array}$$

K

ADDITION
CARD SET

4. If possible, conduct interviews with several children similar to those illustrated here. Describe your interviews.

INTERVIEWS

SUMMARY OF PATTERNS CHART			
Examples	Child 1	Child 2	Child 3
I. 9 + 7	Not Included in the Interview		
II. 9 - 4			
III. 15 - 6			Not Included in the Interview

TEXT OF INTERVIEWS

First Child

INTERVIEWER: Hi, Vern, how are you today?

VERN: Fine.

INTERVIEWER: Tell me what the card says first of all.

$\boxed{9 - 4 = \square}$

VERN: minus four.

INTERVIEWER: Nine minus four. Do you know what nine minus four is?

VERN: Five.

INTERVIEWER: Five. How did you work that one out?

VERN: In my mind.

INTERVIEWER: Can you tell me how you thought it out though? Suppose you'd forgotten it for a moment.

VERN: Like I'm adding five and four.

INTERVIEWER: Oh, like you're adding five and four. Good.

INTERVIEWER: Let's try one more. What's that one say?

$\boxed{15 - 6 = \square}$

VERN: Fifteen take away six.

INTERVIEWER: Okay, I wonder if you can work that one out.

VERN: Nine.

INTERVIEWER: Oh, you got that quick. How did you do it this time?

VERN: I did it in my mind.

INTERVIEWER: How did you do it though?

VERN: I subtracted six away from five and took away one from ten and I got to nine.

INTERVIEWER: You took six away from five. And then...

VERN: And then I had one more left over to take away so I took it away from the ten.

INTERVIEWER: Oh, I see. I think you really took five away from five, didn't you, first of all? And then took away one from the ten that was left. That's great, Vern.

Second Child

INTERVIEWER: Hello, Tracey. How are you?

TRACEY: Fine.

INTERVIEWER: Well, we've got a little bit of work with addition and subtraction combinations. Are you pretty good with those? Let's have a look at this one first. It's an addition one. Okay. Why don't you tell me what it says?

$$9 + 7 = \square$$

TRACEY: Nine plus seven.

INTERVIEWER: Okay. Do you know what nine plus seven is?

TRACEY: Sixteen.

INTERVIEWER: You got that pretty quick. Suppose you had forgotten. Could you tell me how you could work it out?

TRACEY: Seven and nine are sixteen so six and ten would be sixteen. So all you have to do is add one more to this one and less one on that.

INTERVIEWER: Oh, I see. Great!

INTERVIEWER: Well, I want to try a subtraction one. Let's have a look at this one. It's a nice easy one for you,

$$9 - 4 = \square$$

I'm sure. Why don't you tell me what it says?

TRACEY: Nine minus four.

INTERVIEWER: Nine minus four. Well, do you know what this is?

TRACEY: Five.

INTERVIEWER: Five. How did you work that one out?

TRACEY: Five and four equals nine.

INTERVIEWER: Five and four equals nine. You used an addition fact to get it. Okay. You're pretty good, aren't you?

INTERVIEWER: Now, what about the subtraction ones? This one is a bit harder. Why don't you tell me what it says?

15 - 6 = $\boxed{}$

TRACEY: Fifteen minus six.

INTERVIEWER: Okay. Do you know what fifteen minus six is?

TRACEY: Nine.

INTERVIEWER: Nine. How did you work that one out?

TRACEY: Nine and six equals fifteen.

INTERVIEWER: You used nine and six again. Is there any other way you could do it? (Long pause) You don't know exactly. You think you used nine and six. Great, Tracey. Thank you very much indeed. That was good. You're pretty quick on those combinations, I'll agree.

Third Child

INTERVIEWER: Hi, Nathan, how are you today?

NATHAN: Fine.

INTERVIEWER: Well, I'm going to do some work with addition and subtraction combinations here. Are you pretty good at those?

NATHAN: Yeah.

INTERVIEWER: Okay. Why don't we start with this addition one? Why don't you tell me what this says?

$9 + 7 = \square$

NATHAN: Nine plus seven equals box.

INTERVIEWER: Okay. Do you know what nine and seven are?

NATHAN: No.

INTERVIEWER: You don't. I wonder if you can work it out for me, though. See if you can work it out for me. Any way you like. (Long pause)

NATHAN: Sixteen?

INTERVIEWER: Sixteen! How did you do it? You were working away there hard. You must have done something.

NATHAN: I was thinking, see, since there wasn't a ten, I knew that wouldn't be seventeen. And then, so, it would be one more down than seventeen.

INTERVIEWER: One more down than seventeen. That's great, Nathan. Thanks very much. Let's look at another one.

INTERVIEWER: I'm sure you'll find this one a bit easier. It's a subtraction one. Why don't you tell us what it says, first of all.

$9 - 4 = \square$

NATHAN: Nine take away four equals box again.

INTERVIEWER: Okay. Do you know what that is?

NATHAN: Five.

INTERVIEWER: Oh, you worked that one out quickly. How did you work it out?

NATHAN: I already knew how.

INTERVIEWER: You already knew it. Suppose...

NATHAN: Well, four plus five and five plus four is nine so nine take away four is five.

INTERVIEWER: Ah, I see. Well that certainly wasn't too hard for you.

ACTIVITY 13

HELPING CHILDREN DEVELOP THINKING STRATEGIES FOR ADDITION

FOCUS:

The importance of helping children develop insight and strategies before memorization seems clear. This activity will suggest the use of some number properties and "doubling" as techniques that might encourage children to think about the basic number combinations as they get ready to memorize these combinations.

DISCUSSION:

Young children are expected to memorize the number combinations. There are 100 different addition combinations using the numbers 0 through 9. It would be a huge (if not impossible) task for children to memorize these facts without some way of organizing them. The sad truth is that many children have not mastered these facts even by age 12 or 15. While there may be many reasons for this phenomenon, one of the reasons that has been identified is the lack of attention paid to helping young children develop their own thinking strategies and ways of organizing large quantities of information.

DIRECTIONS:

1. Complete and study the addition chart at the right.

    a) How might knowledge of the commutative property (a + b = b + a) reduce the number of combinations?

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

b) How might knowledge of the identity element for addition help reduce the number of addition combinations the child must learn?

2. The associative property can be used to find facts such as 9 + 6. How would you explain this to children? Draw a diagram to show how concrete materials could be used in this explanation.

3. How would you explain or demonstrate the properties for children? Would you use the names "commutative," "associative," etc. with the children or might you use less technical terms? (Consult some textbooks to see how the properties are introduced to young children.)

4. Knowledge of doubles (e.g., 5 + 5) can also reduce the number of addition facts to be learned. Identify the position of the doubles in the addition matrix. How could a child who knows "doubles" find 6 + 7? Identify three other facts which could be found easily from a knowledge of 6 + 6.

5. Study the addition table and list as many patterns as you can find that might be helpful to children in developing thinking strategies.

6. In the upper primary grades, after a child has learned the addition facts, he should be able to handle extensions such as: (a) 30 + 60 and (b) 32 + 7. How might you explain such extensions?

100

ACTIVITY 14

STRATEGIES FOR FINDING SUBTRACTION FACTS

FOCUS:

Subtraction presents more difficulty than addition for children. This activity provides an opportunity to think about different strategies that might be used to help children "get ready" to memorize the basic facts in subtraction.

DISCUSSION:

If one were to ask primary-grade teachers what the most difficult topic in arithmetic was at that level, there would be a fairly unanimous answer: "subtraction." Careful teaching of this topic is essential since it always seems to present difficulty for children.

DIRECTIONS:

Assignment: In preparation for the seminar (Activity 15) outline two games or puzzles which can be used to help children memorize the basic addition or subtraction facts. For each outline provide an objective, approximate grade level, materials, and directions. (These outlines may be prepared on an overhead transparency.)

1. Why do you think that subtraction presents so much difficulty for children as compared with addition? Discuss this question and list as many reasons as you can.

2. Complete and study the subtraction table on the following page.

   a) List patterns that might be useful in helping children cut down the number of subtraction combinations.

   b) What properties (if any) might be helpful to children in reducing the number of combinations?

Second Number

-	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

First Number

3. How are addition and subtraction related? Give an example of how this relationship might help children with their subtraction combinations, assuming that they know most of the addition facts.

4. Study the illustrations of the children's thinking below.

   a) Explain the thinking behind each child's method.

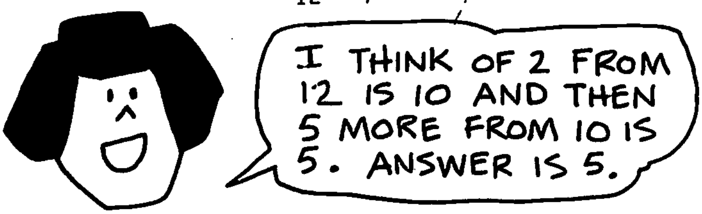
   b) Explain the mathematical principle(s) involved in each method.

   c) Use each method to solve 16 - 6.

   d) Do both methods apply to 9 - 4?

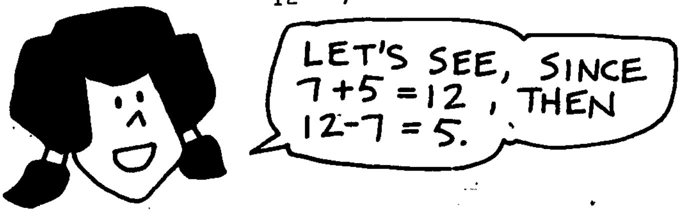
Lisa

12 - 7



Michele

12 - 7



5. Discuss and list one or more additional strategies for finding 12 - 7.

ACTIVITY 15

SEMINAR

FOCUS:

In this activity you will have a chance to ask questions and discuss problems related to teaching the basic number facts of addition and subtraction. You will also have an opportunity to consider activities related to the memorization stage of basic addition and subtraction facts.

DISCUSSION:

The diagram which was presented at the beginning of this section illustrates the stages that prepare the child for eventual memorization of the basic combinations for addition and subtraction.

Introduction	→	Getting Ready to Memorize	→	Memorization

In Activities 12, 13 and 14 some experiences in the "getting ready to memorize" category were presented. The seminar will consider further activities of this kind, as well as methods for progressing from this stage to the memorization stage.

DIRECTIONS:

1. Describe the activities you have prepared to help children memorize basic addition or subtraction number facts. (See assignment, page 79.)

2. Discuss questions or issues you have found concerning the three stages of development of the basic facts.

3. Among the questions the instructor will discuss are the following:

a) What are some examples of activities (other than those considered in Activities 12 and 13) which fit into the category of "getting ready to memorize"?

b) What is the role of drill in learning the facts? How can drill be carried out effectively?

c) What is the role of games in learning the facts? What are the strengths of games? What are their weaknesses?

4. Critique the game which follows by discussing the positive and negative features of it. How might this game be modified to be more effective (interesting, fun, etc.)?

GAME: A Skirmish on the Number Line

Here is a game which could be used to give children practice in recalling addition combinations. Try the game with a partner.

<u>How to Play</u>:

a) The first player selects any addition combination in the matrix on the following page, puts an X in the appropriate square of the matrix and also an X on the number line to indicate the result of the combination, (e.g., at 7 for 5 + 2). If a player gets the combination wrong his mark is removed by the other player.

b) The second player selects any combination not already used in the matrix, puts a $\triangle$ in the appropriate square of the matrix and also a $\triangle$ on the number line to indicate the result of the combination. Should there be an X there he erases it and puts a $\triangle$.

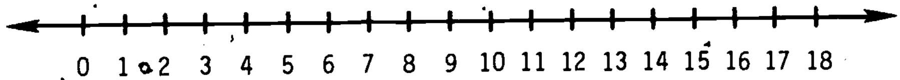
c) The game continues with the first player following a procedure similar to that given in (b). The game ends when the matrix is completely filled. The winner of the game is the one with more markings remaining on the number line. Can you suggest some kind of winning strategy?

<u>Note</u>: The game can be shortened by restricting the number of combinations that can be used in the matrix.

Addition Matrix:

+	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Number Line:



Section IV

ALGORITHMS AND PROBLEM SOLVING

Previous sections have focused on introductory concepts in addition and subtraction, the mathematical foundations associated with these operations, and the development of the basic combinations in addition and subtraction. The development of algorithms in addition and subtraction relies heavily on the work of previous sections.

In this section the rationale and techniques for instruction in the algorithms are presented. Consistent with the approach used throughout this unit, the relation between the real world and mathematics is again emphasized. The need for algorithms arises in real-world problems. Materials are used to model these problems and to provide the link with the symbolic forms of the algorithms. Eventually, as the child comes to use larger numbers, he must rely more and more on the symbolic forms. Skill in adding and subtracting larger numbers requires a clear understanding of the basic algorithms.

Problem solving is given a special focus in this section. Teachers should work on developing problem-solving skills with children every day. Relating the real world and mathematics is not something to be worked upon only when the textbook has a page of problems.

As you work through this section, you should focus on the major questions that follow.

MAJOR QUESTIONS

1. Why does the subtraction algorithm cause more difficulty than the addition algorithm? Give illustrations to support your answer.

2. Three different types of algorithms are discussed in the unit: transitional, standard and nonstandard. Choose addition or subtraction and discuss the role of transitional algorithms with respect to the development of the standard algorithm.

3. Describe some ways in which you plan to help children become effective solvers of addition and subtraction problems.

108

ACTIVITY 16

SEQUENCING ADDITION AND SUBTRACTION ACTIVITIES

FOCUS:

In this activity you will be given a chance to sequence a set of activities which develop the addition algorithm. As a second task you will be asked to create a "card set" of activities which develop the subtraction algorithm.

DIRECTIONS:

1. a) Work alone and put the card set into an appropriate sequence.

   b) Compare your sequence with a classmate; then with a group.

   c) Discuss the sequence as a class with your instructor.

2. Work in pairs. Use 5 x 8 index cards or standard-size paper ($8\frac{1}{2}$ x 11) divided in half and create a "card set" of 10 or more activities which develop the subtraction algorithm. You may consult elementary textbooks.

3. After you have finished creating your card set in subtraction, exchange sets with another pair of classmates. Put your classmates' card set in sequence and discuss differences of opinion.

4. Combine the addition <u>and</u> subtraction card sets into one set, which represents a suitable overall learning order.

ACTIVITY 17

USING MATERIALS TO INTRODUCE ADDITION AND SUBTRACTION

FOCUS:

In this activity you are asked to use bundling sticks, Dienes blocks and an abacus to illustrate steps in demonstrating subtraction (and addition) algorithms.

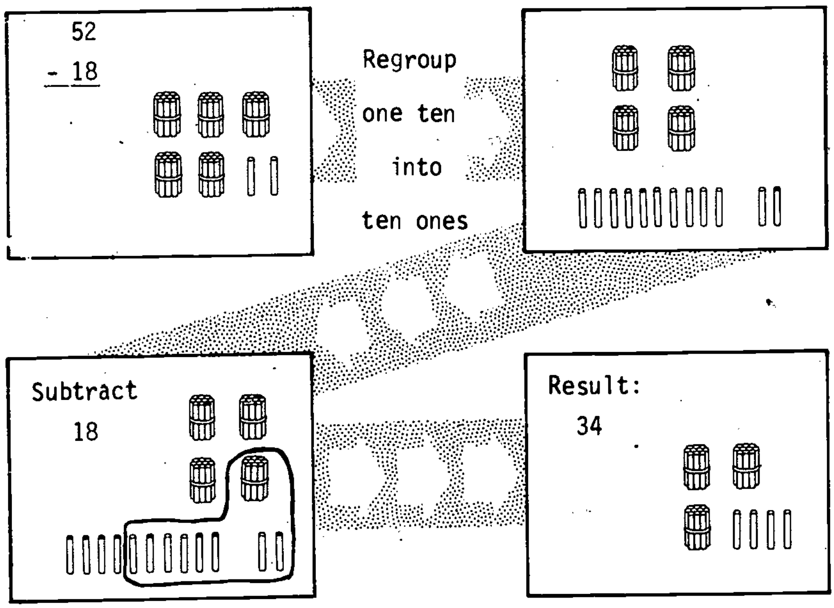
MATERIALS:

Bundling sticks, Dienes blocks, abacus.

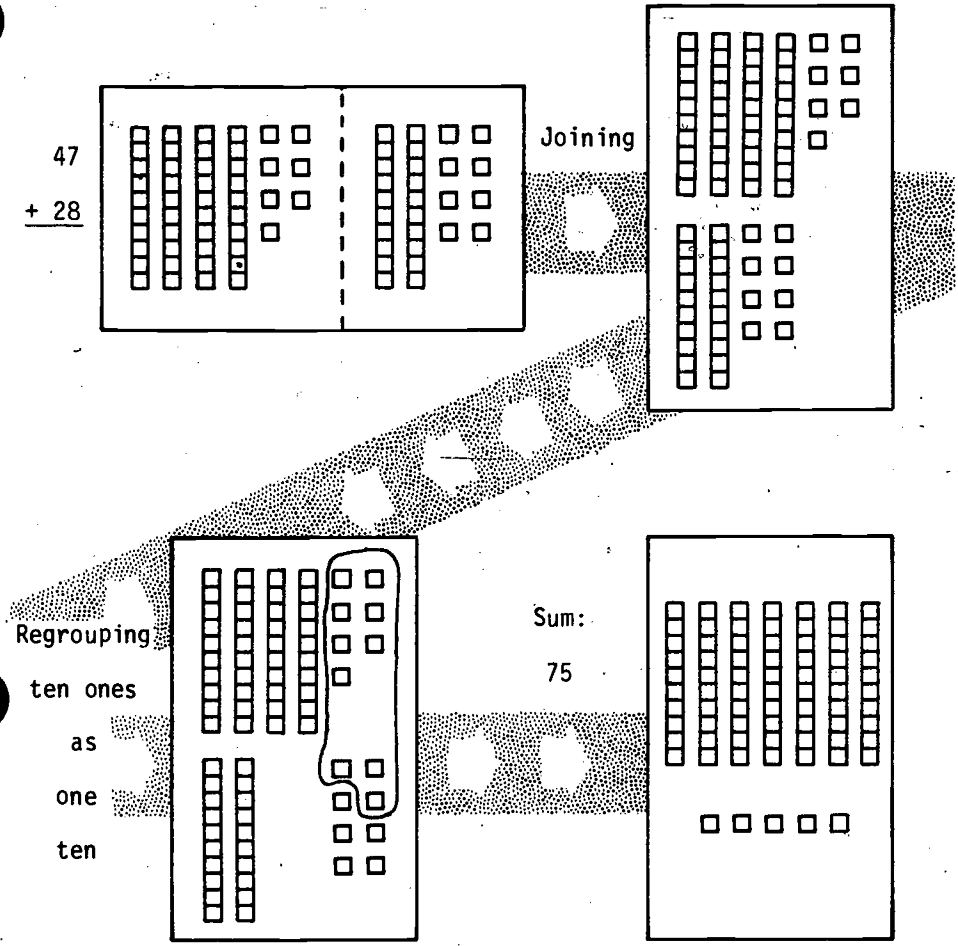
DISCUSSION:

The use of materials in introducing basic concepts of addition and subtraction was developed in Section I. In introducing the algorithms, materials should also play an important role. As the numbers become greater and as the students become more confident in their work, less emphasis should be placed on materials. Study the two examples which follow.

EXAMPLE 1: SUBTRACTION WITH BUNDLING STICKS



EXAMPLE 2: ADDITION WITH DIENES BLOCKS



The introduction of algorithms should follow the same procedure as the introduction of the operations. That is, the example should proceed from a real-world or story situation. Next, a drawing or some concrete aids should be used to illustrate the example. Finally, the appropriate symbols should be related to the aids or the drawing. The illustration of this instructional mode is shown on the following page. The double-headed arrows suggest that the relationship should be emphasized in both directions.

Real-world
Problem

Concrete aids ⟷ Mathematical
or Symbols

Drawing

In Example 1 on page 88, a story such as

"John has 52 cards. He drops 18 of them. How many remain?"

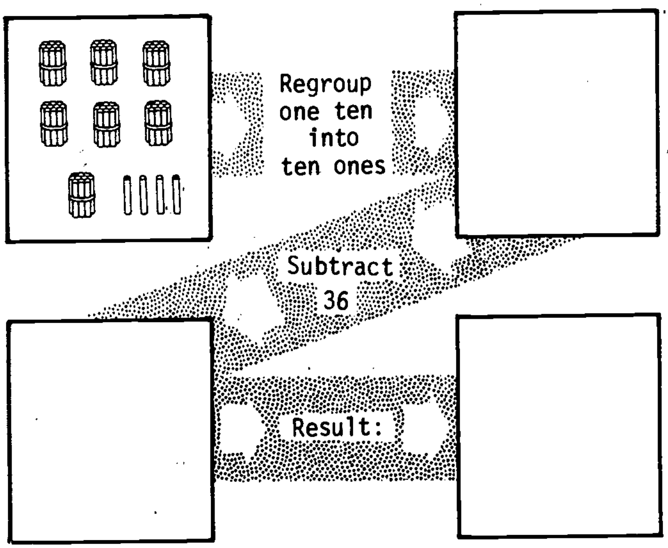
might be used to introduce the illustration. For Example 2, the children might be asked to make up a story from which 47 + 28 might have come.

DIRECTIONS:

1. Use bundling sticks to demonstrate the example:

$$\begin{array}{r} 74 \\ \underline{-\ 36} \end{array}$$

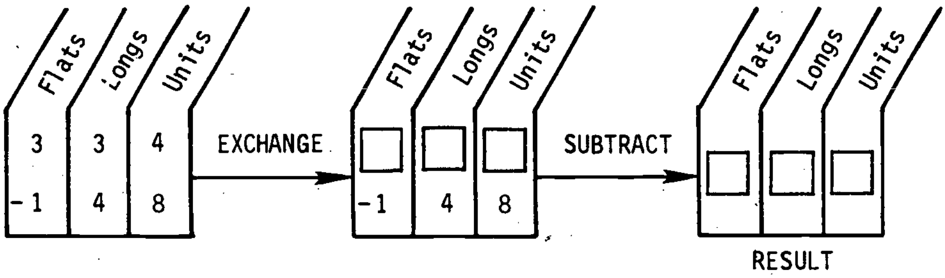
Complete the diagram below to represent each stage in the process, paying particular attention to the regrouping stage.



Discuss the relationship between the standard form of subtraction and that illustrated using bundling sticks.

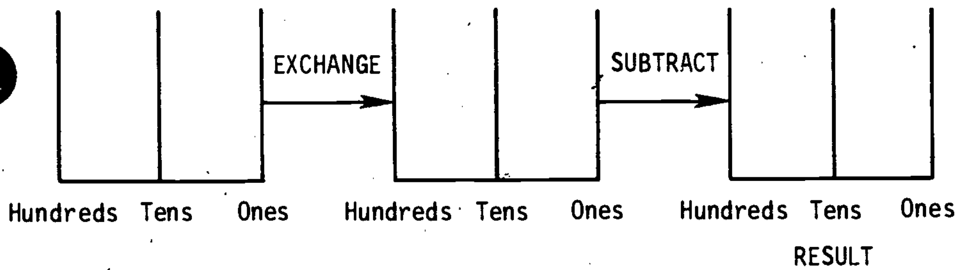
2. Using Dienes blocks, perform the example:

$$\begin{array}{r} 334 \\ -\ 148 \\ \hline \end{array}$$



3. Use an abacus to perform the example:

$$\begin{array}{r} 637 \\ -\ 289 \\ \hline \end{array}$$



4. Work in groups of two to four. Take turns demonstrating the following subtraction and addition examples, using bundling sticks, Dienes blocks or an abacus. In demonstrating the examples, focus on the "regrouping" of 1 ten for 10 ones ("borrowing") and 10 ones for 1 ten ("carrying").

Recall from your work in numeration that regrouping takes place when 10 objects are bundled to form 1 ten (100 objects to form 10 tens or 1 hundred, etc.). Conversely, regrouping may also involve the exchange of a bundle of 1 ten for 10 ones. Renaming, on the other hand, is the term used to describe the symbolic form of exchange. For example, 74 can be renamed as 70 + 4 or 60 + 14, etc.

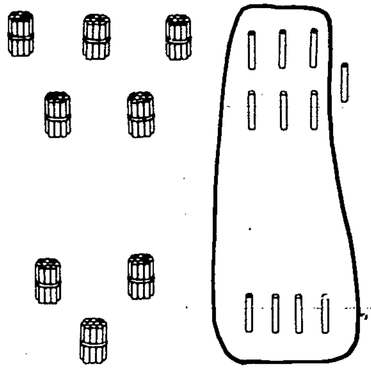
In addition and subtraction, both regrouping and renaming play an important role in understanding the algorithms. Study the illustrations on page 93 before doing the example below. Discuss the merits of each aid.

a) $\begin{array}{r} 342 \\ -\ 178 \\ \hline \end{array}$	b) $\begin{array}{r} 617 \\ +\ 187 \\ \hline \end{array}$	c) $\begin{array}{r} 245 \\ +\ 185 \\ \hline \end{array}$
d) $\begin{array}{r} 608 \\ -\ 139 \\ \hline \end{array}$	e) $\begin{array}{r} 370 \\ +\ 639 \\ \hline \end{array}$	f) $\begin{array}{r} 815 \\ -\ 405 \\ \hline \end{array}$

Don't put the Dienes blocks away. You'll use them in the next activity.

REGROUPING AND RENAMING

Addition

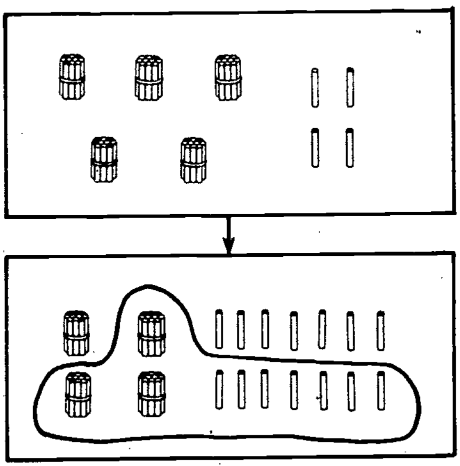


$$\begin{array}{r} ^{1} \\ 57 \\ +\,34 \\ \hline 91 \end{array}$$

After regrouping one obtains 1 ten and 1.

After adding 4 and 7, 11 ones are renamed as 1 ten and 1.

Subtraction



$$\begin{array}{r} 4\ {}^{1} \\ \cancel{5}4 \\ -\,37 \\ \hline \end{array}$$

1 bundle of ten has been regrouped as 10 ones.

1 ten has been renamed as 10 ones, giving 4 tens and 14 ones.

ACTIVITY 18

ADDING AND SUBTRACTING IN OTHER BASES

FOCUS:

This activity provides an opportunity for you to tackle adding and subtracting in bases other than ten.

MATERIALS:

Dienes blocks (one set of base two, base three, base four and base five for each group).

DISCUSSION:

In the upper elementary grades other bases are introduced. While the intention is not to have children gain proficiency in operations with other bases, they may enjoy adding and subtracting in bases other than ten.

DIRECTIONS:

(Groups of four)

1. Use sets of Dienes blocks for bases two, three, four and five. It is suggested that each member of your group choose a different base to work with. You will use only this base throughout the remainder of this activity.

   a) Using the appropriate part of the table on page 95, construct a table of basic addition facts for the base you have selected. (For example, if you were working in base ten, your addition table would contain 100 entries for sums up to 9 + 9.) You may find the appropriate set of Dienes blocks helpful in constructing your addition table.

+	0	1	2	3	4
0					
1					
2					
3					
4					

b) Is the following statement true or false? Why?

If b is any whole number greater than 1, there are $100_{\text{base } b}$ entries in the basic addition table (i.e., basic addition combinations).

2. Select and work the appropriate set of addition and subtraction examples from those given below. Work each example:

a) Using Dienes blocks;

b) Symbolically, using the addition table you have constructed above. (Remember that the addition table will provide you with the subtraction facts you need.)

EXAMPLES

(Select only the set of examples which corresponds to the base you have chosen.)

Base Two:

$$\begin{array}{r} 110 \\ +\ 111 \\ \hline \end{array} \qquad \begin{array}{r} 110 \\ -\ 11 \\ \hline \end{array} \qquad \begin{array}{r} 1001 \\ +\ 1011 \\ \hline \end{array} \qquad \begin{array}{r} 1010 \\ -\ 1 \\ \hline \end{array}$$

Base Three:

$$\begin{array}{r} 122 \\ +\ 102 \\ \hline \end{array} \qquad \begin{array}{r} 121 \\ -\ 12 \\ \hline \end{array} \qquad \begin{array}{r} 1021 \\ +\ 1012 \\ \hline \end{array} \qquad \begin{array}{r} 1020 \\ -\ 2 \\ \hline \end{array}$$

Base Four:

$$\begin{array}{r} 312 \\ +\ 221 \\ \hline \end{array} \qquad \begin{array}{r} 331 \\ -\ 22 \\ \hline \end{array} \qquad \begin{array}{r} 3031 \\ +\ 1203 \\ \hline \end{array} \qquad \begin{array}{r} 3320 \\ -\ 3 \\ \hline \end{array}$$

Base Five:

$$\begin{array}{r} 423 \\ +\ 222 \\ \hline \end{array} \qquad \begin{array}{r} 431 \\ -\ 23 \\ \hline \end{array} \qquad \begin{array}{r} 4032 \\ +\ 1303 \\ \hline \end{array} \qquad \begin{array}{r} 4320 \\ -\ 4 \\ \hline \end{array}$$

3. The difficulty which you had in adding in another base may give you some insight into the difficulties that children have in learning how to add. Discuss some of these difficulties.

ACTIVITY 19

SOME TRANSITIONAL ALGORITHMS

FOCUS:

Some children find it helpful to use a transitional or "helping" algorithm before proceeding to the standard algorithm. This activity will acquaint you with these transitional algorithms. This activity ~~will also~~ provide an opportunity for you to sequence some examples in order of difficulty.

DISCUSSION:

In the development of algorithms one must be careful not to introduce too much formalism too early. Children should be allowed to think and explore ways of finding computational answers before the algorithm is fixed. On the other hand, transitional or alternate algorithms must not be taught for mastery lest the child become confused with too many algorithms. Thus a balance between encouraging divergent thinking and causing confusion must be effected.

DIRECTIONS:

1. Study the four transitional algorithms used by the children to perform the computation 46 + 35.

Lucy	Jim	Carol	Nancy
46 = 40 + 6	46	46	46
35 = 30 + 5	35	35	35
70 + 11	11	51	76
= 81	70	30	5
	81	81	81

a) Explain and discuss each algorithm.

b) List some other transitional algorithms for addition.

c) Choose one of the transitional algorithms and discuss how one might use it as a bridge from basic facts to the standard algorithm.

2. Study the four transitional algorithms used by the children to perform the computation 72 - 39.

Michele	Nancy	Jose	George
$\begin{array}{r} 72 = 60 + 12 \\ -\ 39 = 30 + 9 \\ \hline 30 + 3 \\ = 33 \end{array}$	$\begin{array}{r} 72 \\ -39 \\ \hline 42 \\ -9 \\ \hline 33 \end{array}$	$\begin{array}{r} 72 \\ -39 \\ \hline 63 \\ -30 \\ \hline 33 \end{array}$	$\begin{array}{r} 72 \\ -39 \\ \hline -7 \\ 40 \\ \hline 33 \end{array}$

a) Explain and discuss each algorithm. In particular, what was George's thinking pattern?

b) Choose one of the transitional algorithms and discuss how it might be used as a bridge from the basic facts to the standard subtraction algorithm.

3. There are three addition examples and three subtraction examples shown on the following page.

a) Order the addition examples from easiest to hardest.

b) Give the reason why the harder and hardest addition examples are more difficult than the easiest.

c) Do as in (a) and (b) for subtraction.

Addition:

38 + 24	32 + 51	83 + 49

Subtraction:

78 - 40	72 - 47	74 - 22

ACTIVITY 20

WRITING LESSONS FOR ADDITION AND SUBTRACTION ALGORITHMS

FOCUS:

In this activity you will have an opportunity to plan and write an activity related to addition or subtraction algorithm development. After everyone has written an activity you will be able to share ideas related to lesson development with your classmates.

MATERIALS:

Elementary mathematics textbook series, teaching aids as needed.

DISCUSSION:

A previous activity focused on using materials to introduce the addition and subtraction algorithms. While materials should play an important role in your teaching, they should be incorporated into an overall activity plan. Knowing where you are going and what learning you want the pupils to acquire is most essential to developing an activity. The goal of an instructional activity is often stated in terms of a behavior or objective for the pupil to attain. In planning a lesson it is important not only to have a specific objective, but to see how that objective fits into a sequence of learning experiences and to consider what type of development is needed to attain the objective. One must consider what the pupil knows, what can be built upon, and how this activity will help the future development of the child.

DIRECTIONS:

Each group should consist of two or four members. Study the objectives listed on page 102 and have each person choose one objective. (For a group of two choose one objective for addition and one for subtraction.)

Each person will be responsible for preparing an activity appropriate for children. This activity should be aimed at helping children attain the stated objective. The use of cartoons, pictures, and any other form of motivational material is encouraged. Remember the value for children of relating the real world and mathematics. Be sure to develop your objective by using real-world referents. You may work in pairs, helping and advising each other, or you may work alone. Be sure to have your partner review your activity.

Prepare your activity on overhead transparencies and be ready to present them at the next class meeting. Note questions that you have raised while preparing your activity and ask them in the seminar in which the transparencies are shown.

OBJECTIVES

A sequence of objectives (learning outcomes) for developing the <u>addition</u> algorithms.

1. The pupil will be able to find the sums for all combinations through 9 + 9.

2. The pupil will be able to find the sum in examples such as 30 + 60.

3. The pupil will be able to add with two-digit numbers using concrete and pictorial forms (without regrouping).

4. The pupil will be able to add with two-digit numbers using paper-and-pencil forms (without renaming).

5. As for 3, with one regrouping.

6. As for 4, with one renaming.

7. The pupil will be able to add with three- (or more) digit numbers using paper-and-pencil forms (one or more renamings).

A sequence of objectives (learning outcomes) for developing the <u>subtraction</u> algorithms.

1. The pupil will be able to find the differences for all subtraction facts up to sums of 19.

2. The pupil will be able to find the difference in examples such as 90 - 30.

3. The pupil will be able to subtract with two-digit numbers using concrete and pictorial forms (without regrouping).

4. The pupil will be able to subtract with two-digit numbers using paper-and-pencil forms (without renaming).

5. As for 3, with one regrouping.

6. As for 4, with one renaming.

7. The pupil will be able to subtract with three- (or more) digit numbers using paper-and-pencil forms (one or more renamings).

ACTIVITY 21

SEMINAR

FOCUS:

In this seminar you will present and discuss the teaching activities you have developed. This will allow you to ask questions about addition and subtraction that have occurred to you.

DIRECTIONS:

Various groups will present their activities. The different stages in the development of algorithms, particularly the role of concrete materials and transitional algorithms, should be discussed during the seminar. Some questions which might be raised during the seminar follow.

1. What is the role of transitional algorithms in teaching addition and subtraction?
2. Outline a sequence of algorithm development starting with the basic facts and proceeding to an example like 4563 + 8758 or 7324 - 5075.
3. What role does the use of real-world referents have in the teaching of algorithms?
4. Bill is a student who does not know his basic number facts. As a teacher would you feel he must learn these facts before proceeding to learn some algorithms or would you provide a chart of basic facts and proceed to teach Bill the algorithms?
5. What could be the role of story problems in developing algorithms, or vice versa?

ACTIVITY 22

DIAGNOSIS AND REMEDIATION: ADDITION AND SUBTRACTION ALGORITHMS

FOCUS:

In this activity you will have an opportunity to diagnose pupil errors and make suggestions for remediation. Be sure to share ideas and listen to others in doing this activity.

DISCUSSION:

Diagnosis and remediation are two tasks which are basic to good teaching. Identifying errors is not a difficult task. Determining the cause of these errors and suggesting sound techniques for remediation take more insight. Assigning 15 more subtraction examples to a student who makes subtraction errors is not an appropriate remedy. Analysis of the prerequisite skills and concepts provides a framework for diagnosing the cause of such errors. Using the fact that understanding often develops in proceeding from concrete or pictorial models to the symbolic form may suggest the types of activity that would be helpful to a pupil needing remediation. Use what you have learned about appropriate sequencing of learning experiences, the use of materials, and other ideas developed in this unit to help you diagnose the pupil errors and suggest appropriate activities for remediation.

DIRECTIONS:

1. For each pupil page locate the error(s) by circling or checking.
2. Identify the error pattern exemplified by the child by completing examples E and F as the child would do them.
3. Briefly describe one instructional activity which would help replace the child's error pattern with a correct computational procedure. Space has been left on each pupil page for you to do this.

4. After completing all the pupil pages, read pages 110-112, "An Analysis of Computational Errors with the Decomposition Algorithm at the Time of Initial Learning of the Subtraction Algorithm" and discuss each of the error categories with your group or your instructor.

PUPIL PAGE A

Name: Cheryl

A. 31 − 14 = 23

B. 384 − 238 = 154

C. 632 − 287 = 455

D. 243 − 86 = 243

E. 726 − 372

F. 541 − 96

Suggested Instructional Activity:

PUPIL PAGE B

Name: Tina

A. $\begin{array}{r} \scriptstyle 8\,10 \\ 1\cancel{9}\cancel{7} \\ -\ 43 \\ \hline 147 \end{array}$

B. $\begin{array}{r} \scriptstyle 7\,10 \\ 1\cancel{8}\cancel{6} \\ -23 \\ \hline 157 \end{array}$

C. $\begin{array}{r} \scriptstyle 7\,10 \\ 3\cancel{8}\cancel{4} \\ -59 \\ \hline 321 \end{array}$

D. $\begin{array}{r} \scriptstyle 5\,10 \\ 7\cancel{6}\cancel{2} \\ -148 \\ \hline 612 \end{array}$

E. $\begin{array}{r} 273 \\ -\ 38 \\ \hline \end{array}$

F. $\begin{array}{r} 285 \\ -63 \\ \hline \end{array}$

Suggested Instructional Activity:

PUPIL PAGE C

Name: Mike

A.
```
  28
  35
 513
```

B.
```
  46
  84
1210
```

C.
```
  93
  25
 118
```

D.
```
  86
+ 54
1310
```

E.
```
  43
  72
```

F.
```
  75
  86
```

Suggested Instructional Activity:

130

PUPIL PAGE D

Name: Donna

A.
```
 248
- 30
 210
```

B.
```
 342
 141
 201
```

C.
```
 628
 303
 305
```

D.
```
 707
 405
 302
```

E.
```
 626
 420
```

F.
```
 818
 200
```

Suggested Instructional Activity:

AN ANALYSIS OF COMPUTATIONAL ERRORS WITH THE DECOMPOSITION ALGORITHM AT THE TIME OF INITIAL LEARNING OF THE SUBTRACTION ALGORITHM

The table shown on page 112 is constructed from data collected from 101 third-grade pupils on a test of two-digit subtraction. The ten items of the test are shown in the first column of the table. Other columns in the table show the number of errors which occurred in various categories for each of these items. A brief description of the various categories of errors follows below:

1. Fact Errors

   These errors resulted from incorrect responses to the basic facts of subtraction. For example, in $\begin{array}{r} 44 \\ -\ 25 \\ \hline 18 \end{array}$ the pupil responded incorrectly to 14 - 5.

2. Renaming

   Renaming errors resulted from inability to understand the process of renaming a number; e.g., 45 = 30 + 15. These errors occurred in several different ways:

   a) $\begin{array}{r} {}^{14} \\ 3\cancel{4} \\ -\ 16 \\ \hline 28 \end{array}$ In this case, the pupil renamed 1 ten as 10 ones and correctly formed 14 ones but did not transform 3 tens to 2 tens. In effect he wrote 34 = 30 + 14.

   b) $\begin{array}{r} {}^{2\ 10} \\ \cancel{3}\cancel{4} \\ -\ 16 \\ \hline 28 \end{array}$ In this case, the pupil renamed 1 ten as 10 ones but did not combine the 10 ones with the 4 ones. In effect he wrote 34 = 20 + 10.

   c) $\begin{array}{r} {}^{4\ 16} \\ \cancel{5}\cancel{6} \\ -\ 33 \\ \hline 13 \end{array}$ In this case, the student renamed when it was not necessary. His renaming 56 = 40 + 16 was correct, but he did not proceed to subtract the ones correctly.

   d) $\begin{array}{r} {}^{6\ 2} \\ \cancel{7}\cancel{2} \\ -\ 34 \\ \hline 34 \end{array}$ In this case, the pupil transformed 1 ten but did not rename it as 10 ones. In effect he wrote 72 = 60 + 2.

3. Reversals

Reversal errors occurred when pupils reversed the order of subtraction for the units digits while maintaining the correct order for the tens digits. This could have resulted from inability to appreciate the importance of order in subtraction (they may even have felt subtraction was commutative), inability to recognize the need for renaming, or inability to carry out renaming. It may even have been caused by a combination of all three. Again the errors occurred in several ways.

a) $\begin{array}{r} 72 \\ -\ 34 \\ \hline 42 \end{array}$ This is the general case and is likely to occur wherever renaming is necessary.

b) $\begin{array}{r} 60 \\ -\ 15 \\ \hline 55 \end{array}$ Some pupils only reversed when a zero occurred. A few pupils reversed in every case where renaming was necessary except the zero case.

TABLE SHOWING COMPUTATIONAL ERRORS WITH THE DECOMPOSITION ALGORITHM AT THE TIME OF INITIAL LEARNING OF THE SUBTRACTION ALGORITHM*

(Number of Pupils = 101)

Examples	Total Errors	Fact Errors	Procedural Errors			
			Renaming	Reversal	Miscellaneous	Not Complete
1. 83 - 78	31	21	3	6	0	1
2. 51 - 27	28	7	12	8	0	1
3. 60 - 15	23	3	1	18	1	0
4. 80 - 39	22	4	0	17	0	1
5. 76 - 58	22	14	4	3	0	1
6. 72 - 34	20	6	3	9	1	1
7. 44 - 25	19	6	5	6	1	1
8. 34 - 16	15	6	4	4	0	1
9. 67 - 37	12	2	6	0	3	1
10. 56 - 33	6	1	5	0	0	0

*Paul R. Trafton, "The Effects of Two Initial Instructional Sequences on the Learning of the Subtraction Algorithm in Grade Three," (Ph.D. dissertation, University of Michigan, 1970), pp. 126-127.

ACTIVITY 23

GAME TIME

FOCUS:

In this activity you will study some games and prepare game activities designed to motivate children's understanding of computational algorithms and number properties.

MATERIALS:

(Listed for each game.)

DISCUSSION:

The use of games can provide a strong motivational force for encouraging children to build skill in computation. Games or puzzles can also be used to heighten children's understanding of the basis for the algorithms or properties of operations. This activity presents two examples of such games. Following the games is an optional section on the mini-computer, including discussion and references.

GAME 1

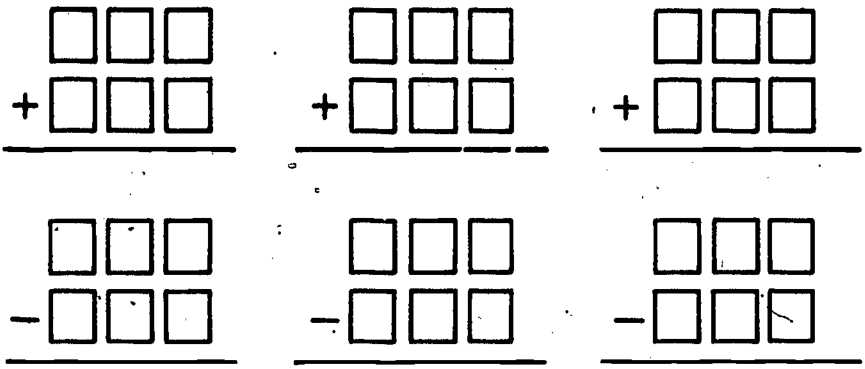
"Pick the Cards"

The following game can be played in a group of three to four.

MATERIALS:

1. Deck of ten cards: $\boxed{0}$ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ • • • • • $\boxed{9}$
2. Worksheet shown below.

EXAMPLES



DIRECTIONS:

1. Decide which of the games is to be played:
   a) Greatest sum
   b) Least sum
   c) Greatest difference
   d) Least difference
2. One person acts as leader. The deck is shuffled and the cards placed face down.
3. The leader draws a card, identifies the number, and all players write the number in any box of the first worksheet example. Once a box is filled the player cannot change it. The card is not replaced in the deck.

4. The second card is chosen, the number identified and all players write the number in any remaining boxes of the first worksheet example.

5. One card is drawn for each box.

6. After all boxes are filled (six, in the examples shown), compute the answer according to the game you have chosen.

7. You can also play two games at once (e.g., games a and b) by using two example boxes and recording each number once in each example box.

<u>Note</u>: Similar games could be made up for multiplication.

QUESTIONS

1. Discuss any helpful strategies you may have found.

2. What important values does such a game have for children?

GAME 2

"Match the Squares"

MATERIALS:

Scissors, tape.

DIRECTIONS:

The square on page 117 has been divided into four smaller squares. On each of the four edges of each square a sum or difference has been written. Cut out the four smaller squares and rearrange them so that the edges with matching sums or differences touch. Then tape the squares together. Turn over when you finish to check your results. Write a similar activity for children.

The Mini-Computer

(Optional)

MATERIALS:

Van Arsdel, J. and Lesky, J. "A Two-Dimensional Abacus--the Papy Mini-Computer," Arithmetic Teacher (October, 1972), pp. 445-451.

Whitney, H. "A Mini-Computer for Primary Schools," Mathematics Teaching, LII (Autumn, 1970), pp. 4-7.

DISCUSSION:

The mini-computer is not a game; it is an aid designed to help children understand place value and to perform computations using a paper-and-pencil computer. Many of you will enjoy studying and using this computer.

The Papy mini-computer was first presented in the United States in 1970 by Madame Frédérique Papy. At that time she demonstrated its use with a number of classes of children. Since then the Papy mini-computer has been used by many classes with fascinating results. This activity will give you an opportunity to construct and work wi[th] a Papy computer. A second mini-computer has been developed by Hassler Whitney. An article on Whitney's mini-computer is suggested so that you can compare it with the Papy mini-computer.

DIRECTIONS:

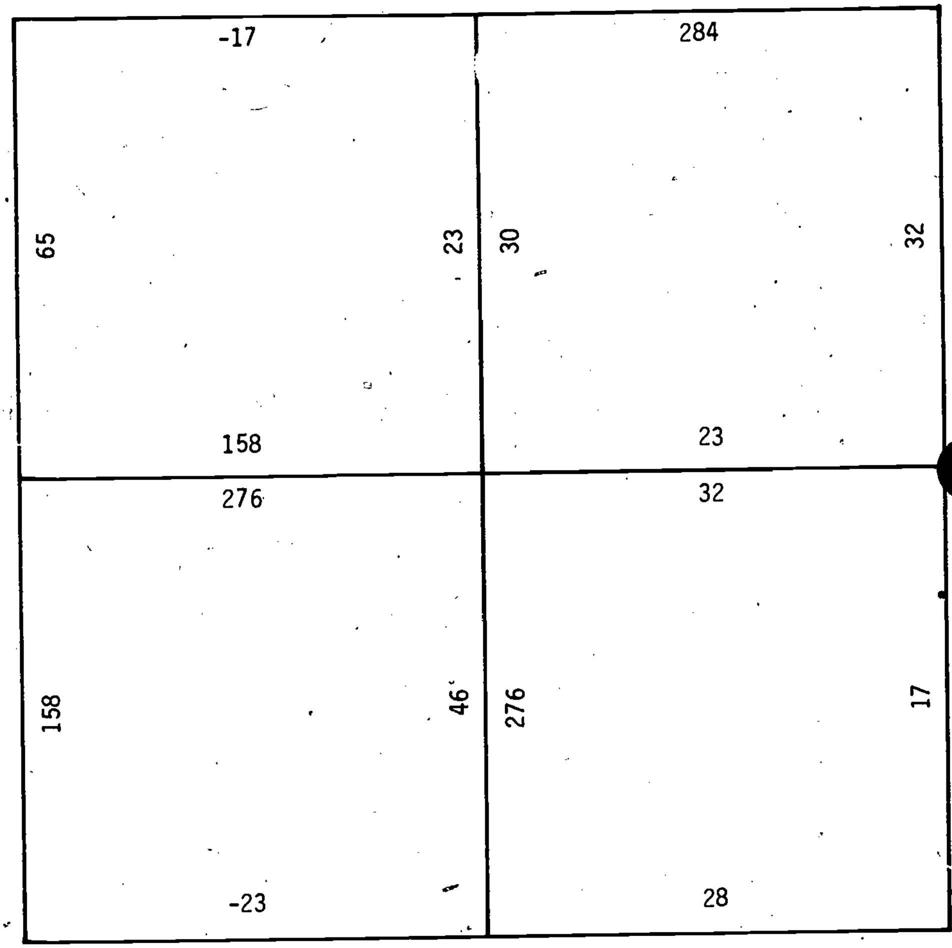
1. Read the article by Van Arsdel and Lesky. Using the article as a guide, construct your own Papy mini-computer.
2. Use your mini-computer to work the following examples:

   a) 376 + 289

   b) 533 - 278

3. Read the article by Whitney and compare the two mini-computers by writing brief answers to the following questions. How are they alike? How are they different? Which one would you prefer to use with a class? Why?

(8 + 92) + (8 + 176) (97 - 48) - 17 (29 - 17) + 18 88 - 65	0 - 17 (100 - 77) 96 - (48 - 17) (158 + 58) - 58
97 - 65 17 - 0 8 + (92 + 176) (29 - 18) + 17	100 + 176 100 - 54 100 + 58 65 - 88


-17
284
65
23
30
32
158
23
276
32
158
46
276
17
-23
28

ACTIVITY 24

NONSTANDARD ALGORITHMS

FOCUS:

In this activity you will have an opportunity to work and explain some examples using nonstandard algorithms for addition and subtraction.

DISCUSSION:

Sometimes children think of different algorithms for performing addition and subtraction. While they may not think of the algorithms presented for you in this activity, perhaps you will enjoy working through these and will encourage children to make up their own.

DIRECTIONS:

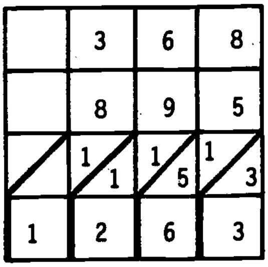
1. Working in groups of three, each person should select one of the three algorithms given on pages 120-121 of this unit. Make sure each algorithm is chosen by someone in the group.

   Read the discussion of the algorithm you have selected, use the algorithm to work the three examples provided, and answer any other questions associated with the algorithm. Each member of the group should then present the algorithm he (she) has selected to the other members in the group. The presentation should include an explanation of <u>how</u> the algorithm is performed, the <u>relation</u> of the algorithm to the standard algorithm, and an explanation of <u>why</u> it works.

2. Discuss: A pupil in your fifth-grade class is subtracting using a nonstandard algorithm. What would you do?

I. Lattice Method (Addition)

A. How It Works:



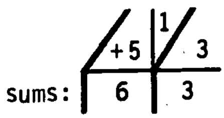
1. Original problem: 368 + 895. Each addend is written as usual.

2. Add:

   5 + 8 = 13
   9 + 6 = 15
   3 + 8 = 11

   Notice how each two-digit numeral (for example, 13) is written.

3. Beginning at the lower right-hand corner, add the number in each diagonal channel and record the sums as shown:



4. The sum is 1263.

B. Questions:

1. Work the following examples using the lattice method.

87	999	2638
+ 468	+ 899	1784
		+ 5386

2. How is this algorithm similar to the standard algorithm? How is it different?

3. Give a mathematical explanation of why it works.

II. "Balancing" Method (Addition)

A. How It Works:

A child solves the problem 209 + 357 by thinking... "I'm going to subtract 9 from 209 and add it to 357. My example then becomes 200 + 366 = 566."

B. Questions:

1. Work the following examples using the above method.

$$\begin{array}{r} 396 \\ +\ 315 \\ \hline \end{array} \qquad \begin{array}{r} 482 \\ +\ 304 \\ \hline \end{array} \qquad \begin{array}{r} 69 \\ +\ 35 \\ \hline \end{array}$$

2. How is this algorithm similar to the standard algorithm? How is it different? Does the above algorithm have any limitations?

3. Could this be used for subtraction? How? Work an example.

4. Give a mathematical explanation of why it works.

III. Austrian Method (Subtraction)

A. How It Works:

$$\begin{array}{r} 67 \\ -\ 39 \\ \hline \end{array}$$

1. In the example 67 - 39, 7 - 9 is not a whole number.

$$\begin{array}{r} 6\,{}^{1}7 \\ -\ \overset{4}{\cancel{3}}9 \\ \hline \end{array}$$

2. Increase 67 by 10, adding the ten to the units. Increase 39 by 10, adding the ten to the tens place.

$$\begin{array}{r} 6\,{}^{1}7 \\ -\ \overset{4}{\cancel{3}}9 \\ \hline 28 \end{array}$$

3. Perform the subtraction as shown.

B. Questions:

1. Work the following examples using the above method.

$$\begin{array}{r} 436 \\ -\ \ 89 \\ \hline \end{array} \qquad \begin{array}{r} 3007 \\ -\ 1782 \\ \hline \end{array} \qquad \begin{array}{r} 8382 \\ -\ \ 947 \\ \hline \end{array}$$

2. How is the algorithm similar to the standard algorithm? How is it different?

3. Give a mathematical explanation of why it works.

TEACHER TEASER

Three men registered at a hotel, paying ten dollars each for their room. The clerk, later realizing that the three rooms constituted a suite for which the charge was only twenty-five dollars, gave five dollars to the bellhop to refund to the guests. Since five dollars is not evenly divisible by three, as well as for other less subtle reasons, the bellhop kept two dollars for himself and returned only three dollars as a refund. On his way back he calculated as follows, "They each paid ten dollars making thirty dollars in all. I returned three dollars, or one dollar to each of them, so that they each really paid nine. Now three times nine is twenty-seven and two dollars I kept, making twenty-nine. Where is the thirtieth dollar?"

Explain the bellhop's dilemma.

ACTIVITY 25

"OPEN-ENDED" PROBLEMS

FOCUS:

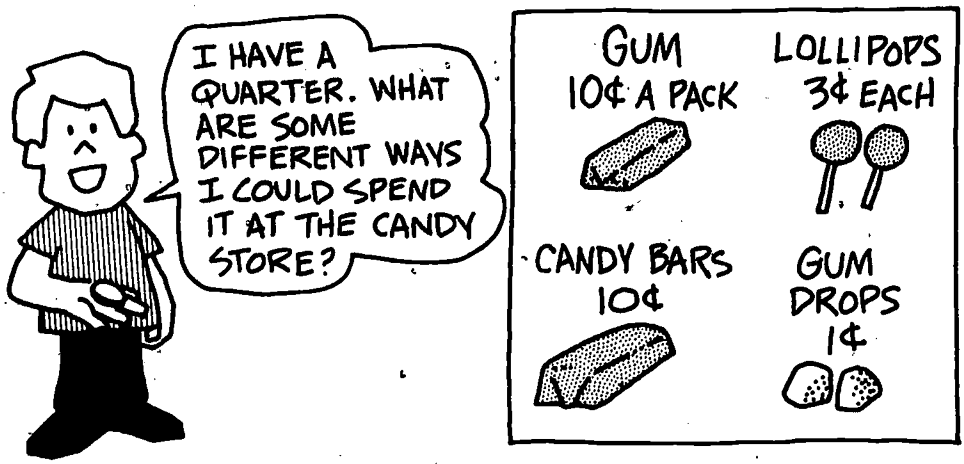
In this activity you will have a chance to discuss certain open-ended problems and their value for children.

MATERIALS:

Several sets of elementary mathematics textbooks (grades 3 through 6).

DISCUSSION:

One of the aims of elementary mathematics is to foster creativity in the child. Mathematics educators have suggested that certain "open-ended" problems may be one way of helping to achieve this. While "open-ended problem" has many interpretations, two simple examples follow which may give you a suggestion of the types of problems that you could use with children. In general, an "open-ended" problem is one in which there is more than one answer. In fact there may be several answers.



MAILGRAM
Mom and Dad, plane will arrive at 7:25 p.m. Can you meet me? Jerry

PLANNING CONSIDERATIONS

Drive to Airport .... 45 minutes
Park Car .... 10 minutes
Walk to Terminal .... 15 minutes

Jerry's parents plan to meet the plane. When should they leave?

DIRECTIONS:

(Do (1) and (2) as a homework assignment, (3) as a group discussion.)

1. Choose a textbook at a grade level (grades 3 - 6) of your own choosing. Study the verbal addition and subtraction problems in the text. Are any of the problems "open-ended"? List three typical addition and subtraction problems.

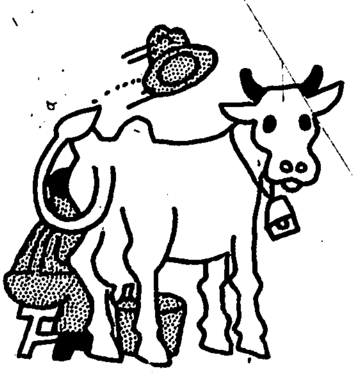
2. Write two open-ended problems which involve addition or subtraction or both. You may be able to adapt one of the problems you found in the text series. Provide solutions for the two problems you have constructed.

3. In your groups, discuss the characteristics of an open-ended problem by comparing your open-ended problems with those found in the text series. Share some of the open-ended problems you have constructed with other members of your group.

4. Working as a group, choose one of the problems written in the group. Formulate a plan to teach the problem to the class. The class members will pretend to be students at the level of the

problem--fifth grade, college, etc. The lesson should present the problem in such a way as to elicit various methods of solving the problem.

TEACHER TEASER



Mr. Smith has 3 pails. One holds 5 quarts, one 3 quarts, and one 8 quarts. The 8-quart pail is filled with milk. With the help of the 5-quart pail and the 3-quart pail, he divided the milk up into equal parts. How did he do it? (Have you got a minimum number of steps?)

ACTIVITY 26
TECHNIQUES FOR IMPROVING PROBLEM SOLVING

FOCUS:

In this activity you will have an opportunity to consider some techniques which can be used by a teacher to help children learn to solve problems.

MATERIALS:

Several elementary mathematics textbook series.

DISCUSSION:

Problem solving is perhaps the most important and yet most difficult area of teaching elementary school mathematics. A "best technique" has not been found for teaching problem solving, but several useful teaching techniques will be presented in this activity. Some of these techniques are illustrated in the pupil pages which follow; others are described in an article by Alan Riedesel.*

DIRECTIONS:

1. In each of the three pupil pages shown on pages 127-129 a different technique is used to present problem solving. Work out each problem as directed and analyze and describe the technique used.
2. Look through several textbook series and outline the problem-solving strategies used in the text. To share the work each member of a group of four may summarize the strategy of one series and describe it to the group. Be sure to record all the summaries.

*Riedesel, C. Alan. "Problem Solving: Some Suggestions from Research," Arithmetic Teacher (January, 1969), pp. 54-55.

PUPIL PAGE A

Name: ______________________________

For each problem:

a) Read the question.

b) Look at the information given.

c) Cross out the information you do not need.

d) Solve the problem.

1. How much did Dennis grow?

Height Last Year: $47\frac{1}{2}$ in.

Height This Year: 49 in.

Weight : 84 lbs.

Answer ________________

2. How many more points than Hank did Tom score in the game?

Hank: 8 points

Pete: 14 points

Tom : 21 points

Answer ________________

PUPIL PAGE B

Name: ____________________

In each problem one number or both numbers are missing. Choose a number for each box. Then solve the problem.

1. In your class there are ☐ boys and ☐ girls. Are there more boys or girls?

   ____

   How many more? ______

2. A book has 150 pages. You have read ☐ pages. How many more pages are left to read?

   ____

3. In her bank Vicky has 3 quarters and ☐ dimes. How much money is in her bank?

   ____

4. You have $1.00 and spend ☐ for a game. How much money do you have left?

   ____

150

PUPIL PAGE C

Name: ____________________

Name	Free Throws
Paul	28
Peter	40
Wally	34
Wilbur	37

Make up a problem to fit each sentence.

1. 40 - 37 = n ____________________

2. 40 + 28 = n ____________________

3. 50 - 34 = n ____________________

4. (37 + 34) - (40 + 28) = n ____________________

ACTIVITY 27

SEMINAR

FOCUS:

In this activity you will have an opportunity to review and discuss some aspects of the teaching and learning of addition and subtraction.

DIRECTIONS:

1. Spend approximately ten minutes in a group preparing some questions for class discussion. Your instructor may ask you to list the questions on the board or present them on cards.

2. If time permits discuss some of the following questions.

   a) What is the significance of the model shown below in teaching addition and subtraction?

      Real-world Problem

      Concrete Aids or Diagrams ⟷ Mathematical Symbols

   b) What is meant by the "getting ready to memorize" stage in the instructional development of the basic facts? What are some activities which might foster growth in that stage?

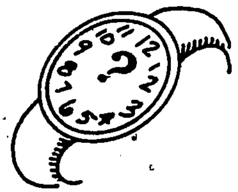
   c) What are the properties that hold for addition of whole numbers? For subtraction? How can these properties be used in the teaching of addition and subtraction to young children?

   d) What role do transitional algorithms have in the teaching of addition and subtraction of whole numbers?

e) How can problem-solving skills and computational skills be taught concurrently?

3. The handheld calculator is becoming popular. What effect do you think the calculator will have on the teaching of addition and subtraction? List some ways in which the calculator might be helpful; some ways it might be harmful.

TEACHER TEASER



Two Watches That Need Adjusting

Charley and Sam were to meet at the railroad station to make the 8 o'clock train. Charley thinks his watch is 25 minutes fast while in fact it is 10 minutes slow. Sam thinks his watch is 10 minutes slow, while in reality it has gained 5 minutes. Now, what is going to happen if both, relying on their timepieces, try to be at the station 5 minutes before the train leaves?

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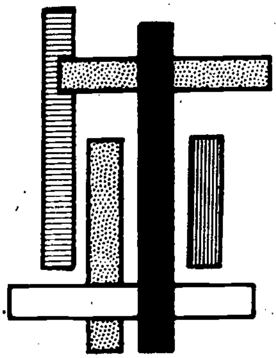
APPENDIX A

OVERVIEW OF CUISENAIRE RODS

(Following the descriptive overview is a worksheet with activities which you might use for some guided experimentation with the rods.)



The Cuisenaire materials were invented by George Cuisenaire, a school teacher in Belgium reportedly to help his own children learn mathematics. Chief exponent of the method has been Caleb Gattegno, a University of London professor of mathematics and eminent psychologist, who has demonstrated the materials in many countries, including Canada and the United States.

The Cuisenaire-Gattegno materials consist of a set of rods that combine color and length to embody algebraic principles and number relationships. Each of the rods is one square centimeter in cross section and they vary in length from one to ten centimeters. The color, which indicates the rod's length, is based on a scheme worked out during years of experimentation. There is a <u>blue-green family</u> (light green, dark green, blue) whose lengths are in the ratio 3:6:9 compared with unity; a <u>red family</u> (red, purple, brown) whose lengths are in the ratio 2:4:8 compared with unity; a <u>yellow family</u> (yellow, orange) whose lengths are in the ratio 5:10 compared with unity; a <u>black rod</u> (7) and the <u>white cube</u> (1).

The rods are designed to give children the opportunity to develop mathematical concepts such as addition, by letting them observe and discover the essential mathematical relationships. Gattegno suggests four stages for children working with the rods:

1) Creative construction and independent exploration

In this free play situation the child can construct a variety of objects from his environment including buildings, cars, people, staircases, etc. The purpose here is to allow the child to become familiar with the rods and to make discoveries which are meaningful to him. The amount of time given over to this stage will vary with the grade level and with the amount of previous experience.

2) Independent exploration and directed activities without written notation

During this stage children will become familiar with rod structures which are essential for developing mathematical relationships at a later stage. For example, "trains" (rods placed end to end) are constructed; this idea is essential for addition.

3) Introduction of written signs and symbols

During this stage the pupils record their discoveries in terms of color only. For example, Red + White = Green. This indicates that a train consisting of a red rod and a white rod is equivalent in length to a light green rod. Gattegno describes this as the "algebra" stage.

4) Assigning number values to the rods

In dealing with whole numbers it is usual to assign 1 to the white rod--the other colored rods then take values from 2 through 10 according to their length relationship with the white rod. In later work with fractions, other rods will be assigned the value 1. For example, if red is 1, white is $\frac{1}{2}$.

The annotated references on the following page provide a fuller description of the rods and also suggest further activities for using the rods in the elementary school.

Davidson, Jessica. *Using the Cuisenaire Rods--A Photo-Text Guide for Teachers*. New Rochelle, New York: Cuisenaire Company of America, 1969.

The book provides the ground rules for using the rods and an explanation of how they can be used to explain mathematical concepts. While the book does not provide lesson plans or activities that can be directly used in the classroom these are implied in many of the suggestions.

Davidson, Patricia. *Student Activity Cards for Cuisenaire Rods*. New Rochelle, New York: Cuisenaire Company of America, 1971.

Activity cards are presented for a variety of arithmetical topics such as whole numbers, fractions and geometry. The first set of activities is designed to introduce children to the rods.

Kunz, John. *Modern Mathematics Made Meaningful*. New Rochelle, New York: Cuisenaire Company of America, 1970.

The book provides explanations on how the rods can be used to explain a variety of mathematical concepts. Useful suggestions for using the rods with children are also presented. This book is essentially written for teachers and parents.

CUISENARIE RODS WORKSHEET

1. Construct a staircase using each color once and only once and identical staircase made entirely of white rods. How many white rods are equivalent in length to:

   a) a light green rod? ______________

   b) a black rod? ______________

   c) a blue rod? ______________

   Find a rod which is three times the length of the red rod. Name two other pairs of rods for which one rod is three times the length of the other.

2. The following code is used to identify the rods:

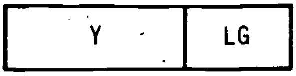
W	White	DG	Dark Green
R	Red	Bk	Black
LG	Light Green	Br	Brown
P	Purple	Bl	Blue
Y	Yellow	O	Orange

   When working with whole numbers it is most convenient to use the white rod as the unit. Given that W = 1, complete the following:

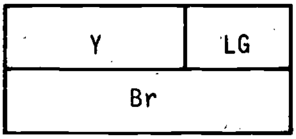
W _____	DG = _____
R _____	Bk = _____
LG = _____	Br = _____
P = _____	Bl = _____
Y = _____	O = _____

   Keep this table for future reference. It may be helpful to construct a colored diagram with the appropriate numerals attached.

3. Put a Y rod and a LG rod end to end as shown on the following page. This is called a train consisting of a Y rod and a LG rod. Trains provide the model for addition using the rods.



Find a single rod equivalent in length to this train. The result is indicated as Y + LG = Br. Numerically this represents $5 + 3 = 8$.



Find the following by making a train and then finding rod(s) equivalent to the train:

$$4 + 1 \quad \text{and} \quad 1 + 4$$

$$6 + 4 \quad \text{and} \quad 4 + 6$$

4. What pattern did you notice when each pair was reversed? What do we call this property?

State another property of addition of whole numbers. Try to exemplify it using the rods.

6. Referring to (3), describe a model for subtraction using the rods. Illustrate for the case $8 - 5 = 3$. How do the rods illustrate the inverse property of addition and subtraction?

7. Solve the following number sentences using the rods:

$$4 + \square = 7$$

$$11 = \square + 3$$

$$\square - 7 = 9$$

8. Construct all the different trains which are as long as the yellow rod. Write down all the addition and subtraction facts involving the number 5. How would activities such as this help children learn basic addition and subtraction facts?

9. Construct a real-world problem in addition and another in subtraction for which the rods provide an ideal means of solution. Present solutions to each of your problems using the rods.

10. How does the approach to addition and subtraction using the rods differ from the set approach?

APPENDIX B

THE PROPERTIES OF NUMBER SYSTEMS

DISCUSSION:

The properties of number systems are associated with the words:

- closure
- commutative
- associative
- identity
- inverse
- distributive

The different ways that these properties are used in teaching, learning, and doing mathematics loosely correspond to different stages in their historical evolution.

The concepts of the counting numbers, such as 1, 2, 3 and so on, arose naturally in the life of primitive humans. One would even have to imagine that the primitive hunter knew that killing one beast and then two more would yield the same amount of meat as killing two and then one more. It seems most doubtful, however, that the concept of commutativity was formally understood. Wouldn't you also guess that the hunter knew that if he killed no more, he had no more? We do know on the other hand that the formal idea of zero (or additive identity) was a long time coming.

As number systems have become more developed and more sophisticated, more properties have been observed and more have been needed. Typically, a property is observed to be true for one set of numbers and then it is used as a part of the definition of a new (usually larger) set of numbers.

In learning basic facts such as $2 + 3 = 5$ or $2 \times 3 = 6$, children can observe the commutativity of $+$ and $\times$. The facts that $a + b = b + a$ and $a \times b = b \times a$ cut the number of basic facts to be learned exactly in half. Whenever a child adds $2 + 3 + 4$, he must use some form of the associative law. Usually the child will add 2 and 3 and then 4, or 3 and 4 and then 2. In this case, the child is using the fact that $(a + b) + c = a + (b + c)$, even though a teacher might feel that making a point of the associative property would only confuse the child.

Another way the basic properties are useful is in explaining why certain algorithms work the way they do. For example, the multiplicative identity is used to explain why one inverts and multiplies when dividing fractions. In more advanced number work and for adult understanding, many of the rules and formulas that are taught to children intuitively and by analogy can be explained precisely in terms of the basic number properties.

A more sophisticated and probably much more important fact about the basic number properties is that they are the roots from which much of the power of abstract mathematics grows. Abstract systems that are the objects of study in mathematics are usually defined using properties that are selections from, or modifications of, the basic number properties. These systems help to uncover many similarities in apparently dissimilar situations, thus giving insight into the structure of phenomena in the universe.

On pages 144-147 we list each of the six basic number properties and provide:

a) a general statement of the property,

b) examples to convey the meaning of the property,

c) a statement of when it does and does not hold.

In this material we refer to four important sets of numbers: whole numbers, integers, rational num' -rs, and real numbers.

The whole numbers (sometimes called the nonnegative integers) are the numbers 0, 1, 2, 3, ... 151, ... 1796...

The integers consist of the whole numbers together with their negatives, e.g., ... -71, ... -15, ... -3, -2, -1, 0, 1, 2, 3, ...

The rational numbers include the integers, and they are the numbers which can be represented by the symbols $\frac{a}{b}$ where a and b are integers, $b \neq 0$. $\frac{-17}{8}$, $\frac{-1}{2}$, 0, 3, $\frac{291}{17}$ are all rational numbers. Rational numbers can also be represented by repeating and terminating decimals.

Real numbers are somewhat more difficult to characterize. The real numbers include the rational numbers, which can be represented by repeating or terminating decimals, and the irrational numbers, which can be represented by nonrepeating, nonterminating decimals. Examples of real numbers are $-\sqrt{71}$, -3, $-\sqrt{2}$, 0, 7, $\sqrt{176}$.

1. The Closure Property

a) This property has to be defined for a specific operation on a specific set of numbers. A set is closed under an operation if, whenever two numbers in the set are combined by the operation, a number in the set results.

b) The whole numbers are closed under + since the sum of any two whole numbers is a whole number. The whole numbers are not closed under ÷ since $2 \div 3$ is not a whole number even though 2 and 3 are.

c)

	+	-	x	÷
Whole numbers	Yes	No	Yes	No
Integers	Yes	Yes	Yes	No
Rational numbers	Yes	Yes	Yes	Yes*
Real numbers	Yes	Yes	Yes	Yes

The above table indicates which sets of numbers are closed under which operations.

2. The Commutative Property**

a) $a + b = b + a$ (Commutative property for addition)

$a \times b = b \times a$ (Commutative property for multiplication)

b) $7 + 16 = 23 = 16 + 7$

$39 \times 751 = 29{,}289 = 751 \times 39$

$\frac{-2}{3} + \frac{3}{5} = \frac{-1}{15} = \frac{3}{5} + (\frac{-2}{3})$

$3 \times (-2) = -6 = (-2) \times 3$

*Actually, the rational numbers excluding 0 are closed under ÷.

**The properties from 2 on are generally discussed only for + and x, since - and ÷ can be defined in terms of + and x and their inverses.

c) The commutative property holds for addition and multiplication on all sets of real numbers. The commutative property does not hold for subtraction or division except in very special cases.

3. The Associative Property

a) $(a + b) + c = a + (b + c)$
$(a \times b) \times c = a \times (b \times c)$

b) $(21 + 3) + 7 = 24 + 7 = 31$
$21 + (3 + 7) = 21 + 10 = 31$
$(-3 \times 2) \times 5 = -6 \times 5 = -30$
$(-3) \times (2 \times 5) = -3 \times 10 = -30$

c) Addition and multiplication are associative on all sets of numbers. (Subtraction and division are not associative on any sets of numbers that are of interest to us.)

4. The Identity Property

a) $0 + a = a + 0 = a$ for each a
(0 is called the additive identity)

$1 \times a = a \times 1 = a$ for each a
(1 is called the multiplicative identity)

b) $0 + 2391 = 2391 + 0 = 2391$
$1 \times \frac{3}{4} = \frac{3}{4} \times 1 = \frac{3}{4}$

c) 0 acts as additive identity for any set of numbers that contains 0, and 1 acts as multiplicative identity for any set of numbers that contains 1.

5. The Inverse Property

a) This property must be discussed relative to a set. A set is said to have the additive inverse property if for each a in the set, there is a b in the set so that
$a + b = b + a = 0$. The multiplicative inverse property

requires that, for each a $(\neq 0)$ in the set, there is a b in the set so that $a \times b = b \times a = 1$.

b) The integers have the additive inverse property, e.g., $3 + (-3) = (-3) + 3 = 0$, $(-71) + 71 = 71 + (-71) = 0$. The integers do not have the multiplicative inverse property since $3 \times b = 1$ cannot happen if b is an integer. The rational numbers do, however, have the multiplicative inverse property since, if b is a nonzero rational number, $\frac{1}{b}$ is a rational number and $b \times \frac{1}{b} = \frac{1}{b} \times b = 1$.

c)

	Additive Inverse Property	Multiplicative Inverse Property
Whole numbers	No	No
Integers	Yes	No
Rational numbers	Yes	Yes
Real numbers	Yes	Yes

6. The Distributive Property

a) $a \times (b + c) = (a \times b) + (a \times c)$

b) $$\begin{aligned} 7 \times (5 + -3) &= (7 \times 5) + (7 \times (-3)) \\ &= 35 + (-21) \\ &= 14 \end{aligned}$$

c) The distributive property holds on any set that is closed under + and x.

7. Summary Table

The table on the following page summarizes the applicability of the six basic number properties to the four number systems we have studied.

	closure		commutativity		associativity		identity		inverse		distributivity of x over +
	+	x	+	x	+	x	+	x	+	x	
Whole numbers	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No	Yes
Integers	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes
Rational numbers	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Real numbers	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

REQUIRED MATERIALS

ACTIVITY	AUDIO-VISUAL AND MANIPULATIVE AIDS	READINGS
1	Slide-tape: "Addition and Subtraction in the Elementary School," cassette recorder and projector. (Optional)	
2	Several first-grade textbooks (one or two per group of four students).	
3	Acetate transparencies, felt tipped pens, overhead projector, kindergarten and first-grade textbooks.	Payne, Joseph N., ed. Mathematics Learning in Early Childhood. NCTM Thirty-seventh Yearbook. Reston, Va.: NCTM, 1975. (Optional)
4	Elementary school textbooks (grades 1 and 2), including teachers' editions; counters, bundling sticks, or other set aids; Cuisenaire rods (optional).	
7		Weaver, J. Fred. "Some Factors Associated with Pupils' Performance Level on Simple Open Addition and Subtraction Sentences," Arithmetic Teacher (November, 1971). (Optional)

ACTIVITY	AUDIO-VISUAL AND MANIPULATIVE AIDS	READINGS
9	Elementary mathematics textbooks, grades K-3, (one series published in 1960's, one in 1970's).	
10	Elementary mathematics textbooks, grades K-6.	
15	Acetate transparencies, felt tipped pens, overhead projector. (Optional)	
17	Bundling sticks, Dienes blocks, abacus.	
18	Dienes blocks (one set of base two, base three, base four and base five for each group).	
20	Acetate transparencies, felt tipped pens; elementary mathematics textbook series, teaching aids as needed.	
21	Overhead projector.	

ACTIVITY	AUDIO-VISUAL AND MANIPULATIVE AIDS	READINGS
23	Deck of ten cards with numerals 0-9, scissors, tape.	Van Arsdel, J. and J. Lesky. "A Two-Dimensional Abacus--the Papy Mini-Computer," Arithmetic Teacher (October, 1972), pp. 445-451. (Optional) Whitney, H. "A Mini-Computer for Primary Schools," Mathematics Teaching, LII (Autumn, 1970), pp. 4-7. (Optional)
25	Several sets of elementary mathematics textbooks (grades 3-6).	
26	Several elementary mathematics textbook series.	Riedesel, C. Alan, "Problem Solving: Some Suggestions from Research," The Arithmetic Teacher. (January, 1969), pp. 54-55. (Optional)

Continued from inside front cover

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Transformational Geometry, Analysis of Shapes, Measurement, Graphs: The Picturing of Information, Number Theory, Probability and Statistics,* and *Experiences in Problem Solving.*

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