DOCUMENT RESUME

ED 220 267 SE 038 261

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TITLE Student Performance in Solving Linear Equations.
PUB DATE 82
NOTE 39p.; Paper presented at the Meeting of the International Group in the Psychology of Mathematics Education (Antwerp, Belgium, July 1982).

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS *Algebra; *Cognitive Processes; Educational Research; Equations (Mathematics); Mathematics Education; *Models; *Problem Solving; Research Needs; *Research Problems; Secondary Education; *Secondary School Mathematics
IDENTIFIERS *Mathematics Education Research

ABSTRACT

This document presents a first approximation for a model for student performance in solving linear equations. First, characteristics of a model of student performance for solving linear equations is discussed. It is noted that there is a need for some theoretical structure to support and organize the bits and pieces of information provided by available research. Next, known research on student performance for solving linear equations is surveyed. The material goes on to discuss modeling performance. A student performance model is presented, and research questions are suggested. The suggested questions are viewed to be ones that need to be addressed in order to elucidate the relevance of the model presented. A hope is expressed that future research will be part of a deeper study and analysis of student performance in solving linear equations. A thorough understanding of performance is seen as a possible aid to pupil improvement in algebra competencies. (MP)

STUDENT PERFORMANCE IN SOLVING LINEAR EQUATIONS[1]

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The purpose of the paper is to present a first apporximation for a model of student performance in solving linear equations. The paper is divided into several sections. First, characteristics of a model of student performance for solving linear equations will be discussed. Second, the extant research on student performance for solving linear equations will be surveyed. Third, modeling student perofrmance will be discusses. Fourth, the research will be related to the aspects of modeling performance. Fifth the model of student performance will be presented. Finally, research questions will be suggested which need to be addressed in order to elucidate the relevance of the model to understanding student performance in solving linear equations.

[1]Paper presented at the 1982 Meeting of the International Group in the Psychology of Mathematics Education, Antwerp, Belgium, July 1982.

Printed in the U.S.A.

Characterizing the Model

Much research attention has been paid to algebra in general, and equation solving in particular, since 1970. the research studies, however, which are pertinent to a model of student performance for solving linear equations are often based upon quite different perspectives, and employ a wide range of research techniques. The accumulated bits and pieces of information do not obviously fall together into a consistent whole. The creation of that whole requires that some theoretical structure be identified to support and organize the pieces. As a prelude to this and to the discussion of the research, we attempt to provide an abstract characterization of parts of that whole by citing aspects essential to any potential model of student performance for solving linear equations.

Foremost, we are considering the solving of linear equations in one unknown in the context of school mathematics. The task then is a mathematics problem solving task; given such an equation, one is to find its solution set or a numerical correspondent of the variable such that, if the variable is replaced by its numerical value and the computations are independently performed on each side, an identity results. School mathematics carries this task specification a step further to include the method of writing a sequence of equivalent equations. Except for the original equation, each equation in the sequence should follow from its antecedent by means of an acceptable algebraic-logical operation or process, with acceptability determined either implicitly or explicitly in terms of mathematical logical principles or skillful manipulations of proficient solvers such as teachers.

The reference to proficient solvers suggests that modeling of correct performance may be very important to the study of correct performance. A performance model must, therefore, include considerations of learning as students progress from one level of sophistication to another. At the same time a useful model should also provide a backdrop for organizing and discussing errors. As progress is made in this direction, it will become obvious that the school mathematics view of the task is inadequate, being too coarse in some ways to capture the thinking patterns of the solvers. Indeed, we must seek perspectives more refined in terms of the task and the solver. In particular, it may at times be necessary to consider equations merely as strings of symbols and to study the cognitive processes whereby solvers interpret or manipulate such symbols. This suggests the appropriateness of enhancing or supplementing the mathematical - school mathematics view of the equation solving task with constructs from related areas such as cognitive psychology, theories of human problem-solving, and performance models in artificial intelligence.

A model of student equations solving performance, then, should take into account a multitude of factors. It must acknowledge different levels of awareness on the part of human solvers, making a distinction between abilities to use various ideas, to express such ideas verbally, and to provide justification for their use. Closely tied to this is the existence of different types of knowledge: the mathematics knowledge which provides the basic characterization of the task, the perceptual and conceptual knowledge about notation and symbolism which gives mathematical interpretation to the latter, and knowledge of the appropriate use of acceptable operations and processes with the ability to measure progress toward the goal. Thus, the model should be a multi-level account ranging across (a) perception and interpretation of algebraic symbolism, (b) conceptual understanding of the problem-solving task, (c) application of intellectural operations and process, and (d) the development of strategies and general methods for solving any equation of a particular type.

Reviewing the Research

Historically, a popular approach for studying students' equation solving performance has been to administer group tests, to compute either average test scores or average success rates for each test item, and then to seek possible causes for common student errors (Bell, O'Brien, & Shive, 1980; Carpenter, Coburn, Reys, & Wilson, 1978; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Davis & Cooney, 1977; Hotz, 1918; Monroe, 1951a, 1915b; Reeve, 1926; Rugg & Clark, 1918). From these studies it is clear that the greater the number of steps needed to solve an equation, the less likely students are to solve the equation. Aspects of solution processes that have been cited as potential areas of difficulty include combining like terms, tranposing terms across the equals sign, clearing fractions, arithmetical computation, and understanding fundamental concepts such as "variable", "equation", or "equivalent equation".

Studies of Fundamental Concepts

The concept, "variable", is of course important in many contexts other than that of equations solving. To focus on the full range of that research would divert attention from the main thrust of this paper. Thorndike, Cobb, Orleans, Symonds, Wald, and Woodyard (1928), Davis (1975), Küchemann (1978), Tonnessen (1980), Rosnick (1981), and Jensen, Rachlin, and Wagner (Note 1) all provide useful information relative to the understanding of "variable" in algebra contexts.

More important for this paper are the concepts, "equation" and "equivalent equation". Thordike, et al., (1928) identified two abilities with respect to equations. The first was to solve the equation, which might mean to get a numerical answer, to solve for one variable in terms of the other, or to find the coefficients (e.g., find a and b in y = ax + b) given sufficiently many x, y pairs. The second was to understand the equation as an expression of a certain relationship; that is, to understand that equality is a relation and neither an operation nor an indicator that something is to be produced. It has only been fairly recently that any research seems to have been done on this second ability.

Of central importance is that students understand that the equals sign is a relation. Davis (1975) stated this, and Matz (Note 2) stated a similar view in talking about the equals sign as a constraint. Kieran (1980) pointed out that the preponderance of evidence is that elementary school students view the equals sign as a signal to write down an answer. Hence, □ = 3 + 4 is backwards. To expand that understanding to include the relational aspects of equality, she organized instruction in three steps.

First, students (6 seventh and eighth graders) were asked to write down true number sentences with more than one operation on each side of the equals sign; e.g., $3 \times 5 + 1 = 2 \times 2 + 12$. (Students tended to evaluate from left to right, without using the standard order of operations.) Second, one number was hidden (first with a finger, then with a "box," and finally with a letter) to generate equations while keeping the corresponding true number sentence always retrievable. Third, the rule, "What you do to one side you have to do to the other," was generated through work with number sentences. For example, from $2 \times 5 = 10$, the sentence $2 \times 5 + 7 = 10 + 7$ was generated.

The other part of "equation" that seems critical is the idea of equivalence of equations. (Two equations are equivalent if the domains of the variables are identical and the solutions are also identical.) Wagner (1981) asked students if the equations $7 \times W + 22 = 109$ and $7 \times N + 22 = 109$ had the same solution. She inferred that "conservation of equation" existed if the response was "Yes." If the response was "No, W is larger since W is later in the alphabet," she inferred that conservation of equation was absent. Students who said that the equations had to be solved to know were classified as transitional. About 50% of 12-year-olds and about 80% of 14- and 17-year-olds conserved. There was also a significant correlation ($p < .05$) between conserving and having had algebra, though it is not clear whether there was age confounding in this analysis. Kieran (1979) also reported that students may have the misconception that the solution to an equation changes if the letter used for the variable changes, but she did not measure conservation of equation directly.

Herscovics and Kieran (1980) in an extension of their thinking claimed that "undoing" the equation (that is, apply inverse operations in the reverse order) "brings to the concept of equivalent equations a dynamic flavor that is lost in a formal definition" (p. 579). No data were presented in support of this. Given the extent to which this technique is used in algebra texts and the difficulties that students seem to have with generation of equivalent equations, however, one suspects that "undoing is not as effective as Herscovics and Kieran suggest.

Kieran (1980) stated that understanding equivalent equations seems essential if the steps in equation solving are to be understood. She noted that step three in the following sequence seems to be a bookkeeping use of the equals sign. Possibly the symbols have personal meaning for the learner.

$$2x + 3 = 5 + x$$

$$2x + 3 - 3 = 5 + x - 3$$

$$2x = 5 + x - x - 3 \qquad \text{[clearly not equivalent]}$$

$$2x - x = 5 - 3$$

$$x = 2$$

Similarly, the next sequence is also not uncommon.

$$y + 5 = 8$$
$$= 8 - 5$$
$$= 3$$

The equals sign serves as a link between steps, but equivalent equations in the mathematical sense are not generated.

Bright and Harvey (1982) in a review of literature on equivalent equations concluded that students do not seem to know when equations are equivalent. Perhaps in light of the information on the use of the equals sign, this should be rephrased as students seem not to be concerned whether the equations they write are equivalent. Depending on the role of the equals sign, this lack of concern may from the student's perspective be appropriate.

Thus, the research indicates that students do not necessarily develop the meaning of equality intended by mathematicians. Students' interpretations may affect performance, at least as interpreted by mathematics teachers, and any model of student performance must take into account the possibilities of misinterpretation.

Studies of Students Errors

Studies of erroneous processes are typically based on an analysis of appropriate procedures to use in solving equations. Swain (1962), Romberg (1975), and Matz (Note 2) have described appropriate procedures as manipulation/reduction, sentential transformations, and deductions/reductions, respectively.

Bundy (Note 3) and Bundy and Welham (Note 4) in developing a computer program to solve equations identified three phases based on algebraic axioms and principles. Isolation is performed if there is a single occurence of x;

for example, if $3x = 12$ then the computer program divides both sides by 3 to produce $x = 4$. Collection is used if the number of occurences of the variable can be reduced; for example, $7x + (-3x)$ can be replaced by $4x$. Attraction is the procedure used to get instances of the variable closer together; for example,

$$12x + 7 = 4x - 1$$

$$12x + 7 + (-4x) = -1$$

and

$$12x + 7 + (-4x) = -1$$

$$12x + (-4x) + 7 = -1$$

would both be illustrations of attraction. This process may reflect what students think as they solve linear equations, but it also may be too formal (mathematical) to be an accurate representation. Heller and Greeno (1979) pointed out that knowledge of Bundy's three operations is not sufficient for solving equations. There must also be a higher-order strategy for choosing which operator to apply. There must be some guiding process.

Byers and Herscovics (1977) also pointed to the variety of guiding processes that students might bring to equation solving, but phrased their discussion in terms of "understanding." Four kinds of understanding were identified: (a) instrumental, in which rules are applied without knowing why, (b) relational, in which specific rules for a particular problem are derived from more general rules, (c) intuitive, in which the problem is solved based on some prior analysis, and (d) formal, in which the symbolism and notation are connected to relevant mathematical ideas to get a deductive chain. In solving the linear equation $x + 3 = 7$, students might exhibit the four levels by (a) transposing the number and changing the sign, (b) adding -3 to (or subtracting 3 from) both sides, (c) guessing, or (d) generating the string

$$x + 3 = 7$$
$$x + 3 + (-3) = 7 + (-3)$$
$$x + 0 = 4$$
$$x = 4$$

Carry, Lewis, and Bernard (Note 5) and Lewis (1981) studied the way college students solved various equations. Their work was influenced by Bundy (Note 3) and Bundy and Welham (Note 4), and although they didn't directly analyze levels of understanding, their data could be used to investigate these levels. Figure 1 shows the 14 equations that were presented to 34 college students and five research mathematicians. (The research mathematicians were called experts.)

INSERT FIGURE 1 ABOUT HERE

Each college students subject was videotaped twice. Seven equations were presented in the first session and each subject was asked to "think aloud." For the second set of seven equations presented in the second session, each subject was asked to explain the method of solution as if to a student asking for help on homework. For problem 2B, there were several differences in choices of strategies among the subjects. In particular, the experts were sometimes much more consistent in their choice of strategies. (See Table 1.)

INSERT TABLE 1 ABOUT HERE

Yet, for equation 2A, designed to be analogous to 2B, a consistent strategy was not used even among experts, and for equation 5A there was notable consistency across all four groups. (See Table 2).

INSERT TABLE 2 ABOUT HERE

Of perhaps equal interest in these data were the categories of errors identified among solution attempts. These include cancellation errors of several types (e.g., $\frac{x}{2 + x}$ becomes $\frac{1}{2}$, $x^2 - x$ becomes x): transposion errors (e.g., $7x + 8 = x + 2$ becomes $8x + 8 = 2$), combination errors (e.g., $\frac{x}{1} + \frac{x + 1}{2}$ becomes $\frac{x + x + 1}{2}$), cross-multiplication errors (e.g., $\frac{1}{x} + \frac{1}{7}$ becomes $7 + x$), splitting-equation errors (e.g., $\frac{5}{10} = \frac{x - 10}{x + 5}$ becomes $5 - x = 10$ and $10 = x + 5$), lack of inversion (e.g, $\frac{1}{x} = \frac{a}{b}$ but not $\frac{x}{1} = \frac{b}{a}$), lack of clearing fractions (e.g., $\frac{2x + 3}{x^2} = 1$ but not $2x + 3 = 1 \cdot x^2$), lack of distributivity (e.g., $ax + bx = c$ but not $(a + b)x = c$), dead ends (e.g., $p = A - prt$ for equation 1A), fraction errors (e.g., $\frac{2}{x}$ becomes $2x$), grouping errors (e.g., $\frac{x + 2(x+2)}{x + 2}$ becomes $\frac{(x+2)(x+2)}{(x+2)}$), and distributivity errors (e.g., $2(x+1)$ becomes $2x + 1$).

Some of the errors related to fractions have correspondence in work with common fractions (e.g., Bright & Harvey, 1982). Some of the cancellation errors (e.g., $x^2 - x$ becomes x) may be language related (e.g., Davis & McKnight, Note 6), and the splitting-equations error may be an overgeneralization of other equation solving techniques (e.g., Matz, Note 2). Jensen, Rachlin, and Wagner (Note 1), noted that students seem rule-bound; this behavior may reflect inadequate repertoires of operators to apply or lack of recognition of features to which to apply operators.

In summary, Carry, et. al., put all the errors into three types: (a) operator, reflecting incorrect or incomplete knowledge, (b) applicability, which was mostly mishandling of grouping and (c) execution, which included partial executions, misreading, and miscopying. The first two types seem amenable to correction by instruction. The third may not be easily remediated.

Lewis (1981) noted that the experts also made errors (e.g., transposition confusion of numerator and denominator, incorrect cancellations) similar to those of college students, though at a lower rate. Many of these errors seemed to occur when more than one operation was done at once. Thus, some of the errors may have been careless. Yet it seems important that the errors were of the same kinds as those made by the college students.

Davis and Cooney (1977) also categorized errors made in solving linear equations, but the errors were from written records only; there were neither videotape records nor "thinking aloud" records to supplement the written work. Data were gathered from 72 regular algebra I students and 38 second-year basic algebra (algebra I in two years) students. The categories of errors were (a) mistake in addition of real numbers either as numbers or as coefficients of x, (b) mistakes in multiplication of real numbers, (c) transposing errors (similar to strategy difficulities discussed earlier) either for addition or multiplication, (d) confusion about additive or multiplicative inverses, (e) incomplete work (similar to operation gaps discussed earlier), (f) miscopying, (g) combination errors (e.g., $-4 + 8x = 4x$), and (h) undecipherable. These errors and those identified by Carry, et. al., seem quite consistent. Too, the students made many computational errors, and there seemed to be no difference in the distribution of errors between the two kinds of algebra students. This reinforces the observation of Lewis (1981) that experts and college students made similar errors. However, the distribution of errors of those students who solved ten or eleven of the equations correctly indicated mostly (75%) computational errors rather than process errors (16%), while the errors of those students who solved two to seven equations correctly were less (50%) computational and more (38%) related to processes for solving.

Numerous researchers have pointed out errors in equation solving. Monroe (1915a) noted arithmetic errors, copying errors, and incomplete solutions, Rugg and Clark (1918) noted arithmetic errors, combination errors (e.g., $4c - 6c$ becomes $2c$, or $3x + 4$ becomes $7x$), incomplete solutions, transposition errors, and inverted divisions (e.g., $5x = 13$ becomes $x = \frac{5}{13}$). Reeve (1926) noted combination errors and mixed operation errors (e.g., $\frac{1}{3}y = 3$ becomes $y = 1$). Algebraic errors noted by Matz (note 2) that may interfere with linear equation solving included (a,b,c,d may be numbers or algebraic expressions) (a) $\frac{a}{b} + \frac{c}{d}$ becomes $\frac{a + c}{b + d}$, (b) $\frac{a}{b + c}$ becomes $\frac{a}{b} + \frac{a}{c}$, and (c) $\frac{a}{b} + \frac{c}{d}$ becomes $ad + bc$. These seem not to be unique to algebra and may be extensions of arithmetic errors (Bright & Harvey, 1982). Gerace and Mestre (Note 7) noted similar kinds of errors in bilingual students.

Meyerson (1976) observed that typical remediation of errors like $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$ often takes the form of either substituting numbers for a, b, c, d to generate a false statement or deriving the true relationship ($\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$) algebraically (or some combintaion of these two). He claimed that this technique is based on two assumptions; first, that the pupil's belief in the mistake is not strong, and second, the error is more random than systematic. Meyerson noted that if one speculates as to why students use incorrect rules, then different and perhaps more effective remediation techniques might result. For example, $\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$ may be derived from an overgeneralized multiplication of fractions rule, or it may be an overgeneralized 'baseball addition' rule. That is, if a batter has 3 hits in 5 attempts on Monday and 1 hit in 2 attempts then the cumulative record is 4 hits in 7 attempts ($\frac{3}{5} + \frac{1}{2} = \frac{4}{7}$). In either case, the incorrect rule is frequently reinforced within the domain that it originated. Remediation, therefore may require careful reanalysis of the mutual interference among mathematical rules and 'everyday' mathematics and may not be accomplished simply.

Davis, Jockusch, and McKnight (1978) used the term, binary confusion, to denote the interference between two rules. (See Figure 2.)

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INSERT FIGURE 2 ABOUT HERE

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If the $S_1 \rightarrow P_1$ chain is learned earlier and well, and if both the stimuli S_1 and S_2 and the products P_1 and P_2 are similar, then the student may generate the incorrect chain $S_2 \rightarrow P_1$. Shevarev (1946) in discussin this same example suggested that the incorrect chain $S_2 \rightarrow P_1$ seemed to be learned at the time of instruction on $S_2 \rightarrow P_2$ because the students were already oriented toward the addition of exponents ($S_1 \rightarrow P_1$).

Occasionally, interference may arise from non-mathematics sources. Kieran (Note 8) observed that junior high school students seemed to perform multiple arithmetic operations from left to right; for example, $3 + 4 \times 5$ is 35 rather than 23. Perhaps this is interference from reading instruction, reinforced by use of simple calculators with 'left-to-right' orientation. If students do generate rules like this before beginning algebra, because of the absence of instruction to the contrarv, then it mav be very difficult to overcome the student's belief in the incorrect rule.

The processes and errors presented in this section suggest several conclusions. First, errors are assumed not to 'e andom, but they also may not be effectively algorithmic. Errors may be i terpretable as overgeneralizations of rules to domains which are inappropriate, but the cause of this overgeneralization may be lack of attention by the teacher to specifying the limits on rules. To assume like Davis and McKnight (1979) that students spontaneously, and perhaps unconsciously, search for 'deeper-level rules' may be a stretch of the information processing view of the world to unreasonable limits. Students may apply learned rules whenever there is not a prohibition to refrain.

Second, the possible interference among concepts or rules should be dealt with directly. Probably this means that a teacher should identify explicitly at least some of the possible ways that the concept or rule being taught is <u>not</u> an instance of earlier-learned concepts or rules.

Third, flexibility in approach, suggested by the lack of consistent use of a single process for solving a given equation (Lewis, 1981) may be the best goal for instruction on equation solving. Explicit attention should be given to helping students recognize what might cause a failure to reach solution. Carry, et al., (Note 5) classified such a wide variety of causes of failure that it is unreasonable to expect instruction to deal with them all. The degree to which instruction deals with equation solving as an algorithm process versus a problem solving process may affect flexibility.

Fourth, the similarity of error categories across a variety of studies suggests that convergence may be occurring regarding an appropriate categorization of these errors. The need now seems to be for some conceptual framework within whch to explain those errors. This framework should bring into relief the potential interplay among not only the concepts, principles, and procedures used in teaching equation solving but also the perceptions of those concepts, principles, and procedures.

Modeling Student Performance

We are not using the term model in the sense of computer simulation. Rather, our model will be a verbal description which tries to address the major features of performance in relation to the task, to the school mathematics learning situation, and to some elements of cognitive psychology. Our purposes is to represent student equation solving performances in such a way that our model can be used to guide curriculum and instruction developments as well as continuing research efforts. Although much will be based upon our understanding of existing research, the model has not been validated. We hope that this presentation will lead to such efforts.

The proposed model is multi-level ranging from perceptual interpreation of the symbolic stimuli used to express an equation, to higher-level, more abstract concepts and their associated cognitive processes. We introduce the terms near features to refer to those aspects which are situation specific and closely tied to the surface symbolism expressing a given equation, and remote features for those qualities which exist only after abstraction, association, or interpretation has occurred. When used in a relative sense, remote versus near, the former suggests greater depth of intellectual processing with links to more extensive and general knowledge of equation solving as a problem solving task. For example, "being solvable by means of an 'undoing' strategy" is a more remote feature of the equation

$$3x + 5 = 11$$

than "having only a single occurence of x", since 'undoing' requires relating several aspects of the equation.

Again regarding terminology, we make a distinction between our use of the terms process and operation. For the latter we refer to Berlyne's (1965) model for directed thinking alternating situational and transformational thoughts. The transformations of Berlyne are derived from observations of one kind of stimulus situation being systematically replaced by another kind of stimulus situation. Operation will be used when the action taken by the solver is guided by a single transformational thought or, at least by relatively few transformational thoughts. Intuitively, we are trying to capture the simplest type of alteration that solvers might make as they write one step following another.

A look at the typical learning situation should help califrfy the notion of an algebraic operation. Such operations are often derived from axioms, definitions, or theorems. Instructors typically identify the proposition upon which an operation is based, and then illustrate the actions and/or applicability of the operation. Corresponding to the

Symmetric Property of Equality

If $a = b$, then $b = a$.

for example, there is the algebraic operation changing

from $a = b$ to $b = a$

which might appropriately be applied to

$$11 = 3x + 5$$

to obtain

$$3x + 5 = 11$$

with the single occurrence of the unknown now to the left of the equals sign. After a few such examples the student solver is typically expected to know and to be able to begin using this operation.

The above example shows a single operation derived from one algebraic principle. It is often the case, however, that two distinct operations will be so derived. This is the case, for example, with the

Definition of Subtraction

$$a - b = a + (-b).$$

One operation is in changing from

$$a - b \quad \text{to} \quad a + (-b)$$

as in changing

$$\text{from} \quad 4 - 3(x+2) \quad \text{to} \quad 4 + -[3(x+2)]$$

and the other operation is in changing from

$$a + (-b) \quad \text{to} \quad a - b$$

as in changing

$$\text{from } 7x + -(3x) \quad \text{to} \quad 7x - 3x.$$

(See Bernard, 1978 for more examples and discussion.)

From the preceding examples it is, perhaps, evident that operations might themselves be cognitiviely complex since the stimuli bounding an intervening transformational thought are often complex. Lack of understanding of such complexities might account for student difficulities with both learning and using legitmate operations. Because of the preponderance of operator related errors (Carry, et. al., Note 5), we suggest that researchers consider this possibility for determining the nature and causes of such errors. For our purposes, however, rather than analyzing operations into subunits, it will be sufficient to treat each as a whole.

Process, on the other hand, will be used to denote a connected sequence of operations. Here we look for episodes of performance which have an integrity of their own. Such sequences are often aimed at attaining strategic subgoals like "removing parentheses" or "getting the x's to one side." A process may be totally determined by an algorithm; that is, the specific operations as well as the order of those operations are given. However, we allow for cases where decisions are made selecting applicable operations.

To avoid semantic issues, we also allow a process to be based upon a single operation. Furthermore, this convention recognizes an important learning phenomena. Aside from automaticity which speeds up the execution of a sequence of operations, students frequently learn to collapse a familiar sequence of several operations into one of fewer operations; the operations in the new sequence being, in part or totally, distinct from those in the original sequence. Frequent use of a process which might have been developed in association with remote features can lead to the derivation of new transformational thoughts as indicated by Berlyne (1965). The process, seen in a situation specific context, might be re-defined in terms of near freatures. Typical of this is the development of a transposition process, "moving a term from one side of an equation to the other with a change in sign."

In the early stages of learning about equation solving, students might see a sequence such as

(a) $9x = 7 \underline{+ 4x}$

$9x - 4x = 7 + 4x - 4x$

$9x - 4x = 7 + 0$

(b) $9x \underline{- 4x} = 7$

The underlining is to draw attention to the features which systematically change in this and other examples like it, leading to development of a transposition operation going directly from a) to b). This is not necessarily the fully developed transposition operation stated in the preceding paragraph, but it might become so either through discovery in connection with other specific cases or through discussion between the instructor and the learner of this instance at the more general level of the complete transposition operation. Thus, this process sequence might collapse to a single operation changing from

$$9x = 7 + 4x$$

to

$$9x - 4x = 7.$$

Relating the Research to the Context of the Model

Understanding fundamental concepts, like "variable" and "equation", is part of learning the remote features of equation solving. For example, without an understanding of what equivalent equations are, students might tend to learn the procedures for solving linear equations in a rote way; that is, understanding might help both to organize the procedures and to provide a rationale for selecting appropriate procedures to apply. The remote features seem likely to be cued by near features of a particular situation, but cognitive accessibility to remote features seems essential.

Kieran's instruction on the relational aspects of equality seems to be an attempt to use near features to develop remote features. The uniqueness of her instruction may be that the interplay between near and remote features is always highlighted. The more typical textbook approach of substituting values for variables to obtain true number sentences does not seem to have that dynamic quality and may not adequately communicate to students either that there is an important relationship between near and remote features or that remote features are important to the task at hand.

Equivalence of equations in general, and conservation of equation in particular, also seem to depend on coordination of near and remote features. Students' errors in writing equivalent equations, for example, may reflect an incomplete grasp of the fact that remote features need to be accurately reflected in their work rather than a serious misconception about the concept of equivalence.

In studies of students errors in equation solving, there seems to be an emphasis on interpreting errors first in terms of procedures on near features, even though there may be relationships to underlying remote features. If students perceive these procedures as only related to near features, then instrumental understanding would seem to be the only possible understanding. Students might also be relatively unable to create processes out of operations. If these near features can be related to remote features, then the cognitive structures necessary for creation of processes would seem to be more accessible.

Students' choices of operations or processes to use are almost certainly tied both to the perception of near features and to the way that these perceptions cue remote features. When multiple near features are present, then students may focus on only one and may not even be aware of the multiple features. Understanding what prompts the focusing behavior may be important in expanding the scope of the proposed model. Instruction itself, for example, could significantly influence focusing behavior.

The information processing perspective on student behavior tends to create explanations of student errors in terms of operations and processes that apply essentially only to near features. While this approach may be effective in modeling students' behavior, its very success may focus attention on aspects of equation solving that do not suggest remediation techniques. More attention may need to be given to the interplay between near and remote features. This approach does suggest, however, that students can operate without relating near and remote features and calls into consideration the necessity for providing instruction that does relate near and remote features.

Overgeneralization errors and binary confusion errors seem to be explanations related to understanding remote features. Those who have categorized errors, however, seem not to have paid much attention to potential difference between error explanations set in the context of near features and those set in the context of remote features. There may not be any automatic carry-over of observations of near feature to structures residing in remote features.

The Model

Many of the basic constructs of the model are taken from information processing psychology and theories of problem solving. Readers familiar with Newell and Simon's work (1972) on simulating human problem solving performance, Greeno's discussion (1973) of the role of memory in problem solving, and Bundy's artifical intelligence approach (Note 3) to solving equations will recognize influence of these. There will also be a use of such basic information processing constructs as selective perception, memory (short-, intermediate-, and long-term), executive control mechanism, and response generator.

Intuitively it seem that when a solver is presented with an equation solving task, perception interacts with memory and expectations to focus the solver's attention on some feature of the given equation. For example, the solver may want to remove some perceived obstacle; e.g., decrease the number of symbols used; or attain a perceived subgoal; e.g., getting all the x'es on one side. But the solver also has some expectations concerning which of available process/operations to use. As the process is executed, the solver further checks the outcome against expectations of appropriateness and must decide whether to continue or to reassess the approach. Reassessment may lead to alterations in expectations, choice of process, or even the desired outcome, but the basic cycle is as indicated above.

Detail needs to be added to the above outline in areas of the selection of features, identification of subgoals, choice of proceses and operations, monitoring of progress, and the direction or re-direction of the solver's efforts. Central to this is an assumed hierarchical system of levels of control for the solver's observable behavior. Figure 3 shows this system as it might relate to solving the equation

$$x + 3(x+5) = 27$$

at the point of writing $3x$ in the step

$$x + 3x + 15 = 27.$$

The branches and question marks are used in two different ways. First, the activity chosen at a particular level may have been one of several competing alternatives. For example, the question mark to the left of the METHOD level might indicate a "Guess and Test" approach. Second, the activity may be one of a series of alternatives, all of which are to be used. At the STRATEGY level, for example, "Removing Parentheses" may simply be one of a sequence of phases that the solver consciously plans to execute in the specified order. Processing limitations being what the are, the solver must keep the others on hold while executing the current phase.

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INSERT FIGURE 3 ABOUT HERE

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The PROCESS and OPERATION levels are of course different, but there may be blending of process and operation levels. A specific process may be composed of a single operation, or a process may be expanded into phases or sub-processes before reaching the operation level. While the strategy level is indicative of the more problematic character of the task, either because the solver is faced with a novel situation or because of memory difficulities trying to re-construct a solution sequence, the process level designates the more routinized aspects that come from successful experiences with similar tasks. Figure 3 at least in this way addresses the problem-versus-exercise issue so often mentioned in relation to skill development in school mathematics.

Relative to accessing and activating the control structure, the range of features, near to remote, is placed parallel to the control hierarchy as shown in Figure 4. This arrangement indicates that the perceived features act as cues accessing the different levels in the control hierarchy. Prior knowledge about solution acitivies (Methods, Strategies,..., Tactics) and expectations about the solution task operate in selecting relevant features from the equation. As features connected with a level are selected, corresponding controlled activities are determined. Features associated with higher levels (e.g. task) dominate influencing choices made at lower levels (e.g., operation). While the field of focus on features tends to narrow on a specific subexpression of the equation, control attention passes from higher to lower levels with each subsequent level containing an expansion or redefinition of the acitivities from the higher levels. Thus activated, the control structure is ready to guide the solution process.

- -

INSERT FIGURE 4 ABOUT HERE

- -

At any particular moment during the solution process, the solver's attention will be focued at some level in the hierarchy. Once activity is completed at that level, attention moves upward to the next higher level at which the solver must deal with a list of yet unfinished acitivty. Figure 5 presents a hypothetical illustration of the flow of attention with respect to time.

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INSERT FIGURE 5 ABOUT HERE

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The basic information processing notion of problem solving includes a loop of activity; identify a subgoal, select and apply processes, evaluate the outcome, and decide what to do next. Such a loop might be maintained at each level of the hierarchy, from the METHOD level on down. If a solver has competing choices at each level, this might not be unreasonable. While we allow for such horizontal looping it seems that the more common application of such a loop might be vertical; that is, monitoring and evaluation are used to guarantee that lower level activities satisfy the constraints set at higher levels.

It should be noted that our model is an attempt to capture the essense of the school mathematics approach to equation solving. It is not an attempt to simulate the performance of any particular individual solver. Thus, differences are to be expected between our model and observed individual behavior. Notably, not all student solvers will be able to exhibit the entire well-defined control structure of the model. Typically, in the early stages of instruction, only very simple equations like

$$x + 1 = 7,$$

$$3x = 6, \text{ and}$$

$$x - 2 = 9$$

are considered so that the fledgling solver can be made aware of both the nature of the solving task at one end of the control spectrum and some of the operations and tactics at the other end. The object of subsequent instruction is to help the learner fill in the intervening levels.

Little has been said explicitly about the role of memory in this model. The descriptors for the levels in the control hierarchy imply specific repertoires of knowledge and the means to access such knowledge. The hierarchy itself represents a knowledge structure which the proficient solver is assumed to be able to access from memory. The choice of Process was to acknowledge that larger procedure units rather than operations can be called from memory and executed without the necessity of interviewing evaluation.

The model certainly has implications for both reinterpreting the existing research and analyzing the school mathematics approach to equation solving. As the model continues to evolve at least some of these implications will become obvious. For the purposes of this paper, however, it seemes more important to discuss some of the research needed to amplify and verify the model.

Identifying Needed Research

The proposed model needs to be validated in the sense of determining whether it addresses aspects of equation solving performance that are in fact important in learning and teaching. The model, when it is more fully developed, should help provide understanding of both what should be done to encourage learning and what students actually do in a given instructional context.

The first area of concern is general focusing behavior of learners. Do they identify, or distinguish between, near and remote features? If so, how are remote features cued? Does the learner's perspective of the task affect perception of features? That is, are learners keeping in mind the goal of obtaining the solution, or has the task become the writing down of the expected steps? On the other hand, is there a point in the development of understanding and skill when a shift might take place to operations on near features with remote feature available to use when necessary?

Related to this concern is whether successful performers focus differently than unsuccessful performers. How can focusing behavior be measured? Certainly self-reporting techniques; e.g., thinking aloud; along with proding questions from an interviewer would seem to play an important role in measuring this behavior, but memory research techniques might also be quite useful. If focusing behavior does in fact distinguish between performance, then how can instruction be designed to foster appropriate focusing?

The second general area of concern is the adequacy of the control structure part of the model to explain or interpret performance. Earlier it was suggested first that initial instruction might simultaneously focus on the nature of the equation solving task and on simple processes and operations to accomplish this task and second that other parts of the control structure are filled in during later instruction. Is this suggestion accurate? Perhaps the development of the control structure depends on the extent to which instruction deals with equation solving as a problem solving task as opposed to an algorithmic task. For example, it is possible to teach linear equation solving as a set of mini-algorithms, each of which is applicable to a narrowly defined equation type. In this context the learner might view the task as an identification task of the particular equation type. Then the appropriate algorithm is to be applied. Alternately, equation solving can be taught as a set of procedures which apply to many "types" of equations. The learner must use some, perhaps, more problem-solving like techniques for selecting the procedures to apply to any particular equation.

In the latter case, the development of a stable control structure might be more important for successful performance. If so, the instruction should perhaps deal explicitly with possible control structures. In particular, it may be important to identify ways that processes and operations can be cued from near or remote features of an equation. That is, how are features and processes connected?

One speculation about the relation of near and remote features to performance is that successful solvers tend to perceive structure in the symbol system while unsuccessful solvers tend to view the symbol manipulations as isolated and unrelated actions. If this is true, then the control structure may be fundamental in determining the approach that students take. An incomplete control structure may only permit the less sophisticated view of manipulations.

Related to the development of the control structure may be the possible interference among operators. Such interference phenomena have general psychological interest, but they also may have specific relevance to understanding the control structure.

A third area of concern is the analysis and interpretation of errors. Are errors related to near features different than those related to remote features? What importance would such a difference have? If particular instruction is interpreted within the model, can errors be predicted? If so, then instruction could presumably be improved before students are exposed to it.

In a real sense the development of the model is predicated on the assumption that appropriate understanding of symbols and correct operations and processes either exist or can be developed in the learner's cognition. If this assumption is violated then it will probably be impossible to teach correct performance.

It is hoped that the presentations of this model of student performance in solving linear equations will promote deeper analysis and further study of that performance. Algebra is, and will probably continue to be, important in the development of mathematical competence. A thorough understanding of performance would seem to aid the improvement of this part of mathematical competence.

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Table 1

Strategies for Problem 2B

Group[a]	Choice of Stategy[b]		Choice of First Step[b]	
	Transpose -- Invert	Other	Transpose	Other
E	4 (80)	1 (20)	5 (100)	0 (0)
T	4 (40)	6 (60)	8 (80)	2 (20)
M	3 (21)	11 (79)	9 (64)	5 (36)
B	0 (0)	10 (100)	0 (0)	10 (100)

[a]E = experts (professional mathematicians)
T = top 10 college students
M = middle 14 college students
B = bottom 10 college students

[b]Entries are numbers (percentages) of subjects in each group.

adapted from Lewis (1981)

Table 2

Strategies for Problems 2A and 5A

Group[a]	Strategy for Problem 2A[b]		Operation for Problem 5A[b]		
	Transpose-Invert	Other	Cross-Multiply	Clear Fractions	Other
E	0 (0)	5 (100)	5 (100)	0 (0)	0 (0)
T	2 (20)	8 (80)	6 (60)	4 (40)	0 (0)
M	1 (7)	13 (93)	11 (79)	2 (14)	1 (7)
B	0 (0)	10 (100)	5 (50)	2 (20)	3 (30)

[a]E = experts (professional mathematicians)
T = top 10 college students
M = middle 14 college students
B = bottom 10 college students

[b]Entries are numbers (percentages) of subjects in each group.

adapted from Lewis (1981)

Figure 1

Equations Used by Carry, Lewis, and Bernard

1A $A = p + prt$, solve for p

1B $2x = x^2$

2A $\frac{1}{3} = \frac{1}{x} + \frac{1}{7}$

2B $\frac{1}{R} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, solve for x

3A $9(x+40) = 5(x+40)$

3B $7(4x-1) = 3(4x-1) + 4$

4A $xy + yz = 2y$, solve for x

4B $\frac{x + 3 + x}{x^2} = 1$

5A $\frac{5}{10} = \frac{x - 10}{x + 5}$

5B $\frac{1 - x^2}{1 - x} = 2$

6A $x + 2(x+1) = 4$

6B $x + 2(x+2(x+2)) = x$ [illegible] 2

7A $x - 2(x+1) = 14$

7B $6(x-2) - 3(4-2x) = x - 12$

adapted from Carry, Lewis, and Bernard (Note 6)

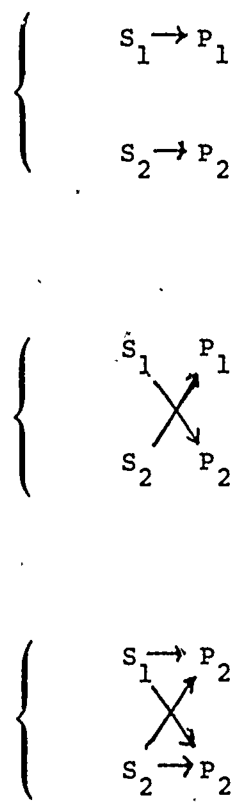
Figure 2

Binary Confusion

Original Learning[a]

Incorrect Associations

Binary Confusion Scheme

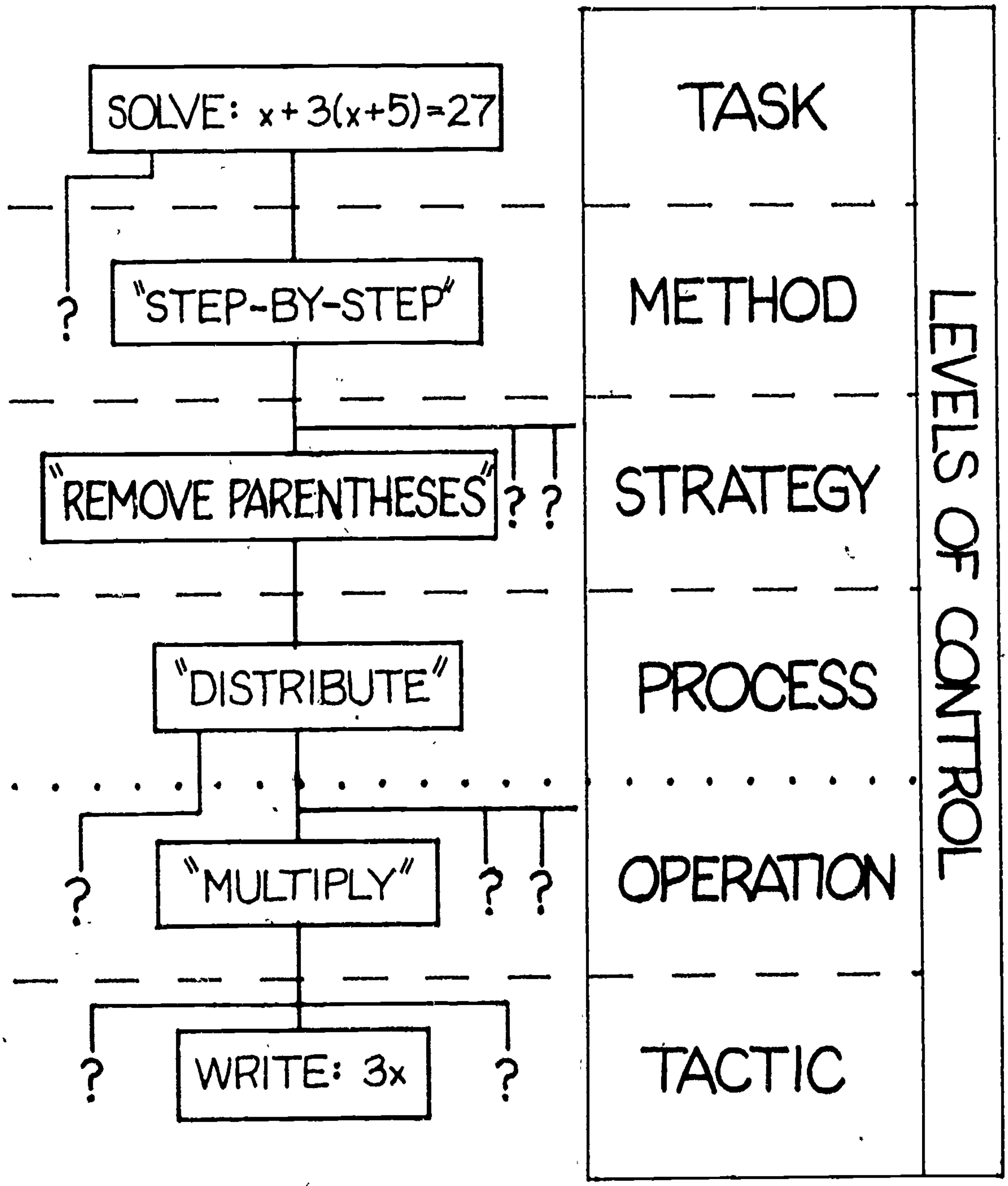


[a] S_i = stimulus i, P_i = product i

adapted from Davis, Jockusch, & McKnight (1978)

FIGURE 3

CONTROL HIERARCHY



AB

FIGURE 4

MODEL of EQUATION SOLVING PERFORMANCE

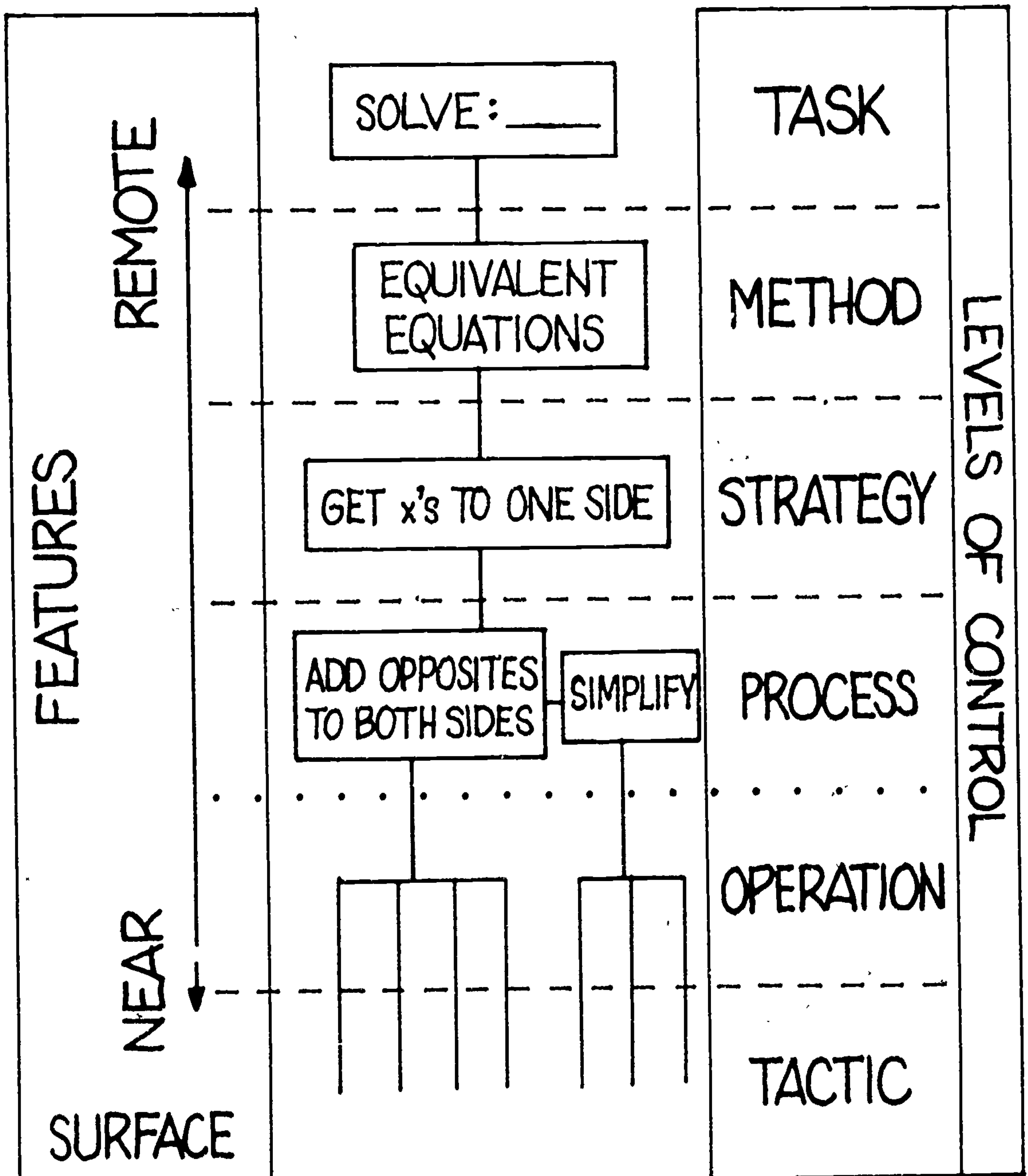
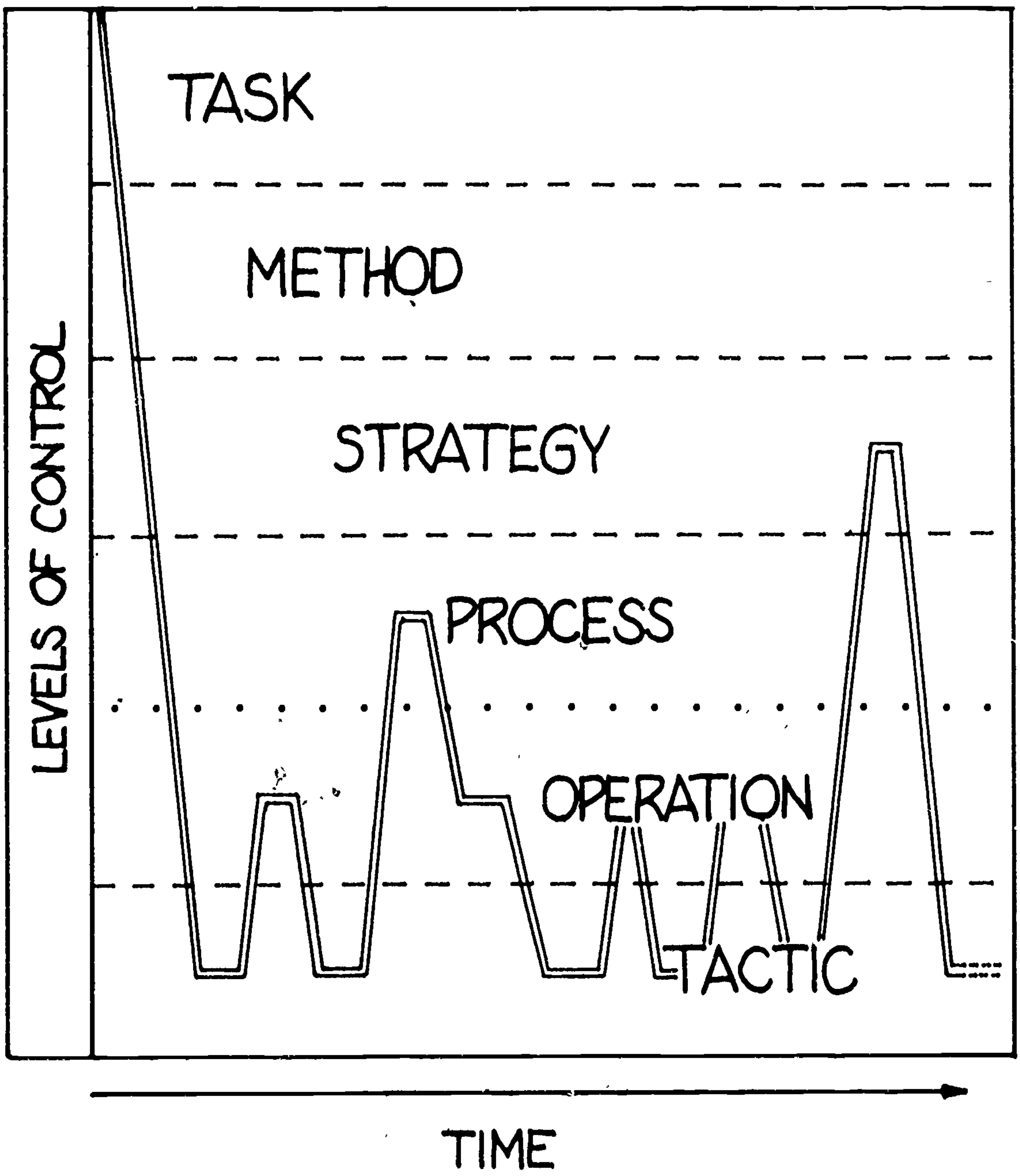


FIGURE 5
FLOW of ATTENTION



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