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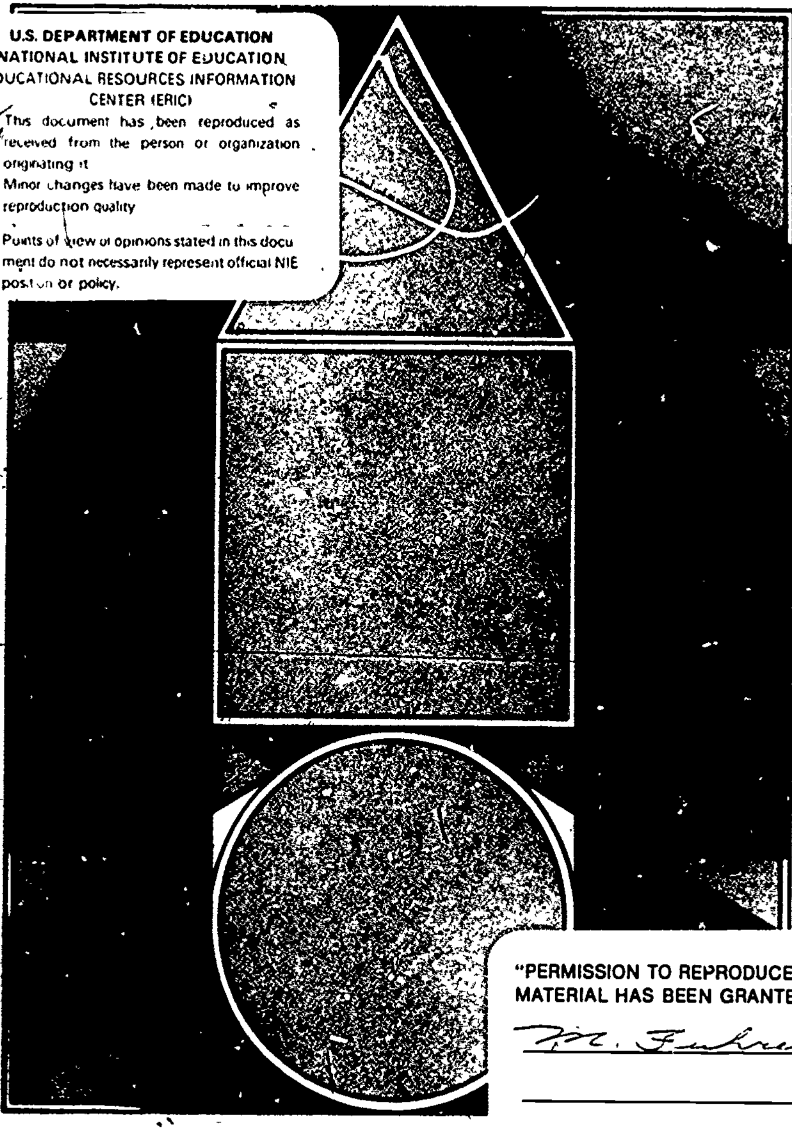
ABSTRACT

This material considers the questions of: 1) What is basic in mathematics? 2) What mathematics do all children need for their lives in a society that depends on technology? and 3) What do pupils need in order to go on and learn the mathematics they will need for access to jobs in business, industry, and careers in the professions? What mathematics is and why people should learn it, is explained. The new mathematics is placed in perspective, and things that can be done to help make mathematics accessible and acceptable to all, are discussed. (MP)

Teaching Mathematics: What Is Basic?

Stephen S. Willoughby



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Teaching Mathematics: What Is Basic?

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FOREWORD

The persistence in our language of that pedagogical trinity, the three Rs, bears witness to almost universal acknowledgment that mathematics is basic in education. Mathematics is one of the givens people accept whether they like it or not. Those whose knowledge of the subject is precarious accept it, as an article of struggling faith. At the same time, those who know a lot bewail the condition of mathematics in American education today.

Although no one questions the subject's importance, developments of the last 25 years have raised a good many questions about the teaching of mathematics in the nation's schools. The shock of Sputnik in the 1950s opened the way for the new math in the 1960s. When the new math proved to be still another shock, popular reaction resulted in a cry to take the subject back to basics. The schools responded by doing just that, and now what is basic is open to serious question. The question, moreover, is sharp and pointed in the light of recent reports that show how far behind others American schools are when it comes to teaching and learning mathematics.

What is basic in mathematics? What mathematics do all children need for their lives in a society that depends on technology? What do they need in order to go on and learn the mathematics they will need — more and more, as time goes on — for access to jobs in industry and business, to careers in the professions?

We put such questions to Stephen Willoughby, who speaks to them with the authority of long experience as a mathematician and teacher of mathematics. Now director of mathematics education at New York University, Dr. Willoughby began his teaching career in 1950 as a tutor in a Boston settlement house when he was a Harvard undergraduate. He has taught in elementary, junior high, and high school, and at the University of Wisconsin. At New York University since 1965 he has concentrated on "improving our understanding of how young children learn mathematics and on developing methods of helping them learn mathematics better and understand the usefulness of mathematics in solving problems in their daily lives and in other academic disciplines."

Dr. Willoughby's publications are extensive. He is now collaborating on a program designed to teach children to think and solve problems while becoming proficient with familiar mathematics.

Presuming that not a few readers of this paper would belong in the category, we asked him to be mindful of those whose knowledge of the subject is precarious.

James Howard
Director of Publications, CBE

CONTENTS

Introduction

An attractive young lady in the airlines waiting area looked over my shoulder and interrupted my train of thought.

"I see you're reading an article on Sherlock Holmes," she ventured.

The long, lonely flight to California began to look more interesting. I closed the magazine.

"Yes," I responded. "It's fascinating."

Only then did she catch a glimpse of the cover of the journal, on which was unmistakably printed, **The American Mathematical Monthly.** A horrified look came over her face as though I had made some sort of indecent proposal.

"Is that about mathematics?"

"Yes," I acknowledged, "but it's really very interesting. If you enjoy Arthur Conan Doyle's mysteries, I'm sure you'd like this. The mathematics is easy, and the mystery is quite good."

The young lady didn't waste a moment more. She picked up her bags and fled to the ticket counter, tossing back over her shoulder the remark, "I could never do mathematics."

When we boarded the jumbo jet, I noted ruefully that she had found a seat as far from me as possible — at a distance quite sufficient to keep her from being infected by what she clearly thought was my perverse preoccupation with mathematics.

Those of us who make our living doing mathematics or teaching it become hardened to this kind of reaction. We realize that in any social situation we are likely to be treated as though we might have a communicable disease. We wish it were not so. Like other mortals, we want to be well regarded. More important, the kind of alienation to which I refer impairs the quality of life of a majority of our citizens and thereby affects the quality of our civilization.

In the following discussion, I attempt to explain what mathematics is and why people should learn it. Trying, first, to put the controversial new math in perspective, I then go on to discuss some things that we can do to help make mathematics accessible and acceptable to all, while ensuring that people learn what is basic and important in mathematics.

I. What Is Mathematics?

Entire books have been written to answer the question, and any short answer must be incomplete at best. No one who has not actually done mathematics can fully appreciate its beauty and power, and therefore people who have not done real mathematics at some level have difficulty understanding what mathematics is, just as people who have not both heard music and produced music themselves would have difficulty fully understanding what music is.

Mathematics involves the manipulation of symbols and abstract concepts to produce information from given data in a form that is more useful in certain circumstances than the original data, even though the information was inherent in the data. Suppose, for example, there are 2 people in a room. Subsequently 4 more people enter the room, and then another 5 people. No one has left the room. By using mathematics we can establish that there are now 11 people in the room. The statement "there are now 11 people in the room" contains less information than the original data, but this conclusion was certainly inherent in the original data. The final statement nevertheless is of substantially greater use to anyone who wishes to serve dinner to all those in the room.

In a similar way, the Pythagorean theorem* is inherent in the postulates of Euclid, but the theorem is more useful than the entire list of postulates if you happen to know that the two short sides of a right triangle are three and four cubits long, respectively, and you want to know how long the hypotenuse is.

The above statement about what mathematics involves can be thought to apply to pure mathematics. In pure mathematics we neither know nor care whether the original data are true. In practice, of course, pure mathemat-

*Remember? In right-angled triangles, the square of the hypotenuse equals the sum of the squares of the other sides.

ics has little utility unless it can be applied to real situations. Indeed, part of the beauty of good mathematics is that it can be applied with equal success to different and apparently unrelated human problems — some of which may never have occurred to the person doing the pure mathematics. One of the most important and creative aspects of mathematics is the abstraction, from a physical situation, of symbols, concepts, and statements that can be manipulated to produce other statements that are more useful. Among the earliest and best known abstractions is the set of postulates produced by Euclid. A still earlier and even better-known example is the number system we use to count and do simple arithmetic. Both are applicable in situations never imagined by their creators.

Today, applied mathematicians regularly abstract mathematical concepts and symbols from problems that are important to human beings, and then deduce mathematical results that help us understand the problems. For example, mathematics can be applied to such problems as designing an automobile, under given conditions which may be set by the laws of physics, the laws of governments, and the laws of economics. Mathematical statements can be abstracted from the given conditions. By manipulating the statements, new ones can be deduced, and these can then be applied to the original situation (in this case, automobile design). If the deduction is done correctly, the results will be true of that situation if the original assumptions are true. Of course if the original assumptions are false, the mathematician may reason flawlessly to false conclusions.

There is another aspect of mathematics that teachers and students often fail to understand, but it is very important to the learning of mathematics. Generally the purpose of mathematics is to do efficiently what would be impossible or very laborious to do without mathematics. Thus, while many schoolchildren and their teachers seem to think of mathematics as a way to make people work harder than they would ordinarily work, the fact is that mathematics is commonly used as a way to reduce work. A simple example should make this point clear.

In an auditorium the rows are labeled from A to T, with no rows or letters omitted. The seats in each row are numbered from 1 to 68, with no numbers or seats omitted. Suppose you want to know how many seats there are in the auditorium. You could walk through the auditorium and count the seats. Chances are you would find this method laborious; you also might come to the wrong answer because of a momentary lapse in attention that caused an error in your counting. If you had studied mathematics as a child to the extent that you knew how to add, you might add 68 + 68 + 68 . . . + 68 until you had added the appropriate number of 68s. But if you had gone still farther, and learned when and how to multiply, you probably would simply multiply 20 × 68. The example is simple, but the general principle it demonstrates has far-reaching importance: mathematics provides a way to solve problems that would otherwise be either unsolvable or difficult to solve.

The one thread running through all of mathematics is human thought. Human thought and creativity are necessary to abstract the appropriate mathematics from the original situations.

Human thought may be necessary for the manipulation of the mathematical symbols (although people can and should be replaced by machines for some symbol manipulation). In any case, human thought and understanding are essential to the interpretation of the results of manipulation. Thinking is the basic skill in mathematics.

II. Why Should People Learn Mathematics?

Two commonly told stories provide insensitive and inappropriate answers to why people should learn mathematics.

The first concerns a student who had begun to learn geometry with Euclid. After learning the first theorem he asked, "But what shall I get by learning these things?" Euclid is said to have replied, "Give him threepence, since he must make gain out of what he learns."

The second purportedly occurred after a prominent specialist in point-set topology had completed a lecture on that topic. A member of the audience asked, "What is the use of all of this?" The professor thought for a moment and replied, "Well, I make a very good living at it."

Because they find mathematics stimulating and enjoyable, mathematicians and mathematics teachers too often believe all other people will find it equally so. Perhaps if mathematics were better taught, this belief would not be so far from the mark; but still it could hardly be used as an excuse for spending large amounts of schooltime on mathematics. The same idea could apply just as well to chess, checkers, bridge, or many other activities that some people find stimulating and challenging.

The most important reason for learning mathematics and for teaching it in the schools is that mathematics is, or ought to be, useful. Mathematics is useful in our everyday life. We are constantly called upon to make decisions based on probability, geometry, statistics, arithmetical estimations, measurement and estimations of measurement, rates, ratios, and the ability to read maps, blueprints, and other scale drawings. Common experience indicates that most people make inadequate and incorrect decisions of this kind, because they are unwilling or unable to apply mathematical thought to such problems: mathematics well learned would help them make better decisions.

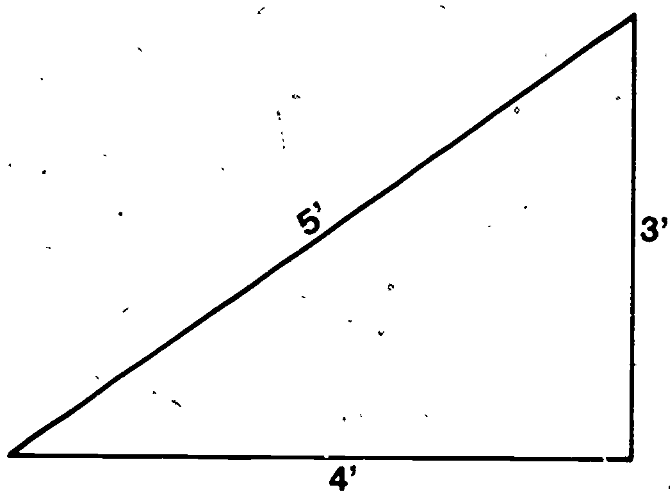
As well as helping the individual to survive from day to day, basic mathematical competence is essential to the functioning of an informed citizen. It has become commonplace for radio and television newspeople to report who has won national elections before all the polls are closed on election day. That there may be errors in these predictions is seldom understood by the audience (and perhaps by the reporters). Such prophecies tend to become self-fulfilling, and at the very least they affect local elections. Better understood, they would probably have less effect, or might even be banned.

Beyond statistical election analysis, we are constantly bombarded by numerical data that presumably should be used in determining public policy. The gross national product, the rate of inflation, the prime interest rate, the trade deficit (or surplus), the cost of education, defense, welfare, transportation, energy, etc. are all reported to us regularly. These data ought to be important in determining our attitudes about various crucial governmental actions. Yet most people seem either not interested or not able to interpret the information in any intelligent fashion. For example, a well-educated professional person recently said to me, "Do you realize the national debt is more than 900 billion dollars — or is it 900 million dollars?" A quick and simple calculation would have shown him that the first figure (approximately correct) makes the average person's share almost $4,000 while the second makes it only about $4.

Creating an informed citizenry that can and will understand and analyze data intelligently should be a prime reason for expecting all students to learn mathematics.

Much of the mathematics taught in courses labeled algebra and trigonometry was designed to prepare students for courses in calculus, which in turn prepared them to be engineers, physicists, mathematics teachers, and so on. However, there are simple applications of these subjects we might make to everyday problems. For example, if three different car rental companies have three different formulas for figuring your cost for the number of days you keep the car and the number of miles you travel, you could solve or graph appropriate equations to determine which company has better rates for various combinations of days and miles. It is unlikely that you would, but you could use trigonometry to determine how tall a tree is and whether it is likely to hit your house if it falls.

High school courses in plane and solid geometry were designed more to teach deductive reasoning and mathematical structure than to provide information that would be useful in everyday life. Most readers of this paper probably believe making people more logical is a desirable goal. Unfortunately, research casts considerable doubt on whether the usual geometry course affects students' logic even in mathematical topics, much less in everyday activities. Beyond the logic and appreciation of mathematical structure presumably learned in geometry courses, there are important techniques and pieces of information that can be used in everyday life. For example, to make a right angle, tie knots in a 12-foot string 3 feet from one end and 5 feet from the other (4 feet between the knots). If the ends of the string are brought together and the string between the knots stretched to make a triangle, the Pythagorean theorem says that the angle between the short sides is a right angle. The activity suggested later in this paper (p. 22)



shows how formulas from solid and plane geometry can be employed to help estimate volumes of grocery store containers, or, from another point of view, how to design such containers so as to fool customers.

It would be unfair not to mention that the applications suggested in the preceding two paragraphs could be accomplished by someone who had no formal courses in algebra, trigonometry, and geometry. Anyone with a good basic ability to do arithmetic, to measure, and, above all, to think. could accomplish them. Whether through formal courses in school or through simpler mathematical learning combined with an ability and willingness to think, mathematical problem solving can certainly be more useful in our everyday lives than most of us allow it to be.

Beyond its usefulness in our daily lives, the importance of mathematics in the professions increases with each passing year. Mathematics has always been essential to the physical sciences, to engineering, and to accounting; but today it is necessary or useful in many other professions and major fields of study. Years ago a person who "liked" science and wished to become a scientist but "didn't like" mathematics would enter the field of biology or one of the related nonphysical sciences. Today, such people discover that they must master calculus and statistics to make any significant progress in such fields. In the social sciences, too, the place of mathematics is constantly growing. This growth is not surprising when we consider fields like economics or the statistics of politics, but mathematics is also used to analyze rational behavior, political structures, and other apparently unmathematical subjects of intense interest to social scientists. Other professions — education, medicine, philosophy, for example -- are tending now to require or reward competence in mathematics.

Beyond the application of mathematics to conventional fields of specialty, new and rapidly expanding professions like systems analysis, computer programming, and energy production require considerable mathematical background. Today, a young person who is ignorant of mathematics excludes himself from many interesting and rewarding occupations. There should be no need to give the student of mathematics threepence today.

III. The New Math

Before the Russians put their first Sputnik into orbit in 1957, there were mathematicians and teachers who believed we could improve school mathematics by putting greater emphasis on thinking and understanding rather than learning by rote. Some suggested incorporating into the school curriculum part of the voluminous mathematics created in the previous 300 years and still neglected in elementary and secondary school, and breaking down the compartmentalization that has existed in American mathematics education. There were also many who believed that the few social applications in seventh- and eighth-grade textbooks were inappropriate and insufficient to show the great value of mathematics and its relationship to the world around us. After 1957, large amounts of money became available to improve mathematics programs, and new projects appeared everywhere.

The popular press hailed this movement as the "new mathematics," or "modern mathematics," or simply "the new math." But members of the educational, mathematical, and lay communities attacked the new math with a vigor equal to that of its proponents. Although one common attribute of new math programs was a careful definition and use of terms, there was never widespread agreement about the definition of the new mathematics. Some remarkable pedagogical and mathematical ideas were foisted upon the schools in the name of new math, and the wilder among them became the center of eloquent attacks upon the movement. It would be neither possible nor profitable to list all the good and bad things attributed to the new math. However, seven major trends were common to many of the programs, and these were often thought to contain the essence of the new math:

1. An increased emphasis on formal logic, rigor, proof, and structure. The emphasis took different forms at different levels and in different programs. In one geometry program, for example, the developers relied on a corrected and augmented form of Euclid's axioms proposed by the great mathematician David Hilbert. According to the project director, his class was unable to proceed beyond the congruency theorems before Easter. Since these are usually covered in the first day or two of an ordinary geometry course, there was some doubt that his students would complete a geometry course before they were ready to collect social security. In some geometry programs, classes spent considerable time and effort showing that if three points are all on the same straight line, one of them must be between the other two — hardly an exercise designed to show the usefulness of mathematics.

At the elementary school level, children were expected to use the axioms of abstract algebra to deduce how to add two simple whole numbers. The following demonstration that 17 + 6 = 23 is taken from one of the better-selling mathematics books of the 1960s:

$$
\begin{aligned}
17 + 6 &= (10 + 7) + 6 \\
&= 10 + (7 + 6) \\
&= 10 + 13 \\
&= 10 + (10 + 3) \\
&= (10 + 10) + 3 \\
&= 23
\end{aligned}
$$

In more advanced versions, the child was expected to supply reasons for the steps or to fill in blanks in a form like this:

$$
\begin{aligned}
17 + 6 &= (10 + \underline{\quad}) + \underline{\quad} \\
&= \underline{\quad} + (\underline{\quad} + \underline{\quad}) \\
&= 10 + \underline{\quad} \\
&= \underline{\quad} + (10 + \underline{\quad}) \\
&= (\underline{\quad} + \underline{\quad}) + \underline{\quad} \\
&= \underline{\quad}
\end{aligned}
$$

Those of us who had the opportunity to watch young children fill in the blanks were intrigued by their procedures. They would begin, as expected, at the top and fill in several blanks. Then, they would skip to the last line, fill in 23, and start working backwards. Clearly, they were not using the laws of algebra and deductive reasoning to discover how to add 6 to 17. Rather, because they knew the answer to begin with, they were able to use their ingenuity to produce a paper that would satisfy the teacher (and presumably the textbook author).

The difficulty with this kind of formalism is that children don't think this way, and neither do mathematicians. Children, that is to say, were learning by rote a new catechism that had no relationship to the way they or anybody else thinks. Thus, in the name of teaching children to think, we were training them to dislike mathematics or develop a totally fallacious concept of what mathematics and thinking are.

2. A closely related trend involved an emphasis on discovery. As with the first trend, the goal was to encourage thinking, and this approach seemed both more realistic and more useful. However, there were unfortunate developments here as well. In many cases, the discovery approach deteriorated into a sort of Socratic dialogue in which the teacher would, through carefully worded questions, trap the student into admitting that a certain theorem must be true even though the student could never have produced the line of reasoning independently nor have believed what the teacher had just trapped him into saying.

3. A third trend involved applications and the relationship of mathematics to practical problems. Good applications turned out to be difficult to create. While there are many excellent applications of the mathematics taught in schools, understanding the applications often requires weeks of studying some other subject. Because mathematics teachers did not have time to teach the necessary physics, political science, biology, etc. for each application, and because the students generally did

not have the necessary backgrounds in these fields, applications tended to be artificial and oversimplified.

One form of application that does not necessarily have this disadvantage is the mathematical game. Unfortunately, when games became part of the mathematics program, they were thought of as supplementary activities to be used if there was nothing more pressing to do. Often the games were not coordinated with other mathematics being studied, and occasionally it was unclear to both teacher and pupil what was to be learned from a game. Such fun-and-games sessions were often left for Friday afternoon, or conducted in a "mathematics laboratory" separated physically as well as temporally from the classroom.

4. A fourth trend closely related to the first was the very careful use of language. A simple example from arithmetic might be the schoolchild's question: "How can you show that half of 18 is 10?" One answer is to take the top half: 18. Another is to take the bottom half. Using Roman numerals, we can easily show that half of 12 is 7, but *only* by using the top half. While this is an amusing example of how we might purposely undermine the distinction between numbers and numerals, in fact there is very little evidence that students of school mathematics have ever been seriously confused about this distinction. Surely, on the few occasions when real confusion existed, it would have been possible to clear the matter up without building an entire curriculum around the distinction. Most textbooks and teachers trying to make the distinction between names and objects were consistently unable to do so, and usually ended up being wrong, pedantic, and confusing.

Other dicta about the use of language propagated in the new mathematics movement had slightly more justification from a pedagogical point of view, because they were intended to prevent confusion. The words "carry" and "borrow," for example, were not supposed to be used in addition and subtraction because the adder did not physically carry an object from one column to the next, nor did the borrower ever repay the debt. The same logic supported the contention that "reduce" should not be used with fractions, for the resulting fraction was no smaller than the original. In all three cases, and in most similar cases, there was probably no great harm in the words themselves that proper development and explanation could not easily correct. Moreover, the words often served a useful purpose, and the replacements were no better. For example, the word "simplify" often replaced "reduce," but who could reasonably argue that 17/25 is really simpler than 68/100? Perhaps the best rule for the use of language in mathematics is to keep the language as simple as possible and in conformity with the language used in everyday conversation, as long as there is no compelling evidence that the language being used confuses students.

5. Introducing new material into the mathematics curriculum was perhaps the most needed and least controversial trend of the new math era. The subject matter of school mathematics books and courses of the 1950s would have been familiar to Galileo, even though it was reliably estimated that by 1950 more new mathematics had been created in the 20th century than in all history before that time. Of course, not all of this new mathematical content was appropriate for school courses, but much was more appropriate and more likely to be useful for students than the mathematics then in the schools. Statistics, probability, functions, graphing, solving inequalities along with equations, work with calculators and computers, and even parts of number theory and topology — these topics were important to introduce. Prior to 1957, they had made remarkably little impact on school mathematics programs.

6. Acceleration is the term usually used to suggest the early teaching of a subject previously taught only to older pupils. It has been changing the way in which we teach mathematics since education began in this country. At the height of the revolution in school mathemat-

ics, a well-known psychologist is purported to have announced that any subject could be taught to any child at any age in some form that is meaningful. This statement might pertain to a two-day-old child and any subject imaginable. Its patent falsity was of less significance than the danger that some educators or parents might try to make good on its promise without asking why a young child might have need for learning the exotic topics that might be chosen for such an experiment. Fortunately, before courses in prenatal ergodic theory were instituted, wiser heads prevailed and the move for acceleration abated somewhat. It is not uncommon still, however, to find principals, department chairmen, and parents who are proud that their schools offer calculus in the eleventh or twelfth grade, even though students who take the course may be completely ignorant of topics like probability or even analytic geometry, to say nothing of the use of calculators and computers — all of which would be of more value in the lives of most and should precede the study of the calculus.

Most educators would agree, however, that before acceleration of any topic is attempted, we should consider whether there are real benefits to be gained by the acceleration, and also whether it is possible to motivate the student to understand the basis of the subject and its relationship to concepts that are intuitively clear and natural. There is considerable psychological evidence available that subjects taught too early may be learned mechanically and in a shallow manner, and thus easily forgotten and unlikely to be applied when appropriate.

7. The seventh major trend of the 1960s was the integration of various new topics in mathematics. Because American colleges began requiring arithmetic in the 1700s and early 1800s but didn't require a year of algebra until after 1820 and a year of geometry until still later, the common high school curriculum in 1910 involved first the study of arithmetic, then a year of algebra followed by a year of geometry. There was no

apparent attempt to relate these subjects to each other, or even to review one while learning another. The one semester of algebra that followed geometry in a minority of our schools in 1910 was apparently intended for a review of the algebra that had been forgotten while students were learning geometry. No major change in this arrangement had occurred by 1957. Since then, there has been an attempt to include some algebra and geometry, and even some graphing, probability, statistics, and other appropriate topics in the elementary school curriculum, with more geometry included in ninth-grade algebra and more algebra in tenth-grade geometry than had been common before 1957. Most knowledgeable observers believe this is a good trend, although the specific ways in which it is achieved in some cases may be rather artificial.

The major improvements that occurred under the banner of new math were the introduction of new content and the integration of the various new topics in the mathematics curriculum. In most cases, the emphasis on thinking and problem solving was beneficial, but all too often these skills were presented in ways too formal and artificial to do any good; indeed, harm may have been done rather than good.

Most recently there has been reaction to the new mathematics in the back-to-basics movement. In many cases the movement — it is widespread and identifiable enough to be called that — seems to be characterized by a stimulus-response, thoughtless, quick, computation-oriented approach to mathematics learning. While knowing facts and algorithms is understandably useful in conjunction with thinking and problem solving, computation was never basic to mathematics; and the wide availability of calculators and computers makes it less basic today than ever. The important, human parts of mathematics are the original abstraction of relevant features from a situation, the decision of what mathematics to apply to this material, and the interpretation of the results. The actual computation can be done by cheap. nonhuman objects such as calculators or computers.

IV. Problem Solving and Teaching Mathematics

Every teacher of mathematics has heard the complaint, "I can do the mathematics; it's just the problems I can't do." This is analogous to saying, "I can do the English; it's just that I can't write anything worthwhile." The main purpose of teaching mathematics is to educate people to use mathematical thought to solve real problems.

Endeavoring first to explain what's wrong with the usual textbook teaching of word problems, I want to suggest (a) how a mathematical environment can be created in which students learn to solve problems naturally, (b) what underlying skills are necessary to solve mathematical problems, and how those skills should be learned to enhance the child's competence in problem solving, and (c) how the success of students, teachers, and commercial programs can be evaluated and enhanced.

I should note at once that there are serious problems in teaching mathematics that this paper does not consider. Among them are math anxiety and the avoidance of mathematics by segments of our population (notably women and minorities). Such problems are important because they affect individuals and deprive society of the potential contributions of many capable people. The kinds of improvements suggested in this paper can help alleviate some of these problems, but there will have to be major changes in the attitudes and behaviors of all of us to eliminate them. Though I cannot treat the problems here, I stress their significance, even as I proceed with my original line of thought.

What's wrong with the usual teaching of textbook word problems?

Let me begin the discussion of what is wrong with the usual textbook teaching of word problems by taking three from well-known textbooks:

(a) Five lights on. Three lights off. How many lights on?

(b) An anthropologist uncovered 17 bones on Monday and three times that many bones on Tuesday. How many bones did she uncover on Tuesday?

(c) Mary took $5 to the circus. She spent $3.85. How much money did she have left?

Lest there be any doubt in your mind, the answer provided in the teacher's guide to problem (a) is two, not five. Presumably, the reader will have no difficulty guessing the answers to (b) and (c).

Look at problem (a). Ignoring the abuse done to the English language and the temporal ambiguity (neither of which should be ignored by textbook writers), we are left with the question of why anybody might want to answer this question by subtracting. If you were in a room in which there were five lights on and somebody wandered around turning off three of those lights, how would you determine how many lights were still on? You would, of course, count them. Indeed, the mind is sorely taxed to try to imagine any situation in which one might have acquired the information given in problem (a) without also having learned the answer. This is a classic example of an "unreal" problem — a problem that gives the learner the impression that mathematics is a way to make work rather than reduce or avoid work. While six-year-old children are much too wise to say so aloud, or even consciously to themselves, surely there are many who, having seen hundreds of problems of this sort, begin subconsciously to think, "If this is mathematics, I want nothing more to do with it."

Consider problem (b). This may have been an attempt to write career education into the mathematics curriculum. Now we can see how mathematics is used in anthropology! Fortunately for the future of anthropology, no normal child will suppose that this is really the way anthropologists discover how many bones to uncover on Tuesday, but again, the mere statement of the problem reinforces the belief that mathematics is a useless subject designed to make people work harder than they would otherwise have to work.

Before continuing, go back and examine problem (c). See if you can decide why I believe this is an unrealistic problem. If you were Mary and you took $5 to the circus, how much money do you think you would spend at the circus? In the unlikely event the circus closed early, or you had a stomachache, or for some other reason you didn't spend the entire $5, how would you determine how much money you had left? Have you ever seen a small child dashing about a circus with a pencil and paper carefully adding the money she is spending so that the sum can be subtracted from the original amount of money to establish how much is left in her pocket? If you were Mary, surely it would occur to you that the simple way to find out how much money you have left is to take the money out of your pocket and count it.

Mathematics textbooks, with rare exceptions, are packed with unrealistic problems nobody would ever solve in real life. Fortunately (or unfortunately, depending on your point of view), children seldom find it necessary to read the problems or think about them. They discover techniques for getting the right answer without having read the problems or understood what is being asked. In problem (c), for example, the pupil will look at the last word — "left." The child will have been taught by teacher or textbook that "left" means subtract. (Indeed, there are programs that actually formalize this approach.) So, the child must look for two numbers and subtract the smaller from the larger.

Because both the teacher and the textbook author are involved in this conspiracy to make everybody think the child has succeeded in solving the problem, it would be considered bad form to supply three numbers in the problem and make the pupil guess which two must be used in the subtraction exercise. Presumably, it would also be bad form to use "left" as the opposite of "right" in a problem where somebody might be misled into subtracting. This method of solving problems is known as the key-word method. If you can find the key word and the numbers, you don't have to understand, or even read, the problem. To see this, try to solve the following word problem even though most of the words are intentionally omitted:

_______ 15 _______ _______

_______ 24 _______ _______

_______ left.

If your answer is 9, you solved the problem correctly and obviously did so without being able to read or understand the problem.

Another method for solving problems without reading them is typified by a true story that has undoubtedly been replicated in many homes. When she was in third grade, my daughter arrived home one day with a sheet of word problems. She worked for about three minutes and then asked me how to do the first. After quick examination I established that this was the easiest problem on the page, so I suggested that we do one of the harder problems, assuming she would then be able to figure out how to do the easier problems. Wendy looked at me, a slight air of disdain creeping into her voice. "Daddy," she said patiently, "if I know how to do the first problem, I'll know how to do them all." Wendy was right. Whenever she met a page of word problems, it always called for the same operation. Such activities may provide practice in performing a single operation, but they do not help students learn to solve problems.

Beyond these two methods children use to solve word problems is a much more sophisticated technique that is often found in the upper elementary grades and in high school. Though it is seldom formally worked out like this, here is an accurate paraphrase of what a child has learned after he has successfully completed much of an individualized learning program, getting 85 percent or more correct on tests: If there are more than two numbers in the problem, you add. If there are two big numbers, you subtract. If there's a small number and a big number, you divide, and if it doesn't come out even (whole number), you multiply. If there are two small numbers, you multiply.

Not only are they often unrealistic and easily solved without having been read; textbook word problems present other difficulties. Most are one-step problems — you perform one operation, like addition, and you have your answer. Most real-life problems require several steps, some of which may not be simple arithmetic operations. Textbook problems commonly provide exactly the right amount of information, never too much or too little. Real-life problems almost always occur with an excess of information; and often we discover that while there is superfluous information, there are also missing pieces. Students should meet problems like these so they learn to decide what information they need, as well as what to do with it when it is known.

Still another difficulty with word problems is that they require a skill not necessarily related to solving mathematical problems — the skill of reading. While we are all agreed that children should learn to read, there is no apparent reason why a child should be excluded from doing the exciting part of mathematics until he learns to read. Few problems originate on the printed page. Let's say we find ourselves in a situation. Because we are in it, there is usually no ambiguity about what is meant. We think about the situation. If more information is needed, we collect the information. If there is extra information, we ignore it. We think some more. We solve the problem.

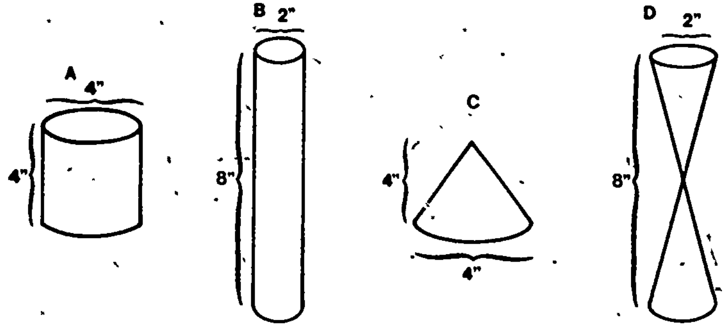
On occasion some reading may be necessary, but reading does not play the same central role in real problem solving that it *always* plays in textbook word problems. By no means should we neglect to teach children to read, but neither should we insist that they learn to read before having the opportunity to solve real mathematics problems. Furthermore, if they approach at least some of their mathematics problems in more realistic situations than those offered by the printed page, students are likely to develop a better understanding of mathematics and a feeling for its power to help them rather than make more work for them.

Activities and games teach children to solve problems naturally

There are activities and games teachers or parents can use to help create an environment where problems occur naturally and children will naturally try to use mathematical thinking to solve them. They are not elaborate, and cost next to nothing. The simplest way to make the point is to illustrate:

1. An activity that develops geometric intuition and estimation skills involves simple grocery store containers. A teacher can line up 10-20 containers in front of the class and ask which would hold the most liquid, which the next most, and so on. Place the containers in any order the class suggests, and then check by pouring water from one into another. If the containers are sufficiently different in shape, most children (and most adults) will be mistaken about several of the volumes. People commonly believe that tall, thin containers hold more than they acutally do, and that short, fat containers hold less. Conical shaped containers contain less than most people expect. This activity demonstrates, on an intuitive and a physical basis, relationships that can be established later when formulas for the various figures

are developed. Consider the figures shown here:



Which of these containers would hold the most water? Which the least? How many times could the contents of container D be poured into container A without overflowing? Put another way, if container D holds one unit of liquid, how many units will container A hold (assume the two halves of D are connected so that shampoo, cooking oil, or any liquid can flow through, like sand in an hourglass)? How many D-units would fit into B? How many D-units would fit into C?

Whether looking at a picture or the actual object, most people tend to forget the depth of three-dimensional objects and base their estimates of capacity on area seen. Thus, it is common to asume that A and B contain about the same amount, and C and D contain half that much. In fact, A holds twice as much liquid as B, three times as much liquid as C, and six times as much liquid as D.

In answer to the questions above, if D contains one unit, then C holds two units, B holds three units, and A holds six units. The fact that we tend to overestimate the capacity of containers like B and D and underestimate the capacity of containers like A helps to explain why so many grocery store containers look more like B and D than like A. For those interested in formulas, the volume of a cylinder is given by the formula $V = \pi r^2h$, and the volume of a cone is given by $V = \frac{1}{3}\pi r^2h$, where r is the length of the radius of the base and h is the height. The formulas are not necessary to the activity, however, and should not be developed until after a good deal of physical/intuitive experience.

Another version of this activity particularly appropriate to the home is to have one person pick up a catsup container (or whatever container is handy), read the label, and ask, "How many fluid ounces do you think came in this?" or "How many milliliters do you think this had in it?" or a similar question about the volume or weight of the product in the container. At first, people tend to give rather poor answers, but with practice their estimates tend to be better. For wrong answers, you can give a hint such as "No, it's more than that," or "No, it's quite a bit less than that." Of course, it's only sporting for the adult to play the guessing game when the child is the first person to get hold of the container.

Another activity involves estimating any measure or number, then checking to see what the correct answer is. How many inches tall is that person? How many meters is it between those two telephone poles? How many grams does that book weigh? What is the area of this tabletop in square inches?

2. To develop two-dimensional intuition, you might show your child the blueprints for your house or apartment, or possibly even make a map of your house with the child's help. This involves work with ratios, but using a scale of one inch to one foot, or another equally simple scale, makes the task easier. Such a map of your house may make rearranging furniture easier if the rearrangements are first checked on the scale drawing. Reading a highway map is a similar activity that helps develop an understanding of geometry and ratio.

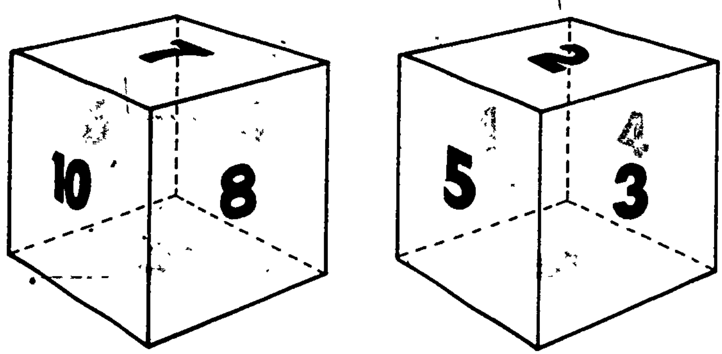
3. Children can also take part in all sorts of projects that require the collection of data. They can collect data on temperature, on rainfall, on sales of certain items in a store — on almost any happening that occurs in their everyday worlds. Data can be collected, analyzed, and used to make predictions and modify behavior.

There are countless activities in which mathematics can be used to help youngsters understand the world.

An alert teacher or parent should try to involve them in the most constructive manner possible, both to show children the usefulness of mathematics and to help them learn the mathematics.

In virtually every species of higher animal, the young learn much that will be useful to them as adults by playing games. Human beings are no exception. We learn much that is useful socially, physically, and mentally through our playing of games. Games like bridge, Monopoly, and backgammon are inherently mathematical, although even adult players of these games are often unaware of the mathematics involved. Playing such games in school is probably undesirable because the mathematics is not sufficiently specific and thus may never be learned. However, it is possible to create games that provide practice in particular skills and also provide a natural way to develop problem-solving skills. Here are two. In each case, you are likely to learn the rules for the game and play it for a while before realizing that the game poses a problem. Usually, the problem is one of finding the best strategy to win the game.

Writing numbers game: In this game and the next, assume that you have two different cubes. One has numbers from 0 to 5, and the other has the numbers from 5 to 10, as shown here. Notice that 5 appears on both cubes, but no other number is duplicated. For this game,



you should also have a form like this:

0	1	2	3	4	5	6	7	8	9	10

Two players take turns rolling a cube and writing the number rolled in the space below that number in the form. The players use different-colored pencils so they can tell at the end of the game who wrote the most numbers. The person who wrote the most numbers is the winner. As with most games, the best way to learn is actually to play, but we will simulate the play here by supposing that John and Mary make the following moves:

John rolls the 0-5 cube, gets a 3, and writes 3 in the appropriate space.

Mary rolls the 5-10 cube, gets a 5, and writes 5 in the correct space.

John rolls the 5-10 cube, gets 7, and writes 7 on the form.

Mary rolls the 5-10 cube, gets 8, and writes 8 on the form.

John rolls the 5-10 cube, gets 8, and can't write a number because the space is already taken.

The form now looks like this (John's numbers are circled):

0	1	2	3	4	5	6	7	8	9	10
			③		5		⑦	8		

The obvious purpose of this game is to practice writing numerals. You can make the form more helpful for very young children by having lightly dotted figures to trace on the first few forms. When young children play this game, they sometimes stay with the same cube even when there are no spaces left corresponding to its numbers. For example, even if all the spaces from 5 to 10 are filled, the child may continue to roll the 5-10 cube, although rolling the 0-5 cube is the only useful choice. If the child realizes the numbers needed are all on the other cube, he will switch to it. In the situation above, the strategy is not quite so obvious, but there is a question of probability that requires a bit of thought. If you were Mary, which cube would you roll? By simply counting, you can establish that there are four numbers on the 0-5 cube that have not yet been used but only three on the 5-10 cube. Therefore, you have a better chance if you roll the 0-5 cube.

Notice how the child discovers both the problem and the solution. There is no need for an adult to tell the child there is a problem or to suggest ways to solve the problem. Sometime during the course of the game, the child is likely to realize that one of the cubes is unlikely to produce a number that isn't already written. He will check this hypothesis by counting, and then try the other cube. I have watched children (and adults) play this game; this is precisely the way they do it, although the adult's thought process is usually less obvious than the child's.

Roll-a-15 game: In this game you use two 0-5 cubes and two 5-10 cubes and try to get a sum as close to 15 as possible by rolling the cubes one at a time and stopping whenever you wish. You may not, however, roll any cube more than once. Suppose John and Mary are playing this game. Mary starts by rolling a 3 on a 0-5 cube, followed by a 7 on a 5-10 cube; next, she rolls a 2 on the remaining 0-5 cube. Her total so far is 3 + 7 + 2, or 12. She has a 5-10 cube left. Her goal is to get as close to 15 as possible. It's all right to go over 15, and 17 would be better than 12, and 18 would be as good as 12. Should she roll the remaining 5-10 cube? If you had been playing the game would you have rolled the cubes in the order Mary did? If not, how would your strategy differ?

Most children and many adults start by rolling whichever cube comes to hand first and continuing with whichever cube is convenient for the second and third rolls. Then, some (but not all) players begin to think about the numbers on the remaining cube and whether rolling it is likely to improve their score. At this point, it may be too late — as it is for Mary. If she rolls the remaining 5-10 cube with a score of 12, she has only one chance in six of improving her score (by rolling a 5). Notice that if she has a score of 12 and a 0-5 cube left, she has five chances in six to improve her score, and the remaining possibility (rolling 0) will not hurt. Children learn quickly that there is an advantage in saving the 0-5 cubes for the end of the game. Furthermore, although the game is designed to practice addition, children get a fair amount of practice in subtraction in order to compare scores. They also do a lot of work with intuitive probability in deciding whether to roll a cube and deciding which cube to roll.

As in the previous game, once a player has learned the rules, there is no need to explain that there lurk in the game mathematical problems that must be solved. The players discover the problems themselves and solve the problems themselves, as one does with real-life problems. Another interesting feature of such games is that nobody tells the child either what the problem is or what the answer is. If the child arrives at the wrong answer and plays the game in a less than optimal fashion, his life will not be seriously affected, since knowing how to play any of these games won't in itself be of any great importance in the child's future. It is the problem-solving skill that is of importance. So, by putting a student into a game situation where problems naturally occur, he learns to recognize and solve problems without undue outside pressure and without being constantly in fear of being told, "You're wrong."

There are teachers and parents who distrust programs that call for playing games in school. They view school as a place for work, not play, and they think of mathematics as a subject that *should* be difficult, if not actually unpleasant. If they believe that learning mathematics is a way to develop the soul rather than the mind, they are wrong.

The goal in teaching mathematics is to get people to think quantitatively and spatially, and if possible to enjoy thinking, so that they will solve problems by thinking even when they are not required to do so in school. By incorporating games and activities into the mathematics curriculum, we can help students see the importance of thinking and discover the enjoyment and benefits that come from thinking mathematically. The games must be carefully constructed, of course, and integrated correctly into the program so as to provide the right kind of practice at the right time, and to provide as much opportunity as possible to solve real problems. If the children have one game-playing session a week that always comes on Friday and they choose only those games they most enjoy, they will not learn much from playing games.

The games suggested here, and others like them, should be played when practicing the principal skill of the game is appropriate; and, of course, every game should provide practice in a specific skill at the same time that it affords real problem-solving practice. It is possible to improvise such a game for virtually every skill in elementary school mathematics.

If children experience the kinds of activities and games proposed in this section regularly and often, and develop the kinds of underlying skills listed in the next section, a far greater portion of the graduates of our schools can be expected to be able and willing to use mathematical thinking to live better lives themselves and to help the rest of us to live better lives as well.

What underlying skills are necessary to solve mathematical problems, and how should those skills be learned to enhance the child's competence in problem solving?

By way of foreword here, let me — at the risk of being repetitious — stress the importance of understanding that thinking is the ultimate basic skill in mathematics.

Without the ability and desire to think, a student, who has all the other skills is unlikely to be successful in problem solving. The willingness to think mathematically is every bi' .s important as the ability. Unfortunately, mathematics instruction has in the past often been so unpleasant that even students who earned straight As in the subject and understood it fully developed such unfavorable attitudes that they resisted using mathematics in their everyday or professional lives.

Any summary of skills must be incomplete and somewhat arbitrary. Moreover, the teaching suggestions I make here are necessarily sketchy, designed more to give a feeling for the kinds of pedagogy that are desirable than to outline a program. With such caveats in mind, consider these fundamental skills that every citizen should develop in school:

1. People should understand whole numbers and rational numbers (decimals and fractions), and the use of whole numbers in counting and rational numbers in measuring. I include here the understanding of rates, ratios, and percents, and the ability to use these in appropriate ways. Scientific notation should also be familiar to everyone as a way to report and understand very large and very small numbers. Real counting and measuring activities, like those suggested in the previous section, should be experienced in order to develop these understandings in ways that will tend to make them useful.

2. They need to know the basic facts for the four common operations of arithmetic (addition, subtraction, multiplication, and division) and understand the relationship of these basic arithmetic facts to physical objects. In the early development of these facts, children should work with physical objects whenever possible and gradually come to realize that the facts .e true no matter what physical objects are considered, thus abstracting the facts as general principles. For example, two children might play the following hidden-counters game:

Five counters (or coins, or ice cream sticks, or other objects) are placed on the table. One child looks away while the other covers several of the objects with a book. The first child now looks back and sees only three counters. How many counters are under the book? This particular round of the game models the subtraction fact 5 − 3 = 2. The answer can be checked by lifting the book and counting.

Variations on this technique provide experience with other skills. For addition, eight counters might be placed on the table and counted, then covered. Now five more counters could be placed on the table. How many are on the table (including those under the book)? Some textbooks have problems that look similar but are really quite different. They might show eight ducks on a pond and five more flying in to join them, then ask how many ducks in all. The difficulty with such a format is that children do not have to add or even "count on" to answer; they can simply look at the picture and count all of the ducks.

After a great deal of physical experience, pupils will know many of the facts because they have used them so often. These should then be organized so that children can see the relationships and use the relationships to remember facts they don't already know. For example, a child who understands the base-10 system of numeration will realize (and probably ought to be encouraged to say, from time to time) that 13 is 10 and 3, 18 is 10 and 8, 35 is three 10s and 5, etc. Such a child will have no trouble realizing that 10 + 6 is 10 and 6, or 16. What then is 9 + 6? It must be one less than 16, or 15. Similarly, if children know the doubles, they can derive the sums that are one more than doubles. For example, if you know that 7 + 7 = 14, it is easy to see that 7 + 8 = 15.

With such experiences and understandings, children can learn the facts well and relate them to the real world so as to use them naturally in solving problems. Then, of course, a great deal of practice is necessary to maintain and sharpen their skills. Games and activities like roll-a-15, for example (see p. 27), can be used to practice these skills with more conventional kinds of practice, including pages of exercises and word problems, flash cards, practice with friends and family members, etc. Practice should not be considered unneccsary (as was sometimes the case in new math programs), nor should it be unpleasant.

3. Paper-and-pencil algorithms (procedures) should be known to all people, but people should also develop the flexibility to use other methods (estimation, mental tricks, etc.) in place of the standard procedures when appropriate. As with other mathematics, these algorithms should be developed from physical situations so that students see the connection between the algorithm and physical reality. For example, the division algorithm might be developed from the following story.

Seven children are out playing in the park. They find $2,938. After turning the money over to the police and getting it back because nobody claims it, they decide to divide it equally. They have 2 $1,000 bills, 9 $100 bills, 3 $10 bills, and 8 $1 bills. Two of the group volunteer to take the 2 $1,000 bills and leave the remaining 20 bills for the others. The proposal is rejected. Instead, the children decide to go to the bank and exchange their 2 $1,000 bills for 20 $100 bills. They now have 29 $100 bills. Each of the 7 children can therefore take 4 $100 bills.

In a class, children can follow this procedure with play money, but they must keep written records of the procedure, once they understand the process. The written record so far would look something like this:

```
              4       (money each child gets)
7 children $ 2 9 3 8  (money found)
```

We write the 4 in the hundreds column to show that each child got 4 $100 bills. How many $100 bills were used up by giving each child 4? Then, how many are left? We will continue to keep records without repeating the description of each number. Now, we have 1 $100 bill remaining. What do you suppose the children will do with it? If they get 10 $10 bills for it, how many $10 bills will they have altogether? How many $10 bills will each child get? Because distribution uses up 7 $10 bills, there will be 6 left. What do you think the children should do with the 6 $10 bills? Then, how many $1 bills will they have? How many should each take? How many will be left? What should they do with the remaining 5 $1 bills? They could change them in for dimes and learn something about decimals, or they could have a party. If we draw in lines to separate the 7 (children) from the 2938 (dollars), and the 2938 (dollars found) from the 419 (dollars each child received), we have the standard division algorithm for a one-digit divisor.

```
    4
7  2938
   28
    1

    41
7  2938
   28
    13
     7
     6

    419
7  2938
   28
    13
     7
     68
     63
      5
```

By relating the development of this algorithm to a possible physical situation, we help children to understand and remember it. More important, we again encourage them to see a relationship between mathematics and the real world, and therefore to see that mathematics is likely to be useful in solving real-world problems.

Rather than move quickly to problems involving two-digit divisors, children should now learn to estimate answers to such problems, using the algorithm developed here, and to check such answers by multiplication. They will find these skills useful when they take standardized tests, as well as in everyday life.

Such treatments are possible for the other standard algorithms, both for whole numbers and rational numbers. As with the basic facts, once they understand the algorithms, pupils should practice using them in many different kinds of situations, so as to understand the general applicability of the procedures as well as to have practice in using them.

4. Measurement and estimation are naturally a part of our everyday lives, and everyone should acquire sufficient skills in these activities to use instruments correctly in making measurements of height, weight, temperature, volume, etc. Estimating before actually taking measures is good experience, and those who make a habit of this become much better estimators after a short time. Students also need to learn how to use measurements appropriately, and how to report measurements in a way that reflects how precise they are.

Children growing up today will have to communicate with people who use both the traditional system of measures and the metric system, and they should be comfortable with both. Only by actually making measurements in both systems and habitually using both systems can they become comfortable. Learning one system and then deriving the second from the first by conversion of units will not produce real understanding of the second.

On the other hand, educators who understand this have carried the idea further and suggested that children not be taught to convert from one system to another. This, of course, is nonsense. Each system should be learned independently, but when both have become part of his intuition, the student should learn to convert from one to the other. For most practical purposes, this can be done through estimates. For example, for temperatures we live with, if you multiply a Celsius reading by 2 and add 30 you will get a good estimate of the Fahrenheit reading; a yard is a little shorter than a meter; a liter and a quart are almost the same size; and a pound is a little less than half a kilogram. Exact conversions can be made when necessary, ordinarily with a calculator.

5. Another basic skill is collecting, organizing, and interpreting data. Deciding what data to collect is impor-

tant and being able to organize the data with a chart, a graph, or other appropriate device is equally necessary. When they are still quite young, children should gain experiences collecting data — temperatures, heights, weights, numbers of cars and other vehicles passing a street corner, and so on. They should then organize the data and try to draw simple inferences and perhaps even make predictions. Older students should study more formal but still simple statistical analyses. In all cases, the mathematical activities should be regularly checked against reality.

6. We often must make decisions with insufficient information, and people need an understanding of probability to do this intelligently. Consider this simple example: If you flip a coin five times and it lands heads all five times, what is the probability that it will land tails next time? Some people behave as though coins have memories. They think the coin must land heads half the time and tails half the time; so if it has landed heads the last five times, it is about due to land tails the next time. In fact, of course, coins don't have memories, and results of the last five throws should not make you change your estimate of what the next result will be unless you have begun to suspect that the coin is not really a fair or an honest coin, in which case you might suppose that it is more likely to land heads than tails in general and therefore is more likely to land heads on the next toss. Situations like this arise in all sorts of practical circumstances — predicting the sex of the next baby born, predicting the results of sporting events, predicting the next blackout or telephone failure, and so on.

7. Although we live in a three-dimensional world, we are often called upon to make decisions involving only two dimensions (in travel, in making models of three dimensions, and in various forms of communication). People should be familiar with and able to use both three- and two- dimensional geometric concepts, and should be aware of the relationship between them. And

since we live on a sphere, a sphere should become familiar as a three-dimensional object.

Activities suggested earlier involving grocery store containers are useful. Beyond these, there are activities with protractors, compasses, rulers, paper for folding, clay, and other measuring and modeling materials that can be helpful in building good intuitive understanding of geometry of two and three dimensions. Deduction and inference should be encouraged with geometric concepts, but in the early years there is no need to emphasize formal logical proofs in geometry. In fact, such formalism tends to be counterproductive if started too early. For the same reason, there should probably be less emphasis on learning special vocabulary in the early grades than is often the case, and more on understanding and using worthwhile concepts.

8. One of the most important and pervasive concepts in modern mathematics and science is the function concept. We use functions to describe and analyze the relation between variables (time and height, size and temperature, size and cost, for example). Everyone should have an understanding of the function concept and some knowledge of the tools with which we study functions. Functions are studied in algebra, trigonometry, analytic geometry, and calculus. There is no reason, however, why children should not begin to understand functions at the age of 5 or 6.

A simple activity involves creating a "function machine" out of a box — large enough to conceal one of the children participating. Make a slot in the box and have the children draw pictures on the box to make it look like their conception of a computer or other machine. Have a child put three ice cream sticks into the box. The box will make strange noises, and maybe even move a bit, then out will come five ice cream sticks. (Inside, of course, is a child with instructions.) Then have a child put two crayons in. Out will come four crayons. Have somebody put in four coins. What do you think will come out? If you said six coins, you understand how the function machine is working. As time goes on, numbers on slips of paper can take the place of actual objects, and then a picture can substitute for the machine, and finally symbolic rules can represent the machine. Using this approach, 9- and 10-year-old children can learn to solve equations with an understanding that 14-year-olds who have not grasped the underlying principles would find difficult.

9. In mathematical modeling we abstract from a physical situation those features we believe relevant, and then carry out arguments and proofs to produce information in a different form. Finally, we interpret these results as they apply to the original situation.

On page 3 of this paper, there are two examples of modeling. Both the Pythagorean example and the example about determining how many people are in a room are simple cases of modeling. In both cases we ignore information we consider nonessential for our purpose (color of hair, heights, ages and names of the people, material from which the triangle is constructed, etc.) and make mathematical statements regarding other information. The statements constitute a mathematical model. When we reach our mathematical conclusions, we reinterpret the results so they apply to the original context. The first and last steps, essential to problem solving, have been discussed elsewhere in this paper. However, understanding proofs, formal and informal, is also an essential part of the mathematical maturation of any person.

Children should be encouraged to understand mathematical arguments produced by others and should be encouraged to produce such arguments themselves. These can include very simple arguments at first, and gradually move to more complex and formal arguments or proofs. Also included should be indirect proofs and counterexamples. Students should be encouraged to understand the place of definitions in mathematical modeling. and to make and apply their own definitions in ways that are convenient and useful. If children regularly see adults argue from known facts to previously unsuspected information, they will begin to get the habit themselves, and they should be encouraged in the habit, even when their arguments are less than airtight.

10. With the wide availability of inexpensive and efficient calculators and computers today, there is no excuse for not knowing how to use these instruments when they are appropriate, and there is even less excuse for not understanding the strengths and weaknesses of such machines.

In the early grades, children should learn to estimate answers and understand the decimal system, since these are both important skills in understanding calculators and computers. After they have mastered the basic arithmetic facts and algorithms, they should have the opportunity to use calculators both in school and at home; and they should learn to see when it is inappropriate or inefficient to use a calculator. For example, have a race between a child using a calculator and a child not using a calculator. Give several problems such as these: 500 + 300 =?, 2,000 + 7,000 =?, 4,000 + 357 =?, 12,000 − 7,000 =?, 4 x 5 =?, 40 x 50 =?, 80 ÷ 20 =? Remember, the child using the calculator must, in fact, use it; and that means pressing all the appropriate keys. See who finishes first and who gets more correct answers. Usually, the child with the calculator will still be trying to push the right keys for the first problem when the other child is finishing, and often the child with the calculator will make several errors while the other — if he has learned his arithmetic — will make no errors.

Whether the student actually learns to use a computer or not, he should at least realize that although computers can compute with great speed, they cannot take the place of serious human thought and judgment. What comes out of a computer depends on what goes into the computer. People should not assume that because a statement is in the form of a computer printout, it must be true.

Computers do only what they are told to do. Moreover, both computers and calculators have peculiar limitations that people should understand. As a simple example, on most calculators, if you divide 1 by 3 and then multiply the result by 3 you will not get 1. Instead you will get 0.9999999. With a long series of calculations, errors of this sort can build up to the extent that even the first or second digit of the answer may be affected. People need to be able to interpret the information put out by a calculator or computer. Today, guided experience with both should be part of everyone's schooling.

Ideally, every high school graduate should have learned at least one computer language (BASIC or FORTRAN, for example) and should have plenty of opportunity and incentive to use a computer to solve problems relating to science, social science, mathematics, language, and other fields of thought. With the great reduction in cost of small computers and computer terminals, this is not an unrealistic prospect now, and it will become even more reasonable with each passing year. Certainly in the future, educated adults will be expected to have at least this much knowledge and experience with computers.

Incomplete as it must be, this summary includes much of what every graduate of our schools should know and understand about mathematics. The most important basic skill is thinking, and the skills described here should be thought of as skills that can help everyone do this and solve problems.

For people with specialized needs, other skills and understandings may be appropriate. These include understanding the real number system (irrational numbers as well as rational), limits, more advanced algebra and geometry, calculus, linear algebra, and other topics. As our society becomes more and more dependent on scientific and technological advances, such subjects will be considered essential in many different occupations and professions. In the meantime, the summary above offers the skeleton of a reliable basic foundation for anyone to learn to solve problems through mathematical thought.

How can the achievement of students and teachers, and the merit of commercial programs, be evaluated and enhanced?

Ideally, to evaluate students' learning of mathematics we should have to follow them through the remainder of their lives and observe the extent to which they use mathematical thought effectively to solve problems. A simpler but still less than satisfactory method of evaluating students' learning is to test them on the subsidiary skills that are necessary to solve problems and also to test them with specific problems to solve.

This is usually done by administering a standardized or a teacher-made test. Unfortunately, such tests have several common faults that stem from our tendency to test what's easy (or possible) to test and that ignore important characteristics that do not lend themselves to testing. Often this weakness in testing is compounded by the tendency of students, teachers, and even curriculum designers to train for the test rather than the more desirable goal of thinking and problem solving.

When subsidiary skills are tested, there is a tendency to stick with basic facts, paper-and-pencil algorithms, definitions, and other easily measured skills, and ignore the skills of estimation, measurement, organization of data, and mathematical modeling. Tests of problem-solving ability almost always consist of word problems. As we have seen, these often have clues that allow the test taker to make a good guess without understanding the problem. At best, word problems are an artificial approximation of the real thing.

There is no simple resolution of the evaluation dilemma. Any process of testing is likely to change both the students and the education they receive. But there are ways to improve evaluation. The first step is to pick tests that evaluate all the underlying skills necessary for mathematical problem solving. Second, choose tests that include as much real problem solving as possible — without giveaway clues like key words or magnitudes of numbers that make the appropriate operations obvious. There ought to be many problems that require more than one operation and others that are solved by using processes quite distinct from the usual arithmetical operation (If one boy can jump a stream five feet wide, how wide a stream can six boys jump?).

Third, distrust test results if there has been an obvious and concerted effort to prepare students to take the test. At best, tests provide only reasonably good simulation of the kinds of activities we really want students to learn. Training for the tests rather than for reality corrupts the system and practically guarantees that students will not learn to solve real problems. In New York State, for example, the Regents' examinations are generally very good multiple-choice tests, but the common practice of starting to train for the June Regents' exams in September does great damage to any concept of education that suggests there might be something worth learning that is not on the exams. Fourth, watch students try to solve problems in real situations. Do they recognize a problem? Do they understand they can use mathematical thinking to solve it? Do they try to avoid using mathematics, or do they seem to enjoy using the mathematics? Are they able to use mathematical thinking effectively in solving the problem?

In general, if teaching and learning have clearly been oriented toward solving real problems using mathematical thought, and if students are willing and able to use such thought to solve problems, and if they do well on good paper-and-pencil tests without having been trained exclusively for those tests, they are probably getting a pretty good mathematics education. What is to be avoided is training for a particular test to the exclusion of real problem solving and presuming that high scores on that test demonstrate effective learning.

Evaluating teachers is even more difficult than evaluating students. One obvious way is to evaluate their students by the criteria suggested in the previous paragraphs. If the students are doing well, we may assume that their teachers are effective. This makes sense so long as we don't give a teacher either credit or blame for someone else's accomplishment. The teacher may be blessed with students who are naturally good problem solvers or who had a great teacher the year before. On the other hand, he may be confronted by students who practically never come to school, who get no encouragement from home, who are undernourished, or who, because they have been transferred from school to school, never had a good start. Evaluating a teacher by student performance may be reliable, but only with careful consideration of all the circumstances.

Another criterion may be the teacher's knowledge of and evident interest in mathematics. The state of the teacher's knowledge and disposition may be hard to judge, and college transcripts do not always reveal either. Today, however, there is a serious shortage of qualified mathematics teachers. Almost anyone qualified to teach high school mathematics can find a job in industry paying considerably more than schools pay. Because there is not a shortage of teachers of most other subjects, teachers of social studies, physical education, English, etc. find that they can keep their jobs by being recertified to teach mathematics. Many state and city certification agencies are allowing teachers to do this by taking courses labeled "mathematics" that have almost no advanced mathematical content.

Universities under financial strain sometimes cater to this market by creating courses of little intellectual challenge for recertification purposes. Such practice is reprehensible. It produces teachers who have neither the ability nor the inclination to teach mathematics although they may have seniority over qualified younger teachers, and it will damage mathematics education in this country for decades to come unless it is modified quickly. Schools and universities that participate in this sham should be threatened with loss of accreditation; city and state certification boards that allow it should be pressed to cease doing so, and school administrators should do everything in their power to avoid hiring poorly trained and motivated teachers or keeping them on the faculty. If the shortage continues (as it almost certainly will), salaries and working conditions will have to be improved to make teaching a more attractive profession. One answer may be to encourage schools and local industries jointly to appoint mathematics teachers with duties and remuneration split appropriately.

I believe that professional organizations, like the National Council of Teachers of Mathematics, should consider a nationwide board certification procedure comparable to that used in professions like medicine. If procedures could be worked out to make sure that only truly qualified mathematics teachers received board certification, there would be justification for paying such teachers more and giving them greater responsibility. In addition, ordinary citizens would have an uncomplicated means of helping to evaluate the quality of school faculties.

Evaluating commercial programs is not easy. Publishers spend substantial amounts of money trying to convince potential customers that theirs is the best possible program. How, then, can you identify a good commercial program?

Some ostensibly objective but obviously fatuous criteria have been propounded, usually starting with copyright date (recent is good) and continuing to include strength of binding, interest level of illustrations, amount of white space (more is good), number of topics per lesson (fewest is best, presumably with one the optimum). Almost always, such lists fail to mention the burning questions: Does the program work? How do you know? After reading a few pages do you think you would enjoy learning from this book? Does the book make provisions for integrating games, activities, and other real problem-solving opportunities into mathematics education? Does the teacher's guide provide enough help so that an elementary teacher who is not a specialist in mathematics can teach the program successfully? Are there indications that author and publisher tested content and procedures thoroughly, or were the books apparently turned out in a hurry to meet a market demand or a formal adoption schedule? All these questions are worth asking about a textbook series. But the crucial question is *Does it work?*

Few commercial programs have been fully tested before they are issued. If it is at all rigorous, testing is costly both in time and money. For example, one would want to observe the effects of an elementary program cumulatively, beginning with the first grade and extending upward to the eighth. If as little as a year is required for initial planning and writing, nine years will be needed for comprehensive development and testing, and nine years is a long time.

It is a particularly long time for authors and publishers if the schools seem cheerfully disposed to accept programs that have been tested only ceremonially. Indeed, such development and testing seem to be actively discouraged by a severely limiting combination of circumstances, with roots in tradition, in state school codes, in administrative convenience, and in the undeniable difficulty of accurately evaluating learning materials (beyond, that is, noticing the date of copyright). To protect themselves, and students and teachers, publishers issue programs that are inconsequential variants of those that came before, on the apparent theory that "new" programs will at least be no worse than their predecessors.

5

SUMMING UP

Thinking is the basic skill to be taught in school mathematics. Students should learn to use mathematical thought to solve problems that are real to them and important to society. To do this effectively, they will need to master many subsidiary skills, but such mastery should never be mistaken for the basic goal. Parents, teachers, administrators, textbook publishers, and other citizens can contribute to improving the quality of mathematics education. We can push aside the false notion that textbook word problems alone, particularly as they typically appear, teach problem solving. We can develop and use activities, games, and other means to create mathematical environments where students learn to solve problems naturally. And we can begin to evaluate and enhance the success of students, teachers, and commercial programs with the idea firmly in mind that real learning involves using and developing the mind. We can remember — always — that at the heart of problem solving is the most basic and human skill of all — thinking.

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