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ABSTRACT

This document contains three modules. The first of these examines applications of algebra to geometry. It is designed to teach students how to algebraically characterize points which may be constructed with a compass and straight edge, and how to use this characterization to obtain classical geometric nonconstructibility results. The second unit features applications of statistics. It is designed to help the student: 1) perform the t-test on appropriate experimental data and interpret the calculated value of "t"; 2) understand the role of a statistical test of significance in the research process; and 3) recognize the relationship between the statistical arithmetic and the design and conduct of the experimental investigation. The final module views applications of probability. The student is taught to: 1) use the Monte Carlo technique to simulate simple experiments; 2) better appreciate the role of approximate solutions to complex problems. Each of the three modules includes exercises, and answers are provided. The second and third units contain model exams, with answer keys included. (JN)

UMAP

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

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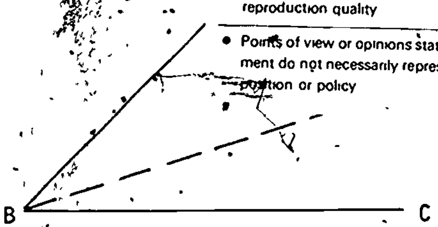
MODULE 267

The Impossibility of Trisecting Angles

by Mark D. Meyerson



Applications of Algebra to Geometry

THE IMPOSSIBILITY OF TRISECTING ANGLES

by

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TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 267*

Title: THE IMPOSSIBILITY OF TRISECTING ANGLES

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Review Stage/Date: IV 4/30/80

Classification: APPL GOEMETRY

References:

Moise, E.E., Elementary Geometry from an Advanced Standpoint, 2nd ed., Addison-Wesley, Reading, MA, 1974.

Peressini, A.L. and D.R. Sherbert, Topics in Modern Mathematics for Teachers, Holt, Rinehart, and Winston, NY, 1971.

Prerequisite Skills:

1. Elementary trigonometry.
2. General equation of a circle.
3. Manipulation of polynomials.
4. Fundamental theorem of algebra.
5. Elementary Euclidean geometry.

Output Skills:

1. To algebraically characterize those points which may be constructed with compass and straightedge.
2. To use this characterization to obtain classical geometric non-constructibility results.

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1. INTRODUCTION

One of the most intriguing geometrical problems of antiquity is to trisect an angle using a compass and straightedge. Although E. Galois proved (around 1830) that it is impossible, in general, to trisect an angle, much effort has since been wasted in futile constructions. Our goal is to give a brief and elementary proof of this nonconstructability. A few related theorems, such as the impossibility of duplicating the cube, are also included.

You might be surprised by all the algebra used in proving these geometric facts. The necessity of approaching these problems algebraically is the reason they were unsolved for so long. In fact, the most striking discoveries in mathematics often result from interplay between apparantly unrelated fields, that is, the application of one branch of mathematics to another branch.

The only background needed for the following material is some elementary high school mathematics, such as factoring polynomials, knowing the equation of a circle, and using some trigonometry.

Here is a precise statement of the problem. Given an arbitrary angle, <ABC, one would like to construct a point D with the measure of <DBC one-third the measure of <ABC. All construction must be done only with compass and straightedge. Given two points E and F, a compass may only be used to draw the circle through E with center F and straightedge may only be used to draw the line through E and F. Points are constructed by intersecting a line or circle with another line or circle. Although certain angles, such as a 90° angle, can be trisected in this manner, we shall see that other angles, such as 60°, cannot be so trisected.

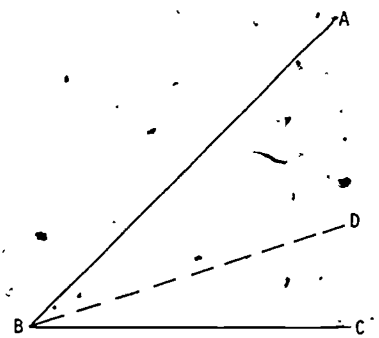

Figure 1. Angle DBC has one third the measure of angle ABC.

The sources for most of this module are the two books, Elementary Geometry from an Advanced Standpoint, by Edwin E. Moise, and Topics in Modern Mathematics for Teachers, by Anthony L. Peressini and Donald R. Sherbert. These books are recommended if you desire to continue with the subject.

2. SUBFIELDS

All our calculations will be done with real numbers. The set of real numbers is denoted by $\mathbb{R}$.

Definition 1. A subset, F, of $\mathbb{R}$, is called a *subfield* (of $\mathbb{R}$) if it contains 0 and 1, and if it is closed under division by non-zero elements of F and subtraction. For example, closed under subtraction means that if a and b are elements of F, so is a - b. Note that a subfield is closed under multiplication and addition, since $ab = a/(1/b)$ and $a + b = a - (0 - b)$. There is a technical definition of *field* which we shall not need.

Examples: 1. $\mathbb{R}$ is a subfield.

2. A number is called *rational* if it can be written as p/q for p and q ($\neq 0$) integers. The set of rational numbers is denoted $\mathbb{Q}$. We show in the aside below that $\mathbb{Q} \neq \mathbb{R}$. But $\mathbb{Q}$ is a subfield, since $0 = 0/1$, $1 = 1/1$, $(p/q)/(r/s) = (ps)/(rq)$ for $r/s \neq 0$ (hence $r \neq 0$), and $p/q - r/s = (ps - qr)/(qs)$.

3. The set of integers is not a subfield, since 1/2 is not an integer.

Aside. $\sqrt{2}$ is not rational.

Proof. Suppose $\sqrt{2}$ is rational. Then we could write it as p/q in reduced form. So $\sqrt{2}q = p$, and squaring, $2q^2 = p^2$.

Since p^2 is even, p must be even. So $p = 2m$ for some integer m. Substituting, we get $2q^2 = (2m)^2 = 4m^2$, or $q^2 = 2m^2$.

Since q^2 is even, q must be even. So p/q is *not* in reduced form, because p and q each have a factor of 2.

Hence $\sqrt{2}$ cannot be a rational number. □

We close this section with a theorem about the roots of an equation.

Theorem 1 (The Rational Root Test). If $a_n x^n + \ldots + a_1 x + a_0 = 0$ is a polynomial equation with integer coefficients and p/q is a rational root, in reduced form, then p divides a_0 and q divides a_n.

Proof: We have $a_n(p/q)^n + a_{n-1}(p/q)^{n-1} + \ldots + a_1(p/q) + a_0 = 0$. Multiplying by q^n we get $a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$. Since p and q each divide the right hand side of this equation, they each divide the left hand side. And since p divides each term on the left, except perhaps $a_0 q^n$, p must divide $a_0 q^n$ also. But p and q have no factors in common, so p divides a_0. Similarly, q divides $a_n p^n$, and so divides a_n. □

3. SURDS

Definition 2. A number is called a *surd* if it can be calculated from 0 and 1 by a finite number of additions, subtractions, multiplications, divisions, and *extractions of square roots.*

Any rational number is a surd. And $\sqrt{\sqrt{2} + 1} - \sqrt[3]{2}$ is a surd. There are many numbers which are not surds. We will see later that $\sqrt[3]{2}$ and $\cos(20^\circ)$ are not surds; also, π is not a surd.

The set of all surds forms a subfield. For 0 and 1 are surds, and if a,b and $c \neq 0$ are surds, so are a - b and a/c.

We now consider the Euclidean plane with a coordinate system.

Definition 3. A *surd-curve* is a circle or line with equation $A(x^2 + y^2) + Dx + Ey + F = 0$, such that all the coefficients are surds. We may assume that A = 1 for a circle (why?) and A = 0 for a line. A *surd-point* is a point (x,y) such that x and y are surds.

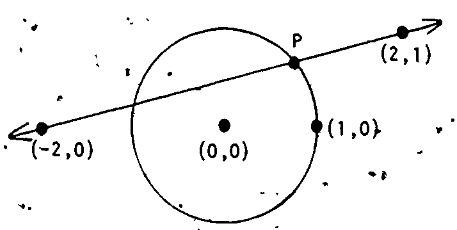

Figure 2. An example of two surd-curves.
$P = ((-2 + 4\sqrt{13})/17, (8 + \sqrt{13})/17)$ is a surd-point.

Theorem 2. If P = (x,y) lies on two distinct surd-curves, then P is a surd-point.

Proof: This can be proven by solving for P, and showing that x and y are surds. We prove only the hardest case, in which both surd-curves are circles.

The two surd-curves have equations $x^2 + y^2 + Dx + Ey + F = 0$ and $x^2 + y^2 + Gx + Hy + I = 0$, with surd coefficients. Subtracting, we get $Jx + Ky + L = 0$, where

J, K, and L are surds. J and K are not both zero, since if they were we would have distinct concentric circles meeting at P.

We now suppose $K \neq 0$. The proof is entirely analogous if $J \neq 0$. So we can solve for y, $y = Mx + N$, where M and N are surds. Subsituting into the very first equation, we get $ax^2 + bx + c = 0$, where a, b and c are surds. Since $a = 1 + M^2$, $a \neq 0$. So $x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$ and $y = Mx + N$, both surds. □

Theorem 3. Given a collection of only surd-points, any point we can construct using compass and straightedge must be a surd-point.

Proof: Let $P = (a,b)$ and $Q = (c,d)$ be surd-points.

It's easy to check directly that the line through P and Q has equation: $(d - b)x + (a - c)y + (bc - ad) = 0$, and that the circle with center P through Q has equation $x^2 + y^2 - 2ax - 2by + (2ac + 2bd - c^2 - d^2) = 0$. All coefficients are surds!

So only surd-curves can be constructed from surd-points. The only way to construct a new point is to consider the intersection of two of these surd-curves, which must be a surd-point by Theorem 2. We can continue constructing curves and points, but only surd-curves and surd-points. □

4. CUBIC EQUATIONS

Definition 4. Let F be a subfield (of $\mathbb{R}$) and let k be a positive number in F such that $\sqrt{k}$ is not in F. Then F(k) denotes the set of all numbers of the form $x + y\sqrt{k}$, where x and y are in F.

For example, if $F = \mathbb{Q}$, $k = 2$, we get $\mathbb{Q}(2)$, which includes $3 + 2\sqrt{2}$, $(1/2) - \sqrt{2}$, and $3 = 3 + 0 \cdot \sqrt{2}$. Or if $F = \mathbb{Q}(2)$, $k = 3$, we get $F(3) = (\mathbb{Q}(2))(3)$ (we shall see in an exercise that $\sqrt{3} \notin \mathbb{Q}(2)$).

Each element of $F(k)$ can be written as $x + y\sqrt{k}$ in only one way. For if $a + b\sqrt{k} = c + d\sqrt{k}$, then $(a - c) = (d - b)\sqrt{k}$. If $b \neq d$, then $\sqrt{k} = (a - c)/(d - b)$ an element of F, contradicting the choice of k. So $b = d$, and hence $a = c$.

Also $F(k)$ is a subfield; let's check the definition of subfield. Now $0 = 0 + 0\cdot\sqrt{k}$ and $1 = 1 + 0\cdot\sqrt{k}$ are in $F(k)$, and $(a + b\sqrt{k}) - (c + d\sqrt{k}) = (a - c) + (b - d)\sqrt{k}$ an element of $F(k)$. So we only need check closure under division by non-zero elements. But

$$\frac{a+b\sqrt{k}}{c+d\sqrt{k}} = \frac{(a+b\sqrt{k})(c-d\sqrt{k})}{(c+d\sqrt{k})(c-d\sqrt{k})} = \left(\frac{ac-bdk}{c^2+d^2k}\right) + \left(\frac{bc-ad}{c^2+d^2k}\right)\sqrt{k}.$$

Note that $\mathbb{Q} \subset \mathbb{Q}(2) \subset$ the set of surds $\subset \mathbb{R}$.

Theorem 4. For $F(k)$ as above, suppose the coefficients of $x^3 + ax^2 + bx + c = 0$ are all in F and that $r + s\sqrt{k}$, an element of $F(k)$, is a root. Then some element of F is a root.

Proof. We may assume that $s \neq 0$, since otherwise we're done.

We have $0 = (r + s\sqrt{k})^3 + a(r + s\sqrt{k})^2 + b(r + s\sqrt{k}) + c = (r^3 + 3rs^2k + ar^2 + as^2k + br + c) + (3r^2s + s^3k + 2ars + bs)\sqrt{k}$. Write this as $A + B\sqrt{k} = 0$. So $A = B = 0$. Putting $r - s\sqrt{k}$ into the polynomial gives us $A - B\sqrt{k} = 0$, since only even powers of s occur in A and odd powers occur in every term of B. So $r - s\sqrt{k}$ is another root.

Now $x^3 + ax^2 + bx + c = (x - x_1)(x - x_2)(x - x_3)$

$= x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3$,

where x_1, x_2, and x_3 are the roots. So let's take $x_1 = r + s\sqrt{k}$, $x_2 = r - s\sqrt{k}$. Then $a = -(x_1 + x_2 + x_3) = -(r + s\sqrt{k} + r - s\sqrt{k} + x_3) = -(2r + x_3)$, so $x_3 = -a - 2r$, an element of F. □

Theorem 5 (Main Theorem) Given cubic equation $x^3 + ax^2 + bx + c = 0$, where the coefficients are rational. If the equation has a surd as a root, then it has a rational root.

Proof. Suppose x_1 is a surd and a root. As a surd, x_1 is in some subfield $(\ldots(\mathbb{Q}(k_1))(k_2)\ldots)(k_n)$. To see this, start to calculate x_1 from 0 and 1. (Recall that by definition, a surd can be calculated from 0 and 1 by additions, subtractions, multiplications, divisions, and abstractions of square roots.) Let $\sqrt{k_1}$ be the first non-rational square root we extract. Continue, until we must extract a square root, $\sqrt{k_2}$, not in $\mathbb{Q}(k_1)$. Continuing in this fashion, we get the above subfield.

By Theorem 4, the given cubic equation has a root in $(\ldots(\mathbb{Q}(k_1))(k_2)\ldots)(k_{n-1})$. Applying Theorem 4 a total of n times, we see that the cubic equation has a root in $\mathbb{Q}$. □

5. NONCONSTRUCTABILITY PROOFS

Theorem 6. The cube cannot be duplicated. In other words, given the edge of a unit cube (a unit segment), we cannot construct (with compass and straightedge) the edge of a cube of twice the volume. (The edge of such a cube would be $\sqrt[3]{2}$.)

Proof. We can think of this as being given surd-points (0,0) and (1,0) and being asked to construct $(\sqrt[3]{2},0)$. So it suffices to show that $\sqrt[3]{2}$ is not a surd.

Suppose it were. Then the cubic equation $x^3 - 2 = 0$ has a surd as a root. By the Main Theorem it has a rational root. But by the Rational Root Test, the only possible rational roots are ± 1 and ± 2 which are not roots. So $\sqrt[3]{2}$ is not a surd. □

Theorem 7. There are angles that cannot be trisected with compass and straightedge.

Proof. We actually show that no 60° angle can be trisected. Given a 60° angle, we can choose a coordinate system so that $A = (1, \sqrt{3})$, $B = (0,0)$, $C = (2,0)$ and the given angle is $\angle ABC$. (Note that A, B, and C are surd-points

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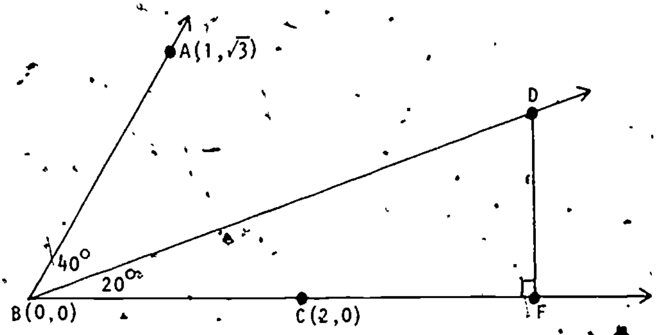

Figure 3.

which form the vertices of an equilateral triangle.) We want to show that there is no surd-point D such that the measure of angle DBC = 20°.

Suppose there were such a D. Let $\overline{DF}$ be the perpendicular to the x-axis. F is a surd-point since it lies on the two surd-curves $y = 0$ and $x =$ (x-coordinate of D). Since the distance between two surd-points is a surd, $\cos 20^\circ = BF/BD$ is a surd. Next we shall use the standard trigonometric identities:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(2A) = 2 \sin A \cos A$$
$$\cos(2A) = \cos^2 A - \sin^2 A$$
$$1 = \sin^2 A + \cos^2 A$$

Now $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta = (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta = (2\cos^2\theta - 1 - 2(1 - \cos^2\theta))\cos\theta = (4\cos^2\theta - 3)\cos\theta$. Since $\cos 60^\circ = 1/2$, we let $\theta = 20^\circ$ to see that $\cos 20^\circ$ is a solution of $1/2 = 4y^3 - 3y$. Letting $y = x/2$, the surd $2\cos 20^\circ$ is a root of $x^3 - 3x - 1 = 0$. By the Main Theorem, $x^3 - 3x - 1 = 0$ has a rational root. But the only possibilities are ± 1, which are not roots. This contradiction implies the Theorem.

<u>Theorem 8</u>. It is impossible to construct a regular seven sided polygon (heptagon) with compass and straightedge.

<u>Proof</u>. Suppose we could. Then we can construct the central angle, $\theta = 360^\circ/7$. And so, as before, $x_0 = \cos\theta$ is a surd.

Now $3\theta + 4\theta = 360^\circ$, so $\cos 3\theta = \cos(360^\circ - 4\theta) = \cos 4\theta$. So $4\cos^3\theta - 3\cos\theta = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1$. Hence x_0 is a solution of $4y^3 - 3y = 2(2y^2 - 1)^2 - 1$, $4y^3 - 3y = 8y^4 - 8y^2 + 1$, $16y^4 - 8y^3 - 16y^2 + 6y + 2 = 0$.

So $2x_0$ is a root of $x^4 - x^3 - 4x^2 + 3x + 2 = 0$. Since 2 is a root of this, we see that the left hand side equals $(x - 2)(x^3 + x^2 - 2x - 1)$. But $x_0 = \cos\theta \neq 1$, so $2x_0 \neq 2$, and $2x_0$ is a surd and a root of $x^3 + x^2 - 2x - 1 = 0$. By the Main Theorem, there must be a rational root. But neither ±1 are roots, so we have a contradiction.□

6. SUMMARY

First some algebraic background. Subfields of the real numbers are subsets of the real numbers that contain 0 and 1, and that are closed under (non-zero) division and subtraction. The Rational Root Test allows us to find all rational roots of a polynomial with integer coefficients.

Next we consider constructions. The subfield of surds is the smallest subfield in which we can take all square roots. The basic property of ruler and compass construction is that if we start with surd-points, then we can construct only surd-points.

In final preparation we need a basic algebraic result. We extend a subfield by including some square roots and the numbers needed to make our new set a subfield. For a cubic equation with rational coefficients, if there is a surd root (which is necessarily in some finite extension of the rationals) then there is a rational root.

Finally, we suppose we could trisect a 60° angle (defined using 3 surd-points) to get a 20° angle. Then, since we can construct only surd-points, we show using standard trigonometric identities that the cubic equation $x^3-3x-1 = 0$ has a surd root. Hence, by the previous paragraph, it has a rational root. But that is contradicted by the Rational Root Test.

7. EXERCISES

SECTION 2

1. Prove that $\sqrt{3}$ is not rational.
2. Find all the roots of
   a. $2x^3 + x^2 - 5x + 2 = 0$
   b. $x^3 - 2x^2 + 1 = 0$
3. Prove that any subfield contains the set of rational numbers.
4. Prove that the set of all numbers of the form $a + b\sqrt{2}$, where a and b are rational, is a subfield.

SECTION 3

5. Complete the proof of Theorem 2. In other words, prove that P is a surd-point if P lies on two distinct surd-curves:
   a. which are lines;
   b. one of which is a line and the other a circle.
6. Show that the curves in Figure 2 are surd-curves.
7. Prove that the distance between two surd-points is a surd.

SECTION 4

8. Prove that $x^3 - x + 2 = 0$ has no surds as roots.
9. Prove that $x^3 - 2 = 0$ has no surds as roots.
10. Characterize all subfields of the form $\emptyset(k)$ as follows:
    a: Show that $\emptyset(p/q) = \emptyset(pq)$, for p and q positive integers.
    b. Show that $\emptyset(a^2p) = \emptyset(p)$, for a and p positive integers.
    c. Show that if p and q are positive integers greater than one, neither of which contains a perfect square (other than 1) as a factor, and $\emptyset(p) = \emptyset(q)$, then p = q.
    d. Conclude that we get a complete non-repititious list of subfields of the form $\emptyset(k)$ by letting k range over the integers greater than one which contain no non-trival squares as factors.

11. Show that $(\emptyset(2))(3)$ is not equal to any $Q(k)$.

SECTION 5

12. Assume without proof that π is not a surd. Then prove that, in general,

   a. We cannot construct a segment whose length is the circumference of a circle with given diameter segment.

   b. We cannot construct a square whose area equals the area inside a circle, with a given diameter segment (known as squaring the circle).

   (Hint: Given a segment and a ray, it is possible to construct a point on the ray whose distance from the endpoint of the ray is the length of the segment.)

13. Develop a scheme which will trisect any given angle if its measure is $p \cdot 90^{\circ}/2^{n}$ where p and n are integers. (Hint: a) It's possible to tell whether angles coincide. b) It's possible to construct an angle with measure 60°. c) It's possible to bisect any angle. d) It's possible to 'copy' an angle in another location.)

14. Prove that it is impossible to trisect an angle of 30° with a compass and straightedge.

15. Construct a regular n-sided polygon with compass and straightedge for n = 3,4,6,8.

8. SOLUTIONS TO MOST EXERCISES

1. Mimic the proof of the Aside in Section 1. Suppose $\sqrt{3} = p/q$, in reduced form. Then $p^2 = 3q^2$, so 3 divides p^2, and hence 3 divides p. So $p = 3m$ for some integer m. Then $9m^2 = 3q^2$, or $q^2 = 3m^2$, and 3 divides q^2, so 3 divides q. But then p/q is not in reduced form.

2. a. By the Rational Root Test, the only possible rational roots are ± 1, ± 2, $\pm 1/2$. Since 1, -2, 1/2 work, they must be all three roots.

   b. As in a., we only need check ± 1. Since 1 is a root, we can factor: $x^3 - 2x^2 + 1 = (x - 1)(x^2 - x - 1)$. By the quadratic formula, the other two roots are $(1 \pm \sqrt{5})/2$.

3. Since any subfield, F, contains 0 and 1 and is closed under subtraction, it contains $-1 = 0 - 1$. Let n be an integer of smallest magnitude not in F, so $n - (\pm 1)$ is in F, and hence $n = n - (\pm 1) - (\mp 1)$ is in F, a contradiction. Hence F contains all integers. Since F is closed under non-zero division, it contains all rational numbers, p/q.

4. One must check that we have closure under subtraction and non-zero division (by rationalizing the denominator). This is done in Section 4 (take $k = 2$).

5. a. Suppose P lies on $Ax + By + E = 0$ and $Cx + Dy + F = 0$ with all coefficients surds. Since these are distinct lines which meet, the difference of their slopes, $(-A/B) - (-C/D) = -(AD - BC)/BD$, is non-zero. So for $d = AD - BC$, d is a non-zero surd. (If one of the lines has infinite slope, then either $d = -BC \neq 0$ or $d = AD \neq 0$, and d is still a non-zero surd.) Solving, we get $x = (BF - DE)/d$ and $y = (CE - AF)/d$, both surds.

   b. Suppose P lies on $Jx + Ky + L = 0$ and $x^2 + y^2 + Dx + Ey + F = 0$, with all coefficients surds. Proceed exactly as in the last paragraph of the proof of Theorem 2.

6. The circle has equation $x^2 + y^2 + (-1) = 0$ and the line equation $-1(x) + 4y + (-2) = 0$.

7. If (x_1, y_1) and (x_2, y_2) are surd points, then x_1, x_2, y_1, y_2 are all surds. So the distance between the points,

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, is a surd.

8. By the Rational Root Test the only possible rational roots are ± 1 and ± 2. These are not roots, so by the Main Theorem there are no surd points.

9. Same argument as 8.

10. a. $Q(p/q)$ consists of all numbers of the form $a + b\sqrt{p/q} = a + (b/q)\sqrt{pq}$ where a and b are in Q. But this gives us all the elements of $Q(pq)$.

 b. An element of $Q(a^2p)$ is the form $c + d\sqrt{a^2p} = c + da\sqrt{p}$. This is an element of $Q(p)$ and every element of $Q(p)$ can be written this way.

 c. Since $Q(p) = Q(q)$, $\sqrt{p}$ is in $Q(q)$, so $\sqrt{p} = a + b\sqrt{q}$ for some a and b in Q. Squaring, we get $p = (a^2 + b^2q) + 2ab\sqrt{q}$. Since every element of $Q(q)$ has a unique representation in the standard form, $ab = 0$. So $a = 0$ or $b = 0$. If $b = 0$, then $p = a^2$, contradicting the assumption that p has no non-trivial square factor (a must be an integer, since p is, and $a \in Q$). So we must have $a = 0$, and $p = b^2q$. Write $b = k/m$ in reduced form. Then $m^2p = k^2q$. Since p and q have no non-trivial square factors, $m = k = 1$ and $p = q$.

 d. From a and b we see that we get all such subfields, and c tells us that our list doesn't repeat.

11. By 10, if $(Q(2))(3) = Q(k)$ for some k, we may assume k is a positive integer greater than one with no non-trivial squares as factors. But $\sqrt{2} \in Q(k)$ and $\sqrt{3} \in Q(k)$. By the proof used in 10 c., we get $2 = k = 3$, an impossibility.

12. a. Choose a coordinate system so that the ends of the given diameter have coordinates (0,0) and (1,0). If we could construct a segment, with length the circumference, we could construct (with the hint) the point $(\pi, 0)$. Since $(\pi, 0)$ is not a surd-point, this contradicts Theorem 3.

b. As in a., we could construct the point $(\sqrt{\pi}/2, 0)$. This is not a surd-point, since $(\sqrt{\pi}/2)(\sqrt{\pi}/2)4 = \pi$, so we have a contradiction.

13. Start to list the pairs (p,n), by listing the pairs whose absolute values of coordinates add to 0,1,2,...,((0,0), (1,0), (0,1), (-1,0), (0,-1), (1,1), (2,0),...). For each pair, (p,n) construct a 60° angle. Bisect (or double if negative) n + 1 times to get a $30^\circ/2^n$ angle. Copy this angle $|p|$ times to get a $|p|\ 30^\circ/2^n$ angle. Copy this angle, α, 3 times and see whether it can be placed to coincide with the given angle. If so, α can be placed to determine the angle trisector.

14. If we could trisect a 30° angle, then we could take a 60° angle, bisect it to get a 30° angle, trisect that to get a 10° angle, double that to get a 20° angle (see 13) and we would have trisected a 60° angle. This is impossible by the proof of Theorem 7.

UMAP MODULE 268

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

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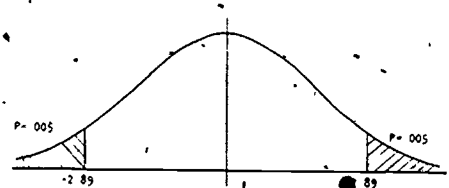
Testing a Hypothesis: t-Test for Independent Samples

by Herbert L. Kayne



Applications of Statistics

TESTING A HYPOTHESIS:

t-TEST FOR INDEPENDENT SAMPLES

by

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Intermodular Description Sheet: UMAP Unit 268

Title: TESTING A HYPOTHESIS: t-TEST FOR INDEPENDENT SAMPLES

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Review Stage/Date: IV 4/29/80

Classification: APPL STAT/t-TEST(U268)

Suggested Support Material: Hand calculator, 't' table, and references to source material.

References:

Bryant, E.C. (1966). Statistical Analysis, 2nd ed. McGraw-Hill, Inc., N.Y.

Dixon, W.J. and F.J. Massey (1969). Introduction to Statistical Analysis. McGraw-Hill, Inc., N.Y.

Hoel, P.G. (1965). Introduction to Mathematical Statistics, 3rd ed. John Wiley and Sons, Inc., N.Y.

Snedecor, G.W. and W.G. Cochran (1967). Statistical Methods, 6th ed. Iowa State University Press, Ames.

Prerequisite Skills:

1. Define: frequency distribution, Gaussian ('normal') distribution, t-distribution for small samples, mean, variance, degrees of freedom, standard deviation and standard error.
2. Given a small sample of measurement data (continuous variable), calculate the mean, variance, standard deviation and standard error.

Output Skills:

1. Perform the t-test on appropriate experimental data and interpret the calculated value of 't'.
2. Understand the role of a statistical test of significance in the research process. Recognize the relationship between the statistical arithmetic and the design and conduct of the experimental investigation.

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1. THE BIOLOGICAL PROBLEM

In evaluating a biological experiment, an investigator often wants to know whether the results obtained under the experimental condition are really different from the results under the control condition. The following statistical test, the t-test for independent groups, allows the experimenter to answer that question with confidence. The experimental procedure, the rationale of the statistical test, the analysis of the data and the interpretation will be presented. This is an application of statistics to biological research as well as to research in several other disciplines.

2. A COMPARATIVE EXPERIMENT

2.1 Setting up the Experiment

Based on prior knowledge and reasoning, an investigator has an idea, for example, that a particular hormone affects the calcium ion concentration of heart muscle. That is, the calcium ion concentration of a group of animals injected with the hormone will differ from the calcium ion concentration in a group of animals not given the hormone. Assume there are 20 animals available for the experiment. By means of a coin flip, each animal is "randomly" allocated to either the experimental group or the control group. The result might be 9 animals in the experimental group and 11 in the control group. The animals are treated for one week. On each day each animal is given an injection -- the experimental animals get the hormone and the control animals are injected with the solvent in which the hormone is dissolved. At the end of the treatment period, the animals are sacrificed and the calcium ion concentration is determined for each heart.

2.2 Features of the Chosen Design

There are two critical features of the design and conduct of this experiment. First, by random allocation of animals to groups, one tries to ensure that there are no systematic differences between the two groups prior to the actual experiment. One assigns to "chance" the task of making the groups comparable. This will be the basis of the statistical arithmetic. Second, the comparison is made with all animals in both groups treated as nearly alike as possible. Thus, it is assumed that the only difference between the two groups is the presence or absence of the hormone. The conclusion depends on this.

An incidental feature is that the sample sizes need not be equal. Under most circumstances, samples of equal size do represent the most efficient utilization of experimental material, but minor inequality does not present a serious problem. In the above example, were the allocation to be rather extreme, e.g., 13 and 7, one would simply discard it and try again.

3. EXPERIMENTAL DATA

TABLE I

Calcium Ion Concentration of Heart Muscle

(micrograms per gram of wet weight)

Control Group (n = 11)	Experimental Group (n = 9)
189	222
172	215
154	206
230	159
193	230
110	211
134	241
174	190
173	199
192	
160	

Symbolically, each value is denoted as X_i, the sample size as n, the sample mean as $\overline{X}$, and the sample variance as s^2. Where

$$\overline{X} = \frac{\Sigma X_i}{n}$$ [1]

and

$$s^2 = \frac{\Sigma(X_i - \overline{X})^2}{n - 1}.$$

4. STATISTICAL RATIONALE

4.1 The Null Hypothesis

Prior to making the calcium measurements, tentatively assume that in terms of calcium ion concentration of heart muscle the two groups of animals are random samples from one population. In common usage, this is expressed in several ways: 1) there is no real difference between the two groups, 2) the mean calcium ion concentration of the control group is equal to the mean of the experimental group, 3) the hormone had no effect on calcium ion concentration, or 4) whatever variation there is among the 20 measured values of calcium is due to sampling variation from one population. Specifically, this tentative assumption is called the null (no difference) hypothesis. Symbolically the *null hypothesis* for the control and experimental samples is denoted,

$$\overline{X}_{control} - \overline{X}_{experimental} = 0.$$

4.2 Test of Significance

Having made the experimental measurements, a statistical test of significance is performed. The test asks, "What is the probability that chance alone is responsible for the discrepancy between the experimental result and the null hypothesis?" If this probability is large, the

[1] In statistics, summations are employed so much that it has become a convention to use Σ instead of the more precise notation $\sum_{i=1}^{n}$. In keeping with this convention "Σ" is used for "$\sum_{i=1}^{n}$" throughout this module.

null hypothesis will be accepted. If the probability is small, the null hypothesis will be rejected. Notice two things. The answer to the test question is not an absolute one; it is a probability statement. Moreover, acceptance or rejection of the null hypothesis requires that an arbitrary decision be made as to what constitutes "large" and "small."

A test of significance is a ratio:

$$\frac{\text{a measure of the effect}}{\text{a measure of the variation}}$$

In effect it asks, "How big is the difference between the two means relative to the uncertainty associated with that difference?" From the context of this experimental situation, it is reasonable that for this so-called t-ratio the numerator is the difference between the mean of the experimental group and the mean of the control group. What is no apparent is the nature of the denominator. The denominator reflects the variation one can expect between two means by random sampling from one population.

The following statements describe what the denominator is. For a rigorous development of why it is that, one should consult a textbook of mathematical statistics.[2] Assume that calcium ion concentration of heart muscle is a continuous variable that tends to follow the Gaussian distribution.[3] The variance (s^2) of a sample is the estimate of theoretical population variance, σ^2. The variance of a sample mean is estimated by $\frac{s^2}{n}$. The variance of the difference between two means is equal to the sum of the variance of the two means -- the denominator -- is equal to the square root of the variance of the difference.[4]

---

[2] Hoel, P.G., Introduction to Mathematical Statistics, 3rd ed., John Wiley & Sons, Inc.

[3] For large samples, the assumption of Gaussian distribution is not necessary; for small samples in which this assumption is unlikely to hold, there are alternatives to the t-test.

[4] For a detailed analysis of testing a hypothesis see [S-1].

4.3 The t Statistic (Ratio)

Were the sample sizes equal, the t-ratio would then be:

$$t = \frac{\bar{X}_C - \bar{X}_E}{\sqrt{\frac{s_C^2}{n_C} + \frac{s_E^2}{n_E}}}$$

When the sample sizes differ, as in our example, a weighted average of s_C^2 and s_E^2 is used. Weight is based on the number of data in each sample. The degrees of freedom are respectively, $n_C - 1$ and $n_E - 1$. The resulting ratio is:

$$t = \frac{\bar{X}_C - \bar{X}_E}{\sqrt{\frac{\Sigma(X_C - \bar{X}_C)^2 + (X_E - \bar{X}_E)^2}{n_C + n_E - 2}\left(\frac{1}{n_C} + \frac{1}{n_E}\right)}}$$

As an exercise in algebra, it is left for the curious reader to show that this expression reduces to the expression above when $n_C = n_E$. (Recall that $s^2 = \Sigma(X_i - \bar{X})^2/n - 1$.)

5. DATA ANALYSIS

Control Group	Experimental Group
$\bar{X} = 171.00$	$\bar{X} = 208.11$
$n = 11$	$n = 9$
$\Sigma(X_i - \bar{X})^2 = 10,244.0$	$\Sigma(X_i - \bar{X})^2 = 4,636.88$

$$t = \frac{171.00 - 208.11}{\sqrt{\frac{10,244.0 + 4,636.88}{18}\left(\frac{1}{11} + \frac{1}{9}\right)}}$$

$$t = -2.87.$$

6. INTERPRETATION

6.1 Importance of Degrees of Freedom

The original question was, "What is the probability that chance alone is responsible for the discrepancy between the experimental result and the null hypothesis?" Now that question becomes, "What is the probability of getting a t-value as large as -2.87 by random sampling?"

The t-distribution is a theoretical probability distribution that is symmetrical and bell-shaped, like the Gaussian curve. In addition, there is a different t-distribution for each *degree of freedom*. (For degrees of freedom above 30, the t-distribution is very similar to the Gaussian distribution; in fact, we can say that the Gaussian distribution is t with infinite degrees of freedom.) The degrees of freedom for the t-test for independent groups is defined as $(n_C - 1) + (n_E - 1)$. The t-values and their corresponding probabilities are tabulated in most statistics texts.[5] Part of a t-table is shown here.

Degrees of Freedom	Probability (two-tailed) .10	.05	.01	.001
5	2.02	2.57	4.03	6.86
10	1.81	2.23	3.17	4.59
18	1.73	2.10	2.89	3.92
30	1.70	2.04	2.75	3.65

For 18 degrees of freedom, the t-value of 2.87 has a probability of about .01.

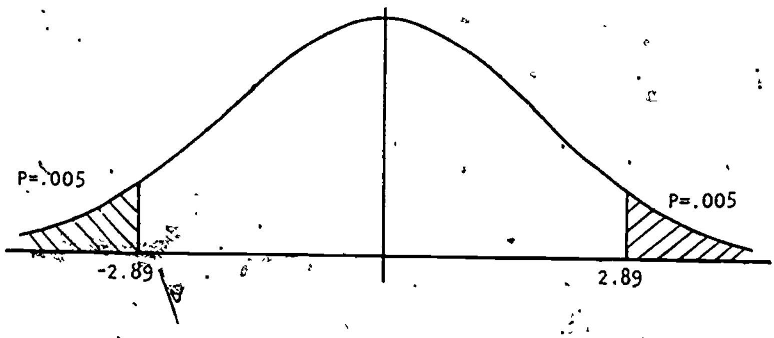
6.2 The Level of Significance

Most biological investigators agree that a probability of .05 or less is "small." Thus, in keeping with the aforementioned ground rules, the null hypothesis is rejected in this example. It is improbable that sampling

[5] Snedecor, G.W. and W.G. Cochran, Statistical Method, 6th ed., Iowa State University Press, Ames.

variation alone is responsible for the experimental results. The two groups differ by more than one would expect by chance, and since the groups are comparable except for the presence or absence of the hormone, the conclusion is that the hormone altered the calcium ion concentration.

Since the alternate hypothesis did not specify the direction of the difference, the two-tailed probability was used for the t-value of -2.87. The tabulated t-distribution for df = 18 shown graphically is:



If the investigator had specified, a priori, that if there were a difference the experimental mean would be larger than the control (as it turned out), then the one-tailed probability associated with the calculated t-value of -2.87 would be .005.

7. CONFIDENCE IN THE CONCLUSION

The statistical t-test is designed to aid the investigator in making a decision. Where treatment effects are small (but perhaps important) and biological variation among individuals is large, the test can be particularly helpful. However, having rejected the null hypothesis in this example, the question remains, "How confident can the investigator be?" At least three aspects of the experiment must be considered -- the allocation of animals to the two groups, the physical conduct of the experiment, and the statistical procedure.

One relies on random allocation to control all extraneous factors. Randomization holds in the long run; but in the short run, as one would expect, it might not do its job completely. Short of checking one or two factors, such as body weight of the animals in this example, the investigator cannot evaluate the vicissitudes of randomization. Whether all factors pertinent to calcium ion concentration of heart muscle are balanced among the two groups remains unknown.

It is in the actual conduct of the experiment that factors other than the presence or absence of hormone are most likely to bias the outcome. It is difficult for the investigator and other participants in the experiment to treat the two groups of animals exactly the same. An essential safeguard is to keep the participants "blind" to the group designation of each animal. This minimizes their subconscious tendency to bias the outcome. Unfortunately some treatments defy masking. In general, the magnitude of investigator bias cannot be evaluated.

The uncertainty in the statistical test can be quantified. By arbitrarily selecting a probability of .05 or less for rejecting the null hypothesis, one knows that the chance of rejecting a true null hypothesis is .051. In this example, t-values of 2.87 or greater can occur by random sampling from one population; they will occur almost 1 in 100 times. According to the ground rules of the test of significance, the null hypothesis will be rejected every time the t-value is $\geq$ 2.87. One out of a hundred times will be an error.

As part of the experimental plan, the investigator can control this error by altering the critical probability. For example, by defining "small" as a probability of .0001 or less, one minimizes the probability of erroneously rejecting the null hypothesis. But by so doing, one maximizes the probability of accepting a null hypothesis that is actually false and should be rejected.

So while the intent of the statistical test was to increase confidence in the conclusion, it is nonetheless true that uncertainty still remains. Hence the word, "re-search." Only by replication of the experiment, particularly by other investigators and in other settings, is one's confidence fortified.

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This material was field-tested and/or student reviewed in preliminary form by Robert G. Goodale, Boston State College, Boston, Massachusetts; Brian J. Winkel, Albion College, Albion, Michigan; W. Hugh Haynsworth, The College of Charleston, Charleston, South Carolina; Gail L. Lange, University of Maine, Portland, Maine; T.R. Hamlett, Arkansas Technical University, Russellville, Arkansas; Reider Peterson, Southern Oregon State College, Ashland, Oregon; Richard M. Meyer, Niagara University, Niagara, New York, and Jay E. Anderson, Idaho State University, Pocatello, Idaho, and has been revised on the basis of data received from these sites.

8. EXERCISES

1. An experiment was designed to test the effect of a vitamin supplement on weight gain in mice. Animals were randomly allocated to two groups. One group was given ordinary chow and the other was given chow to which the vitamin supplement had been added. Each mouse was weighed at the beginning of the experiment and after 10 days. The data are expressed as gain in body weight (grams) in 10 days.

Chow	Chow + Vitamin
4.0	6.4
5.7	4.3
4.2	5.5
5.3	4.5
5.1	

Does the vitamin supplement alter growth?

2. The effect of ambient temperature on food intake was studied in rats. One group of 10 rats was maintained at 25°C. Their average food intake over a period of 21 days was 161 grams with a standard deviation of 9 grams. The other group of 10, maintained at 20°C, had a mean intake of 204 grams with S.D. = 12. Is there a statistically significant difference between the two groups?

3. One of the important questions that an investigator must answer before he begins an experiment is, "How many animals shall I use?" Consideration of the t-ratio allows one to make an educated guess at the answer. Basically one uses preliminary data and/or makes reasonable estimates of all the terms in the t-ratio except 'n' and then solves for 'n'. For example, an investigator was planning an experiment to measure the effect of removal of the testes on developed tension of heart muscle in dogs. In a pilot study on several normal dog hearts, the mean developed tension was 2.4 grams with a standard deviation of .4 grams. If removal of the testes were to have an appreciable effect, the investigator guessed that the mean tension might be reduced by .5 grams. How many animals should be used in the control and experimental groups in order to detect a significant difference at a probability of .05? 10

9. ANSWERS TO EXERCISES

1.

Chow	Chow & Vitamins
$\Sigma X = 24.30$	$\Sigma X = 20.70$
$n = 5$	$n = 4$
$\overline{X} = 4.86$	$\overline{X} = 5.18$
$\Sigma(X_i - \overline{X})^2 = 2.13$	$\Sigma(X_i - \overline{X})^2 = 2.83$

$$t = \frac{\overline{X}_C - \overline{X}_{CV}}{\sqrt{\frac{\Sigma(X_C - \overline{X}_C)^2 + \Sigma(X_{CV} - \overline{X}_{CV})^2}{n_C + n_{CV} - 2}\left(\frac{1}{n_C} + \frac{1}{n_{CV}}\right)}}$$

$$t = \frac{4.86 - 5.18}{\sqrt{\frac{2.13 + 2.83}{7}\left(\frac{1}{5} + \frac{1}{4}\right)}} = -.56$$

$df = (5 - 1) + (4 - 1) = 7.$

By inspection of the abbreviated t-table in this unit, $P > .05$. Accept the null hypothesis; there is insufficient evidence to conclude that the vitamin supplement alters growth. Notice the implication -- *if* there were data on more animals there *might* be evidence to reject the null hypothesis. Accepting the null hypothesis is not the same as 'proving' that the two groups are the same.

2.

$20^{\circ}C$	$25^{\circ}C$
$\overline{X} = 204$	$\overline{X} = 161$
$n = 10$	$n = 10$
S.D. = 12	S.D. = 9
$s^2 = 144$	$s^2 = 81$

$$t = \frac{\bar{X}_{20} - \bar{X}_{25}}{\sqrt{\frac{s^2_{20}}{n_{20}} + \frac{s^2_{25}}{n_{25}}}} = \frac{204 - 161}{\sqrt{\frac{144}{10} + \frac{81}{10}}} = 9.06$$

df = 18, $P < .001$, reject null hypothesis.

3. Assume that the two means will be 2.4 and 1.9 (1.9 is a reduction of .5 from the mean of 2.4 in the pilot study). Assume that the standard deviations will both be .4 (as was found in the pilot study) so that the variances, s^2 will both be .16. Since, at this point, the degrees of freedom are unknown, assume that the t-value at $P = .05$ is 2.0. In the expression for 't', solve for n:

$$2 = \frac{2.4 - 1.9}{\sqrt{\frac{.16}{n} + \frac{.16}{n}}}$$

$n = 5$, roughly.

Were there to be 5 animals in each group, the degrees of freedom would be 8. The 5% value of t at df = 8 is 2.3, so sample sizes of 6 or 7 would be safer. How conservative the investigator guesses relates to the importance associated with a difference between means of a given amount as well as to the resources available for the experiment. At best, guess work can only result in a "ball-park" estimate.

10. MODEL EXAM

1. a. Cerebral blood flow was measured in 10 dogs under anesthesia. Hemorrahagic shock was induced in 5 dogs by removing 20% of their circulating blood volume. The other 5 dogs were the controls. All blood flow measurements were made one hour after induction of anesthesia, and this was also 30 minutes after hemorrhage in the experimental animals. The flow measurements (in ml/min/100 g brain tissue) were:

Control	Experimental
10.7	9.5
12.1	7.1
11.9	8.1
8.6	6.2
10.0	7.0

Does the data imply that there is a significant difference in blood flow to the brain following severe hemorrhage?

b. Could this experiment have been designed in another way that might be more sensitive in detecting the effect of hemmorhage on blood flow?

2. A neurobiologist was studying the incorporation of amino acid into protein in the brain of rats at 15 days of age. She kept one group of 6 rats in a single cage. Another group of 6 rats were kept in individual cages. These animals tend to be more aggressive than those living together. After a 'treatment' period of 2 weeks, radioactive amino acid was injected intravenously into each animal; one hour later the animals were sacrificed and the brain was dissected out for measurement of radioactivity. For each of the two groups, the data are presented in summary form (mean and standard deviation) rather than the uptake of radioactivity for each animal.

	Specific radioactivity (counts/min/mg of tissue)
Single cage	3600 ± 400 (mean ± S.D. in 6 rats)
Individual cages	2300 ± 500 (mean ± S.D. in 5 rats; 1 died)

Does it seem as if the environment has altered the uptake of amino acid?

3. An ornithologist at University A has been studying the body temperature of a particular species of bird that he trapped in the wild. A colleague at near-by University B had trapped this species 2 years before, and he too had recorded body temperature as part of a more complicated study. He decided to compare the body temperature data via t-test, and he was surprised to find a striking difference ($P < .001$) between the data of ornithologists A and B. Why should one not be surprised?

11. ANSWERS TO MODEL EXAM

1. a. $t = 3.59$, $P < .05$, reject the null hypothesis; hemorrhage decreases blood flow to the brain.

   b. Yes. A more sensitive way would be to make measurements of flow before shock and after shock in each animal. Each animal then serves as its own control. The t-test must be modified and is known as a paired t-test. In part (a), technical reasons negated the measurement of brain blood flow twice in the same animal; hence the design involves two independent groups.

2. For the 'single cage' group:

   s(or SD) = 400; therefore $s^2 = 160{,}000$.

$$s^2 = \frac{\Sigma(X_i - \overline{X})^2}{n_i - 1}; \quad 160{,}000 = \frac{\Sigma(X_i - \overline{X})^2}{5};$$

$$\Sigma(X_i - \overline{X})^2 = 800{,}000.$$

For the 'individual cages' group, the above calculations result in $\Sigma(X_i - \overline{X})^2 = 1{,}000{,}000$. Finally,

$$t = \frac{3600 - 2300}{\sqrt{\frac{800{,}000 + 1{,}000{,}000}{9}\left(\frac{1}{6} + \frac{1}{5}\right)}}$$

$$t = 4.80 \ (P < .05).$$

Reject the null hypothesis; animals in individual cages have reduced uptake of amino acid.

3. Any number of factors could be responsible for this result. Ornithologist B might have trapped a different subset of the wild-ranging population. His thermometer might have differed systematically from that of ornithologist A. The ambient conditions might have differed between the two times and in some way influenced body temperature, etc., etc. The fact is that when one can neither employ random allocation nor make measurements under comparable conditions, a variety of explanations can account for an apparent difference. Plugging numbers into a t formula is unwise unless the experimental design warrants it.

However, there are circumstances where one *cannot* randomly allocate individuals to groups. This is particularly true when one wants to compare normal (presumably healthy) human subjects with patients who have a specific disease. One must be extremely cautious in interpreting a test of significance in this case.

SPECIAL ASSISTANCE SUPPLEMENT

[S-1]

We can formalize the procedure for testing a hypothesis by considering seven steps.

1. State the null hypothesis H_o and the alternative H_1.
2. Choose a level of significance. This is usually referred to as 'α'. The choice of α is arbitrary; however, it is standard to use $\alpha = .05$ or $\alpha = .01$. Please consult a statistics text for a detailed discussion of Type I and Type II errors.
3. Decide on the distribution and the statistic that will best analyze your problem. In this case we use the t-distribution, and the statistic is:

$$t = \frac{\bar{X}_C - \bar{X}_E}{\sqrt{\frac{\Sigma(X_C - \bar{X}_C)^2 + \Sigma(X_E - \bar{X}_E)^2}{n_C + n_E - 2}\left(\frac{1}{n_C} + \frac{1}{n_E}\right)}}$$

   It is important to realize that this is not the only way t can be expressed. Again, you may wish to consult a statistics text for other motivations for and forms of the t-distribution.
4. Choose a region of rejection. In the two-tailed example that is employed in this unit we have:

   $t \leq -2.10$ or $t \geq 2.10$

   assuming $\alpha = .05$. For a single (upper) tail test we have $t \geq 1.73$ for $\alpha = .05$.

   If $\alpha = .01$ we have $t \leq -2.89$ or $t \geq 2.89$ for a two-tailed test, and $t \geq 2.55$* for an upper tail test. All of these statements are of course based on 18 degrees of freedom.
5. Computations: Do the proper substitutions to get a calculated t value as presented in Section 5.

*See Bryant E.C., Statistical Analysis, 2nd ed., McGraw-Hill, Inc. for a complete t Table.

6. Reject or do not reject the hypothesis depending on whether the calculated t is in or out of the stated region of rejection. (If the t is in the region of rejection, we reject the hypothesis.)

(Note: A *statistician* would probably not say that we accept an hypothesis no matter how heavy the weight of the evidence. We must keep in mind that we are examining a sample. Even though our analysis of a sample gives us no mathematical grounds to reject, we cannot be sure that an alternative is in fact true. In short, it is better to say "reject H_o" if in essence we believe H_1 should be accepted, and say reject H_1 if we believe we should accept H_o.)

This sixth step is called the statistical decision. In many cases we may still not reject an hypothesis even if the data and statistics suggest so. This brings us to step 7.

7. Make the scientific or management decision. Decisions of this kind are based on experience and other factors outside the experimental design. This is what is being discussed in Section 7.

[S-2]

The t statistic is sometimes written with the weighted average of s_C and s_E displayed explicity, i.e.,

$$t = \frac{\bar{X}_C - \bar{X}_E}{s\sqrt{(1/n_C) + (1/n_E)}}, \quad \text{where } s^2 = \frac{(n_C - 1)s_C + (n_E - 1)s_E}{(n_C - 1) + (n_E - 1)}.$$

s^2 can be further simplified so that we can write:

$$s^2 = \frac{(n_C - 1)s_C + (n_E - 1)s_E}{n_C + n_E - 2}$$

The degrees of freedom are given by the demoninator n_C+n_E-2.

[S-3]

This supplement is here so that students appreciate the fact that there are some other statistical assumptions that must be made if we desire more precision.

The t- formula suggested here should only be used if the variances are equal ($\sigma_1^2 = \sigma_2^2$). To do things properly a test using $F = s_1^2/s_2^2$ should be done first. (Please refer to a statistics text for further details on the F- distribution.) If we do not reject H_o ($\sigma_2^2 = \sigma_2^2$), then it is all right to proceed with a test for the means using the t statistic suggested. If we are forced to reject H_o and assume $\sigma_1^2 \neq \sigma_2^2$ then a *common* approximation for the t- statistic, which is basically the same form as in the module, is used. However the degrees of freedom is calculated by a long involved expression. The t statistic and the degrees of freedom are:

$$t = \frac{\bar{y}_1 - \bar{y}_2 - (u_1 - u_2)}{\sqrt{s^2\bar{y}_1 + s^2\bar{y}_2}}$$

$$\text{with } \frac{(s^2\bar{y}_1 + s^2\bar{y}_1)^2}{\left[(s^2\bar{y}_1)^2/(n_1+1)\right] + \left[(s^2\bar{y}_2)^2/(n_2+1)\right]} - 2 \text{ degrees of freedom}$$

where $s^2\bar{y}_1$ is the variance of the first sample mean and $s^2\bar{y}_2$ is the variance of the second sample mean.

We will not, with the expression above, get an integer in every instance for the degrees of freedom. However, one may obtain a value for the region of rejection by interpolating in the t table.

As it turns out, the example used does provide variances that are not significantly different. This, however, is only seen after a test of the hypothesis ($\sigma_1^2 = \sigma_2^2$) is done.

A test for equality of variances is sometimes called a test of homogeneity. We give that test below.

The Test of Homogeneity of Variances:

$s_1^2 = 1024.4$ and $s_2^2 = 579.61$ from the data.

1) H_o: $\sigma_1^2 = \sigma_2^2$ H : $\sigma_1^2 > \sigma_2^2$ or $\sigma_1^2 < \sigma_2^2$
2) $\alpha = 0.10$ (arbitrary)
3) $F = s_1^2/s_2^2$ with $n_1 - 1$ and $n_2 - 1$ df.
4) Region of rejection: $F \geq 3.35$ with 10 and 8 degrees of freedom.*
5) $F = \frac{1024.4}{579.61} = 1.7667$
6) Do not reject H_o.

Alternatively, one could use Bartlett's test:

Suppose there are K(2 in this case) variances to be compared, denoted s_1^2 and s_2^2 with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. Then the quality

$$\chi^2 = \frac{2.3026 \left[s_p^2 \Sigma (n_i - 1) - \Sigma (n_i - 1) \log s_i^2 \right]}{1 + \frac{1}{3(k - 1)} \left[\Sigma \frac{1}{n_i - 1} - \frac{1}{\Sigma (n_i - 1)} \right]}$$

is distributed as χ^2 with K - 1 degrees of freedom. χ^2 calculated greater than the critical value for a specific α would suggest rejection of homogeneity i.e., $\sigma_1^2 \neq \sigma_2^2$. ($s_p^2$ is a pooled estimate of the variance.)

*See Dixon, W.J. & Massey, F.J. Introduction to Statistical Analysis. Table A-7 for complete set of F values.

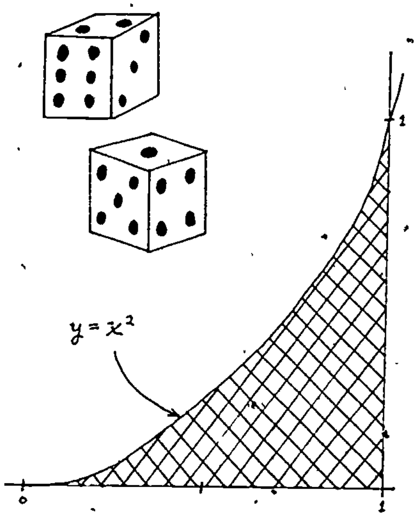
UMAP MODULE 269

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

Monte Carlo:

The Use of Random Digits to Simulate Experiments

Dale T. Hoffman



Applications of Probability

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MONTE CARLO:

THE USE OF RANDOM DIGITS TO SIMULATE EXPERIMENTS

by

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TABLE OF CONTENTS

Intermodular Description Sheet: UMAP Unit 269

Title: MONTE CARLO: THE USE OF RANDOM DIGITS TO SIMULATE EXPERIMENTS

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Review Stage/Date: IV 7/26/79

Classification: APPL PROBABILITY

Prerequisite Skills:

1. Know and understand the "relative frequency" definition of the probability of an event.
2. Construct a frequency histogram.
3. Calculate a mean, median, mode, and standard deviation.

Output Skills:

1. Use the Monte Carlo technique to simulate simple experiments.
2. Realize the strengths and weaknesses of this technique.
3. Better appreciate the role of approximate solutions to complex problems.

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1. PROBLEMS FOR SIMULATION

The world of science and business is full of mathematical problems which cannot be solved exactly or easily. But there is a general procedure which can be used to get workable answers to many of these problems. The following examples show the variety and complexity of the problems which the Monte Carlo technique can solve. They indicate the power and breadth of the application of this straightforward technique. Don't panic if the examples look difficult now, for some of them are. But after completing this module they should seem much easier.

1.1 The Rhythm Method

The rhythm method of birth control is known to be 70% effective. That is, the probability that someone, using this method by itself, will become pregnant in any one year is 30%. What is the expected number of years before someone who uses the method becomes pregnant?

1.2 Lottery

Each ticket in a lottery contains a single "hidden" letter. Among all the tickets, 50% contain a "W," 40% contain an "I," and 10% contain an "N." How many tickets should you expect to buy in order to be able to spell the word "WIN," and thus win a prize? To spell "IWIN?" (Variations of this spelling scheme are used by several state lotteries.)

1.3 Drunkard

A drunkard leans against a lamp post in the middle of a large plaza. He takes one step north or south or east or west and then stops. If he continues to step randomly (each direction is equally likely), how far would you expect him to be from the lamp post after 5 steps? After 10 steps? (A variation of this example is used to model the behavior of a molecule suspended in a liquid. The random motion exhibited by such molecules is called Brownian Motion, after the English botanist, Robert Brown. Brown reported in 1827 that an aqueous suspension of a pollen he was studying contained microscopic particles which carried out a continuous, haphazard zigzag movement. Brown was not the first to notice this phenomenon, but was the first to study it in detail, and was the first to notice that the movement could not be attributed to life in the particles themselves. For more about Brown, see the Encyclopaedia Britannica.)

1.4 Grocery Store

As the owner of a small grocery store you have a choice of hiring

(a) 2 cashiers who do their own bagging, and each of whom can check out a shopper in 2 minutes, or
(b) 1 cashier and 1 boxboy who, working as a team, can check out a shopper in 1 minute.

Based on your experience, you estimate that 30% of the time (minutes) no new shoppers get into the checkout line; 40% of the time, 1 new shopper gets into line; and 30% of the time, 2 new shoppers get into line. Using each checkout system, estimate

(1) the expected waiting time in line per shopper, and
(2) the expected line length a shopper will encounter.

Which checkout system would you adopt for your store? (This type of problem occurs frequently in business. For example, one could simulate the expected cost-performance of proposed inventory plans using data from previous years.)

Exercise 1

For each of the previous problems:
(1) Describe an actual experiment which could be used to obtain an approximate answer, and
(2) List some of the disadvantages of the direct experiments which you proposed in part (1).

---

2. INTRODUCTION

Frequently a scientist or someone in business wants to know how a "system" will behave. If the system is fairly simple, we can sometimes determine mathematically how it will behave. But as the system becomes more complicated, we may not know the necessary mathematics, or the equations may be too complex to solve, or the mathematics may not have been invented yet. At this point we could resort to a series of experiments: If we operated the system long enough, we could get a good idea about how it would behave under different circumstances. Unfortunately, this is not always practical or possible--the system may not have been built yet, the experiments may be very expensive in time or money, or be dangerous or immoral. In these cases we can sometimes run a series of simulated experiments--experiments which behave like the real thing, but which do not have the disadvantages of the real experiments. If designed and run properly, these simulations will mimic the behavior of the real experiments and yield results quicker and cheaper.

In this module we will look at a particular type of discrete simulation called the Monte Carlo technique. It is very powerful and is widely used in science and business. It is attractive because it is easy to use, inexpensive in time and money, and because computers can perform much of the work.

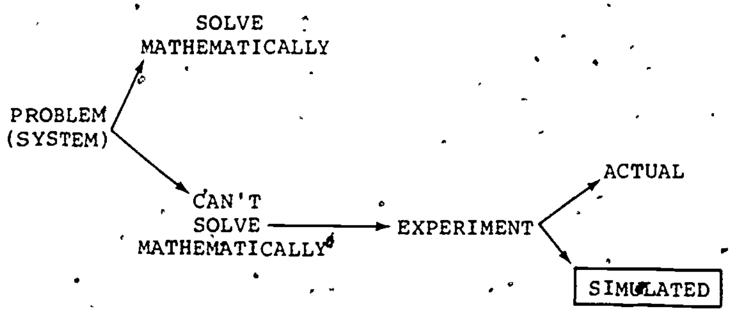

Figure 1. To solve a problem that we cannot solve mathematically, we may be led to simulate an experiment.

3. A WORKED EXAMPLE

3.1 Coin

How many times should we expect to flip a coin before we accumulate three heads?

The actual experiment in this problem would be to flip a coin until a total of 3 heads have appeared, and then to note how many flips were required. This single number, the number of flips, will be an approximation of the answer, but perhaps a poor approximation. To increase your confidence in the accuracy of your approximation, you could repeat the experiment many times, keeping a record of the outcomes.

NOTATION: Pr(A) = probability that event A occurs.

To save time and wear on your thumb, you could use random digits instead of a coin. On each flip there are only two possible outcomes, heads (H) or tails (T), each of which occurs with probability 0.5. If we let the occurrence of an even digit represent H and an odd digit represent T, then a sequence of digits, say 72362, would represent a sequence of outcomes of flips, THTHH. It is crucial to recognize that

Pr(even digit) = 5/10 = Pr(H)

and

Pr(odd digit) = 5/10 = Pr(T).

Starting with the (randomly selected) 29th row of the Random Digit Table in the Appendix, we have the digits 09463 63823 which represent the outcomes HTHHT HTHHT. For our first experiment four flips, HTHHT, were needed to accumulate 3 heads. Starting our second experiment where the first one ended, we have the outcomes T HTHHT. So in the second experiment five flips, T HTHHT, were necessary.

Digits	0946	3 6382	3 2964	3 624	01 06	537 63918 52	056 8	33
H or T	HTHH	T HTHH	T HTHH	T HHH	HT HH	TTT HTTTH TH	HTH H	TT
Experiment Number	1	2	3	4	5	6	7	
Number of Flips to Get 3 Heads	4	5	5	4	4	10	4	

Each vertical line notes the end of one experiment--after accumulating 3 H's a line was drawn. We start the next experiment with the next digit in the table and continue, digit by digit, until 3 more H's are obtained.

Seven experiments are not enough to give much confidence. The results of 100 experiments are given in the frequency histogram in Figure 2. The histogram indicates

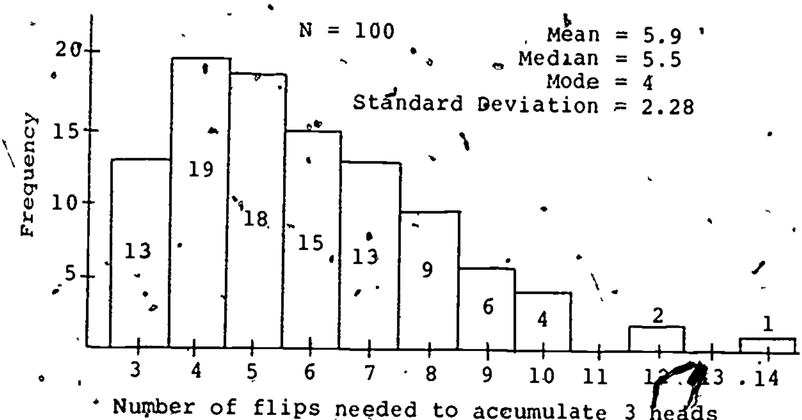

Figure 2. This histogram shows the results of 100 coin-flipping experiments. The goal in each experiment was to accumulate 3 heads. In 13 experiments, the first 3 flips gave 3 heads. In 19 experiments, it took 4 flips to get 3 heads. In 18 experiments, it took 4 flips to total 3 heads, and so on.

that most of the time (65%) it will take 6 or fewer flips to accumulate 3 heads. If we needed more accurate results, we could continue the simulation as long as necessary.

Exercises

2. Verify the values of the mean, median, mode, and standard deviation for the histogram in Figure 2.

3. Use the table of random digits in the Appendix to perform the experiment described above at least 20 times, and construct a frequency histogram of the experimental outcomes.

4. Use a simulation to approximate
   Pr(exactly 3 heads occur when 5 coins are flipped).

4. CORRESPONDENCE BETWEEN DIGITS AND EVENTS

If a cheating gambler had "fixed" the coin in the previous example so that Pr(H) = 0.6, then a different correspondence between the possible outcomes, H or T, and the possible digits, 0 to 9, would be necessary. One possible correspondence would be H $\longleftrightarrow$ {0,1,2,3,4,5} and T $\longleftrightarrow$ {6,7,8,9}. Then

$$Pr(H) = Pr(\{0,1,2,3,4,5\}) = 0.6$$

and

$$Pr(T) = Pr(\{6,7,8,9\}) = 0.4,$$

as required.

If Pr(H) = 0.63, the correspondence becomes only slightly more complex. Instead of using single digits, we can consider pairs of digits. The range of the pairs is 00 to 99, and one possible correspondence would be H $\longleftrightarrow$ {00 to 62} and T $\longleftrightarrow$ {63 to 99}. Then

$$Pr(H) = Pr(\{00 \text{ to } 62\}) = 0.63,$$

as required.

To simulate the rolling of a balanced die (6 sides), we could set up the correspondence

digit "1" in the table	$\longleftrightarrow$	side 1 on the die
digit "2" in the table	$\longleftrightarrow$	side 2 on the die
digit "3" in the table	$\longleftrightarrow$	side 3 on the die
digit "4" in the table	$\longleftrightarrow$	side 4 on the die
digit "5" in the table	$\longleftrightarrow$	side 5 on the die
digit "6" in the table	$\longleftrightarrow$	side 6 on the die
digits "7,8,9,0" in the table	$\longleftrightarrow$	No Event (the die rolled under the desk)

If the digits 7, 8, 9 and 0 are eliminated from the table the remaining digits are still randomly ordered, and each remaining digit occurs with relative frequency approximately 1/6.

Exercises

5. Use the correspondence just given, and the table of random digits in the Appendix, to perform at least 20 experiments to determine how many rolls of a die are usually necessarv for the sum of the roll outcomes to exceed 6.

6. Set up a correspondence for the possible event outcomes and the random digits for
   (a) the rhythm method problem,
   (b) the lottery problem,
   (c) the drunkard problem, and
   (d) the grocery store problem.

---

5. RHYTHM METHOD -- WORKED

For the rhythm method example (page 1) an experiment could consist of keeping track of many women who use the method for a period of time. This is how the original "effectiveness" figures were compiled. But the Monte Carlo technique is much quicker and cheaper.

We will simulate the results for one woman and then repeat the experiment many times. An event will consist of a P (pregnancy) or an N (nonpregnancy) for a given year. An experiment will consist of a sequence of years until the occurrence of the first P. But first we need a correspondence between P and N and the digits. Since Pr(P) = 0.3 and Pr(N) = 0.7, one possible correspondence is P ⟷ {0,1,2} and N ⟷ {3 to 9}.

If we start with the (randomly selected) 16th digit of the 12th row of the Random Digit Table, we find

Digit	52	71	2	2	1	558 36734 2	41	31	9580	7 80	92	2	850	1	0
Event	NP	NP	P	P	P	NNN NNNNN P	NP	NP	NNNP	N NP	NP	P	NNP	P	P
Years to First P	2	2	1	1	1	9	2	2	4	3	2	1	3	1	1

As in the coin example on page 5, each vertical line marks the end of one experiment. Fifteen is not a large number of experiments. The histogram shown in Figure 3 results from performing the experiment 100 times.

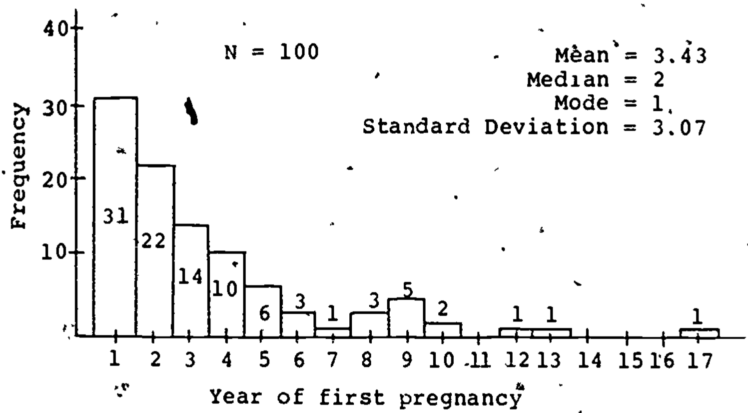

Figure 3. The results of 100 Monte Carlo simulations of the rhythm method experiment.

6. OUTLINE OF THE MONTE CARLO TECHNIQUE

I. List all possible outcomes for each event, e.g., H/T; P/N; or W/I/N.

II. Determine the probability of each outcome, e.g., 0.5/0.5; 0.3/0.7; or 0.5/0.4/0.1.

III. Determine subsets of the integers which have the same relative frequencies as the probabilities listed in II, e.g., {even}/{odd}; {0,1,2}/{3-9}; or {0,1,2,3,4}/{5,6,7,8}/{9}.

IV. Set up a correspondence between the outcomes and the subsets of integers, e.g., H ⟷ {even}/T ⟷ {odd}; P ⟷ {0,1,2}/N ⟷ {3-9}; W ⟷ {0,1,2,3,4}/I ⟷ {5,6,7,8}/N ⟷ {9}.

V. Randomly select a starting point in the Table of Random Digits.

VI. Using each random number to represent the corresponding event outcome, perform the experiment and note the outcome.

VII. Repeat step VI until the desired confidence in the accuracy of the result is obtained.

Exercises

7. Using a certain tire manufacturing process Pr(defective tire) = 0.2, if you randomly select 5 tires as they come off the production line, use the Monte Carlo technique to estimate Pr(exactly 2 of the 5 are defective).

8. The Soggy Cereal Company includes a small toy in each box of Lumpy Lead cereal. There are 3 types of toys, and they are evenly distributed, one to a box. Follow the steps in the out-

line to determine the number of boxes of Lumpy Lead you should expect to have to buy in order to accumulate all 3 types of toys.

7. COMMENTS

7.1 The Name Monte Carlo

The use of random numbers is clearly vital to the Monte Carlo technique. To obtain these numbers, one could use a table like the one included in this module, or a computer (one was used to generate the table), or a suitably random physical device. A die could be used to generate a table of the digits 1 to 6; a roulette wheel could generate the numbers 1 to 36. The whole technique has a strong flavor of gambling, as does much of probability theory.

During World War II, physicists working on the Manhattan Project encountered the problem of describing the behavior of neutrons in various materials. This problem had immediate applications to the construction of shielding and dampers for nuclear bombs and reactors. Direct experimentation would have been time-consuming and extremely dangerous. The basic data about the behavior of single neutrons were known, but there was no practical direct formula for calculating how a whole system would behave.

> "At this crisis the mathematicians John von Neumann and Stanislas Ulam cut the Gordian knot with a remarkably simple stroke. They suggested a solution which in effect amounts to submitting the problem to a roulette wheel. Step by step the probabilities of the separate events are merged into a composite picture which gives an approximate but workable answer to the problem."
>
> (Daniel McCraken, *Scientific American*, May 1955, p. 90)

The basic ideas of the technique had been around for a long time, but for its use in the secret work at Los Alamos, John von Neumann descriptively code named the method "Monte Carlo," after Europe's most famous gambling center.

7.2 The Number of Experiments

A discussion of the number of experiments necessary to attain a predetermined level of confidence in the final estimated answer would require too much space and time as well as background in statistics. However, two general comments can be made:

(i) As the number of experiments increases, our confidence in the accuracy of the estimate should also increase. As a general rule, to double the accuracy of the result (cut the expected error in half), four times as many experiments are necessary.

(ii) If the outcome data are tightly bunched after many experiments (i.e., the standard deviation is small), we should have more confidence in the accuracy of our estimate than if the outcome data are scattered.

Confidence intervals for the mean and median of the outcome data can be found in most introductory statistics books. Introduction to Statistics by G. Noether and Statistics, a First Course by J. Freund both contain rules for determining the number of samples.

7.3 Use of a Computer

The repetition required by the Monte Carlo technique can be tedious and boring. But computers are very fast and not easily bored. Because of their speed, computers can often perform the "busy work" on very complex problems in reasonable amounts of time. Large numbers of experiments can be rapidly performed to detect small changes in probabilities, or to examine the effects of delicate changes in the system. You could compare different betting systems in Black Jack or roulette by having the computer play thousands of games using each strategy and comparing the results.

To use a computer for the "busy work," one must first have a good understanding of the ideas behind the technique, the "thinking" part. Before beginning to program a Monte Carlo experiment, it is a good idea to run a few experiments by hand to be certain of the procedure involved.

The vast majority of computers do not require the use of a random digit table since they are capable of generating their own random numbers.

8. ADDITIONAL EXERCISES

9. A grasshopper sits in the middle of a 7 foot long log. Each minute this grasshopper hops 1 foot to the right or left (with equal probability). How long do you expect it to remain on the log? (This is a 1-dimensional random walk.)

10. Based on at least 10 trials, preferably many more, estimate the solution to the Drunkard problem (page 1). You may find it easier to use graph paper to keep track of the intermediate steps and final stopping position.

Comment: This is sometimes called a random walk. It can also be used to study the mixing of gases or liquids by diffusion. Each labeled molecule in the figure can be treated as a "drunkard" and allowed to wander. Different step sizes would correspond to different temperatures.

B	B	B	B	A	A	A	A
B	B	B	B	A	A	A	A
B	B	B	B	A	A	A	A

Each molecule of Gases A and B is a "drunkard."

11. Based on at least 10 trials, preferably many more, estimate the solution to the Lottery problem (page 1).

Comment: This model can also be used to represent the selfreplication of a DNA strand in a medium which contains the four necessary components, represented by the letters A, T, C, G, in various proportions. One could study how a change in the proportions present will effect the time needed for replication.

9. PROBLEMS

The problems below are more complicated than the ones in the previous exercises. Read them and think about how they could be simulated, but don't perform the simulation unless you have access to a computer. Answers are given.

Problem 1

Determine the best batting order for a Mathball team. Mathball is a simplifed form of baseball and is played by the following rules:

Field:	3 bases; home plate, first base, and last base.
Team:	Each team has 5 players.
Game:	A game consists of 5 innings.
Inning:	A team bats in an inning until 3 outs are made.
Hitting:	A batter who gets a hit goes only to first base. (No doubles or home runs.)
Running:	A runner advances 1 base on a hit.
Your Team:	Batter — Average (probability of a hit)

Batter	Average (probability of a hit)
Tina	.200
Ingrid	.500
George	.200
Elmer	.300
Roger	.300

Problem 2A

Calculate that part of the area of the circle $x^2 + y^2 \le 1$ which lies in the first quadrant ($x > 0$, $y > 0$). (This circle is centered at the origin and has radius 1.)

Comment: If we generate a large number of random points in the circumscribed square (see the diagram), then the ratio of the number of points in the quarter circle to the total number of points in the square will approximate the ratio of the area of the quarter circle to the area of the square. The area of the square is 1, so the area of the quarter circle is readily approximated.

$$\left(\frac{\text{points in quarter circle}}{\text{total points in the square}}\right) = \left(\frac{\text{area of quarter circle}}{\text{area of the square}}\right)$$

This technique of integration (finding area) is less efficient for people and computers in the two-dimensional case than several other approximate integration techniques (e.g. Riemann Sums or Simpson's Rule). But for multiple integration in higher dimensions variations of this Monte Carlo technique are competitive with other techniques and are frequently better.

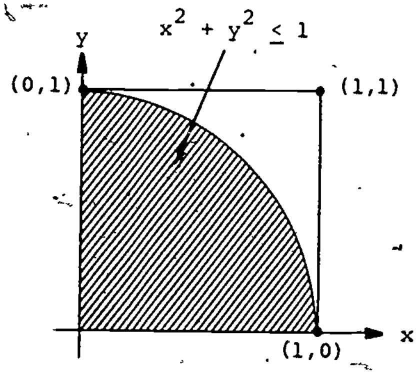

Figure 4. The area of a quarter circle may be estimated by the Monte Carlo technique described in Problem 2A.

Problem 2B

Calculate that part of the volume of the sphere $x^2 + y^2 + z^2 \leq 1$ which lies in the first octant ($x > 0$, $y > 0$, $z > 0$). See the diagram below.

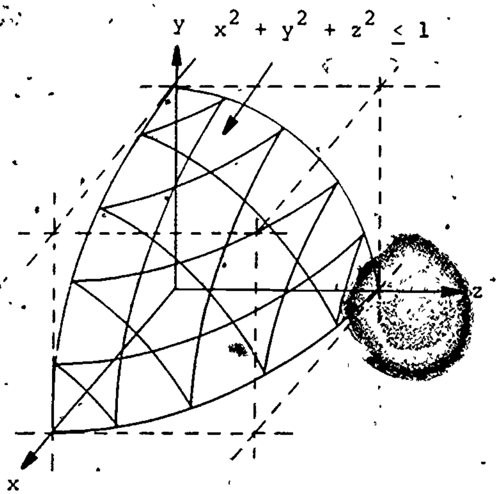

Figure 5. Monte Carlo techniques may be used effectively to calculate volumes.

10. MODEL EXAM

Directions: For each problem:

(1) Set up the correspondence between the possible experiment outcomes and the digits,

(2) perform the experiment at least 20 times using the table of random digits, and

(3) draw a frequency histogram of the experimental results and compute the mean.

1. If you randomly select two 1-digit numbers (0,1,2,...,9), what is the expected distance between them?

2. If you decide to have a family of 4 children, what is the probability that the resulting family is 2 boys and 2 girls? (Assume Pr(boy) = Pr(girl) = 0.5.)

3. The Soggy Cereal Company is running out of whistles to put into their cereal boxes, so they are putting shoulder patches in 70% of the boxes and whistles in the remaining 30%. Assuming that the boxes are well shuffled, about how many boxes should you expect to have to buy in order to get both prizes?

4. If you sit down for lunch with 5 other people, what is the probability that at least 2 of the 6 people at the table were born under the same astrological sign? (Assume that a person is equally likely to have been born under each of the 12 signs.)

5. In a popular board game, the attacking player roles 2 dice and the defending player roles only 1. The attacker wins if the higher of his 2 dice is <u>larger</u> than the number shown on the defender's die. Calculate the probability that the attacker wins.

6. Use the Monte Carlo technique to estimate the area between the curve $y = x^2$ and the x-axis for $0 \leq x \leq 1$. (See the diagram on the front cover.)

11. ANSWERS TO SOME EXERCISES AND PROBLEMS

Exercise 3

Your frequency histogram should have the same general shape as the histogram on page 4, although yours will probably be more jagged.

Exercise 4

- The exact probabilities (using the Binomial Formula) are

P(exactly 0 heads in 5 flips) = 0.031
P(exactly 1 head in 5 flips) = 0.156
P(exactly 2 heads in 5 flips) = 0.312
P(exactly 3 heads in 5 flips) = 0.312 ***
P(exactly 4 heads in 5 flips) = 0.156
P(exactly 5 heads in 5 flips) = 0.031

Your estimate should be close to 0.312.

Exercise 5

These are the results of 2000 experiments.

Number of Rolls Needed to Exceed 6	Frequency	Percent
1	0	0.00%
2	1134	56.70%
3	673	33.65%
4	173	8.65%
5	19	0.95%
6	1	0.05%

Exercise 6

(a) See Section 5.

(b) "W" ↔ {0,1,2,3,4}; "I" ↔ {5,6,7,8}; "N" ↔ {9}

(c) North ↔ {00 to 24}; South ↔ {25 to 49}; East ↔ {50 to 74}; West ↔ {75 to 99}.

(d) 0 new shoppers ↔ {0,1,2}
1 new shopper ↔ {3,4,5,6}
2 new shoppers ↔ {7,8,9}.
There are other correct correspondences for each of these.

Exercise 7

P(exactly 2 are defective) = 0.2048. Your estimate should be close to this value.

Exercise 8

Based on the results of 1000 experiments (shoppers), the mean number of boxes needed to acquire all 3 different toys was 5.6.

P(3 boxes were required) = 0.222
P(4 boxes were required) = 0.222
P(5 boxes were required) = 0.173
P(6 boxes were required) = 0.123
.
.
.

One of the 1000 shoppers needed 28 boxes to get all 3 toys.

Exercise 9

These are the results for 1000 experiments. Mean Number of Hops = 15.8, Median Number of Hops = 12. In this situation the maximum number of hops observed before the grasshopper fell off the log was 68, but the bug could occasionally stay on much longer.

Exercise 10

These are the results for 2000 experiments.

	Mean	Standard Deviation
5 Random Steps	2.03	0.96
10 Random Steps	2.78	1.43

Doubling the number of steps does not double the expected distance from the lamp--some directions tend to bring the drunkard back to the lamp. After one step the drunkard will be exactly 1 unit from the lamp, but after two steps, the expected distance will be $(0 + 2 + \sqrt{2} + \sqrt{2})/4 = 1.207$ units.

Exercise 11

These are the results for 1000 ticket buyers.

Word	Mean Number of Tickets Needed	Median	Standard Deviation
"WIN"	11.07	8	9.40
"IWIN"	11.37	8	8.80

In this simulation one ticket buyer needed 78 tickets.

Problem 1

Runs per game averages are given for several batting orders. Each average is based on 500 games using the order.

Batting Order (by bat. avg.)	Mean Number of Runs Per Game	Standard Deviation
0.5/0.3/0.3/0.2/0.2	4.732	2.96
0.3/0.5/0.3/0.2/0.2	4.496	2.86
0.3/0.3/0.5/0.2/0.2	4.586	2.89
0.2/0.2/0.3/0.3/0.5	4.166	2.92
0.3/0.3/0.2/0.5/0.2	4.434	3.02

A more sophisticated model might include doubles and home runs as well as base stealing and double plays.

Problem 2

Exact area of 1/4 circle = $\pi/4$ = 0.785398.
Exact volume of 1/8 sphere = $\pi/6$ = 0.523599.

Number of Points	Estimated Area (error)		Estimated Volume (error)	
10	0.9	(.115)	0.5	(0.024)
20	0.6	(.185)	0.35	(0.174)
100	0.74	(.045)	0.53	(0.006)
1000	0.776	(.0094)	0.527	(0.0034)
10000	0.7827	(.0027)	0.5239	(0.0003)

12. ANSWERS TO MODEL EXAM

1. The mean distance between 2 randomly selected digits is 3.30. Your estimate should be close to that.

2. Exact probabilities (using the Binomial Formula) are

   Pr(4 boys, 0 girls) = Pr(0 boys, 4 girls) = 0.0625
   Pr(3 boys, 1 girl) = Pr(1 boy, 3 girls) = 0.25
   Pr(2 boys, 2 girls) = 0.375

   Your estimated value should be close to 0.375.

3. Based on 4000 experiments (shoppers), the mean number of boxes needed was 3.78, the median was 3.

   Pr(need only 2 boxes) = 0.42
   Pr(need only 3 boxes) = 0.21
   Pr(need only 4 boxes) = 0.1218

4. Below are the exact probabilities for various numbers of persons at the table.

Number at Table	P(at least 2 share a sign)	Number at Table	P(at least 2 share a sign)
2	0.007	8	0.954
3	0.236	9	0.985
4	0.427	10	0.996
5	0.618	11	0.9996
**6	0.777**	12	0.99996
7	0.889	13	1.0

Your estimate should be close to 0.777.

5. Based on 1000 attacks, the attacker won 582 times. Your estimate should be close to 0.582.

6. The exact area, using elementary calculus, is 1/3. (See Problem 2A.) Estimates will vary.

APPENDIX: RANDOM NUMBERS

A1. Properties and Tests

The "Random Digits" generated by computers are not truly random, but are usually determined by some procedure from a previous number. However, a "good" procedure will generate numbers with properties which truly random numbers would have. The most basic and desirable of these properties are:

A1.1 Uniform Distribution: Each digit occurs with about the same frequency. Sometimes pairs and triples of digits are also used to test for uniform distribution.

A1.2 Independence: The digits do not appear to follow any regular pattern. Since there are so many possible patterns which could occur, it is impossible to test for all patterns. However, it is possible to test for some of the more obvious ones.

Up/Down Test: How often is the next number larger, smaller, or the same as the previous number in the table? For truly random numbers, about 10% of the time the next digit should be the same as the previous digit, about 45% of the time (half of the remaining 90%) it should be smaller, and about 45% of the time it should be larger.

Cycles: How long before the digits start to repeat in the same order? How long until the digits cycle? This is usually very difficult to determine just by examining the procedure or the resulting table. A "good" procedure takes a very long time before starting the cycle.

A technique for generating "random" numbers can be found in "Methods of Random Number Generation" by Edwin Landauer in The Two-Year College Mathematics Journal, November 1977, pages 296-303.

A.2 Use of the Random Digit Table

Starting Point: If you want to get "fancy" you could use a spinner or a pair of dice to generate two random numbers--the first number to be the starting row, and the second to be the starting digit in that row (e.g., the pair 3,6 would direct you to start with the 6th digit of the 3rd row).

But most people use the "blind stab" technique; close your eyes, point to a point on the page and start there.

Continuing: Since the digits in the table are already in random order it is not necessary to select a new starting point for each experiment--simply start the next experiment with the next digit in the table.

A.3 10,000 Random Digits

Frequency of Digit	Page 1	Page 2	Page 3	Page 4	Total	Total Percent
0	249	252	253	243	997	9.97
1	237	266	257	269	1029	10.29
2	281	241	242	247	1011	10.11
3	270	227	252	244	993	9.93
4	253	256	250	226	985	9.85
5	257	244	249	256	1006	10.06
6	229	260	245	250	984	9.84
7	22[illegible]	268	255	250	1002	10.02
8	256	243	225	285	1009	10.09
9	239	243	272	230	984	9.84
Same	251	239	255	248	993	9.93
up	1125	1133	1141	1131	4530	45.30
Down	1124	1128	1104	1121	4477	44.77

82698	26610	90511	08055	80364	70233	91451	34528	30357	27456
93680	27051	67692	57437	08779	81065	50586	20621	28296	43353
45153	17985	74725	08526	09220	89778	59814	02387	78112	16035
65055	40547	20834	50243	23998	59708	12313	89349	25103	43682
80863	76681	73173	48970	91202	81344	89446	60285	12653	95567
65704	35329	80233	67505	22518	58994	63968	79316	53447	65610
16862	82356	69963	61171	96043	56593	73637	82198	51634	71363
76048	34462	57543	98743	80838	42517	42094	98970	07496	22223
92003	32221	39595	99113	43596	90842	87684	80098	54888	32782
74244	90661	80795	20305	92055	54532	99534	34660	41569	88305
38128	35924	55245	97971	52694	92422	15875	18971	20058	78333
33729	56998	99535	52712	21558	36734	24131	95807	80922	85010
63971	68875	13322	07349	73991	41072	31419	29611	10297	85465
57653	56330	22804	71402	62635	33217	85828	69039	77095	57063
36395	30423	96224	53481	23420	44921	30883	56083	32038	63699
90543	52660	09346	76795	89783	87944	92379	34576	18055	67418
58133	19098	70130	16092	43843	80508	96387	42270	35335	18264
57487	88972	50914	65331	87902	42601	85407	19867	77391	48159
77128	23219	48346	02047	63984	66444	83317	40167	39020	00798
13964	87042	24341	25448	30779	30472	92064	71532	47311	33061
03114	30226	65252	72519	11706	72966	95952	93649	64857	57621
41182	02953	20581	46556	03312	24241	54804	29809	04113	75128
94953	59747	35056	70403	17822	04416	08601	45680	69568	35183
26528	96679	08165	34005	90199	48983	99761	51229	31275	27314
71479	66012	23245	49574	10116	41521	06750	29164	63007	55902
42292	82996	86159	79513	84410	45582	38596	55311	04895	13515
63234	72661	32908	22815	30490	01502	52419	97075	95007	03410
95770	34807	06273	59221	42470	68812	28923	28313	16271	06813
09463	63823	29643	62401	06537	63918	52056	83389	86422	88943
37271	74277	85283	81867	66660	40978	80906	50846	32802	18984
29627	48227	75458	13027	68341	24267	89088	40988	53103	79923
64074	46280	63010	53561	12276	25624	43287	38239	56965	03913
41630	85293	87811	97757	34504	72791	18594	21759	58785	72898
49567	12521	70419	45853	84408	65065	47690	61921	43879	15782
16466	91260	29140	78385	54921	16937	33072	60675	18101	77288
78056	96364	45121	63361	65742	03964	01998	36442	97701	85267
53747	76678	84504	88985	58473	24216	20720	99730	44223	81412
35832	52303	07091	97235	22488	73307	63024	42020	82151	67453
00316	98955	28410	62443	65885	14166	61937	71899	05764	73820
03281	21503	85071	43185	37724	27177	54710	86306	83226	05223
88027	73621	54167	61642	15908	74027	32009	40957	00489	50941
58412	57958	10784	91459	05057	09259	04821	48798	94313	22552
26713	91036	37943	39567	35577	92419	27216	48996	88339	11204
74903	84443	30195	38568	91675	53618	40088	24647	70893	99334
63137	40699	42046	10281	57445	67771	00976	20883	79039	54049
45053	55031	75934	73950	89878	20357	03257	14471	24981	42633
39121	28849	82954	36481	10444	85221	55466	27512	03441	57984
45968	12528	55047	03065	63942	45232	22368	05620	22057	13135
62420	23116	23984	50249	42438	47611	68085	84966	08318	18250
69928	18998	17186	08202	28286	60156	27066	95713	47429	64033

68420	91594	18774	99086	97471	11096	83934	54694	99278	53366
10766	65860	45180	01167	45771	87610	05272	85867	82672	68059
59782	87239	59260	65113	59876	10642	79247	45118	65702	96858
50136	54869	77626	25256	27837	49592	37705	01488	05843	88203
24186	44144	99986	26937	10126	47675	36747	22790	65792	22751
32149	32697	40420	71863	11583	02864	25198	15551	90871	10326
71152	46507	83616	93181	68659	77281	18518	27371	68757	18253
98480	39095	31712	53194	51924	06287	57890	12455	01641	85052
88139	98726	64244	13517	03796	92669	46544	77797	63147	47743
83133	03717	59230	24429	89823	69684	22210	56398	71136	22323
53546	43190	65854	17069	11174	88317	85699	13809	67271	94418
48162	65787	83194	80075	59176	32700	38999	41747	54312	69993
84814	97006	58212	24471	00035	20523	67758	63351	69789	62877
17462	44657	39043	05410	13946	13306	73265	42812	81182	47604
69490	13754	57511	73595	60986	91695	36815	68175	46810	17198
44817	89371	87318	64743	96118	62417	34657	53990	18410	85309
78895	96529	26425	94164	79378	85802	35855	47916	07173	33690
61302	09781	24426	50261	49587	09675	11506	48489	86292	32713
37458	11722	96040	26021	43539	68552	16742	38625	30907	03649
41800	34521	69828	47464	16216	72943	00330	93677	38492	08708
14708	41640	22349	89030	62190	14042	13371	62037	33843	04555
08497	13633	90824	32538	31091	81954	91588	80743	90094	31006
19085	81561	47008	01014	23479	26661	70725	77994	95119	28515
05856	76772	94061	87862	56015	86653	82671	06105	50992	52662
38818	06893	03319	32736	25017	70617	49879	73150	12355	30007
74183	52870	70880	87765	25043	05881	69958	33040	06060	99228
71626	80724	43948	82019	56251	40368	63507	10557	74890	25340
60240	60570	16600	16414	70969	69191	33937	47968	29374	93538
05759	07744	12089	90706	94402	10132	75795	27739	88054	67702
24124	49735	10951	60217	65867	16628	80069	31145	42728	72525
49727	52958	52316	95660	66210	64217	20436	32849	24576	40591
81607	30289	71071	02563	32613	66914	00753	60781	09185	76051
60087	18737	05805	27414	42912	77982	68504	67410	87694	49195
93765	51974	86490	26739	16100	58912	99557	29283	52530	14750
87941	61498	16658	05112	29020	34744	25975	59405	88830	48603
04392	42554	34738	97944	69423	22576	36792	02929	35868	45485
77057	75328	28431	68407	96972	18792	53721	48557	94522	22621
56462	32852	12144	00576	24336	97318	40797	73018	91053	70169
39735	44807	82227	46221	60117	04324	05759	45700	13299	65149
83050	41721	29138	38823	34923	93301	98190	27037	72070	56465
62314	32142	24102	69218	20065	76827	04831	07796	34291	50754
38950	28005	63258	67274	33980	68269	89313	38086	13712	95206
52433	45631	05969	75331	64046	29691	13143	55478	53127	64545
64146	82545	12934	80945	73589	33866	10603	70451	66164	44446
09412	55663	59584	83213	69608	28923	66469	17481	71246	89910
04503	72085	84585	52396	99506	26123	61681	38641	65181	34728
91891	38326	29940	45907	45271	45001	97684	24776	73079	91367
91115	19370	48461	77755	42845	87704	48785	96845	62677	84985
34907	68127	68011	77047	78265	87344	65971	04187	40044	73874
99856	31340	10282	20389	65561	25532	40236	48359	90606	33979

27037	72070	56465	62314	32142	24102	69218	20065	76827	04831
07796	34291	50754	38950	28005	63258	67274	33980	68269	89313
38086	13712	95206	52433	45631	05969	75331	64046	29691	13143
55478	53127	64545	64146	82545	12934	80945	73589	33866	10603
70451	66164	44446	09412	55663	59584	83213	69608	28923	66469
17481	71246	89910	04503	72085	84585	52396	99506	26123	61681
38641	65181	34728	91891	38326	29940	45907	45271	45001	96784
24776	73079	91367	91115	19370	48461	77755	42845	87704	48785
96845	62677	84985	34907	68127	68011	77047	78265	87344	65971
04187	40044	73874	99856	31340	10282	20389	65561	25532	40236
48359	90606	33979	09262	40436	09883	01575	68238	27119	17924
07467	09580	91949	40502	88651	71376	75607	04357	90371	55872
49553	14062	00424	02124	37379	05349	75145	03491	39624	85800
33440	18028	58970	77255	79334	17183	16797	99398	21953	62722
94755	55821	45393	22103	24316	91264	00600	80582	09369	12410
40878	94006	39046	35972	34043	37932	18147	85569	27222	78437
76132	77876	78520	68648	06763	27722	33982	07731	09320	41612
68604	38954	98664	75455	11612	39453	75065	14744	59921	71415
50293	93191	28657	26168	37711	39518	95835	44677	36107	36507
07572	15191	30913	31779	31882	90985	61839	49827	24802	60107
21226	86662	23537	76993	20755	68716	04093	17531	53987	29851
25315	35166	39491	66700	23618	10847	71758	05973	63890	23930
62320	60272	14194	25682	22076	78290	11417	96950	64187	44749
33095	96290	79305	94440	15733	97670	60119	33895	25756	68336
58181	97886	19181	72545	45880	92045	45942	97354	24978	44064
79137	85011	25825	77765	48999	27891	41211	97539	29325	73742
74608	11499	79360	04111	45656	74946	35111	80164	47756	71792
73792	92958	81557	58549	41130	81741	79532	01009	95449	36689
14460	65984	03756	10929	16602	56985	00066	82737	88829	89605
65415	23352	22492	61876	33008	84859	39280	36179	72862	56485
42665	91326	49901	98208	32107	71929	80351	72674	21477	92890
73228	68631	21125	69520	54548	27554	24653	07923	80316	20491
42600	44409	28867	46699	87567	93070	24929	47224	50096	61129
47795	044[illegible]	64756	94677	32392	77386	10028	87136	67855	36325
00674	55308	78280	52420	56172	35985	32502	07917	60875	32698
96948	49982	34663	81020	08403	41764	80806	98992	33011	83328
09165	79907	39990	25101	13110	59756	53055	38281	15223	55161
51521	72704	16153	91634	00414	39927	18382	84077	38849	69007
24470	48081	21733	32191	43908	90379	16210	29556	90426	65542
79487	91238	75526	59403	49369	18605	80756	07663	91091	51813
48699	06823	83560	81900	40815	00607	50010	78251	98775	05996
62469	93915	04158	27301	73810	07137	80416	82445	51490	79066
38220	15919	33125	71986	79159	12636	86029	84827	85675	18296
67209	67163	99349	39213	22891	73833	92740	91488	28799	00213
06540	99623	13250	90540	13108	89249	62160	44891	54297	56239
14753	02313	03396	37114	25045	34352	90955	57786	51942	32876
07757	10548	60790	46231	67260	34411	98285	47594	71957	54096
23751	94093	31487	75997	91215	80803	59431	13014	69782	54283
46250	93850	94388	07478	96634	32036	33138	70445	38034	54464
43158	43643	57121	76627	71454	97036	72365	13356	92051	73963

05685	67447	73286	07963	62056	60432	63421	07580	72783	51002
11660	88785	25096	15556	68804	51431	08324	46281	59901	75352
19106	09665	36862	59300	57953	89717	41554	32228	99023	79115
20960	45950	54946	44967	23065	26049	10121	79641	96642	85528
65018	57366	60878	71150	17045	63518	79555	13778	44533	72487
96558	42015	54031	77461	44942	96934	36057	54767	77631	94379
77942	86764	27579	66873	76015	49750	57561	52388	31914	15033
81681	81749	30071	81901	82813	91797	71713	39843	72933	04286
52570	35966	06913	47050	51782	83549	35859	60841	87978	46089
84425	73503	28578	11685	16815	15978	71920	60815	53895	16900
25535	79807	68457	14232	78851	32810	76576	39616	18218	22350
48138	11077	43245	25517	98105	79268	41262	05899	99484	52920
11627	66583	65057	70008	51932	73879	68870	82455	48444	10163
31856	07704	91883	61734	18850	65907	98368	11990	52967	05709
37440	43130	86051	46031	06637	34541	62925	29648	94378	56422
35746	62282	17094	04250	27396	04887	56023	53245	92661	85005
97804	87773	82534	15870	49848	80953	31171	43955	80603	27364
91175	98163	28490	92192	06701	09367	32946	69758	00663	96301
38302	33327	44826	09816	96198	79200	63473	66838	34831	50893
09180	07928	29973	81575	15131	51129	60741	16445	36119	49166
55666	50999	13101	18800	83267	68427	99189	89986	40883	22136
29038	91236	46064	39305	25733	73713	58019	61522	70278	45161
84577	95714	88417	56123	14788	03531	11364	15741	54252	80109
13885	72669	93446	02807	17864	56620	29762	51934	65284	97767
31558	13888	11120	13444	25443	62756	99367	65893	82758	71926
51899	52677	07831	47908	40581	22500	91302	52878	90316	06545
20509	72146	63389	23334	13954	80998	83254	22812	38984	72714
03217	22731	94040	21712	76040	63934	88640	12379	25720	75254
88124	29703	13149	86345	03022	36374	74372	16250	40762	29498
41194	15611	78675	81355	85500	01372	06289	87351	32462	15277
20934	10776	24867	85067	30259	57928	13688	40697	12574	80184
44809	12889	67228	84743	49216	38306	81830	34922	71881	37172
62731	23281	74601	51960	90468	28746	80548	07593	71242	10987
54498	35814	84711	86001	27335	75586	48299	10307	33053	45522
20236	85252	25616	86696	28140	76975	56010	17620	31577	78843
50128	86377	67646	45256	80934	14582	93088	82511	68214	30086
18022	72895	58151	83733	21216	80519	92430	82367	29961	00058
25245	68730	00615	18502	99241	78775	52898	96479	47632	23135
64830	21037	12282	53755	34260	95694	58986	05376	69326	73447
18707	81752	39539	50986	24542	48826	40438	01398	30363	17431
82453	18982	80155	17210	62812	68135	25603	88137	58265	34246
00981	58914	69021	10403	47073	57405	83520	62114	19764	14452
08392	82666	30928	69790	21875	57349	73750	41290	32488	49351
51013	70767	27662	54188	27703	95973	97846	46031	44783	25761
37469	09296	35058	19965	47830	17051	53582	78327	05699	66906
32261	50791	29185	29158	35977	65910	96837	80985	95785	62837
96266	15685	44491	50286	36609	17567	46168	90818	04408	41019
25690	99568	88380	91418	43852	99198	98576	42972	33829	27422
00504	88071	48494	14313	22705	83812	13743	98413	34617	32793
89449	00747	33760	18193	19696	68067	26374	98229	38250	12026

to fight to get her—a woman—in at all. She was fat, roug[illegible] and loud, but so kind, humorous and sociable that all w[illegible] knew her loved her. With the Nazi rise to power she left [illegible] the United States.

"Emmy's early work [illegible] invariants gave no hi[illegible] that she would becom[illegible] one of the creato[illegible] of abstract axiomat[illegible] algebra. She develope[illegible] the axiomatic theory [illegible] [illegible]eals, introducing th[illegible] [illegible]ding chain cond[illegible] tion, [illegible]ve a unified the[illegible] ry o[illegible]n-commutativ[illegible] algebra[illegible]nd their repr[illegible] sentatio[illegible] defined th[illegible] 'verschrän[illegible]e Produk[illegible] and, with [illegible]rauer an[illegible] Hasse, prove[illegible] that ever[illegible] simple algebra over a[illegible] algebraic number field [illegible] cyclic."

Au[illegible]ck's biography, complemented by obituaries an[illegible] [illegible]written by Pavel Sergeevic Alexandroff, B.L. va[illegible] [illegible]en and Hermann Weyl capt[illegible] [illegible] spirit an[illegible] [illegible] her life and work. This tra[illegible]ation has bee[illegible] [illegible] with some previously unpubl[illegible]ed photograph[illegible] and co[illegible]es as a timely tribute to one of the g[illegible]test mathemati[illegible] cians [illegible] our age.

128 Pages, Cl[illegible]bound, August 198[illegible]

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