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ABSTRACT

This document consists of four units. The first of these views calculus applications to work, area, and distance problems. It is designed to help students gain experience in: 1) computing limits of Riemann sums; 2) computing definite integrals; and 3) solving elementary area, distance, and work problems by integration. The second module views calculus applications to atmospheric pressure. The material is geared to aid pupil understanding of the application of calculus to construct a mathematical model of the atmosphere. The third unit covers applications of multivariate calculus to physics. The user is expected to learn how to compute the Gradient, and to be able to use the Gradient to solve certain practical problems involving steepest ascent. The final module covers calculus applications to physics. The material is designed to instruct pupils how to: 1) be able to handle motion problems in polar coordinates, especially those concerning gravitational interaction; 2) state the relationships between Kepler's Laws and the Inverse Square Law; and 3) discuss the history and thought going into scientific development of an idea and the interaction between experiment and theory, at least in the context of planetary motion. All units include exercises with solutions; the first, third, and fourth provide model exams with answers. (MP)

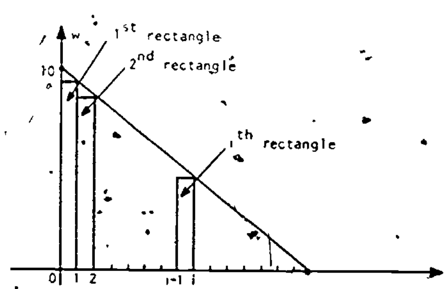
ED218132

UMAP MODULE 323

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

Developing the Fundamental Theorem of Calculus

by Peter A. Lindstrom



Applications of (and Distance Pr

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DEVELOPING THE FUNDAMENTAL THEOREM OF CALCULUS

by

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Intermodular Description Sheet: UMAP Unit 323

Title: DEVELOPING THE FUNDAMENTAL THEOREM OF CALCULUS

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Review Stage/Date: IV 4/16/80

Classification: APPL CALC/WORK, AREA & DISTANCE PROBLEMS

Prerequisite Skills:

1. Familiarity with the basic formulas for area, distance, and work.
2. Familiarity with summation notation and formulas
   $\sum_{i=1}^{n} 1 = n$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
3. Familiarity with antiderivatives of polynomial functions.
4. Ability to perform elementary algebraic operations.

Output Skills:

1. To gain experience in computing limits of Riemann sums.
2. To gain experience in computing definite integrals.
3. To gain experience in solving elementary area, distance, and work problems by integration.

Other Related Units:

Measuring Cardiac Output *(Unit 71)*
Determining Constants of Integration *(Unit 162)*

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DEVELOPING THE FUNDAMENTAL THEOREM OF CALCULUS

1. INTRODUCTION

One of the most useful tools of mathematics is the Fundamental Theorem of Calculus. Although beginning calculus students learn to use the Fundamental Theorem of Calculus to find the value of a definite integral, they are often left with only a manipulative tool and not a thorough understanding of the tool itself, its proof, and its numerous applications. The purpose of this module is to develop, by means of some applications, some of the basic concepts of the Fundamental Theorem of Calculus.

2. THREE SIMILAR PROBLEMS

Consider the three following problems, followed by their solutions:

EXAMPLE 1. Find the area of the rectangle that is 15 feet by 10 feet.

EXAMPLE 2. Find the distance traveled by a car if its velocity is 10 feet per second and the length of time it travels is 15 seconds.

EXAMPLE 3. Find the work required to lift a 10-pound bag of salt a distance of 15 feet.

SOLUTIONS. All three problems have common numerical quantities, namely the 10 units and the 15 units. Figure 1 and Table 1 show the relationship between these quantities and the solution of each example.

It should be noted in each of these examples that the vertical quantity is a constant function of the horizontal quantity, i.e., in EXAMPLE 1, the width w is always 10 feet for $0 \leq \ell \leq 15$, in EXAMPLE 2, the velocity $v = 10$ ft/sec for $0 \leq t \leq 15$, and in EXAMPLE 3, the force $f = 10$ lb for $0 \leq d \leq 15$.

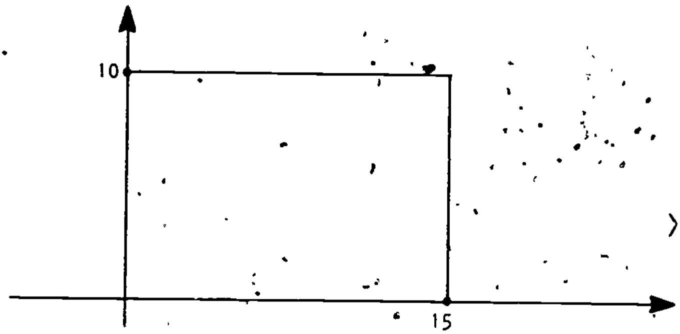

Figure 1. Solution to EXAMPLES 1, 2, and 3.

TABLE 1

Solution to EXAMPLES 1, 2, and 3.

EXAMPLE	Horizontal Axis	Vertical Axis	Formula	Solution
1	$\ell = 15$ ft	$w = 10$ ft	$A = \ell \times w$	Area = $15 \times 10 = 150$ ft^2
2	$t = 15$ sec	$v = 10$ ft/sec	$D = v \times t$	Dist. = $15 \times 10 = 150$ ft
3	$d = 15$ ft	$f = 10$ lb	$W = f \times d$	Work = $15 \times 10 = 150$ ft-lb

Exercises

1. A car moving with a velocity of 10 ft/sec begins to reduce its speed uniformly and comes to rest in 15 seconds. Explain why the product of the velocity and time does not produce the distance it travels.

2. A 10-pound bag of salt is lifted up steadily to a height of 15 feet. The bag has a hole in it and salt leaks out at a uniform rate, so that the bag is empty when it reaches the destined height. Explain why the product of the force and the distance does not produce the work necessary to lift the bag.

3. THREE MORE SIMILAR PROBLEMS

3.1 Statement of the Problems

Consider the three following problems:

EXAMPLE 4. Find the area of a right triangle whose base is 15 feet and whose altitude is 10 feet.

EXAMPLE 5. Find the distance traveled by a car whose velocity at time t is $v(t) = [(-2/3)t + 10]$ ft/sec when it travels from t = 0 to t = 15 seconds. (See Exercise 1.)

EXAMPLE 6. Find the work done in lifting a bag of salt a distance of 15 feet above the ground, assuming that the bag has a hole in it so that at height d above the ground its weight (magnitude of downward force) is $g(d) = [(-2/3)d + 10]$ lb. (See Exercise 2.)

3.2 Solution of One of the Problems

All three examples (EXAMPLES 4, 5, and 6) have been purposely set up to make their solutions similar. (This might not be noticeable at a first glance!) For EXAMPLE 4, consider Figure 2 to aid you in its solution.

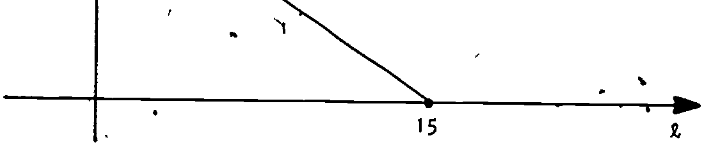

Figure 2. Graphical aid for solution to EXAMPLE 4.

Since the area of a triangle = (1/2)(length of base)(length of altitude), then the area = (1/2)(15)(10) = 75 square feet. It should be remembered that this problem was solved rapidly by using an <u>appropriate formula</u>.

Notice that in Figure 2, w is <u>not</u> a constant function of ℓ. Since the line segment in Figure 2 has a slope of $(10-0)/(0-15) = (-2/3)$ and a w-intercept of 10, then $w(\ell) = (-2/3)\ell + 10$, where $0 \leq \ell \leq 15$. For the functions in EXAMPLES 5 and 6, their graphs are given in Figures 3 and 4 respectively, where $v(t) = (-2/3)t + 10$ for $0 \leq t \leq 15$ and $g(d) = (-2/3)d + 10$ for $0 \leq d \leq 15$.

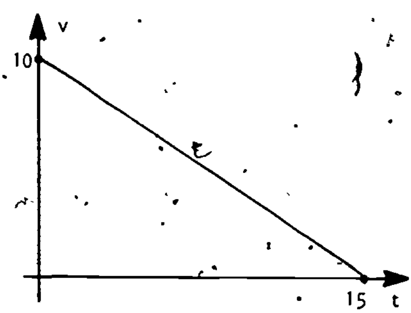

Figure 3. Graph of function v.

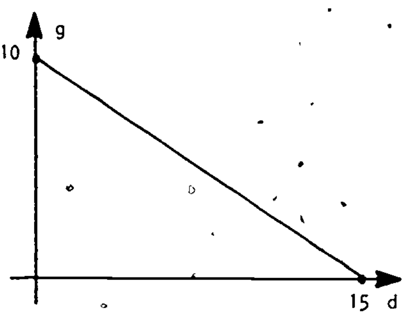

Figure 4. Graph of function g.

The fog that may have hung over EXAMPLES 4, 5, and 6 should now be clearing and one should begin to see the similarity between these three examples. For all three examples, we have the same function and the same domain, but each with different labels:

EXAMPLE 4: $w(\ell) = (-2/3)\ell + 10$, where $0 \leq \ell \leq 15$.
EXAMPLE 5: $v(t) = (-2/3)t + 10$, where $0 \leq t \leq 15$.
EXAMPLE 6: $g(d) = (-2/3)d + 10$, where $0 \leq d \leq 15$.

QUESTION: Since the solution to EXAMPLE 4 is 75 square feet and each example has the same function (with different labels), can we assume that EXAMPLES 5 and 6 will have the same numerical solution but with different labels?

To answer this question, we will go back and solve EXAMPLE 4 by a different method, where we do not use the area of a triangle formula. This will enable us to also solve the two other examples.

3.3 Approximate Solutions of EXAMPLES 4, 5, and 6

We will start by first approximating the area of the triangle in Figure 2 by a sum of areas of rectangles. First we divide [0,15] into 15 subintervals of equal length, each of the form [i-1,i], where i = 1, 2, 3, . . . , 15 as shown in Figure 5. Over the ith subinterval, [i-1,i], we

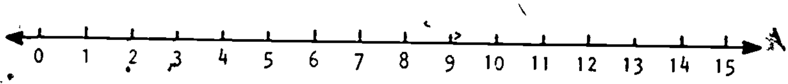

Figure 5. Subdivision of [0, 15].

construct the ith rectangle whose "length" is the length of the ith subinterval and whose "width" is determined by the right endpoint i, as shown in Figure 6. The area of

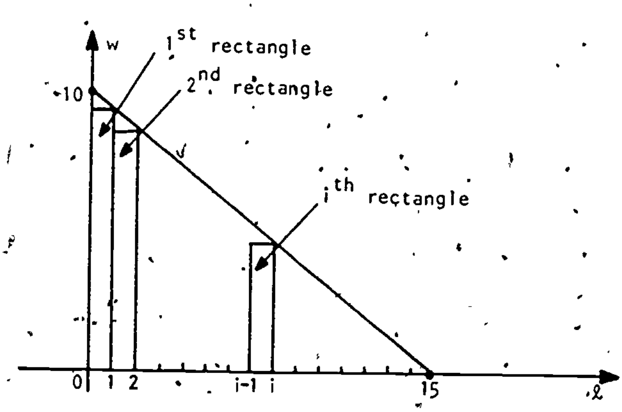

Figure 6. Approximation of area of triangle.

the ith rectangle is the product of the width w(i) and the length of the ith subinterval, namely i - (i - 1) = 1, or

(A) $w(i)\cdot 1 = [(-2/3)i + 10](1)$

for $i = 1, 2, 3, \ldots, 15$. Since there are 15 such rectangles, then the area of all of these is

(B) $\sum_{i=1}^{15} w(i)\cdot 1 = \sum_{i=1}^{15} [(-2/3)i + 10](1)$, as $w(i) = (-2/3)i + 10$,

$$= \sum_{i=1}^{15} (-2/3)i + \sum_{i=1}^{15} (10)\cdot 1,$$

$$= (-2/3)\sum_{i=1}^{15} i + (10)\sum_{i=1}^{15} 1,$$

$$= (-2/3)\frac{(15)(15+1)}{2} + 10(15),$$

as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} 1 = n$,

(C) $= 70.$

Hence, the area of the triangle is approximately 70 square feet.

Let's go back and look at (A), (B), and (C) in terms of EXAMPLE 5 and Figures 3 and 6. Changing the labels, we have the approximate distance over the ith subinterval represented by

(A') $v(i)\cdot 1 = [(-2/3)i + 10](1)$,

where $v(i) = [(-2/3)i + 10]$ is the velocity over the ith subinterval and 1 represents 1 second of elapsed time. Notice that we are assuming that the velocity is a constant value over the ith subinterval. In turn,

(B') $\sum_{i=1}^{15} v(i)\cdot 1$

is an approximation to the distance traveled over the 15-second time period. This sum works out to be

(C') $\sum_{i=1}^{15} v(i)\cdot 1 = 70$ ft.

Exercises

3. State what (A), (B), and (C) represent with the appropriate labels when the width function w is replaced by weight function g of Example 6.

4. Approximate the area of the triangle in EXAMPLE 4 by dividing the interval [0,15] into 45 subintervals. Use the left endpoint $(i - 1)(1/3)$ to determine the height of each rectangle.

5a. Interpret the results of Exercise 4 above in terms of EXAMPLE 5.

b. Interpret the results of Exercise 4 above in terms of EXAMPLE 6.

Since 70 (with the proper label) is an approximation for the solution of EXAMPLES 4, 5, and 6, lets solve all of these problems by a more general method. Let $f(x) = (-2/3)x + 10$, where $0 \le x \le 15$, where function f represents any one of the three functions in EXAMPLES 4, 5, and 6. Generalizing Figure 5, we divide [0,15] into n equal length subintervals (each of length 15/n) and denote the ith subinterval by $[(i-1)(15/n), (i)(15/n)]$, for i = 1, 2, 3, . . . , n as shown in Figure 7.

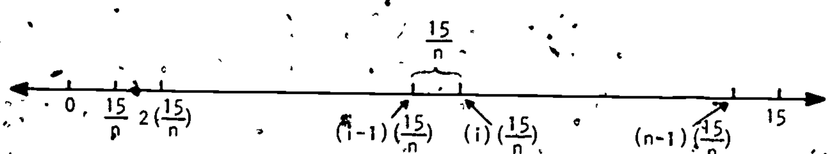

Figure 7. Subdivision of [0, 15] into n equal length subintervals.

Consider then the product

(D) $f[(i)(15/n)](15/n)$,

which may be interpreted in the following three ways:

1. For EXAMPLE 4, (D) is the area of the ith rectangle with width $f[(i)(15/n)]$ and length $(15/n)$, where the width is determined by the right endpoint, $(i)(15/n)$, of the ith subinterval.

2. For EXAMPLE 5, (D) is the approximate distance traveled over the ith subinterval of time length $(15/n)$ and where the velocity, $f[(i)(15/n)]$, is constant over the interval and it is determined by the right endpoint, $(i)(15/n)$, of the ith subinterval.

3. For EXAMPLE 6, (D) is the approximate work done over the ith subinterval of distance $(15/n)$ and where the force, $f[(i)(15/n)]$, is constant over the interval and it is determined by the right endpoint, $(i)(15/n)$, of the ith subinterval.

Adding up all n of these products, we then have

$$\text{(E)}\quad \sum_{i=1}^{n} f[(i)(15/n)](15/n) = \sum_{i=1}^{n} [(-2/3)(i)(15/n) + 10](15/n),$$

$$= \sum_{i=1}^{n} [(-10i/n) + 10](15/n),$$

$$= \sum_{i=1}^{n} [(-150i/n^2) + 150/n],$$

$$= (-150/n^2)\left[\sum_{i=1}^{n} (i)\right]$$

$$+ (150/n)\left[\sum_{i=1}^{n} (1)\right],$$

$$= (-150/n^2)\left(\frac{(n)(n+1)}{2}\right) + (150/n)(n),$$

or

$$\text{(F)}\quad \sum_{i=1}^{n} f[(i)(15/n)](15/n) = (-75)(1 + 1/n) + 150.$$

Notice that the expression in (F) is a function only of n, the number of subintervals of [0,15]. By letting n take on specific values in (F), we obtain various approximations to the solutions of EXAMPLES 4, 5, and 6. For the case that n = 15, we obtain the numerical value of 70, the result we had previously seen.

Returning to (F), we will let $n \to \infty$, so that

$$\text{(G)} \quad \lim_{n\to\infty} \sum_{i=1}^{n} f[(i)(15/n)](15/n) = \lim_{n\to\infty}[(-75)(1 + 1/n) + 150],$$

$$= (-75)(1) + 150,$$

$$= 75.$$

Notice that the 75 is the same numerical value that we had previously obtained (see page 4) for the area of the triangle when we had used the specific formula for determining the area of a triangle. Notice also, that we have now answered the question that was posed on page 4.

Exercises

6a. Find the value of the expression in (F) for the case that n = 25; n = 75; n = 300.

b. Why is each successive value of n a better approximation than each previous value of n?

7. The sum of products in (F) was determined by using the right endpoint, (i)(15/n), of each subinterval.

a. Determine a similar sum of products by using the left endpoint, (i - 1)(15/n), of each subinterval. Simplify the result as much as possible.

b. In your results of 7a above, let n = 25; n = 75; n = 300.

c. In your results of 7a above, let $n \to \infty$. Then compare the result with (G).

8. As in Exercise 7 above,

a. Determine a sum of products by using the midpoint of each subinterval. Then simplify it as much as possible.

b. In your results of 8a above, let $n = 25$; $n = 75$; $n = 300$.

c. In your results of 8a above, let $n \to \infty$ and compare the result with (G).

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4. RIEMANN SUMS AND THE DEFINITE INTEGRAL

4.1 Definition of Riemann Sum

We have been looking at some special cases of what is called a Riemann Sum and the definite integral. Let us now look at these more general concepts.

Let f be a function defined on a closed interval $[a,b]$. Let $a = x_0$ and $b = x_n$ and select $(n-1)$ points $x_1, x_2, x_3, \ldots, x_{n-1}$ between x_0 and x_n so that $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$. These $(n+1)$ points are said to partition the interval $[a,b]$ into n subintervals, where the ith subinterval is denoted by $[x_{i-1},x_i]$, for $i = 1, 2, 3, \ldots, n$. These n subintervals are said to form a Partition P of $[a,b]$. See Figure 8.

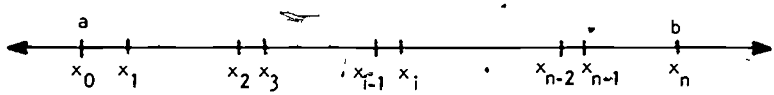


Figure 8. A partition P of [a,b].

Denoting the length of the ith subinterval by Δx_i, we have

(H) $$\Delta x_i = x_i - x_{i-1}$$

For each value of i, select one point c_i from the ith subinterval, so that $x_{i-1} \le c_i \le x_i$. We now form the product

(I) $f(c_i)\cdot\Delta x_i$,

and then the sum of these n products,

(J) $$\sum_{i=1}^{n} f(c_i)\cdot\Delta x_i .$$

This sum of products (J) is called a Riemann Sum of the function f over [a,b] for the partition P and the choice of c_i. It should be noted that a Riemann Sum depends upon

1. the function f,
2. the closed interval [a,b] over which f is defined,
3. the partition P of [a,b], where each subinterval need not be of the same length, and
4. the point c_i selected from each subinterval $[x_{i-1}, x_i]$.

Returning to the Riemann Sum in (E), we see that

1. the function f is given by $f(x) = (-2/3)x + 10$,
2. [0,15] is the closed interval over which f is defined,
3. the partition P of [0,15] consists of n subintervals of equal length, and
4. the point $c_i = (i)(15/n)$ is the right endpoint of each subinterval.

Exercises

9. Set up and simplify a Riemann Sum for the function $f(x) = 2x + 4$ over [1,8], using a partition with subintervals of equal length and selecting the left endpoint of each subinterval for c_i.

10. In Exercise 9 above, let n take on various positive integer values.

11. Interpret the numerical results of Exercises 9 and 10 above in terms of EXAMPLES 4, 5, and 6.

4.2 Definition of the Definite Integral

In a partition P of [a,b], the length of the longest subinterval is called the norm of the partition P, and is denoted by $\|P\|$. For the Riemann Sum in (E), since each subinterval is of the same length, then $\|P\| = (15/n)$. We now come to one of the main topics of this module, the Riemann, or definite integral.

> Definition: (1) Let f be a function defined over [a,b]. (2) Let P be a partition of [a,b] having n subintervals, where Δx_i is the length of the ith subinterval and c_i is any point in the ith subinterval. (3) If there exists a number L such that
>
> $$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\cdot\Delta x_i = L,$$
>
> then L is called the Riemann Integral or the definite integral of f over [a,b] and is denoted by $\int_a^b f(x)dx$; i.e.,
>
> $$L = \int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\cdot\Delta x_i .$$

Returning to Riemann Sum (E) or (F) and the partition P associated with this Riemann Sum, we have $\|P\| = (15/n)$. Since $\|P\|\to 0$ as $n\to\infty$, we then have (G) which now can be written as

$$\text{(K)}\quad 75 = \int_0^{15} [(-2/3)x + 10]dx = \lim_{n\to\infty} \sum_{i=1}^{n} f[(i)(15/n)](15/n).$$

Exercises

12. In the results of Exercise 9 above, let $n\to\infty$ to find the value of $\int_1^8 (2x + 4)dx$.

13a. Write a Riemann Sum to approximate the value $\int_2^5 (3x + 5)dx$.

b. Find the value of $\int_2^5 (3x + 5)dx$ by using the results of 13a above.

c. Interpret the results of 13b above, in terms of EXAMPLES 4, 5, and 6.

14a. Find the value of $\int_{-2}^4 (5x + 18)dx$.

b. Find the value of $\int_{-1}^4 (x^2 + 4x + 5)dx$.

---

5. THE FUNDAMENTAL THEOREM OF CALCULUS

QUESTION: Does one always have to evaluate a definite integral by calculating a limit of Riemann Sums? The answer to this question is usually NO. For functions f that are continuous over [a,b], the value of $\int_a^b f(x)dx$ can often, but not always, be determined by the Fundamental Theorem of Calculus. We will present this theorem in Section 5.3, but before we look at it we will look at some of its underlying principles in EXAMPLES 7 and 8.

5.1 A Definite Integral with a Variable Endpoint

In (K), we saw that $\int_0^{15} [(-2/3)x + 10]dx = 75$. We now consider a more general form of this integral, namely

(L) $$\int_0^x [(-2/3)t + 10]dt,$$

where we will assume that $0 \le x \le 15$. Note that in (L), we have changed $f(x) = (-2/3)x + 10$ to $f(t) = (-2/3)t + 10$; had we considered $\int_0^x [(-2/3)x + 10]dx$, then we would have used x for two different purposes, namely, (1) to denote the right endpoint of [0,x], and (2) to denote the independent variable of the function f.

Exercises

15a. If x = 0 in (L), what is the value of $\int_0^x [(-2/3)t + 10]dt$?

b. If x = 15 in (L), what is the value of $\int_0^x [(-2/3)t + 10]dt$?

c. If x = 10 in (L), what is the value of $\int_0^x [(-2/3)t + 10]dt$?

16. Interpret $\int_0^{10} [(-2/3)t + 10]dt$ in terms of EXAMPLES 4, 5, and 6.

By letting x, $0 \le x \le 15$, take on various values in (L), we obtain one and only one definite integral, so that we can consider the expression in (L) to be a function of x, namely

$$(M) \qquad F(x) = \int_0^x [(-2/3)t + 10]dt\ ,$$

where $0 \le x \le 15$ and $0 \le t \le x$.

5.2 Antiderivatives and the Definite Integral

EXAMPLE 7. Find another representation of the function F that is represented in (L) and (M).

Solution: Since F(x) is a definite integral with lower limit 0 and upper limit x, we will find its value as a limit of Riemann Sums. By dividing [0,x] into n equal length subintervals, then $\Delta x_i = x/n$ and the ith subinterval is $[(i - 1)(x/n),(i)(x/n)]$, for i = 1, 2, . . . , n. Selecting the right endpoint of each subinterval as c_i, then $c_i = (i)(x/n)$. See Figure 9.

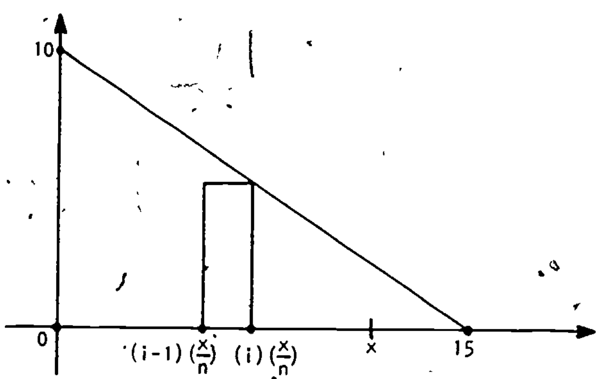


Figure 9. A partition of [0, x].

For this partition P of $[0,x]$ and for $f(t) = [(-2/3)t + 10]$, the Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\cdot\Delta x_i = \sum_{i=1}^{n} [(-2/3)(c_i) + 10](\Delta x_i),$$

$$= \sum_{i=1}^{n} [(-2/3)(i\cdot x/n) + 10)](x/n),$$

$$= \sum_{i=1}^{n} [(-2x^2/3n^2)(i) + (10x/n)(1)],$$

$$= (-2x^2/3n^2)\left[\sum_{i=1}^{n}(i)\right] + (10x/n)\left[\sum_{i=1}^{n}(1)\right],$$

$$= (-2x^2/3n^2)\left[\frac{(n)(n+1)}{2}\right] + (10x/n)(n),$$

(N) $$= (-x^2/3)(1 + 1/n) + 10x,$$

where the value of the Riemann Sum in (N) is a function of <u>both</u> x (the right endpoint of $[0,x]$) and n (the number of equal lengthed subintervals of $[0,x]$). Since $\|P\| = (x/n)\to 0$ as $n\to\infty$, then

$$F(x) = \int_0^x [(-2/3)t + 10]\,dt,$$

$$= \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\cdot\Delta x_i,$$

$$= \lim_{n\to\infty}[(-x^2/3)(1 + 1/n) + 10x],$$

$$= (-x^2/3)(1) + 10x,$$

$$= (-x^2/3) + 10x.$$

Hence, another representation of the function F is

(P) $$F(x) = (-x^2/3) + 10x.$$

Upon investigating some of the properties of this function F in (P) and the given function f, we see that

1) $F'(x) = \frac{d}{dx}[(-x^2/3) + 10x] = (-2/3)x + 10,$

2) $f(t) = (-2/3)t + 10$, or $f(x) = (-2/3)x + 10$.

Here we see an extremely important concept, namely that $F(x) = (-x^2/3) + 10x$ is an antiderivative of $f(x) = (-2/3)x + 10$; i.e.,

$$F'(x) = f(x), \text{ when } F(x) = \int_0^x [(-2/3)t + 10]dt.$$

Note: Remember this fact when we investigate the Fundamental Theorem of Calculus in Section 5.3.

Exercises

17. As in EXAMPLE 7, find another representation of the function F in (M) by letting $c_i = (i - 1)(x/n)$, the left endpoint of each subinterval. Compare this result with (P).

18. As in EXAMPLE 7, find another representation of the function F in (M) by the following method:

  a. In Figure 9, let $G(x)$ = the area of the trapezoid whose two parallel sides are 10 and $(-2/3)x + 10$ units in length and whose base is x units in length.

  b. Use the area formula for a trapezoid to determine $G(x)$.

  Compare $G(x)$ with the results of (P) in EXAMPLE 7.

19. As in EXAMPLE 7, give another justification of the function F in (P) on the basis that velocity is the derivative of a position function.

Before looking at the statement of the Fundamental Theorem of Calculus, we will consider another example.

EXAMPLE 8. Find the distance traveled from $t = 2$ to $t = 8$ seconds by a car whose velocity is $v(t) = (-2/3)t + 10$ ft/sec.

Solution: As a definite integral, this distance can be represented as $\int_2^8 [(-2/3)t + 10]dt$. (Take a moment to consider why this is so.) Since $F(x) = \int_0^x [(-2/3)t + 10]dt$ is the distance the car travels over $[0,x]$, then

1) $F(8) = \int_0^8 [(-2/3)t + 10]dt$ is the distance it travels over $[0,8]$, and

2) $F(2) = \int_0^2 [(-2/3)t + 10]dt$ is the distance it travels over $[0,2]$.

Since we are concerned with the distance traveled over $[2,8]$, we then seek $F(8) - F(2)$. That is,

$$\int_2^8 [(-2/3)t + 10]dt = F(8) - F(2).$$

But in (P) we saw that $F(x) = (-x^2/3) + 10x$, so that

$$\int_2^8 [(-2/3)t + 10]dt = [(-8^2/3)+10(8)] - [(-2^2/3)+10(2)]$$

$$= 40 \text{ feet.}$$

5.3 Statement of the Fundamental Theorem of Calculus

We are now ready to state the Fundamental Theorem of Calculus.

> **FUNDAMENTAL THEOREM OF CALCULUS**
>
> If (1) function f is continuous over $[a,b]$, and
> (2) $F(x)$ is any antiderivative of $f(x)$ over $[a,b]$,
> then
>
> $$\int_a^b f(x)dx = F(b) - F(a).$$

Returning to EXAMPLE 8, we see that we have actually used the Fundamental Theorem of Calculus in the solution of the problem, as

1) $v(t) = (-2/3)t + 10$ (or, $f(x) = (-2/3)x + 10$) is continuous over [2,8] as function v (or, function f) is a linear function.

2) $F(t) = (-t^2/3) + 10t$ (or, $F(x) = (-x^2/3) + 10x$) is an antiderivative of $v(t) = (-2/3)t + 10$ (or, $f(x) = (-2/3)x + 10$) over [2,8].

Hence, the two parts of the hypothesis of the theorem have been satisfied. In conclusion then,

(Q) $$\int_2^8 [(-2/3)t + 10]dt = F(8) - F(2) = 40 \text{ feet}$$

the same result that we had previously obtained.

Exercises

20. In the second hypothesis of the Fundamental Theorem of Calculus, it is stated that F(x) is any antiderivative of f(x) over [a,b].
 a. Evaluate (Q) using $(-t^2/3 + 10t + 5)$ as an antiderivative.
 b. Evaluate (Q) using $(-t^2/3 + 10t - 4/3)$ as an antiderivative.
 c. Evaluate (Q) using $(-t^2/3 + 10t + k)$ as an antiderivative, where k is any constant.

21. Return to item (K) and Exercise 14 and evaluate each of these definite integrals by using the Fundamental Theorem of Calculus.

22. Use the Fundamental Theorem of Calculus to evaluate each of the following definite integrals:

 a. $$\int_2^5 (x^2 + 2x + 1)dx$$

 b. $$\int_2^5 (x^2)dx + 2\int_2^5 (x)dx + \int_2^5 (1)dx$$

c. $\int_{2}^{5}(x+1)^2dx$

d. $\left[\int_{2}^{5}(x+1)dx\right]^2$

e. $\int_{2}^{25}(x+1)dx.$

23. Explain why the Fundamental Theorem of Calculus cannot be used to evaluate the following integral:

$$\int_{-1}^{6}(1/x^2)dx.$$

WARNING: Do not attempt to use the Fundamental Theorem of Calculus unless all conditions of the hypothesis have been satisfied.

24. Evaluate each of the following by using the Fundamental Theorem of Calculus:

a. $\int_{-1}^{4}(-3x^2+5x-7)dx$

b. $\int_{4}^{8}(x+1/x^2)dx$

c. $\int_{0}^{7}(1/x^3+x^{-2})dx.$

d. $\int_{1}^{9}[(x^2+1)/x^2]dx.$

25. Find the area of the region bound by the x-axis and the parabola $y = 6 - x - x^2$.

26. Let $v(t) = t^2 - 3t + 2$ be the velocity function of a car when $0 \leq t \leq 3$, where the velocity is measured in ft/sec and the time t is measured in terms of seconds. Find the distance it travels. WARNING: What does a negative velocity indicate?

27. A bag of salt originally weighing 144 pounds is lifted upward. The salt leaks out uniformly at a rate so that half of the salt is lost when the bag has been lifted 18 feet. Find the work done in lifting the bag this distance.

6. CONCLUSION

The Fundamental Theorem of Calculus has a long and fascinating history behind it. Prior to its development, mathematicians worked for centuries with the derivative, the antiderivative, and sums of products. Isaac Barrow (1630-1677), a teacher of Isaac Newton (1642-1727), discovered and proved the Fundamental Theorem of Calculus, although his method and terms were quite different from those used in this module. Using Barrow's results, both Newton and Gottfried Leibnitz (1646-1716), working independently of each other, developed many of the concepts of calculus, although much of the calculus that we know and use today is attributed to Georg B. Riemann (1826-1866).

In this module, we have solved area, distance, and work problems by the Fundamental Theorem of Calculus. It can also be used to solve problems dealing with the volume of a solid of revolution, arc length, moments, center of gravity, hydrostatic force, product cost, growth (or decay) of a substance, etc. Good luck on your usage of this remarkable tool!

7. ANSWERS TO EXERCISES

1. The formula, distance = velocity x time, can be used only when the velocity is a constant, which is not the case here.

2. The formula, work = force x distance, can be used only when the force is a constant, which is not the case here.

3. (A'') $g(i) \cdot 1 = [(-2/3)i + 10](1)$ is the approximate work done over the ith subinterval, where g(i) is the constant force over the ith subinterval and 1 represents 1 foot, the length of the ith subinterval.

   (B'') $\sum_{i=1}^{15} g(i) \cdot 1$ represents an approximation of the work done in lifting the bag of salt a distance of 15 feet.

(C'') 70 represents the numerical approximation of the work done in terms of foot-pounds.

4. The length of each subinterval is (15/45) = 1/3 units, so that the left endpoint is $(i - 1)(1/3)$, for $i = 1, 2, 3, \ldots, 45$. The area of the ith rectangle is $w[(i - 1)(1/3)](1/3)$. The area of the 45 rectangles is

$$\sum_{i=1}^{45} w[(i - 1)(1/3)](1/3) = 76\ 2/3 \text{ sq. ft.}$$

5a. The approximate distance traveled is 76 2/3 feet.

b. The approximate work done is 76 2/3 foot-pounds.

6a. If $n = 25$, then $(-75)(1 + 1/25) + 150 = 72$.
If $n = 75$, then $(-75)(1 + 1/75) + 150 = 74$.
If $n = 300$, then $(-75)(1 + 1/300) + 150 = 74\ 3/4$.

b. In terms of the area of the triangle, as n becomes larger, there are more rectangles being used, with these rectangles forming a "closer fit" to the shape of the triangle.

7a. $$\sum_{i=1}^{n} f[(i - 1)(15/n)](15/n) = \sum_{i=1}^{n} [(-2/3)(i - 1)(15/n) + 10](15/n)$$
$$= 75 + 75/n.$$

b. If $n = 25$, then 78.
If $n = 75$, then 76.
If $n = 300$, then 75 1/4.

c. $\lim_{n\to\infty}[75 + 75/n] = 75$, the same numerical value that was obtained in (G).

8a. Since the midpoint of the ith subinterval is given by

$[(i - 1)(15/n) + (i)(15/n)]/2$, or, $(i)(15/n) - (15/2n)$, so

$\sum_{i=1}^{n} f[(i)(15/n) - (15/2n)](15/n)$, which = 75 for all values of n.

b. If $n = 25$, then 75.
If $n = 75$, then 75.
If $n = 300$, then 75.

c. $\lim_{n\to\infty}(75) = 75$, the same numerical value that was obtained in (G).

9. Since the length of each subinterval is $(8-1)/n = 7/n$, then $\Delta x_i = 7/n$ and the left endpoint is $c_i = 1 + (i - 1)(7/n)$, so that

$$\sum_{i=1}^{n} f[1 + (i - 1)(7/n)]\cdot(7/n) = 49(1 - 1/n) + 42.$$

10. If $n = 7$, then $49(1 - 1/7) + 42 = 84$.
If $n = 49$, then 90.
etc.

11. Consider the graph of $f(x) = 2x + 4$ over $[1,8]$.
EXAMPLE 4: If $n = 7$, then the area of the trapezoid in the graph is approximately 84 square units.
EXAMPLE 5: If $n = 7$, then from $t = 1$ to $t = 8$ seconds, 84 feet is the approximate distance traveled by a car whose velocity is given by $v(t) = 2t + 4$ feet/sec.
EXAMPLE 6: If $n = 7$, then from $d = 1$ to $d = 8$ feet, 84 foot-pounds is the approximate work done in lifting a bag of salt whose weight at distance d is given by $g(d) = 2d + 4$.

12.
$$\int_1^8 (2x + 4)dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f[1 + (i - 1)(7/n)](7/n),$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f[1 + (i - 1)(7/n)](7/n)$$
$$= \lim_{n\to\infty}\left[49(1 - \frac{1}{n}) + 42\right],$$
$$= 91.$$

13a. Partition $[2,5]$ into n equal lengthed subintervals, and select c_i as the right endpoint. Then $\Delta x_i = 3/n$ and $c_i = 2 + i(3/n)$, for $i = 1, 2, 3, \ldots, n$. A Riemann Sum for $f(x) = 3x + 5$ over $[2,5]$ is

$$\sum_{i=1}^{n} f[2 + (i)(3/n)](3/n) = (27/2)(1 + 1/n) + 33.$$

b. $$\int_2^5 (3x + 5)dx = \lim_{n\to\infty} [(27/2)(1 + 1/n) + 33] = 93/2.$$

c. EXAMPLE 4: 93/2 sq. units is the area of the region bound by the lines $x = 2$, $x = 5$, $y = 0$ and $y = 3x + 5$.

EXAMPLE 5: 93/2 feet is the distance traveled by a car whose velocity is $v(t) = 3t + 5$ feet/second, when $2 \le t \le 5$.

EXAMPLE 6: 93/2 foot-pounds is the work done in lifting a bag of salt from $d = 2$ feet to $d = 5$ feet, where the weight of the salt is $g(d) = 3d + 5$.

14. Using the right endpoint of each equal lengthed subinterval,

a. $$\int_{-2}^{4} (5x + 18)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f[-2 + (i)(6/n)](6/n)$$

$$= \lim_{n\to\infty}(90)(1 + 1/n) + 48 = 138.$$

b. Using the right endpoint of each equal lengthed subinterval,

$$\int_{-1}^{4} (x^2 + 4x + 5)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f[-1 + (i)(5/n)](5/n),$$

$$= \lim_{n\to\infty}[(125/6)(1+1/n)(2+1/n) + 25(1+1/n) + 10]$$

$$= 230/3.$$

15a. 0, as $\Delta x_i = 0$ for all values of i.

b. 75, by (G).

c. $$\int_{0}^{10} [(-2/3)t + 10]dt = \lim_{n\to\infty} \sum_{i=1}^{n} [(-2/3)(i)(10/n) + 10](10/n),$$

$$= \lim_{n\to\infty}[(-100/3)(1 + 1/n) + 100]$$

$$= 200/3.$$

16. EXAMPLE 4: 200/3 is the area of the region bound by the lines $x = 0$, $x = 10$, $y = 0$, and $y = (-2/3)x + 10$.

EXAMPLE 5: 200/3 is the distance traveled by a car whose velocity is $v(t) = (-2/3)t + 10$ feet/second when $0 \le t \le 10$.

EXAMPLE 6: 200/3 is the work done in lifting a bag of salt from $d = 0$ to $d = 10$ and where the weight of the salt is given by $g(d) = (-2/3)d + 10$.

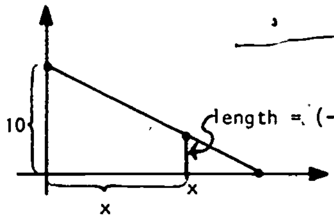
17. $F(x) = \int_0^x [(-2/3)t + 10]dt$

$= \lim_{n\to\infty} \sum_{i=1}^{n} [(-2/3)(i - 1)(x/n) + 10](x/n)$

$= \lim_{n\to\infty} [(-x^2/3)(1 + 1/n) + (2x^2/3n) + 10x]$

$= (-x^2/3) + 10x$, which is the same as (P).

18a.



b. $G(x) = (1/2)(x)[10 + (-2/3)x + 10]$, or, $G(x) = (-x^2/3) + 10x$, which is the same as (P).

19. In (M), let $f(t) = (-2/3)t + 10$ be the velocity of a car over $[0,x]$. Since the velocity is the derivative of a position function, say $F(x) = (-x^2/3) + 10x$, where $F'(x) = (-2/3)x + 10$ which $= f(x)$.

20. a., b., and c. $F(8) - F(2) = 40$.

21. (K) $\int_0^{15} [(-2/3)x + 10]dx = F(15) - F(0)$, where $F(x) = (-x^2/3) + 10x$,

$= 75$.

(14a.) $\int_{-2}^{4} (5x + 18)dx = F(4) - F(-2)$, where $F(x) = (5x^2/2) + 18x$,

$= 138$.

(14b.) $\int_{-1}^{4} (x^2 + 4x + 5)dx = F(4) - F(-1)$, where $F(x) = x^3/3 + 2x^2 + 5x$,

$= 230/3$.

22a. 63; b. 63; c. 63; d. $(27/2)^2$; e. 677/2.

23. The function $f(x) = 1/x^2$ is not continuous over $[-1,6]$.

24. a. $-62\ 1/2$.

b. $24\ 1/8$.

c. Did you heed the WARNING in Exercise 23 above?

d. $80/9$.

25. $$\text{Area} = \int_{-3}^{2} (6 - x - x^2)dx = 125/6 \text{ square units.}$$

26. $$\text{Distance} = \int_0^1 (t^2 - 3t + 2)dt + \int_1^2 -(t^2 - 3t + 2)dt + \int_2^3 (t^2 - 3t + 2)dt = 11/6.$$

27. Since slope $= (144 - 72)/(0 - 18) = -4$ and the vertical intercept is 144, then $g(x) = -4x + 144$.

$$\text{Work} = \int_0^{18} (-4x + 144)dx = 1944 \text{ ft-lbs.}$$

8. MODEL EXAM

1a. Set up and simplify a Riemann Sum to approximate the area of the region bound by the lines $x = 2$, $x = 5$, $y = 0$, and $y = 2x + 3$.

b. In the simplified Riemann Sum in 1a. above, let $n \to \infty$.

c. Set up the definite integral that will yield the area of the region in 1a. above.

d. Evaluate the definite integral in 1c. above by using the Fundamental Theorem of Calculus.

2. Suppose that a particle travels along a straight line and its velocity is given by $v(t) = t^2 + 8t + 17$ feet/sec. Find the distance it travels over $[1,5]$ by (a) using the limit of a Riemann Sum, and (b) by using the Fundamental Theorem of Calculus.

3. Consider the following limit of a Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\{[1 + (i)(6/n)]^2 + 5[1 + (i)(6/n)] - 2\}(6/n)).$$

Write this as a definite integral and evaluate it by using the Fundamental Theorem of Calculus.

4. Evaluate each of the following by using the Fundamental Theorem of Calculus:

a. $\int_{-2}^{4} (x^3 - 3x + 1)dx$

b. $\int_{3}^{5} (x^{-2} + x)dx$

c. $\int_{-1/2}^{3/4} (x^{-1/3} + x)dx.$

5. According to Hooke's Law, the force F required to stretch a spring x units beyond its natural length is $F(x) = kx$, where k is called the "modulus of the spring." Suppose that it takes a 2-pound force to stretch a spring from 15 inches (its natural length) to 20 inches, so that $F(5) = k(5) = 2$, or $k = 2/5$. With the same spring, what is the work required to stretch the spring from 15 inches to 21 inches?

9. ANSWERS TO MODEL EXAM

1a. $\sum_{i=1}^{n} \{2[2 + (i)(3/n)] + 3\}(3/n).$

b. 30.

c. $\int_{2}^{5} (2x + 3)dx.$

d. 30.

2a. $\lim_{n\to\infty} \sum_{i=1}^{n} \{[1 + (i)(4/n)]^2 + 8[1 + (i)(4/n)] + 17\}(4/n) = 616/3.$

b. $\int_1^5 (t^2 + 8t + 17)dt = 616/3$ feet.

3. $\int_1^7 (x^2 + 5x - 2)dx = 222.$

4a. 48.

b. 8 2/15.

c. $f(x) = x^{1/2} + x$ is <u>not</u> continuous over [-1/2, 3/4].

5. $\int_0^6 [(2/5)x]dx = 36/5$ inch-pounds.

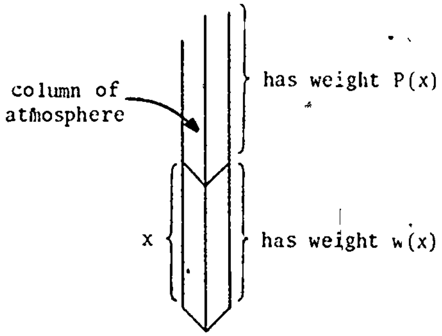
umap

UNIT 426

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

ATMOSPHERIC PRESSURE IN RELATION TO HEIGHT AND TEMPERATURE

by Arnold J. Insel



APPLICATIONS OF CALCULUS TO ATMOSPHERIC PRESSURE

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ATMOSPHERIC PRESSURE IN RELATION TO HEIGHT AND TEMPERATURE

by

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Intermodular Description Sheet: *UMAP Unit 426*

Title: ATMOSPHERIC PRESSURE IN RELATION TO HEIGHT AND TEMPERATURE

Author: Arnold J. Insel
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Normal, Illinois 61761

Review Stage/Date: III 4/16/80

Classification: APPL CALC/ATMOSPHERIC PRESSURE

Suggested Support Material:
Access to a calculator is important.

Prerequisite Skills:
1. Knowledge of the calculus as applied to logarithms and exponentials.
2. An acquaintance with high school chemistry is useful.

Output Skills:
1. To understand the application of calculus to construct a mathematical model of the atmosphere.

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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ATMOSPHERIC PRESSURE IN RELATION TO HEIGHT AND TEMPERATURE

1. INTRODUCTION

Atmospheric pressure on the Earth's surface is due to the weight of the atmosphere above. Imagine a vertical column whose cross section is an inch square and which extends upwards from the Earth's surface without bound. (See Figure 1.)

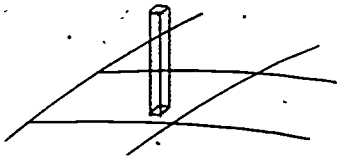

Figure 1.

The weight of this column in pounds (as weighed in a vacuum) is numerically equal to the atmospheric pressure in pounds per square inch at the surface. This is so since the pressure at the surface is the force per unit area due to the weight of the atmosphere. More generally, at a height x above the Earth's surface the atmospheric pressure is numerically equal to the weight of that portion of the air column above height x.

We make use of this simple observation along with certain well known facts about ideal gasses to create two mathematical models of atmospheric pressure. The first model is somewhat simplified since it does not take temperature variation with altitude into account. Its introduction serves the purpose of preparing the way for the study of the more complicated second model which does take temperature variation into account. We apply this second model to study conditions under which the atmosphere is unstable.

2. THE SIMPLIFIED MODEL

2.1 Derivation of the Formula

Consider the column of atmosphere as described in the introduction. Let P_0 denote the weight of this column in pounds. For any $x \geq 0$ let $P(x)$ denote the atmospheric pressure in pounds per square inch, x inches above the Earth's surface. Likewise for any $x \geq 0$ let $w(x)$ denote the weight of that portion of the column, in pounds, from the Earth's surface to the height of x inches above the surface. Figure 2 illustrates the relationship between $P(x)$ and $w(x)$.

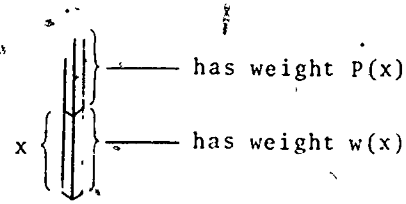

Figure 2.

The following equations are now clear.

$$P(0) = P_0$$

and

(1) $$P(x) = P_0 - w(x) \quad \text{for any } x \geq 0.$$

Observe that as functions of x, P is decreasing and w is increasing. Let us suppose that the functions P and w are each continuously differentiable.

If we were to weigh samples of air at various altitudes we would normally find that for a fixed volume, samples of air taken at low altitudes are heavier than samples taken at higher altitudes. For any $x \geq 0$ let $\rho(x)$ denote the weight of a cubic inch of air in the column at height x inches above the Earth's surface. We shall call $\rho(x)$ the *density* of the atmosphere at height x.

Ordinarily ρ is a decreasing function in x. Let $\rho_0 = \rho(0)$, the density of air at the surface of the column.

Next we assert that for any $x \geq 0$

(2) $$\frac{d}{dx} w(x) = w'(x) = \rho(x).$$

To see this, first observe that for any $x \geq 0$ and $\Delta x > 0$, $w(x + \Delta x) - w(x)$ is the weight in pounds of the air in the column from height x to height $x + \Delta x$. So the quotient

$$\frac{w(x + \Delta x) - w(x)}{\Delta x}$$

is the average density of the air in pounds per cubic inch in that portion of the column. Thus

$$w'(x) = \lim_{\Delta x \to 0} \frac{w(x + \Delta x) - w(x)}{\Delta x}$$

represents the density of the atmosphere at height x, thus justifying Equation (2).

Next, differentiating both sides of Equation (1) and applying Equation (2) we obtain

(3) $$P'(x) = -\rho(x) \quad \text{for any } x.$$

We now introduce two assumptions used to construct the simplified model:

(a) the chemical composition of the atmosphere is uniform and independent of the height. (The ratios of the various gasses making up the atmosphere are independent of height.)

(b) The temperature of the atmosphere is independent of height.

If we apply these assumptions along with the assumption that the atmosphere is an ideal gas we may invoke a variant of Boyle's law which states that the density of a gas is proportional to its pressure. So for any $x \geq 0$

$$\frac{P(x)}{P_0} = \frac{\rho(x)}{\rho_0}$$

or

(4) $$\rho(x) = \frac{\rho_0}{P_0} P(x)$$

Combining Equations (3) and (4) yeilds the equation

$$P'(x) = \frac{-\rho_0}{P_0} P(x)$$

whose solution is evidently

(5) $$P(x) = P_0 \exp\left[\frac{-\rho_0 x}{P_0}\right] \quad \text{for any } x \geq 0.$$

Equation (5) is a formula relating atmospheric pressure with height. For the sake of practicality we modify (5) so that x is in units of feet rather than inches. Thus (5) yeilds

(6) $$P(x) = P_0 \exp\left[\frac{-12\rho_0 x}{P_0}\right] \quad \text{for x in units of feet.}$$

2.2 An Example

Let us now apply Formula (6) based on the assumptions of Section 2.1.

Given the assumptions of this section and that $P_0 = 14.7$ lbs./sq. inch and $\rho_0 = 4.34 \times 10^{-5}$ lbs./cu. inch, find the atmospheric pressure at 20,000 feet above the Earth's surface. By (6)

$$P(20{,}000) = (14.7) \exp\left[\frac{-(12)(4.34\times10^{-5})}{14.7}(20{,}000)\right]$$

$$= 7.24 \text{ lbs./sq. inch.}$$

Exercise 2.1

Given the assumptions of this section and the values of P_0 and ρ_0 as in the example above, at what height is the atmospheric pressure 1/2 of its value at the surface?

Exercise 2.2

Given the assumptions of this section, suppose that at the Earth's surface the atmospheric pressure is 15.00 pounds per square inch while at a height of 1,000 feet above the surface the atmospheric pressure is 14.47 pounds per square inch. Find ρ_0.

Exercise 2.3

Given the assumptions of this section and the values of P_0 and ρ_0 in Example 2.2, assume that the Earth's radius is 4,000 miles. What is the total weight of the Earth's atmosphere in pounds?

3. THE MORE COMPLICATED MODEL

3.1 Derivation of the Formula

Let us now delete the assumptions that atmospheric temperature is independent of height.

We shall measure temperature on the absolute scale of the Kelvin system. Recall that the Kelvin and the Celsius systems are related since an increase in temperature of one degree K (Kelvin) is identical to an increase in temperature of one degree Celsius, and each correspond to a temperature increase of 1.8 degrees on the Fahrenheit scale. However, under the Kelvin system, 0° K is absolute zero, water freezes at 273.1° K, and water boils at 373.1° K. To convert from the Celsius to the Kelvin system, simply add 273.1.

The relationship between temperature on the Fahrenheit and Kelvin scales are given by the equations

$$F = 1.8K - 459.58$$

and

$$K = (F + 459.58)/(1.8)$$

where F and K are the temperatures on the Fahrenheit and the Kelvin scale respectively.

For any $x \geq 0$ let $T(x)$ denote the temperature in degrees Kelvin of the atmosphere at height x in inches above the Earth's surface. Let $T_0 = T(0)$, the temperature at the surface. In Section 2 we applied Boyle's law of ideal gasses to obtain Equation (4). In this section we complicate the model by superimposing a variation of Charles's law of ideal gasses which states that the gas density varies inversely with temperature when measured on an absolute scale, such as the Kelvin scale. Consequently we must introduce the factor $T_0/T(x)$ to the right hand side of Equation (4) to obtain

$$\rho(x) = \frac{\rho_0 T_0 P(x)}{P_0 T(x)} \tag{7}$$

Hence

$$\frac{P'(x)}{P(x)} = \frac{-\rho_0 T_0}{P_0 T(x)}$$

Hence for any $x \geq 0$ we have

$$\int_0^x \frac{P'(t)}{P(t)}\,dt = \frac{-\rho_0 T_0}{P_0}\int_0^x \frac{1}{T(t)}\,dt$$

$$\ln\frac{P(x)}{P_0} = \frac{-\rho_0 T_0}{P_0}\int_0^x \frac{1}{T(t)}\,dt$$

and hence

$$P(x) = P_0 \exp\left[\left(\frac{-\rho_0 T_0}{P_0}\int_0^x \frac{1}{T(\ )}\,dt\right)\right]$$

where x is the height in inches above the Earth's surface. Finally, let us adjust this formula, as in Section 2, so that x is in units of feet. Suppose that the function T gives the temperature at height x, where x is in units of feet rather than inches. Then the expression T(x/12) gives the temperature at height x, where x is in units of inches. Hence at x feet above the ground

$$P(x) = P_o \exp\left[\left(\frac{-\rho_o T_o}{P_o}\right)\int_0^{12x} \frac{1}{T(t/12)}\, dt\right].$$

So we must simplify the expression

$$\int_0^{12x} \frac{1}{T(t/12)}\, dt \ .$$

Introducing the substitution u = t/12 we have

$$\int_0^{12x} \frac{1}{T(t/12)}\, dt = 12 \int_0^{x} \frac{1}{T(u)}\, du$$

$$= 12 \int_0^{x} \frac{1}{T(t)}\, dt \ .$$

Thus we finally arrive at

(8) $$P(x) = P_o \exp\left[\left(\frac{-12\rho_o T_o}{P_o}\right)\int_0^{x} \frac{1}{T(t)}\, dt\right]$$

where x is in units of feet and T(x) is the temperature at a height of x feet above the Earth's surface.

3.2 An Example, T Varies Linearly with x

General aviation pilots use the rule: the temperature of the atmosphere decreases linearly with height at a rate of 2° C (Celsius) per 1,000 feet of altitude. Since the difference of a degree Celsius is identical to the difference of a degree Kelvin we may translate this

rule into the formula

$$T(x) = T_o - (2/1000)x$$

where x is the height in units of feet above the Earth's surface. Combining this formula with Equation (8) we have

$$\int_0^{x} \frac{1}{T(t)}\, dt = \int_0^{x} \frac{1}{T_o - (2/1000)t}\, dt$$

$$= -(500)\ \ln(1 - x/(500T_o)) \ ,$$

and so by (8) we have

$$P(x) = P_o \exp\left[\frac{12\rho_o T_o}{P_o}(500)\ \ln(1 - x/(500T_o))\right]$$

or equivalently

(9) $$P(x) = P_o(1 - x/(500T_o))^{\left(\frac{6000\rho_o T_o}{P_o}\right)}$$

<u>Example</u>:

Assuming Equation (9) with P_o = 14.7 lbs./sq. inch, $\rho_o = 4.34 \times 10^{-5}$ lbs./cu. inch and $T_o = 293^o$ K

(a) What is the atmospheric pressure at 20,000 feet above the Earth's surface?

(b) ~~At what height is the atmospheric pressure half of the pressure at the surface?~~

<u>Solution of (a)</u>:

$$P = 14.7\left(1 - \frac{20,000}{(500)(293)}\right)^{\frac{(6000)(4.34 \times 10^{-5})(293)}{14.7}}$$

$$= 6.86 \text{ lbs./sq. inch.}$$

Comparing this answer to the answer of 7.24 lbs./sq. inch of the problem as treated in Section 2.2 we see that there is a difference of only about 5.5%. This is not surprising if we observe that the first order Taylor approximation of Equation (9) is independent of T_o.

Solution of (b):

Assuming $P(x)/P_o = 1/2$ Equation (9) becomes

$$\frac{1}{2} = \left(1 - \frac{x}{(500)(293)}\right)^{\frac{(6000)(4.34 \times 10^{-5})(293)}{14.7}}$$

$$= \left(1 - \frac{x}{(500)(293)}\right)^{5.19} \quad \text{(computing to three significant digits)}$$

So

$$\frac{1}{2}^{\left(\frac{1}{5.19}\right)} = 1 - \frac{x}{(500)(293)}$$

or

$$x = (500)(293)\left(1 - \frac{1}{2}^{\left(\frac{1}{5.19}\right)}\right)$$

$$= 18{,}300 \text{ feet} \quad \text{(to three significant digits)}$$

3.3 An Application to Meteorology

Let $\rho(x)$ denote the density of the atmosphere at height x in feet. We would normally expect that ρ decreases as x increases. Under these conditions we shall say that the atmosphere is *stable*. Otherwise we shall call the atmosphere *unstable*. In an unstable atmosphere a given volume of air above would weigh at least as much as an equal volume of air below. Under these circumstances there would be a vertical motion of air causing winds and down draughts. Mathematically the air is unstable if

(10) $\rho'(x) \geq 0.$

Let us study the conditions of instability under the assumptions and results of Section 3.1. Combining Equations (7) and (8) we have

$$\rho(x) = \frac{\rho_o T_o}{T(x)} \exp\left[\left(\frac{-12\rho_o T_o}{P_o}\int_0^x \frac{1}{T(t)}\,dt\right)\right]$$

where x is in units of feet and T(t) is the temperature (measured on the Kelvin scale) t feet above the surface. Thus by Equation (10) the atmosphere is unstable if

$$\rho'(x) = \frac{-\rho_o T_o}{[T(x)]^2}\left[T'(x) + \frac{12\rho_o T_o}{P_o}\right]\exp\left[\left(\frac{-12\rho_o T_o}{P_o}\int_0^x \frac{1}{T(t)}\,dt\right)\right] \geq 0.$$

This inequality reduces to

(11) $T'(x) \leq \dfrac{-12\rho_o T_o}{P_o}.$

Example

Assume that T drops linearly with height, $T_o = 293°$ K, $\rho_o = 4.34 \times 10^{-5}$ lbs./cu. inch, and $P_o = 14.7$ lbs./sq. inch. Find the maximum temperature at 1000 feet above the surface of the Earth so that the atmosphere is unstable. The assumption of linearity requires that

$$T(x) = T_o - kx$$

for some positive constant k. Thus by inequality (11) we have

$$-k \leq \frac{-12\rho_o T_o}{P_o}$$

or

$$0.0104 \leq k \quad \text{(approximating to three significant digits)}$$

To achieve the maximum temperature we require that k be as small as possible, i.e. $k = 0.0104$. Thus

$$T(x) = 293 - (0.0104)x.$$

So for $x = 1{,}000$

$$T(x) = T(1{,}000) = 293 - 10.4 = 282.6.$$

Notice that there is a temperature drop of 10.4 degrees Celsius per 1,000 feet. This corresponds to a drop of approximately 18.7 degrees Fahrenheit per 1,000 feet.

One final note. The second model is still over-simplified since it does not consider the possibility of variation of atmospheric composition with altitude. In particular, the atmosphere may vary in altitude with respect to the amount of water vapor it contains. The atmosphere may also contain such pollutants as smoke and smog. These all contribute to its density and hence to its pressure.

Exercise 3.1

Assuming Equation (9), $P_0 = 14.7$, $\rho_0 = 4.34 \times 10^{-5}$, and $T_0 = 300°$ K, what is the atmospheric pressure at 10,000 feet and at 20,000 feet?

Exercise 3.2

Assume the model of this section and that temperature decreases linearly with height. Suppose $P_0 = 14.7$, and $T_0 = 283$. Suppose, in addition, that at 10,000 feet $T = 293$ and $P = 10.2$. Find ρ_0.

Exercise 3.3

Assuming T is a constant (and therefore $T(x) = T_0$ for all x) show that Equation (8) reduces to Equation (6).

Exercise 3.4

Assuming that T decreases linearly with height and assuming the model of this section, generalize the formula of Equation (9), given that T decreases by k degrees Celsius per 1,000 feet.

---

4. ANSWERS TO SELECTED EXERCISES

2.1. 19,600 feet.

2.2. $\rho_0 = 4.497 \times 10^{-5}$

2.3. 1.19×10^{19}

3.1. 10.2 lbs. per square inch at 10,000 feet.
6.87 lbs. per square inch at 20,000 feet.

3.2. $\rho_0 = 4.40 \times 10^{-5}$

3.4. $$P(x) = P_0\left(1 - \frac{kx}{1000T_0}\right)^{\frac{12000\rho_0 T_0}{P_0 k}}$$

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name____________________ Unit No.______

Page______ ○ Upper ○ Middle ○ Lower	OR	Section______ Paragraph______	OR	Model Exam Problem No.______ Text Problem No.______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

70

48

Instructor's Signature____________________

Please use reverse if necessary.

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STUDENT FORM 2

Unit Questionnaire

Name______________________ Unit No.____________ Date____________

Institution______________________ Course No.______________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain______________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain______________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

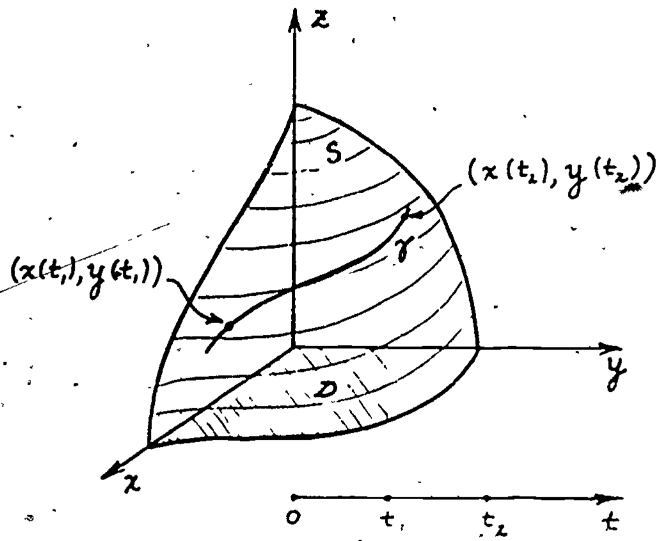
umap

UNIT 431

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

THE GRADIENT AND SOME OF ITS APPLICATIONS

by Joan R. Hundhausen and Robert A. Walsh



APPLICATIONS OF MULTIVARIATE CALCULUS TO PHYSICS

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THE GRADIENT AND SOME OF ITS APPLICATIONS

by

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TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 431*

Title: THE GRADIENT AND SOME OF ITS APPLICATIONS

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Review Stage/Date: III 9/15/80

Classification: APPL MULTIVARIATE CALC/PHYSICS

Prerequisite Skills:
1. Know what is meant by a surface.
2. Know what is meant by a space curve
3. Know how to compute partial derivatives.
4. Know how to compute the dot product of two vectors.

Output Skills:
1. Know how to compute the Gradient.
2. Be able to use the Gradient to solve certain practical problems involving steepest ascent.

Other Related Units:
Unconstrained Optimization: Refinement of Local Extrema by the Gradient Search Procedure for 3-D Surfaces *(Unit 522)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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1. INTRODUCTION

In this module we consider some properties and illustrations of space curves that lie on given surfaces. We explore the following questions:

1) How is the instantaneous rate of climb along a curve on the surface related to the equation of the surface?

2) Is it possible to find a path on the surface for which the rate of climb is optimized at each point along the way?

As physical examples, consider the following situations:

Example 0-1. A party of mountaineers can ascend to a summit by a leisurely process of "switchbacking". A few hardier members of the group wish to test their mettle by starting from the same location as the others, but reaching the summit by a path that is always the steepest possible. Certainly such a course, if it can be found, will be much more strenuous! If the topography of the mountain is known, the principles of the gradient may be used to chart such a course.

Example 0-2. An oil tanker has met with disaster at night, on a calm sea, and is left without radio communication. A rescue vessel that is able to monitor continuously the concentration C of the spreading oil slick tries to locate the tanker by moving in the direction of greatest *increase* of the concentration. What is its path?

In the situation of Example 0-2, we are thinking of the concentration of oil, C, as the dependent variable, and we have $C = f(x,y)$ which is of the form $z = f(x,y)$, (a standard designation for a 3-dimensional surface).

2. PRELIMINARIES

Let us begin by considering what is meant by "a curve lying on a surface". Suppose S is a surface in three-dimensional space E_3 defined by

$$z = f(x,y) \text{ for all } (x,y) \text{ in } D,$$

where D is a set in the xy-plane. We assume that S is a "smooth" surface; i.e., the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are continuous at each point in D.

Let γ be a curve that is defined by the parametric equations

$$\text{(A)} \qquad \gamma: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

where t varies over an interval I on the real line. If x(t) and y(t) lie in D for each t in I, and if z(t) satisfies the equation $z(t) = f[x(t),y(t)]$, then we say that γ lies on the surface S. See Figure 1.

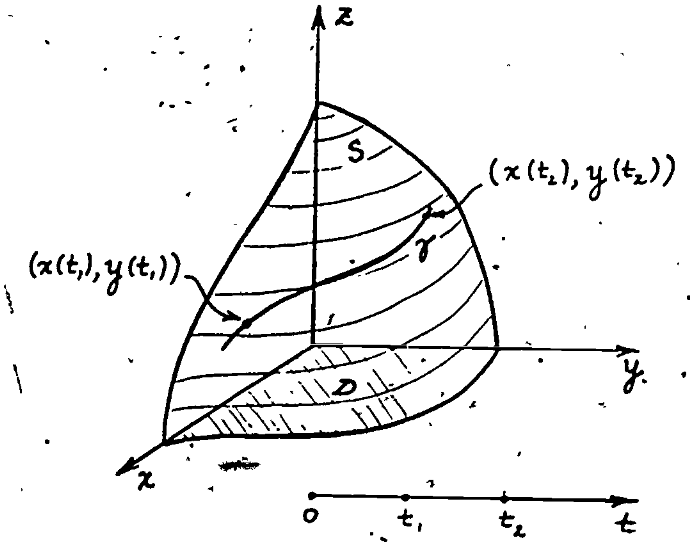

Figure 1.

We also stipulate that γ be a differentiable curve; i.e., γ will possess arc length for all finite intervals (t_1, t_2), in I. This is equivalent to the requirement that the function $[\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2$ be continuous for all t in (t_1, t_2) and the arclength s of γ from t_1 to t_2 be given by

$$s = \int_{t_1}^{t_2} \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2}\, dt .$$

The symbols $\dot{x}(t)$, $\dot{y}(t)$, $\dot{z}(t)$ indicate, respectively, dx/dt, dy/dt, dz/dt.

In applications, D may be a bounded or unbounded domain in the xy-plane and I may be a finite or infinite interval.

Example 1. Consider the surface S given by

(1) $$z = x^2 + xy + 2y^4 ,$$

or

$$f(x,y) = x^2 + xy + 2y^4 .$$

Note that

$$\frac{\partial f}{\partial x} = 2x + y ,$$

$$\frac{\partial f}{\partial y} = x + 8y^3 .$$

One specific curve γ lying on S is given by

$$\gamma: \begin{cases} x = x(y) = t \\ y = y(t) = t^2 \\ z = z(t) = t^2 + t^3 + 2t^8 \end{cases} \quad t \text{ in } [1,2]$$

where we may verify that $z(t) = [x(t)]^2 + x(t)y(t) + 2[y(t)]^4$ by substituting into Equation (1). The arclength s of this curve is given by

$$s = \int_1^2 \sqrt{1 + 4t^2 + (2t + 3t^2 + 16t^7)^2}\, dt .$$

(The value of s can be found by numerical integration; we used a Romberg procedure to obtain $s \approx 520.0302$.)

Exercise 1: Let S be the surface defined by

$z = f(x,y) = x^2 + 2y^2$, for all (x,y).

a) Sketch the portion of S that lies over the first octant.

b) Find $\partial f/\partial x$, $\partial f/\partial y$.

c) Verify that the curve $\gamma: \begin{cases} x(t) = t \cos t \\ y(t) = \frac{t}{\sqrt{2}} \sin t \\ z(t) = t^2 \end{cases}$

d) Find the arclength of the curve γ for t in $[0,2\pi]$.
(Hint: You should see that γ is a helix winding up around the elliptical paraboloid $z = x^2 + 2y^2$.)

---

3. DEFINITION OF THE GRADIENT

The partial derivatives of f(x,y) determine an important vector field in the xy-plane.

The *Gradient* $\vec{\nabla} f(x_0, y_0)$ of the surface S defined by the function $z = f(x,y)$ at any point (x_0, y_0) in D is the vector

$$\vec{\nabla} f(x_0, y_0) \equiv \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\,\Bigg|_{(x_0, y_0)} ,$$

where $\hat{i}$ and $\hat{j}$ are unit vectors along the positive x-axis and the positive y-axis, respectively, and the notation

$$\cdot\,\Big|_{(x_0, y_0)}$$

indicates that we substitute x_0 for x and y_0 for y in the expressions $\partial f/\partial x$ and $\partial f/\partial y$.

Example 2. To find the Gradient of the function in Example 1 at the point $(x_0, y_0) = (1,2)$, we find the partial derivatives

$$\frac{\partial f}{\partial x} = 2x + y , \quad \frac{\partial f}{\partial y} = x + 8y^3 ,$$

form the general expression

$$\vec{\nabla} f(x_0, y_0) = (2x + y)\hat{i} + (x + 8y^3)\hat{j}\Big|_{(1,2)} ,$$

then substitute 1 for x and 2 for y to find

$$\vec{\nabla} f(1,2) = 4\hat{i} + 65\hat{j} .$$

Note: The Gradient of S depends only on $f(x,y)$ and on (x_0, y_0) but *not* on z_0, where $z_0 = f(x_0, y_0)$. This fact, though readily apparent, deserves emphasis! Even though S is *3-dimensional*, the gradient of S is a *2-dimensional* vector. Often the Gradient $\vec{\nabla} f(x,y)$ is confused with the Normal to S at the point (x_0, y_0, z_0). The Normal $N_S(x_0, y_0, z_0)$ to S, however, is given by

$$\vec{N}_S(x_0, y_0, z_0) = \frac{\partial f}{\partial x}(x_0, y_0)\hat{i} + \frac{\partial f}{\partial y}(x_0, y_0)\hat{j} - 1\hat{k}$$

which is a vector with three components, while the Gradient has only two.

4. FINDING THE TANGENT VECTOR TO A CURVE ON A SURFACE USING THE GRADIENT

Suppose that a surface S is defined by $z = f(x,y)$, and that γ is a curve on S. Then recalling the parametric form (A) for γ we may define $T(t_0)$, the tangent vector to γ at $t = t_0$ by the equation

$$\vec{T}(t_0) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}\Big|_{t = t_0}$$

$\dot{z}(t)$ may be found directly from the parametric forms, but since we have a chain

$$z = f(x,y): \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

we may also use the appropriate chain rule, as follows:

$$\text{(B')} \quad \dot{z}(t_0) = \frac{\partial f}{\partial x}[x(t),y(t)]\dot{x}(t) + \frac{\partial f}{\partial y}[x(t),y(t)]\dot{y}(t)\Big|_{t = t_0} ;$$

or, what is the same thing,

$$\dot{z}(t_0) = \frac{\partial f}{\partial x}(x,y)\Big|_{\substack{x=x(t_0) \\ y=y(t_0)}} \dot{x}(t_0) + \frac{\partial f}{\partial y}(x,y)\Big|_{\substack{x=x(t_0) \\ y=y(t_0)}} \dot{y}(t_0) .$$

The chain rule is also commonly written as

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} ;$$

see Thomas & Finney, p. 401 or Greenspan & Benney, p. 496. An example should help to clarify the two distinct ways of finding $\dot{z}(t)$.

Example 3. From Example 1, we know that the curve

$$\gamma: \begin{cases} x = x(t) = t \\ y = y(t) = t^2 \\ z = z(t) = t^2 + t^3 + 2t^8 \end{cases} \quad t \text{ in } [1,2]$$

lies on the surface

$$z = f(x,y) = x^2 + xy + 2y^4 .$$

From the definition of γ we have

$$\dot{z}(t_0) = 2t_0 + 3t_0^2 + 16t_0^7 ;$$

so that tangent vector to γ at t_0 is given by

$$\vec{T}(t_0) = 1\hat{i} + 2t_0\hat{j} + (2t_0 + 3t_0^2 + 16t_0^7)\hat{k} .$$

We can also calculate $\vec{T}(t_0)$ by using the chain rule, as follows:

$$\frac{\partial f}{\partial x}(x(t_0), y(t_0)) = 2x + y\Big|_{(x = t_0, y = t_0^2)}$$

$$= 2t_0 + t_0^2$$

$$\frac{\partial f}{\partial x}(x(t_0),y(t_0)) = x + 8y^3 \Big|_{(x = t_0, y = t_0^2)}$$

$$= t_0 + 8t_0^6 ,$$

from which, using the chain rule (B),

$$\dot{z}(t_0) = (2t_0 + t_0^2)\dot{x}(t_0) + (t_0 + 8t_0^6)\dot{y}(t_0)$$

$$= (2t_0 + t_0^2)1 + (t_0 + 8t_0^6)(2t_0)$$

$$= 3t_0 + 3t_0^2 + 16t_0^7 .$$

Hence, the tangent vector to γ at t_0 is given by

$$\vec{T}(t_0) = 1\hat{i} + 2t_0\hat{j} + (2t_0 + 3t_0^2 + 16t_0^7)\hat{k} ,$$

which checks with the above calculation.

<u>Exercise 2</u>. For the curve

$$\gamma: \begin{cases} x(t) = t + 1 \\ y(t) = t - 1 \\ z(t) = t^2 - 1 \end{cases}$$

which lies on the surface $z = xy$, find the tangent vector $\vec{T}(1)$ in two distinct ways.

The connection between the Gradient and this alternate calculation which uses the Chain Rule will be explored after we recall some concepts and properties of vectors.

Let a be the vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

1. The *length* of the vector a is denoted by $|\vec{a}|$ and is given by

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} .$$

2. The *projection* of a on the xy plane will be denoted by

$$\vec{a}_p \equiv a_1\hat{i} + a_2\hat{j} ; \quad \text{See Figure 2a.}$$

3. The *dot product* of $\vec{a}$ with vector $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is denoted by $\vec{a}\cdot\vec{b}$ and is given by $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$. We also have $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$; see Figure 2b.

4. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be *orthogonal* (or perpendicular) if $\vec{a}\cdot\vec{b} = 0$, i.e., if the angle θ between them is $\pi/2$. From 3, a condition for orthogonality also becomes $a_1b_1 + a_2b_2 + a_3b_3 = 0$, which is satisfied in two dimensions (note $a_3 = 0$, $b_3 = 0$) if and only if we have $\vec{b} = c(-a_2\hat{i} + a_1\hat{j})$, where c is a proportionality constant.

So far we have seen that if the curve γ *lies on the surface S*, (given by $z = f(x,y)$) and if $x = x(t)$ and $y = y(t)$ are given, then the z-coordinate for γ must satisfy the "surface requirement" $z(t) = f[x(t),y(t)]$.

More importantly we notice that, as in (B),

$$\dot{z}(t) = \frac{\partial f}{\partial x}[x(t),y(t)]\dot{x}(t) + \frac{\partial f}{\partial y}[x(t),y(t)]\dot{y}(t)$$

which may be written as the dot product

$$\dot{z}(t) = \vec{\nabla}f(x(t),y(t)) \cdot \vec{T}_p(t),$$

where $\vec{T}_p(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$ is the projection of the tangent vector $\vec{T}(t)$ in the xy-plane.

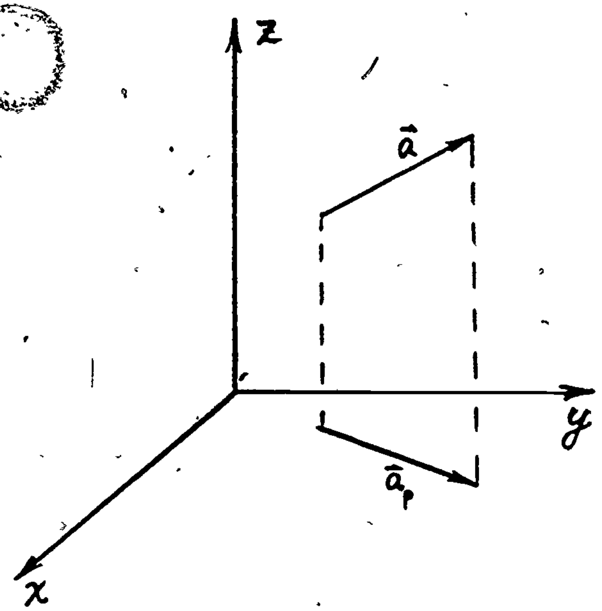

Figure 2a.

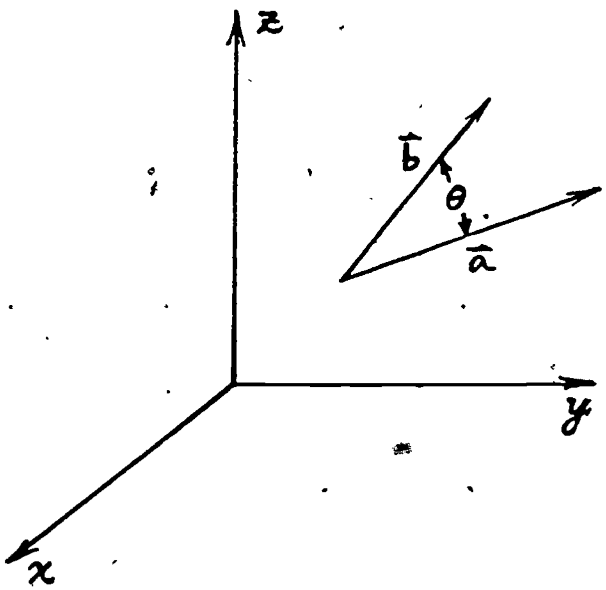

Figure 2b.

Thus the rate of change of the coordinate $z(t)$ depends on two vectors, $\vec{\nabla} f(x(t),y(t))$ and $\vec{T}_p(t)$, *both of which are located in the xy-plane.* By the definition of the dot product we can also write

$$\dot{z}(t_0) = |\vec{\nabla} f(x(t),y(t))|\ |\vec{T}_p(t)|\cos\psi(t)\Big|_{t = t_0}$$

where $\psi(t_0)$ is the angle between the Gradient and the projected vector $\vec{T}_p(t)$ at $t = t_0$. We see, then, that the *sign* of $\dot{z}(t_0)$ is determined by the sign of $\cos\psi(t)$ and thus that the coordinate $z(t)$ increases when $\cos\psi(t) > 0$ and decreases when $\cos\psi(t) < 0$.

And so $\dot{z}(t_0)$ may be found from the two vectors in the xy-plane (Figure 3) merely by forming their dot product.

Of particular significance is the situation where $\psi(t) = \pi/2$ for all values of t in some interval. In that case, $\cos\psi(t) \equiv 0$, and $\dot{z}(t) \equiv 0$, (from above) for all t in the interval. Since the rate of change of $z(t)$ is 0, $z(t)$ is *constant* for all t in the interval. This leads us to consider only the family of *level curves*,

$$\tilde{\gamma}: \quad f(x,y) = c, \quad \text{or} \quad f[x(t),y(t)] = c$$

which are projections of curves γ lying on S such that each curve γ is *parallel* to the xy-plane (all along the curve γ, z has the same value c). The notation γ_L will be used to refer to a particular level curve of the form $f(x,y) = c$; see Figure 4.

Thus $\cos\psi(t) \equiv 0$ implies:

At each point (x_0,y_0) on a level curve $f(x,y) = C_0$,

$$\vec{\nabla} f(x_0,y_0) \cdot \vec{T}_p(t_0) = 0$$

where $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $\vec{T}_p(t_0)$ is constructed from the parametric equations of the level curve (in the case they are available). If parametric equations for the level curve $f(x,y) = C_0$ are not readily available, a

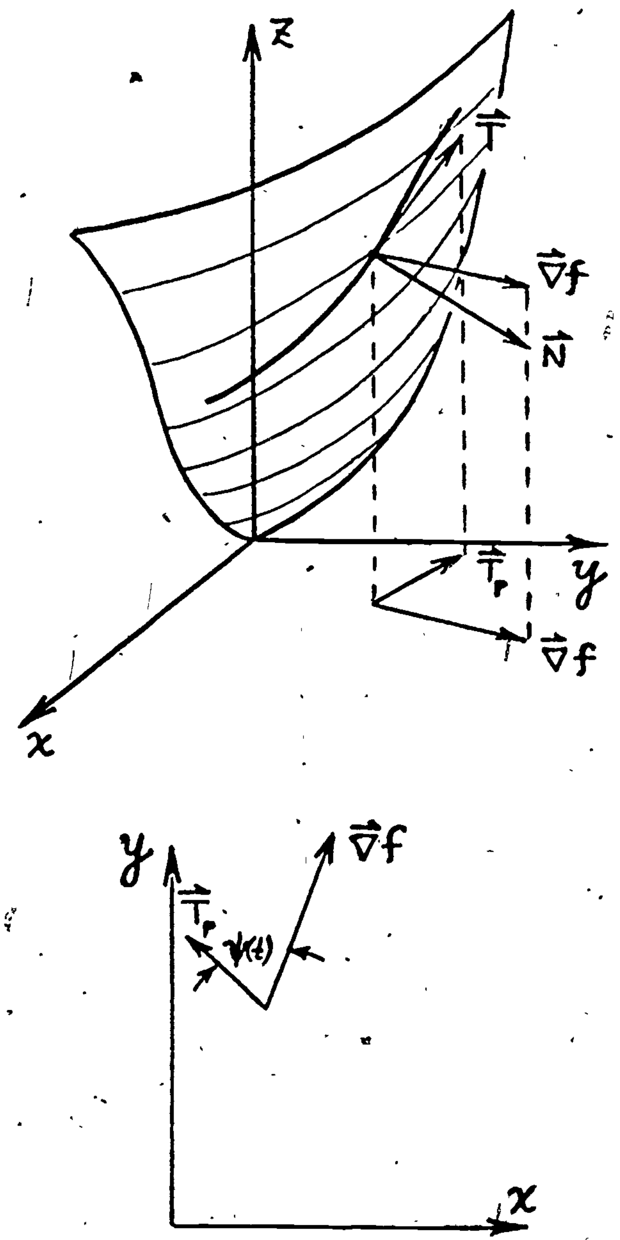

Figure 3.

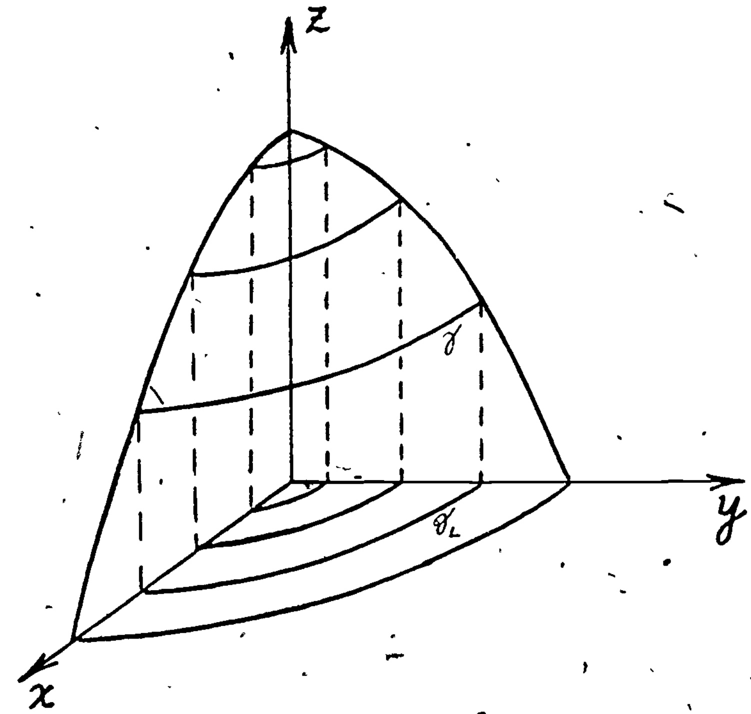

Figure 4.

vector proportional to $\vec{T}_p(t_0)$ may be constructed simply by noticing that $\vec{\nabla}f(x_0,y_0)$ and $\vec{T}_p(t_0)$ are *perpendicular*, so that by property 4 above,

$$\vec{T}_p(t_0) = c\left[-\frac{\partial f}{\partial y}(x_0,y_0)\hat{i} + \frac{\partial f}{\partial x}(x_0,y_0)\hat{j}\right].$$

Some examples will clarify these concepts.

Example 4. One level curve γ_L for the surface S: $z = x^2 + 4y^2$ is $5 = x^2 + 4y^2$, and it contains the point $(x_0,y_0) = (1,1)$. See Figure 5. For this level curve, we write some parametric equations with *relative* ease:

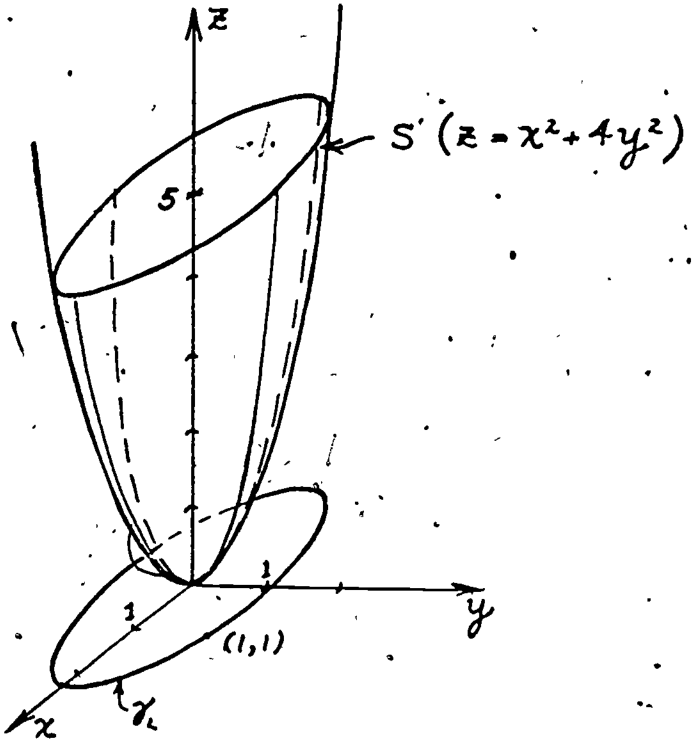

Figure 5a.

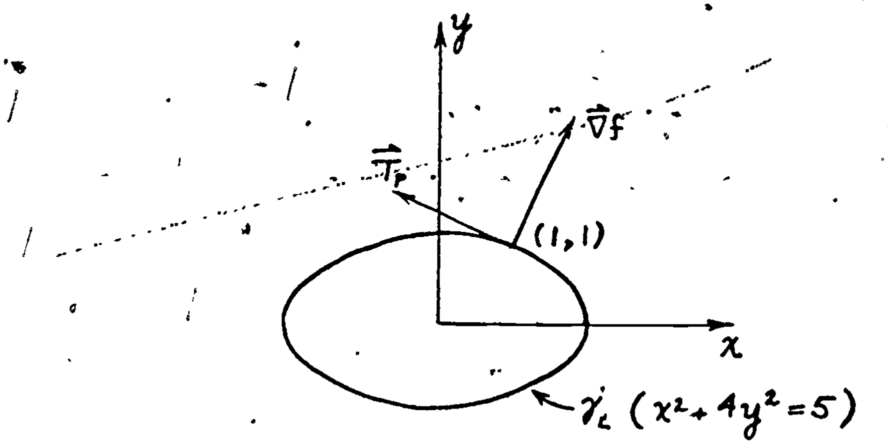

Figure 5b.

$$\gamma_L: \begin{cases} x(t) = \sqrt{5}\cos t \\ y(t) = \frac{\sqrt{5}}{2}\sin t, \end{cases} \quad t \text{ in } [0,2\pi]$$

and at $(x_0,y_0) = (1,1)$, $\cos t_0 = 1/\sqrt{5}$, $\sin t_0 = 2/\sqrt{5}$. Now

$$\vec{\nabla} f(x_0,y_0) = 2x_0\hat{i} + 8y_0\hat{j} = 2\hat{i} + 8\hat{j};$$

$$\vec{T}_p(t_0) = \dot{x}(t_0)\hat{i} + \dot{y}(t_0)\hat{j} = -\sqrt{5}\sin t_0\hat{i} + \frac{\sqrt{5}}{2}\cos t_0\hat{j}$$

$$= -\sqrt{5}\cdot\frac{2}{\sqrt{5}}\hat{i} + \frac{\sqrt{5}}{2}\cdot\frac{1}{\sqrt{5}}\hat{j}$$

$$= -2\hat{i} + \frac{1}{2}\hat{j} = \frac{1}{2}(-4\hat{i} + \hat{j}).$$

And we check that on this level curve,

$$\vec{\nabla} f(x_0,y_0)\cdot\vec{T}_p(t_0) = (2\hat{i} + 8\hat{j})\cdot\frac{1}{4}(-8\hat{i} + 2\hat{j})$$

$$= \frac{1}{4}(-16 + 16) = 0.$$

Example 5. Next, treat the same surface as in Example 3, but without parametrizing γ_L. The level curve $5 = x^2 + 4y^2$ contains the point (1,1). As before,

$$\vec{\nabla} f(x_0,y_0) = 2\hat{i} + 8\hat{j},$$

and by inspection we construct

$$\vec{T}_p(t_0) = c[-8\hat{i} + 2\hat{j}],$$

which has the *same direction* as the $\vec{T}_p(t_0)$ in the previous example.

5. SUMMARY

Let (x_0,y_0) be a point on the *level curve* γ_L: $f(x,y) = C_0$. Then the vector

$$\vec{\nabla} f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\hat{i} + \frac{\partial f}{\partial y}(x_0,y_0)\hat{j}$$

is *normal* to γ_L at (x_0,y_0), and the tangent vector to γ_L

at $(x_0,y_0) = (x(t_0),y(t_0))$ is

$$\vec{T}_p(t_0) = -\frac{\partial f}{\partial y}(x_0,y_0)\hat{i} + \frac{\partial f}{\partial x}(x_0,y_0)\hat{j}$$

or

$$\vec{T}_p(t_0) = \dot{x}(t_0)\hat{i} + \dot{y}(t_0)\hat{j}.$$

A final example will illustrate the use of the Gradient in the case where the level curve is *not* easy to parametrize.

Example 6. Let $z = f(x,y) = x^2 + xy + 2y^4$. A level curve γ_L for this surface S is $x^2 + xy + 2y^4 = 4$, and $(x_0,y_0) = (1,1)$ is a point on γ_L. As can be seen, γ_L is *not* easy to parametrize, but from Example 1 and the above summary, we may write $\vec{\nabla} f(x_0,y_0) = 3\hat{i} + 9\hat{j}$ and, by construction, $\vec{T}_p(t_0) = (-3\hat{i} + \hat{j})$. Incidentally, this latter could also be obtained by noticing that *direct* differentiation of the level curve yields

(2) $$(2x + y)dx + (x + 8y^3)dy = 0$$

or, at the point (1,1),

$$3dx + 9dy = 0 ;$$

thus forcing the tangent vector,

$$\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

to have a direction parallel to $-3\hat{i} + \hat{j}$.

Exercise 3. A point P is moving along the level curve of Example 6. At a certain moment its acceleration is known to be $\vec{a} = 4\hat{i} - 2\hat{j}$, and it is located at the point (1,1). Resolve the acceleration vector into two components, a tangential component and a component normal to the path. (Hint: If a vector $\vec{a}$ is resolved into tangential and normal components $\vec{T}$ and $\vec{N}$, such as $\vec{a} = c_1\vec{T} + c_2\vec{N}$, then taking the dot product of both sides with $\vec{T}$ yields $\vec{T} \cdot \vec{a} = c_1\vec{T} \cdot \vec{T}$ since $\vec{T} \cdot \vec{N} = 0$; thus

$$c_1 = \frac{\vec{T} \cdot \vec{a}}{|\vec{T}|^2}$$

and in a similar manner we find

$$c_2 = \frac{\vec{N} \cdot \vec{a}}{|\vec{N}|^2} .)$$

Question: How does Equation (2), p. 20 differ from the chain rule (B) on page 6 for $\dot{z}(t)$ mentioned earlier in this section?

Review Question: What is the role of the Gradient in constructing the tangent vector $\vec{T}(t)$ to a curve lying on a surface S?

Answer: (i) for an arbitrary curve γ, the planar components $x(t)$ and $y(t)$ would have to be given, from which the components $\dot{x}(t)$ and $\dot{y}(t)$ are obtained; then the Gradient may be used (via the chain rule (B)) to determine the component $\dot{z}(t)$.

(ii) for a level curve γ_L, we may find a vector proportional to $\vec{T}(t)$ by merely choosing a vector *perpendicular* to the Gradient vector.

6. APPLICATION OF THE GRADIENT TO FINDING CURVES OF STEEPEST ASCENT (DESCENT)

Consider again the preliminary example mentioned in the introduction. Suppose that $f(x,y)$ in Figure 6 represents the concentration of oil, $C = f(x,y)$. Now if the rescue ship is somehow able to monitor the concentration of oil at each point (x_0,y_0) and wishes to go in the direction of greatest increase of C (from left to right) *proceeding* from level curve to level curve, we would expect from the results of Section 4 that it would continuously follow the *Gradient direction* from the point, say (x_0,y_0), to the right. Actually, this is true, and we will provide a proof of this before continuing with some examples.

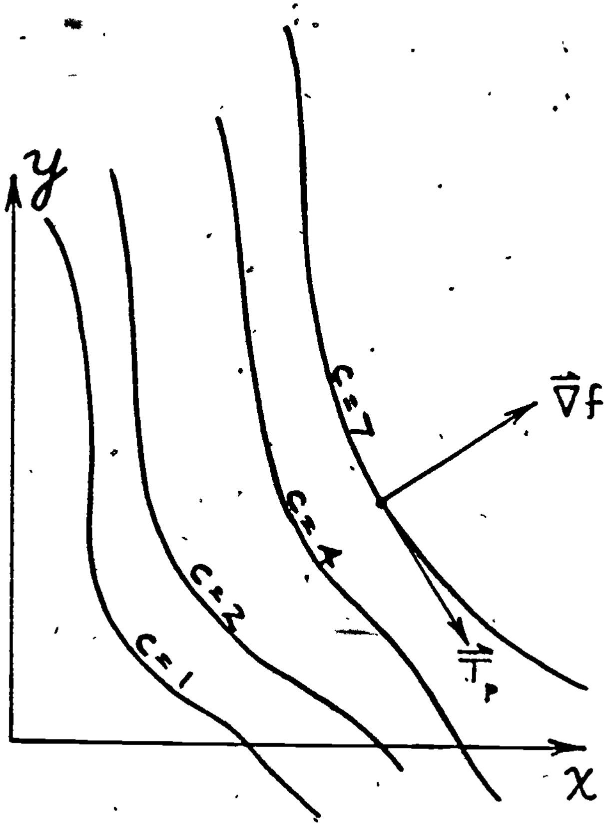

Figure 6.

Keep in mind that we are seeking a path in the xy-plane which will lead the rescue ship to the disabled tanker by the method described in the preceding paragraph. Let us translate our goal into more formal mathematical language. The desired path in the xy-plane could be interpreted as the projection $\gamma_p(t)$ onto the xy-plane of a special curve $\gamma(t)$ which lies on the surface $C = f(x,y)$ or $z = f(x,y)$, (where we see that the roles of C and z are interchangeable!). The special property of $\gamma(t)$ is that its projection $\gamma_p(t)$ in the xy-plane *continually moves in the direction of greatest increase* of $C = f(x,y)$ *from level curve to level curve.*

Our method of solution will be to discover the curve $\gamma(t)$ and then to return attention to its projection $\gamma_p(t)$ in the xy-plane, which is the path we seek.

The curve $\gamma(t)$ can be described parametrically as follows:

$$\gamma(t): \begin{cases} x = X(t) \\ y = Y(t) \\ z = Z(t) \end{cases} \quad a \le t \le b \Bigg\};$$

here, since $\gamma(t)$ lies on the surface $C = f(x,y)$ (or $z = f(x,y)$), we have $Z(t) = f[X(t),Y(t)]$, and the projected curve $\gamma_p(t)$, as in Section IV, is

$$\gamma_p(t): \begin{cases} x = X(t) \\ y = Y(t) \end{cases} \quad a \le t \le b \Bigg\}.$$

As before, the symbol $\vec{T}_p(t)$ denotes the tangent vector to the curve $\gamma_p(t)$, and as in (B), we also have

$$\text{(C)} \qquad \dot{Z}(t) = \vec{\nabla} f[X(t),Y(t)] \cdot \vec{T}_p(t) .$$

As you are aware, any curve may have several (and even infinitely many!) different parametrizations; we wish to avoid any complications which may arise along this line by somehow "normalizing" the curve $\gamma_p(t)$. It is also for convenience that we wish to make $\gamma_p(t)$ independent of the parameter t, and thus we introduce the arclength s which is independent of the particular parametrization used for $\gamma_p(t)$.

$$\text{(D)} \qquad s \equiv \int_0^t |\vec{T}_p(t)| dt = \int_0^t \sqrt{[\dot{X}(t)]^2 + [\dot{Y}(t)]^2}\, dt$$

or (by the Fundamental Theorem of Calculus)

$$\frac{ds}{dt} = |\vec{T}_p(t)| = \sqrt{[\dot{X}(t)]^2 + [\dot{Y}(t)]^2} .$$

Note that, regardless of the form of parametrization of $\gamma(t)$, we can solve (D) (symbolically!) for t as a function of s, say $t = t(s)$. Now, dividing both sides of (C) by $|\vec{T}_p(t)|$, we obtain

$$\frac{\dot{Z}(t)}{|\vec{T}_p(t)|} = \vec{\nabla} f[X(t),Y(t)] \cdot \frac{\vec{T}_p(t)}{|\vec{T}_p(t)|}$$

or

$$\frac{\frac{dZ}{dt}}{\frac{ds}{dt}} = \frac{dZ}{ds} = \vec{\nabla} f[X(s),Y(s)] \cdot \hat{u}_{T_p}(s)$$

where

$$\hat{u}_{T_p}(s) \equiv \frac{\vec{T}_p(t)}{|\vec{T}_p(t)|}$$

is (conveniently!) a unit vector in the direction of $\vec{T}_p(t)$, and we have replaced t by the expression $t = t(s)$ which was found (symbolically) from (D).

Once again we recall the rules for a dot product and note that

$$\frac{dZ}{ds} = \left|\vec{\nabla} f[X(s),Y(s)]\right| \left|\hat{u}_{T_p}(s)\right| \cos\psi(s)$$

$$= \left|\vec{\nabla} f[X(s),Y(s)]\right| \cos\psi(s) \ .$$

What have we achieved here? This expression for dZ/ds relates the rate of change of Z with respect to distance (arc length) along a curve lying on a surface S and the Gradient of that surface.

From this expression, we can see that

(i) dZ/ds is maximized or minimized by allowing $\psi(s)$ to be 0 or π, respectively, and

(ii) $\max dZ/ds = |\vec{\nabla} f[X(s),Y(s)]|$,
$\min dZ/ds = -|\vec{\nabla} f[X(s),Y(s)]|$;

either (i) or (ii) *will be attained* if the direction of the unit vector $\hat{u}_{T_p}(s)$ is precisely the same (or opposite!) direction as the Gradient; more precisely, if

$$\hat{u}_{T_p}(s) = \pm \frac{\vec{\nabla} f[X(s),Y(s)]}{|\vec{\nabla} f[X(s),Y(s)]|}$$

Thus we see that our assertion in the first paragraph of Section 6 is indeed valid: the path to follow to achieve the greatest local increase (decrease) in $C = f(x,y)$ is precisely a path which takes the *direction of the Gradient vector* (or opposite that direction) at any point $(X,Y) = [X(s),Y(s)]$: i.e., at each point the decision on which direction to move next is made on the basis of examining the Gradient. We will refer to this path as the "path of steepest ascent (descent)".

Another way of stating this important result is to note that the tangent vector $\hat{u}_{T_p}(s)$ to the path of steepest ascent (descent) $\gamma_p(t)$ is *parallel* to the Gradient, $\vec{\nabla} f[X(s),Y(s)]$; thus the components dX/ds and dY/ds of the tangent vector can be expressed in direct proportion to those of the Gradient.

$$\frac{dX}{ds} = \lambda \frac{\partial f}{\partial X}[X(s),Y(s)] \ ,$$

$$\frac{dY}{ds} = \lambda \frac{\partial f}{\partial Y}[X(s),Y(s)] \ ,$$

where $\lambda = \lambda(s)$ is a function of proportionality.

Example 7. If $B \neq 0$ and $A \neq 0$, then the level curves of the surface S: $z = f(x) = Ax + By + D$, which is a plane, are straight lines

$$Ax + By = C \ ,$$

each of whose slopes is $m = -A/B$. We will construct $\gamma_p(t)$, the curve of steepest ascent in the xy-plane through the point (0,0), and $\gamma(t)$ the curve of steepest ascent lying in the plane S. We have

$$\frac{\partial f}{\partial x} = A \ , \quad \frac{\partial f}{\partial y} = B \ .$$

(3) $$\frac{dX}{ds} = \lambda A \ , \quad \frac{dY}{ds} = \lambda B \ .$$

and since the curve is to pass through the point (0,0) we may choose to measure arc length s from that point, so that an additional condition on the curve of steepest ascent becomes

(4) $$X(0) = 0 , \quad Y(0) = 0 .$$

Then from (3) we have

$$\frac{dY}{dX} = \frac{B}{A} .$$

Now integrate and apply conditions (4).

$$Y(s) = \frac{B}{A} X(s) .$$

Hence we see that a set of parametric equations for the curve of steepest ascent in the xy-plane is

$$\gamma_p(s) = \left\{ \begin{array}{ll} X(s) = s & \\ Y(s) = \frac{B}{A} s , & s \geq 0 \end{array} \right\} .$$

and a set of parametric equations for the curve of steepest ascent lying on the surface S is

$$\gamma(s): \left\{ \begin{array}{ll} X(s) = s & \\ Y(s) = \frac{B}{A} s & s \geq 0 \\ Z(s) = (A + \frac{B^2}{A})s + D & \end{array} \right\} .$$

It is now evident that $\gamma_p(t)$ (where we replace the parameter s with the parameter t to facilitate correlation with concepts developed earlier) is perpendicular to *every* level curve γ_L in the xy-plane. This is no accident, and we leave it to you to prove:

> A steepest ascent curve $\gamma_p(t)$ is perpendicular (orthogonal) to any level curve at a point of intersection.

Example 8. Find both steepest ascent curves $\gamma(s)$ and $\gamma_p(s)$, for the surface $z = \frac{1}{2}(x^2 + y^2)$, through the point (x_0, y_0).

$$\left\{ \begin{array}{l} \frac{dX}{ds} = f_x = X \\ \frac{dY}{ds} = f_x = Y \end{array} \right. \Rightarrow \frac{dY}{dX} = \frac{Y}{X} ;$$

Upon separating variables,

$$\frac{dY}{Y} = \frac{dX}{X} ,$$

and integrating, $\ln Y = \ln X + \ln C = \ln CX$ or $Y = CX$. Also since $Y = y_0$ when $X = x_0$ we have $Y = (y_0/x_0)X$.

A parametrization for $\gamma_p(s)$ would be:

$$\gamma_p(s): \left\{ \begin{array}{ll} X(s) = s & \\ Y(s) = \left(\frac{y_0}{x_0}\right)s , & 0 \leq s \leq x_0 \end{array} \right\} .$$

and

$$\gamma(s): \left\{ \begin{array}{ll} X(s) = s & \\ Y(s) = \left(\frac{y_0}{x_0}\right)s & 0 \leq s \leq x_0 \\ Z(s) = \frac{1}{2}[1 + \left(\frac{y_0}{x_0}\right)^2]s^2 & \end{array} \right\} .$$

You will notice that "Gradient" curves $\gamma_p(s)$ are straight lines whenever the level curves $f(x,y) = C$ are circles (see Figure 7); keep this in mind as you proceed to the next example.

Example 9. Find both "Gradient" curves $\gamma(s)$ and $\gamma_p(s)$ through the point (x_0, y_0) when $z = f(x,y) = \frac{1}{2}(x^2 + \mu y^2)$, $\mu > 1$.

$$\left\{ \begin{array}{l} \frac{dX}{ds} = X \\ \frac{dY}{ds} = \mu Y \end{array} \right. \Longrightarrow \frac{dY}{dX} = \frac{\mu Y}{X}$$

from which we form the differential equation:

$$\frac{dY}{Y} = \mu \frac{dX}{X} ; \quad X(0) = x_0 , \quad Y(0) = y_0 .$$

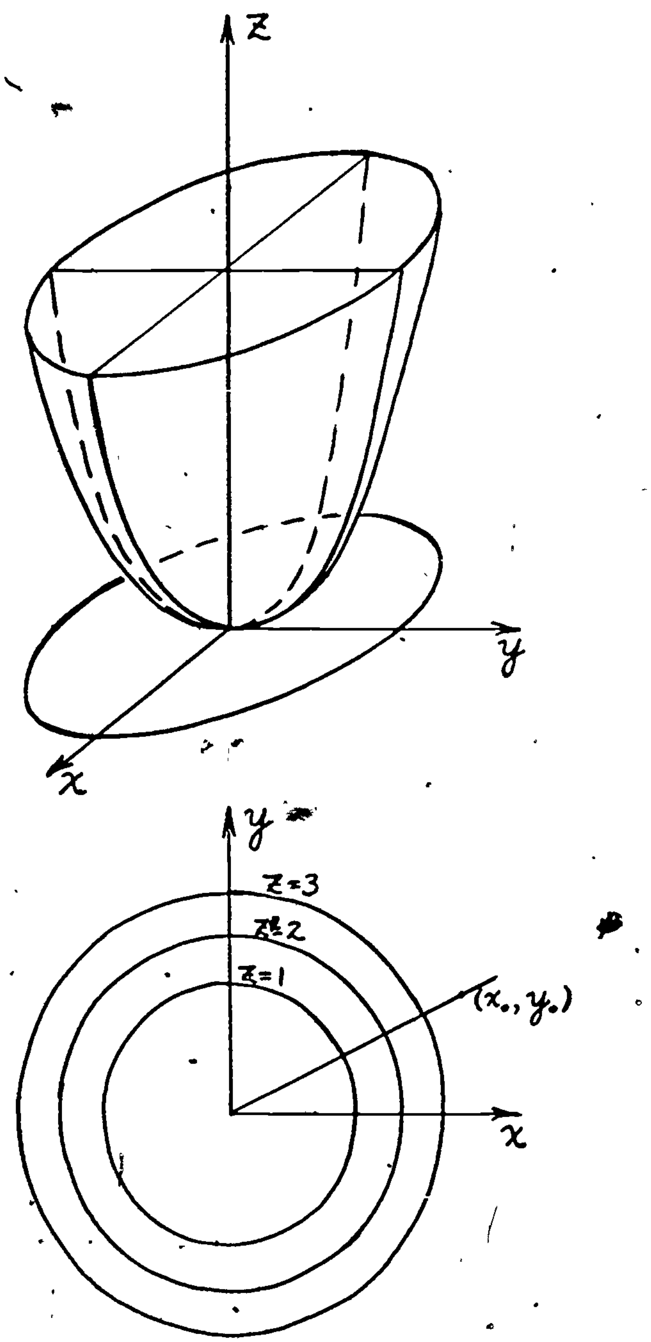

Figure 7.

Integrating, we obtain

$$\ln Y = \mu \ln X + \ln C$$
$$= \ln X^{\mu} + \ln C$$
$$= \ln C X^{\mu}$$

or

$$Y = C X^{\mu}$$

Again inserting the point (x_0, y_0) we have $Y = y_0(X/x_0)^{\mu}$. Thus parametrizations for $\gamma(s)$ and $\gamma_p(s)$, respectively, would be

$$\gamma_p(s): \left\{ \begin{array}{ll} X(s) = s & \\ Y(s) = y_0\left(\frac{s}{x_0}\right)^{\mu} & 0 \le s \le x_0 \end{array} \right\}.$$

$$\gamma(s): \left\{ \begin{array}{ll} X(s) = s & \\ Y(s) = y_0\left(\frac{s}{x_0}\right)^{\mu} & 0 \le s \le x_0 \\ Z(s) = \frac{1}{2}\left(s^2 + \mu^2 y_0{}^2\left(\frac{s}{x_0}\right)^{\mu}\right) & \end{array} \right\}.$$

In the present case, we see that the steepest ascent curves in the xy-plane resemble parabolas passing through the origin whenever the level curves of $f(x,y) = C$ are ellipses centered at the origin.

Exercise 4. Sketch the level curves and the gradient curves $\gamma_p(s)$ for the previous example in the cases where $\mu = 2$ and $\mu = 3$.

A remark concerning the solution of the differential equations is in order. In finding the gradient curve $\gamma_p(s)$, you will always be using expressions of the form

$$\frac{dX}{ds} = f_x(X,Y), \quad \frac{dY}{ds} = f_y(X,Y),$$

and consequently

(E) $$\frac{dY}{dX} = \frac{f_y(X,Y)}{f_x(X,Y)}$$

The general problem of solving this differential equation for the actual curve $Y = Y(X)$ which we have been referring to as $\gamma_p(s)$ can be quite formidable. In our examples, you will notice, the expression (E) gave rise to a rather simple separable differential equation. In every case, this was the result of special choices for $f(x,y)$, and a little imagination should convince you that a more complicated surface $z = f(x,y)$ will result in a more challenging differential equation. Along these lines, we suggest that those of you who are better versed in solving first-order differential equations explore the following:

- <u>Problem</u>: Find the Gradient curves for the surface

$$z = F(x,y) = C_0 e^{-\frac{1}{2}[Ax^2 + 2by + Cy^2]}$$

where $AC - B^2 > 0$, $B > 0$, such that $\gamma_p(s)$ passes through $x = x_0$, $y = y_0$. You will find that (E) becomes a homogeneous differential equation. A further hint: Perhaps an initial rotation of axes to eliminate the xy term would prove convenient!

We conclude with an additional exercise which relates our development again to a physical setting.

<u>Exercise 5</u>. A tanker located at coordinates (x_1,y_1) has capsized, leaving an oil slick floating on the calm surface of the ocean with concentration given by the law

$$C(x,y) = C_0 e^{-\alpha[(x-x_1)^2 + \mu(y-y_1)^2]}, \quad \alpha > 0, \quad \mu > 1.$$

A small rescue vessel proceeding from location (x_0,y_0) moves slowly in the direction of *increasing* concentration according to the above law for a calm sea. What is the equation of the rescue vessel's path?

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7. MODEL EXAM

A "sugarloaf" mountain has the equation $z = H - \alpha(x^2 + \mu y^2)$, $a > 0$, $\mu > 1$. Find the steepest ascent curve a mountain climber must take if he or she wishes to ascend the mountain by Gradient methods! The climber starts at the point $(x_0,y_0,0)$. (Follow the indicated steps to the solution.)

a) Find the level curves for the surface $z = f(x,y)$. What type of curves are they?

b) Find the Gradient vector for the surface at the point (1,2), and also at an arbitrary point (x,y)

c) Find the level curve passing through the point (1,2).

d) Find a tangent vector to the level curve in c) at the point (1,2).

e) Using the result of b), find the differential equation for the steepest ascent curve.

f) Solve the differential equation inserting the given condition at $X = x_0$, $Y = y_0$.

g) Find parametric expressions for the steepest ascent curve $\gamma(s)$ which actually lies on the mountain surface.

h) Choose your own values for α and μ, and make sketches of the mountain surface, the level curves, and the steepest ascent curve $\gamma_p(s)$.

8. ANSWERS TO EXERCISES AND MODEL EXAM

Exercises

1. a) The surface is an elliptical paraboloid, with lowest point at the origin; its sketch is very similar to that of Figure 5a.

   b) $\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 4y$

   c) $$[x(t)]^2 + 2[y(t)]^2 = t^2\cos^2 t + 2\left(\frac{t^2}{2}\right)\sin^2 t$$
   $$= (t^2 \cos^2 t + \sin^2 t)$$
   $$= t^2 = z(t)$$

   d) Using the formula for arc length given in Section II, and grouping terms,
   $$s = \int_{t=0}^{t=2\pi} \sqrt{\left(t + \frac{1}{2}\right)\sin^2 t + \left(\frac{t^2}{2} + 1\right)\cos^2 t - t \sin t \cos t + 4t^2}\, dt$$

2. From the definition of γ and $\vec{T}(t_0)$, with $t_0 = 1$, we have,
   $$\vec{T}(1) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}\Big|_{t=1}$$
   or $\vec{T}(1) = 1\hat{i} + 1\hat{j} + 2\hat{k}$. Alternately, we can use the chain rule; here $f(x,y) = xy$. Since $x(1) = 2$ and $y(1) = 0$,
   $$\frac{\partial f}{\partial x}(x(1),y(1)) = y\Big|_{\substack{x=2\\y=0}} = 0,$$
   and
   $$\frac{\partial f}{\partial y}(x(1),y(1)) = x\Big|_{\substack{x=2\\y=0}} = 2,$$

from which
$$z(1) = 0\,\dot{x}(1) + 2\dot{y}(1)$$
$$= 0(1) + 2(1) = 2$$
so that
$$\vec{T}(1) = 1\hat{i} + 1\hat{j} + 2\hat{k}.$$

4. From analytic geometry, you should find that the level curves for both the cases $\mu = 2$ and $\mu = 3$ are ellipses, with the major and minor axes maintaining a constant proportion to each other in each case.
   The gradient curves $\gamma_p(s)$ are parabolas for the case $\mu = 2$ and cubic curves for the case $\mu = 3$. Sketch the corresponding families of curves on the same graph, observing the orthogonality property!

5. $$\frac{dX}{ds} = \frac{C_0(-2\alpha(X-x_1))}{M}$$
   $$\frac{dY}{ds} = \frac{C_0(-2\alpha\mu(Y-y_1))}{M},$$
   where we have used M to represent
   $$e^{-\alpha[(X-x_1)^2 + \mu(Y-y_1)^2]}.$$
   Then
   $$\frac{dY}{dX} = \frac{\mu(Y-y_1)}{X-x_1};$$

h) For simplicity, let us choose $\alpha = 1$, $\mu = 4$.
$z = f(x,y) = H - (x^2 + 4y^2)$, an elliptical paraboloid. The level curves are $H - C = x^2 + 4y^2$, $0 \le C \le H$; a family of ellipses.

The steepest ascent curve $\gamma_p(s)$ is

$$Y = y_0\left(\frac{X}{X_0}\right)^4.$$

See Figures 8a, 8b, and 8c.

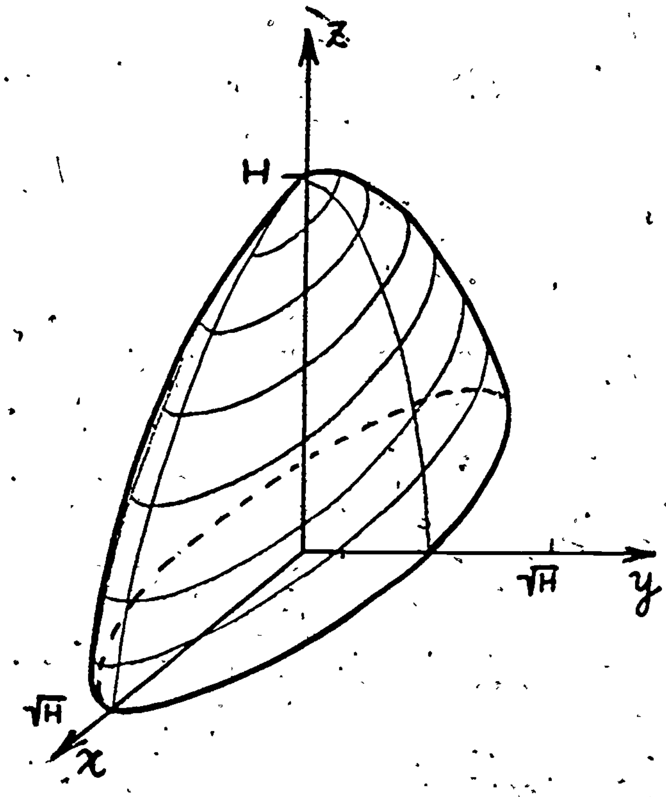

Figure 8a.

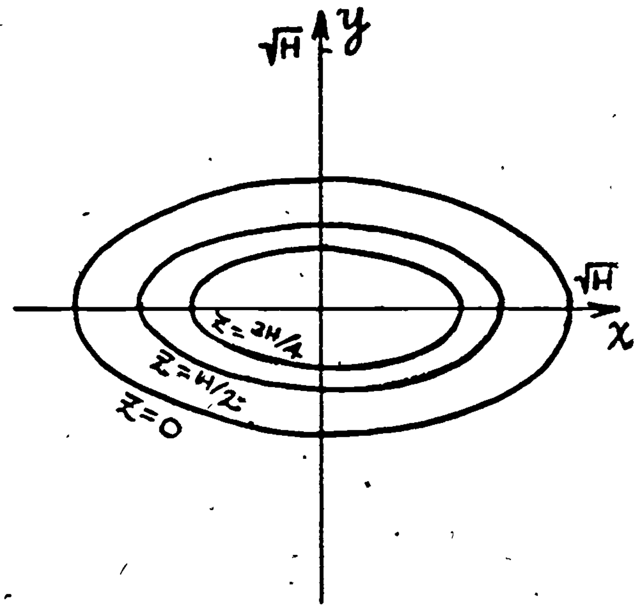

Figure 8b.

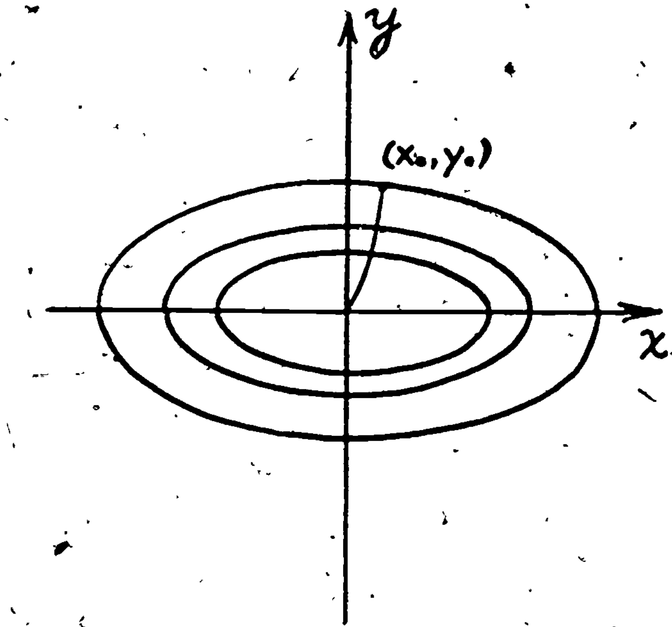

Figure. 8c.

separating variables we find

$$\frac{dY}{y-y_1} = \frac{\mu\, dX}{x-x_1},$$

and integrating,

$$\ln(y-y_1) = \mu\ln(x-x_1) + \ln C$$
$$= \ln[C(x-x_1)^{\mu}]$$

Taking antilogs, we obtain

$$y - y_1 = C(x-x_1)^{\mu}$$

Let us insert the given condition, which is the starting point (x_0,y_0) for the rescue vessel. Then $y_0 - y_1 = C(x_0-x_1)^{\mu}$,

$$C = \frac{y_0 - y_1}{(x_0-x_1)^{\mu}}.$$

so that finally

$$y = y_1 + (y_0-y_1)\left(\frac{x - x_1}{x_0 - x_1}\right)^{\mu}$$

What kind of path is this? For example, if $\mu = 2$, this path would be a parabola, leading directly from the point (x_0,y_0) to the point of disaster, (x_1,y_1). (The vertex of the parabola is located at (x_1,y_1).)

Model Exam

a) From the form of the surface equation, the largest value z can possibly have occurs when $x = y = 0$; indeed $z = H$ is the height of the mountain! Also $z = 0$ would indicate ground level. Thus the family of level curves would be

$$C = H - \alpha(x^2 + \mu y^2), \quad 0 \le C \le H,$$

or

$$\frac{H - C}{\alpha} = x^2 + \mu y^2.$$

This is a family of ellipses, with the ratio of the major axis to minor axis remaining constant.

b) $\vec{\nabla} f(x,y) = -2\alpha x\hat{i} - 2\alpha\mu y\hat{j}$;

$\vec{\nabla} f(1,2) = -2\alpha\hat{i} - 4\alpha\mu\hat{j}$.

c) To find the level curve passing through (1,2), insert $x = 1$, $y = 2$ into the family of level curves, obtaining $(H - C)/\alpha = 1 + 4$; using this value for $(H - C)/\alpha$, the particular member of the family passing through (1,2) can be identified as

$$1 + 4\mu = x^2 + \mu y^2.$$

d) The tangent vector along a level curve is perpendicular to the Gradient, so from b),

$$\vec{T}_p = 4\alpha\mu\hat{i} - 2\alpha\hat{j}$$

e)

$$\frac{dX}{ds} = -2\alpha X, \qquad \frac{dY}{ds} = -2\alpha\mu Y$$

$$\frac{dY}{dX} = \frac{\frac{dY}{ds}}{\frac{dX}{ds}} = \frac{-2\alpha\mu Y}{-2\alpha X} = \frac{\mu Y}{X}.$$

f)

$$\frac{dY}{Y} = \frac{\mu dX}{X}; \qquad \ln Y = \mu\ln X + \ln C;$$

$$\ln Y = \ln CX^{\mu};$$

$$Y = CX^{\mu}.$$

Inserting the condition, $y_0 = Cx_0^{\mu}$, $C = \frac{y_0}{x_0^{\mu}}$ and thus

$$Y = y_0\left(\frac{X}{x_0}\right)^{\mu}.$$

g)

$$Y(s): \begin{cases} X(s) = s \\ Y(s) = y_0\left(\frac{s}{x_0}\right)^{\mu} \\ Z(s) = H - \alpha\left(s^2 + \mu\left[y_0\left(\frac{s}{x_0}\right)^{\mu}\right]^2\right) \end{cases} \quad 0 \le s \le x_0$$

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STUDENT FORM 1

Request for Help

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

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Name______________________ Unit No.__________ Date__________

Institution______________________ Course No.__________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain______________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain______________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

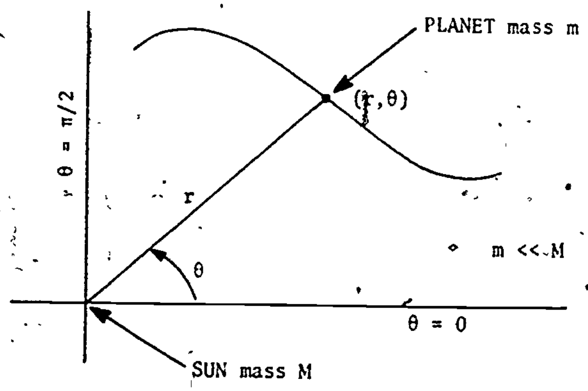
umap

UNIT 473

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

KEPLER'S LAWS AND THE INVERSE SQUARE LAW

by A.M. Fink



APPLICATIONS OF CALCULUS TO PHYSICS

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KEPLER'S LAWS AND THE INVERSE SQUARE LAW

by

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Iowa State University
Ames, Iowa 50011

TABLE OF CONTENTS

Intermodular Description Sheet: UMAP Unit 473

Title: KEPLER'S LAWS AND THE INVERSE SQUARE LAW

Author: A.M. Fink
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Iowa State University of Science and Technology
Ames, IA 50011

Review Stage/Date: III 6/15/80

Classification: APPL CALC/PHYSICS

Prerequisite Skills:

1. Know the elementary facts about vector algebra and vector differentiation.
2. Know how to compute area and arclength in polar coordinates.
3. Know the global features of the graphs of the conic sections.
4. Be conversant with elementary integration.
5. Have the ability to change units of measurements and know how to write numbers in scientific notation.

Output Skills:

1. Be able to handle motion problems in polar coordinates, especially those concerning gravitational interaction.
2. State the relationships between Kepler's Laws and the Inverse Square Law.
3. Discuss the history and thought going into scientific development of an idea and the interaction between experiment and theory, at least in the context of planetary motion.

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank Joel Greenstein of New York City Community College, J.W. Goldston of Pembroke State University, Andrew C. Pak of G.E. Aircraft Engine Group, and Paul C. Rogers of the University of Southern Maine for their reviews, and all others who assisted in the production of this unit.

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1. DESCRIPTION OF THE PROBLEM

The orbit of a satellite around the earth may be considered to be determined by the gravitational interaction between it and the earth alone. Since the sun also affects the motion of the satellite, a small amount of error is introduced. However, the earth's gravity is the most important gravitational factor.

In the same way, the description of the motion of a planet around the sun may be viewed as the result of these two bodies' mutual gravitational attraction. Again other planets affect this motion, but the sun is the major influence on the orbit of a planet. The dynamics of the history of the solution of the problem of describing the motion of a planet can be recreated in a short time using the modern conveniences of vector differentiation. The central ideas and facts are the Inverse Square Law and Kepler's three Laws.

The common thread of the satellite and planetary motions reappears in modern physics on the sub-molecular level in the form of Coulomb potentials.

2. STATEMENT OF THE LAWS

We imagine the situation of two point masses, one mass very much larger than the other. The effect of this assumption is that we let the position of the larger mass be fixed. Put the origin of the coordinate system at the larger mass M. For convenience, call it the Sun. The motion of the smaller mass m, now called a planet describes a curve in 3-space. In Section 6 we will show that the motions of interest are planar. We will describe the motions in polar coordinates. The path of the planet is written in parametric form $(r(t),\theta(t))$ where t is time.

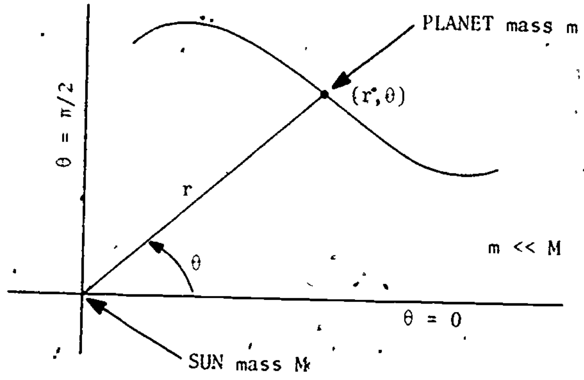

Figure 1. A planet moving around the sun in a fixed coordinate system. The heliocentric view of the solar system provided by Aristarchus c. 310-230 B.C. and resurrected by Copernicus, 1473-1543 led to the discovery of Kepler's Laws and the Inverse Square Law.

Two of Newton's Laws are germane to our discussion. The first is

(1) [illegible], $F = mA$

that is, force F and acceleration A are vector quantities and they are proportional with the proportionality constant being the mass. In our problem, the acceleration that accounts for the vector motion (r,θ) is due to the external forces via equation (1).

Secondly, we have the Inverse Square Law

(2) $$F = \frac{-GmM}{r^2} U$$

where G is a constant which depends on the units of measurement, but not on the solar system, and U is a unit vector directed from the origin to the mass M.

We will discuss three of Kepler's Laws.

Kepler's First Law. The radius vector to the planet sweeps out area at a constant rate with respect to time, that is, the area of the shaded region is $\lambda|t_2 - t_1|$ where λ is a constant.

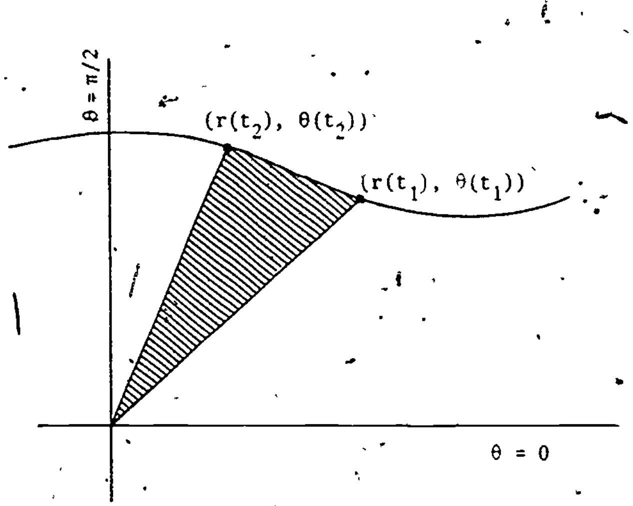

Figure 2. The radius vector sweeping out area. The first of Kepler's Laws was the computation of the rate in which this area is swept out. It is a constant, even for comets that pass the sun only once.

Kepler's Second Law. The planet's orbit is an ellipse with the sun at a focus.

Kepler's Third Law. If a is the semi-major axis of an elliptical orbit and T the time to complete one orbit, then T^2/a^3 is a solar system constant. In other words, if the quantity T^2/a^3 is computed for two different planets of the solar system, it is the same in both cases. It only depends on the mass of the sun and units.

3. HISTORY OF THE PROBLEM

The problem of trying to explain the planetary motion goes back to antiquity. The part of the history which we describe begins in the sixteenth century with Copernicus (1473-1543) who was a proponent of the heliocentric view of astronomy, i.e. that the "sun is at the center of things."

Tycho Brahe (1546-1601) opposed the Copernican theory on religious grounds. It was thought that anything that

did not put the earth at the center degraded humanity. Nevertheless, this astronomer's careful work was a major contribution in validating the Copernican approach. He had received a commission from King Frederich II of Denmark to update astronomical tables. His observatory on the island of Hven contained no telescope (it was invented in 1609) but he was nevertheless able to record a great deal of accurate information.

This accurate information was put to good use by Johann Kepler (1571-1630), who was Tycho's assistant for a short time. Trained as a mathematician he took as his task, the study of the orbit of Mars. He was an ardent supporter of the Copernican theory and his life long ambition was to find the mystical harmony in the skies. His detailed study of Mars led to his publishing his first two laws in 1609 and the third some ten years later.

Galileo Galilei (1564-1642), who is well known for his experiments on particles moving under the influence of gravity, dismissed Kepler's astronomy because in introducing ellipses he was departing from the more perfect circular motion. Thus the scientist who was to be branded a heretic for his scientific views rejected, out of hand, the work of Kepler for very unscientific reasons.

In the year that Galileo died, another scientist, Sir Isaac Newton (1642-1727) was born. Newton was well acquainted with the work of both Kepler and Galileo and of course with the Copernican approach. At the age of 25 he discovered that the only gravitational force consistent with Kepler's laws was the Inverse Square Law. He did not publish his result immediately because he attempted to validate it by doing calculations on the orbit of the moon. Unfortunately they did not check because some of the data on distance to the moon was incorrect. The correct data indeed did verify the Inverse Square Law. He published his result only when it was begun to be proposed by other scientists.

An amusing sidelight is that when asked about the posibility of the Inverse Square Law as being correct, Newton replied that he had once done the calculations. When asked to reproduce them, he could not! Eventually he found an error in his second calculation which when corrected gave the correct answer.* He is generally credited with the discovery of the Inverse Square Law.

In succeeding sections we try to recreate the scientific process of going from Tycho's empirical data to Kepler's Laws to Newton's Laws. We will also show that starting with Newton's Laws, we can recover Kepler's Laws.

4. MOTIONS DESCRIBED IN POLAR COORDINATES

Suppose that we have a motion described in polar coordinates and R(t) is the position vector. In order to isolate certain aspects of the motion we can introduce a local coordinate system as follows.

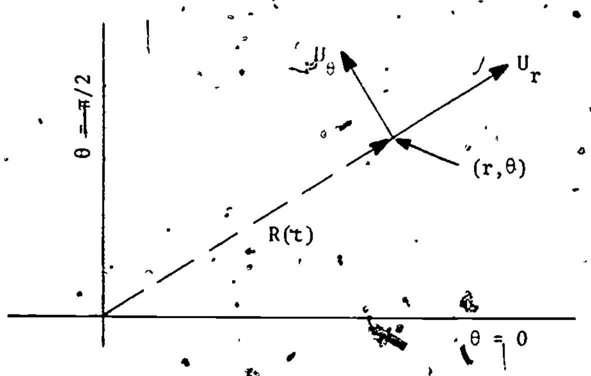

Figure 3. The standard unit vectors for parametric polar coordinate motions. These form a "moving coordinate system" which depend on the position of the particle. This modern tool, not available to Kepler and Newton, allows one to almost completely dispense with geometric and/or trigonometric arguments.

*This is a warning to the scientific neophyte. Keep your notebooks orderly.

The vectors U_r and U_θ are to be unit vectors oriented as indicated. They do not depend on r but it is easy to see that $U_r(\theta) = (\cos\theta, \sin\theta)$ and $U_\theta(\theta) = (-\sin\theta, \cos\theta)$,

(3) $$\frac{d}{d\theta} U_r = U_\theta \;; \quad \text{and} \quad \frac{d}{d\theta} U_\theta = -U_r .$$

Now $R(t) = r(t)U_r(\theta(t))$ is the equation of the motion in polar coordinates. We let V and A be the velocity and acceleration vectors, the prime notation means differentiation with respect to time. Then

$$V = R' = r'U_r + r\frac{dU_r}{d\theta}\frac{d\theta}{dt} = r'U_r + rU_\theta\theta'$$

and

$$A = V' = (r''U_r + r'U_\theta\theta') + (r'U_\theta\theta' + rU_\theta\theta'' - r(\theta')^2U_r)$$

Therefore

(4) $$A = [r'' - r(\theta')^2]U_r + (2r'\theta' + r\theta'')U_\theta .$$

The coefficients in this vector

(5) $$a_r = r'' - r(\theta')^2$$

and

(6) $$a_\theta = 2r'\theta' + r\theta''$$

are called the radial and angular components of acceleration respectively. These are the usual tangential and normal components only in special cases, e.g. when the motion is on a circle centered at the origin.

We recall also that in polar coordinates the area element is rdr dθ so that the area of the shaded region in Figure 2 is given by

(7) $$S(t) = \int_{\theta_1}^{\theta_2} d\theta \int_0^{r(\theta)} r\,dr = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 d\theta = \int_{t_1}^{t_2} \frac{1}{2}r^2\theta'\,dt$$

5. DEDUCTION OF THE INVERSE SQUARE LAW FROM KEPLER'S LAWS

Kepler's Laws are empirical results based on careful observations. If one assumes that only forces external to the planet account for its motion, then what must this force be? Let us assume Kepler's Laws.

First observe that the first law applied to the formula (7) gives

$$S'(t_2) = \frac{1}{2} r^2\theta' = \frac{\lambda}{2}$$

where λ is a constant.

The constant is divided by 2 to make the next equation and its further uses simpler. Thus

(8) $\quad r^2\theta' = \lambda$.

If we differentiate this relation then

(9) $\quad 0 = 2rr'\theta' + r^2\theta'' = r[2r'\theta' + r\theta''] = ra_\theta$

We see that Kepler's first law implies that $a_\theta = 0$, that is the acceleration and therefore the force is purely in the radial direction. Such forces are called *central force fields*.

We further assume that the planet moves in a conic section (see Appendix 1), that is

(10) $\quad r(1 + e\cos(\theta + \alpha)) = B$.

Differentiating with respect to time we get

$$r'(1 + e\cos(\theta + \alpha)) - re\sin(\theta + \alpha)\theta' = 0$$

If we multiply this equation by r and use (8) and (10) we get

$$Br' - \lambda e\sin(\theta + \alpha) = 0 .$$

Now we differentiate again to get

(11) $\quad Br'' - \lambda e\cos(\theta + \alpha)\theta' = 0$

We solve (10) for $\cos(\theta + \alpha)$ and use in (11) to get

$$Br'' - \lambda(\frac{B}{r} - 1)\theta' = 0 .$$

Rearranging, we have

$$r'' - \frac{\lambda}{r}\theta' = -\frac{\lambda}{B}\theta'$$

On the left hand side replace λ by (8) and on the right side replace $\theta' = \lambda r^{-2}$ by (8) to get:

$$a_r = r'' - r(\theta')^2 = -\frac{\lambda^2}{Br^2}$$

Since $a_\theta = 0$ we have, (compare (1) and (4))

$$F = -\frac{\lambda^2 m}{Br^2} U_r \quad , \quad \text{the Inverse Square Law.}$$

We have shown that the Kepler's First and Second Laws imply the Inverse Square Law. The above calculation is not what Newton did. Something closer to what he did is outlined in Exercise 1 where it is shown that the Second and Third Laws imply the Inverse Square Law. In any case, this section shows that experimental evidence well used can lead to nice and powerful theoretical results.

Exercise 1. Assume, as Newton did, that the moon is in a circular orbit and that Kepler's Second and Third Laws hold. Show that θ' is a constant and therefore it is uniform circular motion. Introduce the linear speed $v = ds/dt$ along the circle. Show that $a_r = -(v^2/r)$. This is the usual formula for *centripetal acceleration*. Combine with the Third Law to get the Inverse Square Law.

Exercise 2. Justify the steps in the following without doing any integrals.

$$\int_{-a}^{a} 2\sqrt{b^2(1 - x^2/a^2)}\,dx = ab\int_{-1}^{1} 2\sqrt{1 - u^2}\,du = ab\pi$$

Why does this show that the area of an ellipse is πab? We need this formula for the area of an ellipse in deriving the Third Law. Hint: Evaluate the last integral by interpretation rather than calculation.

---

6. KEPLER'S LAWS AS CONSEQUENCES OF NEWTON'S LAWS

If one decides that F = mA and the Inverse Square Law are correct, do Kepler's Laws follow? This is an important question. Kepler's Laws were empirical results based on fanciful hope and data which had unavoidable inaccuracies. Suppose one can conduct other experiments which verify the Inverse Square Law. If Kepler's Laws are a consequence of the Inverse Square Law, then Kepler's Laws will no longer be empirical results from a single set of datum. A physical theory is strengthened by logical implications between various empirical results. In this section we will show that Kepler's Laws can be deduced from the Inverse Square Law.

First we give an argument that *for any central force field, the motions are planar.* Suppose that the planet is at a point P and that its motion has a tangent vector T at that point. Draw the plane through the sun, through P, and containing the tangent vector T. If Z is the coordinate normal to this plane, then the force and hence the acceleration in the Z direction is zero. That is $Z(t_0) = 0$, $Z'(t_0) = 0$ and $Z''(t) = 0$ for all t. This implies $Z(t) \equiv 0$; the motion is planar.

Secondly, if the force is a central force field we show Kepler's First Law holds. In the planar polar coordinates, $a_\theta = 0$. Thus we may start at equation (9) and retrace our steps to (8) and (7) giving the equal area results.

Exercise 3. Do it.

To get the Second and Third Laws of Kepler we must specilize the central force field to the Inverse Square Law. We have

(12) $$r'' - r(\theta')^2 = -\frac{GM}{r^2}$$

and of course (8). The solution of these differential equations is difficult, but is made easier by looking at the goal. We want r to be a conic as in (10). Notice that $1/r(\theta)$ is very simple. This suggests introducing the function $w(\theta) = 1/r(\theta)$ and deriving a differential equation that it satisfies. We have

$$\frac{dw}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{r'}{\theta'} = -\frac{r'}{\lambda} \quad \text{using (8).}$$

Then

$$\frac{d^2w}{d\theta^2} = -\frac{d}{dt}\left(\frac{r'}{\lambda}\right)\frac{dt}{d\theta} = -\frac{r''}{\lambda\theta'} = -\frac{1}{\lambda\theta'}\left((r(\theta')^2 - \frac{GM}{r^2}\right)$$

$$= -\frac{r\theta'}{\lambda} + \frac{GM}{\lambda r^2\theta'} = -w + \frac{GM}{\lambda^2} \quad \text{using (12).}$$

Then from (8)

(13) $$\frac{d^2w}{d\theta^2} + w = \frac{GM}{\lambda^2}.$$

It turns out that every solution of (13) is given in the form

$$w(\theta) = \frac{GM}{\lambda^2} + \beta\cos(\theta + \alpha)$$

for some constants α and $\beta > 0$ (See Appendix 2).

Thus

(14) $$r(\theta) = \frac{B}{1 + e\cos(\theta + \alpha)}$$

where

$$B = \frac{\lambda^2}{GM} \quad \text{and} \quad e = \frac{\beta\lambda^2}{GM}$$

Hence any such motion describes a conic section. Notice that parabolas and hyperbolas are possible. In fact the constants β, α depend on some initial condition. If they are selected so that $e \geq 1$, then the mass leaves the solar system. The only bounded orbits are therefore ellipses (and circles as a special case). This is Kepler's Second Law.

To derive Kepler's Third Law we need to compute the period T and semi-major axis a of an ellipse. Let b be the semi-minor axis and $c = \sqrt{a^2 - b^2}$ be the focal length.

The formula for T is easy. Since the area of an ellipse is πab, (see Exercise 2) we have by (7) and (8)

$$\pi ab = \int_0^T \frac{1}{2} r^2\theta' dt = \frac{\lambda}{2} T ,$$

or

(15) $$T^2 = \frac{4\pi^2}{\lambda^2} a^2 b^2 .$$

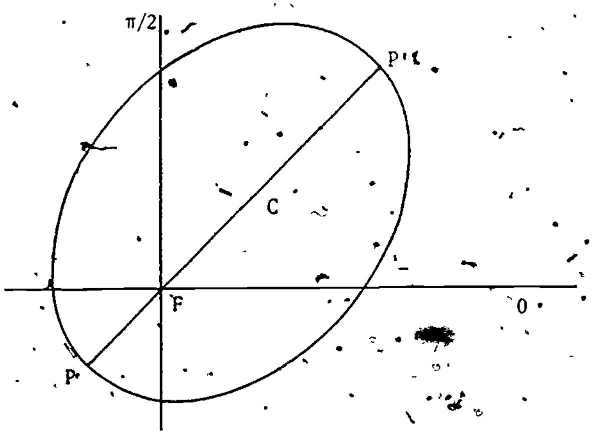

Figure 4. An ellipse with the focus at the origin. This is the picture that Kepler saw in his mind's eye. It is one we can describe neatly in polar coordinates.

The major axis of an ellipse with ends P and P' has length 2a. The length PF is the minimum distance from F to the ellipse, P'F is maximum, while the focal distance c from the center C is $c = 1/2|PP'| - |PF| = a - |PF|$.

Looking at (14) we see that r is smallest when $\cos(\theta + \alpha) = 1$ and largest when $\cos(\theta + \alpha) = -1$. Thus

(16) $$a = \frac{1}{2}\left(\frac{B}{1 + e} + \frac{B}{1 - e}\right) = \frac{B}{1 - e^2}$$

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Moreover

$$b^2 = a^2 - c^2 = a^2 - \left(a - \frac{B}{1 + e}\right)^2$$

$$= 2a \frac{B}{1 + e} - \frac{B^2}{(1 + e)^2}$$

Using (16)

$$\frac{B^2}{(1 + e)^2} = \frac{B}{1 + e} a(1 - e)$$

Thus

$$b^2 = \frac{Ba}{1 + e} [2 - (1 - e)] = Ba$$

Putting this into (15) gives

(17) $$T^2 = \frac{4\pi^2}{\lambda^2} Ba^3 = \frac{4\pi^2}{\lambda^2} \frac{\lambda^2}{GM} a^3 = \frac{4\pi^2}{GM} a^3$$

But $4\pi^2/GM$ is a constant that only depends on the units and the mass of the sun, a Solar System Constant! This is Kepler's Third Law.

7. COMMENTS

We have discussed the relationship between Kepler's Laws and the Inverse Square Law in the context of two masses. We also assumed that the larger mass was fixed. In fact, it will wobble slightly. What is fixed is the common center of mass. When more than two masses are involved, exact description of the motions can be ascertained only in very special cases. This is called the n-body problem. It is the focus of a great deal of mathematical research.

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8. FURTHER EXERCISES

Exercise 4. Suppose you observe that for the earth $a = 1.495 \times 10^8$ km. and $T = 365.25$ days. If $G = 6.670 \times 10^{-11}$ in $m^3/kg\text{-}sec^2$ find the mass of the Sun.

Exercise 5. If the eccentricity of the earth's orbit is given to be 0.167322, and a as in Exercise 4 find the exact equation of the earth's orbit.

Exercise 6. You know that the usual expression for gravity at sea level is $g = -9.807$ m/sec^2. Use the Inverse Square Law and the fact that we may replace the earth by a point mass at its center to get an exact expression for the Inverse Square Law with the earth being the large mass. Use 6371 km. as the radius of the earth. Find the mass of the earth. Hint: Compare the two formulae at the surface of the earth.

Exercise 7. If a satellite going around the earth remains in a circular orbit, then centripetal acceleration must balance the acceleration due to gravity. Using this equation, compute the linear speed v the satellite must have if its altitude is 100 miles above sea level. (Ignore air resistance.) 1 mile = 1609.35 meters.

Exercise 8. The Intelsat series geosynchronous satellite remains above a fixed point on the earth's equator. If it is in a circular orbit you can deduce its altitude. Please do.

Exercise 9. OSO 4, launched October 18, 1967 was in an orbit with radius between 5.375 and 5.697×10^5m. above sea level. What is the period of its motion?

Exercise 10. Take the general formula $ma_r = f(r)$ with θ' removed by (8). Multiply by r' and integrate once. Then replace λ by (8) and derive the formula

$$\frac{1}{2} m \left(\frac{ds}{dt}\right)^2 + G(r) = \text{constant}$$

where $dG/dr = -f$. (You may recall the formula for ds in polar coordinates) G is called the potential energy; $\frac{1}{2} m(ds/dt)^2$ is kinetic energy. The result of this Exercise is the Conservation of Energy.

Exercise 11. Describe a laboratory experiment that would show that centripetal acceleration is $K(v^2/r)$. Newton did this with a thought experiment. He reasoned that the moon must "fall" in its circular orbit in one second the same distance as if it were dropped from a stationary position.

9. HINTS TO THE SOLUTIONS OF THE EXERCISES

Exercise 1. From (8) $\theta' = \lambda/r^2$ is a constant. For a circle $s = r\theta$ so $v = ds/dt = r\theta' = \lambda/r$. Since $r'' = 0$ $a_r = -r(\theta')^2 = -v^2/r$. From $2\pi r = vT$ and Kepler's Third Law $a_r = C/r^2$.

Exercise 2. The last integral is the area of a unit circle.

Exercise 4. Solve (17) for M.

Exercise 5. Use (16) to compute B.

Exercise 6. Write $f(r) = mC/r^2$, $f(6371 \times 10^6) = -mg$. Compare with (2) to find M.

Exercise 7. The equation $C/r^2 = v^2/r$ can be solved for v in terms of known quantities.

Exercise 8. Compute $v = \omega r$, ω the rate of spin of the earth and apply the equation in Exercise 6 to find r, or use (17).

Exercise 9. Apply the derivation of (16) to find a, and then use (17).

Exercise 10. $ds^2 = dr^2 + r^2 d\theta^2$

Exercise 11. Spin a weight at the end of a string. Measure the tug of the string. Keeping r fixed, double the speed, remeasure the tug. Keeping v fixed, double r.

10. MODEL EXAM

1. Suppose we are considering a planet's motion around the Sun.

   a) At what point on the orbit is the vector from the Sun to the planet turning the most rapidly? (First Law)

   b) Recalling that speed $v = ds/dt$, find a formula for the speed of the planet which involves the polar coordinates r and other constants, but not θ. (Second and First Laws).

   c) On the basis of b), when is the speed the maximum?

2. Using the Third Law (or otherwise) find the relationship between the period of a satellite in a circular orbit and its distance from the center of the earth.

3. Describe why Kepler's Laws were more acceptable after Newton's work.

4. Given that the acceleration due to gravity at sea level on the earth is $g = -32$ ft/sec^2. Find the constant in the Inverse Square Law in the units feet and seconds. Use the radius of the earth as 4000 miles and 5280 feet in 1 mile.

5. Suppose that the gravitational force were $F = k/r\, U_r$. Would it still be true that in a satellite's motion, the radius vector sweeps area out in a constant rate?

APPENDIX 1

Conics in Polar Coordinates. If one of the foci is at the origin then the equation of any conic section in polar coordinates is of the form

$$r = \frac{B}{1 + e\cos(\theta + \alpha)}$$

The number e is called the eccentricity and the various conic sections are given by the table

$e = 0$:	circle
$0 < e < 1$:	ellapse
$e = 1$:	parabola
$e > 1$:	hyperbola

The table is easy to remember. If $e < 1$ then the denominator is never zero and r is bounded so we have an ellipse. If $e = 1$ then the denominator is zero precisely once as θ traverses a complete rotation, the point (r, θ) "jumping across the open end of the parabola at infinity." If $e > 1$, then r is infinite twice "jumping from one branch of the hyperbola to the other" each time.

APPENDIX 2

Solutions of Certain Differential Equations. We want to show that the differential equation

$$* \qquad v'' + v = 0$$

with $v(0) = \alpha_0$, $v'(0) = \alpha_1$ has a unique solution. Suppose v_1 and v_2 are two solutions and let h (h for a function pulled out of a hat) be defined by

$$h(\theta) = (v_1 - v_2)^2 + (v_1' - v_2')^2$$

Then

$$\frac{1}{2} h'(\theta) = (v_1 - v_2)(v_1 - v_2)' + (v_1' - v_2')(v_1'' - v_2'')$$

$$= (v_1' - v_2')[(v_1 - v_2) + (v_1'' - v_2'')]$$

$$= (v_1' - v_2')[D - D] = 0 \quad \text{(Using * twice)}$$

so h is a constant. But $h(0) = 0$ means $(v_1 - v_2)^2 + (v_1' - v_2')^2 \equiv 0$. Thus $v_1 - v_2 \equiv 0$. That is $v_1 = v_2$.

Now one solution of the problem is (just plug it in)

$$v(\theta) = D - \frac{\alpha_1}{\sin \alpha} \cos(\theta + \alpha)$$

with

$$\cot \alpha = \frac{D - \alpha_0}{\alpha_1}$$

if $\alpha_1 \neq 0$. If $\alpha_1 = 0$ take $v(\theta) = D - (D - \alpha_0)\cos \theta$. Since every solution of the differential equation satisfies the extra conditions at 0 for some α_0 and α_1, it is of the form $v = D + \beta \cos(\theta + \alpha)$. To make $\beta > 0$ replace α by $\alpha + \pi$ since $\cos(\theta + \alpha + \pi) = -\cos(\theta + \alpha)$ changes the sign of β.

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DETAILED SOLUTIONS TO EXERCISES

Note that we are using rounded off data so that these results may not agree exactly with published tables of data.

Exercise 1. From (8) $\theta' = \lambda/r^2$ is constant. On a circle arclength $s = \theta r$ so $v = ds/dt = r\theta'$. Now $r'' = 0$ so $a_r = -r(\theta')^2 = -(v^2/r)$. Compute the circumference of the circle in two ways. Then $2\pi r = vT$ and use the Third Law to write

$$a_r = -\frac{v^2}{r} = -\left(\frac{2\pi r}{T}\right)^2 \frac{1}{r} = -\frac{4\pi^2 r}{T^2} = -4\pi^2 r\left[\frac{K}{r^3}\right] = -\frac{4\pi^2 K}{r^2}$$

Exercise 2. The first integral is the usual area integral for the ellipse $x^2/a^2 + y^2/b^2 = 1$. Let $x = au$ to get the last integral. This integral computes the area of the unit circle so is π.

Exercise 4. By (17), $M = 4\pi^2 a^3/GT^2$. Put into m, kg, sec. Then $G = 6.670 \times 10^{-11}$, $a = 1.495 \times 10^{11}$ m, $T = (365.25)(24)(3600)$ sec $= 3.156 \times 10^7$ sec. Thus $M = 4\pi^2(1.495)^3 10^{33}/(6.670)(3.156)^2 \times 10^3 = 1.986 \times 10^{30}$ kg.

Exercise 5. By (16), $B = a(1 - e^2) = (1.495) \times 10^8(1 - (.1673)^2) = 1.453 \times 10^8$. Now put into (14).

Exercise 6. $f(r) = -(Cm/r^2)$. At sea level $-(Cm/r^2) = gm$. Writing everything in m and kg., $C = -gr^2 = (9.807)(6.371 \times 10^6)^2 = 3.981 \times 10^{14}$. But $mC = mMG$ so $M = C/G = 3.981 \times 10^{14}/6.670 \times 10^{-11} = 5.968 \times 10^{24}$ kg.

Exercise 7. From $v^2/r = C/r^2$, $v^2 = C/r$, with C as in Exercise 6. But $r = (6.371) \times 10^6 + 100(1609.35)$m, so $v^2 = [(3.981) \times 10^{14}]/[6.532 \times 10^6] = .6095 \times 10^8$. Thus $v = 7.807 \times 10^3$ m/sec.

Exercise 8. a) Let ω be the rate of spin of the earth. Then $\omega = 2\pi/24(3600) = .727 \times 10^{-4}$ rad/sec. Thus $v = \omega r$ and $C/r^2 = v^2/r$ as in Exercise 7 lead to $r^3 = C/\omega^2 = 3.981 \times 10^{14}/(.727)^2 \times 10^{-8} = 75.32 \times 10^{21}(m)^3$. Then $r = 4.223 \times 10^7$ m.

b) Alternately, using (17) $r^3 = GMT^2/4\pi^2$ and the results of Exercise 6,

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$$r^3 = (6.670)\times 10^{-11}(5.968)\times 10^{24}\,(24\times 3600)^2/4\pi^2$$

$$= 75.27\times 10^{21}$$

$$r = 4.222\times 10^{9}\text{m}.$$

Exercise 9. Look at the derivation of (16).

$$a = \frac{1}{2}(\text{max distance} + \text{min distance}).$$

The maximum distance is $(5.697\times 10^5 + 6.371\times 10^6)$m and the minimum distance is $(5.375\times 10^5 + 6.371\times 10^6)$m. Thus $a = 6.925\times 10^6$m. Apply (17).

$$T^2 = 4\pi^2 a^3/GM$$

$$= 4\pi^2(6.925\times 10^6)^3/(6.670\times 10^{-11})(5.968\times 10^{24})$$

$$= 32.94\times 10^6\,\text{sec}^2.$$

$$T = 5.739\times 10^3\,\text{sec}.$$ (The listed period is 95.7 min.)

Exercise 10. From $m[r'' - r(\theta')^2] = f(r)$ we get

$$m[r'' - \frac{\lambda^2}{r^3}] = f(r)$$

so

$$m[r'r'' - \lambda^2\frac{r'}{r^3}] - f(r)r' = 0.$$

Notice that.

$$\frac{1}{2}[m(r')^2 + \frac{m\lambda^2}{2r^2}]' = m[r'r'' - \lambda^2\frac{r'}{r^3}]$$

and

$$\frac{d}{dt}G(r) = \frac{d}{dr}G(r)r' = -f(r)r'.$$

Thus

$$\frac{d}{dt}[\frac{1}{2}m(r')^2 + \frac{m\lambda^2}{2r^2} + G(r)] = 0.$$

This makes the bracket a constant. Now replace λ by $r^2\theta'$ to get

$$\frac{1}{2}m[(r')^2 + r^2(\theta')^2] + G(r) = \text{constant}$$

but

$$(\frac{dr}{dt})^2 + r^2(\frac{d}{dt})^2 = (\frac{ds}{dt})^2 = v^2$$

so we get

$$\frac{1}{2}mv^2 + G(r) = \text{constant}.$$

ANSWERS TO MODEL EXAM

1. a) From the first law $\theta' = \lambda/r^2$, so θ' is largest when r is smallest.

b) $$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

$$= \left[\left(\frac{dr}{dt}\right)^2 + r^2\right]\left(\frac{d\theta}{dt}\right)^2 = \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\left(\frac{\lambda}{r^2}\right)^2.$$

Now

$$r = \frac{B}{1 + e\cos\theta}$$

so

$$\frac{dr}{d\theta} = \frac{B\,e\sin\theta}{(1 + e\cos\theta)^2}$$

Thus

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{B^2}{(1 + e\cos\theta)^4}[(1 + e\cos\theta)^2 + e^2\sin^2\theta]$$

$$= \frac{r^4}{B^2}(1 + e^2 + 2e\cos\theta).$$

But

$$e\cos\theta = \frac{B}{r} - 1$$

so

$$v^2 = \frac{(e^2 - 1)\lambda^2}{B^2} + \frac{2\lambda^2}{Br}.$$

Recall that $B > 0$, so v^2 is largest when r is smallest.

2. a) The third law is $T^2 = kr^3$. Compute the distance travelled in one period two ways, $vT = 2\pi r$. Now eliminate T to get $v^2 = \frac{4\pi^2}{k}\,\frac{1}{r}$; or

b) Centripetal acceleration must balance the inverse square acceleration. Thus

$$\frac{v^2}{r} = \frac{GM}{r^2}$$

from which

$$v^2 = \frac{GM}{r}.$$

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3. Newton showed that Kepler's Laws follow from the Inverse Square Law. So if one accepts the latter then Kepler's Laws are also acceptable.

4. If $f(r) = -\frac{k}{r^2}$, then

   $f(4000 \cdot 5280) = -32$,

   thus,

   $k = 32(5280)^2(4000)^2$.

5. Yes, the first law on the area swept out by the radius vector holds in any central force field.

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STUDENT FORM 1

Request for Help

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ____________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ____________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

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Name________________________ Unit No.____________ Date____________

Institution________________________ Course No.________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)