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ABSTRACT

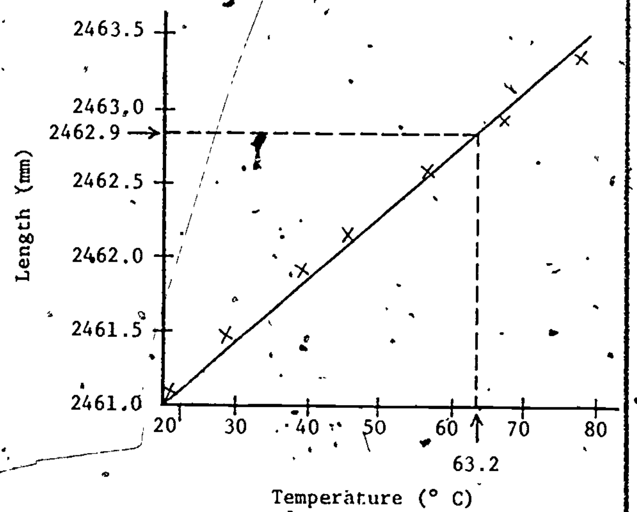
This document consists of two modules. The first of these views applications of algebra and elementary calculus to curve fitting. The user is provided with information on how to: 1) construct scatter diagrams; 2) choose an appropriate function to fit specific data; 3) understand the underlying theory of least squares; 4) use a computer program to do desired curve fitting; and 5) use augmented matrix approach to solve simultaneous equations. The second unit provides techniques to formulate simple linear programing problems and solve them graphically. Both modules contain exercises and provide exams. Answers to all problems are supplied. (MP)

umap

UNIT 321

CURVE FITTING VIA THE CRITERION OF LEAST SQUARES

by John W. Alexander, Jr.



APPLICATIONS OF ALGEBRA AND ELEMENTARY CALCULUS TO CURVE FITTING

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CURVE FITTING VIA THE CRITERION OF LEAST SQUARES

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Intermodular Description Sheet: *UMAP Unit 321*

Title: CURVE FITTING VIA THE CRITERION OF LEAST SQUARES

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Review Stage/Date: III 9/3/79

Classification: APPL ALG & ELEM CALC/CURVE FITTING

Suggested Support Materials: A computer terminal on line to a system with BASIC compiler (to be used for the appendix).

Prerequisite Skills:

1. Be able to do partial differentiation.
2. Be able to maximize functions.
3. Know how to solve simultaneous equations by elimination or substitution for 2 x 2 cases.
4. Know how to graph elementary, exponential, and logarithmic functions.

Output Skills:

1. Be able to construct scatter diagrams.
2. Be able to choose an appropriate function to fit specific data.
3. To understand the underlying theory of the method of least squares.
4. To be able to use a computer program to do desired curve-fitting.
5. Be able to use augmented matrix approach to solve simultaneous equations.

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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CURVE FITTING VIA THE CRITERION OF LEAST SQUARES

1. INTRODUCTION

In many instances, we wish to be able to predict the outcome of certain phenomena. For example, we may want to know which students in a graduating high-school class will do well in their first year of college.

One way to get a measure or at least an indication would be to observe the high-school grades in English of 20 or so students who have gone to college. If we match the students' English grades with their grade point average after one semester, we would be able to see if good grades in English matched with high grade point averages.

If the "correlation" is high, then, we might wish to assert that students who do well in high-school English do well in college. There may be exceptions of course. We may want to look at other indicators (e.g., math grades) but, the point is, we wish to look at two or more statistics on the same individual, and we are interested to know how these statistics relate.

Ideas of the sort alluded to above are the subject of this module.

2. SCATTER DIAGRAMS

Many statistical problems are concerned with more than a single characteristic of an individual. For instance, the weight and height of a number of people could be recorded so that an examination of the relationship between the two measurements could be made. As a further example, consider how the length of a copper rod relates to its temperature.

TABLE 1

Temperature (° C) x	Length (mm) y
20.1	2461.16
28.2	2461.49
38.5	2461.88
44.6	2462.10
57.4	2462.62
66.2	2462.93
78.1	2463.38

When we draw a scatter diagram letting the horizontal axis be the scale for the temperature and the vertical axis the scale for the length, we note that the plotted points lie very close to a straight line. It is, therefore, reasonable to make a quick and accurate estimate of the length of the rod for any temperature between 20.1° and

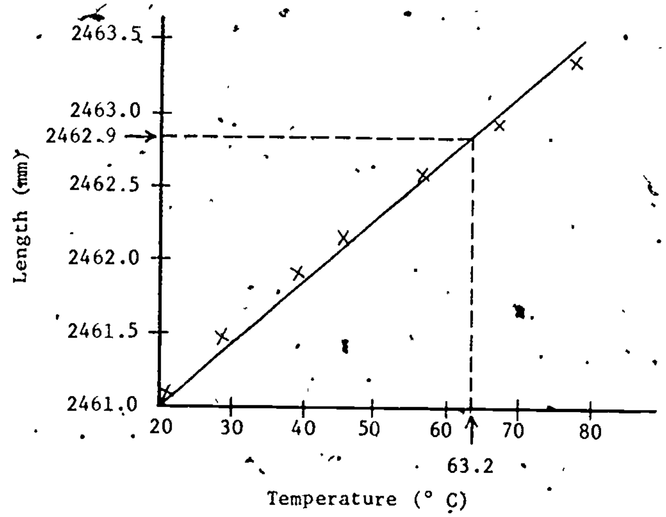


Figure 1.

78.1°.* For example, if the temperature was 63.2° the dotted lines in Figure 1 indicate that the corresponding point on the line gives a length of approximately 2462.9 mm.

Let us explore another example that gives us a scatter diagram where the points are more scattered. Table 2 gives us the weight in grams, x, and the length of the right hind foot in millimeters, y, of a sample of 14 adult field mice.

TABLE 2

Weight (g) x	Length (mm) y
22.3	23.0
16.0	22.6
18.8	23.2
18.2	22.5
16.0	22.2
20.4	23.3
17.9	22.8
19.4	22.4
16.9	21.8
17.6	22.4
16.5	22.4
18.8	21.5
17.2	21.9
20.4	23.3

The point in Figure 2 that is circled indicates where two points of the data coincide. The points here are much more scattered than those of the previous set. It would be extremely difficult to determine which straight line best fits this set of points. In fact, if a number of people were to attempt to fit a line to these points, there is little doubt that each person would come up with a different line. What we need is a mathematical method for determining the line that comes "closest" to all of the points.

*We are not in a position to speculate about values outside of this range.

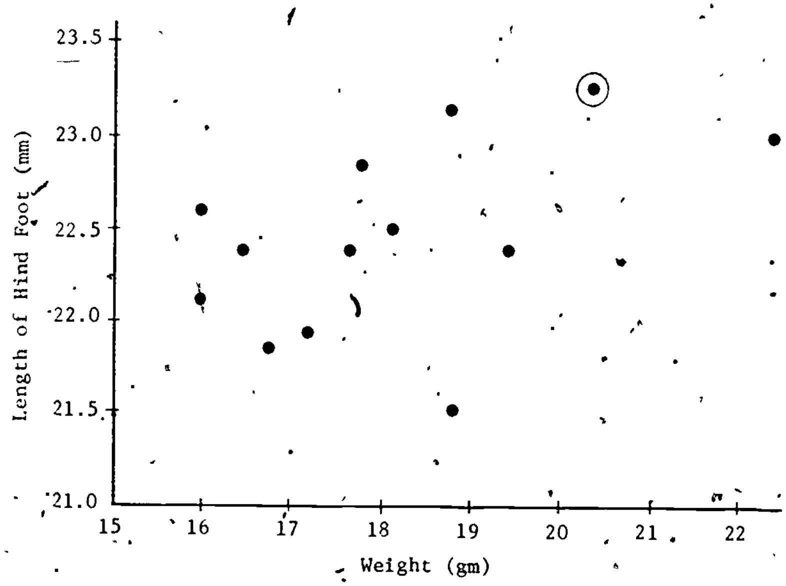


Figure 2.

3. THE LINE OF REGRESSION

The criterion traditionally used to define a "best" fit dates back to the nineteenth century French mathematician Adrien Legendre. It is called the *criterion, or method, of least squares*. This criterion requires the line of regression which we fit to our data to minimize *the sum of the squares of the vertical deviations (distances) from the points to the line*. In other words, the method requires the sum of the squares of the distances represented by the solid line segments of Figure 3 to be as small as possible.

From the figure, we see that the actual grade received for a student who studied 11 hours was 79. Reading from the line of regression we predict a grade of about 71.

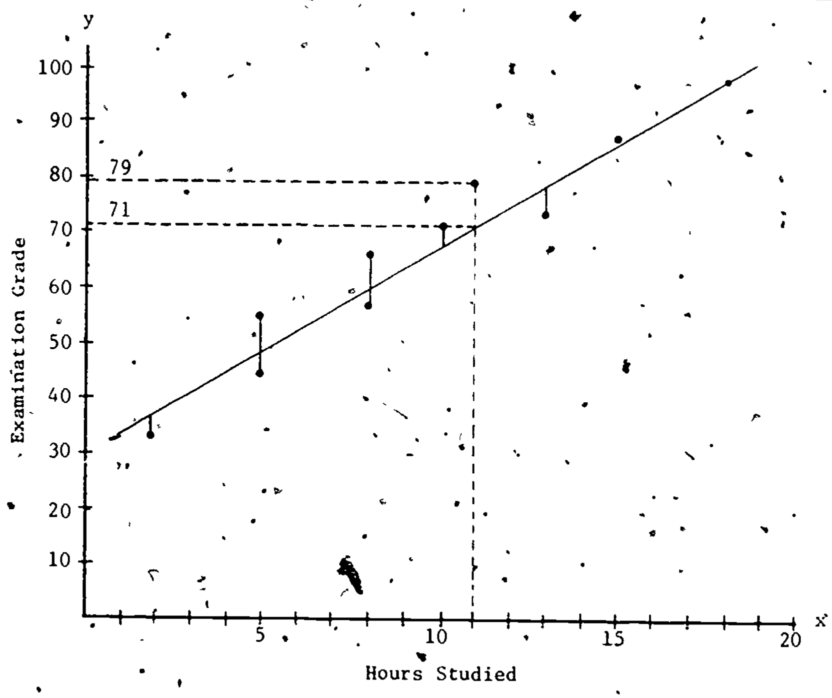


Figure 3. Line of regression fitted to data on hours studied and examination grades.

Observe that any line can be expressed:

(1) $y = bx + c$

or

(2) $x = b'y + c'$,

where the b's represent the slope of the line and the c's are interpreted as the intercept of the axis.

If we consider Equation (1), knowing the values of b and c will allow us to compare the actual values in the y column with bx + c. We take the difference in each case and square the result. Consider the values of x and y in Table 3.

TABLE 3

x	y	bx + c	[Difference]^2
25	30	b25 + c	$[30 - (25b + c)]^2$
30	46	b30 + c	$[46 - (30b + c)]^2$
50	51	b50 + c	$[51 - (50b + c)]^2$
20	28	b20 + c	$[28 - (20b + c)]^2$
70	48	b70 + c	$[48 - (70b + c)]^2$
80	88	b80 + c	$[88 - (80b + c)]^2$
91	75	b91 + c	$[75 - (91b + c)]^2$
46	52	b46 + c	$[52 - (46b + c)]^2$
35	35	b35 + c	$[35 - (35b + c)]^2$
25	28	b25 + c	$[28 - (25b + c)]^2$
80	95	b80 + c	$[95 - (80b + c)]^2$

We add up all of these squared differences. It must then be determined what values of b and c must be used in order to have a line such that the sum of the vertical distances from the line to the data points is at a minimum.

<u>The Problem</u>: Find the values of b and c such that the sum indicated below is a minimum.

$$\sum D^2 = (30-25b-c)^2 + (46-30b-c)^2 + (51-50b-c)^2 + (28-20b-c)^2 + (48-70b-c)^2 + (88-80b-c)^2 + (75-91b-c)^2 + (52-46b-c)^2 + (35-35b-c)^2 + (28-25b-c)^2 + (95-80b-c)^2.$$

The symbol sigma can be employed on both sides of the equation above (i.e., $\Sigma D^2 = \sum_{i=1}^{n}(y_i - bx_i - c)^2$). Since $\sum D^2$ is a function of b and c we can write

$$\sum D^2 = f(b,c) = \sum_{i=1}^{n}(y_i - bx_i - c).$$

To find our desired minimum we find the partial derivatives with respect to b and c and set the results equal to zero. We obtain two equations in two unknowns which we solve simultaneously. This gives us the desired values of b and c and thus our line of best fit (the line of regression).

Trace through the actual development given below.

$$f(b,c) = \sum_{i=1}^{n} (y_i - bx_i - c)^2$$

$$\frac{\partial f}{\partial b} = \sum_{i=1}^{n} 2(y_i - bx_i - c)(-x_i) = 0$$

$$= \sum_{i=1}^{n} (-2y_i x_i + 2bx_i^2 + 2cx_i) = 0,$$

finally,

$$\sum_{i=1}^{n} bx_i^2 + \sum_{i=1}^{n} cx_i = \sum_{i=1}^{n} x_i y_i .$$

To continue with the other derivatives:

$$\frac{\partial f}{\partial c} = \sum_{i=1}^{n} 2(y_i - bx_i - c)(-1) = 0$$

$$= 2\sum_{i=1}^{n} (-y_i + bx_i + c) = 0$$

and

$$\sum_{i=1}^{n} bx_i + \sum_{i=1}^{n} c = \sum_{i=1}^{n} y_i .$$

Thus our two equations which are traditionally called *normal equations* are:

(3) $$b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

(4) $$b\sum_{i=1}^{n} x_i + nc = \sum_{i=1}^{n} y_i .$$

In order to solve these equations, we must calculate the indicated sums as is done in Table 4. We have also included the table of y_i^2's because we can use the sum $\sum_{i=1}^{n} y_i^2$ to find the line of regression $x = b'y + c'$.

The normal equations for this line are obtained merely by interchanging x and y in the original two equations (3) and (4).

TABLE 4

x	y	x^2	y^2	xy
25	30	625	900	750
30	46	900	2116	1380
50	51	2500	2601	2550
20	28	400	784	560
70	48	4900	2304	3360
80	88	6400	7744	7040
91	75	8281	5625	6825
46	52	2116	2704	2392
35	35	1225	1225	1225
25	28	625	784	700
80	95	6400	9025	7600
552	576	34372	35812	34382

As an example, for $x = b'y + c'$ we have:

(3') $$b'\sum_{i=1}^{n} y_i^2 + c'\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i x_i .$$

(4') $$b'\sum_{i=1}^{n} y_i + nc' = \sum_{i=1}^{n} x_i$$

From Table 4, our equations become:

(3) $$34372b + 552c = 34382.$$

and

(4) $$552b + 11c = 576.$$

Thus,

$$11c = 576 - 552b$$

$$c = \frac{576 - 552b}{11}.$$

Substituting the value of c into (3) we get

$$34372b + 552\left(\frac{576 - 552b}{11}\right) = 34382$$

$$378092b + 317952 - 304704b = 378202$$

$$73388b = 60250$$

$$\therefore \quad b = 0.8209789$$

$$\text{and} \quad c = 11.165422.$$

Hence, $y = 0.8209789x + 11.165422$. Similarly,

$$(3') \qquad 35812b' + 576c' = 34382$$

$$(4') \qquad 576b' + 11c' = 552$$

$$\therefore \quad 11c' = 552 - 576b'$$

$$c' = \frac{552 - 576b'}{11}$$

Substituting in (3') we get

$$35812b' + 576\left(\frac{552 - 576b'}{11}\right) = 34382$$

$$393932b' + 317952 - 331776b' = 378202$$

$$62156b' = 60250$$

$$\therefore \quad b' = 0.9693352$$

and

$$c' = -0.5760972$$

$$\therefore \quad x = 0.9693352y - 0.5760972.$$

So, we now have the lines of best fit with respect to y and with respect to x. (See Figures 4a and 4b.) We can use either one, depending on our needs. Further than that, having the two lines allows us to calculate what is called the coefficient of correlation.

4. COEFFICIENT OF CORRELATION

In order to get a numerical indicator of how well the two sets of scores compare, we take the geometric mean of the slopes of the two lines of regression (i.e., $r = \pm\sqrt{bb'}$).

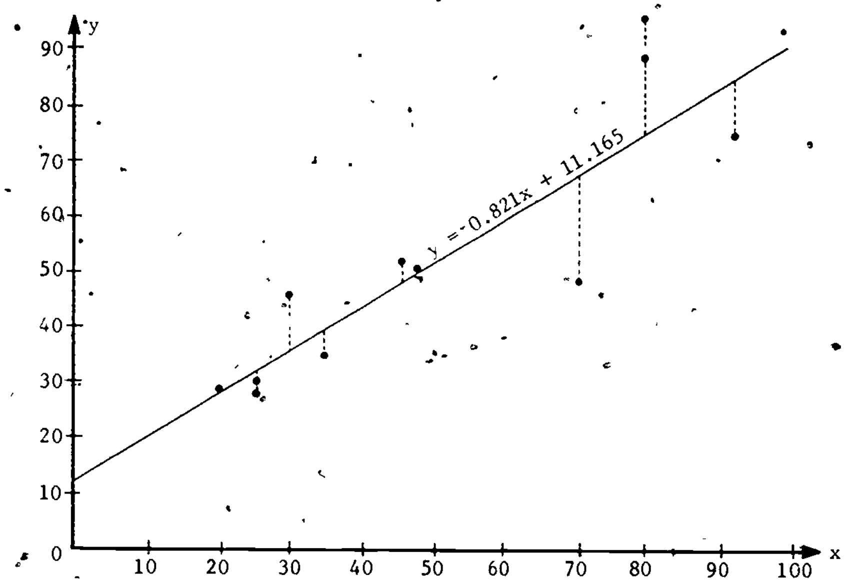


Figure 4a. Regression of y on x.

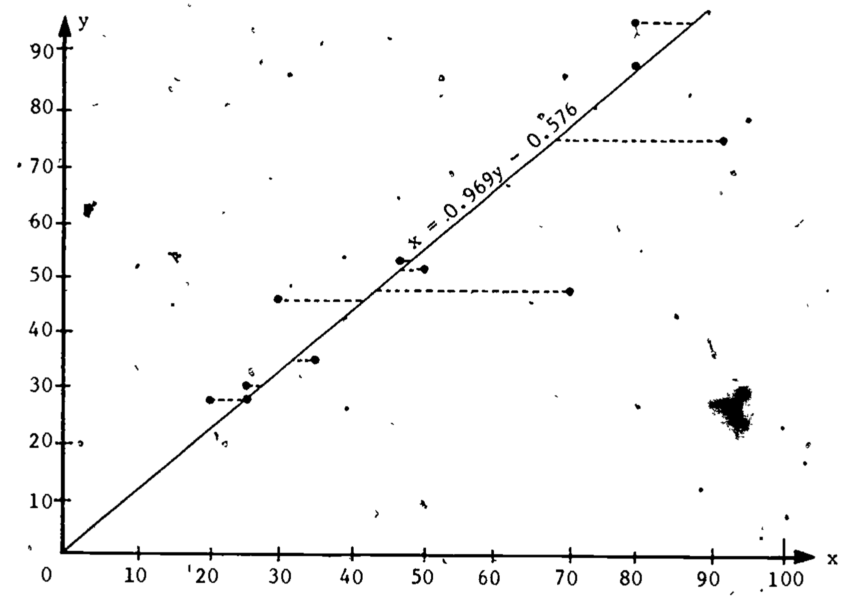


Figure 4b. Regression of x on y.

The sign is chosen to be negative if both slopes are negative, and positive if both slopes are positive. This value, which ranges from -1 to 1 is called the coefficient of correlation. If we have good correlation the value r is close to 1. Poor correlation is indicated by a value near 0. If high values of one characteristic are associated with low values of the other, the correlation is considered negative. Observe the distribution of points in the graphs of Figure 5.

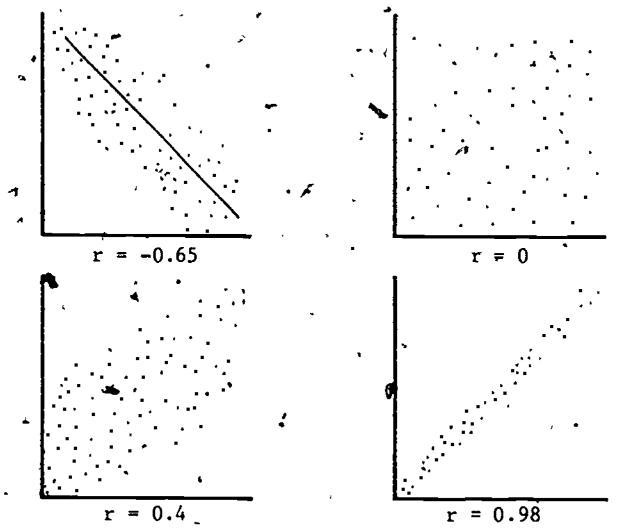


Figure 5.

Using the data from the example in the previous section we have:

$$r = \sqrt{bb'} \doteq \sqrt{(0.8209789)(0.9693352)}$$

$$\doteq \sqrt{0.7958037}$$

$$\doteq 0.8921.$$

The value of r indicates a reasonably good correlation.

We can determine b and b' directly from the two normal equations. This allows us to calculate r without

the trouble of finding the lines of regression. With a little algebra we can write:

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

and

$$b' = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}$$

Exercise 1.

Given the normal equations (3), (4), (3'), and (4'), use algebra to obtain b and b' above.

Since $r = \sqrt{bb'}$ we can write:

$$r = \sqrt{\frac{\left[n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\right]^2}{\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]\left[n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2\right]}}$$

More simply:

$$r = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sqrt{\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right]\left[n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2\right]}}$$

As a check we substitute the indicated sums in this new formula:

$$\therefore r = \frac{11(34382) - 552(576)}{\sqrt{(11(34372) - (552)^2)(11(35812) - (576)^2)}}$$

$$= \frac{378202 - 317952}{\sqrt{(73388)(62156)}} = \frac{60250}{67540}$$

$$\therefore r \doteq 0.8920639 \doteq 0.8921.$$

This agrees with the results obtained by employing the explicit slopes, b and b' of the two lines of regression.

5. REGRESSION FOR LOGARITHMIC SCATTERS

Consider the graph in Figure 6 of a man's growth measured every three years after birth. Notice that there is a great deal of growth between birth and 15 years. After that time, growth tapers off. Table 5 gives the data used in plotting the graph.

TABLE 5

Age in Yrs.	Birth	3	6	9	12	15	18	21	24	27	...
Height in Ft.	1.5	3	3.75	4.5	5	5.8	6.1	6.15	6.17	6.18	...

Using the techniques developed in Section 3, we can easily fit a line to the data. See the calculations below in Table 6.

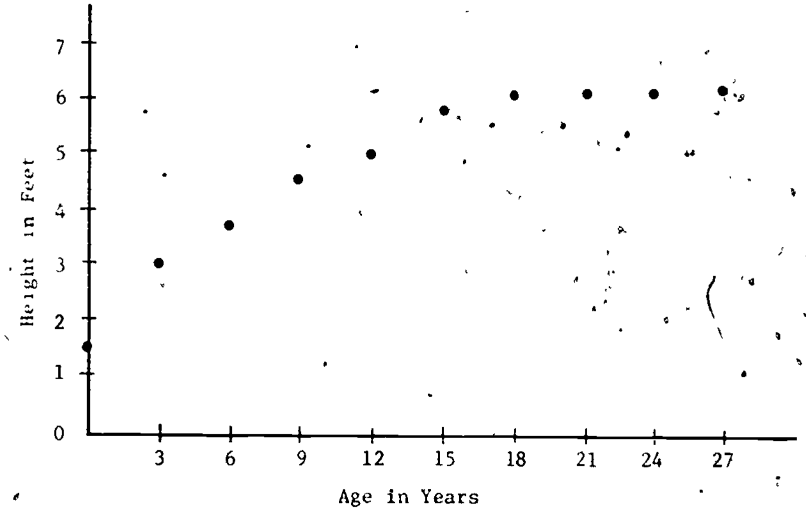


Figure 6.

TABLE 6

x	x^2	y	xy
0	0	1.60	0
3	9	2.90	8.70
6	36	3.75	22.50
9	81	4.50	40.50
12	144	5.00	60.00
15	225	5.80	87.00
18	324	6.10	109.80
21	441	6.15	129.15
24	576	6.17	148.08
27	729	6.18	166.86
135	2565	46.55	772.59

Using $b\Sigma x^2 + c\Sigma x = \Sigma xy$

$b\Sigma x + cn = \Sigma y.$

Therefore we can write:

$$2565b + 135c = 772.59$$

$$135b + 10c = 46.55$$

$$c = \frac{46.55 - 135b}{10}$$

and we can further write

$$2565b + 135\left(\frac{46.55 - 135b}{10}\right) = 772.59$$

$$25650b + 135(46.55) - (135)^2 b = 7725.9$$

$$25650b + 6284.25 - 18225b = 7725.9$$

$$7425b = 1441.65$$

$$b = 1441.65/7425$$

$$b = 0.1942$$

$$\therefore \quad c = 2.0333$$

and we have our line of regression:

$$y = 0.1942x + 2.0333.$$

In order to draw the line we need only locate two points.

For $x = 0$, $y = 2.0333$,

for $x = 3$, $y = 0.1942(3) + 2.0333 = 2.6159$.

While this is not a bad fit, we can do better. It turns out that the data will fit a logarithmic curve much better than a straight line. In general, logarithmic curves look like the one shown in Figure 8.

We can take the $\log_e$* of each of the x values (age in this example). We then use the same technique of least squares to find a log line of best fit. The calculations are given below. Notice how much closer this curve is to the actual data.

*$\log_e$ is also written ln. We are using $\log_e$ here to emphasize the general nature of logarithms. We can arbitrarily use any base.

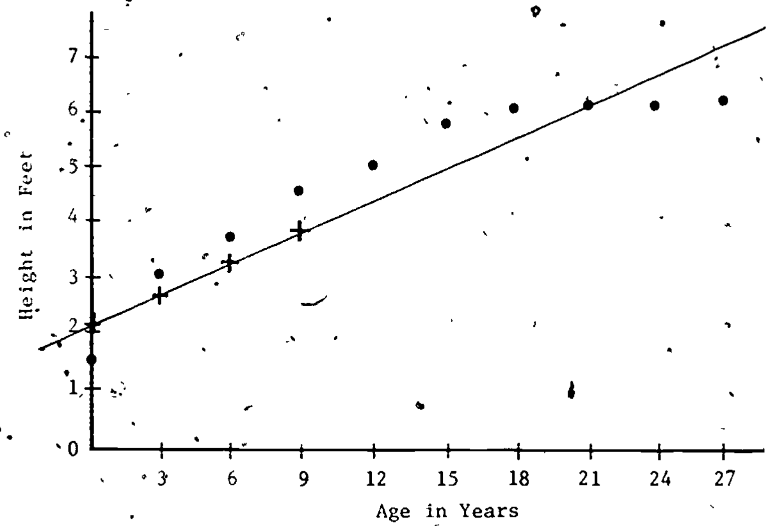


Figure 7.

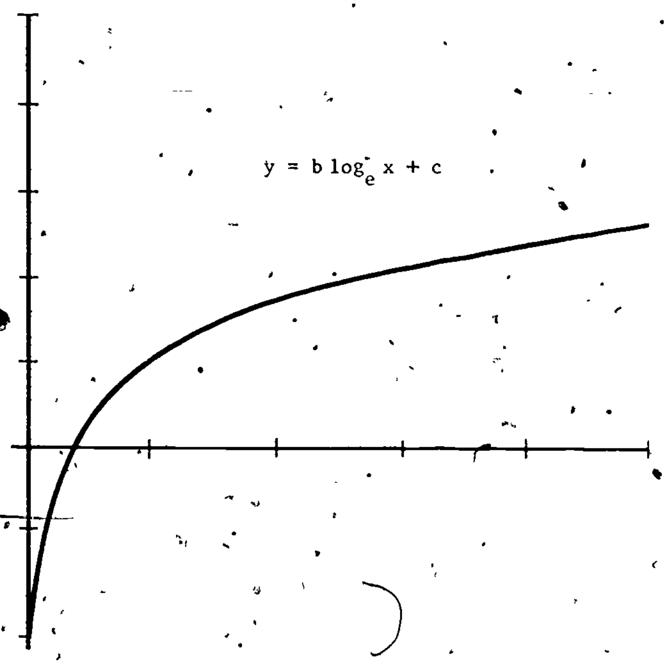


Figure 8.

TABLE 7

x	$\log_e x$	$(\log_e x)^2$	y	$(\log_e x)y$
0	$-\infty$			
3	1.0986	1.2069	2.90	3.1859
6	1.7918	3.2105	3.75	6.7193
9	2.1972	4.8277	4.50	9.8874
12	2.4849	6.1743	5.00	12.4245
15	2.7081	7.3338	5.80	15.5707
18	2.8904	8.3544	6.10	17.6314
21	3.0445	9.2690	6.15	18.7237
24	3.1781	10.1003	6.17	19.6089
27	3.2958	10.8623	6.18	20.3680
	22.6894	61.3392	46.55	124.1198

We use

$$b\Sigma(\log_e x)^2 + c\Sigma\log_e x = \Sigma(\log_e x)y$$

$$b\Sigma\log_e x + cn = \Sigma y.$$

Therefore we can write:

$$61.3392b + 22.6894c = 124.1198$$

$$22.6894b + 9c = 46.55$$

$$c = \frac{46.55 - 22.6894b}{9}$$

therefore

$$61.3392b + 22.6895\left(\frac{46.55-22.6894b}{9}\right) = 124.1198$$

$$9(61.3392b) + 22.6894(46.55) - (22.6984)^2 b = 9(124.1198)$$

$$552.0528b + 1056.1915 - 515.2174b = 1117.0782$$

$$36.8354b = 60.8867$$

$$b = 1.6529$$

$$c = \frac{46.55 - 22.6894(1.6529)}{9} = \frac{46.55 - 37.5033}{9}$$

$$= \frac{9.0467}{9} = 1.0052$$

$$y = 1.6529 \log_e x + 1.0052.$$

Let $x = 3$, then $y = 1.6529(1.0986) + 1.0052 = 2.821.$

$x = 6$, then $y = 1.6529(1.7918) + 1.0052 = 3.9669.$

$x = 9$, then $y = 1.6529(2.1972) + 1.0052 = 4.637.$

$x = 12$, then $y = 1.6529(2.4849) + 1.0052 = 5.1125.$

$x = 15$, then $y = 1.6529(2.7081) + 1.0052 = 5.4814.$

$x = 18$, then $y = 1.6529(2.8904) + 1.0052 = 5.7827.$

$x = 21$, then $y = 1.6529(3.0445) + 1.0052 = 6.0375.$

$x = 24$, then $y = 1.6529(3.1781) + 1.0052 = 6.2583.$

$x = 27$, then $y = 1.6529(3.2958) + 1.0052 = 6.4528.$

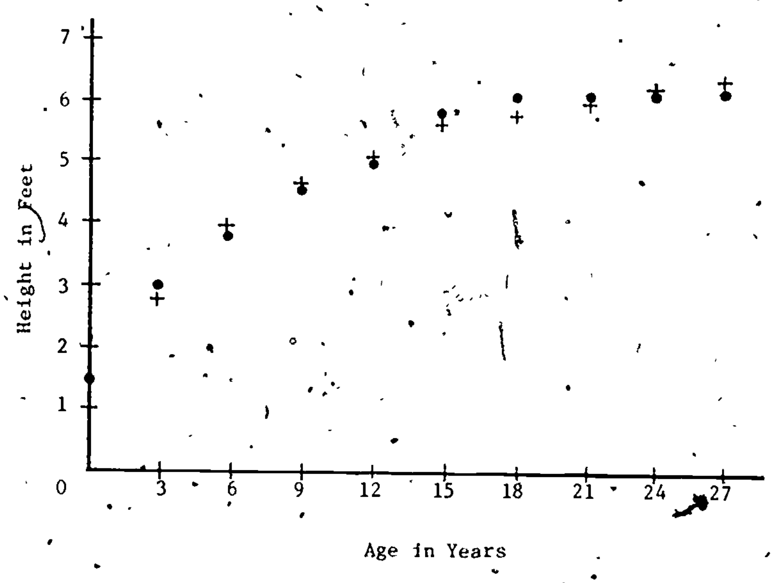


Figure 9.

Exercise 2.

Fit a logarithmic curve to the data given in the table below.

x	1	6	11	16	21	...
y	12	42	53	71	76	...

6. REGRESSION FOR EXPONENTIAL SCATTERS

Consider now, an experiment where a large number of corn seedlings were grown under favorable conditions. Every two weeks a few plants were weighed, and the average of their weights was recorded. (See Table 8.) We also give a graph in Figure 10. It would be difficult to find a straight line that would fit very well. The logarithmic curve does not fit so well, either.

TABLE 8

Age in Weeks	2	4	6	8	10	12	14	16	18	20	...
Average Weight in Grams	21	28	58	76	170	422	706	853	924	966	...

This set of data is probably best fit to an exponential curve. The general shape of such curves ($y = e^x$) is given in Figure 11. Algebraically $y = e^x$ can be written $\log_e y = x$. For a general exponential we can write:

$$y = ce^{bx}.$$

With a little algebra, we can get a form that will allow us to use the least squares method. Analyze the development below.

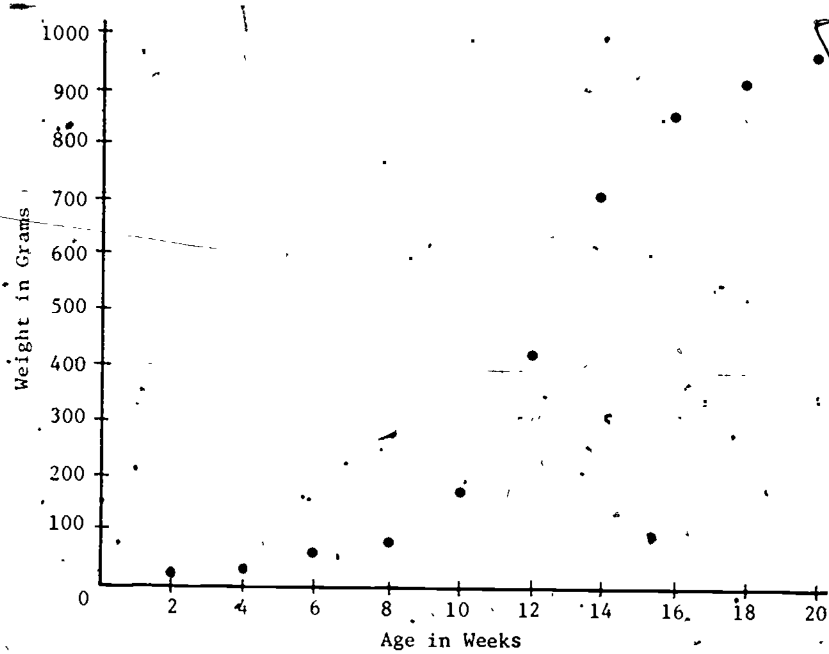


Figure 10.

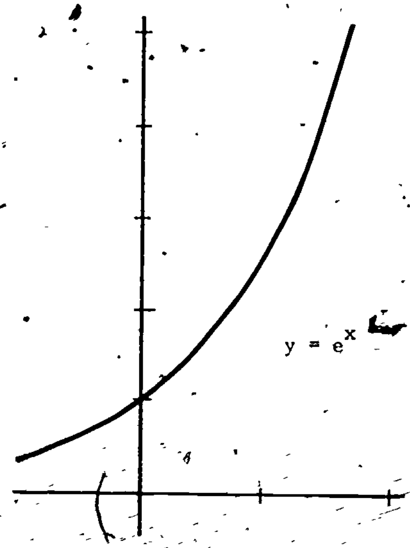


Figure 11.

$$y = ce^{bx} \rightarrow \frac{y}{c} = e^{bx} \rightarrow \log_e \frac{y}{c} = bx.$$

And further we have:

$$\log_e y - \log_e c = bx \quad \text{or} \quad \log_e y = bx + \log_e c.$$

We can find the "line" of best exponential fit by taking the $\log_e$ of the y values and then proceeding with the least squares technique. (See Table 9.)

TABLE 9

x	y	$\log_e y$	x^2	$x(\log_e y)$
2	21	3.0445	4	6.0890
4	28	3.3322	16	13.3288
6	58	4.0604	36	24.3624
8	76	4.3307	64	34.6456
10	170	5.1358	100	51.3580
12	422	6.0450	144	72.5400
14	706	6.5596	196	91.8344
16	853	6.7488	256	107.9808
18	924	6.8287	324	122.9166
20	966	6.8732	400	137.4640
110		52.9597	1540	662.5196

Keeping the equation

$$\log_e y = bx + \log_e c$$

in mind, we calculate

$$1540b + 110 \log_e c = 662.5196$$

$$110b + 10 \log_e c = 52.9597$$

$$b = \frac{52.9597 - 10 \log_e c}{110}$$

$$\log_e c = \frac{52.9597 - 110b}{10}$$

$$1540b + 110\left(\frac{52.9597 - 110b}{10}\right) = 662.5196$$

$$1540b + 5825.567 - 12100b = 6625.196$$

$$3300b = 799.629$$

$$b = 0.2423$$

$$\log_e c = \frac{52.9597 - 26.653}{10} = 2.63067$$

$$\log_e y = 0.2423x + 2.63067.$$

For $x = 2$, $\log_e y = 0.2423(2) + 2.63067$

$= 3.11527$

therefore

$$y = e^{3.11527} \doteq 2.71828^{3.11527}$$

$= 22.5395.$

For $x = 4$, $\log_e y = 0.2423(4) + 2.6307$

therefore

$y = 36.5946.$

For $x = 6$, $y = 59.4122$
for $x = 8$, $y = 96.4573$
for $x = 10$, $y = 156.6008$
for $x = 12$, $y = 254.2454$
for $x = 14$, $y = 412.7739$
for $x = 16$, $y = 670.1489$
for $x = 18$, $y = 1088.0038$
for $x = 20$, $y = 1766.4019$.

We could have solved for c when we obtained $\log_e c = 2.63067$.

If $\log_e c = 2.63067$, then $c \doteq 13.8831$, therefore

$$y = 13.8831e^{0.2423x}.$$

If we substitute 2 for x we get a value which is virtually the same as we got using the other form. That is,

$$y = 13.8831e^{0.2423(2)} = 22.5395.$$

Observe the fitted curve in Figure 12.

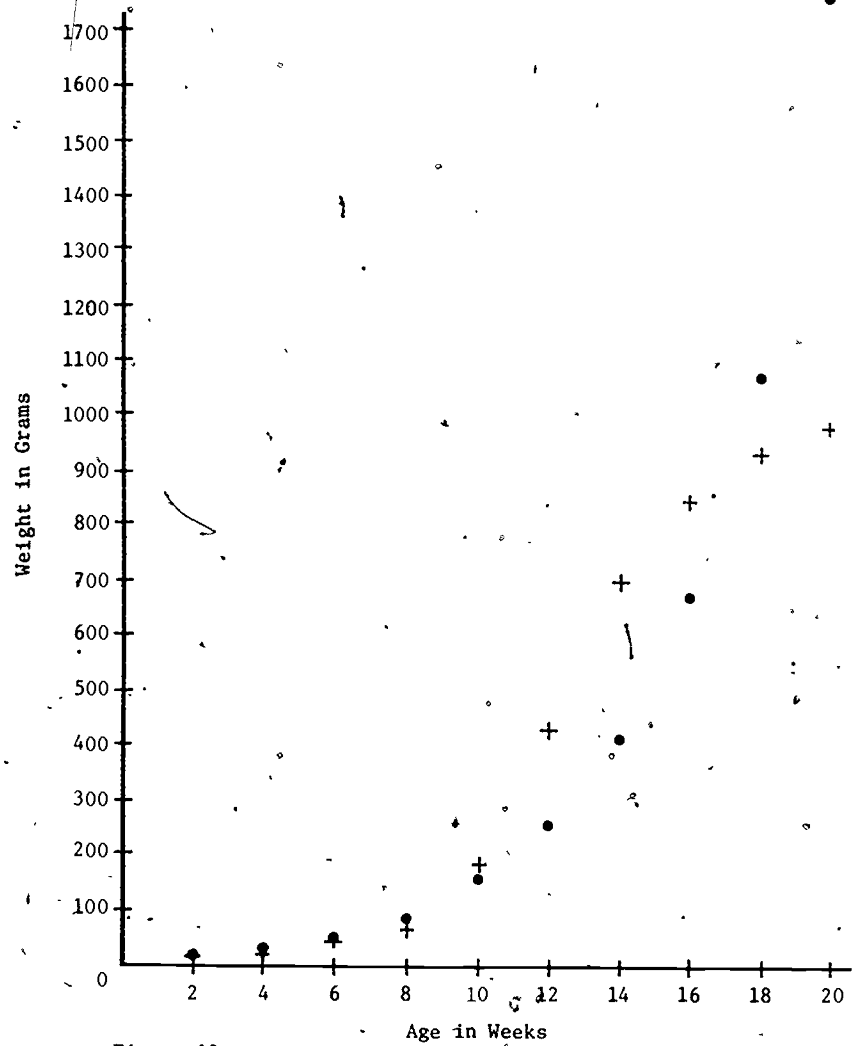


Figure 12.

Exercise 3.

Try to fit the data for the growth of the corn seedlings using 15 as a base instead of 10.

7. POLYNOMIAL SCATTERS

A disc was rolled down an inclined plane and the distance it travelled was measured after 0, 2, 4, ..., seconds. The results are organized in Table 10.

TABLE 10

Time (x)	0	2	4	6	8	10	12	14	16
Distance (y)	0	1	3	5	8	12	17	23	29

We give a graph of the data in Figure 13. Notice that it looks as if it could be fitted to an exponential. However, this data fits closer to a second degree polynomial or a parabola, $y = ax^2 + bx + c$.

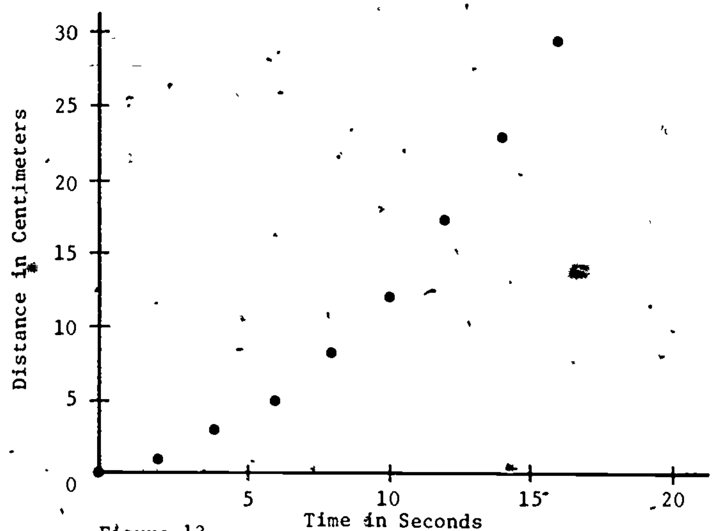


Figure 13.

In order to fit a polynomial we must do a little more mathematics. Notice, that we now have three constants to identify, namely a, b, and c.

We must consider minimizing the sum

$$\sum_{i=1}^{n} [y_i - (ax_i^2 + bx_i + c)]^2.$$

This means that we must calculate the partial derivatives of this sum with respect to a, b, and c. We set these derivatives equal to 0 and come up with three equations in three unknowns a, b, and c. That development is given below:

$$\frac{\partial}{\partial a} = \sum_{i=1}^{n} -2[y_i - ax_i^2 - bx_i - c]x_i^2.$$

$$= \sum_{i=1}^{n} (-2x_i^2 y_i + 2ax_i^4 + 2bx_i^3 + 2cx_i^2) = 0.$$

$$a\Sigma x_i^4 + b\Sigma x_i^3 + c\Sigma x_i^2 = \Sigma x_i^2 y_i.$$

$$\frac{\partial}{\partial b} = \sum_{i=1}^{n} -2[y_i - ax_i^2 - bx_i - c]x_i = 0.$$

$$a\Sigma x_i^3 + b\Sigma x_i^2 + c\Sigma x_i = \Sigma y_i x_i.$$

$$\frac{\partial}{\partial c} = \sum_{i=1}^{n} -2[y_i - ax_i^2 - bx_i - c](1) = 0.$$

$$a\Sigma x_i^2 + b\Sigma x_i + c\Sigma(1) = \Sigma y_i.$$

$$a\Sigma x_i^2 + b\Sigma x_i + nc = \Sigma y_i.$$

From these *normal* equations we can obtain a best parabolic fit. We must find the indicated sums Σx_i^4, Σx_i^3, Σx_i^2, Σx_i, $\Sigma x_i^2 y$, $\Sigma x_i y_i$, and Σy_i. We give these calculations below using the data from Table 11.

$$a(140352) + b(10368) + c(816) = 16324.$$
$$a(10368) + b(816) + c(72) = 1218.$$
$$a(816) + b(72) + c(9) = 98.$$

TABLE 11

x_i	x_i^2	x_i^3	x_i^4	y_i	$x_i y_i$	$x_i^2 y$
0	0	0	0	0	0	0
2	4	8	16	1	2	4
4	16	64	256	3	12	48
6	36	216	1296	5	30	180
8	64	512	4096	8	64	512
10	100	1000	10000	12	120	1200
12	144	1728	20736	17	204	2448
14	196	2744	38416	23	322	4508
16	256	4096	65536	29	464	7424
72	816	10368	140352	98	1218	16324

To solve this system, we can use an augmented matrix.*

$$\begin{bmatrix} 140352 & 10368 & 816 & 16324 \\ 10368 & 816 & 72 & 1218 \\ 816 & 72 & 9 & 98 \end{bmatrix}.$$

We first divide the top row through by 140352 to obtain 1 in the first row and first column:

$$\begin{bmatrix} 1 & 0.0739 & 0.0058 & 0.1163 \\ 10368 & 816 & 72 & 1218 \\ 816 & 72 & 9 & 98 \end{bmatrix}.$$

Next, we multiply the top row by -10368 and add it to the second row. Then, we multiply the top row by -816 and add it to the bottom row. The resulting matrix is given below:

*For a more detailed discussion on matrix manipulations see *Elementary Differential Equations with Linear Algebra* by Ross. L. Finney and Donald R. Ostberg.

$$\begin{bmatrix} 1 & 0.0739 & 0.0058 & 0.1163 \\ 0 & 423.0528 & 11.8656 & 12.2016 \\ 0 & 41.0736 & 4.2672 & 3.0992 \end{bmatrix}.$$

To continue we divide the second row by 423.0528 to obtain 1 in the second row, second column.

$$\begin{bmatrix} 1 & 0.0739 & 0.0058 & 0.1163 \\ 0 & 1 & 0.0280 & 0.0288 \\ 0 & 41.0736 & 4.2672 & 3.0992 \end{bmatrix}.$$

We now multiply the second row by -41.0736 and add it to the third row.

$$\begin{bmatrix} 1 & 0.0739 & 0.0058 & 0.1163 \\ 0 & 1 & 0.0280 & 0.0288 \\ 0 & 0 & 3.1172 & 1.9163 \end{bmatrix}.$$

We divide the last row by 3.1172 and obtain c from the system above:

$$\begin{bmatrix} 1 & 0.0739 & 0.0058 & 0.1163 \\ 0 & 1 & 0.0280 & 0.0288 \\ 0 & 0 & 1 & 0.6148 \end{bmatrix}.$$

Therefore,

$$c \doteq 0.6148.$$

$$b + 0.028(0.6148) = 0.0288$$

$$b + 0.0172 = 0.0288$$

$$b \doteq 0.0116.$$

$$a + 0.0739(0.0116) + 0.0058(0.6148) = 0.1163$$

$$a + 0.0008572 + 0.003566 = 0.1163$$

$$a = 0.1119.$$

Therefore the parabola of best fit is:

$$y = 0.1119x^2 + 0.0116x + 0.6148.$$

We obtain the y values below:

$x = 0,\quad y = 0.6148.$

$x = 2,\quad y = 0.1119(4) + 0.0116(2) + 0.6148$

$= 0.4475 + 0.0232 + 0.6148$

$= 1.0855.$

$x = 4,\quad y = 1.6683.$

$x = 6,\quad y = 4.9684.$

$x = 8,\quad y = 7.8692.$

$x = 10,\quad y = 11.9208.$

$x = 12,\quad y = 16.8676.$

$x = 14,\quad y = 22.1104.$

$x = 16,\quad y = 29.4468.$

When this data is graphed on the original set of axes, we see that we have a very close fit. (See Figure 14.)

Exercise 4.

Fit the data to an exponential. It should be convincing that the exponential does not fit as well as the parabola.

There are sets of data that produce scatters that fit higher order polynomials than 2. For example, the

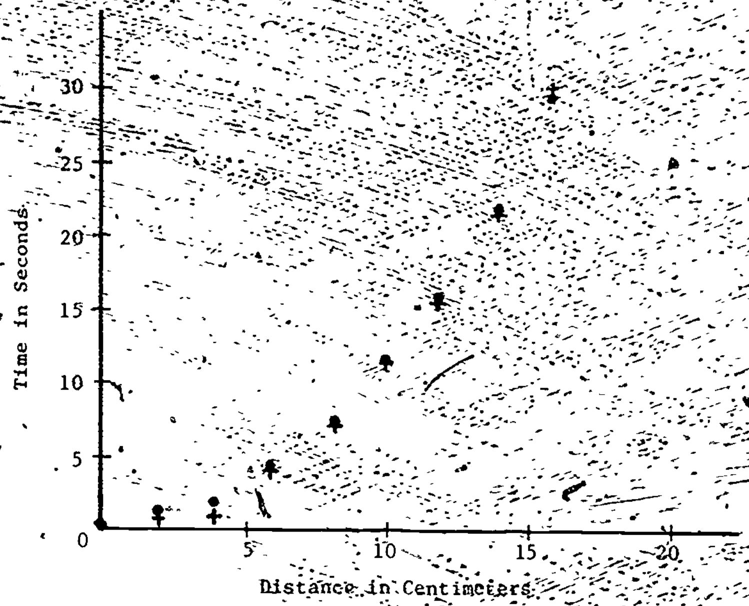


Figure 14.

corn seedling example in Section 6 might be fit with a cubic (i.e., $y = ax^3 + bx^2 + cx + d$).

However, this means that we would have to solve four equations in four unknowns (a, b, c, d). This is no small task. There are methods for finding the coefficients without going through all the work of the partial derivations, namely, the square root method and Gauss's method.

There is still a great deal of calculation to do even with these methods. In fact, all curve fitting requires a good deal of calculation. Now that we have computers, we can write programs to deal with any type of scatter.

We present, as an appendix, a BASIC program called "Super Fit." After going through this unit the reader should be comfortable with using the program. The

program uses the same procedures, but you are spared the calculations.

It should also be pointed out that in practice, the amount of data collected would more than likely be more extensive. We have also kept the numbers reasonably small.

With a computer program to do the work, we can enter a large number of data and the numbers can be either very large, or very small.

There are other functions such as powers and powers raised to powers that can be employed, and data fitted to them (i.e., $y = cx^n$, $y = cx^{nb}$, etc.). Appropriate manipulation of the data can be employed to handle these situations. The basic mathematics of the least square method can still be used. Hopefully, this material has given enough background so that virtually any type of scatter can be fitted.

8. MODEL EXAM

1. Given the data in the table below construct a scatter diagram:

x	2	8	7	10	15	18	15	18	23	23	25	25	30	32	35
y	2	2	8	15	10	5	20	16	10	15	20	25	23	27	25

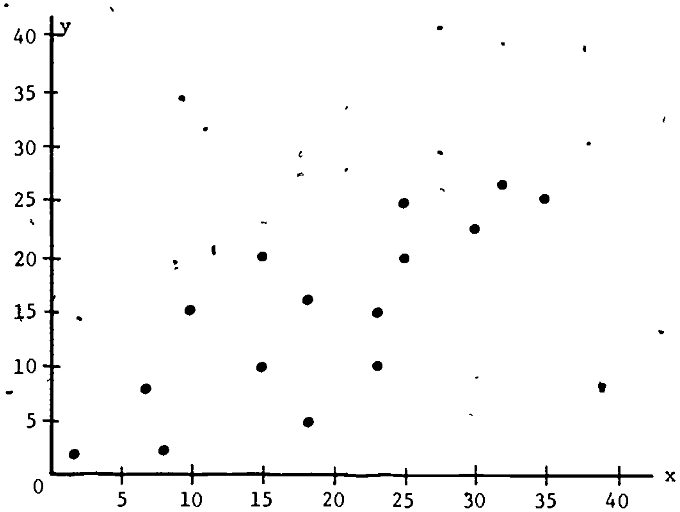
2. For the data given in Question 1, is the coefficient of correlation positive, negative, or zero? Fit a line by eye through the points of the scatter diagram that was constructed for Question 1. Fit a line through the data using the least square technique.

3. Given the parabola $y = 2x^2 + 3$, let x take on the values 1, 2, 3, 4, 5, 6, and 7. Find the corresponding y values. Which type of function—logarithmic or exponential—will best fit the given parabola?

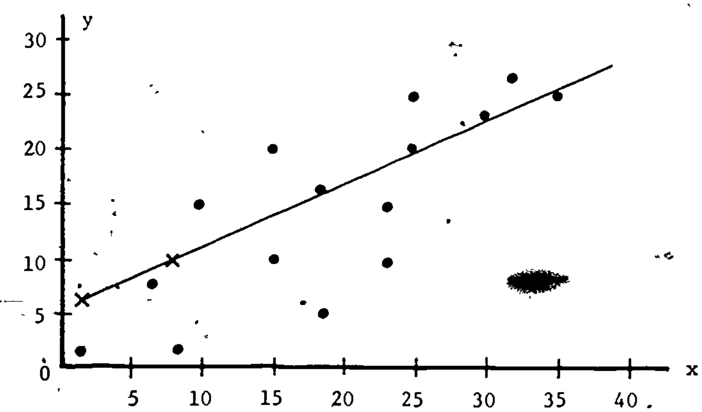
4. Fit the data in Question 3 to either a logarithmic or exponential curve depending on your choice from Question 3.

9. ANSWERS TO MODEL EXAM

1.



2. Positive.



x	y	x^2	xy
2	2	4	4
8	2	16	16
7	8	49	56
10	15	100	150
15	10	225	150
18	5	324	90
15	20	225	300
18	16	324	288
23	10	529	230
23	15	529	345
25	20	625	500
25	25	625	625
30	23	900	690
32	27	1024	864
35	25	1225	875
253	223	6724	5183

$$6724b + 253c = 5183$$
$$253b + 15c = 223$$

$$c = \frac{223 - 253b}{15}$$

$$6724b + 253\left(\frac{223 - 253b}{15}\right) = 5183$$
$$6724(15)b + 253(223) - 253^2 b = 5183(15)$$
$$100860b + 56419 - 64009b = 77745$$
$$36851b = 21316$$
$$b = 0.5787.$$

Since $c = \frac{223 - 253b}{15}$ we have $c = \frac{223 - 253(0.5787)}{15}$

therefore

$$c = 5.1059.$$

The line of regression is

$$y = 0.5787x + 5.1059.$$

3. $y = \{5,11,21,35,53,75,101\}$

An exponential would fit best.

4.

x	y	$\log_e y$	x^2	$x \log_e y$
1	5	1.6094	1	1.6094
2	11	2.3979	4	4.7958
3	21	3.0445	9	9.1335
4	35	3.5553	16	14.2212
5	53	3.9703	25	19.8515
6	75	4.3175	36	25.9050
7	101	4.6151	49	32.3057
28		23.5100	140	107.8221

$$140b + 28\log_e c = 107.8221$$

$$28b + 7\log_e c = 23.5100$$

$$\log_e c = \frac{23.51 - 28b}{7}$$

$$140b + 28\left(\frac{23.51 - 28b}{7}\right) = 107.8221$$

$$7(140)b + 28(23.51) - 28^2 b = 7(107.8221)$$

$$980b + 658.28 - 784b = 754.7547$$

$$196b = 96.4747$$

$$b = 0.4922.$$

$$\log_e c = \frac{23.51 - 28(0.4922)}{7} = \frac{9.7284}{7} = 1.3898$$

therefore

$$c \doteq 4.0139.$$

We have $y = ce^{bx}$, which yields $y = 4.0139e^{0.4922x}$

so if

$x = 1$

$y = 4.0139e^{0.4922(1)} = 4.0139(1.6359) = 6.5664$

$x = 2$

$y = 4.0139e^{0.4922(2)} = 4.0139(2.6762) = 10.742$

$x = 3$

$y = 4.0139e^{0.4922(3)} = 4.0139(4.378) = 17.573$

$x = 4$

$y = 4.0139e^{0.4922(4)} = 4.0139(7.162) = 28.7479$

$x = 5$

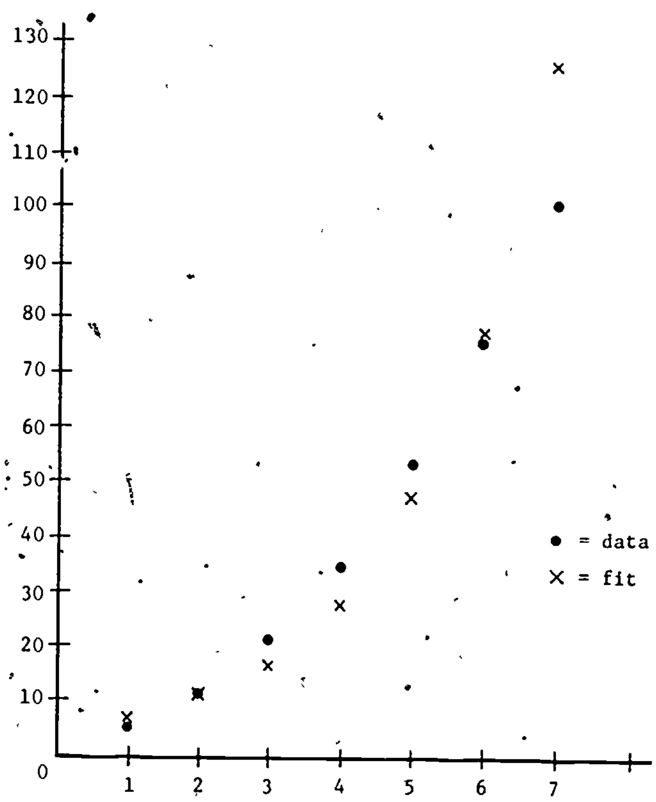
$y = 4.0139e^{0.4922(5)} = 4.0139(11.7165) = 47.0289$

$x = 6$

$y = 4.0139e^{0.4922(6)} = 4.0139(19.1672) = 76.9352$

$x = 7$

$y = 4.0139e^{0.4922(7)} = 4.0139(31.3558) = 125.8491.$



10. ANSWERS TO EXERCISES

1. Given

(3) $$b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

(4) $$b\sum_{i=1}^{n} x_i + nc = \sum_{i=1}^{n} y_i,$$

using (4) we can write:

$$nc = \sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i$$

$$\therefore \quad c = \frac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n}$$

Substitute this value of c into (3):

$$b\sum_{i=1}^{n} x_i^2 + \left(\frac{\sum_{i=1}^{n} y_i - b\sum_{i=1}^{n} x_i}{n}\right)\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i .$$

Multiply the equation obtained by n and remove the parentheses using the distributive law:

$$nb\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i = n\sum_{i=1}^{n} x_i y_i .$$

Further we can write:

$$b\left(n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right) = n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i$$

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i}{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)^2}$$

To solve for b' we do a similar procedure:

Given

(3') $$b'\sum_{i=1}^{n} y_i^2 + c\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i x_i$$

(4') $$b'\sum_{i=1}^{n} y_i + nc = \sum_{i=1}^{n} x_i,$$

using (4') we can write:

$$nc = \sum_{i=1}^{n} x_i - b'\sum_{i=1}^{n} y_i$$

$$\therefore \quad c = \frac{\sum_{i=1}^{n} x_i - b'\sum_{i=1}^{n} y_i}{n}$$

Substitute the value of c into (3'):

$$b'\sum_{i=1}^{n} y_i^2 + \left(\frac{\sum_{i=1}^{n} x_i - b'\sum_{i=1}^{n} y_i}{n}\right)\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i x_i$$

$$nb'\sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - b'\sum_{i=1}^{n} y_i \sum_{i=1}^{n} y_i = n\sum_{i=1}^{n} y_i x_i$$

$$b'\left(n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2\right) = n\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i$$

$$b' = \frac{n\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}$$

2.

x	$\log_e x$	$(\log_e x)^2$	y	$(\log_e x)y$
1	0	0	12	0
6	1.7918	3.2104	42	75.2556
11	2.3979	5.7499	53	127.0887
16	2.7726	7.6872	71	196.8546
21	3.0445	9.2691	76	231.3820
55	10.0068	25.9166	254	630.5809

Use

$$b\sum(\log_e x)^2 + c\sum\log_e x = \sum(\log_e x)y$$

$$b\sum\log_e x \quad + \quad cn = \sum y.$$

Therefore we can write:

$$25.9166b + 10.0068c = 630.5809$$
$$10.0068b + 5c = 254$$

$$c = \frac{254 - 10.0068b}{5}$$

$$\therefore \quad 25.9166b + 10.0068\left(\frac{254 - 10.0068b}{5}\right) = 630.5809$$

$$5(25.9166b) + 10.0068(254 - 10.0068b) = 5(630.5809)$$

$$129.583b + 2541.7272 + 100.136 = 3152.9045$$

$$129.583b = 511.0413$$

$$b = 3.9437.$$

Hence,

$$c = \frac{254 - 10.0068(3.9437)}{5}$$

$$= 42.9072$$

and

$$y = 3.9437\log_e x + 42.9072.$$

Let

$x = 1$, then $y = 3.9437(0) + 42.9072$

$= 42.9072$

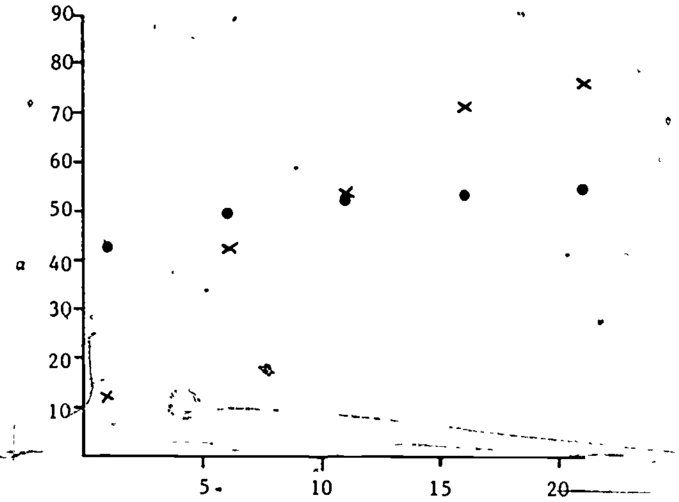
$x = 6$, $y = 3.9437(1.7918) + 42.9072$

$= 49.9735$

$x = 11$, $y = 52.3638$

$x = 16$, $y = 53.8415$

$x = 21$, $y = 54.9138.$



Graph for solution to Exercise 2.

3. We could use any base and obtain the same fit. The calculations are given for a base 10 (common logs) fit. Notice that the y values are virtually the same as those obtained for $\log_e$.

x	y	$\log_{10}y$	x^2	$x(\log_{10}y)$
2	21	1.3222	4	2.6444
4	28	1.4472	16	5.7888
6	58	1.7634	36	10.0580
8	76	1.8808	64	15.0464
10	170	2.2304	100	22.3040
12	422	2.6253	144	31.5036
14	706	2.8488	196	39.8832
16	853	2.9309	256	46.8944
18	924	2.9657	324	53.3826
20	966	2.9850	400	59.7000
110		22.9997	1540	287.2054

$$1540b + 110 \log c = 287.2054$$

$$110b + 10 \log c = 22.9997$$

$$b = \frac{22.9997 - 10 \log c}{110} = \frac{22.9997 - 11.5968}{110} = 0.10366$$

therefore

$$1540\left(\frac{22.9997 - 10 \log c}{110}\right) + 110 \log c = 287.2054$$

$$35419.538 - 15400 \log c + 12100 \log c = 31592.594$$

$$3300 \log c = 3826.944$$

$$\log c = 1.15968$$

$$c \doteq 14.444.$$

$x = 20$

$y = 14.444(10)^{0.10366(20)} = 14.444(10)^{2.0732}$

$= 14.444(118.3586) = 1709.5716.$

$x = 18$

$y = 1060.6879$

$x = 16$

$y = 658.0932$

$x = 14$

$y = 408.2134$

$x = 12$

$y = 253.2727$

$x = 10$

$y = 157.1406$

$x = 8$

$y = 97.4956$

$x = 6$

$y = 60.49$

$x = 4$

$y = 37.5226$

$x = 2$

$y = 23.2808$

4.

x	y	$\log_e y$	x^2	$x(\log_e y)$
0	0	$-\infty$		
2	1	0	4	0
4	3	1.0986	16	4.3944
6	5	1.6094	36	9.6564
8	8	2.0794	64	16.6352
10	12	2.4849	100	24.8490
12	17	2.8332	144	33.9984
14	23	3.1354	196	43.8956
16	29	3.3673	256	53.8768
72	98	16.6082	816	187.3058

Keeping the equation

$$\log_e y = bx + \log_e c$$

$$816b + 72 \log_e c = 187.3058$$

$$72b + 8 \log_e c = 16.6082$$

$$\log_e c = \frac{16.6082 - 72b}{8}$$

$$816b + 72\left(\frac{16.6082 - 72b}{8}\right) = 187.3058$$

$$6528b + 1195.7904 - 5184b = 1498.4464$$

$$1344b = 302.656$$

$$b = .2252$$

$$\log_e c = \frac{16.6082 - 72(.2252)}{8}$$

$$= .0492$$

$\therefore$ the fitted curve is:

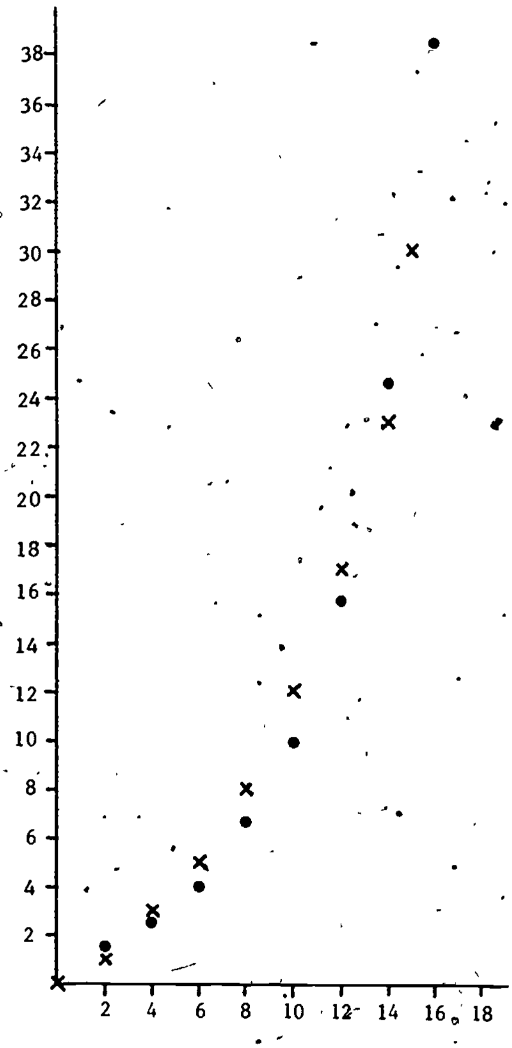
$$\log_e y = .2252x + .0492.$$

for $x = 2$, $\log_e y = .2252(2) + .0492$

$= .4996$

$y = e^{.4996} \doteq 1.6481$

x = 4,	y = 2.5857	x = 12,	y = 15.6677
x = 6,	y = 4.0568	x = 14,	y = 24.5818
x = 8,	y = 6.6859	x = 16,	y = 38.5673
x = 10,	y = 9.9742		



Graph for solution to Exercise 4.

APPENDIX*

Once the program listed here is loaded into a computer, it will be a simple matter to do curve fitting. The program is interactive. The user will be prompted to give the needed information (i.e., all x and y values).

This program fits given data to the following types of curves and plots them:

(1) Linear $y = mx + b$;

(2) Exponential $y = ce^{mx}$;

(3) Logarithmic $y = m \log x + b$;

(4) Power $y = cx^n$; and

(5) Polynomial $y = a_0 + a_1x + a_2x_2 + a_nx^h$.

Note that Equation (2) can be written as $\log y = mx + \log c$ and Equation (4) can be written as $\log y = n \log x + \log c$. Thus, Equation (2), (3) and (4) can be reduced to linear equations by simple substitutions.

*The material in this appendix is adapted from *Technical Data for BASIC Programs*, Preliminary Version, July 1974, developed by Project CALC/Education Development Center, Inc.

SUPER FIT

```
LIST
1 LET F8=0
2 LET N5=200
3 LET P1=3.14159
4 DIM C$(3),D$(1)
5 PRINT
6 PRINT "SUPER FIT."
13 PRINT
14 PRINT "A• MINIMUM X=";
15 INPUT I9
16 IF I9<>998 THEN 19
17 GOSUB 980
18 GOTO 14
19 IF I9=999 THEN 14
20 IF I9=997 THEN 900
21 LET L9=I9
22 PRINT "   MAXIMUM X=";
23 INPUT I9
24 IF I9<>998 THEN 27
25 GOSUB 980
26 GOTO 22
27 IF I9=999 THEN 22
28 IF I9=997 THEN 900
29 LET R9=I9
30 IF R9>L9 THEN 33
31 PRINT "ERROR: MAXIMUM X MUST BE GREATER THAN MINIMUM X."
32 GOTO 14
33 PRINT "   MINIMUM Y=";
34 INPUT I9
35 IF I9<>998 THEN 38
36 GOSUB 980
37 GOTO 33
38 IF I9=999 THEN 33
39 IF I9=997 THEN 900
40 LET B9=I9
41 PRINT "   MAXIMUM Y=";
42 INPUT I9
43 IF I9<>998 THEN 46
44 GOSUB 980
45 GOTO 41
46 IF I9=999 THEN 41
47 IF I9=997 THEN 900
48 LET T9=I9
49 IF T9>B9 THEN 70
50 PRINT "ERROR: MAXIMUM Y MUST BE GREATER THAN MINIMUM Y."
51 GOTO 33
70 GOSUB 800
94 PRINT
95 PRINT "   X GLITCH=";G8
96 PRINT "   Y GLITCH=";G9
97 GOSUB 920
```

```
98 LET E8=1
100 REM
101 REM         PREPARE FOR DATA
102 REM
109 LET E9=(R9-L9)/200
110 LET E8=(T9-E9)/200
150 DIM S(4),T(4),U(4),V(4),N(4),M(4),P(4)
160 DIM A(11,12),O(11),R(21)
165 LET D8=10
175 FOR I6=1 TO 4
176 LET S(I6)=0
177 LET T(I6)=0
178 LET U(I6)=0
179 LET V(I6)=0
180 LET M(I6)=0
181 LET P(I6)=0
182 LET N(I6)=0
183 NEXT I6
185 FOR I6=1 TO 21
186 LET R(I6)=0
187 IF I6>11 THEN 189
188 LET O(I6)=0
189 NEXT I6
198 REM
199 REM       INPUT DATA
200 REM
201 PRINT
202 PRINT
203 LET W9=1
204 PRINT "D: INPUT DATA."
205 PRINT
206 PRINT "   X=";
207 INPUT I9
208 IF I9<>998 THEN 211
209 GOSUB 920
210 GOTO 206
211 IF I9=999 THEN 225
212 IF I9=997 THEN 900
213 LET X9=I9
214 PRINT "   Y=";
215 INPUT I9
216 IF I9<>998 THEN 219
217 GOSUB 920
218 GOTO 214
219 IF I9=999 THEN 225
220 IF I9=997 THEN 900
221 LET Y9=I9
222 GOSUB 600
223 GOTO 205
224 LET W9=1
225 LET W9=-W9
226 IF W9=1 THEN 249
227 PRINT
228 PRINT "E: ERASE DATA."
229 GOTO 205
248 GOSUB 500
249 PRINT
```

```
383 PRINT
391 PRINT "     DEGREE OF POLYNOMIAL=";
392 INPUT I9
393 IF I9=999 THEN 249
394 IF I9=998 THEN 391
395 IF I9=997 THEN 391
398 IF I9<=D8 THEN 401
399 PRINT "ERROR: DEGREE MUST BE AN INTEGER BETWEEN 0 AND";D8
400 GOTO 391
401 IF INT(I9)<>I9 THEN 399
402 LET D7=I9
403 IF R(1)>=D7+1 THEN 410
404 PRINT "ERROR: NOT ENOUGH DATA FOR DEGREE";D7;"POLYNOMIAL FIT."
405 GOTO 391
407 REM
408 REM       FIT POLYNOMIAL
409 REM
410 FOR I6=1 TO D7+1
411 FOR J6=1 TO D7+1
412 LET A(I6,J6)=R(I6+J6-1)
413 NEXT J6
414 LET A(I6,D7+2)=Q(I6)
415 NEXT I6
419 IF D7=0 THEN 470
420 FOR I6=1 TO D7+1
421 IF A(I6,I6)<>0 THEN 450
429 IF I6=D7+1 THEN 399
430 FOR J6=I6+1 TO D7+1
431 IF A(J6,I6)<>0 THEN 440
432 NEXT J6
433 GOTO 399
440 FOR K6=1 TO D7+2
441 LET T7=A(I6,K6)
442 LET A(I6,K6)=A(J6,K6)
443 LET A(J6,K6)=T7
444 NEXT K6
450 LET T7=A(I6,I6)
451 FOR J6=1 TO D7+2
452 LET A(I6,J6)=A(I6,J6)/T7
453 NEXT J6
455 FOR J6=1 TO D7+1
456 IF J6=I6 THEN 465
460 LET R7=A(J6,I6)
461 FOR K6=1 TO D7+2
462 LET A(J6,K6)=A(J6,K6)-R7*A(I6,K6)
463 NEXT K6
465 NEXT J6
466 NEXT I6
470 IF A(D7+1,D7+1)=0 THEN 399
471 LET A(D7+1,D7+2)=A(D7+1,D7+2)/A(D7+1,D7+1)
473 PRINT "        THE POLYNOMIAL IS:"
474 PRINT "        Y=";A(1,D7+2)
475 IF D7=0 THEN 383
476 LET D$="+"
477 LET A6=ABS(A(2,D7+2))
478 IF A6=A(2,D7+2) THEN 480
479 LET D$="-"
```

```
250 PRINT "F: FIT DATA WITH CURVE."
251 PRINT "   TYPE OF CURVE";
252 INPUT C$
253 IF C$="LIN" THEN 307
254 IF C$="EXP" THEN 328
255 IF C$="LOG" THEN 345
256 IF C$="POW" THEN 366
257 IF C$="POL" THEN 381
258 IF C$<>"998" THEN 261
259 GOSUB 920
260 GOTO 251
261 IF C$="999" THEN 1010
262 IF C$="997" THEN 900
263 GOTO 251
300 REM
301 REM          FIT CURVES
302 REM
307 LET G6=1
308 PRINT "   LINEAR FIT."
317 GOSUB 1100
318 IF F9=0 THEN 327
319 LET D$="+"
320 LET A6=ABS(B(1))
321 IF A6=B(1) THEN 323
322 LET D$="-"
323 PRINT "        Y=";M(1);"*X";
324 PRINT D$;
325 PRINT A6
327 GOTO 248
328 LET G6=2
329 PRINT "   EXPONENTIAL FIT."
338 GOSUB 1100
339 IF F9=0 THEN 344
340 PRINT "        Y=";EXP(B(2));"*EXP(";M(2);"*X)"
341 PRINT "       WHICH IS EQUIVALENT TO:"
342 PRINT "        Y=";EXP(B(2));"*(";EXP(M(2));"↑X)"
344 GOTO 248
345 LET G6=3
346 PRINT "   LOGARITHMIC FIT."
355 GOSUB 1100
356 IF F9=0 THEN 365
357 LET D$="+"
358 LET A6=ABS(B(3))
359 IF A6=B(3) THEN 361
360 LET D$="-"
361 PRINT "        Y=";M(3);"*LN(X)";
362 PRINT D$;
363 PRINT A6
365 GOTO 248
366 LET G6=4
367 PRINT "   POWER FIT."
376 GOSUB 1100
377 IF F9=0 THEN 380
378 PRINT "        Y=";EXP(B(4));"*(X↑";M(4);")"
380 GOTO 248
381 LET G6=5
382 PRINT "   POLYNOMIAL FIT."
```

```
632 LET V(2)=V(2)+W9*X9
633 LET U(2)=U(2)+W9*Y8
634 LET T(2)=T(2)+W9*X9*Y8
635 LET S(2)=S(2)+W9*X9*X9
636 LET N(2)=N(2)+W9
640 IF X9<=0 THEN 650
641 IF Y9<=0 THEN 650
642 LET X8=LOG(X9)
643 LET Y8=LOG(Y9)
644 LET V(4)=V(4)+W9*X8
645 LET U(4)=U(4)+W9*Y8
646 LET T(4)=T(4)+W9*X8*Y8
647 LET S(4)=S(4)+W9*X8*X8
648 LET N(4)=N(4)+W9
650 IF X9<=0 THEN 660
651 LET X8=LOG(X9)
652 LET V(3)=V(3)+W9*X8
653 LET U(3)=U(3)+W9*Y9
654 LET T(3)=T(3)+W9*X8*Y9
655 LET S(3)=S(3)+W9*X8*X8
656 LET N(3)=N(3)+W9
660 LET X8=Y9*W9
661 FOR I6=1 TO D8+1
662 LET Q(I6)=Q(I6)+X8
663 LET X8=X8*X9
664 NEXT I6
670 LET X8=W9
671 FOR I6=1 TO 2*D8+1
672 LET R(I6)=R(I6)+X8
673 LET X8=X8*X9
674 NEXT I6
700 REM
701 REM PLOT POINT
702 REM
710 GOSUB 940
711 GOSUB 950
720 LET H9=X9-E9
721 LET V9=Y9-(1-W9)*E8/2
722 GOSUB 960
730 LET H9=X9+E9
731 LET V9=Y9+(1-W9)*E8/2
732 GOSUB 970
740 LET H9=X9+(1-W9)*E9/2
741 LET V9=Y9-E8
742 GOSUB 960
750 LET H9=X9-(1-W9)*E9/2
751 LET V9=Y9+E8
752 GOSUB 970
760 GOSUB 990
765 RETURN
797 REM
798 REM       SUBROUTINE TO DRAW AXES
799 REM
800 LET Z7=INT(LOG((R9-L9)/10)/LOG(10)+100)-100
801 LET Z8=(R9-L9)/(10↑(Z7+1))
802 LET Z5=INT(LOG((T9-B9)/10)/LOG(10)+100)-100
803 LET Z6=(T9-B9)/(10↑(Z5+1))
```

```
480 PRINT "        ";
481 PRINT D$;
482 PRINT A6;"X"
483 IF D7=1 THEN 383
484 FOR I6=3 TO D7+1
485 LET A6=ABS(A(I6,D7+2))
486 LET D$="+"
487 IF A6=A(I6,D7+2) THEN 489
488 LET D$="-"
489 PRINT "        ";
490 PRINT D$;
491 PRINT A6;"X↑";I6-1
492 NEXT I6
494 GOSUB 500
495 GOTO 383
500 REM
501 REM          PLOT FITTED CURVE
502 REM
510 GOSUB 940
511 GOSUB 950
522 FOR H9=L9 TO R9 STEP (R9-L9)/N5
530 IF G6>1 THEN 540
531 LET V9=M(1)*H9+B(1)
532 GOTO 580
540 IF G6>2 THEN 550
541 LET V9=EXP(M(2)*H9+B(2))
542 GOTO 580
550 IF G6>3 THEN 560
551 LET V9=T9+1
552 IF H9<=0 THEN 580
553 LET V9=M(3)*LOG(H9)+B(3)
554 GOTO 580
560 IF G6>4 THEN 570
561 LET V9=T9+1
562 IF H9<=0 THEN 580
563 LET V9=EXP(M(4)*LOG(H9)+B(4))
564 GOTO 580
570 LET X8=1
571 LET V9=0
572 FOR I6=1 TO D7+1
573 LET V9=V9+A(I6,D7+2)*X8
574 LET X8=X8*H9
575 NEXT I6
580 GOSUB 970
590 NEXT H9
592 GOSUB 990
595 RETURN
600 REM
601 REM          CALCULATE SIGMAS
602 REM
620 LET V(1)=V(1)+W9*X9
621 LET U(1)=U(1)+W9*Y9
622 LET T(1)=T(1)+W9*X9*Y9
623 LET S(1)=S(1)+W9*X9*X9
624 LET N(1)=N(1)+W9
630 IF Y9<=0 THEN 640
631 LET Y8=LOG(Y9)
```

```
866 LET H9=H9+G7
867 GOSUB 970
868 LET H9=H9-G7
869 IF L9>=1 THEN 873
870 LET H9=H9-G7
871 GOSUB 970
872 LET H9=H9+G7
873 GOSUB 970
874 NEXT V9
875 LET V9=T9
876 GOSUB 970
880 GOSUB 990
890 RETURN
900 PRINT
901 IF F3<>0 THEN 904
902 PRINT "WARNING: SINCE YOU HAVEN'T ENTERED YOUR AXIS LIMITS"
903 PRINT "  (FOR THE FIRST TIME), YOU'D BETTER ENTER A."
904 PRINT
905 PRINT "GO TO A,D,E,F OR X";
906 INPUT C$
907 IF C$(1,1)="A" THEN 13
908 IF C$(1,1)="D" THEN 200
909 IF C$(1,1)="E" THEN 224
910 IF C$(1,1)="F" THEN 249
911 IF C$(1,1)="X" THEN 1010
912 GOTO 905
920 GOSUB 980
921 GOSUB 830
939 RETURN
940 REM  TURN ON PLOTTER/AXIS LIMITS
949 RETURN
950 REM  LIFT PEN
959 RETURN
960 REM  PLOT POINT
969 RETURN
970 REM  PLOT LINE
979 RETURN
980 REM  ERASE PICTURE
989 RETURN
990 REM  TURN OFF PLOTTER
999 RETURN
1010 REM
1011 REM          SUBROUTINE TO ERASE ALL DATA
1012 REM
1020 PRINT
1021 PRINT "X: ERASE ALL DATA";
1022 INPUT C$
1023 IF C$(1,1)="N" THEN 200
1024 IF C$(1,1)="Y" THEN 1033
1025 IF C$<>"998" THEN 1028
1026 GOSUB 920
1027 GOTO 1021
1028 IF C$="999" THEN 1021
1029 IF C$="997" THEN 900
1032 GOTO 1021
1033 FOR I6=1 TO 4
1034 LET S(I6)=0
```

```
804 LET G8=10↑Z7
805 IF Z8>5.0001 THEN 813
806 IF Z8>2.0001 THEN 811
807 IF Z8>1.0001 THEN 809
808 GOTO 814
809 LET G8=2*G8
810 GOTO 814
811 LET G8=5*G8
812 GOTO 814
813 LET G8=10*G8
814 LET G9=10↑Z5
815 IF Z6>5.0001 THEN 823
816 IF Z6>2.0001 THEN 821
817 IF Z6>1.0001 THEN 819
818 GOTO 824
819 LET G9=2*G9
820 GOTO 824
821 LET G9=5*G9
822 GOTO 824
823 LET G9=10*G9
827 RETURN
830 GOSUB 940
831 LET V9=0
832 IF B9<=0 THEN 834
833 LET V9=B9
834 IF T9>=0 THEN 836
835 LET V9=T9
836 LET H9=R9
837 GOSUB 960
838 LET G7=(T9-B9)/200
839 FOR H9=INT(L9/G8-.1)*G8 TO R9+G8/10 STEP G8
840 GOSUB 970
841 IF T9<=0 THEN 846
842 LET V9=V9+G7
843 GOSUB 970
844 LET V9=V9-G7
845 IF B9>=0 THEN 849
846 LET V9=V9-G7
847 GOSUB 970
848 LET V9=V9+G7
849 GOSUB 970
850 NEXT H9
851 LET H9=R9
852 GOSUB 970
853 LET V9=T9
854 LET H9=0
855 IF L9<=0 THEN 857
856 LET H9=L9
857 IF R9>=0 THEN 859
858 LET H9=R9
859 GOSUB 960
860 LET V9=B9
861 GOSUB 970
862 LET G7=(R9-L9)/200
863 FOR V9=INT(B9/G9-.1)*G9 TO T9+G9/10 STEP G9
864 GOSUB 970
865 IF B9<=0 THEN 869
```

```
1035 LET T(I6)=0
1036 LET U(I6)=0
1037 LET V(I6)=0
1038 LET M(I6)=0
1039 LET B(I6)=0
1040 LET N(I6)=0
1041 NEXT I6
1042 FOR I6=1 TO D8+1
1043 LET Q(I6)=0
1044 NEXT I6
1045 FOR I6=1 TO 2*D8+1
1046 LET P(I6)=0
1047 NEXT I6
1049 GOTO 200
1100 REM
1101 REM          SUBROUTINE TO FIT CURVE
1102 REM
1109 PRINT
1110 IF N(G6)>1 THEN 1120
1111 LET F9=0
1112 PRINT "NOT ENOUGH DATA FOR FIT"
1113 GOTO 1130
1120 LET D9=N(G6)*S(G6)-V(G6)*V(G6)
1121 LET M(G6)=(N(G6)*T(G6)-V(G6)*U(G6))/D9
1122 LET B(G6)=(S(G6)*U(G6)-T(G6)*V(G6))/D9
1123 LET F9=1
1130 RETURN
2000 END
```

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name______________________________ Unit No.______

Page______ ○ Upper ○ Middle ○ Lower	OR	Section______ Paragraph______	OR	Model Exam Problem No.______ Text Problem No.______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Name______________________ Unit No.__________ Date__________

Institution______________________ Course No.______________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain______________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain______________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

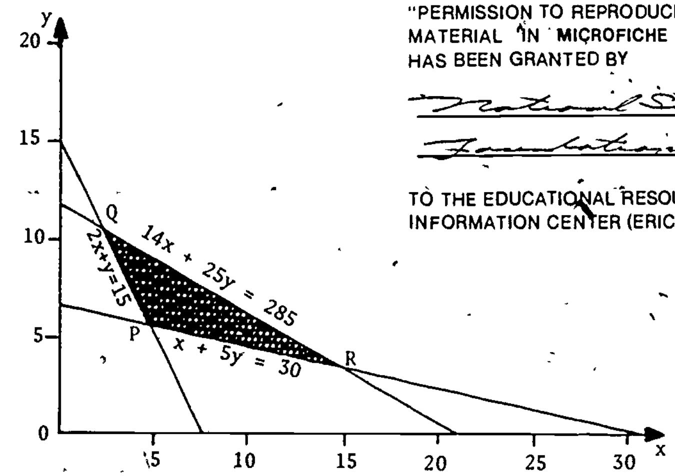
umap

UNIT 453

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

LINEAR PROGRAMMING IN TWO DIMENSIONS: I

by Nancy S. Rosenberg



APPLICATIONS OF HIGH SCHOOL ALGEBRA TO OPERATIONS RESEARCH

edc/umap 55chapel st. newton, mass 02160

LINEAR PROGRAMMING IN TWO DIMENSIONS: I

by

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Riverdale Country School
Bronx, N.Y. 10471

TABLE OF CONTENTS

Intermodular Description Sheet: UMAP Unit 453

Title: LINEAR PROGRAMMING IN TWO DIMENSIONS: I

Author: Nancy S. Rosenberg
Riverdale Country School
Bronx, NY 10471

Review Stage/Date: 2/8/80

Classification: APPL HIGH SCHL ALG/OPERATIONS RESEARCH

Prerequisite Skills:

1. Understand what is meant by "first degree equation."
2. Be able to graph linear equations and inequalities.
3. Be able to graph and solve simultaneous sets of linear equations in two unknowns.

Output Skills:

1. Be able to formulate simple linear programming problems and solve them graphically.

This module was written with support from The Klingenstein Center, Teachers College, Columbia University.

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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1. LINEAR PROGRAMMING PROBLEMS

1.1 Examples of Linear Programming Problems

Several years ago, a major grain supplier decided to produce chicken feed from a mixture of grains and food supplements. Each of the possible ingredients had a different price, and each contained different proportions of the various nutrients that chickens need each day. The question was this: Which ingredients, in which proportions, should be combined to meet the nutritional needs of the chickens as inexpensively as possible?

The producers of a Broadway musical were designing an advertising campaign. They planned to advertise through several different media. Each type of advertisement was known to reach different numbers of people in various income brackets, and each had a different cost. The producers knew how many people they had to reach in each income bracket if the campaign were to be successful. How should they distribute their advertising dollars among the various media in order to have an effective campaign at the minimum possible cost?

A farmer planned to grow several crops, each of which required different amounts of irrigation and acreage. In addition, the labor costs associated with each crop were different, as were the selling prices. Naturally, the farmer had limited amounts of water, land, and capital available. How much of each crop should she plant in order to maximize her profits?

1.2 The Characteristics of Linear Programming Problems

What do these three problems have in common? First, they all involve quantities that can be assigned a whole range of possible values at the will of the problem solver. The grain supplier can decide which ingredients he will use and in which proportions he will use them. The

Broadway producers can choose to run different numbers of advertisements on radio and television or in newspapers and magazines. The farmer can plant varying amounts of many possible crops. These are called *controllable variables*. Second, all three problems involve conditions that limit the range of values that these variables can assume. The grain supplier must meet the nutritional needs of the chickens, the producers must reach certain numbers of people, and the farmer must stay within the limits of the available water, capital, and land. These are the *constraints*. Third, each problem has as its object the minimization or maximization of a critical quantity. The grain supplier and the producers wish to minimize their costs; the farmer wants to maximize her profits. Taken together, these are some of the major characteristics of linear programming problems.

1.3 What is Linear Programming?

Linear programming is a mathematical technique for achieving the best possible results in a situation that is governed by restrictions. It is not to be confused with computer programming, which is programming of an entirely different sort. Of the many quantitative procedures that are now used as aids in decision making, linear programming is one of the most successful. It is applicable to a wide variety of situations, and it has already helped to save many millions of dollars.

The word "linear" refers to the fact that the mathematical equations used in a linear program are equations of the first degree. In two dimensions, these are the equations of straight lines. Anyone who can graph linear equations and inequalities in a two-dimensional coordinate system and solve them simultaneously can learn to solve simple linear programming problems.

2. A SIMPLE PROBLEM IN LINEAR PROGRAMMING

2.1 Formulating the Problem

Let us return to the problem of producing an economical feed for chickens. For the sake of simplicity, we will consider just two of the feed's ingredients, corn and alfalfa. (Although the reasoning used here is similar to that used in solving real life problems, the numbers have been altered to simplify the computations.)

Suppose that corn is priced at 6¢ a pound, alfalfa at 8¢ a pound. Each pound of corn contains 2 mg of protein, 1 mg of thiamine, and 14 mg of fat. (Mg stands for milligram, a very small unit of weight. There are 1000 milligrams in a gram and 28.4 grams in an ounce.) Each pound of alfalfa contains 1 mg of protein, 5 mg of thiamine, and 25 mg of fat. Animal nutritionists have determined that chickens require, at a minimum, 15 mg of protein per week and 30 mg of thiamine. It is also known that chickens will not eat more than 285 mg of fat per week. This information is summarized in Table I below.

TABLE I

	protein	thiamine	fat	cost
corn	2 mg/lb	1 mg/lb	14 mg/lb	6¢/lb
alfalfa	1 mg/lb	5 mg/lb	25 mg/lb	8¢/lb
minimum required	15 mg	30 mg		
maximum allowed			285 mg	

Given these conditions, how many pounds of corn and how many pounds of alfalfa must be mixed together to meet the chicken's weekly requirements at the lowest possible cost?

The first step in formulating a linear programming problem is to assign symbols to the controllable

variables, in this case the number of pounds of corn and the number of pounds of alfalfa that are to be used in the chicken's weekly feed.

Let x = the number of pounds of corn to be used.
Let y = the number of pounds of alfalfa to be used.

Now the constraints can be stated in terms of x and y. We will start with the protein constraint. Since each pound of corn contains 2 milligrams of protein, the number of milligrams of protein in x pounds of corn will be 2x. In the same way, the number of milligrams of protein in y pounds of alfalfa will be 1y, or simply y. Then the total amount of protein in the corn and alfalfa mix will be 2x + y. And since each chicken needs at least 15 milligrams of protein every week, we know that 2x + y must be at least 15. In algebraic terms,

$$2x + y \geq 15.$$

Similarly, since each gram of corn contains 1 milligram of thiamine and each gram of alfalfa contains 5 milligrams of thiamine, in order to have at least 30 milligrams of thiamine in the chicken's weekly feed we must be sure that

$$x + 5y \geq 30.$$

Unlike the constraints on the protein and thiamine, which set minimum values, the constraint on the fat sets a maximum value. The fat content in the chicken's weekly feed cannot exceed 285 milligrams. Since the corn will contain 14x milligrams of fat and the alfalfa 25y milligrams of fat, it is necessary that

$$14x + 25y \leq 285.$$

It is also important to realize that neither x nor y can be negative, that is,

$$x \geq 0 \text{ and } y \geq 0.$$

Having formulated the constraints, we must state the object of the program, which is to minimize the cost of the feed. This cost will be the sum of the cost of the corn and the cost of the alfalfa. We know that x pounds of corn at 6¢/lb will cost 6x cents; y pounds of alfalfa at 8¢/lb will cost 8y cents. The total cost of the mix, in cents, will therefore be

$$C = 6x + 8y$$

where C stands for cost. Because it is our object to minimize the value of C, this equation is called the *objective function.*

The linear program for this problem is summarized below.

Letting x = the number of pounds of corn to be used
and y = the number of pounds of alfalfa to be used

Minimize $C = 6x + 8y$

subject to

$2x + y \geq 15$ (protein)

$x + 5y \geq 30$ (thiamine)

$14x + 25y \leq 285$ (fat)

where $x \geq 0$ and $y \geq 0$.

Example 1. Formulate the constraints and the objective function for the following problem. A bakery must plan a day's supply of eclairs and napoleons. Each eclair requires 3 ounces of custard and 7½ minutes of labor. Each napoleon requires 1 ounce of custard and 15 minutes of labor. The bakery makes 40¢ on each eclair that it sells and 30¢ on each napoleon. If 120 ounces of custard are available, and 10 hours of labor, how many eclairs and how many napoleons should the bakery make to maximize its profits?

Step 1: Assign symbols to the controllable variables.

Let x = the number of eclairs the bakery should make.
Let y = the number of napoleons the bakery should make.

Step 2: Formulate the constraints in terms of x and y.

Since each eclair requires 3 ounces of custard, x eclairs require 3x ounces of custard. Similarly, y napoleons require y ounces of custard. 120 ounces of custard are available, so

$$3x + y \leq 120.$$

Eclairs require 1/8 of an hour of labor, napoleons 1/4 of an hour. With 10 hours of labor available, this means that

$$1/8x + 1/4y \leq 10.$$

In addition, $x \geq 0$ and $y \geq 0$.

Step 3: Formulate the objective function.

The profit on x eclairs is .40x; the profit on y napoleons is .30y. The total profit on x eclairs and y napoleons is therefore

$$P = .40x + .30y.$$

Formulate linear programs for the following problems.

Exercise 1. A dry cleaning company is buying up to 30 new pressing machines and is considering both a deluxe and a standard model. The deluxe model occupies 2 square yards of floor space and presses 3 pieces per minute. The standard model occupies 1 square yard of floor space but presses only 2 pieces per minute. If 44 square yards of floor space are available, how many machines of each type should the company buy to maximize its output?

Exercise 2. The producers of a Broadway musical plan to advertise on New York City buses and on a local radio station. Each bus advertisement costs $3000; each radio commercial costs $1000. The producers want to have at least one third as many bus advertisements as radio commercials. Bus advertisements are known to reach 400 upper income families, 400 middle income families, and 500 lower income families each week. The radio commercials reach 100 upper income families, 1100 middle income families, and 100 lower income families each week. If the producers want to reach at least 2100 upper income families and 9100 middle income families and no more than 5000 lower income families every week, how should they distribute their advertising between the two media in order to minimize the cost of the campaign?

Exercise 3. A farmer has 30 acres on which to grow tomatoes and corn. 100 bushels of tomatoes require 1000 gallons of water and 5 acres of land, 100 bushels of corn require 6000 gallons of water and $2\frac{1}{2}$ acres of land. Labor costs are \$1 per bushel for both corn and tomatoes. The farmer has available 30,000 gallons of water and \$750 in capital. He knows that he cannot sell more than 500 bushels of tomatoes or 475 bushels of corn. If he makes a profit of \$2 on each bushel of tomatoes and \$3 on each bushel of corn, how many bushels of each should he raise in order to maximize his profits?

2.2 Graphing the Problem

It is not hard to find pairs of values for x and y that will satisy all the constraints listed in the program formulated in Section 2.1. $x = 5$ and $y = 7$ is one such pair; $x = 8$ and $y = 6$ is another. (Try them.) The possibilities are, in fact, unlimited. The question is, which of these pairs will give the lowest possible value for C? When the problem has only two unknowns, as this one does, one way to answer this question is to make a graph.

Since $x \geq 0$ and $y \geq 0$, we shall be interested in points in the first quadrant only. This is always the case when the variables in a linear programming problem represent physical quantities that cannot be negative.

We will start by graphing the protein constraint. The line $2x + y = 15$ is shown in Figure 1, as well as the shaded region where $2x + y > 15$. The points in this region, together with those on the line, are the only ones for which it is true that $2x + y \geq 15$. These points are hence the only ones that satisfy the protein constraint.

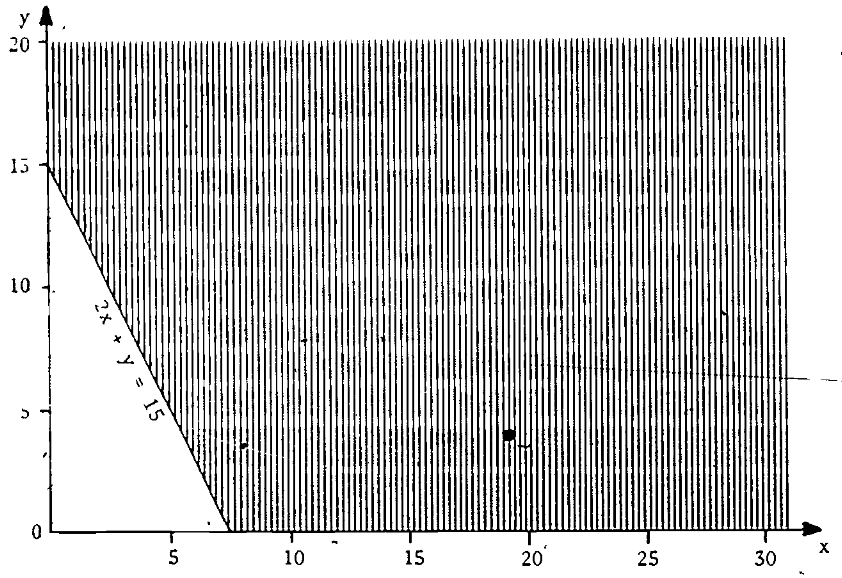


Figure 1. The points that satisfy the protien constraint.

Which of these points also satisfy the thiamine constraint? To find out, we must draw the line $x + 5y = 30$ on the same set of axes and shade in the region above it. Only those points which lie in the intersection of the two sets of points satisfy both the protein and the thiamine constraints. (See Figure 2.)

Because the fat constraint sets a maximum condition, it will be satisfied only by points on or below the line $14x + 25y = 285$. In Figure 3, this constraint is combined with the other two, and the shaded region now shows those points that satisfy all three of the constraints together. A region like this one is called a convex set. The points labelled P, Q and R are its vertices.

A set of points is convex if the line joining any two points of the set lies within the set. Convex sets have no holes in them, and their boundaries are straight or bend outward. The intersection of any two convex sets is itself a convex set.

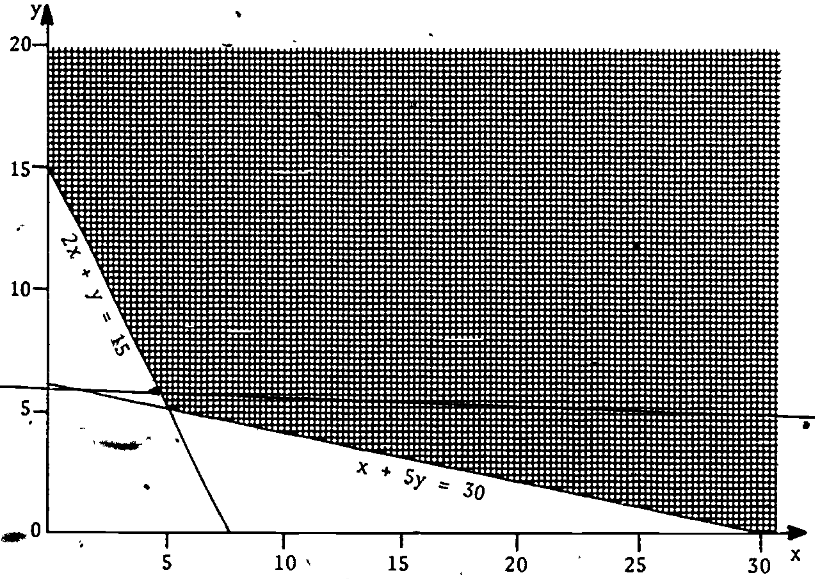


Figure 2. The points that satisfy the protein and the thiamine constraints.

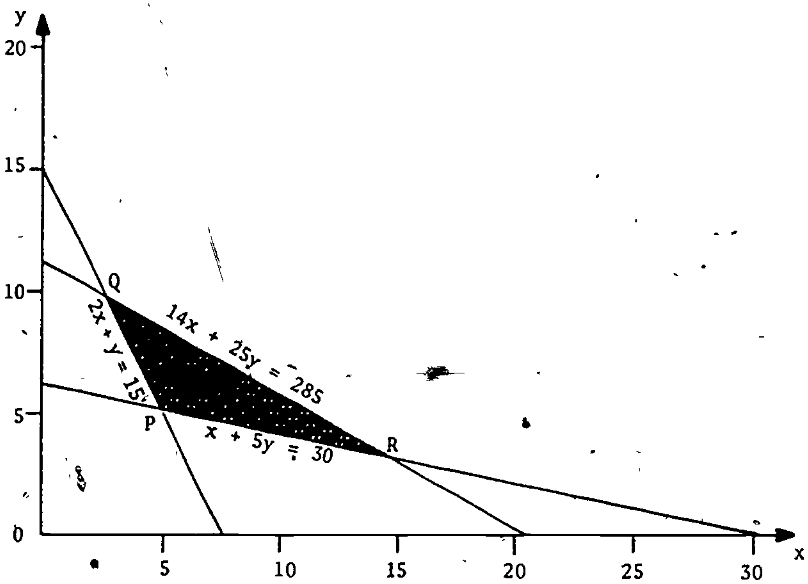


Figure 3. The points that satisfy the protein, thiamine, and fat constraints.

In Figure 4 below, a, b and c are convex sets; d and e are not.

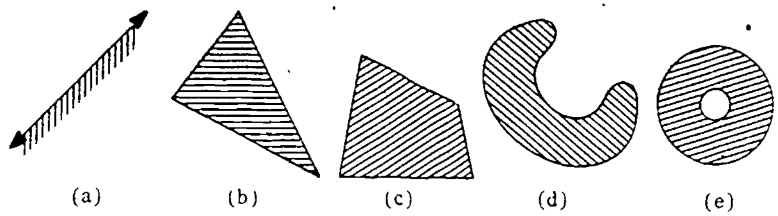


Figure 4.

Set a, consisting of a straight line and all the points on one side of it, is called a half-plane. Since the triangle in Figure 3 is formed by the intersection of three such half-planes, it too is a convex set. The non-negative solutions to any linear programming problem, no matter how complex, lie in a convex set.

2.3 Solving the Problem

Now that we have a picture of all the points whose coordinates are possible solutions, we are ready to solve the problem, that is, to find the point whose coordinates minimize the cost, C, of the feed. To do this, we must interpret the equation of the objective function, $C = 6x + 8y$, as the equation of a line in the xy-plane. In slope-intercept form, this equation becomes

$$y = -\frac{6x}{8} + \frac{C}{8}.$$

Thus, the slope of the objective function is -6/8, or -3/4, and the value of C determines the y-intercept, C/8. In particular, the smaller the value of C, the smaller the y-intercept will be.

All lines with slopes of -3/4 belong to a family of parallel lines, some of which are shown in Figure 5. These lines can be viewed as possible positions of a single line with a slope of -3/4 moving across the coordinate system parallel to itself. In some of these positions it will pass through the region that is shaded

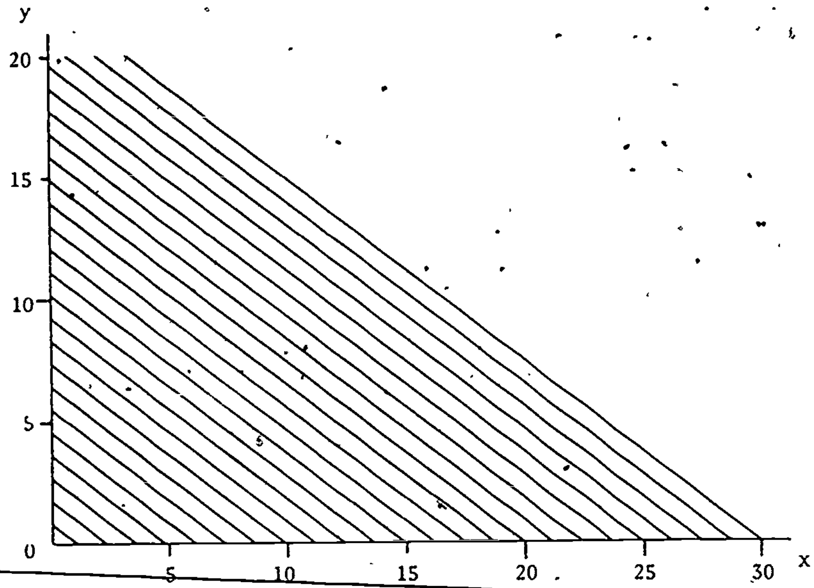


Figure 5. The family of lines whose slopes are -3/4.

in Figure 3; in others it will not. Figure 6 shows Figure 5 superimposed on Figure 3. In Figure 6, the lowest line in the family to pass through the shaded region appears to be the one that passes through the

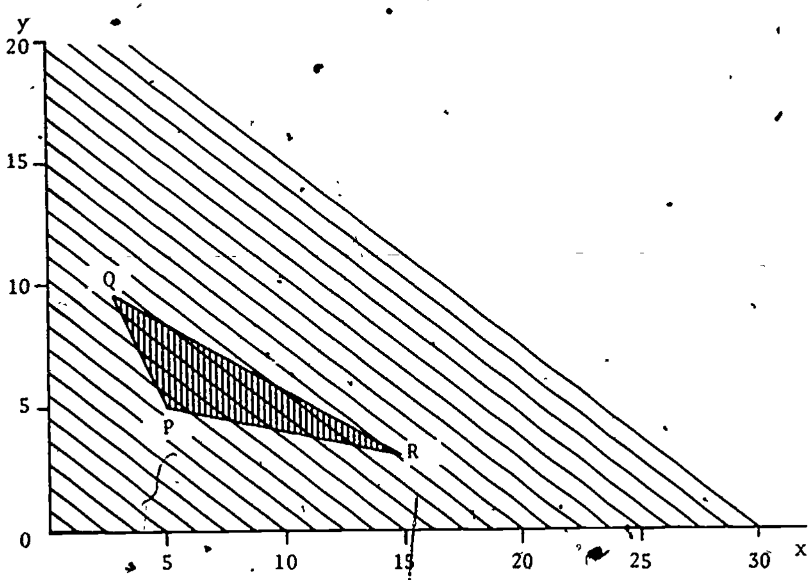


Figure 6. Figure 5 superimposed on the shaded region of Figure 3.

point marked P. Of all the lines that have slopes of -3/4 and contain at least one point that satisfies the constraints of the problem, this is the one with the smallest y-intercept. Point P is therefore the point in the shaded region whose coordinates minimize C.

It can be shown that whenever a linear equation such as the objective function C = 6x + 8y is defined on a region bounded by a convex polygon, it will assume its minimum and maximum values at vertices of the polygon. To minimize or maximize the objective function of a linear program, it is therefore necessary to evaluate it only at the vertices of the convex polygon determined by the constraints. The vertex that gives the best value of the objective function is then the solution of the program. (In the special case where a side of the polygon has the same slope as the objective function, two consecutive vertices and all the points between them may minimize or maximize the function.)

Since the vertices of the convex polygon are points at which its sides intersect, the coordinates of these vertices can be found by solving the appropriate equations simultaneously. Point P is the intersection of the protein and thiamine constraints; its coordinates are therefore found by solving simultaneously the equations

$$2x + y = 15$$

and

$$x + 5y = 30.$$

In the same way, point Q is the simultaneous solution of

$$2x + y = 15$$

and

$$14x + 25y = 285$$

and point R the simultaneous solution of

$$x + 5y = 30$$

and

$$14x + 25y = 285.$$

The coordinates of P, Q and R, together with the values of C they determine, are given in Table II below.

TABLE II

	Coordinates of Vertex	Value of Objective Function at Vertex
P	(5, 5)	C = 6(5) + 8(5) = 70
Q	(5/2, 10)	C = 6(5/2) + 8(10) = 95
R	(15, 3)	C = 6(15) + 8(3) = 114

Table II shows that P is indeed the vertex whose coordinates minimize the objective function. Thus, the chicken's weekly feed should contain 5 pounds of corn and 5 pounds of alfalfa, and the cost of this mix will be 70¢.

To solve a linear program in two dimensions, it is therefore necessary to:

1. Formulate the constraints and the objective function.
2. Graph the constraints.
3. Shade in the convex polygon they determine.
4. Find the coordinates of the vertices of this polygon.
5. Evaluate the objective function at each of these vertices.

The vertex whose coordinates give the best value of the objective function (a maximum or a minimum as the case may be) is the solution to the linear program.

Exercise 4. Solve the linear program for Example 1, Section 2.1.
Exercise 5. Solve the linear program for Exercise 1.
Exercise 6. Solve the linear program for Exercise 2.
Exercise 7. Solve the linear program for Exercise 3.

3. CONCLUSION

The problem with which this module began was a real problem, but the version of it given in Section 2.1 was greatly simplified. When the grain supplier actually made its chicken feed, it used nearly thirty different ingredients which, taken together, fulfilled the chicken's requirements for several dozen vitamins, minerals, and other nutrients. Correspondingly, the linear program which was formulated to solve the problem contained several dozen constraints, each involving up to thirty different variables.

Like this one, most real world problems in linear programming involve a large number of variables which are subject to many different constraints. Although similar in form to those for two variable problems, their linear programs are far more complex and are usually solved by computers. The method used for their solution, however, is entirely analagous to the one presented here.

4. SAMPLE EXAM

Formulate and solve the following problems.

1. The manager of a watch company is planning a month's production schedule. The company manufactures both quartz and regular watches and wishes to produce at least as many quartz watches as regular ones. An order for 225 regular watches has already been received, but no more than 500 regular watches are sold in any one month. Quartz watches require 3 hours of production time, regular watches 2. 3150 production hours are available, and there are 1150 sets of straps on hand and 870 quartz assemblies. If the company makes $15 on each quartz watch and $7 on each regular one, how many of each should it manufacture to maximize its profits?

2. Dog food is made from a mixture of horsemeat and beef. The manufacturers want to use at least half as much horsemeat as beef and must use 75 pounds of beef already on hand. Each pound of beef contains 1 gram of calcium, 5 grams of ash, and 1 gram of moisture. Each pound of horsemeat contains 1 gram of calcium, 1 gram of ash, and 7 grams of moisture. The mixture must contain at least 225 grams of calcium and may contain no more than 1100 grams of ash and 1580 grams of moisture. If beef costs $2 per pound and horsemeat costs $1 per pound, how many pounds of each should the manufacturers use to minimize their cost?

5. ANSWERS TO EXERCISES

Exercise 1.

Letting x = the number of deluxe models to buy
and y = the number of standard models to buy

Maximize $OP = 3x + 2y$

subject to
$x + y \le 30$ (number of machines)
$2x + y \le 44$ (floor space)

where $x \ge 0$ and $y \ge 0$.

Exercise 2.

Letting x = the number of bus advertisements to be run
and y = the number of radio advertisements to be run

Minimize $C = 3000x + 1000y$

subject to
$3x \ge y$ (ratio of bus ads to radio ads)
$400x + 100y \ge 2100$ (upper income families)
$400x + 1100y \ge 9100$ (middle income families)
$500x + 100y \le 5000$ (lower income families)

where $x \ge 0$ and $y \ge 0$.

Exercise 3.

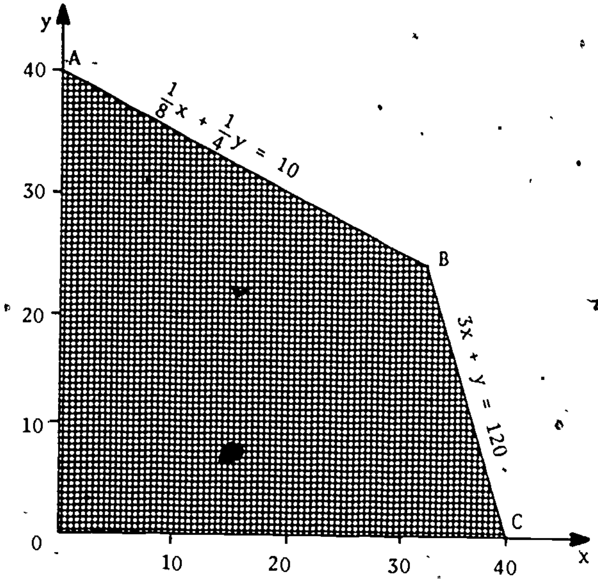
Letting x = the number of bushels of tomatoes to be raised
and y = the number of bushels of corn to be raised

Maximize $P = 2x + 3y$

subject to
$10x + 60y \le 30000$ (water)
$5/100x + 5/200y \le 30$ (acreage)
$x + y \le 750$ (capital)
$x \le 500$ (limit on tomatoes)
$y \le 475$ (limit on corn)

where $x \ge 0$ and $y \ge 0$.

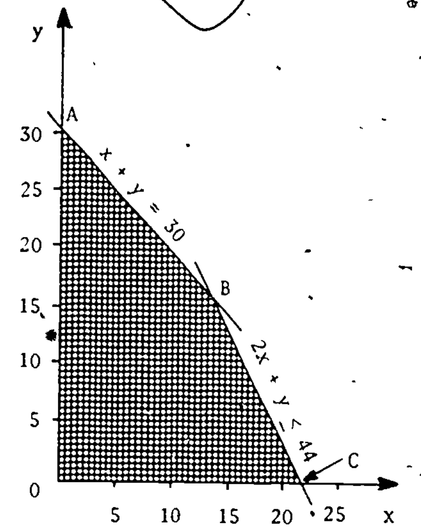
Exercise 4.



Vertex	Coordinates of Vertex	Value of Objective Function at Vertex
A	(0, 40)	P = (.40)(0) + (.30)(40) = 12.00
B	(32, 24)	P = (.40)(32) + (.30)(24) = 20.00
C	(40, 0)	P = (.40)(40) + (.30)(0) = 16.00

B is the vertex whose coordinates maximize the objective function. The bakery should make 32 eclairs and 24 napoleons. Its profit will then be $20.00.

Exercise 5.

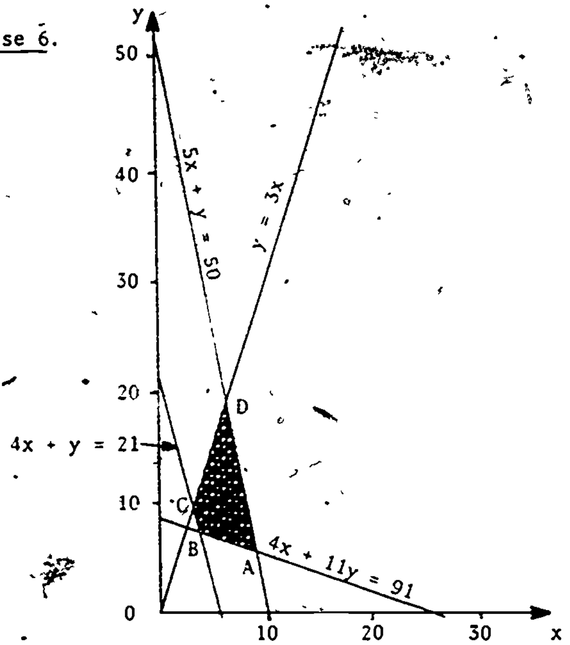


Vertex	Coordinates of Vertex	Value of Objective Function at Vertex
A	(0, 30)	OP = (3)(0) + (2)(30) = 60
B	(14, 16)	OP = (3)(14) + (2)(16) = 74
C	(22, 0)	OP = (3)(22) + (2)(0) = 66

B is the vertex whose coordinates maximize the objective function. The company should buy 14 deluxe machines and 16 standard ones. Its output will then be 74 pieces per minute.

79 80

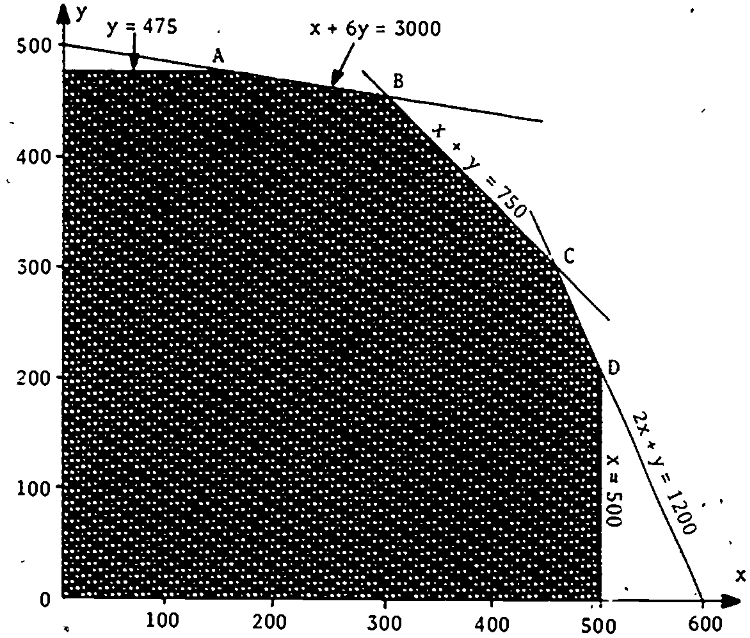
Exercise 6.



	Coordinates of Vertex	Value of Objective Function at Vertex
A	(9, 5)	C = (3000)(9) + (1000)(5) = 32,000
B	(3½, 7)	C = (3000)(3.5) + (1000)(7) = 17,500
C	(3, 9)	C = (3000)(3) + (1000)(9) = 18,000
D	(6.25, 18.75)	C = (3000)(6.25) + (1000)(18.75) = 37,500

B is the vertex whose coordinates minimize the objective function. The producers should run 3½ bus advertisements and 7 radio advertisements every week. (Since they cannot run half an advertisement, this means that they will run 7 bus advertisements every two weeks.) The weekly cost of this campaign will be $17,500.

Exercise 7.



	Coordinates of Vertex	Value of Objective Function at Vertex
A	(150, 475)	P = (2)(150) + (3)(475) = 1725
B	(300, 450)	P = (2)(300) + (3)(450) = 1950
C	(450, 300)	P = (2)(450) + (3)(300) = 1800
D	(500, 200)	P = (2)(500) + (3)(200) = 1600

B is the vertex whose coordinates maximize the objective function. The farmer should raise 300 bushels of tomatoes and 450 bushels of corn. His profit will then be $1950.

6. ANSWERS TO SAMPLE EXAM

1. Letting x = the number of quartz watches to be manufactured
   and y = the number of regular watches to be manufactured

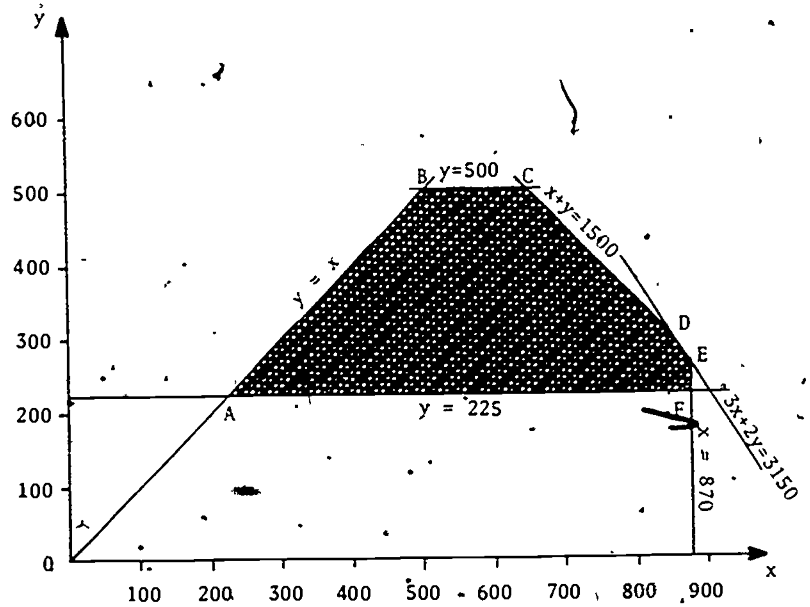
   Maximize $P = 15x + 7y$

   subject to
   - $x \geq y$ (ratio of quartz to regular)
   - $225 \leq y \leq 500$ (limits on regular)
   - $3x + 2y \leq 3150$ (production hours)

$x + y \leq 1150$ (straps)

$x \leq 870$ (quartz assemblies)

where $x \geq 0$ and $y \geq 0$.



	Coordinates of Vertex	Value of Objective Function at Vertex
A	(225, 225)	P = (15)(225) + (7)(225) = 4950
B	(500, 500)	P = (15)(500) + (7)(500) = 11000
C	(650, 500)	P = (15)(650) + (7)(500) = 13250
D	(850, 300)	P = (15)(850) + (7)(300) = 14850
E	(870, 270)	P = (15)(870) + (7)(270) = 14940
F	(870, 225)	P = (15)(870) + (7)(225) = 14625

E is the vertex whose coordinates maximize the objective function. The company should manufacture 870 quartz watches and 270 regular watches. Its profit will then be $14,940.

2. Letting x = the number of pounds of beef to be used
   and y = the number of pounds of horsemeat to be used

Minimize $C = 2x + y$

subject to

$2y \geq x$ (ratio of horsemeat to beef)

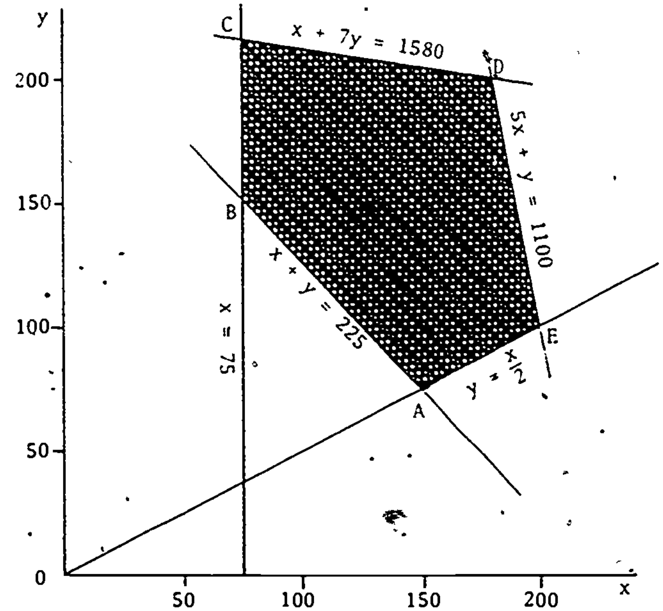
$x + y \geq 225$ (calcium)

$x \geq 75$ (beef)

$x + 7y \leq 1580$ (moisture)

$5x + y \leq 1100$ (ash)

where $x \geq 0$ and $y \geq 0$.



	Coordinates of Vertex	Value of Objective Function at Vertex
A	(150, 75)	C = (2)(150) + 75 = 375
B	(75, 150)	C = (2)(75) + 150 = 300
C	(75, 215)	C = (2)(75) + 215 = 365
D	(180, 200)	C = (2)(180) + 200 = 560
E	(200, 100)	C = (2)(200) + 100 = 500

B is the vertex whose coordinates minimize the objective function. The manufacturers should use 75 pounds of beef and 150 pounds of horsemeat.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name________________________ Unit No.________

Page________ ○ Upper ○ Middle ○ Lower	OR	Section________ Paragraph________	OR	Model Exam Problem No.______ Text Problem No.______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature________________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
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Name____________________ Unit No.__________ Date__________

Institution____________________ Course No.________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain____________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain____________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

ERRATA

UNIT 453 - LINEAR PROGRAMMING IN TWO DIMENSIONS: I

page 21: line segment CD in the graph should be labelled

$$x + y = 1150$$

as illustrated below.