DOCUMENT RESUME

ED 218 129 SE 038 239

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TITLE A Unified Method of Finding Laplace Transforms, Fourier Transforms, and Fourier Series. [and] An Inversion Method for Laplace Transforms, Fourier Transforms, and Fourier Series. Integral Transforms and Series Expansions. Modules and Monographs in Undergraduate Mathematics and Its Applications Project. UMAP Units 324 and 325.
INSTITUTION Education Development Center, Inc., Newton, Mass.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 78
GRANT SED-76-19615; SED-76-19615-A02
NOTE 59p.

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *Algorithms; Answer Keys; *College Mathematics; Higher Education; Instructional Materials; *Learning Modules; *Mathematical Concepts; *Problem Solving; Supplementary Reading Materials
IDENTIFIERS Fourier Series; *Fourier Transformation; *Laplace Transforms

ABSTRACT

This document contains two units that examine integral transforms and series expansions. In the first module, the user is expected to learn how to use the unified method presented to obtain Laplace transforms, Fourier transforms, complex Fourier series, real Fourier series, and half-range sine series for given piecewise continuous functions. In the second unit, the student is expected to use the method presented to find a function when given the Laplace transform, the Fourier transform, the coefficient transform, or the Fourier series expansion of a function. Each module contains exercises and a model exam. Answers to all questions are provided. (MP)

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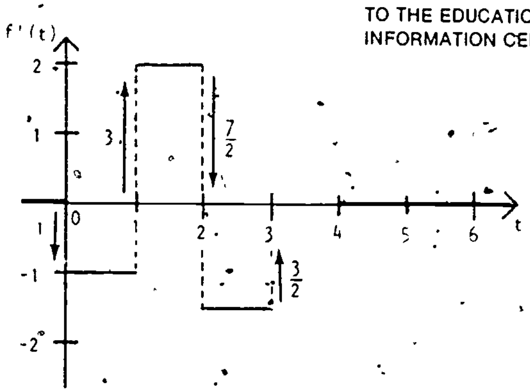
UNIT 824

A UNIFIED METHOD OF FINDING LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

by C.A. Grimm



INTEGRAL TRANSFORMS AND SERIES EXPANSIONS

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A UNIFIED METHOD OF FINDING LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

by

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Intermodular Description Sheet

Title: A UNIFIED METHOD OF FINDING LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

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Classification: INT TRANS & SERIES EXP

Suggested Support Materials: Standard textbooks on engineering mathematics, applied advanced calculus textbooks

Prerequisite Skills:
1. Differentiate and integrate elementary functions.
2. Sketch graphs of elementary functions.
3. Use Euler's formula.
4. Have a basic understanding of integral transforms and orthogonal functions
5. Understand odd and even functions.
6. Understand piecewise continuous functions.
7. Sum geometric series.

Output Skills:
1. Use the unified method described in this unit to obtain
   a. Laplace transforms
   b. Fourier transforms
   c. complex Fourier series
   d. real Fourier series
   e. half range sine series
   for given piecewise continuous functions.

Other Related Units:

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank members of the UMAP Analysis and Computation Panel for their reviews and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation grant No. SED76-19615. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF nor of the National Steering Committee.

1. INTRODUCTION

Integral transforms and orthogonal functions provide the basis for widely used techniques in solving a large number of physical and engineering problems. In this unit we present a method which facilitates the finding of the Laplace transform, the Fourier transform, and the Fourier series when the given function is piecewise continuous (as are most functions that are encountered in practice).

The transforms and the series expansions that can be obtained by the method presented here may be obtained by various other techniques that are frequently used. Normally the two transforms and the series expansion are presented as three separate (but related) topics, and the techniques for handling piecewise continuous functions include integration, use of tables, and manipulation with unit step functions. The unified method has the following desirable characteristics:

- it avoids the use of tables;
- it avoids almost all integration;
- unit step functions are unnecessary;
- the method is quick;
- it provides a single, unified approach to all three problems;
- it employs graphical techniques.

2. THE METHOD EXPLAINED

We begin by recalling the basic definitions of the transforms and the series in question. The Fourier series expansion of a function f(t) of period 2p is given by

(1) $$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{p} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{p},$$

where

(2) $$a_n = \frac{1}{p}\int_0^{2p} f(t)\cos\frac{n\pi t}{p}\,dt, \quad b_n = \frac{1}{p}\int_0^{2p} f(t)\sin\frac{n\pi t}{p}\,dt.$$

The Laplace transform of a function f(t) is given by

(3) $$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt.$$

Note that f(t) need be defined only for $t \geq 0$, however we throughout this unit we shall regard f(t) as being identically zero for $t < 0$ when we are finding the Laplace transform. The Fourier transform of f(t) is given by

(4) $$F\{f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt.$$

There are, of course, basic questions that arise concerning conditions under which the series (1) and the improper intergrals in (3) and (4) exist. There is also the question of whether the series in (1) actually represents f(t) and if so, in what sense. Answers to these questions can be found in textbooks on engineering mathematics and applied advanced calculus, and we shall limit our study to functions for which appropriate conditions are satisfied.

The similiarity between expressions (3) and (4) is apparent. In addition, the integrals for a_n and b_n in (2) also resemble (3) and (4). In order to connect (2) with (3) and (4), we define the coefficient transform

(5) $$C\{f(t)\} = \int_0^{2p} e^{-in\pi t/p} f(t)\,dt.$$

The structural similarity of (3), (4), and (5) is apparent and the connection with (2) can be seen from the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$. The coefficient transform may be regarded as arising from the complex form of (1), which is

(6) $$f(t) \sim \sum_{n=-\infty}^{\infty} C_n e^{in\pi t/p},$$

where

(7) $$C_n = \frac{1}{2p}\int_0^{2p} f(t)e^{-in\pi t/p}dt.$$

Formulas for C_0 and C_{-n} may be obtained from (6) by substituting $n = 0$ or by replacing n with $-n$. Expressions (1) and (5) may each be obtained from the other via the relationships

(8) $$C_n = \tfrac{1}{2}(a_n - ib_n) \qquad C_0 = \tfrac{1}{2}a_0 \qquad C_{-n} = \tfrac{1}{2}(a_n + ib_n)$$

$$a_0 = 2C_0 \qquad a_n = C_n + C_{-n} \qquad b_n = i(C_n - C_{-n})$$

(The Euler formula, of course, provides the basis for (8).)

Suppose now that $f(t)$ is a piecewise continuous function. In Figures 1 - 3 we show functions of this type that are appropriate for the three transforms in question. The graphs also show the information needed to apply the unified method, namely:

- the location of the discontinuities of $f(t)$ (indicated by a_i);
- the "jump" in $f(t)$ at a discontinuity (indicated by M_i);
- the direction of the jump (indicated by an arrow up or down).

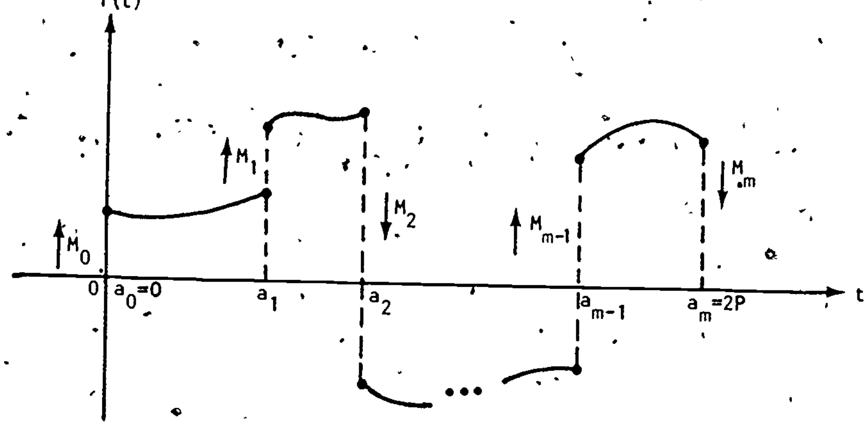


Figure 1. One period of a 2p-periodic function $f(t)$; this illustration is appropriate for the Fourier series expansion.

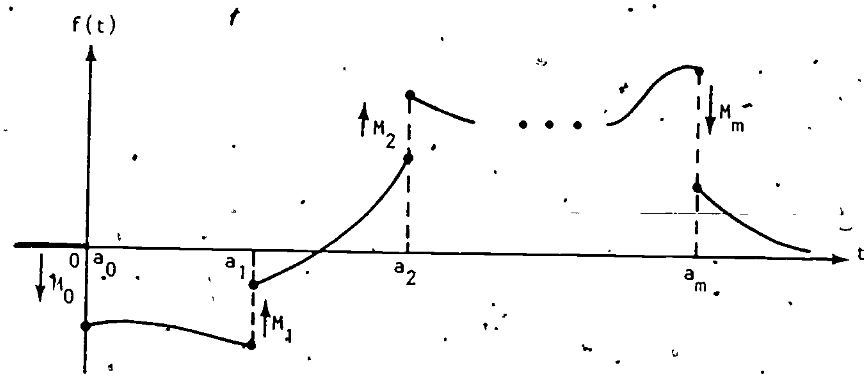


Figure 2. An illustration that is appropriate for the Laplace transform. Note that $f(t) = 0$ for $t < 0$; however, $f(t)$ need not be discontinuous at 0 in general.

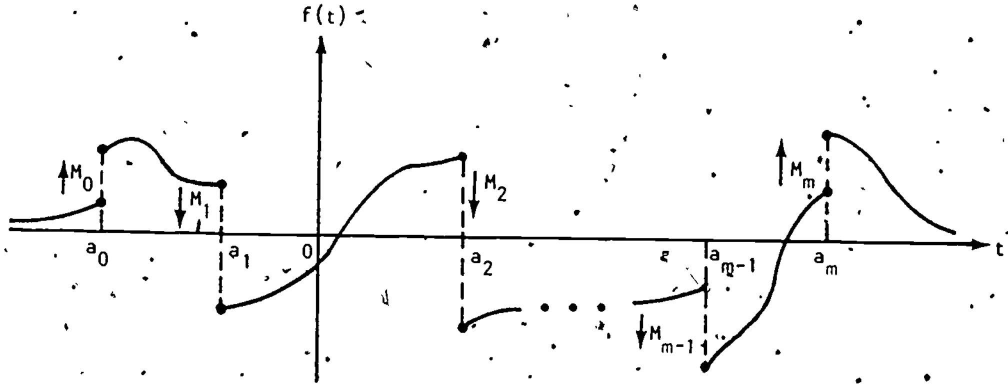


Figure 3. Illustration of $f(t)$ appropriate for the Fourier transform.

Under the conventions that M_i will be positive if the jump at a_i is up and negative if the jump is down, we have the following unified formulas:

(9) $$C\{f(t)\} = \frac{p}{in\pi}\sum_{k=0}^{M} M_k\, e^{-in\pi a_k/p} + \frac{p}{in\pi}C\{f'(t)\}$$

(10) $$L\{f(t)\} = \frac{1}{s}\sum_{k=0}^{M} M_k\, e^{-a_k s} + \frac{1}{s}L\{f'(t)\}$$

$$F\{f(t)\} = \frac{-1}{i\alpha} \sum_{k=0}^{M} M_k e^{-i\alpha a_k} + \frac{1}{i\alpha} F\{f'(t)\}. \quad (11)$$

Formulas (9), (10), (11) may now be used iteratively to find $C\{f'(t)\}$, $L\{f'(t)\}$, $F\{f'(t)\}$, and so forth. With a little practice the iterative process becomes very quick, and graphical techniques make it easy to implement the algorithm for many elementary functions, as we shall see next.

We note that Formulas (9) - (11) can be obtained by elementary methods. A derivation is carried out in detail for the coefficient transform in the appendix.

3. THE METHOD IN ACTION

3.1 Example 1

To find $L\{f(t)\}$ for the function

$$f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 3, & 2 \le t < 4 \\ t, & 4 \le t \end{cases}$$

we first graph f(t) and its distinct nonzero derivatives, indicating all jumps (see Figure 4).

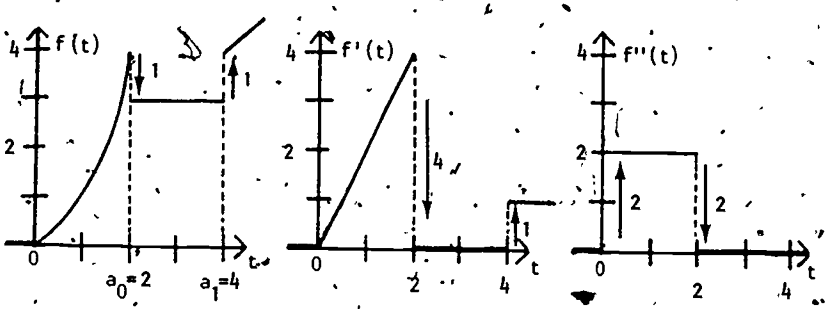


Figure 4. Graphs of f(t), f'(t), f''(t) for Example 1, showing jumps at discontinuities.

Apply formula (10) to f(t), f'(t), f''(t):

$$L\{f(t)\} = \frac{1}{s}\left(-e^{-2s} + e^{-4s}\right) + \frac{1}{s} L\{f'(t)\}$$

$$= \frac{1}{s}\left(-e^{-2s} + e^{-4s}\right) + \frac{1}{s}\left[\frac{1}{s}\left(-4e^{-2s} + e^{-4s}\right) + \frac{1}{s} L\{f''(t)\}\right]$$

$$(*) \quad = \frac{1}{s}\left(-e^{-2s} + e^{-4s}\right) + \frac{1}{s^2}\left(-4e^{-2s} + e^{-4s}\right) + \frac{1}{s^2}\left[\frac{1}{s}\left(2 - 2e^{-2s}\right)\right]$$

$$= \frac{2}{s^3} + e^{-2s}\left(-\frac{1}{s} - \frac{4}{s^2} - \frac{2}{s^3}\right) + e^{-4s}\left(\frac{1}{s} + \frac{1}{s^2}\right).$$

At first, this method seems to offer little, if any, advantage over a direct integration or the use of unit step functions. However, the calculation which we carried out in detail above can be vastly shortened -- we need only observe, as can be seen from Equation (*), that *the k^{th} power of 1/s multiplies the jumps in the $(k-1)^{st}$ derivative, each weighted with the appropriate exponential function.* Therefore, merely by keeping track graphically of the magnitude, direction, and location of the jumps for each derivative, we may write down the transform in essentially final form as soon as the sketches are drawn. We use this shortened procedure in all subsequent examples, and ask you to do the same for the exercises.

3.2 Example 2

Suppose we wish to find $L\{f(t)\}$ for the function which is defined graphically in Figure 5.

Since we need only the segment slopes (which are obtainable from the end point coordinates) in order to graph f'(t), we do not even need explicit formulas. (See Figure 6.)

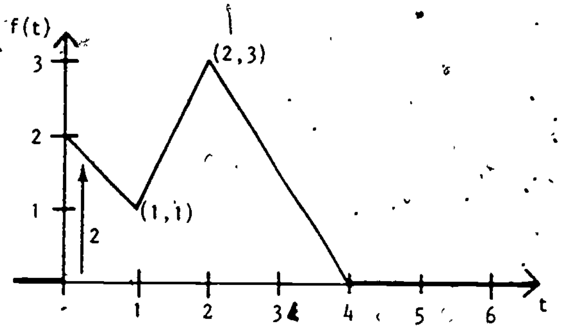


Figure 5. The function f(t) in Example 2 consists of the straight line segments shown, and is zero for $t \geq 4$.

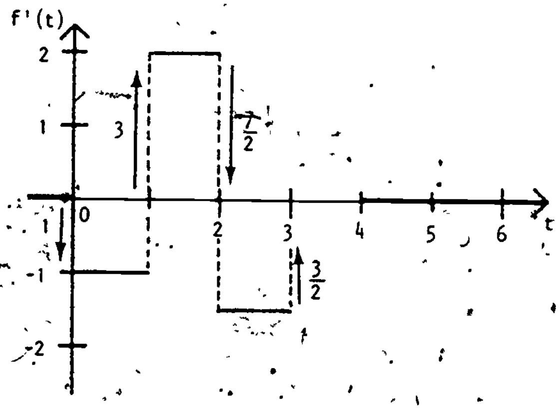


Figure 6. The graph of f'(t) for Example 2, showning jumps and their locations.

From Figures 5 and 6, and using our understanding of Formula (10), we may write immediately

$$L\{f(t)\} = \frac{2}{s} + \frac{1}{s^2}\left[-1 + 3e^{-s} - \frac{7}{2}e^{-2s} + \frac{3}{2}e^{-4s}\right].$$

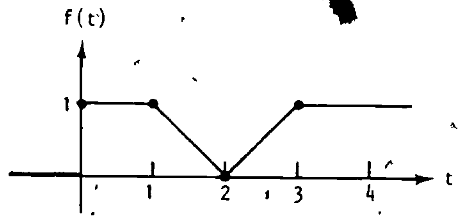
Exercise 1

Use the unified method (the abbreviated form illustrated in Example 2) to find L{f(t)} for the following functions:

a.
$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 2-t, & 1 \leq t < 2 \\ 0, & 2 < t. \end{cases}$$

b.
$$f(t) = \begin{cases} t-1, & 0 \leq t < 2 \\ 6-t^2, & 2 \leq t < 3 \\ 0, & 3 \leq t. \end{cases}$$

c. The function shown in the diagram to the right.



3.3 Example 3

Find the coefficient transform C{f(t)} for the periodic function whose formula over one period is

$$f(t) = \begin{cases} t^2 + 1, & 0 \leq t < 1 \\ 0 & 1 \leq t < 2. \end{cases}$$

Here we have 2p = 2, so p = 1. We proceed as in Example 2, graphing f(t) and its distinct nonzero derivatives (see Figure 7).

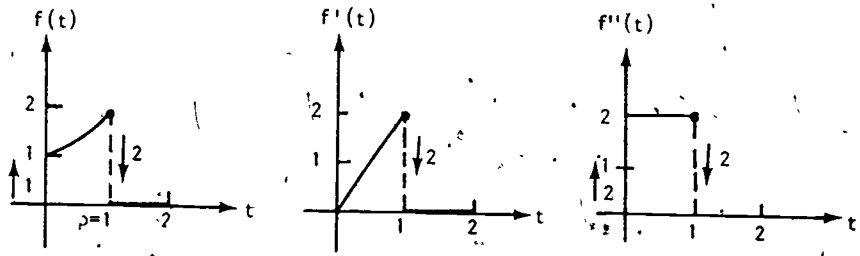


Figure 7. Graphs of f(t), f'(t), and f''(t) for Example 3.

Using the data recorded in Figure 7, and applying Formula (9), we may write down immediately:

$$C\{f(t)\} = \frac{1}{in\pi}\left[1 - 2e^{-in\pi}\right] + \frac{1}{(in\pi)^2}\left[-2e^{-in\pi}\right] + \frac{1}{(in\pi)^3}\left[2 - 2e^{-in\pi}\right]$$

$$= \frac{1}{in\pi} + \frac{2}{(in\pi)^3} + e^{-in\pi}\left[-\frac{2}{in\pi} - \frac{2}{(in\pi)^2} - \frac{2}{(in\pi)^3}\right].$$

Since $e^{-in\pi} = \cos n\pi = (-1)^n$, we may simplify:

$$C\{f(t)\} = -\frac{i}{n\pi} - \frac{2i}{n^3\pi^3} + 2(-1)^n\left[\frac{i}{n\pi} - \frac{1}{n^2\pi^2} - \frac{2i}{n^3\pi^3}\right].$$

(This result holds, of course, for $n \neq 0$.)

3.4 Example 4

Find the real form of the Fourier series expansion for the function $f(t)$ from Example 3. Since

$$C_n = \frac{1}{2p} C\{f(t)\} \qquad \text{(see (5) and (7))},$$

and since $p = 1$ (see Example 3), we have for this example

$$C_n = \frac{1}{2} C\{f(t)\}.$$

Thus,

$$C_n = -\frac{i}{2n\pi} + \frac{i}{n^3\pi^3} + (-1)^n\left[\frac{i}{n\pi} + \frac{1}{n^2\pi^2} - \frac{i}{n^3\pi^3}\right],$$

from which

$$C_{-n} = \frac{i}{2n\pi} - \frac{i}{n^3\pi^3} + (-1)^n\left[-\frac{i}{n\pi} + \frac{1}{n^2\pi^2} + \frac{i}{n^3\pi^3}\right].$$

In addition,

$$C_0 = \frac{1}{2}\int_0^1 (t^2 + 1)\,dt = 2/3.$$

From (8) we have

$$\frac{a_0}{2} = C_0 = 2/3,$$

$$a_n = C_n + C_{-n} = \frac{2(-1)^n}{n^2\pi^2} \quad \text{for } n \neq 0,$$

$$b_n = i(C_n - C_{-n}) = i\left[-\frac{i}{n\pi} + \frac{2i}{n^3\pi^3} + 2(-1)^n\left(\frac{i}{n\pi} - \frac{i}{n^3\pi^3}\right)\right]$$

$$= \frac{1}{n\pi} - \frac{2}{n^3\pi^3} + (-1)^{n+1}\frac{2}{n\pi} + (-1)^n\frac{2}{n^3\pi^3}.$$

The Fourier series expansion is

$$f(t) = \frac{2}{3} + \sum_{n=1}^{\infty} a_n \cos n\pi t + \sum_{n=1}^{\infty} b_n \sin n\pi t,$$

with a_n, b_n as given above.

3.5 Example 5

Find the half range sine expansion for the function

$$f(t) = t^2 - 2t, \quad 0 \leq t < 1.$$

We first make an odd extension of $f(t)$ to include the interval $-1 \leq t < 0$. Since we need only information on the jumps, the extension may be carried out graphically with no formulas necessary. We save additional effort by noting that the derivative of an odd function is even and the derivative of an even function is odd. See Figure 8.

Applying formula (9) with the information displayed in Figure 8, and noting that for the extended function $2p = 2$ so that $p = 1$, we obtain

$$C\{f(t)\} = \frac{1}{in\pi}\left[e^{in\pi} + e^{-in\pi}\right] + \frac{1}{(in\pi)^3}\left[-2e^{in\pi} + 4 - 2e^{-in\pi}\right]$$

$$= e^{in\pi}\left[\frac{1}{in\pi} - \frac{2}{(in\pi)^3}\right] + \frac{4}{(in\pi)^3} + e^{-in\pi}\left[\frac{1}{in\pi} - \frac{2}{(in\pi)^3}\right]$$

$$= (-1)^n\left[-\frac{2i}{n\pi} - \frac{4i}{n^3\pi^3}\right] + \frac{4i}{n^3\pi^3}.$$

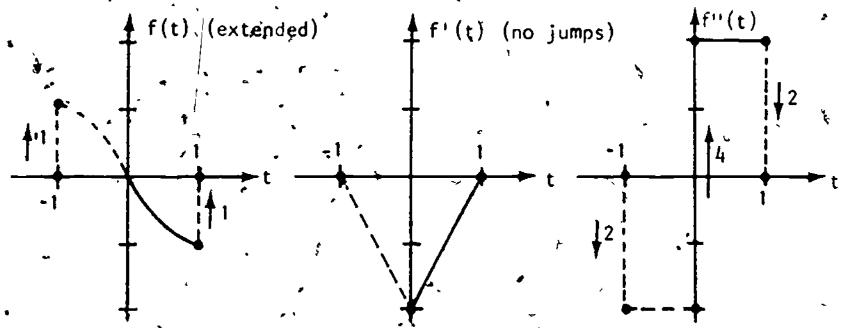


Figure 8. The graph of f(t) extended to form an odd function, and the graphs of f'(t) and f''(t).

Then we have for the Fourier coefficients

$$C_n = \frac{1}{2p} C\{f(t)\} = \frac{1}{2} C\{f(t)\}$$

$$= (-1)^n \left(-\frac{i}{n\pi} - \frac{2i}{n^3\pi^3}\right) + \frac{2i}{n^3\pi^3}$$

and

$$C_{-n} = (-1)^n \left(\frac{i}{n\pi} + \frac{2i}{n^3\pi^3}\right) - \frac{2i}{n^3\pi^3}$$

In the half range sine expansion $a_n = 0$ for all n, and we have

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t,$$

where

$$b_n = i(C_n - C_{-n}) = (-1)^n\left(\frac{2}{n\pi} + \frac{4}{n^3\pi^3}\right) - \frac{4}{n^3\pi^3},$$

or

$$b_n = (-1)^n \frac{2}{n\pi} + \left[(-1)^n - 1\right] \frac{4}{n^3\pi^3}.$$

Exercise 2

Find the complex Fourier series expansion for the following periodic functions, where the definition over one period is given by:

a. $$f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & 3 \le t < 6. \end{cases}$$

b. $$f(t) = \begin{cases} \frac{1}{2}t, & 0 \le t < 2 \\ 2 - \frac{1}{2}t, & 2 \le t < 4. \end{cases}$$

Exercise 3

Find the half range sine series expansion for the function

$$f(t) = \frac{1}{2}t, \quad 0 \le t \le 5.$$

3.6 Example 6

Find the Fourier transform of the function

$$f(t) = \begin{cases} 1 - t^2, & |t| \le 1 \\ 0, & |t| > 1. \end{cases}$$

The process is the same as before: sketch the function and its distinct nonzero derivatives, recording the relevant data on all jumps (see Figure 9).

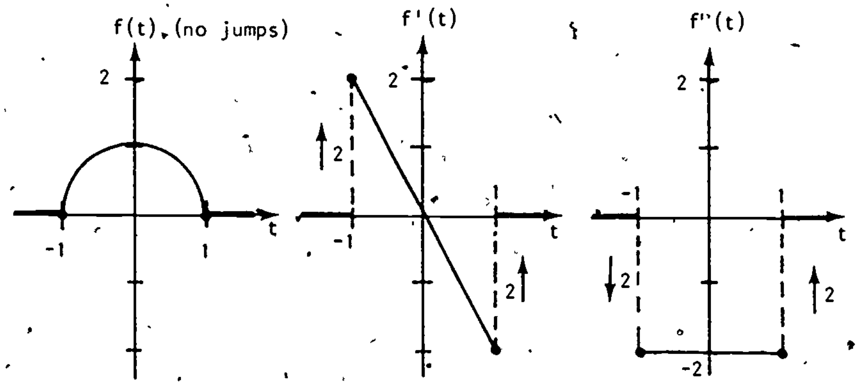


Figure 9. The graphs of f(t), f'(t), and f''(t) for Example 6, with information on jumps.

Applying Formula (11) and the information displayed in Figure 9, we obtain

$$F\{f(t)\} = \frac{1}{(i\alpha)^2}\left(2e^{i\alpha} + 2e^{-i\alpha}\right) + \frac{1}{(i\alpha)^3}\left(-2e^{i\alpha} + 2e^{-i\alpha}\right)$$

$$= -\frac{2}{\alpha^2}\left(e^{i\alpha} + e^{-i\alpha}\right) - \frac{2}{\alpha^3}\left(e^{i\alpha} - e^{-i\alpha}\right)$$

$$= -\frac{4\cos\alpha}{\alpha^2} + \frac{4\sin\alpha}{\alpha^3}$$

Exercise 4

For the following functions f(t) find the Fourier transform $F\{f(t)\}$:

a. $f(t) = \begin{cases} 1, & |t| \le 2 \\ 0, & |t| > 2. \end{cases}$

b. $f(t) = \begin{cases} 1+t, & -1 \le t \le 0 \\ 1-t, & 0 < t \le 1 \\ 0, & |t| > 1. \end{cases}$

3.7 Example 7

Find the Fourier cosine transform of the function $f(t) = e^{-mt}$. We first note that by definition the Fourier cosine transform of the given function is

$$\int_0^\infty e^{-mt} \cos\alpha t \, dt = R\int_0^\infty e^{-i\alpha t} e^{-mt} dt = R\, F\{f(t)\},$$

where f(t) in the latter expression is redefined as

$$f(t) \begin{cases} e^{-mt}, & t > 0 \\ 0, & t < 0, \end{cases}$$

and R denotes the real part of the transform. (See Figure 10.)

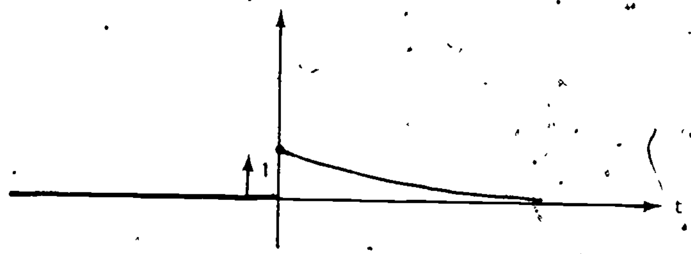


Figure 10. Graph of the redefined f(t) for Example 7.

Since $f'(t) = -mf(t)$, we have by Formula (11)

$$F\{f(t)\} = \frac{1}{i\alpha} + \frac{1}{i\alpha} F\{-mf(t)\}.$$

Therefore

$$F\{(m + i\alpha) f(t)\} = 1,$$

from which

$$F\{f(t)\} = \frac{1}{m + i\alpha} = \frac{m - i\alpha}{m^2 + \alpha^2},$$

and the cosine transform is

$$Co\{f(t)\} = \frac{m}{m^2 + \alpha^2}.$$

3.8 Example 8

As a final example we expand the function

$$f(t) = \cos t, \quad 0 \le t \le 2\pi$$

in a half range sine series. The graphs of the odd extension of f(t) and its first derivative are shown in Figure 11.

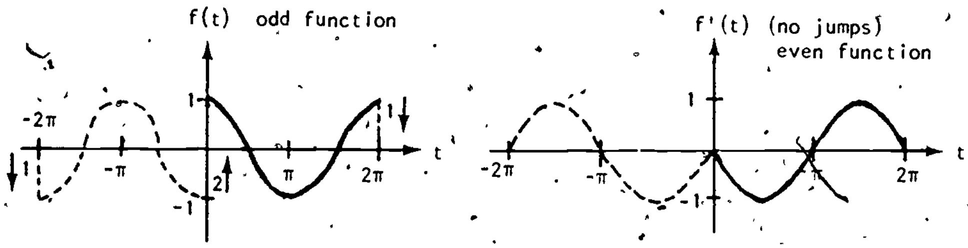


Figure 11. Graphs of the odd extension of f(t) and its first derivative.

In this example, $2p = 4\pi$, $p = 2\pi$, hence $\frac{p}{in\pi} = \frac{2}{in}$. Since $f''(t) = -f(t)$, we have by Formula (9)

$$C\{f(t)\} = \frac{2}{in}\left(-e^{in\pi}+2-e^{-in\pi}\right)+\frac{2}{(in)^2}\,C\{-f(t)\}.$$

Therefore,

$$C\{(1-\frac{4}{n^2})f(t)\} = (-1)^n\frac{4i}{n} - \frac{4i}{n},$$

from which

$$C\{f(t)\} = \begin{cases} -\dfrac{8ni}{n^2-4}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Since

$$2p = 4\pi,$$

we have for n odd

$$C_n = -\frac{2ni}{(n^2-4)\pi},$$

$$C_{-n} = \frac{2ni}{(n^2-4)\pi},$$

so that

$$b_n = i\left(C_n - C_{-n}\right) = \frac{4n}{(n^2-4)\pi},$$

and

$$f(t) = \sum_{n=1,3,5..} \frac{4n}{(n^2-4)\pi}\sin\frac{nt}{2}.$$

Examples 7 and 8 illustrate that it is not necessary that f(t) have some derivative which vanishes in order to apply the unified method -- it is possible to use this method also when there is an algebraic relationship between f(t) and its first few derivatives. This fact will be useful in some of the following exercises.

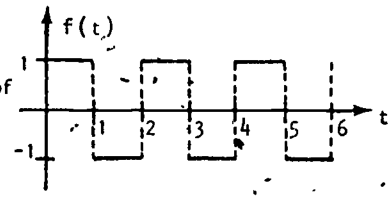
The next exercises will conclude the application portion of this unit. For those who wish to learn how the unified method can be derived, we carry out the derivation in the Appendix for the coefficient transform. Derivation of Formulas (9) and (11) can be carried out in a similar way.

Exercise 5

Use the unified method to find the Laplace transform of $f(t) = e^{-2t}$.

Exercise 6

Find the Laplace transform of the periodic square wave shown to the right. This problem will require an extension of the unified method to a case where the number of jumps discontinuities in f(t) is countably infinite.



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4. ANSWERS TO EXERCISES

1. a. $\frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$.

b. $\frac{1}{s^2} - \frac{1}{s} + e^{-2s}\left\{\frac{1}{s} - \frac{5}{s^2} - \frac{2}{s^3}\right\} + e^{-3s}\left\{\frac{3}{s} + \frac{6}{s^2} + \frac{2}{s^3}\right\}$.

c. $\frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{2e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2}$.

2. a. $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi t/3}$, where

$C_n = \frac{2}{in\pi}[1 - (-1)^n]$ for $n \neq 0$, $C_0 = 0$.

b. $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi t/2}$, where

$C_n = \frac{[illegible]}{n^2\pi^2}[1 - (-1)^n]$ for $n \neq 0$, $C_0 = \frac{1}{4}$.

3. $f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{5}$, where $b_n = (-1)^{n+1}\frac{5}{n\pi}$.

4. a. $\frac{2\sin 2\alpha}{\alpha}$.

b. $\frac{2}{\alpha^2}(1 - \cos\alpha)$.

5. $L\{e^{-2t}\} = \frac{1}{s} + \frac{1}{s}L\{-2e^{-2t}\}$, from which $L\{e^{-2t}\} = \frac{1}{s+2}$.

6. $L\{f(t)\} = \frac{1}{s}\left[1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + - \ldots\right]$

$= \frac{1}{s}\left[1 - 2\frac{e^{-s}}{1+e^{-s}}\right]$

$= \frac{1}{s}\tanh\frac{s}{2}$.

5. MODEL EXAM

Find the following transforms and series expansions using the unified method.

1. Find the Laplace transform $L\{f(t)\}$ for the functions:

a. $f(t) = \begin{cases} t^2, & 0 \leq t \leq 1 \\ 1, & 1 < t. \end{cases}$

b. $f(t) = \cos t$.

2. Find the Fourier transform $F\{f(t)\}$ for the functions:

a. $f(t) = \begin{cases} 4-t^2, & |t| \leq 2 \\ 0, & |t| > 2. \end{cases}$

b. $f(t) = e^{-2|t|}$, all t.

3. Find the half range sine series for function

$f(t) = 1 - \frac{1}{2}t, \quad 0 \leq t \leq 1.$

6. ANSWERS TO MODEL EXAM

1. a. $\frac{2}{s^3} - 2e^{-s}\left(\frac{1}{s^2} + \frac{1}{s^3}\right)$

   b. $L\{\cos t\} = \frac{1}{s} + \frac{1}{s}\left(\frac{1}{s} L\{-\cos t\}\right)$, from which

   $L\{\cos t\} = \frac{s}{s^2+1}$.

2. a. $\frac{4}{\alpha^3}\sin 2\alpha - \frac{8}{\alpha^3}\cos 2\alpha$

   b. $F\{e^{-2|t|}\} = \frac{-4}{(i\alpha)^2} + \frac{1}{(i\alpha)^2} F\{4e^{-2|t|}\}$, from which

   $F\{4e^{-2|t|}\} = \frac{4}{4+\alpha^2}$.

3. $f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$, where

   $b_n = \frac{1}{n\pi}\left[2 + (-1)^{n+1}\right]$.

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7. APPENDIX: THE METHOD DERIVED

In this section we carry out the derivation of formula (9). The formal proof for a function with a finite number of jump discontinuities requires an induction argument, but the idea can be seen by considering a function f(t) with jumps only at t=0, t=2p and one intermediate point t=a (see Figure A1). Thus we take f(t) of the form

$$f(t) = \begin{cases} f_1(t), & 0 \le t < a \\ f_2(t), & a < t < 2p. \end{cases}$$

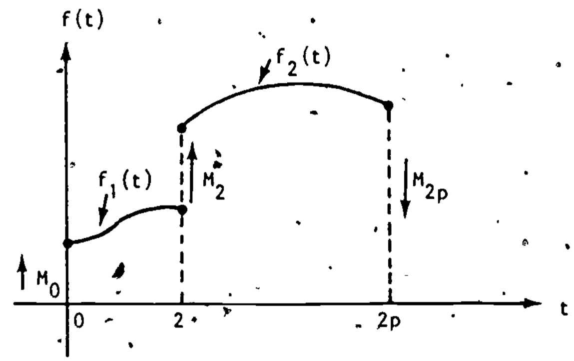

Figure A1. Graph of f(t) with one intermediate discontinuity.

By the definition of C{f(t)}, (see Formula (5)), we have

$$C\{f(t)\} = \int_0^a e^{-in\pi t/p} f_1(t)dt + \int_a^{2p} e^{-in\pi t/p} f_2(t)dt.$$

We integrate by parts, with $u=f_1(t)$, $f_2(t)$ and $dv = e^{-in\pi t/p}$, so that $du=f_1'(t)dt$, $f_2'(t)dt$ and $v= -\frac{p}{in\pi} e^{-in\pi t/p}$. Thus

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$$C\{f(t)\} = \frac{p}{in\pi}\left[-f_1(t)e^{-in\pi t/p}\Big|_0^a - f_2(t)e^{-in\pi t/p}\Big|_a^{2p}\right]$$

$$+ \frac{p}{in\pi}\left[\int_0^a e^{-in\pi t/p} f_1'(t)dt + \int_a^{2p} e^{-in\pi t/p} f_2'(t)dt\right].$$

Collecting terms we find

$$C\{f(t)\} = \frac{p}{in\pi}\{f_1(0) + [f_2(a+) - f_1(a-)]\, e^{-in\pi a/p} - f_2(2p)\}$$

(A1)

$$+ \frac{p}{in\pi}\int_0^{2p} e^{-in\pi t} f'(t)dt,$$

since $e^{-in\pi t/p}=1$ for $t=2p$. Let M_0, M_a, M_{2p} denote the "jumps" in $f(t)$ as shown in Figure A1; moreover we assume that the value is positive when the jump is up and negative when down. (Thus, for the function pictured in Figure A1, $M_0>0$, $M_a>0$, $M_{2p}<0$.) We may therefore write the expression (A1) as

(A2) $$C\{f(t)\} = \frac{p}{in\pi}\{M_0 + M_a e^{-in\pi a/p} + M_{2p}\} + \frac{p}{in\pi} C\{f'(t)\}.$$

The first term in (A2) is more systematic that it appears, since it can be written as

$$M_0 e^{-in\pi 0/p} + M_a e^{-in\pi a/p} + M_{2p} e^{-in\pi 2p/p}.$$

Thus, in actuality, each signed jump is multiplied by the exponential $e^{-in\pi t/p}$ evaluated at the value of t where the jump is made, and the resulting products are summed. Finally, as may be verified by an easy induction argument, when $a_0=0$, $a_m=2p$, and the function $f(t)$ has $m-1$ jump discontinuities in between, at $a_1, \ldots, a_{m-1}$, we have

$$C\{f(t)\} = \frac{p}{in\pi}\sum_{k=0}^{M} M_k e^{-in\pi a_k/p} + \frac{p}{in\pi} C\{f'(t)\},$$

where M_k is the signed jump at a_k.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________________

Please use reverse if necessary.

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STUDENT FORM 2

Unit Questionnaire

Name________________________ Unit No.__________ Date__________

Institution________________________ Course No.____________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much	Somewhat	About	Somewhat	Much
____Longer	____Longer	____the Same	____Shorter	____Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

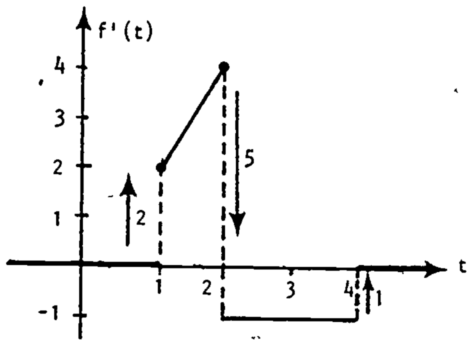
umap

UNIT 325

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

AN INVERSION METHOD FOR LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

by C.A. Grimm



INTEGRAL TRANSFORMS AND SERIES EXPANSION

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AN INVERSION METHOD FOR LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

by

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Rapid City, South Dakota 57701

TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 325*

Title: AN INVERSION METHOD FOR LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

Author: C.A. Grimm
Department of Mathematics
South Dakota School of Mines and Technology
Rapid City, South Dakota 57701

Classification: INT TRANS & SERIES EXP

Review Stage/Date: III 12/1/78

Prerequisite Skills:
1. Differentiate and integrate elementary functions.
2. Sketch graphs of elementary functions.
3. Use Euler's formula.
4. Have a basic understanding of integral transforms and orthogonal functions.
5. Identify odd and even functions.
6. Identify piecewise continuous functions.
7. Apply basic theorems on uniformly convergent series of functions..

Output Skills:
1. Use the method described in this unit to find a function f(t) when given
   a) the Laplace transform L{f(t)}
   b) the Fourier transform F{f(t)}
   c) the coefficient transform C{f(t)}
   d) the Fourier series expansion of f(t).

Other Related Units:
A UNIFIED METHOD OF FINDING LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES *(Unit 324)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank members of the UMAP Analysis and Computation Panel, Carroll O. Wilde, Chairman, Richard J. Allen, Louis C. Barrett, G.R. Blakely, and B. Roy Leipnik, for their reviews and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615 A02. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF nor of the National Steering Committee.

1. INTRODUCTION

In Unit 324 we presented a unified method of finding Laplace transforms, Fourier transforms, and Fourier series. The present unit is a sequel to Unit 324, and we shall freely carry over the notation and terminology used there. In particular, we are concerned primarily with the transforms L{f(t)}, F{f(t)}, and C{f(t)}.

Some applied problems require only the use of the forward transforms. In such problems, the calculation of the transform represents passage from time domain to the frequency domain, and the information obtained by studying frequency-related properties is all that is required. Other problems (such as solution of differential equations by transform techniques), however, require determination of inverse transforms; that is, recovery of f(t) from L{f(t)} or from F{f(t)}.

In this unit we build upon the ideas presented in Unit 324 to attack the problem of finding a function from its given transform or series expansion. Given a transform L{f(t)}, F{f(t)} or C{f(t)} we attempt to reconstruct first the derivatives of f(t), and finally f(t) itself, by reversing the process used to find the forward transform. Since the forward process is based on integration by parts, the method is generally applicable. Hence the inverse methods presented here are also generally applicable -- theoretically!! The problem is primarily that in finding a forward transform we may cancel terms which, if present, would provide clues to the nature of the derivative. For this reason, the inverse process does require patience and practice; nevertheless

it does offer most of the same advantages listed in Unit 324 for the unified (forward) method.

2. INVERSE LAPLACE TRANSFORMS

The process will be developed by examples. Again, it is assumed that you are familiar with the unified method presented in Unit 324.

Example 1

Suppose we wish to find f(t) if

$$L\{f(t)\} = \frac{2}{s} e^{-3s} - \frac{2}{s} e^{-4s}.$$

We first note that the right side can be written in slightly more revealing form as

$$2(\tfrac{1}{s})\, e^{-3s} - 2(\tfrac{1}{s})\, e^{-4s}.$$

The factor $\frac{1}{s}$ in each term indicates a jump in the function f(t), the multipliers 2 and -2 show the magnitude and the direction of each jump and, finally, the factors e^{-3s} and e^{-4s} show that the jumps occur at t = 3 and t = 4. Therefore the function must be given by

$$f(t) = \begin{cases} 2, & 3 < t < 4 \\ 0, & \text{elsewhere.} \end{cases}$$

(See Figure 1.)

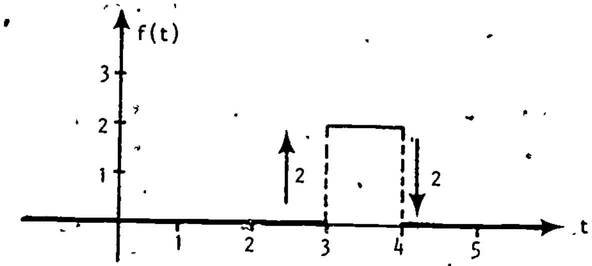

Figure 1. The graph of f(t) for Example 1.

Example 2

Find f(t), if

$$L\{f(t)\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{2}{s}\right) + e^{-2s}\left(-\frac{2}{s^3}-\frac{5}{s^2}-\frac{3}{s}\right) + \frac{e^{-4s}}{s^2}.$$

The exponentials e^{-s}, e^{-2s}, and e^{-4s} tell us to look for jumps at t = 1, 2, and 4, so we must watch these positions. However, we begin construction of the function with the terms $2(1/s^3)e^{-s}$ and $-2(2/s^3)e^{-2s}$, which indicate jumps of +2 at t=1 and +2 at t=2 in the second derivative. Hence, we have

$$f''(t) = \begin{cases} 2, & 1 < t < 2 \\ 0, & \text{elsewhere} \end{cases}$$

as shown in Figure 2.

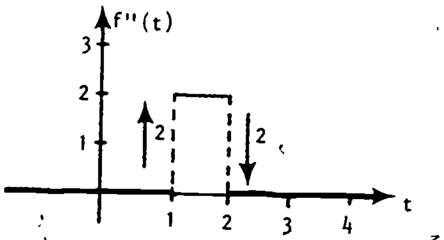

Figure 2. Graph of f"(t) for Example 2.

By integrating f"(t) we obtain the following expression:

$$f'(t) = \begin{cases} C_1, & 0 < t < 1 \\ 2t + C_2, & 1 < t < 2 \\ C_3, & 2 < t \end{cases}$$

where C_1, C_2, C_3 are constants to be determined.

Now the detective story begins! We must look to L{f(t)} to evaluate these constants. Since there is no term of the form

$$\frac{1}{s^k} = \frac{1}{s^k}e^{-0s},$$

there can be no jump in f(t) or any of its derivatives at the origin, so $C_1 = 0$. The term $2(1/s^2)e^{-s}$ reveals a jump of +2 in f'(t) at t=1 and since $C_1 = 0$, we must have

$$2 = f'(1) = 2 + C_2,$$

from which $C_2 = 0$. Next, the term $-5(1/s^2)e^{-2s}$ shows a jump of +5 at t=2. But f'(t) = 2t to the left of t=2, and f'(t) = C_3 to the right of t=2. Hence at t=2 we jump 5 units from 4 down to C_3, so that $C_3 = -1$. Finally, the term $1(1/s^2)e^{-4s}$ is consistent with the value $C_3 = -1$, since it shows a unit jump back to the t-axis at t=4. Therefore

$$f'(t) = \begin{cases} 2t, & 1 < t < 2 \\ -1, & 2 < t < 4 \\ 0, & \text{elsewhere} \end{cases}$$

(Figure 3).

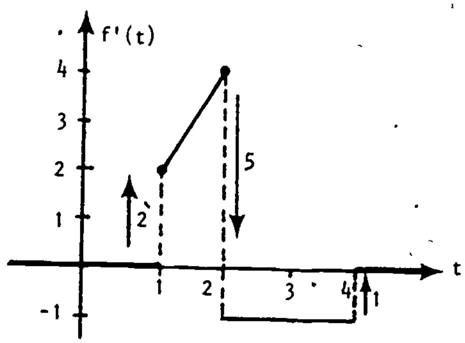

Figure 3. Graph of f'(t) for Example 3.

While the above explanation of how to find f'(t) may seem complicated, in actuality by observing the transform carefully and proceeding from left to right, we can (after a little practice) sketch f'(t) section by section rather quickly. We illustrate by obtaining f(t) from f'(t) graphically. The result is (See Figure 4):

$$f(t) = \begin{cases} t^2 + 1, & 1 < t < 2 \\ -t + 4, & 2 < t < 4 \\ 0, & \text{elsewhere.} \end{cases}$$

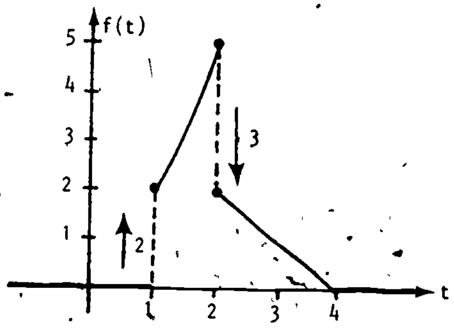

Figure 4. Graph of f(t) for Example 3.

The result was obtained section by section, as follows. First, we have previously observed that $f(t) = 0$ for $0 < t < 1$. Then, from the graph of $f'(t)$ we obtain $f(t) = t^2 + b_1$ for $1 < t < 2$, but the term $2(1/s)e^{-s}$ indicates a jump of $\uparrow 2$ at $t=1$. Hence, $2 = f(1+) = 1 + b_1$, so that $b_1 = 1$, and $f(t) = t^2 + 1$. Now for $2 < t < 4$, the graph of $f'(t)$ yields $f(t) = -t + b_2$, and the term $-3(1/s)e^{-2s}$ yields a jump of $\downarrow 3$ at $t=2$. Since $f(2-) = 5$ (from $t^2 + 1$), and $f(2+) = -2 + b_2$, we have $3 = 5 - (-2 + b_2)$, from which $b_2 = 4$ and $f(t) = -t + 4$ for $2 < t < 4$. Since there is no jump in $f(t)$ at $t=4$ we have $f(t) = 0$ for $t > 4$.

Exercise 1

For each of the following, find $f(t)$ from the given expression for $L\{f(t)\}$:

a) $\frac{1}{s} e^{-s}$;

b) $\frac{2}{s} e^{-s} - \frac{1}{s} e^{-3s}$;

c) $\frac{2}{s} + e^{-2s}\left(\frac{1}{2s^2} - \frac{4}{s}\right)$,

d) $e^{-2s}\left(-\frac{3}{s} - \frac{5}{s^2} - \frac{2}{s^3}\right) + e^{-6s}\left(\frac{2}{s} + \frac{1}{s^2}\right) + \frac{1}{s} + \frac{2}{s^3}$.

---

3. INVERSE FOURIER TRANSFORMS

We illustrate the procedure for finding the inverse Fourier transform by an example. Again, a familiarity with the forward transform from Unit 324 is assumed.

Example 3

Suppose we wish to find $f(t)$, if

$$F\{f(t)\} = \frac{2}{\alpha^2}(-1 + \cos\alpha) + \frac{2}{\alpha}\sin\alpha .$$

We begin by converting $F\{f(t)\}$ to exponential form so that we can identify the location, magnitude and direction of all jumps. Hence we have

$$F\{f(t)\} = \frac{1}{\alpha^2}\left(e^{i\alpha} + e^{-i\alpha}\right) - \frac{2}{\alpha^2} + \frac{1}{i\alpha}\left(e^{i\alpha} - e^{-i\alpha}\right).$$

We now recall that information on jumps for the forward transform is recorded in terms of powers of $1/i\alpha$; hence we must make a further adjustment to obtain

$$F\{f(t)\} = e^{i\alpha}\left(-\frac{1}{(i\alpha)^2} + \frac{1}{i\alpha}\right) + \frac{2}{(i\alpha)^2}$$

$$+ e^{-i\alpha}\left(-\frac{1}{(i\alpha)^2} - \frac{1}{i\alpha}\right).$$

The terms

$$-\frac{1}{(i\alpha)^2} e^{i\alpha}, \quad \frac{2}{(i\alpha)^2}, \quad -\frac{1}{(i\alpha)^2} e^{-i\alpha}$$

indicate jumps of $\downarrow 1$ at $t = -1$, $\uparrow 2$ at $t = 0$, and $\downarrow 1$, at $t = 1$. Hence we may now sketch the graph of $f'(t)$ (Figure 5). Then working from left to right as before,

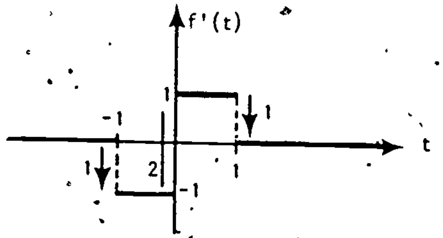

Figure 5. Graph of $f'(t)$ for Example 3.

we find first that $f(t) = -t + C_1$, $-1 < t < 0$. But the term $\frac{1}{i\alpha}e^{i\alpha}$ shows a jump of $+1$ in $f(t)$ at $t = -1$; hence $f(-1+) = 1 + C_1$, from which $C_1 = 0$. For $0 < t < 1$, $f(t) = t + C_2$; but the term $-\frac{1}{i\alpha}e^{-i\alpha}$ shows a jump of -1 in $f(t)$ at $t = 1$, and in addition, no jump in $f(t)$ is indicated at $t = 0$. Therefore, $C_2 = 0$; hence we obtain

$$f(t) = \begin{cases} |t|, & |t| < 1 \\ 0, & |t| > 1. \end{cases}$$

(See Figure 6.)

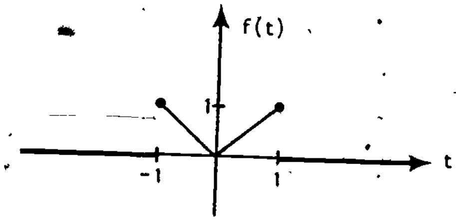

Figure 6. Graph of $f(t)$ for Example 3.

Exercise 2

For each of the following, find $f(t)$ from the given expression for $F\{f(t)\}$:

a. $\frac{1}{\alpha}(i\cos 2\pi\alpha - \sin 2\pi\alpha - i)$;

b. $\frac{2i}{\alpha^2}(\sin\alpha - \sin 2\alpha)$;

c. $\frac{2}{\alpha^3}(\alpha^2 \sin\alpha + 2\alpha\cos\alpha - 2\sin\alpha)$.

4. INVERSE COEFFICIENT TRANSFORMS

For our final examples we shall find the "inverse" of three Fourier series. That is, the problem we solve is the following: given a Fourier series, find the function to which the series converges. Those of you who are familiar with Fourier series may well be surprised to learn that this problem may have a reasonable solution.

The problem of recovering a function from its Fourier series representation may well require considerable ingenuity, insight and, perhaps, even some experimentation. The reason for this is three-fold. The first reason is the nature of the expansion itself -- very simple functions may yield expansions with coefficients of considerable complexity. The second difficulty arises from the terms of the form

$$e^{-in\pi a_k/p}$$

[see formula (9) of Unit 324]. The problem is that for $a_k = 0$ and for $a_k = 2p$, we have

$$e^{-in\pi a_k/p} = 1.$$

Therefore, we may not know whether the jump is at $t = 0$

or at $t = 2p$, or perhaps both. Similarly, if terms of the form $(-1)^n$ occur in the expansion, we may have either $a_k = -p$ or $a_k = p$, since $(-1)^n = \cos n\pi = e^{in\pi} = e^{-in\pi}$. Therefore, in this case the actual integration in $C\{f(t)\}$ would have been from $t = -p$ to $t = p$. But again, we may not know whether the jump was at $-p$ or at p.

The third difficulty is related to the second. Since the exponentials involved are equal at the end points of the interval in question, it follows that if the corresponding coefficients are equal in magnitude but opposite in sign, then the sum of these terms will vanish! Hence, we may be looking at a situation in which there is actually a jump present, but no indication of it. It may well require some patience to overcome these difficulties!

Example 4

Suppose we wish to find the function $f(t)$ whose Fourier series expansion is

$$\frac{8}{3} + \sum_{n=1}^{\infty} \left[\frac{3}{n^2\pi^2}\left(\cos\frac{4n\pi}{3} - 1\right)\cos\frac{2n\pi t}{3} + \left(-\frac{4}{n\pi} + \frac{3}{n^2\pi^2}\sin\frac{4n\pi}{3}\right)\sin\frac{2n\pi t}{3}\right].$$

We observe immediately that

$$a_0 = \frac{16}{3},$$

and for $n = 1, 2, 3, \ldots$ we have

$$a_n = \frac{3}{n^2\pi^2}\left(\cos\frac{4n\pi}{3} - 1\right),$$

$$b_n = \left(-\frac{4}{n\pi} + \frac{3}{n^2\pi^2}\sin\frac{4n\pi}{3}\right).$$

The approach we use is: first find C_n, next find $C\{f(t)\}$, and then recover the function $f(t)$. Since

$$C_n = (a_n - ib_n)/2 \quad \text{(formula (8) from Unit 324)}$$

we have

$$C_n = \frac{1}{2}\left[\frac{3}{n^2\pi^2}\left(\cos\frac{4n\pi}{3} - i\sin\frac{4n\pi}{3}\right) - \frac{3}{n^2\pi^2} + \frac{4i}{n\pi}\right]$$

$$= \frac{1}{2}\left(\frac{3}{n^2\pi^2}e^{-4n\pi i/3} - \frac{3}{n^2\pi^2} + \frac{4i}{n\pi}\right).$$

From the general form of the Fourier expansion we obtain that for this example

$$\cos\frac{n\pi t}{p} = \cos\frac{2n\pi t}{p}$$

from which

$$p = 3/2.$$

Since

$$C\{f(t)\} = 2pC_n = 3C_n,$$

we have

$$C\{f(t)\} = \frac{9}{2n^2\pi^2}e^{-4n\pi i/3} - \frac{9}{2n^2\pi^2} + \frac{6i}{n\pi}.$$

Because the coefficients in the C-transform involve powers of $\frac{p}{in\pi} = \frac{3}{2in\pi}$ and exponentials of the form $e^{-in\pi a_k/p}$, we write

$$C\{f(t)\} = -2\left(\frac{3}{2in\pi}\right)^2 e^{-\frac{2in\pi}{3}\cdot 2} + \left(\frac{3}{2in\pi}\right)^2 = 4\left(\frac{3}{2in\pi}\right).$$

The first term indicates a jump of -2 in the first derivative, $f'(t)$, at $t = 2$. The second term is the bothersome one -- it could indicate a jump of $+2$ at $t = 0$ or a jump of $+2$ at $t = 2p = 3$, or it could be the result of a combination of jumps at both places.

In order to allow for the various possibilities, we write for one period:

$$f'(t) = \begin{cases} 2-a, & 0 < t < 2 \\ -a, & 2 < t < 3. \end{cases}$$

(See Figure 7.) From the expression for $f'(t)$ we

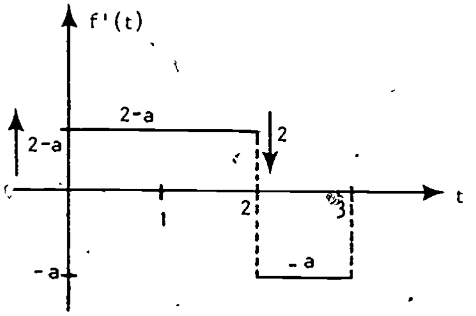

Figure 7. Graph of f'(t) for Example 4.

obtain for one period:

$$f'(t) = \begin{cases} (2-a)t + b, & 0 < t < 2 \\ -at + c, & 2 < t < 3. \end{cases}$$

The expression for $C\{f(t)\}$ above shows no jump in $f(t)$ at $t = 2$, and therefore the left and right sections of $f(t)$ agree at $t = 2$. Thus,

$$(2 - a)(2) + b = (-a)(2) + c,$$

from which

$$c = b + 4.$$

The last term in the expression for $C\{f(t)\}$ could arise from a jump of -4 at $t = 0$ or at $t = 3$, or from a combination of jumps at both ends. Now observe that the jump at $t = 0$ is $f(0+)$ and the jump at $t = 3$ is

$-f(3-)$. Since these jumps must combine to produce the value -4, we have

$$\begin{aligned} -4 = f(0) - f(3) &= b - (-3a + c) \\ &= (b - c) + 3a \\ &= -4 + 3a, \end{aligned}$$

from which

$$a = 0.$$

So far we have for one period

$$f(t) = \begin{cases} 2t + b, & 0 < t < 2 \\ b + 4, & 2 < t < 3. \end{cases}$$

Finally, since

$$a_0 = \frac{16}{3} = \frac{1}{p}\int_0^{2p} f(t)\,dt,$$

we have

$$\frac{16}{3} = \frac{2}{3}\left[\int_0^2 (2t + b)\,dt + \int_2^3 (b + 4)\,dt\right],$$

from which

$$8 = \left[t^2 + bt\right]_0^2 + \left[(b + 4)t\right]_3^2$$

$$= 8 + 3b,$$

so that

$$b = 0.$$

Hence the given series converges to a function of period 3 whose definition for one period is

$$f(t) = \begin{cases} 2t, & 0 < t < 2 \\ 4, & 2 < t < 3. \end{cases}$$

(See Figure 8.)

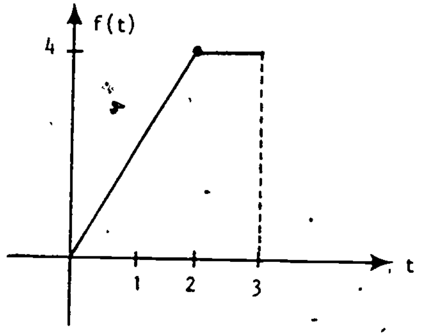

Figure 8. Graph of f(t) for Example 4.

Example 5

Suppose we wish to find the function f(t) whose Fourier series expansion is given by

$$f(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{12}{n^3\pi^3} \sin n\pi t.$$

We first note that from the given expansion, $a_n = 0$ for $n = 0, 1, 2, \ldots$, so that f(t) is an odd function. In addition, from the terms $\sin n\pi t$ we have $p = 1$, hence f(t) has period 2, and we must find an expression for f(t) over any interval of length 2; we choose $-1 \leq t \leq 1$.

Next we find C_n:

$$C_n = \frac{1}{2}(a_n - ib_n) = \frac{(-1)^{n+1}(-bi)}{n^3\pi^3} = \frac{6(-1)^n}{(in\pi)^3}.$$

Therefore,

$$C\{f(t)\} = 2p\,C_n = \frac{12(-1)^n}{(in\pi)^3}.$$

In general, we must express C{f(t)} in powers of $p/in\pi = 1/in\pi$, but this task is already accomplished here. The single term in C{f(t)} indicates a jump of

magnitude 12 in f"(t), and the factor $(-1)^n = e^{in\pi} = e^{-in\pi}$ shows that the jump is at t = 1 or t = -1 or that, perhaps, the whole term results from a combination of jumps at both ends. However, a little reflection shows that we simply cannot have a positive jump at just one end of the interval $-1 \leq t \leq 1$ or, for that matter, any combination of jumps at both ends with sum total positive if, after a jump is made the function remains constant until the next jump.

The following function is a possibility for f"(t) and we take it as our starting point:

$$f''(t) = -6t + a, \qquad -1 < t < 1.$$

(See Figure 9.) If we try the above function for

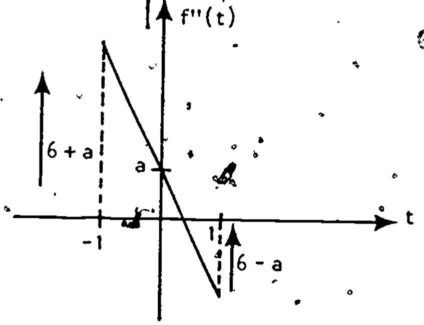

Figure 9. Graph of a possible f"(t) for Example 5.

f"(t) as a point of departure, then

$$f'''(t) = -6, \qquad -1 < t < 1.$$

(This is where a bothersome point arises: with f'''(t) as above, C{f(t)} would contain the expression

$$-\frac{6}{(in\pi)^4} e^{-in\pi(-1)} + \frac{6}{(in\pi)^4} e^{-in\pi(1)},$$

which results from jumps of +6 at t = -1 and +6 at t = 1. However, since $e^{in\pi} = e^{-in\pi} = (-1)^n$, the sum reduces to zero. Because of this cancellation the jumps made by $f'''(t)$ are lost from view in the transform $C\{f(t)\}$.)

We proceed from our point of departure. From the expression for $f''(t)$ we obtain

$$f'(t) = -3t^2 + at + b, \qquad -1 < t < 1 .$$

To evaluate the constants it helps if we know as much about the nature of the function as possible. The following argument will be very useful.

Both the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{12}{n^3\pi^3} \sin n\pi t$$

and the series of its derivatives with respect to t

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{12}{n^2\pi^2} \cos n\pi t$$

converge uniformly by the Weienstrass M-test, applied with the series of constants

$$\frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^3}, \qquad \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

respectively, used for comparison. Since the sum of a uniformly convergent series of continuous functions is continuous, we have continuity of $f(t)$. In addition, since the series for $f(t)$ converges and the series of derivatives converges uniformly, we have that

$$f'(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{12}{n^2\pi^2} \cos n\pi t,$$

and hence $f'(t)$ is also continuous.

Now, there are two easy arguments we can use to find the constant a in $f'(t)$. First, since $f'(t)$ is given by a cosine series, it is an even function,

and hence $a = 0$. An alternative argument which is also useful in general is that by continuity and periodicity we have

$$f'(-1) = f'(1),$$

so that

$$-3 - a + b = -3 + a + b,$$

from which $a = 0$. Hence, either way we find

$$f(t) = -t^3 + bt + c.$$

Similarly, to find c we may observe that $f(t)$ is an odd function, so that $c = 0$. We could also argue that since $f(t)$ is a sine series, we must have $f(0) = 0$, from which $c = 0$. Finally, to find b, we observe that by the series definition of $f(t)$ and its continuity, which precludes jumps from one period to the next, we have $f(1) = 0$, hence $-1 + b = 0$, and $b = 1$. Alternatively, by the continuity and the periodicity of $f(t)$ we have $f(-1) = f(1)$, so that

$$1 - b = -1 + b, \quad \text{and} \quad b = 1.$$

Thus, for one period

$$f(t) = -t^3 + t, \qquad -1 \le t \le 1.$$

(See Figure 10.)

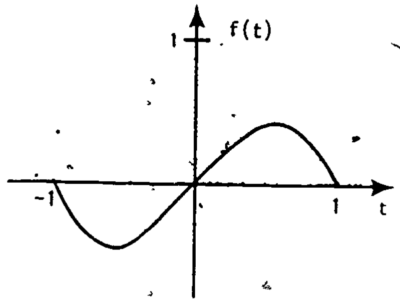

Figure 10. Graph of $f(t)$ for Example 5.

Example 6

Suppose we wish to find the function f(t) whose Fourier series expansion is given by

$$f(t) = \frac{8}{\pi^2} \sum_{n=1,3,5\ldots} \frac{(-1)^{(n-1)/2}}{n^2} \sin nt.$$

We first note that $n\pi t/p = nt$, from which $p = \pi$, $2p = 2\pi$. In addition, $a_n = 0$ for $n = 0, 1, 2, \ldots$, so that f(t) is an odd function. The coefficients b_n are given by

$$b_n = \begin{cases} \frac{8}{n^2\pi^2}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Since

$$C_n = \frac{1}{2}(a_n - ib_n),$$

we have

$$C_n = \begin{cases} -\frac{4i}{n^2\pi^2}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

from which

$$C\{f(t)\} = 2p\,C_n = \begin{cases} -\frac{8i}{n^2\pi}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Since the transform coefficients in formula (9) of Unit 324 are expressed in powers of $p/in\pi$, we rewrite the preceding expression in the form

$$C\{f(t)\} = \begin{cases} \left(\frac{\pi}{in\pi}\right)^2 \cdot \frac{8i}{\pi}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

We now face the crucial problem of finding the way in which the two distinct expressions (for the odd and the even coefficients) in C{f(t)} can be unified into a single form. Your ability to make this step requires careful observation in working with the forward transform, and with trigonometric functions in general. We simply note the result

$$(-1)^{(n-1)/2} = \sin\frac{n\pi}{2}, \quad n = 1, 2, 3, \ldots.$$

(Check out a few values of n for yourself!) We therefore have

$$C\{f(t)\} = \left(\frac{\pi}{in\pi}\right)^2 \cdot \frac{8i}{\pi} \sin\frac{n\pi}{2}, \quad n = 1, 2, 3, \ldots$$

We now convert to exponential form:

$$C\{f(t)\} = \left(\frac{\pi}{in\pi}\right)^2 \frac{4}{\pi}\left(e^{in\pi/2} - e^{-in\pi/2}\right).$$

With $p = \pi$, we now obtain the coefficient transform in the form of (9) from Unit 324:

$$C\{f(t)\} = \left(\frac{\pi}{in\pi}\right)^2 \left(\frac{4}{\pi} e^{-in\pi(-\pi/2)/\pi} - \frac{4}{\pi} e^{-in\pi(\pi/2)/\pi}\right).$$

This form reveals jumps of $\uparrow 4/\pi$ at $-\pi/2$ and $\downarrow 4/\pi$ at $\pi/2$ in the first derivative f'(t). However, we must also be alert to possible cancellation of terms, especially at $-\pi$ and π. The simplest type of function whose behavior agrees with what we have so far is one with derivative of the form

$$f'(t) = \begin{cases} a, & -\pi < t < -\pi/2 \text{ and } \pi/2 < t < \pi \\ \frac{4}{\pi} + a, & -\pi/2 < t < \pi/2 \end{cases}$$

(See Figure 11. We note that because the period is π, and contributions at $-\pi$ and π would cancel:

$$ae^{in\pi} - ae^{-in\pi} = 0.$$

Such cancellation does not occur at $\pi/2$.)

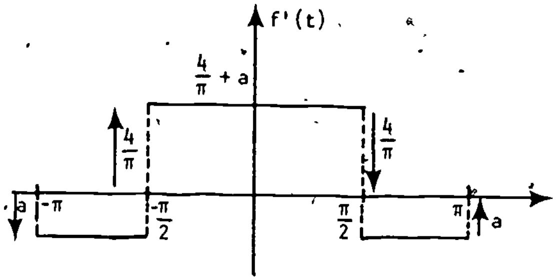

Figure 11. A possible form of f'(t) for Example 6.

Proceeding from our point of departure, we have

$$f(t) = \begin{cases} at + b, & -\pi < t < -\pi/2 \\ \left(\frac{4}{\pi} + a\right)t + c, & -\pi/2 < t < \pi/2 \\ at + d, & \pi/2 < t < \pi . \end{cases}$$

To evaluate the constants, we first apply the M-test (as in Example 5), using $(8/\pi^2) \sum_{n=1,3,5...} 1/n^2$ for comparison to see that f(t) is continuous everywhere. Again, we illustrate two alternative methods for determining the constants.

First, since f(t) is an odd function, $f(-t) = -f(t)$, so that

$$f(-t) = \begin{cases} -at + b, & -\pi < -t < -\pi/2, \text{ i.e., } \pi/2 < t < \pi \\ -\left(\frac{4}{\pi} + a\right)t + c, & -\pi/2 < -t < \pi/2, \text{ i.e., } -\pi/2 < t < \pi/2 \\ -at + d, & \pi/2 < -t < \pi, \text{ i.e., } -\pi < t < -\pi/2 \end{cases}$$

$$= -f(t) = \begin{cases} -at - b, & -\pi < t < -\pi/2 \\ -\left(\frac{4}{\pi} + a\right)t - c, & -\pi/2 < t < \pi/2 \\ -at - d, & \pi/2 < t < \pi \end{cases}$$

from which

$$b = -d \quad \text{and} \quad c = 0.$$

By continuity of f(t) at $t = \pi/2$,

$$a\pi/2 + d = \left(\frac{4}{\pi} + a\right)\frac{\pi}{2} ,$$

from which $d = 2$ and hence $b = -2$. From the series definition of f(t) and continuity, $f(\pi) = 0$, hence $a\pi + 2 = 0$ and $a = -2/\pi$.

Alternatively, we could have used the series definition of f(t) to obtain $f(t) = 0$, from which $c = 0$. Continuity of f(t) at $t = \pi/2$ now yields the equation

$$a\pi/2 + d = \left(\frac{4}{\pi} + a\right)\frac{\pi}{2} ,$$

from which $d = 2$. Similarly, from continuity of f(t) at $-\pi/2$ we obtain $b = -2$. The constant a is determined as above.

By either approach we obtain

$$f(t) = \begin{cases} -\frac{2}{\pi}t - 2, & -\pi < t < -\pi/2 \\ \frac{2}{\pi}t , & -\pi/2 < t < \pi/2 \\ -\frac{2}{\pi}t + 2, & \pi/2 < t < \pi . \end{cases}$$

(See Figure 12.)

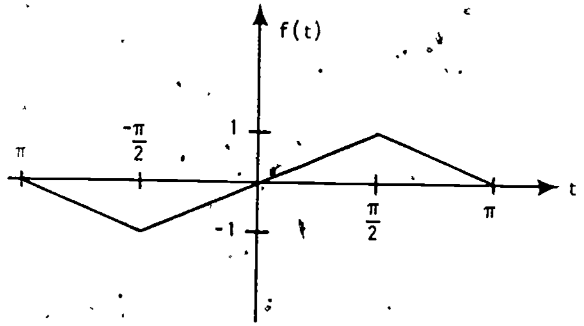

Figure 12. Graph of f(t) for Example 6.

Exercise 3

For each of the following, find the function f(t) whose Fourier series expansion is given:

a. $f(t) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin \frac{n\pi t}{2}$

(it may be helpful to note that $\cos n\pi = (-1)^n$ and that

$$1 - (-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases});$$

b. $f(t) = \frac{1}{3} + 4 \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n^2\pi^2} \cos n\pi t + \frac{(-1)^{n+1}}{n\pi} \sin n\pi t\right].$

---

5. MODEL EXAM

1. Find f(t) if

a. $L\{f(t)\} = \frac{1}{2s^2}\left(3 - e^{-2s} - 4e^{-3s} + 2e^{-5s}\right) - \frac{1}{s}\left(1 + e^{-3s}\right);$

b. $L\{f(t)\} = e^{-3s}\left(\frac{1}{2s^2} - \frac{1}{s}\right) - e^{-s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right)$

$$+ \frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}.$$

2. Find f(t) if

$$F\{f(t)\} = \frac{4 \sin \alpha}{\alpha} (\cos \alpha - 1).$$

3. Find f(t) if f(t) is periodic of period 3, f(3/2) = 0 and if

$$C\, f(t) = \begin{cases} -\frac{3i}{2n\pi} + \frac{9}{n^2\pi^2}, & n \text{ odd} \\ -\frac{3i}{2n\pi}, & n \text{ even.} \end{cases}$$

6. ANSWERS TO EXERCISES

1\. a. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t. \end{cases}$

b. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & 1 < t < 3 \\ 1, & 3 < t. \end{cases}$

c. $f(t) = \begin{cases} 2, & 0 < t < 2 \\ \frac{1}{2}t - 3, & 2 < t. \end{cases}$

d. $f(t) = \begin{cases} t^2 + 1, & 0 < t < 2 \\ -t + 4, & 2 < t < 6 \\ 0, & 6 < t. \end{cases}$

2\. a. $f(t) = \begin{cases} -1, & -2\pi < t < 0 \\ 0, & \text{elsewhere.} \end{cases}$

b. $f(t) = \begin{cases} t + 2 & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & 2 < |t|. \end{cases}$

c. $f(t) = \begin{cases} t^2 & |t| < 1 \\ 0 & |t| > 1. \end{cases}$

3\. a. $f(t) = \begin{cases} -1, & -2 < t < 0 \\ 1, & 0 < t < 2. \end{cases}$

b. $f(t) = t^2 + 2t, \quad -1 < t < 1.$

7. ANSWERS TO MODEL EXAM

1\. a. $f(t) = \begin{cases} \frac{3}{2}t - 1, & 0 < t < 2 \\ t, & 2 < t < 3 \\ 5 - t, & 3 < t < 5 \\ 0, & 5 < t. \end{cases}$

b. $f(t) = \begin{cases} t^2 + t + 1, & 0 < t < 1 \\ 2, & 1 < t < 3 \\ \frac{1}{2}t - \frac{1}{2} & 3 < t. \end{cases}$

2\. $f(t) = \begin{cases} t + 2, & -2 < t < -1 \\ |t|, & -1 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & 2 < |t|. \end{cases}$

3\. $f(t) = \begin{cases} \frac{3}{2}t + 1, & -\frac{3}{2} < t < 0 \\ -\frac{4}{3}t + 2, & 0 < t < \frac{3}{2}. \end{cases}$

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name____________________ Unit No.________

Page______ ○ Upper ○ Middle ○ Lower	OR	Section______ Paragraph______	OR	Model Exam Problem No._____ Text Problem No._____

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature____________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
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Newton, MA 02160

Name________________________Unit No.____________Date____________

Institution________________________Course No.________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)