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ABSTRACT

This document consists of three modules. The first looks at applications of analysis to medical radiology. The goals are to provide: 1) acquaintance with a significant applied mathematics problem utilizing Fourier Transforms; 2) generalization of the Fourier Transforms to two dimensions; 3) practice with Fourier Transforms; and 4) introduction to the Mankel Transform. Exercises and problem solutions are included. The second unit looks at applications of probability to medicine. Output skills are aimed to provide an authentic application of elementary probability, and to introduce binomial probability through a real-life application. The next module examines applications of differential equations to biology. Stated goals are to aid the user in understanding: 1) the geometric structure of a honeycomb cell; and 2) how minimization techniques from calculus apply to minimum surface area problems for a bee's cell. All units include exercises, with answers provided. The second module contains sample exams, with answers posted at the conclusion of the unit. (MP)

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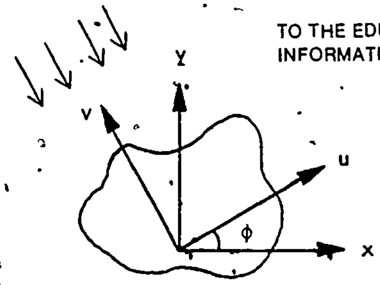
UNIT 318

TOMOGRAPHY:

THREE DIMENSIONAL IMAGE RECONSTRUCTION

by Frederick Solomon



APPLICATIONS OF ANALYSIS TO MEDICAL RADIOLOGY

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TOMOGRAPHY:
THREE-DIMENSIONAL IMAGE RECONSTRUCTION

by

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Intermodular Description Sheet: UMAP Unit 318

Title: TOMOGRAPHY: THREE DIMENSIONAL IMAGE RECONSTRUCTION

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Review Stage/Date: III 1/23/79

Classification: APPL ANALYSIS/MEDICAL RADIOLOGY

Suggested Support Material:

References: See Section 9 of text.

Prerequisite Skills:
1. Familiarity with the definition and basic properties of Fourier Transforms.
2. Familiarity with contour integration and the Residue Theorem in the complex plane.

Output Skills:
1. Acquaintance with a significant applied math problem utilizing Fourier Transforms.
2. Generalization of the Fourier Transforms to two dimensions.
3. Practice with Fourier Transforms.
4. Introduction to the Hankel Transform.

Other Related Units:

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank R. Bruce Meriole, William Glessner and Robert L. Baker, Jr. for their reviews, and all others who assisted in the production of this unit.

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TOMOGRAPHY:

THREE-DIMENSIONAL IMAGE RECONSTRUCTION

1. THE PROBLEM

Suppose we want to map out the interior of an opague object. To be specific suppose we want to determine the interior of part of a human body—an arm, say. An x-ray photograph reveals the two-dimensional projection of the arm on a viewing screen. This projection indicates points of high and low mass density. However, given a point of high density, we still do not know where in the interior of the arm the point is located—it could be anywhere on the line perpendicular to the screen and hence parallel to the x-ray beam. The problem is an important one. A lesion in the brain will register on the viewing screen; and it is important to know just where along the line parallel to the x-ray beam the lesion is located. So the problem is in constructing a three-dimensional map from two-dimensional pictures. (Actually, the phrase mass density is used in a suggestive sense here; it is only one factor affecting x-ray impenetrability.)

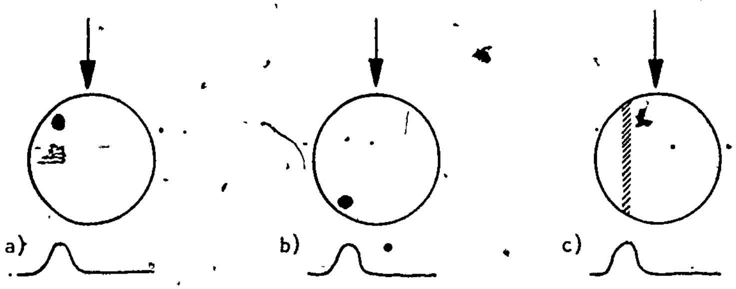


Figure 1. Arrows indicate direction of x-ray beam on a circular object. The dark area inside the object corresponds to a region of high mass density. All three have the same projection on the viewing screen, but the locations of the high mass density are quite different. It might even be uniformly spread along the line parallel to the x-ray beam as in (c).

As we shall see, if x-ray photographs are taken at many different angles, then the information from all of them can be synthesized into the desired three-dimensional map. The purpose of this unit is to show how this is done.

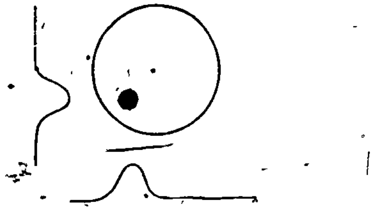


Figure 2. Two x-ray projections indicate the point of high mass density.

Although the goal is to construct a three-dimensional map, the problem is actually a two-dimensional one. If x, y, z-axes are situated so that the x-ray beams are perpendicular to the z-axis, it is enough to construct a map of each cross section parallel to the xy plane. Thus we will be concerned solely with a two-dimensional object in the xy plane with incident x-rays in this plane.

2. NOTATION

Let $f(x,y)$ be the mass density of the object and let $\phi+\frac{\pi}{2}$ be the angle between the x-axis and the incident beam of x-rays. Draw u,v-axes as in Figure 3, that is, the

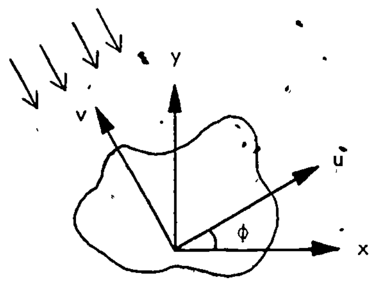


Figure 3. Arrows in upper left indicate the direction of x-rays.

v-axis points in the direction of the x-ray source while the u-axis is perpendicular to the v-axis with the u,v-axes in the same orientation as the x,y-axes. Finally we define the *ray-sum* $R(u,\phi)$ as the *total* mass registered u units from the v-axis when $\phi+\frac{\pi}{2}$ is the angle of the x-ray beam. That is, $R(u_0,\phi_0)$ is the total mass on line ℓ in Figure 4(b).

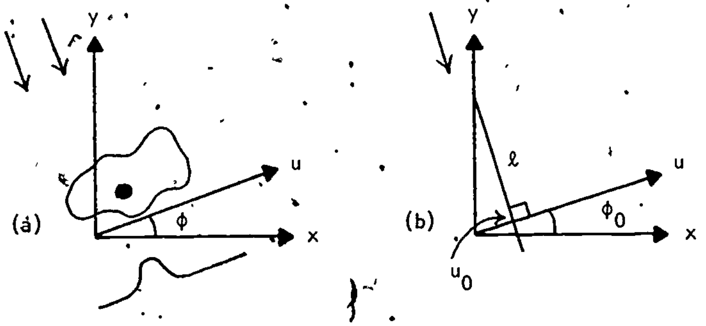


Figure 4(a) Graph of $R(u,\phi)$ for fixed ϕ. (b) Line ℓ is u_0 from origin.

Now we assume that all ray-sums $R(u,\phi)$ (for all u and all ϕ) are known. (In actuality we would know the function $R(u,\phi)$ for a finite number of angles ϕ as we direct beams from different directions at the object; and $R(u,\phi)$ would be approximated by interpolation for all other values of ϕ.) Notice that $R(u,\phi+\pi) = R(-u,\phi)$ (Why?); so it is only necessary to find $R(u,\phi)$ for $0 \le \phi \le \pi$.

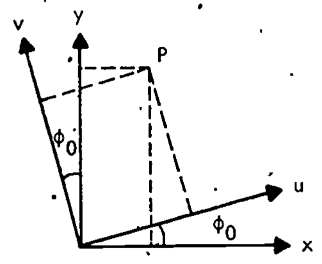
The definition for fixed u_0 and ϕ_0 of $R(u_0,\phi_0)$ is related to the mass density $f(x,y)$ by

$$R(u_0,\phi_0) = \text{total mass along } \ell \text{ in Figure 4(b)} = \int_\ell f(x,y)\,ds$$

where s is arc-length along ℓ. The equation of ℓ is, of course, just $u = u_0 = \text{constant}$. But trigonometry yields

$$(1)\quad \begin{aligned} u &= x\cos\phi_0 + y\sin\phi_0 \\ v &= -x\sin\phi_0 + y\cos\phi_0 \end{aligned} \qquad \begin{aligned} x &= u\cos\phi_0 - v\sin\phi_0 \\ y &= u\sin\phi_0 + v\cos\phi_0. \end{aligned}$$

Exercise 1. Derive these equations: Let P be a point with coordinates (x,y) in the x,y coordinate system and (u,v) in the u,v system. Show that (x,y) and (u,v) are related as above.



So

$$(2)\quad R(u_0,\phi_0) = \int_\ell f(x,y)\,ds = \int_{-\infty}^{\infty} f(u_0\cos\phi_0 - v\sin\phi_0,\ u_0\sin\phi_0 + v\cos\phi_0)\,dv.$$

(Note that if the object has finite extent [so that $f(x,y) = 0$ for (x,y) not in the object], then the above integral actually has finite limits.) The problem is to solve this integral equation for $f(x,y)$ given the ray-sums $R(u,\phi)$. In other words, in a real situation the ray-sums would be known, but the actual mass density would be of interest.

3. BACK PROJECTION: AN APPROXIMATION

Let's first obtain an approximate solution to the problem of reconstructing the unknown mass density $f(x,y)$ from the known ray-sums $R(u,\phi)$. This is the technique of back projection. Suppose in addition to the ray-sums we also know that the object is contained in the disk of radius L about the origin. Now $R(u,\phi)$ is the integral of f along ℓ; and the length of ℓ (or rather the length of the intersection of ℓ with the disk of radius L) is $2\sqrt{L^2-u^2}$ (see Figure 5). Suppose that (x_0,y_0) is a point

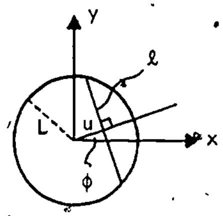


Figure 5. ℓ has length $2\sqrt{L^2-u^2}$.

on ℓ. Under the assumption that all the mass on ℓ is *uniformly* distributed along the line segment ℓ, then the mass density along ℓ would be

$$(3) \qquad \delta(x_0,y_0) = \frac{R(u,\phi)}{2\sqrt{L^2-u^2}}$$

where $u = x_0 \cos\phi + y_0 \sin\phi$ as in Exercise 2. Finally we approximate $f(x_0,y_0)$ as the average of all $\delta(x_0,y_0)$ as ϕ varies from 0 to 2π.

Exercise 2. Show that the equation of line ℓ through (x_0,y_0) at angle ϕ as in Figure 5 is

$$y = -(\tan\phi)^{-1}x + [y_0 + (\tan\phi)^{-1}x_0].$$

Show that the distance from this line to the origin is

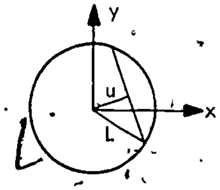
$$u = x_0\cos\phi + y_0\sin\phi.$$

Using Exercise 2 we see that the back projection estimate of f is

$$(4) \qquad F(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R(x_0\cos\phi + y_0\sin\phi,\phi)}{2\left[L^2 - (x_0\cos\phi + y_0\sin\phi)^2\right]^{\frac{1}{2}}}\,d\phi .$$

Example: Suppose the object is actually the disk of radius L of uniform density $f(x,y) = $ constant c. Then we would observe the ray-sums (from Equation (2))

$$R(u,\phi) = \begin{cases} 2c\sqrt{L^2-u^2}, & |u| \le L \\ 0, & |u| > L. \end{cases}$$



Thus for every u,ϕ

$$\frac{R(u,\phi)}{2\sqrt{L^2-u^2}} = c \text{ for } |u| \le L$$

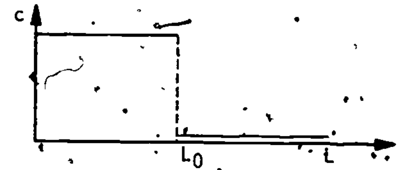
and back projection yields approximation

$$F(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} c\,d\phi = c$$

for (x_0,y_0) inside the disk of radius L. And this agrees with the exact value of $f(x_0,y_0)$.

Exercise 3. Suppose the object is contained in the disk of radius L about the origin with density $f(x,y) = c(x^2 + y^2)$, c a positive constant. Find the ray-sums $R(u,\phi)$ and the back projection $F(x,y)$. How does F compare with f?

Exercise 4. Suppose the object is known to be contained in the disk of radius L about the origin, but is actually the disk of radius $L_0 < L$ of constant density c.



Abscissa represents $r = \sqrt{x^2+y^2}$ = distance from origin.

What are the ray-sums $R(u,\phi)$ and the back projection approximation $F(x,y)$? (Leave your answer as an integral.) Draw a *rough* sketch of F as a function of the distance r from the origin.

In general the back projection approximation overestimates regions of low mass density and underestimates regions of high mass density. This is due to the fact that projection for each fixed ϕ of $R(u,\phi)$ *uniformly* back through the object will overestimate regions of low density and underestimate regions of high density.

4. FOURIER TRANSFORM NOTATION

Let $f(x)$ be a function of the real variable x, $-\infty < x < \infty$. We use this convention for the *Fourier Transform* $\hat{f}$ of f:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\omega x}dx.$$

Under suitable conditions the *Fourier Inversion Theorem* holds

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{2\pi i\omega x}d\omega.$$

(Suitable conditions are, for example, that $\int_{-\infty}^{\infty} |f| < \infty$, $\int_{-\infty}^{\infty} |f|^2 < \infty$, f is continuous and is piecewise differentiable.) These results can easily be extended to functions $f(x,y)$ of two variables. Thus we set

$$\hat{f}(\omega,\sigma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(\omega x+\sigma y)}dxdy.$$

And under suitable conditions the Fourier Inversion Theorem holds

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\omega,\sigma)e^{2\pi i(\omega x+\sigma y)}d\omega d\sigma.$$

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Exercise 5. Let $f(x,y) = e^{-a(x^2+y^2)}$ where a is a positive constant. Use contour integration to show that

$$\hat{f}(\omega,\sigma) = \frac{\pi}{a}\exp\left[-\frac{\pi^2}{a}(\omega^2+\sigma^2)\right].$$

Show that the Fourier Inversion Theorem holds for this f. (Note: the second part of this problem requires no new integrations.)

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5. THE EXACT SOLUTION

We now show how the ray-sums $R(u,\phi)$ completely determine $\hat{f}(\omega,\sigma)$, the Fourier Transform of the mass density. The trick is to change the variables of integration in the integral defining $\hat{f}$ above. Switch to the u,v coordinates defined in Equations (1). Orient ω,σ axes—the domain variables for $\hat{f}$—so that $\phi = \arctan\left(\frac{\sigma}{\omega}\right)$. Then

$$\cos\phi = \sec^{-1}\phi = (1+\tan^2\phi)^{-\frac{1}{2}}$$

$$= (1+\sigma^2/\omega^2)^{-\frac{1}{2}} = \frac{\omega}{\sqrt{\omega^2+\sigma^2}}$$

$$\sin\phi = \frac{\sigma}{\sqrt{\omega^2+\sigma^2}}$$

$$\omega x+\sigma y = \omega(u\cos\phi - v\sin\phi) + \sigma(u\sin\phi + v\cos\phi)$$

$$= (\sqrt{\omega^2+\sigma^2})u.$$

Also $dudv = dxdy$ since (u,v) and (x,y) are each Cartesian coordinate systems. So

$$\hat{f}(\omega,\sigma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(\omega x+\sigma y)}dxdy \tag{5}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dv\,\exp\left[-2\pi i(\sqrt{\omega^2+\sigma^2})u\right]du.$$

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where $x = u\cos\phi - v\sin\phi$

$y = u\sin\phi + v\cos\phi$.

But the inner integral is $R(u,\phi)$ by Equation (2). Thus

$$\hat{f}(\omega,\sigma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(u,\phi)\exp\left[-2\pi i(\sqrt{\omega^2+\sigma^2})u\right]du.$$

However, this integral is by definition the Fourier Transform of $R(u,\phi)$ with respect to the u variable at $\sqrt{\omega^2+\sigma^2}$ and with ϕ fixed at arctan (σ/ω). In other words, $f(\omega,\sigma)$ is obtained by fixing ϕ at arctan (σ/ω), taking the Fourier Transform of $R(u,\phi)$ with respect to the single variable u, and evaluating this Fourier Transform at $\sqrt{\omega^2+\sigma^2}$. For a function $g(x,y)$ of two variables we denote the Fourier Transform with respect to the first variable by $\hat{g}_1(\omega,y)$:

$$\hat{g}_1(\omega,y) = \int_{-\infty}^{\infty} g(x,y)e^{-2\pi i\omega x}dx.$$

Summarizing yields

6. THE BASIC THEOREM

$$(6)\qquad \hat{f}(\omega,\sigma) = \hat{R}_1\left[\sqrt{\omega^2+\sigma^2},\ \arctan\left(\frac{\sigma}{\omega}\right)\right].$$

The Theorem shows us how, in theory, to reconstruct the mass density $f(x,y)$ from the ray-sums $R(u,\phi)$. Namely, from $R(u,\phi)$ calculate the Fourier Transform with respect to u, $\hat{R}_1(\tau,\phi)$. Then $\hat{f}(\omega,\sigma)$ can be calculated by the Theorem from which $f(x,y)$ can be found by Fourier Inversion. Of course in practice $R(u,\phi)$ is much too complicated to find its Fourier Transform without a computer approximation of the integral defining $\hat{R}_1(\tau,\phi)$ and similarly for the step from $\hat{f}$ to f. Still, the Theorem indicates the general technique.

Example: Suppose the ray-sums are found to be

$$R(u,\phi) = e^{-au^2}$$

for positive constant a. (Noting that $R(u,\phi)$ is independent of ϕ, we expect $f(x,y)$ to be a function of the distance x^2+y^2 from the origin only.) Now

$$\hat{R}_1(\tau,\phi) = \int_{-\infty}^{\infty} e^{-au^2}e^{-2\pi i\tau u}du$$

$$= \sqrt{\frac{\pi}{a}}\exp\left[-\frac{\pi^2}{a}\tau^2\right]$$

either using tables or by a contour integration as in Exercise 5. So

$$\hat{f}(\omega,\sigma) = \hat{R}_1(\sqrt{\omega^2+\sigma^2}, \arctan(\sigma/\omega))$$

$$= \sqrt{\frac{\pi}{a}}\exp\left[-\frac{\pi^2}{a}(\omega^2+\sigma^2)\right].$$

Thus

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(\omega,\sigma)e^{2\pi i(\omega x+\sigma y)}d\omega d\sigma$$

$$= \sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty}\exp\left[-\frac{\pi^2}{a}\omega^2 + 2\pi i\omega x\right]d\omega \cdot \int_{-\infty}^{\infty}\exp\left[-\frac{\pi^2}{a}\sigma^2 + 2\pi i\sigma y\right]d\sigma$$

$$= \sqrt{\frac{\pi}{a}}\cdot\sqrt{\frac{a}{\pi}}e^{-ax^2}\cdot\sqrt{\frac{a}{\pi}}e^{-ay^2}$$

$$= \sqrt{\frac{a}{\pi}}\exp\left[-a(x^2+y^2)\right]$$

where the second to the last equality follows by a contour integration as in Exercise 5.

Exercise 6. Suppose $R(u,\phi)$ is found to be $e^{-|u|}$ independently of ϕ. Find $\hat{R}_1(\tau,\phi)$, $\hat{f}(\omega,\sigma)$, set up the integral for $f(x,y)$.

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7. CIRCULAR SYMMETRY

Many corollaries follow from the basic theorem of Equation (6). We indicate one which generalizes Exercise 6 and the Example before it.

Suppose $R(u,\phi)$ exhibits some sort of periodic behavior in ϕ. That is, suppose the x-ray photographs begin to repeat themselves as we move the angle of the incident x-ray beam. (Of course such a repetition always occurs by moving the beam by 2π. We have smaller periods of repetition in mind here.) Then it is natural to use Polar rather than Cartesian Coordinates. Thus let

$$x = r\cos\theta$$
$$y = r\sin\theta$$

as always for Polar Coordinates. And in the (ω,σ)-plane let

$$\omega = t\cos\beta$$
$$\sigma = t\sin\beta$$

denote the Polar representation. That is, $t = \sqrt{\omega^2+\sigma^2}$, $\beta = \arctan(\sigma/\omega)$. Now

$$(7)\qquad f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\omega,\sigma)e^{2\pi i(\omega x+\sigma y)}d\omega d\sigma$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{R}_1\left[\sqrt{\omega^2+\sigma^2}, \arctan(\sigma/\omega)\right]e^{2\pi i(\omega x+\sigma y)}d\omega d\sigma$$

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$$= \int_0^{\infty}\int_0^{2\pi} \hat{R}_1(t,\beta)e^{2\pi irt\cos(\beta-\theta)}t\,d\beta dt$$

where the second equality follows from the Basic Theorem and the last by using Polar Coordinates: the fact that

$$\omega x+\sigma y = rt\cos\beta\cos\theta + rt\sin\beta\sin\theta$$
$$= rt\cos(\beta-\theta).$$

Note that $t\,d\beta dt$ is the area element in polar coordinates. And by Exercise 7 below we conclude that

$$(8)\qquad f(r,\theta) = \int_0^{\infty}\int_0^{2\pi} \hat{R}_1(t,\beta+\theta)e^{2\pi irt\cos\beta}t\,d\beta dt.$$

Exercise 7. Let g be a real valued function of a real variable which is periodic of period 2 :

$$g(\beta+2\pi) = g(\beta)$$

for all β. Let c be a fixed number. By considering the graph of g show that

$$\int_c^{2\pi+c} g(\beta)d\beta = \int_0^{2\pi} g(\beta)d\beta.$$

Noting that both $\cos\beta$ and $\hat{R}_1(t,\beta+\theta)$ are periodic of period 2π (so that their product is as well) derive Equation (8) from the last expression in Equation (7).

So far we have been completely general. To understand the above equation more, let's take the case $R(u,\phi) = R(u)$ independently of ϕ. Then $\hat{R}_1(\tau,\phi) = \hat{R}_1(\tau)$ is independent of ϕ (Why?). So

$$(9)\qquad f(r,\theta) = \int_0^{\infty}\int_0^{2\pi} \hat{R}_1(t)e^{2\pi irt\cos\beta}t\,d\beta dt.$$

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Noting that the integral is independent of θ, we have just proved the obvious fact: *If the ray-sums* $R(u,\phi)$ *are independent of the angle of the incident x-rays, then* f *must be circularly symmetric*, i.e., independent of θ.

Exercise 8. Let λ be a real constant. The value of

$$\int_0^{2\pi} e^{i\lambda \cos s} ds$$

is denoted $2\pi J_0(\lambda)$. (J_0 is called the *Bessel Function* of order zero.) Evaluate this integral as an infinite sum using contour integration. (Hint: Convert to a contour integral over the counterclockwise unit circle centered at the origin using $z = e^{is}$. Show that the integral is

$$\int_{C} \exp\left[i \frac{\lambda}{2} \left(z + \frac{1}{z}\right)\right]\frac{dz}{iz}$$

which is $(2\pi i)$ Residue of $\frac{1}{iz} \exp\left[i \frac{\lambda}{2} \left(z + \frac{1}{z}\right)\right]$ at $z = 0$. Now evaluate this residue as an infinite series.)

Using Exercise 8 and its solution we see that if $R(u,\phi)$ is independent of ϕ, then (from Equation (9))

(10) $$f(r,\theta) = f(r)$$

$$= \int_0^\infty \hat{R}_1(t)\left[\int_0^{2\pi} e^{2\pi i r t \cos\beta} d\beta\right] t\,dt$$

$$= \int_0^\infty \hat{R}_1(t) \cdot 2\pi J_0(2\pi r t)\, t\,dt$$

where $J_0(\lambda)$ is the Bessel Function of order zero.

$$J_0(\lambda) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2}\left(\frac{\lambda}{2}\right)^{2j}.$$

(See the Answer to Exercise 8.) This formula exhibits $f = f(r)$ as what is called the *Hankel Transform of order zero* of $\hat{R}_1(t)$. Hankel Transforms occur quite naturally in two-dimensional Fourier Analysis problems.

Exercise 9. Suppose the ray-sums are found to be independent of ϕ with.

$$R(u,\phi) = R(u) = \frac{2}{1 + (2\pi u)^2}.$$

By a contour integration show that $\hat{R}_1(t) = e^{-|t|}$. Show that

$$\int_0^\infty e^{-t} \cdot t^{2j+1} dt = (2j + 1)!$$

and that $f(r,\theta) = 2\pi(1 + (\pi r)^2)^{-3/2}$. (Note: You will need the fact that

$$\binom{2j}{j} = \binom{-1/2}{j}(-1)^j$$

and the Binomial Theorem in the special case

$$(1 + x)^{-1/2} = \sum_{j=0}^{\infty} \binom{-1/2}{j} x^j$$

where by definition

$$\binom{-1/2}{j} = \frac{(-1/2)(-1/2-1)\ .\ .\ .\ (-1/2 - j + 1)}{j!}.)$$

Although this formula exhibiting $f(r)$ as the Hankel Transform of $\hat{R}_1(t)$ solves the case when $R(u,\phi) = R(u)$ is independent of ϕ, the actual integration would normally require a computer approximation. However, the Hankel Transform of some common functions are listed in mathematical tables.

8. THE CIRCULARLY ASYMMETRIC CASE

A similar analysis can be carried out when $R(u,\phi)$ *does* depend on ϕ. The solution exhibits $f(r,\theta)$ as combinations of higher order Hankel Transforms. The solution is elegant, but we relegate it to

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Exercise 10. Now $\hat{R}_1(t,\phi)$—the Fourier Transform of $R(u,\phi)$ with respect to the u variable—is *for each fixed* t a periodic function of ϕ of period 2π. This is so since the ray-sums $R(u,\phi)$ repeat as ϕ changes by 2π. Thus $\hat{R}_1(t,\phi)$ can be expanded as a Fourier Series for each fixed t:

$$\hat{R}_1(t,\phi) = \sum_{n=-\infty}^{\infty} \hat{c}_n(t)e^{in\phi}$$

where by the Fourier Series formula

$$\hat{c}_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \hat{R}_1(t,\phi)e^{-in\phi}d\phi.$$

a. Show that $\hat{c}_n(t)$ is the Fourier Transform of the nth Fourier Coefficient of $R(u,\phi)$

$$c_n(u) = \frac{1}{2\pi}\int_0^{2\pi} R(u,\phi)e^{-in\phi}d\phi.$$

b. Plug the new expression for $\hat{R}_1(t,\phi)$ into Equation (8) to show that

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta}\cdot\int_0^{\infty}\hat{c}_n(t)d_n(rt)\,tdt$$

where

$$d_n(rt) = \int_0^{2\pi} \exp\,[ins + 2\pi irt\cdot\cos s]\,ds.$$

By definition $J_n(\lambda) = (2\pi i^n)^{-1}d_n(\lambda/2\pi)$ is called the *Bessel Function of order n.*

c. Now fix r and consider $f(r,\theta)$ as a function of θ. Conclude from 10a and 10b above, that the nth Fourier Coefficient of $f(r,\theta)$ when expanded in a Fourier Series as a function of θ is

$$2\pi i^n\int_0^{\infty}\hat{c}_n(t)J_n(2\pi rt)\,tdt$$

which is i^n-times the *nth Hankel Transform* of $\hat{c}_n$.

---

9. HISTORICAL AND BIBLIOGRAPHICAL NOTE

Analytic techniques in tomography were first used in radioastronomy. Antennae were unable to focus on the individual points of the solar surface, but were able to focus on thin striplike segments crossing the sun. Thus, in making a map of the microwave radiation density of the sun, astronomers had to reconstruct it from "strip sums." In medical radiology x-ray CAT (Computer assisted tomography) scanners were introduced in 1972. These machines take x-ray pictures at about one degree intervals between 0° and 180°. While CAT scanners are expensive (of the order of half a million dollars per machine), they afford a revolutionary improvement in medical diagnosis.

For an interesting nontechnical source of information see "Computerized Tomography" by W. Swindell and H. Barrett in *Physics Today* (December 1977). For a deeper mathematical treatment see "Principles of Computer Assisted Tomography" by R. Brooks and G. DiChiro in *Physics in Medicine and Biology*, Volume 21 (1976, pp. 689-732). For an applied, very readable account of Fourier Series, Fourier Transforms and Hankel Transforms see *The Fourier Transform and its Applications* by R.M. Bracewell (McGraw-Hill, 1965). An informative, nonmathematical article on a similar method of image reconstruction is "Ultrasound in Medical Diagnosis" by G. Devey and P. Wells found in

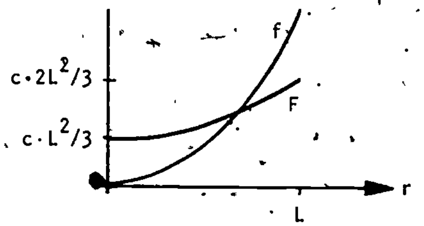
Scientific American (May 1978, pp. 98-112). In this technique, sound rather than x-rays are used to avoid organ damage.

10. ANSWERS AND SUGGESTIONS TO EXERCISES

3. $R(u,\phi) = \frac{2c}{3}\sqrt{L^2-u^2}(L^2+2u^2)$ from Equation (2).

$F(x_0,y_0) = \frac{c}{3}(L^2+x_0^2+y_0^2)$ from Equation (4).

As functions of $r = \sqrt{x^2+y^2}$ the graphs of f and F are



4. $R(u,\phi) = \begin{cases} 2c\sqrt{L_0^2-u^2}, & |u| \le L_0 \\ 0, & L_0 < |u| \le L. \end{cases}$

So by Equation (2)

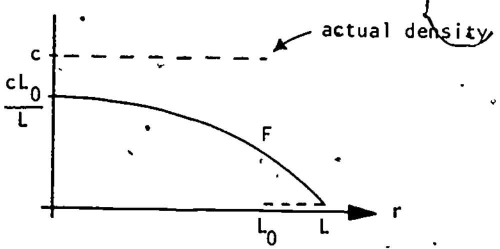
$$F(x_0,y_0) = \frac{1}{2\pi}\int c\sqrt{\frac{L_0^2-u^2}{L^2-u^2}}\,d\phi$$

where $u = x_0\cos\phi + y_0\sin\phi$ and the range of integration is over all $0 \le \phi \le 2\pi$ so that $|u| \le L_0$. Note that

$$F(0,0) = \frac{cL_0}{L}$$

$$F(x_0,y_0) = 0$$

for $\sqrt{x_0^2+y_0^2} = L$. A *rough* sketch of F as a function of the distance from the origin is



5. $$\hat{f}(\omega,\sigma) = \iint e^{-2\pi i(\omega x+\sigma y)}\cdot e^{-a(x^2+y^2)}\,dx\,dy$$

$$= \int e^{-2\pi i\omega x}e^{-ax^2}dx\cdot\int e^{-2\pi i\sigma y}e^{-ay^2}dy.$$

But the one-dimensional Fourier Transform of e^{-ax^2} is $\sqrt{\frac{\pi}{a}}\exp\left[-\frac{\pi^2}{a}\omega^2\right]$ from tables or by contour integration. Now with $\hat{f}(\omega,\sigma) = \frac{\pi}{a}\exp\left[-\frac{\pi^2}{a}(\omega^2+\sigma^2)\right]$,

$$\hat{\hat{f}}(x,y) = \frac{\pi}{a}\cdot\frac{\pi}{\pi^2/a}\exp\left[-\frac{\pi^2}{\pi^2/a}(x^2+y^2)\right]$$

$$= e^{-a(x^2+y^2)}$$

by the above integration with $\frac{\pi^2}{a}$ replacing a. So $\hat{\hat{f}}(-x,-y) = \hat{\hat{f}}(x,y) = f(x,y)$—which is the content of the Fourier Inversion Theorem.

6. $\hat{R}_1(\tau,\phi) = 2(1+4\pi^2\tau^2)^{-1}$.

Thus

$$\hat{f}(\omega,\sigma) = 2\left[1+4\pi^2(\omega^2+\sigma^2)\right]^{-1}.$$

8. On the unit circle

$$z = e^{is},\ \cos s = \frac{1}{2}\left(z+\frac{1}{z}\right),\ \frac{dz}{iz} = ds.$$

Now

$$e^{a(z+\frac{1}{z})} = \sum_{j=0}^{\infty}\frac{(az)^j}{j!}\cdot\sum_{k=0}^{\infty}\frac{(az^{-1})^k}{k!}.$$

The coefficient of z^0 in the product series is

$$\sum_{j=0}^{\infty} \frac{a^j}{j!}\frac{a^j}{j!} = \sum_{j=0}^{\infty} \frac{a^{2j}}{(j!)^2} .$$

Thus the coefficient of z^{-1} in $\frac{1}{iz}\exp\left[i\frac{\lambda}{2}(z+\frac{1}{z})\right]$ is

$$\frac{1}{i}\sum_{j=0}^{\infty}\left(\frac{i\lambda}{2}\right)^{2j}\cdot\frac{1}{(j!)^2} = \frac{1}{i}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j!)^2}\left(\frac{\lambda}{2}\right)^{2j} .$$

$2\pi i$ times this is the answer by the Residue Theorem.

9. With $\hat{R}(t) = e^{-|t|}$ the formulas above the statement of this problem imply

$$f(r,\theta) = 2\pi\int_0^{\infty} e^{-t} \cdot \sum_{j=0}^{\infty}\frac{(-1)^j}{(j!)^2}(\pi r)^{2j}t^{2j+1}dt$$

$$= 2\pi\sum_{j=0}^{\infty}\frac{(-1)^j}{(j!)^2}(\pi r)^{2j} \cdot (2j+1)!$$

$$= 2\pi\sum \binom{2j}{j}(-1)^j(2j+1)(\pi r)^{2j}$$

$$= 2\pi\frac{d}{d(\pi r)}\sum\binom{2j}{j}(-1)^j(\pi r)^{2j+1}$$

$$= 2\pi\frac{d}{d(\pi r)}\pi r\sum\binom{-1/2}{j}\left((\pi r)^2\right)^j$$

$$= 2\pi\frac{d}{d(\pi r)}\frac{\pi r}{\sqrt{1+(\pi r)^2}} .$$

STUDENT FORM 1

Request for Help

Return to:
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55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

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Name__________ Unit No.__________ Date__________

Institution__________ Course No.__________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much	Somewhat	About	Somewhat	Much
____Longer	____Longer	____the Same	____Shorter	____Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain__________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain__________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

umap

Unit 456

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

GENETIC COUNSELING

by Nancy S. Rosenberg

Allele from Mother

Allele from Father	A	a
[illegible]	AA	Aa
a	aA	aa

Genotype of Child

APPLICATIONS OF PROBABILITY TO MEDICINE

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GENETIC COUNSELING

by

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TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 456*

Title: GENETIC COUNSELING

Author: Nancy S. Rosenberg
Riverdale Country School
Bronx, NY 10471

Review Stage/Date: III 3/26/80

Classification: APPL PROBABILITY/MEDICINE

Prerequisite Skills:
1. Elementary Probability
2. The binomial theorem.

Output Skills:
1. To provide an authentic application of elementary probability.
2. To introduce binomial probability through a real-life application.

This module was written with support from The Klingenstein Center, Teachers College, Columbia University.

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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1. GENETIC DISEASE

1.1 Introduction

Genetics, the science of heredity, is barely one hundred years old. Although people have long been aware that parents pass many of their own traits on to their children, it is only recently that scientists have understood exactly how and why this happens. In the process of learning the mechanisms of heredity, they have discovered a large number of genetic diseases which, like other physical characteristics, are passed down from one generation to the next.

The gene is the unit of heredity. A complex protein molecule called deoxyribonucleic acid, or DNA, determines the color of the eyes, the texture of the hair, and most of the other traits that distinguish one person from another. In addition, genes control a multitude of biochemical reactions that take place inside cells. When they fail to do this correctly, the result can be a serious disruption of normal metabolism.

1.2 Cystic Fibrosis and Huntington's Disease

Cystic fibrosis, the most common inherited killer of children, is caused by just such an inborn error of metabolism. This error causes cells to produce a thick mucous secretion that clogs the passageways of many vital organs. Although treatment for cystic fibrosis has improved in recent years, few of its victims live to be adults. If a couple has one child with cystic fibrosis, the chances that each of their other children will be born with the disease are one out of four.

Another genetic disorder, Huntington's disease, does not make its appearance until the middle adult years. At that time, its victims exhibit odd postures, involuntary motions, and bizarre mental changes. Eventually they succumb to the disease. Many victims of Huntington's disease have children before they discover that they are ill. When this happens, the children have one chance in two of developing the disease themselves.

2. MECHANISMS OF INHERITANCE

2.1 Recessive and Dominant Diseases

To understand why the probability associated with Huntington's disease is higher than the one associated with cystic fibrosis, it is necessary to understand the different ways in which these two diseases are inherited. Genes come in pairs of two. In reproduction these pairs split, and the child receives one member of each pair, chosen at random, from each of his parents.

Each gene has a variety of different forms called alleles. Certain genetic diseases are caused by a single abnormal allele. Some of these diseases occur only in people who have two copies of this abnormal allele, one from each parent. These are called recessive diseases. Cystic fibrosis is a disease of this type. Others occur in people who have only one copy of the abnormal allele. If either parent passes down this abnormal allele, the child will be afflicted. Diseases like this are said to be dominant. Huntington's disease is an example of a dominant disorder. Once it is known that a hereditary disease is either recessive or dominant, simple laws of probability can be used to determine the chances that parents with particular combinations of genes, called genotypes, will transmit the disease to their children.

2.2 Genotypes and Their Probabilities

A child with cystic fibrosis has two abnormal alleles, one from each parent. Thus, if two healthy parents have a child with a recessive disease like cystic fibrosis, both of them must carry the abnormal allele. If a symbolizes this abnormal allele and A symbolizes the normal one, the parents' genotypes must therefore be Aa. People who have one normal allele and one abnormal one for a recessive disease are called carriers; they usually show no signs of the disease, but they can pass the abnormal allele on to their children. A child with a recessive disease has genotype aa. The diagram in Figure 1 shows how parents with genotype Aa can have a child with genotype aa.

Allele from Mother

Allele from Father	A	a
A	AA	Aa
a	aA	aa

Genotype of Child

Figure 1. The possible genotypes of a child with two carrier parents.

Because the parents transmit each of their genes with equal probability, the four possible outcomes are equally likely. There is therefore one chance in four that a child of these parents will get two normal alleles, two chances in four, or one in two, that he will be a carrier, and one chance in four that he will be born with the disease. These probabilities are of little use in predicting the outcome of any one particular birth. However, of 1000 children born to parents who are both carriers of cystic fibrosis, approximately 250 can be expected to be victims of the disease. Of the remaining 750, about 500 will be carriers. The chances that the healthy child of two carrier parents is himself a carrier are therefore 500 out of 750 or two out of three.

Because it is a dominant disorder, Huntington's disease will manifest itself in people with only one abnormal allele, that is, in people of genotype Aa. Since Huntington's disease is very rare, its victims are almost always married to people with two normal alleles. As shown in Figure 2, the children of such a marriage will be of two genotypes, Aa and AA. (In this case the mother is the normal parent and the father is the carrier.)

Allele from Mother

Allele from Father	A	A
A	AA	AA
a	aA	aA

Genotype of Child

Figure 2. The possible genotypes of a child with one carrier parent.

Since Aa individuals will develop the disease and AA individuals will not, the chances are two out of four, or one out of two, that the child of a victim of Huntington's disease will eventually develop it himself.

3. GENETIC COUNSELING

3.1 Predicting the Occurrence of Genetic Disease

Although few hereditary diseases can be treated effectively at present, tests have been developed to detect carriers of many genetic disorders and to identify a number of defects in unborn children. By this kind of screening, and by assessing the liklihood that genetic disease will arise in particular families, many hereditary diseases can now be prevented. Central to this effort is the genetic counselor, one of whose jobs it is to calculate the risk that particular parents will transmit a hereditary disease. To do this, she relies on two fundamental principles of probability.

1. Probability has no memory. The genotype of each

child of a particular marriage is independent of the genotypes of any previous children. Thus, the chances that the child of two carriers of cystic fibrosis will be afflicted with the disease are one in four for every child regardless of the outcomes of previous births.

2. If two or more events are independent, their probabilities are multiplied to get the probability that they will occur in sequence.

3.2 Some Typical Problems

Here are some problems of the sort that a genetic counselor might be called upon to solve.

Both Mr. and Mrs. B. had sisters afflicted with cystic fibrosis. What is the probability that they will have a child with the disease?

Solution: Because they produced afflicted children, both Mr. B's parents and Mrs. B's parents were carriers of cystic fibrosis. Of the normal children of carriers, two out of three are carriers themselves. Therefore, the probability that Mr. B is a carrier is 2/3 and the probability that Mrs. B is a carrier is also 2/3. Since the probability that two carriers of cystic fibrosis will produce an afflicted child is 1/4, the probability that Mr. B is a carrier, Mrs. B is a carrier, and their child will be born with the disease is $2/3 \times 2/3 \times 1/4$ or 1/9. It is worth noting that the probability of the complementary event, that their child will not be born with the disease, is 1 - 1/9 or 8/9.

The mother of Mr. L's father died of Huntington's disease. What are the chances that Mr. L will get it?

Solution: The probability that Mr. L's grandmother passed the abnormal allele on to her son is 1/2. The probability that he in turn passed it on to Mr. L is also 1/2. So the probability that Mr. L will develop Huntington's disease is $1/2 \times 1/2$ or 1/4.

Exercise 1. Both Mr. L and Mrs. L had brothers afflicted with cystic fibrosis. If they have two children, what is the probability that both will be afflicted with the disease?

Exercise 2. Mr. and Mrs. P are tested and both are found to be carriers of Tay-Sachs disease, a rare recessive disorder. What is the probability that their first child will be born with the disease? That their first two children will be healthy? That their first three children will be carriers?

Exercise 3. Albinism is a recessive disorder characterized by a marked reduction in pigmentation throughout the body. A woman who is an albino marries a normal man whose brother is an albino. What are the chances that their first child will be an albino? If their first child is an albino, what are the chances that their second child will be too?

Exercise 4. Two first cousins marry. If the sister of their grandfather had cystic fibrosis, what are the chances that they will have a child with the disease?

3.3 Conditional Probability

Example 1. Both Mr. and Mrs. A had sisters who died of cystic fibrosis. If the A's have two children, what are the chances that neither will be afflicted with the disease?

There are two possibilities to be considered here. Unless Mr. and Mrs. A are both carriers, their children run no risk of getting cystic fibrosis. If, on the other hand, both Mr. A and Mrs. A are carriers, the chances that each of their children will be healthy are three out of four. Because both of these conditions must be considered, this problem is said to involve conditional probability.

In the diagram in Figure 3, C represents the event that Mr. and Mrs. A are both carriers, and $\overline{C}$ is its complement. Since both of their parents were carriers, Mr. A and Mrs. A each have a 2/3 chance of being carriers themselves (see Section 3.2). The probability of event C is hence $2/3 \times 2/3$ or 4/9, and the probability of $\bar{C}$ is 1 - 4/9 or 5/9.

H stands for the event that Mr. and Mrs. A have two healthy children, $\bar{H}$ for its complement. If C is true, the probability of H is $3/4 \times 3/4$ or 9/16, and the probability of $\bar{H}$ is 1 - 9/16 or 7/16. If $\bar{C}$ is true, the probability of H is 1 and the probability of $\bar{H}$ is 0. The probabilities of the four compound events, found by multiplication, are given at the bottom of the diagram.

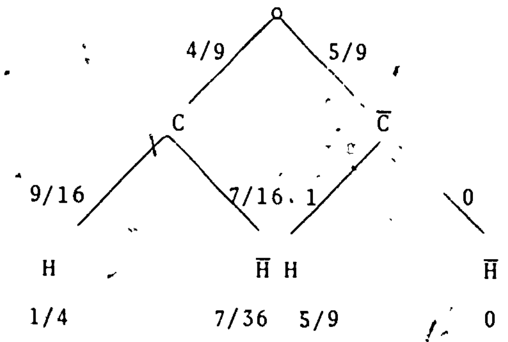


Figure 3. Diagram for Example 1.

The probability of event H, that Mr. and Mrs. A's two children are healthy, is 1/4 + 5/9 or 29/36.

In more conventional notation,

$$P(H) = P(C)\cdot P(H|C) + P(\bar{C})\cdot P(H|\bar{C})$$
$$= 4/9\cdot 9/16 + 5/9\cdot 1 = 29/36.$$

Example 2. Although Mr. P's mother died of Huntington's disease, Mr. P is now forty years of age and healthy. If three-quarters of all carriers of Huntington's disease develop symptoms by the time they are forty, what is the probability that Mr. P will eventually develop Huntinton's disease?

This is another problem in conditional probability,

because there are two conditions under which the son of a woman who died of Huntington's disease can be healthy at forty. Either he did not inherit the abnormal allele, or he did inherit the abnormal allele but has not yet developed the disease. Both of these possibilities must be considered.

In Figure 4, C represents the event that Mr. P is a carrier, $\bar{C}$ that he is not. The probability for each of these is 1/2. H is the event that Mr. P is healthy (that is, free of Huntington's disease) at forty, $\bar{H}$ that he is not. Since three-fourths of all carriers develop the disease by age forty, the probabilities of these in the event of C are 1/4 and 3/4 respectively. In the event of $\bar{C}$, the respective probabilities are 1 and 0.

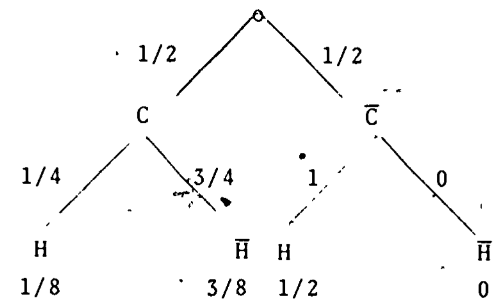


Figure 4. Diagram for Example 2.

Since 1/8 + 1/2 or 5/8 of the children of a woman who died of Huntington's disease can be expected to be healthy at forty, and 1/8 to be healthy at forty and carriers, the probability that Mr. P is a carrier given that he is healthy at forty is $\frac{1/8}{5/8}$ or 1/5. In conventional notation,

$$P(C\ H) = \frac{P(H,C)}{P(H)} = \frac{1/8}{5/8} = 1/5.$$

Exercise 5. Bill's father's mother had Huntington's disease. Although half the carriers of Huntington's disease show symptoms by age thirty-five, Bill's father just celebrated his thirty-fifth birthday and is still healthy. If Bill has a child someday, what is the probability that this child will get Huntington's disease?

Exercise 6. Willy's maternal grandmother has Huntington's disease. His mother is thirty-six and in good health. If 60% of the carriers of Huntington's disease develop symptoms by the time they are thirty-six, what are the chances that Willy will be free of the disease?

Exercise 7. Although both Mr. and Mrs. D have siblings who are albinos, all three of their children are normal. What are the chances that their fourth child will be an albino?

---

4. THE INCIDENCE OF HEREDITARY DISEASE

4.1 Patterns of Inheritance

There are now 2000 recognized genetic diseases. If they are to be investigated and controlled, it is essential that doctors and counselors understand the mechanisms by which they are inherited. In some cases these mechanisms are very complex, involving the interactions of large numbers of different genes. Dominant and recessive disorders, however, exhibit simple patterns of inheritance that make them easy to identify.

In the case of a dominant disorder, every affected individual will have an affected parent. Further, as we have seen, the chances that the child of an affected parent will inherit the disease are one in two, assuming that the other parent is normal. In a large group of children of such marriages, therefore, close to 50% will get the disease.

4.2 Predicting the Incidence of Hereditary Disease with Binomial Probability

What can be said about a single family? Suppose that a normal individual marries a carrier of Huntington's disease and they have three children. What are the possible outcomes? The diagram in Figure 5 can answer this question. Here N(½) indicates the birth of a normal child

with a probability of 1/2 and A(½) indicates the birth of an affected child with this same probability. To get the probabilities in the right hand column, the individual probabilities have been multiplied together.

Of the eight possible outcomes, one is for three normal children with a probability of 1/8, three are for two normal children and one affected child with a combined probability of 3 × 1/8 or 3/8, three are for one normal child and two affected children, also with a combined probability of 3/8, and one is for three affected children with a probability of 1/8. Note that these same results could more easily have been obtained by multiplication since

$$(\tfrac{1}{2}N + \tfrac{1}{2}A)^3 = C(3,0)(\tfrac{1}{2}N)^3(\tfrac{1}{2}A)^0 + C(3,1)(\tfrac{1}{2}N)^2(\tfrac{1}{2}A) + C(3,2)(\tfrac{1}{2}N)(\tfrac{1}{2}A)^2 + C(3,3)(\tfrac{1}{2}N)^0(\tfrac{1}{2}A)^3 = 1/8N^3 + 3/8N^2A + 3/8NA^2 + 1/8A^3.$$

Here the coefficients of the product give the probabilities of the corresponding combinations of N's and A's. Because the binomial theorem is used to obtain these results, the problem is said to involve binomial probability.

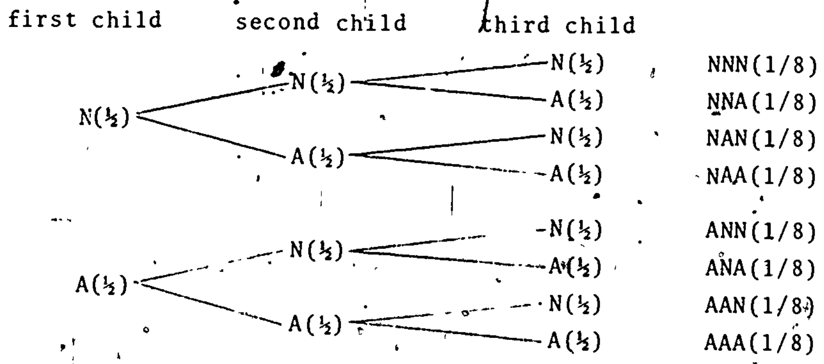


Figure 5. Possible outcomes for a three child family in which one parent carries a dominant disease.

For recessive diseases the probabilities differ. A child will be afflicted with a recessive disorder only if both of his parents are carriers. When this is the case, he has one chance in four of inheriting the disease and three chances in four of being normal. The possible outcomes for the three child family of two carriers of a recessive disorder are shown in Figure 6.

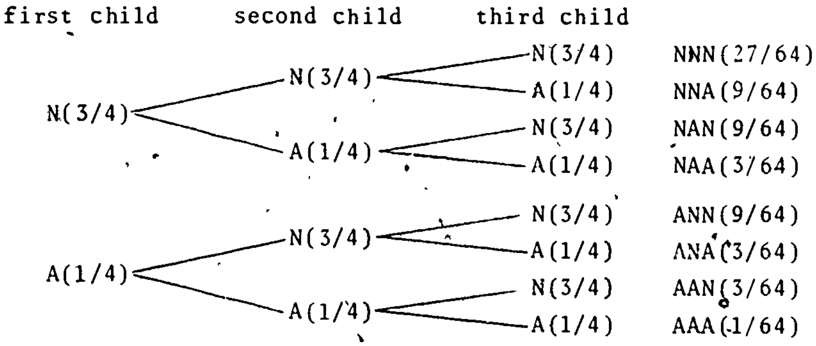


Figure 6. Possible outcomes for a three child family in which both parents carry a recessive disease.

Of the eight possible outcomes, one is for three normal children with a probability of 27/64, three are for two normal children and one affected child with a combined probability of 3 × 9/64 or 27/64, three are for one normal child and two affected children with a combined probability of 3 × 3/64 or 9/64, and one is for three affected children with a probability of 1/64. As before, a binomial expansion would have produced these same results, in this case the expansion of $(3/4N + 1/4A)^3$.

By using the binomial expansion in this way, a variety of questions can be answered about the incidence of hereditary disorders. Consider a large number of families of four children in which both parents are carriers of cystic fibrosis. In what percent of these families will two or more children be affected?

Solution: The expansion of $(3/4N + 1/4A)^4$ yields

$$C(4,0)(3/4N)^4(1/4A)^0 + C(4,1)(3/4N)^3(1/4A) + C(4,2)(3/4N)^2(1/4A)^2 + C(4,3)(3/4N)(1/4A)^3 + C(4,4)(3/4N)^0(1/4A)^4.$$

These last three terms correspond to two, three and four affected children respectively. Summing their coefficients, the probability of two or more children being affected is found to be

$$C(4,2)(3/4)^2(1/4)^2 + C(4/3)(3/4)(1/4)^3 + C(4,4)(1/4)^4 = \frac{54 + 12 + 1}{256} = \frac{67}{256} \approx .26,$$

so two or more children will be affected in about 26% of the families.

Exercise 8. A man with Huntington's disease has four children. a) What is the probability that all four will develop Huntington's disease? b) What is the probability that two children will be affected and two normal? c) That no more than one will be affected?

Exercise 9. Two parents are carriers of the same recessive gene for deafmutism. a) What is the probability that their first child will be deaf? b) That all five of their children will be normal? c) That two of their five children will be affected? d) That at least two will be affected?

Exercise 10. A statistical study is done on families of five children in which both parents are carriers of the same recessive disorder. In what percent of these families should no more than two children be affected?

Exercise 11. A couple's first child has cystic fibrosis. a) What is the probability that their next child will have it too? b) What is the probability that of their next three children only one will be affected?

---

5. SAMPLE EXAM

1. The parents of both Mr. L and Mrs. L were carriers of cystic fibrosis. What is the probability that neither Mr. L nor Mrs. L is a carrier? That both of them are carriers? That both of them are carriers but they will have two normal children?

2. 90% of the carriers of Huntington's disease have symptoms by age forty-five. If a man whose father died of Huntington's disease is healthy at forty-five, what is the probability that he is not a carrier?

3. A study is made of three groups of families in which both parents are carriers of a recessive disorder. The families in the first group have two children, those in the second group have three, and those in the third have four. If the three groups are of equal size, in what percent of the total number of families should two or more children be affected?

6. ANSWERS TO EXERCISES

1. P(each parent is a carrier) = 2/3.
   P(the child of two carriers is affected) = 1/4.
   2/3 × 2/3 × 1/4 = 1/9.

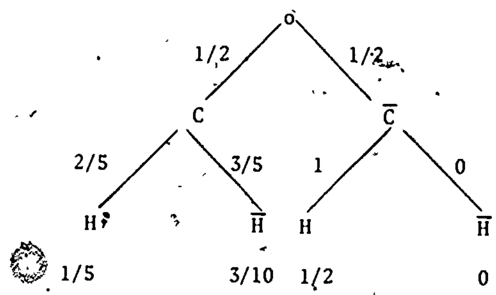
2. P(first child will be born with the disease) = 1/4.
   P(first two children will be healthy) = 3/4 × 3/4 = 9/16.
   P(first three children will be carriers) = 1/2 × 1/2 × 1/2 = 1/8.

3. P(father is a carrier) = 2/3 so P(first child is an albino) = 2/3 × 1/2 = 1/3. If first child is an albino, father is a carrier and P(each subsequent child is an albino) = 1/2.

4. P(grandfather is a carrier) = 2/3. Assuming his spouse is of genotype AA, P(each of his children is a carrier) = 1/2 and P(each of his grandchildren is a carrier) = 1/4. P(two carriers produce a defective child) = 1/4.
   2/3 × 1/4 × 1/4 × 1/4 = 1/96.

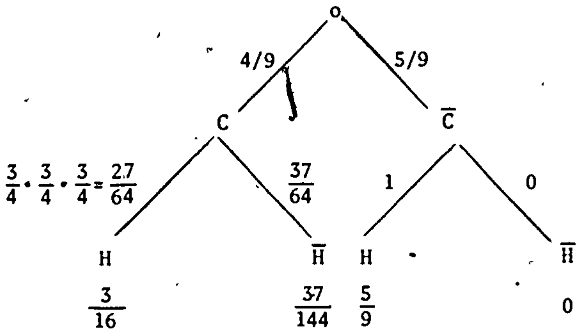
5. P(Bill's father is a carrier) = $\frac{1/4}{1/2 + 1/4}$ = 1/3.
   P(Bill is a carrier) = 1/3 × 1/2 = 1/6.
   P(Bill's child will be a carrier) = 1/6 × 1/2 = 1/12.

6.



P(Willy's mother is a carrier) = $\frac{1/5}{1/5 + 1/2}$ = 2/7.
P(Willy is a carrier) = 2/7 × 1/2 = 1/7.
P(Willy is not a carrier) = 1 - 1/7 = 6/7.

7.



P(Mr. and Mrs. D are carriers given that they have three normal children) $= \dfrac{3/16}{3/16 + 5/9} = \dfrac{27}{107}$

P(fourth child will be an albino) = 27/107 × 1/4 = 27/428 or approximately .06.

8. a) $C(4,4)(\frac{1}{2})^4 = 1/16$
   b) $C(4,2)(\frac{1}{2})^4 = 6/16 = 3/8$
   c) P(one will be affected) $= C(4,1)(\frac{1}{2})^4 = 4/16$
      P(none will be affected) $= C(4,0)(\frac{1}{2})^4 = 1/16$
      P(no more than one will be affected) = 1/16 + 4/16 = 5/16

9. a) 1/4
   b) $C(5,0)(3/4)^5 = 243/1024$
   c) $C(5,2)(3/4)^3(1/4)^2 = 270/1024$
   d) P(none will be affected) $= C(5,1)(3/4)^4(1/4) = 405/1024$
      P(none will be affected) $= C(5,0)(3/4)^5 = 243/1024$
      P(at least two will be affected) = 1 - (405/1024 + 243/1024)
      = 376/1024

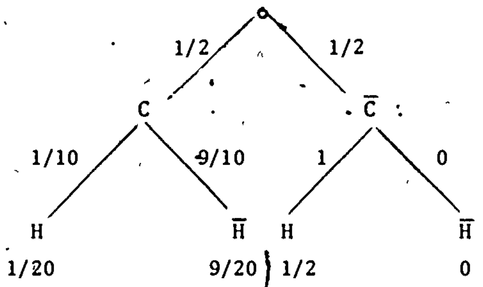
10. P(two will be affected) $= C(5,2)(3/4)^3(1/4)^2 = 270/1024$
    P(one will be affected) $= C(5,1)(3/4)^4(1/4) = 405/1024$
    P(none will be affected) $= C(5,0)(3/4)^5 = 243/1024$
    $\dfrac{270 + 405 + 243}{1024} = \dfrac{918}{1024} \approx .90 = 90\%$

11. a) 1/4
    b) $C(3,1)(3/4)^2(1/4) = 27/64$

7. ANSWERS TO SAMPLE EXAM

1. P(neither Mr. L nor Mrs. L is a carrier) = 1/3 × 1/3 = 1/9.
   P(both are carriers) = 2/3 × 2/3 = 4/9.
   P(both are carriers and they have two normal children)
   = 4/9 × 3/4 × 3/4 = 1/4.

2.



P(he is not a carrier given that he is healthy at forty-five)
$= \dfrac{1/2}{1/2 + 1/20} = 10/11$

3. $C(2,2)(1/4)^2 = 1/16$
   $C(3,2)(3/4)(1/4)^2 + C(3,3)(1/4)^3 = 9/64 + 1/64 = 10/64$
   $C(4,2)(3/4)^2(1/4)^2 + C(4,3)(3/4)(1/4)^3 + C(4,4)(1/4)^4$
   $= 54/256 + 12/256 + 1/256 = \dfrac{67}{256}$
   1/3 × 1/16 + 1/3 × 10/64 + 1/3 × 67/256 = 123/256 = .48
   so two or more children should be affected in about 48% of the families.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name________________________ Unit No.______

Page____ ○ Upper ○ Middle ○ Lower	OR	Section______ Paragraph______	OR	Model Exam Problem No.____ Text Problem No.____

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature________________________

Please use reverse if necessary.

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STUDENT FORM 2

Unit Questionnaire

Name______________________________ Unit No.__________ Date______________

Institution__________________________ Course No.__________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much Longer	Somewhat Longer	About the Same	Somewhat Shorter	Much Shorter
____	____	____	____	____

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain______________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain______________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

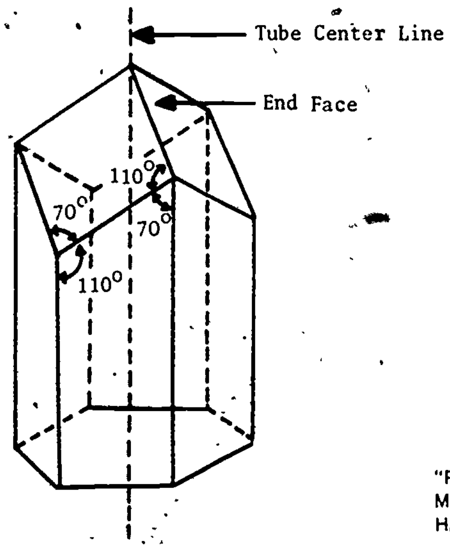
umap

UNIT 502

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

THE DESIGN OF HONEYCOMBS

by Anthony L. Peressini



APPLICATIONS OF DIFFERENTIAL EQUATIONS TO BIOLOGY

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THE DESIGN OF HONEYCOMBS*

by

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University of Illinois ar Urbana-Champaign
Urbana, Illinois 61801

TABLE OF CONTENTS

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Intermodular Description Sheet: *UMAP Unit 502*

Title: THE DESIGN OF HONEYCOMBS

Author: Anthony L. Peressini
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

Review Stage/Date: III 11/1/80

Classification: APPL DIFF CALC/BIOLOGY

Prerequisite Skills:

1. Plane geometry.
2. Determining minimum values of trigonometric functions by differentiation.

Output Skills:

1. To understand the geometric structure of a honeycomb cell.
2. To understand how minimication techniques from calculus apply to a minimum surface area problem for a bee's cell.

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank Gary R. Bates of the University of Central Arkansas; Robert L. Baker, Jr. of Compatibility Analysis Center, Annapolis, Maryland; and Richard Balomeno of the University of New Hampshire for their reviews, and all others who assisted in the production of this unit.

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1. BACKGROUND INFORMATION

The bee's honeycomb is one of the most beautiful geometric configurations in nature. Its intricate geometric form and structural economy have been objects of scientific investigation for centuries.

The architecture of the "ideal" honeycomb has the following basic features:

a) It consists of two stacks of congruent tubes.

b) Each tube is open at one end and closed at the other.

c) The two stacks of tubes are joined together at their closed ends.

d) Each tube has an hexagonal cross-section.

e) The closed end of each tube consists of three end faces that are congruent equilateral paralellograms joined as in Figure 1.

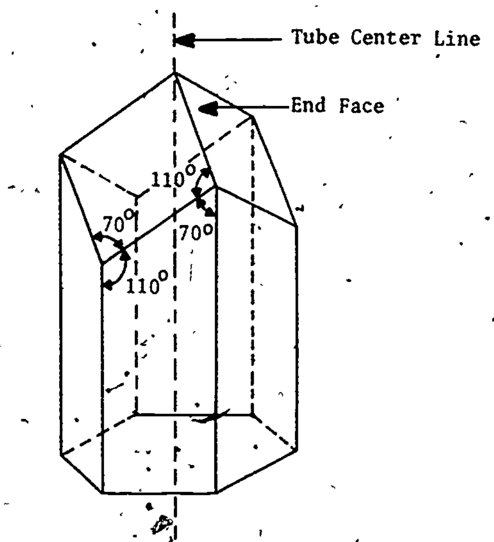

Figure 1. The plane surfaces and surface angles of an ideal honeycomb tube.

f) The measures of the interior angles of the parallelogram faces are approximately 110° and 70°. These same angles occur on the trapezoidal faces of the tubes. Finally, the acute angle between the center line of the tube and the end faces is approximately 54°.

Of course, the honeycombs actually constructed of wax by the bees deviate from the design described above in a variety of ways: The tubes are not exactly the same size and the edges and faces are often curved a bit. However, these deviations from the basic plan are remarkably small so that the design features listed above conform quite closely to reality.

Why and how do bees follow this exacting design pattern for the construction of their honeycombs? As you might expect, these are difficult questions to answer. Some observers have attributed the remarkable uniformity of honeycombs to Divine Guidance, others to genetic natural selection and still others to the action of physical forces.

There is strong evidence to support the conclusion that the architecture of a honeycomb is not merely a random selection from a list of possible multiple tube structures of this sort. This evidence is based largely on a number of theoretical properties of the honeycomb structures described above that suggest that it is the "best possible" choice from several points of view. In this paper, we will discuss one of these theoretical properties in detail and mention some others briefly.

As we pointed out above, the acute angle between the center line of the tube in a honeycomb and each of its end faces is approximately 54°. Why 54°? To provide at least one plausible reason for this choice of angle, suppose that we assume that the bees construct the tubes so that as little wax as possible is used to store a given amount of honey. We could then seek the angle between the tube center line and end faces that would achieve this minimum. More precisely, we could pose the following theoretical problem: *Suppose that a hexagonal tube is capped at one end with three equilateral parallelogram faces and that the other end is open. Find the angle between the tube center line and end faces that will minimize the surface area of the tube for a given volume.* We will solve this problem below and we will see that the correct theoretical value for the angle is approximately 54.7°. This remarkable agreement with the approximate measured angle of 54° strongly suggests that the bees are "doing the right thing" by some mechanism or other!

Scientific observation and study of the geometric structure of the honeycomb can be traced back at least to Pappus of Alexandria near the end of the third century A.D. In the seventeenth century A.D., Johannes Kepler was able to explain the hexagonal cross-section and end face structure of the honeycomb tubes on the basis of packing tubes subject to internal pressures. (This explanation will be outlined in the remarks in Section 3 of this paper.) In 1712, Miraldi made some accurate measurements of the angles on the caps of the honeycomb tubes and conjectured that the measured angle of approximately 54° was the solution to the minimum problem stated in the last paragraph. This conjecture was verified shortly later by Koenig.

2. THE MATHEMATICAL ANALYSIS OF THE BEE'S CELL CAP

For the purposes of our analysis, we shall regard the individual tubes in a honeycomb to be hexagonal cylinders open at one end and capped at the other end with three parallelograms with equal sides as in Figure 2.

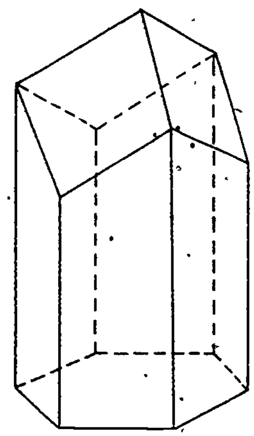

Figure 2. The tube is a regular hexagonal cylinder, open at one end, capped at the other end by congruent parallelograms.

This solid figure can be constructed in the following way:

1. Begin with a solid hexagonal cylinder with end faces perpendicular to the center line of the cylinder. (See Figure 3.)

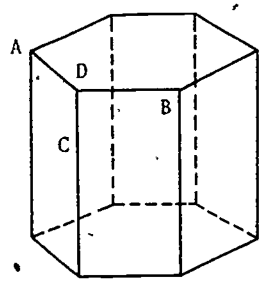

Figure 3. A solid, regular, hexagonal right cylinder.

2) Let A and B denote the upper left-hand and upper right-hand corners of two adjacent lateral faces and let C be a point down the edge joining these faces.

3) Pass a plane through the points A, B, C to cut off a tetrahedral "corner" ABCD of hexagonal cylinder.

4) Using the line segment joining A and B as a hinge, swing the tetrahedron upward through an angle of 180°. (See Figure 4.)

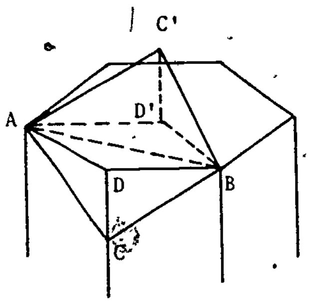

Figure 4. Tetrahedron ABCD relocated as tetrahedron ABC'D'.

The point C will move to the new position C' and D to a new position D'. The points A,C,B,C' will be successive vertices of a parallelogram and C', D' will lie on the axis of the cylinder. (These statements can be verified by checking the angles involved in the tetrahedral corner. You should carry out the details!)

5) Repeat the above procedure with the remaining two pairs of adjacent faces to obtain the solid in Figure 1.

Suppose that θ is the acute angle at C' between the center line of the hexagonal cylinder and the face ACBC'. (By construction, θ is also the acute angle at C between the vertical edge of the hexagonal cylinder and the face ACBC'.) Let h denote the height of the original cylinder (before the three tetrahedral corners were cut) and let s be the width of each face.

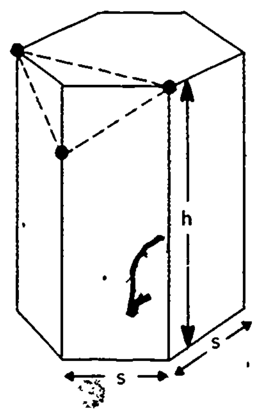

Figure 5. The dimensions of the cylinder as it was before the tetrahedral corners were cut off and moved.

Then, by virtue of the construction described above, the volume V of the hexagonal cylinder capped with the three equilateral parallelogram faces is equal to the volume cylinder of height h and side width s. Since the central angle subtended by each face is $\pi/3$, it follows that

(1) $$V = 6 \cdot [\tfrac{1}{2}s \cdot \tfrac{\sqrt{3}}{2}s]h = \frac{3\sqrt{3}}{2}s^2h.$$

Next, we shall derive an expression for the total surface area of the capped hexagonal cylinder. The diagonals $\overline{AB}$ and $\overline{CC'}$ of the parallelogram face ACBC' bisect each other at a point E and, since ACBC' has equal sides, it follows that ACBC' is divided into four congruent right triangles with side lengths equal to the lengths of AE and CE.

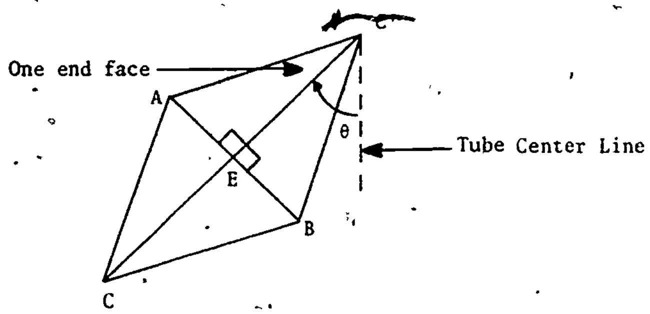

Figure 6. The quadrilateral ACBC' is a parallelogram whose diagonals meet at right angles.

Notice that the length of AE is $s \cdot (\sqrt{3}/2)$ and that the length of CE is $(s/2) \cdot \csc\theta$. Consequently, the area of each face of the cap is

$$4 \cdot \frac{1}{2}\left(s\frac{\sqrt{3}}{2}\right)\left(\frac{s \csc\theta}{2}\right) = s^2\left(\frac{\sqrt{3}}{2}\csc\theta\right)$$

and so the total surface area of the cap is

(2) $$\frac{3\sqrt{3}}{2}(s^2 \csc\theta).$$

The capped hexagonal cylinder has six congruent trapezoidal faces. The width of these faces is s, one vertical edge has height h and the other has height

$$h - \text{length } \overline{CD} = h - \frac{s}{2}\cot\theta.$$

Therefore, the area of each lateral face is

$$s \cdot \frac{1}{2}(h + [h - \frac{s}{2}\cot\theta]) = hs - \frac{s^2}{4}\cot\theta$$

and so the total surface area of the lateral faces is

(3) $$6hs - \frac{3s^2}{2}\cot\theta$$

We now add (2) and (3) to obtain the total surface area S of the capped hexagonal cylinder (open at the other end):

(4) $$S = 6hs + \frac{3}{2}s^2(-\cot\theta + \sqrt{3}\csc\theta).$$

Thus, the surface area S depends on three variables: h, s, θ. If, for a fixed choice of h and s, we seek to minimize S as a function of θ, we differentiate (4) with respect to θ and equate the result to 0 to find the critical points:

(5) $$\frac{3}{2}s^2(\csc^2\theta - \sqrt{3}\csc\theta\cot\theta) = 0$$

(6) $$\csc\theta - \sqrt{3}\cot\theta = 0$$

(7) $$\cos\theta = \frac{\cot\theta}{\csc\theta} = \frac{1}{\sqrt{3}}.$$

The acute angle satisfying the last equation is

(8) $$\theta_0 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \doteq 54.7°.$$

The First Derivative Test can be used to check that this critical point is actually a minimum.

It is very important to notice at this point that the value of θ that yields the minimum surface area S does not depend on the variables h and s. In other words, although we fixed s and h before we minimized S with respect to θ *the value of θ of approximately 54.7° that minimized the surface area S is the same for any choice of the dimensions s and h.* This is actually a stronger conclusion than the one we set out to establish: *For a given volume, the value of θ that minimized the surface area S of the hexagonal cylinder capped at one end with three equilateral parallelograms is approximately 54.7°.*

3. SOME ADDITIONAL COMMENTS

Some of the other features of the geometric structure of honeycombs can be explained on the basis of internal forces within the tubes. We shall now outline these ideas briefly.

Suppose we are given a system of circular cylinders that is closely packed so that each cylinder is in contact with six others along lines parallel to the center line of the cylinders. A cross-sectional view of a central cylinder and its immediate neighbors is given in Figure 7.

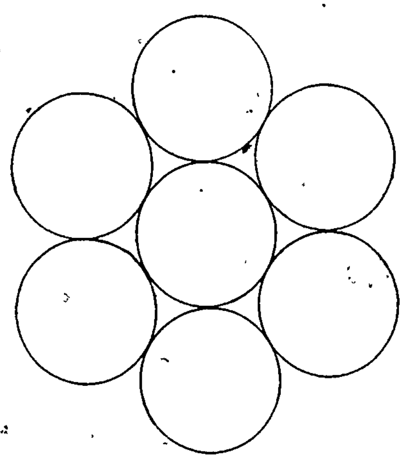

Figure 7. Cross section of seven congruent circular cylinders packed together.

Suppose that the system is subjected to a uniform outward pressure from within each cylinder. This pressure will push the cylindrical walls out into the empty spaces between the cylinders. *It can be shown that when all this empty space has been filled, the system will consist of hexagonal cylinders!* Thus, it is quite conceivable that the hexagonal cross-sectional structure of the honeycomb is due to the efforts of the bees to pack as much honey as possible in each tube.

A similar explanation can be given for the geometric structure of the caps of the tubes in a honeycomb. Consider a system of two stacks of circular cylinders, each cylinder open on one end and capped on the other with a hemispherical cap like a test tube. Place the stacks so

that the capped ends of the cylinders in one stack are in close contact with the capped ends of the other stack. In the closest possible arrangement of the two stacks, each hemispherical end of a cylinder in one stack is in contact with three hemispherical ends of cylinders in the other stack.

If this system is subjected to uniform outward pressure from within each cylinder, the walls of the cylinders push out to fill the empty space between the cylinders. We have already pointed out that the final lateral surfaces of the cylinders will have equal hexagonal cross-sections. *It can also be shown that the end surfaces will consist of three equilateral parallelogram faces when the empty spaces at the ends are completely filled.* Thus, the basic geometry configuration of the honeycomb could result from the action of a uniform internal pressure.

Do the bees construct the best possible caps for their honeycomb tubes? We have shown that they do if the caps are required to consist of three equilateral parallelogram faces. But are there better choices of polyhedral surfaces for the cap? The answer is: Yes! For example, it has been shown that a cap consisting of two hexagonal and two equilateral parallelogram surfaces joined as indicated in Figure 8.

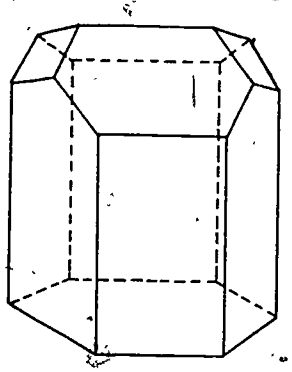

Figure 8. Honeycombs shaped like this would be slightly more efficient.

would be more efficient than a cap with three equilateral parallelogram faces. However, the reduction in surface area per unit volume for this more complicated cap is very small. The bees probably decided that these fancy caps are not cost effective when both labor and materials are considered!

4. PROBLEMS

1. Verify the following details of the construction described in Figure 4:
   a) The points A, C, B, C' are successive vertices of a plane paralellogram.
   b) C', D' lie on the axis of the hexagonal cylinder.
2. Carry out the details of the First Derivative Test to verify that $\theta_0 \doteq 54.7°$ is actually a minimum of the surface area function. (See Equations (5), (6), (7).)
3. Among all regular polygons, only the square, the equilateral triangle and the hexagon can be used to "tile" the plane (that is, to cover the plane with congruent non-overlapping pieces). Show that of these three polygons, the hexagon has the least perimeter for a given enclosed area.

5. SOME REFERENCES FOR FURTHER READING

1. Thompson, D'Arcy W., On Growth and Form Cambridge University Press, 1917. (Available in the Mathematics Library under the call number 577.3 T 37o.) This book is a very readable and fascinating treatise on geometric forms in nature.
2. Tóth, L. Fejes. What the bees know and what they do not know. Bulletin of the American Mathematical Society, Vol 70 (1964) pp 468-481. (Mathematical Library call number: 510.6 Am B2.) This paper is the text of an invited address presented at a national meeting of the American Mathematical Society in 1964. It is a very readable presentation of some minimum surface area problems related to honeycombs. In particular, a full discussion of the comparative economy of the tube caps in Figures 1 and 8 is given.

6. SOLUTIONS TO PROBLEMS

1. (a) Since the point D is rotated through an angle of 180° to the new position D', the points A, B, C and C' lie in a plane. The triangles ABC' and ABC are congruent by construction, so A, C, B, C' are successive vertices of a plane parallelogram.

1. (b) Since angle $\measuredangle ADB = 120^\circ$, it follows that angle $\measuredangle AD'B = 120^\circ$ so D' is on the axis of the hexagonal cylinder. Since the line segment DC is parallel to this axis and since the segment DC is rotated through 180° to the new position D'C', it follows that D'C' is parallel to this axis also. Since the point D' is on this axis, so is the point C'.

2. Note that

$$\frac{ds}{d\theta} = \frac{3}{2} s^2(\csc^2\theta - \sqrt{3}\csc\theta\cot\theta)$$
$$= \frac{3}{2} s^2 \csc\theta\,(\csc\theta - \sqrt{3}\cot\theta).$$

Since $\frac{3}{2}s^2 > 0$ and $\csc\theta > 0$ for $0 < \theta < \frac{\pi}{2}$, it is only necessary to show that $\csc\theta - \sqrt{3}\cot\theta$ changes from negative to positive values as θ increases through $\theta_0 = \cos^{-1}(1/\sqrt{3})$. This sign change can be verified by inspecting the graphs of $\csc\theta$ and $\sqrt{3}\cot\theta$.

3. Suppose that the area enclosed by the polygon is A, that the side length is s, and that the perimeter is P. Then for the:
   (a) square, $A = s^2$, $P = 4s$
   (b) equilateral triangle, $A = \frac{\sqrt{3}s^2}{4}$, $P = 3s$
   (c) regular hexagon, $A = \frac{3\sqrt{3}s^2}{2}$, $P = 6s$

If we express P in terms of A, we obtain for the
(a) square, $P = 4\sqrt{A}$

(b) equilateral triangle, $P = \frac{6}{\sqrt[4]{3}}\sqrt{A}$

(c) regular hexagon, $P = \frac{6\sqrt{2}}{\sqrt[4]{3}\sqrt{3}}\sqrt{A}$

From these calculations, one can conclude that for a given area A, the perimeter of the equilateral triangle is greater than that of the square which in turn is greater than that of the regular hexagon.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

Return to:
EDC/UMAP
55 Chapel St.
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STUDENT FORM 2

Unit Questionnaire

Name______________________________ Unit No.______________ Date______________

Institution______________________________ Course No.______________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much	Somewhat	About	Somewhat	Much
____Longer	____Longer	____the Same	____Shorter	____Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain______________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain______________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)