DOCUMENT RESUME

ED 218 110 SE 038 111

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TITLE	Evaluating Definite Integrals on a Computer Theory and Practice. Applications of Numerical Analysis. Modules and Monographs in Undergraduate Mathematics and Its Applications Project. UMAP Unit 432.
INSTITUTION	Education Development Center, Inc., Newton, Mass.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	80
GRANT	SED-76-19615-A02
NOTE	42p.
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Answer Keys; *Calculus; *College Mathematics; *Computer Programs; Educational Technology; Higher Education; Instructional Materials; *Learning Modules; Mathematical Applications; Problem Solving; *Supplementary Reading Materials; Tests; Textbooks
IDENTIFIERS	BASIC Programing Language; *Integration (Mathematics); *Numerical Analysis

ABSTRACT

This document explores two methods of obtaining numbers that are approximations of certain definite integrals. The methods covered are the Trapezoidal Rule and Romberg's method. Since the formulas used involve considerable calculation, a computer is normally used. Some of the problems and pitfalls of computer implementation, such as roundoff error, are discussed. A sample computer program in BASIC is provided, and computer graphs are provided to illustrate the expected error range. Exercises, a model exam, and references are included, with answer keys provided for both the text problems and the sample test. (MP)

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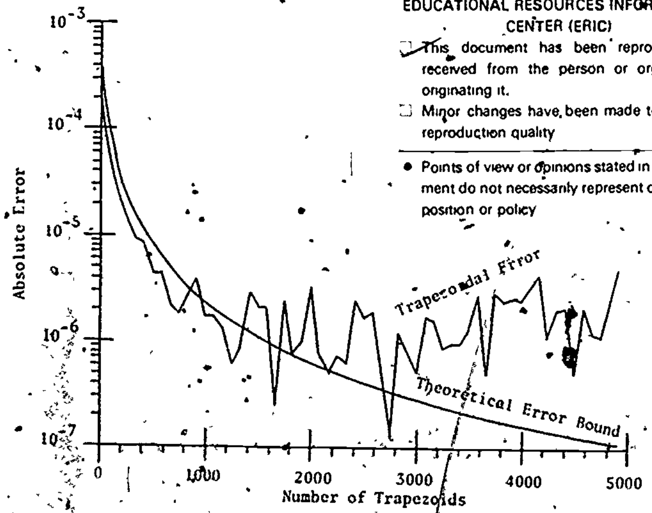
UNIT 432

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

EVALUATING DEFINITE INTEGRALS ON A COMPUTER: THEORY AND PRACTICE

by Stanley Wagon



Trapezoidal Error for sin(x) from 0 to 3.141592

APPLICATIONS OF NUMERICAL ANALYSIS

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EVALUATING DEFINITE INTEGRALS ON A COMPUTER: THEORY AND PRACTICE

by

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Intermodular Description Sheet: UMAP Unit 432

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Review Stage/Date: III 5/25/80

Classification: APPL NUM ANAL

Prerequisite Skills:
1. Knowledge of the definite integral.
2. Some programming experience helpful, but not necessary.

Output Skills:
1. To be able to approach the problem of doing definite integrals on a computer with some knowledge of roundoff error and efficient programming.
2. To be able to estimate the truncation error in the trapezoidal rule.

Other Related Units:

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank Carroll O. Wilde of the Naval Postgraduate School and Chair of the UMAP Analysis and Computation Panel, Richard J. Allen of St. Olaf College and member of the UMAP Analysis and Computation Panel, and Wendell F. Motter of Florida A & M University for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the partial support of National Science Foundation Grant No. SED76-19615 A02. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF or the copyright holder.

1. INTRODUCTION

It is not always easy (or possible) to evaluate the definite integral $\int_a^b f(x)\,dx$ by finding an antiderivative, F, of f and computing F(b)-F(a). It may happen that, although the integrand f is a simple function, there is no simple function F such that F' = f. This happens, for example, if $f(x) = \sin(x^2)$. Or there might be a suitable F but the methods of integration needed to determine F are excessively tedious. The evaluation of definite integrals occurs often enough in those disciplines where calculus is used that this is an important problem, and it can be solved by realizing that what is desired is not the whole antiderivative function F(x), but just the numerical quantity F(b)-F(a).

We shall describe two methods for obtaining numbers that are approximations to $\int_a^b f(x)\,dx$. How close an approximation is needed is of course determined by the context in which the integral arises, but it is important in any approximation method to have a way of knowing how far, at worst, the approximation may be from the true value of the integral. For example, knowing that 1.0999 is an approximation to

$$\int_1^3 \frac{1}{x}\,dx$$

is a relatively useless piece of information compared to being able to say that

$$\int_1^3 \frac{1}{x}\,dx = 1.099 \pm 0.0059 .$$

For the latter assertion allows one to say with certainty that

$$1.0940 \le \int_1^3 \frac{1}{x}\,dx \le 1.1058 ,$$

while the original assertion really says nothing about the value of the integral.

The methods covered in this unit are the Trapezoidal Rule and Romberg's Method. The Trapezoidal Rule is a good place to begin for many reasons. The derivation of the trapezoidal formula is relatively straightforward; the formula itself is easy to program on a computer or programmable calculator; the method is accurate enough for many applications; and there is often an easy way to determine the magnitude, at worst, of the error. Romberg's Method provides a clever way of slightly modifying the results from the Trapezoidal Rule thereby improving the accuracy by a tremendous amount. Because these formulas involve considerable calculation, a computer is usually used to perform these tasks. We discuss some of the problems and pitfalls of computer implementation, such as roundoff error, using computer-drawn graphs for illustration.

2. THE TRAPEZOIDAL RULE - THEORY

Recall that the definite integral $\int_a^b f(x)\,dx$ is defined (using limits of some sort - see Exercise 4) to coincide (when $f(x) \ge 0$ on [a,b]) with our intuitive notion of the area between the x-axis and the graph $y = f(x)$, and between x = a and x = b. The Trapezoidal Rule is based on the observation that trapezoids may easily be used to approximate the area as follows.

Fix a positive integer n and divide the interval [a,b] into n equal subintervals using points $x_0, x_1, \ldots, x_n$, with $x_0 = a$ and $x_n = b$. Note that $x_i = a + i\Delta x$ where $\Delta x = (b-a)/n$. Then by "connecting the dots" from $(a, f(a))$ to $(x_1, f(x_1))$ to $(x_2, f(x_2))$, and so on to $(b, f(b))$, one gets n trapezoids, the sum of whose areas approximates $\int_a^b f(x)\,dx$. Since the area of the i^{th} trapezoid is $\frac{1}{2}(f(x_{i-1}) + f(x_i))\Delta x$, the total trapezoidal area is:

$$\Delta x((\tfrac{1}{2}f(a)+\tfrac{1}{2}f(x_1))+(\tfrac{1}{2}f(x_1)+\tfrac{1}{2}f(x_2))+\ldots+(\tfrac{1}{2}f(x_{n-1})+\tfrac{1}{2}f(b))).$$

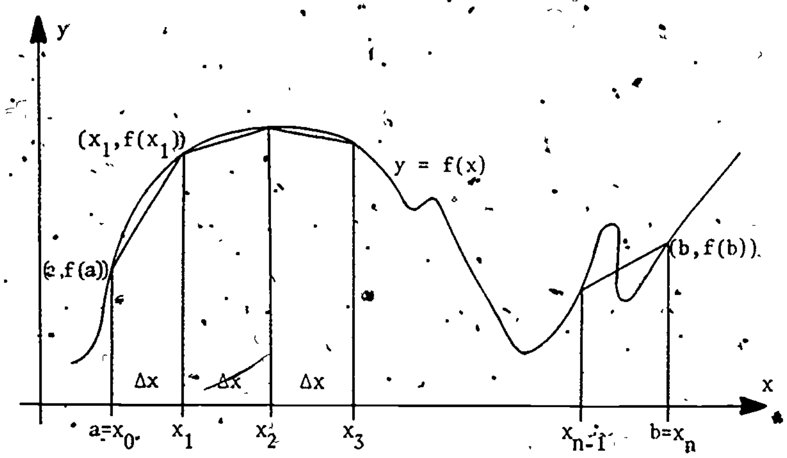


Figure 1.

Combining terms leads to the following definition.

Definition. The *n^{th} trapezoidal approximation*, T_n, for an integral $\int_a^b f(x)\,dx$ is

(1) $$T_n = \Delta x\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right) ,$$

where $x = (b-a)/n$, and the x_i are as stated above.

As an example, consider the integral

$$\int_1^3 \frac{1}{x}\,dx$$

For T_{10},

$$\Delta x = \frac{3-1}{10} = 0.2 ,$$

and the points x_i are 1,1.2,1.4,...,2.8,3. Thus,

$$T_{10} = 0.2\left(\frac{1+\frac{1}{3}}{2} + \frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \ldots + \frac{1}{2.8}\right) = 1.10156 .$$

In this case we may use antiderivatives to get

$$\int_1^3 \frac{1}{x}\,dx = \ln x\Big]_1^3 = \ln 3 - \ln 1 = \ln 3 = 1.09861 .$$

Thus the Trapezoidal Rule gives an approximation to ln 3. Here T_{10} is not very accurate (the error is about 0.003). With more work one can calculate T_{100}, which turns out to be

$$T_{100} = 1.09864 ,$$

with an error of about 0.00003.

It is evident from the geometric nature of the Trapezoidal Rule that T_n is a perfect approximation whenever the integrand is linear (i.e., has the form $ax + b$). For non-linear functions the trapezoidal approximation is, in general, not perfect, but we can predict how great the error can be in terms of the size of the second derivative, $f''(x)$, which is a measure of how close $f(x)$ is to being linear. (If $f(x)$ is linear then $f''(x) = 0$.)

Theorem. If $f''(x)$ exists on $[a,b]$ and n is a positive integer, then there is a number c between a and b such that

(2) $$T_n - \int_a^b f(x)\,dx = \frac{f''(c)\,(b-a)^3}{12\,n^2} .$$

The point c in this theorem is unspecified. After all, if we knew c then the formula above would allow us to calculate the exact value of the integral from T_n. There is no general way of telling where between a and b c might lie. However, the following corollary is an immediate consequence and has a form that is easier to apply.

Corollary. (*Theoretical error bound for T_n*) If $f''(x)$ exists on $[a,b]$ and K is chosen so that $|f''(c)| \le K$ whenever $a \le c \le b$, then

(3) $$\left|T_n - \int_a^b f(x)\,dx\right| \le \frac{K(b-a)^3}{12\,n^2}$$

The proof of the theorem above is relatively long and the interested reader may consult [DM, p.233 or p.305].

Note that this corollary gives only an upper bound on the error. Usually (although see Exercise 5) the error is below this bound, sometimes by a considerable amount.

Consider how the theoretical error bound applies to $\int_1^3 \frac{1}{x}\,dx$. In this case $f''(x)$ is $2/x^3$, which is positive. In fact, $2/x^3$ is a decreasing function on [1,3] (sketch its graph), and so its maximum value for this interval occurs at $x = 1$, and is $2/1^3$ or 2. Thus we may take $K = 2$ and apply the corollary to deduce that the error in T_n will be at most $2(2)^3/12(n)^2$, or $4/3n^2$. So the error in T_{100} is at most 4/30000 or 0.000133.... We stated above that $T_{100} = 1.09864$ and so we may now conclude that

$$(4) \qquad 1.09864 - .00014 \le \int_1^3 \frac{1}{x}\,dx \le 1.09864 + .00014$$

or equivalently, that $1.09850 \le \ln 3 \le 1.09878$. (Exercise 2 shows how a bit more information can be extracted in this case.)

We may use the formula for an error bound to find out which T_n we should compute to obtain a given accuracy. Suppose we wished to calculate $\int_0^\pi \sin x\,dx$ with an error of no more than 0.001. (Of course it is routine to evaluate this integral exactly as 2, but it still serves to illustrate the method.) Since $|f''(x)| = |-\sin x|$, and since $\sin x$ always lies between -1 and 1, we may conclude that $|f''(x)| \le 1$ and so take $K = 1$. Then by the theoretical error bound (3), we know that the error T_n is at most $1 \cdot \pi^3/12n^2$. Thus if n is so large that

$$\pi^3/12n^2 \le 0.001 ,$$

then T_n will be within the required tolerance. In solving this latter inequality for n, we find that

$$12n^2 \ge 1000\pi^3$$

$$n \ge \sqrt{1000\pi^3/12}$$

$$n \ge 50.8$$

Thus for $n = 51$ (or greater), T_n has the desired accuracy. In fact, $T_{51} = 1.99936$, which differs from 2 by 0.00064.

An important consequence of the presence of the n^2 in the denominator of the theoretical error bound in (3) is that usually (though there are exceptions) doubling n, the number of trapezoids, cuts the error by a factor of 4. Similarly, multiplying n by 10 should cut the error by a factor of 100. For

$$\int_1^3 \frac{1}{x}\,dx ,$$

the error in T_{10} is about 0.003. The error in T_{100} is about 0.00003.

In the previous two examples, it was not very difficult to determine a value for K. The determination is sometimes a bit more intricate, and one may have to use the methods of locating maxima and minima learned in elementary calculus (see Exercises 7 and 8). And there are times when the theoretical error bound cannot be used. For instance, if $f(x)$ is complicated, $f''(x)$ may be even more so, and obtaining an upper bound K for $|f''(x)|$ on [a,b] may be extremely difficult. For example, consider

$$\int_1^8 \frac{x^3}{e^x - 1}\,dx ,$$

an integral that arises in thermodynamics. Here $f''(x)$ is very complicated (check for yourself) and a value for K is difficult to obtain. In such a situation you might wish to compute many values of T_n, halting when they seem to have stopped changing in the decimal places you care about. This is discussed further in the next section.

For some integrals the error bound can be lowered by reducing the loss sustained in passing from $f''(c)$ in equation (2) to K in the inequality (3). For if it is possible to divide [a,b] into two subintervals so that on one of them f'' is substantially less than the bound, K, for all of $[a,b]$, then error bounds for the subintervals may be computed, and added. As an example, consider $\int_1^3 \frac{1}{x}\,dx$. The second derivative $f''(x) = 2/x^3$ is larger near 1 than it is near 3. If n is even, we may consider T_n as the sum of $T_{\frac{1}{2}n}$ for $\int_1^2 \frac{1}{x}\,dx$ and $T_{\frac{1}{2}n}$ for $\int_2^3 \frac{1}{x}\,dx$. Now, if K_1, K_2 denote bounds on $2/x^3$ on $[1,2]$ and $[2,3]$ respectively, then

$$K_1 = 2/1^3 = 2,\quad K_2 = 2/2^3 = 0.25\ ,$$

and the error in T_n is no greater than

$$\frac{K_1\,1^3}{12\,(\frac{1}{2}n)^2} + \frac{K_2\,1^3}{12\,(\frac{1}{2}n)^2} = \frac{0.75}{n^2}\,.$$

Thus the error in T_{100} is at most 0.000075 and the bound on the integral given in (4) can be improved to

$$1.09857 \le \int_1^3 \frac{1}{x} \le 1.09872\,.$$

The method of Exercise 2 improves this to

$$1.09857 \le \int_1^3 \frac{1}{x}\,dx \le 1.09864\,.$$

Another context in which the trapezoidal rule is useful, but the theoretical error bound less so, is when the integrand is described by a table of values, as opposed to a mathematical expression. For instance, values of a function, f, might be obtained in a laboratory experiment and a definite integral of f is required. See [HK] (parts of which are in [TF, p.216]) or [DM, p.266] for specific applications, the former to the problem of measuring cardiac output, the latter to a problem in thermodynamics. For example, suppose the following five values of a function f(x) were determined in an experiment, and an approximation to $\int_0^1 f(x)\,dx$ is required.

x	0.00	.25	.50	.75	1.0
f(x)	0.000	.235	.388	.420	.349

Since the Trapezoidal Rule needs values only at equally spaced points, we can immediately compute

$$T_4 = 0.25\left[\frac{0+0.349}{2} + 0.235 + 0.388 + 0.420\right] = 0.304\,.$$

Note that since the integrand is not given by an equation, no estimate K on the second derivative is available, and so the theoretical error bound is not applicable. Because we have no idea of the behavior of the function between the tabulated points, we have no way of telling how close $T_4 = 0.304$ is to the true value of the integral. In practice one should obtain enough data points to eliminate the possibility of bizarre changes in the function between the points. Note that, in this example, $T_1 = 0.175$ and $T_2 = 0.381$. In Section 4 we shall see how these values of T_1, T_2, and T_4 may be combined to yield a closer approximation to the integral.

Exercises

1. Compute T_4 and T_8 for (a) $\int_0^1 \frac{4}{1+x^2}\,dx$; (b) $\int_0^{\pi} \cos x\,dx$; (c) $\int_0^{2\pi} \frac{e^{\frac{\sin x}{\sqrt{2}}}}{\pi}\,dx$.

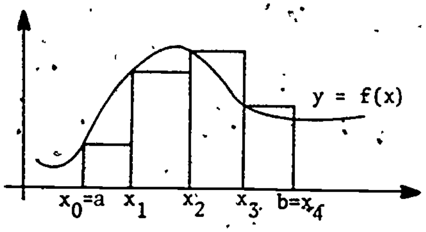
2. Consider the use of T_n to approximate $\int_1^3 \frac{1}{x}\,dx$. Why is it that for every n, the value of T_n is larger than the true value of the integral? This fact implies that the inequality in (4) can be improved to $1.09850 \le \ln 3 \le 1.09864$.

3. Give an example (by sketching a graph if you like) of a function f and two integers m and n with the following property: $n > m$ but T_m is closer to $\int_0^1 f(x)\,dx$ than T_n is.

4. The value of a definite integral can also be approximated by sums of areas of rectangles. Fix n, and let $x_1,\ldots,x_n$ partition $[a,b]$ as on page 3. Let the sum of the areas of the n rectangles be denoted by

$$R_n = \Delta x\left(\sum_{i=0}^{n-1} f(x_i)\right)$$

(Figure 2 illustrates the case n = 4.)



$$R_4 = \Delta x f(x_0) + \Delta x f(x_1) + \Delta x f(x_2) + \Delta x f(x_3)$$

Figure 2.

Show that $T_n = R_n + (b-a)(f(b)-f(a))/2n$.

5. Compute T_1 for $\int_0^1 x^2\,dx$. How much less than the error bound given by (3) is the error in T_1?

6. Which T_n should one compute when approximating $\int_0^2 e^{(x^2)}\,dx$ to be sure that the error is at most 0.0005? Show that a much smaller n will do by computing the error bound separately on [0, 1½] and [1½, 2], as was done in the text for $\int_1^3 \frac{1}{x}\,dx$.

7. Use the inequality in (3) to show that

$$\int_0^{10} e^{x-\frac{x^2}{6}}\,dx = T_{100} \pm 0.015 .$$

If you have a computer available, calculate T_{100} to show that

$$18.631 \leq \int_0^{10} e^{x-\frac{x^2}{6}}\,dx \leq 18.662 .$$

8. Consider the integral of Exercise 1(c). Use the inequality (3) to say by how much, at worst, T_8 differs from the true value of the integral (Hint: In (3), K need not be the exact maximum of $|f''(x)|$ on [a,b] (which in this case is hard to find): any upper bound on $|f''(x)|$ will do.).

9. The temperature outdoors was taken every 3 hours during a 24-hour period, with the following results:

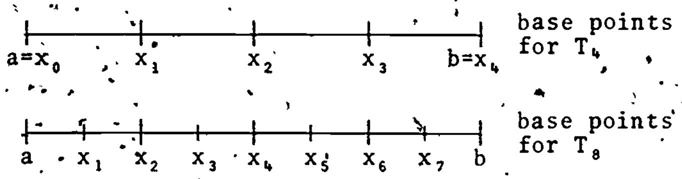
Time	mid-night	3	6	9	noon	3	6	9	mid-night
Temp.	10.0°	9.1°	12.4°	18.6°	25.9°	32.7°	31.5°	20.0°	18.9°

Recalling that the average value of a function f(t) defined on [0,24] is $\frac{1}{24}\int_0^{24} f(t)\,dt$, use the trapezoidal rule to approximate the average temperature during the day.

3. THE TRAPEZOIDAL RULE - PRACTICE

Because of the many calculations required to compute T_n, one normally uses a computing machine - either a large scale digital computer, or a programmable calculator. If all that is wanted is T_n for a single value of n, it is very easy to write a program to compute it. However, it is often desirable to compute many different values of T_n. For instance if, as we saw in Section 2, a theoretical error bound is not readily obtainable, one would like to produce a sequence of values of T_n until the values cease changing by much. If one adopts a straightforward approach to computing $T_1, T_2, \ldots, T_{100}$ then (2+3+...+101) = 5150 evaluations of f(x) will be made (since T_n requires n+1 evaluations of f), many of them more than once.

A more efficient way to compute many values of T_n with no duplication of function evaluations is to compute $T_1, T_2, T_4, T_8, T_{16}, \ldots$. To see why this is efficient note that having just computed T_4, say, the values of $f(x_i)$ needed to compute T_8 are exactly the 5 values just used for T_4, plus the 4 values occurring midway between them.



This observation leads to the following formula for calculating T_{2n} from T_n:

$$(5) \qquad T_{2n} = \frac{b-a}{2n}\left(\frac{T_n}{\left(\frac{b-a}{n}\right)} + \sum_{i=0}^{n-1} f\left(a + (2i+1)\left(\frac{b-a}{2n}\right)\right)\right),$$

where the values of f at the evenly indexed x_i's, i.e.,

$$f\left(a + 2i\left(\frac{b-a}{2n}\right)\right),$$

all appear in

$$T_n/\left(\frac{b-a}{n}\right).$$

Using this method one first decides the number of evaluations of f(x) there is time for, say 513, and, instead of computing just T_{512}, or computing $T_1, T_2, T_3, \ldots, T_{30}$, one can compute T_{512} and obtain the values of T_1, T_2, T_4, T_8, T_{16}, T_{32}, T_{64}, T_{128}, T_{256} along the way: It is useful to have all this output for comparison. Moreover, in Section 4 we shall develop a method which begins with such a sequence of values of T_n, and modifies them to obtain still closer approximations.

Here is a program written in BASIC which uses Equation (5) to generate a sequence of values of T_n. The program was written for the integrand $x^3/(e^x-1)$. For other functions, line 130 must be modified.

```
100 REM  TRAPEZOIDAL APPROXIMATION OF INTEGRALS
110 REM    FIRST DEFINE INTEGRAND; THIS LINE
120 REM    MUST BE MODIFIED FOR NEW INTEGRANDS
130 DEF FNA(X)=X↑3/(EXP(X)-1)
```



```
140 DIM T(30)
150 PRINT "WHAT ARE A AND B?"
160 INPUT A,B
170 PRINT "HOW MANY T(N)'S WOULD YOU LIKE?"
180 INPUT M
190 REM
200 REM    WE FIRST USE ONE TRAPEZOID, STORING
210 REM    THE RESULT IN T(1)
220 D=B-A
230 T(1)=D/2*(FNA(A)+FNA(B))
240 REM
250 REM    NOW COMPUTE THE (2↑N)TH TRAPEZOIDAL
260 REM    APPROXIMATION FROM THE 2↑(N-1)ST,
270 REM    USING FORMULA (5) ABOVE, AND
280 REM    STORING IT IN T(N)
290 FOR N=2 TO M
300 D=D/2
310 S=0
320 REM
330 REM    USE S TO FORM THE SUMMATION IN (5)
340 FOR J=1 TO 2↑(N-1) STEP 2
350 S=S+FNA(A+J*D)
360 NEXT J
370 T(N)=T(N-1)/2+D*S
380 NEXT N
390 REM
400 REM    NOW PRINT THE RESULTS
410 FOR N=1 to M
420 PRINT "T(";2↑(N-1);")=";T(N)
430 NEXT N
440 END
```

Here is a sample output (on a 12 significant digit computer) for

$$\int_1^8 \frac{x^3}{e^x-1}\,dx:$$

```
RUN
WHAT ARE A AND B?
1,8
HOW MANY T(N)'S WOULD YOU LIKE?
11
T(1)=2.63826923395
T(2)=4.90201237702
T(4)=5.76289887395
T(8)=5.95440195063
T(16)=5.99988421985
T(32)=6.01109575704
T(64)=6.01388856817
T(128)=6.01458613933
T(256)=6.01476049262
T(512)=6.01480407847
T(1024)=6.01481497477
```

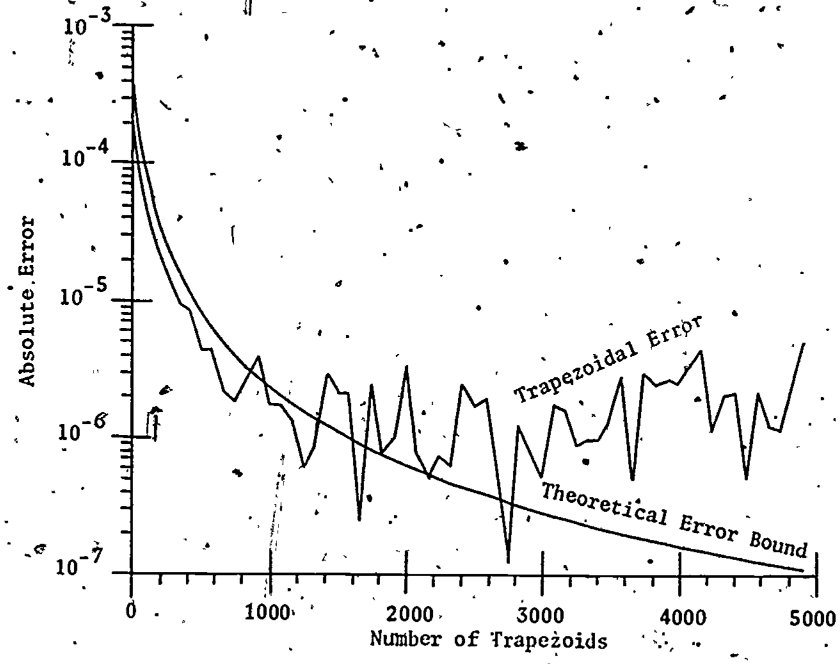
It appears from this output that

$$\int_1^8 \frac{x^3}{e^x-1}\,dx = 6.0148$$

to 5 significant digits (in fact, the true value is 6.0148186...). While it is generally true that, when the values get closer to each other, they are close to the limit, i.e., the true value of the integral, there are some examples of innocent looking integrands where the values are momentarily quite close, but then change significantly as they approach their limit. (An example of this is discussed in Exercise 17). Experience and some conservatism help out; e.g., from the data above one might merely conclude that the true value is 6.014... If a 5^{th} significant digit is required, more values might be computed, or another method used. However 1000 is often a practical upper bound on the number of trapezoids that can be used: the computer used above, working with so many significant digits, takes a lot of time, while on a machine with fewer significant digits the results are

less trustworthy because of roundoff error, which we now discuss.

So far everything that we have done (for instance, concluding from (3) that T_n approaches $\int_a^b f(x)\,dx$ as $n \to \infty$) is based on the fact that the real numbers have infinitely many decimal digits. However, any computer can work only with a fixed finite number of digits. This creates unexpected difficulties, which often require great ingenuity to avoid. Consider Figure 3, which gives a pictorial representation of the results of using a computer* to obtain T_n for $\int_0^{3.141592} \sin x\,dx$.



Trapezoidal Error for sin(x) from 0 to 3.141592

Figure 3.

*A Digital Equipment VAX, which uses 7 significant digits, was used. Equation (1) was used in this example and the one of Figure 4. When Equation (5) is used, there is less roundoff error, though the same general behavior is exhibited.

The absolute value of the error is plotted logarithmically on the vertical axis, against the number of trapezoids. Since $\int_0^{3.141592} \sin x\,dx = 2.00000$, this means that the graph labelled "TRAPEZOIDAL ERROR" is the graph of $|T_n - 2|$. The theoretical error bound for this integral was worked out in Section 2, and is $(3.141592)^3/12n^2$. For small values of n, say $n \leq 500$, the observed or computed error agrees quite well with the theoretical prediction - it is less than the error bound, and the two graphs are roughly parallel. As is the case with the theoretical error bound graph, the actual error is quartered when n is doubled. As the number of trapezoids reaches and passes about 700, however, something quite strange happens. The actual error curve becomes less smooth and is soon greater than the theoretical error bound, in clear contradiction to (3)! If one decided to compute T_{2900}, the theoretical error bound would indicate an error of at most 4×10^{-7} but the error in the computed value would be about 3×10^{-6}, almost 10 times greater then expected.

The reason for this deviation from the theoretically predicted behavior is that the computer uses only 7 significant digits, and the rounding necessary to perform additions in this mode can, when repeated very often, cause a substantial buildup of what is commonly known as roundoff error.

For example, consider how a computer working in 6 significant digit floating point arithmetic adds 123.456 and .987654. It first converts to exponential (power of ten) notation with a common exponent, say .123456 E 3 and .000988 E 3, rounding the smaller number (it cannot store .00987654 E 3 since this requires 9 significant digits). Now it adds to get .124444 E 3 or 124.444, when the true sum is 124.443654. The error here (in the amount of .000346) may seem inconsequential since it does not affect the first 6 significant digits of the result, but

when this occurs repeatedly the error buildup can be quite substantial (even though rounding may cause some cancellation of error, since it sometimes increases and sometimes decreases the result). For a drastic example consider a sum of the form 1.00000 + 0.000004 + 0.000004. The first addition is done as .100000 E 1 + .000000 E 1, yielding a result of 1.00000, and similarly, the next addition yields a final result of 1.00000. But the true answer is 1.000008 which, rounded to 6 digits is 1.00001. This is exactly what happens in the computation of T_n. For instance, when computing T_{400} for $\int_0^{3.141592} \sin x\,dx$ from Equation (1) we calculate the sum

$$\sum_{i=1}^{399} \sin(x_i)$$

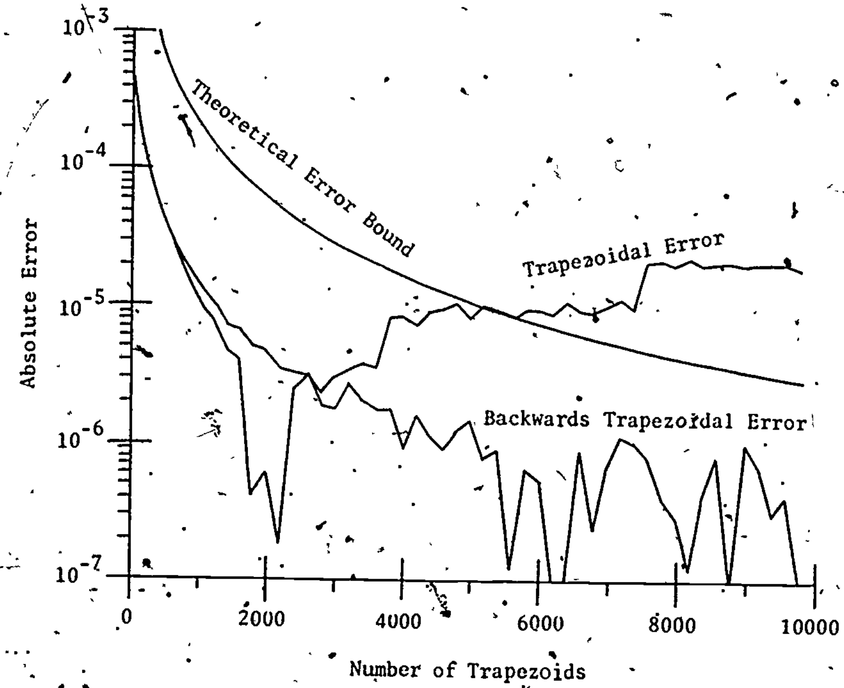
The subtotal of the first 200 terms is about 127, and in completing the sum we add to this subtotal 199 more numbers that are each between -1 and 1. The repeated rounding causes a significant loss in accuracy.

Note that we are using the term accuracy in a slightly new sense here, as we are not talking about the difference between T_{400} and $\int_0^{3.141592} \sin x\,dx$ (which is often called *truncation error*), but rather the difference between the real value of T_{400} and the computed value of T_{400}. It is not generally possible to determine whether the roundoff error will cause the computed value of T_n to be closer to or farther from the actual value of the integral. One should therefore distinguish between the two types of error and, once it is determined that a certain T_n will have a sufficiently small truncation error, T_n should be computed in a way that minimizes the roundoff error. We shall discuss one such technique shortly, but note that one should avoid dealing with T_n at all when n is so large that the machine being used cannot carry enough decimal places to calculate T_n accurately. While a 13-digit computer or programmable calculator can compute T_n where $n \geq 1000$ with little loss of accuracy, a 6-digit computer cannot. Thus if the theoretical error bound says that T_n is needed where $n \geq 800$ (as in Exercise 6) and one is working on a 6-digit machine, it is probably worthwhile to seek another, more efficient method of estimating the integral, like Romberg's Method, discussed in the following section.

Consider again the examples cited above of a computer addition. Note that roundoff loss is much less likely if the numbers to be added are of the same order of magnitude, for then the shifting necessary to get the exponents to agree is unnecessary, and the concomitant roundoff loss avoided. (Rounding error is not entirely eliminated; in adding .44444 and .88888, the result, 1.333332 will have to be rounded to 1.33333.) It follows that when many numbers which vary substantially in size are to be added, roundoff error can be lessened by *adding them in order from smallest to largest*. This is because the smaller numbers may then contribute to the sum and more of the intermediate additions will be of two numbers of the same order of magnitude. If the larger ones were added first, then very few of the additions would involve numbers of the same order of magnitude. For a simple example, note that when 1, .000004, .000004 (see above) are added in reverse order, one first gets .000008, which is rounded to .000001 E 1 so it can be added to .100000 E 1, giving a result of 1.00001, correct to 6 significant digits. In decending order the computed sum is 1.00000.

This fact is dramatically brought home by considering the trapezoidal approximation to $\int_0^{15} e^{-x}\,dx$. The values of e^{-x} on the interval [0, 15] range from 1 down to 3×10^{-7}, so substantial roundoff error is expected for T_n, the computation of which involves summing values of e^{-x} as x ranges from 0 to 15. A simple way to add the numbers in reverse order, i.e., from smallest to largest,

is to consider T_n for $\int_{15}^{0} -e^{-x}\,dx$, where the minus sign is included to account for the interchange of limits. For the latter integral Δx will be negative, and so T_n will be computed from right to left. Theoretically T_n for these two integrals are equal, but as demonstrated in Figure 4, the computed values differ by a lot when $n \leq 1000$. Recall that the error (vertical) scale is logarithmic, and so the difference is really quite great.



Trapezoidal Error for Exp(-X) from 0 to 15

Figure 4.

The mathematical study of roundoff error is generally quite difficult. Techniques of probability theory come into play, to account for the probable cancellation of rounding errors. For instance, it can be shown that in the addition of n numbers, roundoff error will usually build up proportionately to $\sqrt{n}$. This helps to explain the shape of the right hand sections of the graphs in Figures 3 and 4.

Sometimes the way in which roundoff error creeps into a calculation is quite subtle, and anyone using a computer to perform many arithmetical operations must be aware of how it can destroy certain types of calculations.

Exercises

10. In Figure 4 why is the theoretical error bound the same for both $\int_0^{15} e^{-x}\,dx$ and $\int_{15}^{0} -e^{-x}\,dx$?

11. Write a program in a language other than BASIC, e.g., in the language of a programmable calculator, that computes $T_1, T_2, T_4, T_8, \ldots$ for a given integral.

12. Apply the program of the previous exercise to the integral of Exercise 7. Then do it "backwards" as in the text and compare the results.

---

4. IMPROVING THE TRAPEZOIDAL RULE BY ROMBERG'S METHOD

Often rules for bounding the error in a numerical method are ignored in practice because they are too difficult to apply. Still, a detailed analysis of error can be useful in another way, for it may lead to a new method which is better than the original one. In this section we will develop *Romberg's Method*, the basic ideas of which have a wide range of application in numerical analysis.

Suppose a,b, and f are given, and let

$$E_n = T_n - \int_a^b f(x)\,dx = \frac{f''(c)(b-a)^3}{12\,n^2}$$

As you know from Section 2, the point c depends on n. If we changed n to n + 1, then most likely c would change as well. It turns out that E_n can be represented in a different, rather more useful way; namely, if f is sufficiently differentiable, then

(6)* $$E_n = \frac{A}{n^2} + \frac{B}{n^4} + \frac{C}{n^6} + \ldots$$

where A, B, C, ... are constants that depend on a, b and f, *but are independent of n.*

In other words,

(7) $$T_n = \int_a^b f(x)\,dx + \frac{A}{n^2} + \frac{B}{n^4} + \frac{C}{n^6} + \ldots$$

This representation of the trapezoidal error is a consequence of the Euler-Maclaurin summation formula - see [DR, p.108 and p.327]. With no further assumptions on A, B, C,... it seems that the term A/n^2 will be the largest contributor to the error. The key idea in Romberg's Method is to eliminate this term in the following way. We first replace n by 2n in (7) to get

(8) $$T_{2n} = \int_a^b f(x)\,dx + \frac{A}{4n^2} + \frac{B}{16n^4} + \frac{C}{64n^6} + \ldots$$

We then subtract T_n from $4T_{2n}$ to get

(9) $$4T_{2n} - T_n = 3\int_a^b f(x)\,dx + \frac{\frac{B}{4} - B}{n^4} + \frac{\frac{C}{16} - C}{n^6} + \ldots$$

When we divide both sides of this last equation by 3, we obtain

---

*Actually the right-hand expression is not simply an infinite series, but is an "asymptotic series." There is nothing really lost in thinking of it as an ordinary infinite series.

$$(10) \quad \frac{4T_{2n} - T_n}{3} = \int_a^b f(x)dx + \frac{-3B/12}{n^4} + \frac{-15C/48}{n^6} + \dots$$

Equation (10) shows that $(4T_{2n} - T_n)/3$ is a better approximation to the integral than either T_n or T_{2n}, in the sense that its error has terms involving n^{-4}, n^{-6}, ..., but none involving n^{-2}.

If we let $T_n' = (4T_{2n} - T_n)/3$, it follows that the error for T_n' can be written as a series beginning with n^{-4}, and we can repeat this procedure starting with T_n' and T_{2n}' to eliminate the n^{-4} term from the error. The numbers 4 and 3 will be replaced by 16 and 15, and so we let $T'' = (16T_{2n}' - T_n')/15$. There is no need to stop here. The following diagram shows that by beginning with the n values, $T_1, T_2, T_4, T_8, \dots, T_{2^{n-1}}$ one gets a triangular array, ending with a row

T_1	T_2	T_4	T_8	T_{16}	...	$T_{2^{n-1}}$
T_1'	T_2'	T_4'	T_8'			
T_1''	T_2''	T_4''				
T_1'''	T_2'''					
$T_1^{(4)}$						
⋮						
$T_1^{(n-1)}$						

Figure 5

consisting of just $T_1^{(n-1)}$, which is usually the best approximation in the array.

There are many reasons why this method is an excellent one to use. Unlike many methods (Trapezoidal Rule, Rectangle Rule, Simpson's Rule) which are perfect for polynomials of a certain degree (1,0,3 respectively), Romberg's method is perfect for polynomials of arbitrarily high degree, if one computes sufficiently many rows.

More precisely, the first row is perfect for linear functions, the second for cubics, and in general $T_n^{(i)}$ is perfect for polynomials of degree 2i + 1.

Moreover, in terms of computing effort, the amount of work needed to apply Romberg's method once the trapezoidal values are obtained is really quite minimal compared to the gain in accuracy. Indeed, if one begins with the program in Section 3, it is a simple matter to add a few instructions to produce the Romberg array. *No further evaluations of the integrand are needed,* only some straightforward arithmetical manipulations with the $T^{(i)}$'s. Constructing a suitable program for Romberg's method is left as an exercise.

Examples

1. $\int_1^3 \frac{1}{x} dx$	1.33333	1.16667	1.11667	1.10321
	1.11111	1.10000	1.09873	
True value:	1.09926	1.09864		
1.09861	1.09863			

Note that $T_1^{(3)}$, which needed T_8, is closer to the true value than T_{100} (see page).

2. $\int_0^\pi \sin x\, dx$	0	1.57080	1.89612	1.97423
	2.09440	2.00456	2.00027	
True value:	1.99857	1.99998		
2	2.00001			

In this case, it takes 600 trapezoids to obtain a value for T_n as accurate as $T_1^{(3)}$, which required only 9 function evaluations.

3. $\int_1^8 \frac{x^3}{e^x - 1} dx$	2.63827	4.90201	5.76290	5.95440
	5.65659	6.04986	6.01824	
True value:	6.07608	6.01613		
6.01482	6.01518			

Comparing with the results in Section 3, we see that here $T_4^{(3)}$, is about as accurate as T_{128}.

This method is particularly useful when the integrand is given by a table of values obtained in an experiment. In the example at the end of Section 3, we obtained

$$T_1 = 0.175, \quad T_2 = 0.281, \quad T_4 = 0.304 \quad .$$

Romberg's method may be applied to yield

$$T_1' = 0.316, \quad T_2' = 0.312, \quad T_1'' = 0.311 \quad .$$

Without redoing the experiment, that is, without obtaining more function values, we improve our original approximation of this integral from 0.304 to 0.311.

The field of approximate integration is vast, with many methods suited to special types of integrands. Romberg's method is an excellent one to use when the situation requires equally spaced x-values.

Exercises

13. Write a program in BASIC which starts with the program in Section 3 and produces the Romberg array.

14. Use your program you wrote to compute the Romberg array for

$$\int_0^1 \frac{1}{1+x}\, dx \; .$$

15. Compute the Romberg array for $\int_0^{\pi} \cos\,(8 \sin x - x)\, dx$ (which equals 0.73713182...). Does Romberg's method improve on the trapezoidal results?

16. If you know about Simpson's Rule, show that the entries in the second row of the Romberg array (i.e., the T_n') are the approximations one gets from Simpson's Rule.

17. Consider the integral

$$\int_{-1}^{+1} \left(\frac{23}{25} \cosh x - \cos x\right) dx \quad ,$$

where

$$\cosh x = \frac{1}{2}\,(e^x + e^{-x}) \quad .$$

Compute T_1, T_2, and T_4. Then compute two more rows of the Romberg array. Note that T_1' agrees with T_1'' to 6 decimal places, and so one might cease the computations here, expecting the true value of the integral to be 0.479555... . But this is not so, as the true value is 0.4794282... (compute some more rows of the Romberg array, or do the integral by finding an antiderivative). This points out the inaccuracy that can arise if one stops computing when the values agree to a certain number of digits. (This example is adapted from [DR, p.317].)

18. Compute the Romberg array for some integrals of your own choosing, preferably some for which you can determine the true value.

---

5. REFERENCES

[DM] W. S. Dorn and D. D. McCracken, Numerical Methods With Fortran IV Case Studies, New York: Wiley, 1972.

[DR] P. J. Davis and P. Rabinowitz, Methods of Numerical Integration, New York: Academic Press, 1975.

[HK] B. Horelick and S. Koont, Measuring Cardiac Output, Project UMAP, Education Development Center, Inc., 1978.

[TF] G. B. Thomas, Jr. and R. L. Finney, Calculus and Analytic Geometry, Fifth Edition, Reading, MA: Addison-Wesley, 1979.

6. MODEL EXAM

(A calculator is necessary for problems 1, 2, 3, and 9.)

1. With the help of a calculator, evaluate T_5 for

$$\int_{0.1}^{1.1} \frac{1}{e^x - 1}\,dx.$$

2. Show that if T_{10} is used to approximate

$$\int_{0.2}^{1} \frac{1}{x}\,dx$$

then the truncation error is no greater than 0.107.

3. If you were to use T_n to approximate

$$\int_{1}^{2} (\ln x)^2\,dx,$$

and you wished the truncation error to be no greater than 0.0005, which n should you use?

4. From the point of view of evaluating the trapezoidal rule approximation with minimal roundoff error, which of the following four forms of the definite integral is best to use?

a) $\int_{-10}^{10} \frac{1}{x^6+1}\,dx$ b) $-\int_{10}^{-10} \frac{1}{x^6+1}\,dx$

c) $2\int_{0}^{10} \frac{1}{x^6+1}\,dx$ d) $-2\int_{-10}^{0} \frac{1}{x^6+1}\,dx.$

The graph of $\frac{1}{x^6+1}$ on [-10,10] is shown in Figure 6.

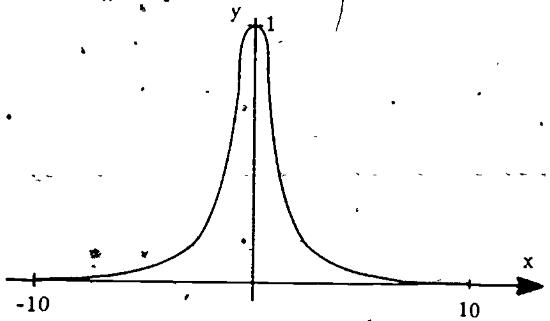


Figure 6. The graph of $y = 1/(x^6+1)$, $-10 \le x \le 10$.

5. The following two integrals are equal.

$$\int_{0}^{\pi} \sin x\,dx \quad \text{and} \quad \int_{0}^{9\pi} \sin x\,dx.$$

For which one will the trapezoidal approximation, T_n, be closer to the true value of the integral?

6. Why, when computing a sequence of trapezoidal approximations to an integral, is it better to compute $T_1, T_2, T_4, T_8, T_{16}, \ldots$ rather than $T_1, T_2, T_3, T_4, T_5, \ldots$?

7. $T_1^{(3)}$ denotes the first entry of the fourth row of the Romberg array for

$$\int_a^b f(x)\,dx.$$

How many times must the integrand, f(x), be evaluated in order to compute $T_1^{(3)}$?

8. Which row of the Romberg array always gives the true value of the integral when the integrand is a polynomial of degree 5?

9. Suppose all that is known about a function, f, is that f(0) = 0, f(1) = 1, f(2) = 3, f(3) = 5, and f(4) = 0, and that an approximation to

$$\int_1^4 f(x)\,dx$$

is desired. Compute as much of the Romberg array as is possible from the given set of f-values.

10. Show that, when the Romberg technique is applied to T_n and T_{3n}, rather than to T_n abd T_{2n}, the resulting improved approximation has the form

$$\frac{9\,T_{3n} - T_n}{8},$$

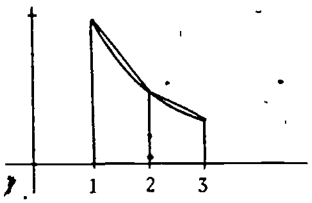
instead of

$$\frac{4\,T_{2n} - T_n}{3}.$$

7. SOLUTIONS TO EXERCISES

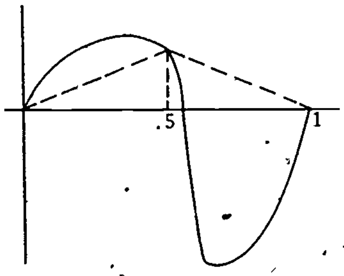
1. (a) $T_4 = 3.13118$, $T_8 = 3.13899$
   (b) $T_4 = 0$, $T_8 = 3.13899$ (Why?)
   (c) $T_4 = 1.03229$, $T_8 = 1.98235$

2. Because the graph of $y = 1/x$ is concave up on $[1,3]$, the trapezoids completely enclose more than the area under the graph. Thus, T_n is too large.



Alternatively, $f''(x) = 2/x^3$ is positive on $[1,3]$, so that the difference $T_n - \int_a^b f(x)\,dx$ in Equation (2) is positive.

3.



If $f(x)$ as in the diagram is such that $\int_0^1 f(x)\,dx = 0$, that is if the areas above and below the x-axis exactly cancel, then $T_1 = 0$ will be perfect while T_2 is positive.

4. $$T_n = \frac{b-a}{n}\left(\frac{f(a)}{2} + \frac{f(b)}{2} + f(x_1) + \dots + f(x_{n-1})\right)$$

$$= \frac{b-a}{n}\left(f(a) + f(x_1) + \dots + f(x_{n-1}) - \frac{f(a)}{2} + \frac{f(b)}{2}\right)$$

$$= R_n + \frac{(b-a)(f(b)-f(a))}{2n} .$$

5. $T_1 = 1\left(\frac{f(0)+f(1)}{2}\right) = \frac{1^2}{2} = \frac{1}{2}$. Since $|f''(x)| = 2$, a constant, we may take $K = 2$ in (3). The error is at most $\frac{2 \cdot 1^3}{12 \cdot 1^2} = \frac{1}{6}$.

Since $\int_0^1 x^2\,dx = \frac{1}{3}$, the error in T_1 is exactly $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

6. $f(x) = e^{(x^2)}$, $f'(x) = 2xe^{(x^2)}$, $f''(x) = (2x)(2x)(e^{(x^2)}) + 2e^{(x^2)} = (4x^2 + 2)e^{(x^2)}$.

Since $f''(x)$ is a positive increasing function on $[0,2]$, $|f''(x)| \le f''(2) = 18e^4$.

Using (3), we want n so that

$$\frac{18e^4 \cdot 2^3}{12n^2} \le 0.0005, \text{ or } \frac{12e^4}{.0005} \le n^2, \text{ or } 1144.7 \le n .$$

Accordingly, n should be at least 1145.

However we may improve this estimate by noting that, for n even, T_n is the sum of $T_{\frac{1}{2}n}$ for

$$\int_0^{1.5} e^{(x^2)}dx$$

and $T_{\frac{1}{2}n}$ for

$$\int_{1.5}^{2} e^{(x^2)}dx .$$

Since $|f''(x)| \le f''(1.5) = 104.365$ on $[0,1.5]$, and $|f''(x)| \le f''(2) = 18e^4 = 982.768$ on $[1.5,2]$, the error T_n is at most

$$\frac{104.365(1.5-0)^3}{12\,(\frac{1}{2}n)^2} + \frac{982.768(2-1.5)^3}{12\,(\frac{1}{2}n)^2}$$

which equals $\frac{158.359}{n^2}$. So if n is large enough to make $\frac{158.359}{n^2} \le 0.0005$, then T_n will be within the desired tolerance.

Therefore it suffices to take

$$n \geq \sqrt{\frac{158,359}{.0005}} = 562.777$$

This suggests taking n = 563, but n must be even for the argument above to be valid. We are safe, however, in concluding that 564 trapezoids will suffice.

7. $f(x) = e^{x-x^2/6}$, $f'(x) = \left(1-\frac{x}{3}\right)e^{x-x^2/6}$

$$f''(x) = \left(\frac{x^2}{9} - \frac{2x}{3} + \frac{2}{3}\right)e^{x-x^2/6}$$

To find K, we apply the methods of elementary calculus to find the maximum and minimum of $f''(x)$ on [0,10]. We first compute $f'''(x)$ to find the critical points:

$$f'''(x) = \left\{\left(\frac{x^2}{9} - \frac{2x}{3} + \frac{2}{3}\right)\left(1 - \frac{x}{3}\right) + \left(\frac{2x}{9} - \frac{2}{3}\right)\right\} e^{x-x^2/6}$$

and so

$$f'''(x) = 0$$

if

$$\frac{-x^2}{27} + \frac{x^2}{3} - \frac{2x}{3} = 0$$

$$x^2 - 9x + 18 = 0$$

$$(x-3)(x-6) = 0$$

$$x = 3 \text{ or } x = 6 .$$

The maxima and minima of $f''(x)$ must occur at x = 0, 3, 6, or 10. The values of f'' at these points are respectively, 0.667, -1.494, 0.667, and 0.0065, whence the maximum value of $f''(x)$ on [0,10] is 1.494. Taking K = 1.494 in (3) shows that the error in T_{100} is at most

$$\frac{1.494 \times 10^3}{12 \times 10^2} = 0.01245 .$$

8. $f(x) = \frac{e^{(\sin x)/\sqrt{2}}}{2\pi}$, $f'(x) = \frac{(\cos x)e^{(\sin x)/\sqrt{2}}}{2\sqrt{2}\pi}$

$$f''(x) = \frac{e^{(\sin x)/\sqrt{2}}}{2\sqrt{2}\pi}\left[\frac{\cos^2 x}{\sqrt{2}} - \sin x\right]$$

Rather than searching for critical points of $f''(x)$, we may obtain a value for K by noting that

$$\left|\frac{\cos^2 x}{\sqrt{2}} + \sin x\right| \leq \left|\frac{\cos^2 x}{\sqrt{2}}\right| + |\sin x| \leq \frac{1}{\sqrt{2}} + 1 \leq 1.7072$$

and that, on [0,2],

$$\left|\frac{e^{(\sin x)/\sqrt{2}}}{2\sqrt{2}\pi}\right| \leq \frac{e^{1/\sqrt{2}}}{2\sqrt{2}\pi} \leq 0.2283 .$$

So $|f''(x)| \leq 0.2283 \times 1.7072 \leq 0.3898$ and we may let K = 0.3898. Thus the error in T_8 is at most

$$\frac{0.3898(2\pi)^3}{12 \cdot 64} \leq 0.126 .$$

9. For $\int_0^{24} f(t)\,dt$,

$$T_8 = \frac{24}{8}\left(\frac{10.0+18.9}{2} + 9.1 + 12.4 + \ldots + 20.0\right) = 493.95 .$$

Thus T_8 for

$$\int_0^{24} \frac{f(t)}{24}\,dt$$

is 493.95/24 = 20.6, which approximates the average temperature during the day.

10. The second derivatives of the two integrands differ only in sign. Since K must bound $|f''(x)|$, K is the same for both integrands.

11. Here is a program in the language of the Texas Instruments SR-52:

```
000 LBL A STO 01 HLT
006 LBL B STO 02 1 STO 06
015 RCL 02 - RCL 01 = STO 03
```

```
026 X(RCL 01 E + RCL 02 E)
038 ÷2=
041 LBL C STO 04 HLT
047 2 INV PROD 03 PROD 06
055 0 STO 05
059 RCL 06 STO 00
065 LBL D RCL 01 + RCL 03 X
075 (RCL 06 + 1 · RCL 00 =
086 E SUM 05
090 1 INV SUM 00 dsz D
097 RCL 05 X RCL 03 +
105 RCL 04 ÷ 2 = GTO C
113 LBL E ... rtn
```

Register usage:

- R_{01}: a
- R_{02}: b
- R_{03}: b-a, $\frac{b-a}{2}$, $\frac{b-a}{4}$, $\frac{b-a}{8}$, ...
- R_{04}: $T_1, T_2, T_4, T_8, \ldots$
- R_{05}: accumulates summation in (5)
- R_{06}: 1, 2, 4, 8, ...
- R_{00}: index for loop to compute summation

User Instructions

1. Fill-in instructions 115 on, to provide a subroutine that computes f(x), the integrand. Use register 07 if necessary. Don't use =; use (...) instead.
2. Enter a, press A, enter b, press B.
3. T_1 is in display. Press run.
4. T_2 is in display. Press run repeatedly to see T_4, T_8, T_{16}, ...

12. Example:

For $\int_0^{10} e^{x-x^2/6}\,dx$, first fill in 115-132 as follows:

115 STO 07 (RCL 07 - RCL 07 x^2 ÷ 6)INV lnx rtn

Then press 0, A, 10, B, and run repeatedly to get:

5.006363169	(T_1)
14.00806104	
18.1343902	
18.5166875	
18.61450407	
18.63898675	
18.64510821	(T_{64})
18.64663861	(T_{128})

Due to the 12-digit accuracy of programmable calculators, one gets the same results for the integral backwards, i.e., $\int_{10}^{0} -e^{x-x^2/6}\,dx$.
Discrepancies would come in with the use of thousands of trapezoids. On a 6-digit machine however, the forward and backward results would already differ at T_{128}.

13. Line 140 has been changed. Lines 440 - 560 compute the Romberg array from the trapezoidal data.

```
100 REM  TRAPEZOIDAL APPROXIMATION OF INTEGRALS
110 REM   FIRST DEFINE INTEGRAND; THIS LINE
120 REM   MUST BE MODIFIED FOR NEW INTEGRANDS
130 DEF FNA(X)=X↑3/(EXP(X)-1)
140 DIM T(30)
150 PRINT "WHAT ARE A AND B?"
160 INPUT A,B
170 PRINT "HOW MANY ROWS OF THE ROMBERG ARRAY?"
180 INPUT M
190 REM
200 REM   WE FIRST USE ONE TRAPEZOID, STORING
210 REM   THE RESULT IN T(1)
220 D=B-A
230 T(1)=D/2*(FNA(A)+FNA(B))
```

```
240 REM
250 REM    NOW COMPUTE THE (2↑N)TH TRAPEZOIDAL
260 REM    APPROXIMATION FROM THE 2↑(N-1)ST,
270 REM    USING FORMULA (5) ABOVE, AND
280 REM    STORING IT IN T(N)
290 FOR N=2 TO M
300 D=D/2
310 S=0
320 REM
330 REM    USE S TO FORM THE SUMMATION IN (5)
340 FOR J=1 TO 2↑(N-1) STEP 2
350 S=S+FNA(A+J*D)
360 NEXT J
370 T(N)=T(N-1)/2+D*S
380 NEXT N
390 REM
400 REM    NOW PRINT THE RESULTS
410 FOR N=1 TO M
420 PRINT "T(";2↑(N-1);")=";T(N)
430 NEXT N
440 PRINT
450 F=1
460 FOR I=1 TO M-1
470 F=4*F
480 REM    COMPUTE AND PRINT THE ITH ROW
490 REM    OF THE ROMBERG ARRAY
500 FOR J=1 TO M-I
510 T(J)=(F*T(J+1)-T(J))/(F-1)
520 PRINT T(J)
530 NEXT J
540 PRINT
550 NEXT I
560 END
```


14.

3.00000	3.10000	3.13118	3.13899
3.13333	3.14157	3.14159	
3.14212	3.14159		
3.14159			

15.

0.0000000	1.5540803	.12506210	.73627360	.73713182
2.0721071	-.35007730	.93971077	.73741790	
-.51155626	1.0256966	.72393171		
1.0500975	.71914179			
.71784392				

In this rather unusual example, Romberg's method provides worse approximations than the original trapezoidal approximations.

16. Using (5) in the form

$$T_{2n} = \frac{T_n}{2} + \frac{b-a}{2n}\,\Sigma^{odd}$$

where

$$\Sigma^{odd} = \sum_{\substack{j\ odd \\ 1\le j\le 2n-1}} f(x_j)\ ,$$

we have

$$T'_n = (4T_{2n}-T_n)/3$$

$$= \left(2T_n + \frac{4(b-a)}{2n}\,\Sigma^{odd} - T_n\right)/3$$

$$= \frac{1}{3}\left(\frac{b-a}{n}\left(\frac{f(a)+f(b)}{2} + \Sigma^{even}\right) + \frac{4(b-a)}{2n}\,\Sigma^{odd}\right)$$

$$= \left(\frac{1}{3}\right)\left(\frac{b-a}{2n}\right)\left(f(a) + f(b) + 2\,\Sigma^{even} + 4\,\Sigma^{odd}\right),$$

which is precisely Simpson's rule with 2n subintervals.

17.

1.758664	.799332	.559499	.499453
.479555	.479555	.479438	
.479555	.479430		
.479428			

8. SOLUTIONS TO MODEL EXAM

1. 2.215.

2. $|f''(x)| = 2/x^3$ which is decreasing on [.2, 1], so K may be taken to be $2/(.2)^3$, which equals 250. Thus

$$\text{ERROR} \leq \frac{250}{12}\frac{(.8)^3}{100} = 0.10666\ldots \leq 0.107.$$

3. $|f''(x)| = \frac{12-2(\ln x)}{x^2} \leq \frac{2}{1}$ since $.613 \leq 2-2(\ln x) \leq 2$ and $1 \leq x^2 \leq 4$ on [1,2]. Taking K = 2 in the error formula yields $n \geq 51.6$, or $n \geq 52$.

4. (d) is best suited since it adds f-values in order from smallest to largest.

5. $\int_0^\pi \sin x\,dx$. For n fixed, a smaller $b - a$ leads to a smaller error.

6. The first sequence reuses the f-values already computed. Moreover, the first sequence can be used to start Romberg's method.

7.

8. The third row.

9. $T_1 = 0$ $\quad T_2 = 6$ $\quad T_4 = 9$

$T_1' = 8$ $\quad T_2' = 10$

$T_1'' = 10.1333$

10. $$T_n = \int_a^b f(x)\,dx + \frac{A}{n^2} + \frac{B}{n^4} + \ldots$$

$$T_{3n} = \int_a^b f(x)\,dx + \frac{A}{9n^2} + \frac{B}{81n^4} + \ldots$$

So

$$\frac{9T_{3n} - T_n}{8} = \int_a^b f(x)\,dx + \frac{B'}{n^4} + \frac{C'}{n^6} + \ldots,$$

and the term of the error involving n^2 has been eliminated.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ________

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

STUDENT FORM 2

Unit Questionnaire

Name________________________ Unit No.______________ Date______________

Institution________________________ Course No.____________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)