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ABSTRACT

A group is viewed to be one of the simplest and most interesting algebraic structures. The theory of groups has been applied to many branches of mathematics as well as to crystallography, coding theory, quantum mechanics, and the physics of elementary particles. This material is designed to help the user: 1) understand what groups are and why they are important; 2) to identify certain groups, subgroups, cyclic groups, and isomorphic groups; and 3) to construct groups. Problems and a sample exam are included in the unit. Answers to the exercises in the text and to the test questions are provided. (MP)

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UNIT 461

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

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AN INTRODUCTION TO GROUPS

by Nancy S. Rosenberg

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x	R_1^4	R_1^1	R_1^2	R_1^3
R_1^4	R_1^4	R_1^1	R_1^2	R_1^3
R_1^1	R_1^1	R_1^2	R_1^3	R_1^4
R_1^2	R_1^2	R_1^3	R_1^4	R_1^1
R_1^3	R_1^3	R_1^4	R_1^1	R_1^2

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AN INTRODUCTION TO GROUPS

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Title: AN INTRODUCTION TO GROUPS

Author: Nancy S. Rosenberg
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Prerequisite Skills:
1. Elementary algebra.
2. Plane geometry.
3. Modular arithmetic of integers.

Output Skills:
1. To understand what groups are and why they are important.
2. To be able to identify certain groups, subgroups, cyclic groups, and isomorphic groups.
3. To be able to construct groups.

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MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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1. WHY STUDY GROUPS?

1.1 The Power of Generalization

Mathematicians like to be as general as they can. When they say that a + b = b + a, they are making a statement about all real numbers. This is the familiar commutative property of addition; it asserts that for any pair of numbers the order in which they are added makes no difference. Such a statement is much more powerful than the assertion that 2 + 3 = 3 + 2 or that $\sqrt{7} + \sqrt{5} = \sqrt{5} + \sqrt{7}$. The value of the more general statement is this: used in conjunction with other such statements, it can serve as a basis for deducing the properties of all real numbers.

The commutative property is not confined to real numbers under addition. Sets are commutative under intersection too, and so are two knight's moves performed in succession on a chessboard. A consideration of situations as diverse as these has prompted mathematicians to be even more general in their thinking and to study algebraic structures, sets of "elements" - objects of any sort at all - which exhibit certain properties when combined under a binary operation. The results of such studies are completely general in their application.

One of the simplest and most interesting algebraic structures is the group. The theory of groups has been applied to many branches of mathematics as well as to crystallography, coding theory, quantum mechanics, and the physics of elementary particles. In this last case, it was used to predict the existence and properties of certain particles well before they were actually discovered.

1.2 What is a Group?

A group is a set of elements that can be combined under a binary operation which we shall call * and which exhibits four properties under this operation. (The result of combining two elements a and b of the set under the operation will be written as a * b, pronounced "a star b.")

Property 1. The set is closed under *, that is, if a and b are members of the set then a * b is also a member.

Property 2. The operation * is associative, that is, a * (b * c) = (a * b) * c.

Property 3. The set contains an element e such that for any element a of the set a * e = e * a = a. e is called the identity element.

Property 4. For every element a of the set there exists a unique element b such that a * b = b * a = e. b is called the inverse of a and is often written a^{-1}.

A group need not be commutative. If it is, however, it is called a commutative group or an Abelian group.

1.3 Examples of Groups

The integers form a group under addition. To prove this, we must show that integers exhibit each of the four group properties when they are added. First, the integers are closed under addition, that is, the sum of any two integers is an integer, Second, addition is an associative operation. Third, there is an identity element, namely zero, since the addition of zero to any integer leaves that integer unchanged. And fourth, each integer has an inverse under addition, namely its negative. Thus 3 + (-3) = (-3) + 3 = 0. Since addition is commutative, the integers under addition form an Abelian group. Since there are an infinite number of integers, it is also called an infinite group.

A second example of a group is the equivalence classes 2, 4, 6 and 8 modulo 10 under multiplication. This group is best investigated by the construction of a table.

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x	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

Inspection of this table reveals that the group is closed. It is also associative, since multiplication is an associative operation. Six is seen to be the identity element since the row next to the six in the column to the left of the table is the same as the row at the top of the table and the column under the six in the top row is the same as the column to the left of the table. The numbers 4 and 6 are their own inverses; 2 and 8 are inverses of each other. That this group is also commutative can be seen from the fact that there is symmetry about the main diagonal of the table, that is, the diagonal that goes from the upper left corner of the table to the lower right corner. Commutative groups will always exhibit symmetry of this sort when their elements are listed in the same order going across as they are going down. Because this group has four elements, it is said to be of order 4. In general, a finite group with n elements is said to be of order n.

Exercise 1. State whether each of the following is a group. If your answer is no, explain which properties of a group fail to hold and why. If your answer is yes, give the group's identity element and one pair of inverses.

a. The equivalence classes of the integers 0, 1, 2 and 3 modulo 4 under addition.

b. The rational numbers, excluding zero, under multiplication.

c. The positive integers under subtraction.

d. The numbers i, -1, -i, and 1 under multiplication, where $i = \sqrt{-1}$.

e. The set of integers with the operation * defined as $a * b = a + b - 2$.

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f. The set $S = \{A, B, C, D\}$ where $A = \emptyset$, $B = \{x\}$, $C = \{y\}$ and $D = \{x,y\}$, under intersection.

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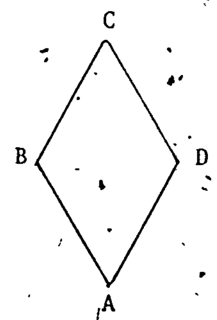
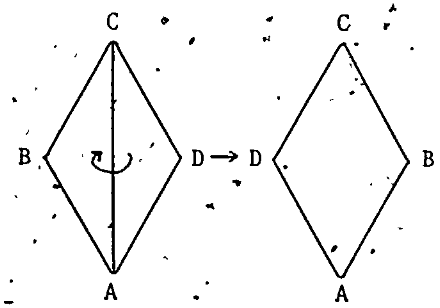
2. SYMMETRY GROUPS

2.1 The Symmetry Group of a Plane Figure

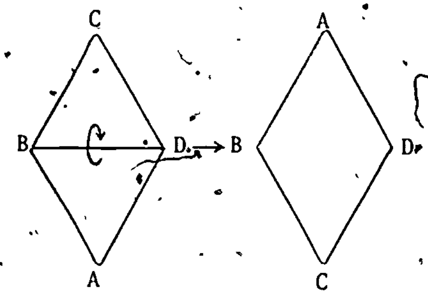
An interesting way to generate groups is to perform operations on a plane figure that leave the appearance of that figure unchanged. Such operations are called symmetry operations, and the groups they generate are symmetry groups. Take, for example, the diamond shown below. For convenience, its vertices are labelled A, B, C and D.



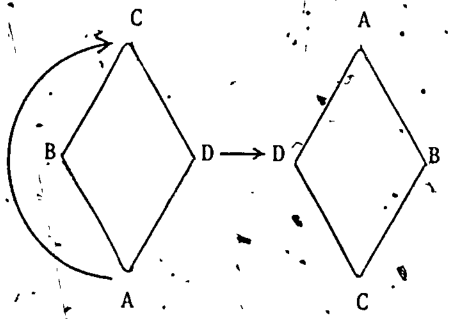
If this diamond is flipped over the line joining A to C, the only effect is to interchange B and D.



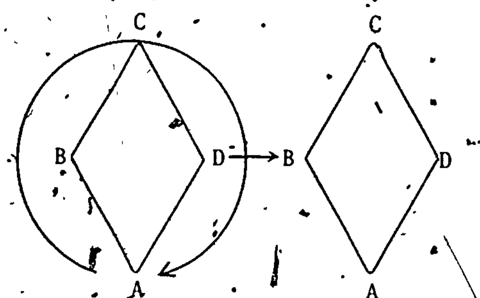
Similarly, if it is flipped over the line joining B to D, the only effect is to interchange A and C.



It is also possible to rotate the diamond through 180°. This has the effect of interchanging both A and C, and B and D.



Finally, the diamind can be rotated through 360° which is, of course, the same as leaving it alone.



Since these four operations are the only ones that leave the appearance of the diamond unchanged, performing any two of them in succession must be equivalent to performing a third. The system is therefore closed under

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the operation of performing first one symmetry operation and then another. This operation is also, by its nature, associative. That is, performing one symmetry operation, and then another, and then another, produces the same result regardless of how the operations are considered to be grouped. Rotating the figure through 360° or, alternatively, leaving it alone, is the identity. And since each operation can be reversed or "undone," there are inverses under this identity. The symmetry operations of a plane closed figure like this one therefore form a group under the operation of performing two of them in succession.

It can be shown that every symmetry operation on a plane closed figure is either a flip or a rotation. We will denote the flip over AC by F_1, the flip over BD by F_2, and the rotation through 180° by R_1. Since the rotation through 360° leaves the figure unchanged, we will call it I, for "identity".

The flip F_1 changes the sequence of the vertices from A B C D to A D C B. One way of representing this is to write

$$F_1 = \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}$$

In the same way,

$$F_2 = \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix} \text{ and } R_1 = \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}.$$

Now consider the effect of performing first F_2 and then R_1. The flip F_2 turns A B C D into C B A D. The rotation R_1 then interchanges the first and third elements and the second and fourth to change C B A D into A D C B. But this is the equivalent of simply performing F_1. Thus $F_2 \times R_1 = F_1$, where $F_2 \times R_1$ means the operation of first performing F_2 and then R_1. The table below shows the results of performing any two of the four symmetry operations in succession. Note the symmetry about the main diagonal that shows that this group is commutative.

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x	I	R_1	F_1	F_2
I	I	R_1	F_1	F_2
R_1	R_1	I	F_2	F_1
F_1	F_1	F_2	I	R_1
F_2	F_2	F_1	R_1	I

Such a table can be made either by considering the permutations shown in the parentheses above or by actually manipulating a cardboard diamond with its vertices labelled.

Let us make a similar table for the triskelion shown below.

This figure has no flips among its symmetry operations. There are only three rotations, through 120°, 240°, and 360°. We will call them R_1, R_2, and I respectively. The combination table for this group appears below.

x	I	R_1	R_2
I	I	R_1	R_2
R_1	R_1	R_2	I
R_2	R_2	I	R_1

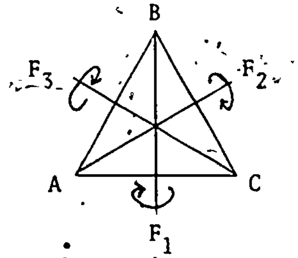
Exercise 2. Define symmetry operations for the following figures and construct conbination tables for them.

a) the letter A **b) the gammadion**  **c) a rectangle (one that is not also a square)**

Which of these groups is commutative?

2.2 The Symmetry Group of a Regular Polygon

A polygon is regular if all of its sides and all of its angles are equal. The first two regular polygons are the equilateral triangle, with three sides, and the square with four. Consider the symmetry operations of the equilateral triangle. There are six of them, three flips and three rotations, as shown below.



R_1 = 120° clockwise rotation
R_2 = 240° clockwise rotation
R_3 = I = 360° rotation

The equilateral triangle has six symmetry operations because it has three sides and three angles. Each of the angles can be rotated to the position occupied by either of the other two, and each of the three altitudes is a line of symmetry about which the triangle can be flipped. In addition, there is the identity.

Here is the combination table for the symmetry operations of an equilateral triangle. Note that this group is not commutative.

x	I	R_1	R_2	F_1	F_2	F_3
I	I	R_1	R_2	F_1	F_2	F_3
R_1	R_1	R_2	I	F_3	F_1	F_2
R_2	R_2	I	R_1	F_2	F_3	F_1
F_1	F_1	F_2	F_3	I	R_1	R_2
F_2	F_2	F_3	F_1	R_2	I	R_1
F_3	F_3	F_1	F_2	R_1	R_2	I

$$R_1 = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$
$$R_2 = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$
$$F_1 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$
$$F_2 = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$$
$$F_3 = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

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It can be seen from this table that the product of two rotations is a rotation, the product of a flip and a rotation is a flip, and the product of two flips is a rotation.

Exercise 3. Find the symmetry group of the square. (It is called the octic group.)

Exercise 4. How many symmetry operations can be performed on a regular pentagon? On a regular octagon? On a regular n-sided polygon?

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3. THE STRUCTURE OF GROUPS

3.1 Subgroups

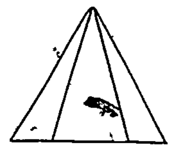
You may have noticed that the three elements I, R_1, and R_2 in the upper left hand corner of the symmetry group of the equilateral triangle are themselves a group that is identical to the symmetry group of the triskelion, whose table is given on page 7. In the same way, the four elements in the upper left-hand corner of the symmetry group of the square form a group that is identical to the symmetry group of the gammadion. A group that is contained within a larger group, as these are, is known as a subgroup. Specifically, given a set S which is a group under the operation *, any subset of S which is also a group under * is called a subgroup. This definition allows S to be a subgroup of itself. A subgroup of S that is not all of S is called a proper subgroup of S.

Finding subgroups of a given group is made easier by the use of LaGrange's Theorem, which states that the number of elements in any subgroup of a group G must be a factor of the number of elements in G. Thus, since there are $6 = 1 \cdot 2 \cdot 3$ elements in the symmetry group of the equilateral triangle, its proper subgroups must contain either one, two, or three elements. (The identity element of a group always constitutes a subgroup of one element. It is

sometimes called the trivial subgroup.)

An easy way to find the subgroups of the symmetry group of a plane figure is to draw on that figure other figures that share some, but not all, of its symmetry operations. When a triskelion is drawn on an equilateral triangle, for example, it can be seen that it shares the triangle's three rotations but none of its flips.

An isosceles triangle can also be drawn on an equilateral triangle.



Besides the identity, the isosceles triangle shares only one of the equilateral triangle's symmetry operations, in this case F_1. Thus I and F_1 also constitute a subgroup, as do I and F_2 and I and F_3.

x	I	F_1
I	I	F_1
F_1	F_1	I

x	I	F_2
I	I	F_2
F_2	F_2	I

x	I	F_3
I	I	F_3
F_3	F_3	I

Exercise 5. Find the proper subgroups of the octic group. (See Exercise 3.)

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3.2 The Division Property

The tables we have seen so far have an interesting feature in common: each element of the group appears just once in every row and every column. Is this always the case? Suppose that it were not. That is, suppose that the element P appeared twice in the A row, once in the B column and once in the C column, as in the following table.

x	A	B	C
A		P	P
B			
C			

Then,

(1) $A \times B = P$ and $A \times C = P$.

If we multiply both sides of the equations in (1) on the left by A^{-1}, we get

(2) $A^{-1} \times (A \times B) = A^{-1} \times P$ and $A^{-1} \times (A \times C) = A^{-1} \times B$.

The associative property then gives

(3) $(A^{-1} \times A) \times B = A^{-1} \times P$ and $(A^{-1} \times A) \times C = A^{-1} \times P$.

But since

$$A^{-1} \times A = I,$$

the two equations in (3) become

(4) $I \times B = A^{-1} \times P$ and $I \times C = A^{-1} \times P$.

Since I is the identity,

(5) $I \times B = B$ and $I \times C = C$,

and we are led to the conclusion that

(6) $B = A^{-1} \times P$ and $C = A^{-1} \times P$.

But $A^{-1} \times P$ has a unique result; it cannot possibly equal both B and C. The premise is therefore false; P can appear only once in the row next to A and, by extension, only once in any row or column in the table. This is to so-called <u>division property</u>. It says that the equation $A \times X = P$ has a unique result, which we call $P \div A$.

Let us use the division property to construct some simple group tables. We have already seen the structure of the table of a two element group. It looks like this:

x	I	A
I	I	A
A	A	I

Any element which is its own inverse forms, with the identity, a two element group.

What about a three element group? Can we, for example, construct a group table for three elements I, A and B in which each element is its own inverse? Such a table would look, in part, like this:

x	I	A	B
I	I	A	B
A	A	I	
B	B		I

Is it possible to fill in the rest of the table so that each element occurs only once in every row and every column? If you try, you will find that it is not. Let us therefore try the only alternative, to construct a table in which A and B are inverses of each other.

x	I	A	B
I	I	A	B
A	A		I
B	B	I	

There is just one way to fill in the blanks in this table. Therefore, the structure of the three element group shown below is the only one possible.

x	I	A	B
I	I	A	B
A	A	B	I
B	B	I	A

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Can every element of a four element group be its own inverse? In other words, can the table of a four element group look like this?

x	I	A	B	C
I	I	A	B	C
A	A	I		
B			I	
C	C			I

There is only one way, shown below, to fill in the remaining spaces of this table so that each element appears only once in every row and every column.

x	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

The only other possibility for a four element group is to have one of the elements, say A, be its own inverse and B and C be inverses of each other.

x	I	A	B	C
I	I	A	B	C
A	A	I		
B	B			I
C	C		I	

Again, there is only one way to fill in the rest of the table.

x	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

We conclude, therefore, that there are two distinct structures for a four element group.

3.3 Cyclic and Dihedral Groups

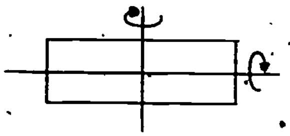
It can be shown that if the symmetry group of a bounded plane figure is finite, it must take one of two forms. When at least one of the symmetry operations is a flip, then there will be as many flips as there are rotations. In this case the symmetry group is called a dihedral group. A group is cyclic if all of its members can be generated by raising a single one of them to successively higher powers.

The symmetry group of the gammadion is a cyclic group. To see this, observe that $R_1^2 = R_2$, $R_1^3 = R_1 \times R_1^2 = R_1 \times R_2 = R_3$, and $R_1^4 = R_1 \times R_1^3 = R_1 \times R_3 = I$. Thus, the elements of this group can all be expressed as powers of R_1. The table can be written like this:

x	R_1^4	R_1^1	R_1^2	R_1^3
R_1^4	R_1^4	R_1^1	R_1^2	R_1^3
R_1^1	R_1^1	R_1^2	R_1^3	R_1^4
R_1^2	R_1^2	R_1^3	R_1^4	R_1^1
R_1^3	R_1^3	R_1^4	R_1^1	R_1^2

Observe that this group, like all cyclic groups, is commutative.

The symmetry group of a rectangle, (one that is not also a square), by contrast, is a dihedral group with two flips,



a 180^o-rotation, R_1, and a 360^o-rotation, I. That the group is not cyclic may be seen by observing that $F_1^2 = I$, $F_2^2 = I$, and $R_1^2 = I$. It follows that $F_1^3 = F_1 \times F_1^2 = F_1 \times I = F_1$ and similarly that $F_2^3 = F_2$ and $R_1^3 = R_1$. Thus, each of these elements generates a cyclic subgroup of two elements; none generates the entire group.

Exercise 6. Of the two possible structures of a four element group found in Section 3.2, which is cyclic? Express the members of this group as powers of a single element. Can this be done in more than one way?

3.4 Isomorphisms

We have studied a number of four element groups:

x	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

The equivalence classes 2, 4, 6 and 8 modulo 10 under multiplication.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

The equivalence classes 0, 1, 2, and 3 modulo 4 under addition.

x	i	-1	-i	1
i	-1	-i	1	i
-1	-i	1	i	-1
-i	1	i	-1	-i
1	i	-1	-i	1

i, -1, -i and 1 under multiplication, where $i = \sqrt{-1}$.

x	I	R_1	F_1	F_2
I	I	R_1	F_1	F_2
R_1	R_1	I	F_2	F_1
F_1	F_1	F_2	I	R_1
F_2	F_2	F_1	R_1	I

The symmetry group of the diamond.

x	I	R_1	R_2	R_3
I	I	R_1	R_2	R_3
R_1	R_1	R_2	R_3	I
R_2	R_2	R_3	I	R_1
R_3	R_3	I	R_1	R_2

The symmetry group of the gammadion.

If you look at them closely, however, you will see that each has one of the two structures we discovered in Section 3.2. Compare, for example, the group consisting of i, -1, -i and 1 under multiplication with the symmetry group of the gammadion. If the table for i, -1, -i and 1 is rearranged so that 1, the identity, is the first element listed, these tables are seen to be identical in structure with I corresponding to 1, R_1 to i, R_2 to -1, and R_3 to -i.

x	I	R_1	R_2	R_3
I	I	R_1	R_2	R_3
R_1	R_1	R_2	R_3	I
R_2	R_2	R_3	I	R_1
R_3	R_3	I	R_1	R_2

x	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Such a correspondence is called an isomorphism.

Two groups, G with group operation * and H with group operation ^{o}, are said to be isomorphic if a one-to-one correspondence can be established between their elements such that if g_1 corresponds to h_1 and g_2 corresponds to h_2, then $g_1 * g_2$ corresponds to $h_1 \, ^o \, h_2$. The tables of two finite, isomorphic groups will be identical in structure if corresponding elements are given corresponding locations in the table.

Exercise 7. Examine the tables for the equivalence classes 2, 4, 6 and 8 modulo 10 under multiplication, the equivalence classes 0, 1, 2 and 3 modulo 4 under addition, and the symmetry group of the diamond. If any two of these are isomorphic, find an appropriate correspondence between their elements and arrange these elements correspondingly in tables.

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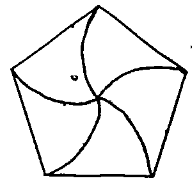
4. SAMPLE EXAM

1. Form the symmetry group of the figure shown below and find its proper subgroups.



2. Give two reasons, based on different group properties, why the table below does not define a group.

*	I	A	B	C	D
I	I	A	B	C	D
A	A	C	D	B	I
B	B	I	C	D	A
C	C	D	A	I	B
D	D	B	I	A	C

3. Show that the set of numbers of the form $a + b\sqrt{2}$ forms a group under the usual multiplication of real numbers, where a and b are integers but cannot both equal zero at the same time.

5. ANSWERS TO EXERCISES

Exercise 1.

a. This is a group, the table for which is given below.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

The identity is 0. 1 and 3 are inverses.

b. This is also a group. The identity is 1 and the reciprocal of each number is its inverse. For example, 2/3 and 3/2 are inverses. (It is necessary to exclude zero because it does not have a multiplicative inverse.)

c. The positive integers are not a group under subtraction. All four of the group properties fail to hold. First, the positive integers are not closed under subtraction since, for example, 3 - 7 is not a positive integer. They are not associative since (3 - 7) - 2 is not the same as 3 - (7 - 2). There is no identity. (The identity for subtraction is zero, but zero is not a positive integer.) Without an identity there can be no inverses.

d. This is a group, the table for which is given below.

x	i	-1	-i	1
i	-1	-i	1	i
-1	-i	1	i	-1
-i	1	i	-1	-i
1	i	-1	-i	1

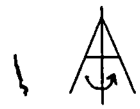
The identity is 1. i and -i are inverses.

e. This is a group. The identity is 2. The inverse of a is 4 - a.

f. This is not a group. D acts as an identity, but there are no inverses.

Exercise 2.

a. The letter A has only one symmetry operation besides the identity, and that is a flip over the vertical line shown below.



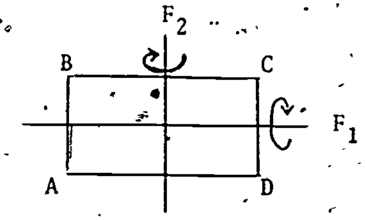
The flip is denoted as F in the table.

x	I	F
I	I	F
F	F	I

b. The gammadion can be rotated through angles of 90°, 180°, 270°, and 360°. In the table these rotations are denoted as R_1, R_2, R_3, and I respectively.

x	I	R_1	R_2	R_3
I	I	R_1	R_2	R_3
R_1	R_1	R_2	R_3	I
R_2	R_2	R_3	I	R_1
R_3	R_3	I	R_1	R_2

c. The rectangle can be rotated through angles of 180° and 360°, called R_1 and I in the table, and flipped over the lines shown as F_1 and F_2 in the figure.



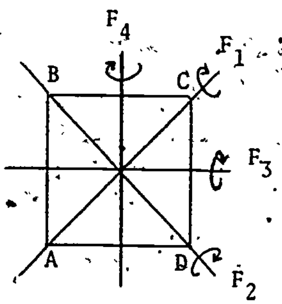
x	I	R_1	F_1	F_2
I	I	R_1	F_1	F_2
R_1	R_1	I	F_2	F_1
F_1	F_1	F_2	I	R_1
F_2	F_2	F_1	R_1	I

All three of these groups are commutative.

Exercise 3.

The square has eight symmetry operations, four flips and four rotations. The rotations, through 90°, 180°, 270°, and 360°, are called, respectively R_1, R_2, R_3, and I in the table. The four flips are shown in the figure.

x	I	R_1	R_2	R_3	F_1	F_2	F_3	F_4
I	I	R_1	R_2	R_3	F_1	F_2	F_3	F_4
R_1	R_1	R_2	R_3	I	F_4	F_3	F_1	F_2
R_2	R_2	R_3	I	R_1	F_2	F_1	F_4	F_3
R_3	R_3	I	R_1	R_2	F_3	F_4	F_2	F_1
F_1	F_1	F_3	F_2	F_4	I	R_2	R_1	R_3
F_2	F_2	F_4	F_1	F_3	R_2	I	R_3	R_1
F_3	F_3	F_2	F_4	F_1	R_3	R_1	I	R_2
F_4	F_4	F_1	F_3	F_2	R_1	R_3	R_2	I



Exercise 4.

A regular pentagon has 10 symmetry operations, a regular octagon has 16, and, in general, a regular n-sided polygon has 2n.

Exercise 5.

The subgroups of the octic group are:

- I, R_1, R_2, and R_3
- I, R_2, F_1 and F_2
- I, R_2, F_3 and F_4
- I and R_2
- I and F_1
- I and F_2
- I and F_3
- I and F_4
- I

Exercise 6.

The latter of the two groups found in Section 3.2 is cyclic and can be generated by either B or C.

x	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

$B^1 = B$ $C^1 = C$

$B^2 = A$ $C^2 = A$

$B^3 = C$ $C^3 = B$

$B^4 = I$ $C^4 = I$

Exercise 7.

The equivalence classes 2, 4, 6, and 8 modulo 10 under multiplication are isomorphic to the equivalence classes 0, 1, 2, and 3 modulo 4 under addition. 6 corresponds to 0, 2 to 1, 4 to 2, and 8 to 3.

x	6	2	4	8
6	6	2	4	8
2	2	4	8	6
4	4	8	6	2
8	8	6	2	4

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

6. ANSWERS TO SAMPLE EXAM

1. This figure has no flips and five rotations, through 72°, 144°, 216°, 288°, and 360°. These are called R_1, R_2, R_3, R_4 and I, respectively, in the table below.

x	I	R_1	R_2	R_3	R_4
I	I	R_1	R_2	R_3	R_4
R_1	R_1	R_2	R_3	R_4	I
R_2	R_2	R_3	R_4	I	R_1
R_3	R_3	R_4	I	R_1	R_2
R_4	R_4	I	R_1	R_2	R_3

Since this group has five elements, and five is a prime number, it can have no proper subgroups other than the identity.

2. This system lacks unique inverses. Notice that $A * D = I$, but that $D * B = I$ also. In addition, the system is not associative since, for example, $(A * D) * B$ does not equal $A * (D * B)$.

3. The set is closed under multiplication since $(a + b\sqrt{2}) \times (c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$. The multiplication is associative because the usual multiplication of real numbers is associative. The number $1 + 0 \times \sqrt{2}$ serves as an identity. The inverse of $a + b\sqrt{2}$ is

$$\frac{a}{a^2 - 2b^2} + \frac{b}{2b^2 - a^2}\sqrt{2}.$$

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STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

Return to:
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55 Chapel St.
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STUDENT FORM 2

Unit Questionnaire

Name________________ Unit No.________ Date________

Institution________________ Course No.________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)