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ABSTRACT

The Rice Index is the first measure of voting unity devised. It is noted that this measure was developed on an ad hoc basis for use in rather rough comparisons. This measure is carefully assessed, and is noted to have the virtue of simplicity, useful in certain situations, but associated with certain problems when it is used for a more detailed analysis. Two other measures of voting unity are called The Probability of Agreement Measure and The Alpha-Index. Each measure is presented with exercises, theoretical problems, and a bibliography. Solutions to the exercises and problems are provided. (MP)

umap

UNIT 271a,b,c

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

MEASURES OF VOTING UNITY

by Christopher H. Nevison

The Rice Index

The α-Index

The Probability of Agreement Measure

APPLICATIONS OF PROBABILITY TO THE SOCIAL SCIENCES

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MEASURES OF VOTING UNITY

by

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Intermodular Description Sheet: *UMAP Units 271a*, 271b, 271c*

Title: MEASURES OF VOTING UNITY

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Prerequisite Skills:
1. Understanding of elementary probability.

Output Skills:
1. To develop a better understanding of the interaction between math and social science.
2. To develop skills in applying and calculating probabilities and expectations.
3. To learn to utilize the binomial distribution.

Other Related Units:

*Unit 271a is also available as a self-contained unit.

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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MEASURES OF VOTING UNITY I:
THE RICE INDEX

1. INTRODUCTION

We will study a problem which frequently occurs in the social sciences: the development of a quantitative measure. Our example is taken from political science: We shall study methods for measuring the unity of a group based on how it votes. This example will exhibit some of the issues which are common to the development of any quantitative measure.

The unity of a group may be discussed verbally. However, when comparisons between different situations are to be made, precision demands some systematic method of quantification. As is frequently the case, the first measure which we shall study was developed on an *ad hoc* basis for use in rather rough comparisons. As more sophisticated analyses have been made, it has become necessary to examine carefully the properties of the measure: Is it a meaningful measure? What factors influence the measurement? Can statistical comparisons legitimately be made? These are questions which must always be answered when a careful quantitative analysis is to be done.

In this module we will study how the first measure of voting unity, the Rice Index, was devised. Then we shall assess this measure in terms of the questions raised above. We shall see that the Rice Index has the virtue of simplicity and can be a useful measure for certain situations, but some problems occur when it is used for a detailed analysis.

This module is the first in a sequence of three. The subsequent modules in this series will study other measures of voting unity which have been developed. This module may be used alone, or in conjunction with Unit 271b, Measures of Voting Unity II: The Probability of Agreement Measure and Unit 271c, Measures of Voting Unity III: The α-Index.

2. THE RICE INDEX OF UNITY

2.1 Introduction

The most obvious and natural way to measure how united a group is when it votes on an issue is to use the size of the majority on the vote. For comparison purposes the *proportion* of the group in the majority can be used as the measure of unity. If we denote the group size by n and the number of the majority by M, this is M/n. If the group votes unanimously, this proportion will be 1, whereas an even split in the group will yield a proportion of 0.5.

It is convenient to put an index which measures extremes, as this one does, ranging from least united to most united, on a scale which ranges from zero to one. (In some circumstances, a measure is constructed so that it ranges from -1 to +1.) The proportion in the majority does not do this. Consequently, we *normalize*, or rescale, the measure so that the ordering is preserved but the range is converted from 0.5 to 1 to 0 to 1.

The easiest method of normalization and one which preserves the relative differences along the scale is to use a linear transformation which consists of two steps: (1) subtract the lowest possible value of the raw measure (the majority proportion) from the observed value; (2) multiply by the reciprocal of the length of the range of the raw measure. The first step will move the lowest possible value to zero and the second step will yield a range of length one, so the derived measure will range from zero to one.

The majority proportion ranges from 0.5 to 1, so the normalization is the following:

$$RI = 2(M/n - 0.5) = (2M/n) - 1$$

where M is the observed majority. We call the derived index the *Rice Index*, after Stuart Rice [1924] who first used this measure of unity.

Rice originally defined the index in terms of the majority proportion and the minority proportion, m/n:

$$RI = M/n - m/n.$$

Since $m/n = 1 - M/n$, this is equivalent to our formulation.

The Rice Index is an absolute measure of unity in terms of majority size. It does not take into account whether people vote yes or no, or for one candidate or his opponent. It is based on the assumption that there are two alternatives for the voter.

The Rice Index has the virtue of being easy to interpret and easy to calculate. It is certainly an appropriate measure of unity for a simple analysis of a situation.

2.2 Calculating the Rice Index

The calculation of the Rice Index is straightforward: if Y is the number who vote yes, N the number who vote no, and n is the size of the group, then the majority proportion will be the larger of Y/n and N/n. RI will simply be the difference between the larger and the smaller of the two numbers.

If a computer is used to calculate RI for a large number of cases, then it is useful to formulate RI in terms of Y, N, and n since data are often available in this form. Rather than test for the larger proportion, it is easiest to formulate RI as

$$RI = |Y - N|/n = |2Y - n|/n,$$

where || denotes absolute value.

Example 1: Suppose a group of eight votes five yes and three no. Then the Rice Index will be

$$RI_8(5) = \frac{|5 - 3|}{8} = 0.25 .$$

Example 2: Suppose a group of sixteen votes one yes and fifteen no. Then the calculation is

$$RI_{16}(15) = \frac{|1 - 15|}{16} = 0.875 .$$

We shall use the notation indicated in these examples: if n is the size of the group and k is the number voting in the majority, then the Rice Index is denoted by

$$RI_n(k).$$

The complete set of values of the Rice Index for groups of eight, sixteen, and twenty-four is shown in Table 1.

TABLE 1

Values of the Rice Index for Groups of 8, 16, and 24.

%m	n = 8	n = 16	n = 24	RI
50.0	k = 4	k = 8	k = 12	0.000
54.2	-	-	13	0.083
56.3	-	9	-	0.125
58.3	-	-	14	0.167
62.5	5	10	15	0.250
66.7	-	-	16	0.333
68.8	-	11	-	0.375
70.8	-	-	17	0.417
75.0	6	12	18	0.500
79.2	-	-	19	0.583
81.3	-	13	-	0.625
83.3	-	-	20	0.667
87.5	7	14	21	0.750
91.7	-	-	22	0.833
93.8	-	15	-	0.875
95.8	-	-	23	0.917
100	8	16	24	1.000

2.3 Exercises

1. Calculate the Rice Index for all possible majority sizes for a group of n = 5.
2. Repeat Exercise 1 for n = 7.
3. Repeat Exercise 1 for n = 10.

3. GROUP SIZE

The usefulness of a measure like the Rice Index depends on the context where it is used. The Rice Index will be accurate when it is used to compare the unity of groups of the same size voting on similar issues. Broader comparisons, however, raise some problems.

3.1 Different Size Groups

The Rice Index is designated so that it always ranges on a standard 0 - 1 scale no matter what the group, with 0 indicating maximum disagreement on a vote and 1, complete agreement. What do intermediate values represent? If two groups of different sizes both have a Rice Index of 0.4, can we say that they are equally united? In the absolute sense of equal proportions, they are equally united. However, if the purpose of measuring unity on a vote is to say something about the common purpose or the underlying cohesiveness of the group, then it is not clear what the Rice Index shows. We have all experienced the phenomena that it is easier to reach agreement in a small group than in a large one. Perhaps it would be reasonable to say that a moderately large Rice Index, say 0.6 indicates more cohesiveness for a large group than for a small group.

3.2 Establishing a Norm

In order to address this problem effectively, we must make precise the idea that it is more difficult for a large group to reach a high Rice Index than for a small group. This suggests that we adopt some assumption about the behavior of the individuals involved. Rice himself suggested that a reasonable neutral basis for comparison would be that each individual member of the group is equally likely to vote yes or no, independent of the others. He observed that under such an assumption an even split in the vote was the most likely outcome. This would be signified by a Rice Index near zero. Rice did not carry his analysis any further.

We will use Rice's idea to establish a norm for the way groups vote. We assume, as Rice did, that each voter is equally likely to vote yes or no, independent of the others. We then consider any deviation from the behavior indicated by this norm to be an indication of unity or disunity. This is not to say that we expect a small group to vote according to the norm, but rather to assert that other behavior indicates something beyond neutral random behavior, something which may, in fact, be associated with the formation of the group under study.

When we use this norm for behavior, we may calculate the probability under it that any particular value for the Rice Index is achieved by a group of a particular size. We may also calculate the average values of the Rice Index under this norm for groups of different sizes. When we do these calculations below, we shall see, in fact, that it is harder, in the sense of less likely, for large groups to achieve high values of the Rice Index.

3.3 The Probability Distribution

The probability distribution of the values of the Rice Index derives from the binomial distribution. Under the assumption that we have made for our norm, each voter is equally likely to vote yes or no independent of the others. This implies that the number of yes votes, Y, is binomially distributed with parameters n, the group size, and $p = 1/2$. The Rice Index is a simple function of Y:

$$RI = \frac{|2Y - n|}{n}$$

The calculation for the probabilities that RI takes on its various possible values is easy if we observe that each value of RI derives from two possible values for Y, each having the same probability, except in the case $RI = 0$. We illustrate these calculations with the following example.

Example 3: We calculate the probability distribution for the Rice Index for a group of four voters.

$$P[RI = 0] = P[2 \text{ yes}, 2 \text{ no}] = \binom{4}{2}\left(\frac{1}{2}\right)^4 = 3/8.$$

$$P[RI = 0.5] = P[3 \text{ yes}] + P[1 \text{ yes}] = 2\binom{4}{3}\left(\frac{1}{2}\right)^4 = 1/2.$$

$$P[RI = 1] = P[4 \text{ yes}] + P[0 \text{ yes}] = 2\binom{4}{0}\left(\frac{1}{2}\right)^4 = 1/8$$

The probability distributions for the Rice Index for groups of eight, sixteen, and twenty-four are shown in Table 2.

TABLE 2

Probability Distributions of the Rice Index

	n = 8		n = 16		n = 24	
RI	M	p	M	p	M	p
0.000	4	0.273	8	0.196	12	0.161
0.083					13	0.298
0.125			9	0.349		
0.167					14	0.234
0.250	5	0.438	10	0.244	15	0.156
0.333					16	0.088
0.375			11	0.133		
0.417					17	0.041
0.500	6	0.219	12	0.056	18	0.016
0.583					19	0.005
0.625			13	0.017		
0.667					20	0.001
0.750	7	0.063	14	0.004	21	0.000
0.833					22	0.000
0.875			15	0.000		
0.917					23	0.000
1.000	8	0.008	16	0.000	24	0.000

3.4 Comparison of Groups Differing in Size

The distributions of the values of the Rice Index, as for example those shown in Table 2, enable us to compare different size groups. In the following example we repeat some of these calculations.

Example 4: We calculate the probability that groups of eight and sixteen achieve RI values of 0.625 or greater.

For the group of eight, $RI \geq 0.625$ when the majority is 7 or 8:

$$P[RI_8 \geq 0.625] = P[7 \text{ yes}] + P[1 \text{ yes}] + P[8 \text{ yes}] + P[0 \text{ yes}]$$

$$= 2\binom{8}{7}\left(\frac{1}{2}\right)^8 + 2\binom{8}{8}\left(\frac{1}{2}\right)^8$$

$$= 0.071.$$

For the group of sixteen, $RI \geq 0.625$ for majorities of 13, 14, 15, or 16:

$$P[RI_{16} \geq 0.625] = P[13 \text{ yes}] + P[3 \text{ yes}] + P[14 \text{ yes}]$$

$$+ P[2 \text{ yes}] + P[15 \text{ yes}] + P[1 \text{ yes}]$$

$$+ P[16 \text{ yes}] + P[0 \text{ yes}]$$

$$= 2\binom{16}{13}\left(\frac{1}{2}\right)^{16} + 2\binom{16}{14}\left(\frac{1}{2}\right)^{16} + 2\binom{16}{15}\left(\frac{1}{2}\right)^{16}$$

$$+ 2\binom{16}{16}\left(\frac{1}{2}\right)^{16}$$

$$= 0.021.$$

We see that it is much less likely for the group of sixteen to achieve an RI value of 0.625 or higher than it is for a group of eight. In addition we note that this outcome is relatively unlikely for either group and it is most likely that the Rice Index will take a value less than one-half. Both of these characteristics of the Rice Index are confirmed if we refer to the distributions in Table 2.

These obeservations confirm two facts about the Rice Index. It does make sense to say that it is harder for large groups to achieve a high degree of unity if we mean by harder, less likely under the neutral behavior reflected by our norm. Consequently, it is difficult to interpret comparisons of the Rice Index for different size groups. In addition, for any group the values of the Rice Index will tend to be in the lowest part of the range. Although the possible values range from zero to one, it should not be surprising if most of our observations yield values less than 0.5. Thus the range of values is in this sense non-uniform.

3.5 Exercises

4. Calculate the probability distribution for the Rice Index for a group of $n = 5$.
5. Repeat Exercise 4 for $n = 7$.
6. Repeat Exercise 4 for $n = 10$.

4. EXPECTED VALUE

4.1 Calculation

One way to assess the tendencies discovered in the previous section is to calculate the expected value for the Rice Index under the norm which we are using. Since many assessments of unity will be made by using averages of a number of observations, the expected value should be a good benchmark. It should further illustrate the tendencies referred to above.

Example 5: We calculate the expected value of RI for a group of eight:

$$E(RI_8) = 0 \cdot P[RI_8 = 0] + 0.25 \cdot P[RI_8 = 0.25]$$

$$+ 0.5 \cdot P[RI_8 = 0.5] + 0.75 \cdot P[RI_8 = 0.75]$$

$$+ 1 \cdot P[RI_8 = 1]$$

$$= 0 + (0.25)(0.438) + (0.5)(0.219)$$

$$+ (0.75)(0.063) + 1(0.008)$$

$$= 0.273.$$

Here we have used the probabilities which we already calculated and are displayed in Table 2.

4.2 The General Form

The following calculation establishes the general formula for the expected value of the Rice Index. The possible values of RI_n will be $\frac{2k-n}{n}$, for k ranging through the integers from $[(n+1)/2]$ to n. $[\cdot]$ denotes the greatest integer function. Then we have

$$E(RI_n) = \frac{2[(n+1)/2] - n}{n} \cdot P[M = [(n+1)/2]]$$

$$+ \frac{2([(n+1)/2] + 1) - n}{n} \cdot P[M = [(n+1)/2] + 1]$$

$$+ \cdots + \frac{2n-n}{n} \cdot P[M = n],$$

where M is the number in the majority. In a more compact form

$$E(RI_n) = \sum_{k=(n+1)/2}^{n} \frac{2k-n}{n} \cdot P(M = k)$$

$$= \sum_{k=(n+1)/2}^{n} \frac{2k-n}{n} (P(k \text{ yes}) + P(n-k \text{ yes}))$$

$$= \sum_{k=0}^{n} \frac{|n-2k|}{n} P(k \text{ vote yes}).$$

The last expression can be written as follows using the binomial distribution for the yes votes:

$$E(RI_n) = \sum_{k=0}^{n} \frac{n-2k}{n} \binom{n}{k} \left(\frac{1}{2}\right)^n$$

(1)
$$= \frac{1}{2^{n-1}} \binom{n-1}{[(n-1)/2]}$$

The last step is a rather tricky calculation which we leave as a challenging problem.

This formula enables us to demonstrate that the expected value of the Rice Index gets smaller as the size of the group gets larger. Suppose, for example, that n is odd, $n = 2r+1$. Then

$$E(RI_{n+1}) = \frac{1}{2^n} \binom{n}{[n/2]}$$

$$= \frac{1}{2^{2r+1}} \binom{2r+1}{r}$$

$$= \frac{1}{2^{2r+1}} \frac{(2r+1)!}{r!(r+1)!}$$

$$= \frac{1}{2} \left(\frac{2r+1}{r+1}\right) \frac{1}{2^{2r}} \left(\frac{2r!}{r!r!}\right)$$

$$= \frac{2r+1}{2r+2} \frac{1}{2^{2r}} \binom{2r}{r}$$

$$= \left(\frac{n}{n+1}\right) \frac{1}{2^{n-1}} \binom{n-1}{[(n-1)/2]}$$

(2)
$$= \left(\frac{n}{n+1}\right) E(RI_n).$$

Thus the value of $E(RI)$ is slightly smaller for a group including one more voter. If n is even, a similar calculation will show that

(3) $$E(RI_{n+1}) = E(RI_n).$$

Thus, whenever we increase the size of the group by one, the expected value of the Rice Index either stays the same (even to odd) or decreases (odd to even).

4.3 An Alternate Form

We can derive a simpler formula for the expected value of the Rice Index from the one given above. First we observe that

$$E(RI_1) = 1,$$

since a group of one always votes as a majority of one. Then by formulas 2 and 3 we can successively derive the following:

$$E(RI_2) = \frac{1}{2} E(RI_1) = \frac{1}{2}$$

$$E(RI_3) = E(RI_2) = \frac{1}{2}$$

$$E(RI_4) = \frac{3}{4} E(RI_3) = \frac{3 \cdot 1}{4 \cdot 2}$$

$$E(RI_5) = E(RI_4) = \frac{3 \cdot 1}{4 \cdot 2}$$

$$E(RI_6) = \frac{5}{6} E(RI_5) = \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2}$$

The general form will be, for n even,

$$E(RI_{n+1}) = E(RI_n) = \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2} \quad (4)$$

This formula makes it easy to calculate values of E(RI) for different size groups. These values are given for n = 1 - 33 in Table 3.

TABLE 3

Values of E(RI) for n = 1, ..., 33

n	$E(RI_n)$	n	$E(RI_n)$
1	1.000		
2,3	0.500	18,19	0.186
4,5	0.375	20,21	0.176
6,7	0.313	22,23	0.168
8,9	0.273	24,25	0.161
10,11	0.246	26,27	0.155
12,13	0.226	28,29	0.149
14,15	0.210	30,31	0.145
16,17	0.196	32,33	0.140

The trend displayed in this table will continue: as the group size gets larger, the expected value of the Rice Index will approach zero.

This result indicates that the size of the group is a major determinant of the Rice Index. Consequently, comparisons between groups of different sizes which are intended to analyze factors other than group size on voting unity are difficult. The use of the Rice Index for such comparisons is questionable.

4.4 Exercises

7. Calculate the expected value of the Rice Index for a group of n = 5 *directly* using the distribution found in Exercise 4. Compare your answer to the result given by Equation (4).

8. Repeat Exercise 7 for n = 7.

9. Repeat Exercise 7 for n = 10.

10. Suppose a group of eight votes 5 - 3 twice, 6 - 2 once, and 8 - 0 once. What is the average (mean) of the Rice Index on these four votes? How does this compare to the expected value? Would you say that this group is relatively united compared to our norm?

5. THEORETICAL PROBLEMS

1. Suppose our objective were to measure agreement of a group with a particular stand on an issue by observing the number who voted for that view. If n is the number in the group and Y is the number who vote yes--in favor of the stand--we can take Y as a raw measure of agreement and Y/n as a proportional measure. What is the range of the proportion Y/n? How can we normalize Y/n to a measure which ranges from -1, indicating complete disagreement with our stand on the issue, to zero, indicating a neutral group, to +1, indicating complete agreement?

2. Show that for n even, $E(RI_{n+1}) = E(RI_n)$.

*3. Show that

$$\sum_{k=0}^{n} \frac{|n-2k|}{n} \binom{n}{k} \left(\frac{1}{2}\right)^n = \frac{1}{2^{n-1}} \binom{n-1}{[(n-1)/2]}.$$

6. BIBLIOGRAPHY

Born, Richard, and Christopher H. Nevison (1975) "A Probabilistic Analysis of Roll Call Cohesion Measures" Political Methodology, v. 2, pp. 131-149.

Casstevens, Thomas (1970) "Linear Algebra and Legislative Voting Behavior: Rice's Indices" Journal of Politics, v. 32, p. 775.

Rice, Stuart A. (1924), Farmers and Workers in American Politics, New York: Columbia University Studies in the Social Sciences #253.

_______________ (1928), Quantitative Methods in Politics, New York: Alfred A. Knopf.

Schubert, Glendon (1959), Quantitative Analysis of Judicial Behavior, Glencoe: Free Press.

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*This problem is essentially the same as Problem A-4 on the 1974 Putnam Examination. Solutions to that examination are in the November, 1975, Mathematical Monthly, p. 910.

MEASURES OF VOTING UNITY II:
THE PROBABILITY OF AGREEMENT MEASURE

1. INTRODUCTION

In Unit 271a we described the Rice Index of unity, a measure of how united a group is based on how it votes. Although the Rice Index is appropriate for simple analysis, we saw that there were difficulties with using it for a more detailed study. In particular, the Rice Index seems to depend on the size of the group. If we assume that each voter is equally likely to vote yes or no independent of the others as a neutral basis for comparison, then the expected value of the Rice Index decreases as the size of the group gets larger.

The fact that this major difficulty with the Rice Index is brought to light by the use of a norm based on probabilities suggests that we might use the ideas of probability theory to develop an alternate measure of unity. In this module we will describe the Probability of Agreement measure which has been discovered or rediscovered by several people (Rae and Taylor, 1970; Rieselback, 1960; Schubert, 1959; Brams and O'Leary, 1970).

2. THE PROBABILITY OF AGREEMENT

2.1 Definition

The basis for the Probability of Agreement measure is a probability calculation based on the results of voting by a small group. We calculate the *a posteriori* probability that two members of the group selected at random agreed on the vote. In other words, knowing how they voted, we calculate the probability that if we randomly select two voters, then they both voted yes or both voted no. We denote this measure of agreement by PA.

Because the PA measure is itself a probability, it will automatically fall on the interval 0 - 1. A value of 1 would indicate that no matter which two individual voters were selected, they were sure to have agreed, meaning the whole group must have been completely united. A low value of PA, which can never be exactly 0, would indicate that the chances of the two randomly selected individuals having agreed is low, meaning the group must have been divided on the vote. Thus we have a 0 - 1 scale for the PA measure which has a natural interpretation.

We can use the ideas of *conditional* probability to concisely describe the PA measure. First, we will let n denote the size of the group under consideration, k the number voting in the majority, and $PA_n(k)$ the resulting value for the Probability of Agreement. Since the measure is based on agreement between two randomly selected members of the group, we denote this event by AG_2. Recall that the notation

$$P[A|B]$$

means the probability of event A, given that event B occurs. Then we may define the Probability of Agreement by the conditional probability:

$$PA_n(k) = P[AG_2|M = k], \tag{1}$$

where M is the size of the majority. We may read this as follows: The Probability of Agreement for a group of size n with k in the majority is the probability that two randomly selected voters will agree, given that the size of the majority on the vote is k. In this formulation we regard the size of the majority, M, as a random variable which may take on different values, k, with different probabilities. Those probabilities may be based on an assumption about behavior such as that introduced in the first module: each voter is equally likely to vote yes or no independent of the others. This formulation of PA is conceptually helpful.

2.2 Calculation of PA

In order to calculate the Probability of Agreement measure, we must calculate the *a posteriori* probabilities which define it. Although the alternate definition of PA involves conditional probabilities based on assumptions about the way individuals vote, these assumptions play no role for these calculations. The calculation of $PA_n(k)$ can be rephrased as a standard probability problem: given a group of n things (the votes) divided into one group of k things (majority voters) and another group of n-k things (minority voters), what is the probability that when we choose two things randomly from the group, they are both in the same subgroup. Consequently,

$$(2) \qquad PA_n(k) = \frac{\binom{k}{2} + \binom{n-k}{2}}{\binom{n}{2}},$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is the number of combinations of b things selected from a things.

Example 1: We calculate the PA measure for the different possible votes in a group of eight:

$$PA_8(4) = \frac{\binom{4}{2} + \binom{4}{2}}{\binom{8}{2}} = \frac{12}{28} = 0.428$$

$$PA_8(5) = \frac{\binom{5}{2} + \binom{3}{2}}{\binom{8}{2}} = \frac{13}{28} = 0.464$$

$$PA_8(6) = \frac{\binom{6}{2} + \binom{2}{2}}{\binom{8}{2}} = \frac{16}{28} = 0.571$$

$$PA_8(7) = \frac{\binom{7}{2} + \binom{1}{2}}{\binom{8}{2}} = \frac{21}{28} = 0.750$$

$$PA_8(8) = \frac{\binom{8}{2}}{\binom{8}{2}} = 1.000$$

Table 1 shows the values of the Probability of Agreement for all possible votes for groups of eight, sixteen, and twenty-four.

TABLE 1

Probability of Agreement

	n = 8		n = 16		n = 24	
k/n	k,	PA	k,	PA	k,	PA
0.5	4	0.429	8	0.467	12	0.478
0.542					13	0.482
0.563			9	0.475		
0.583					14	0.493
0.625	5	0.464	10	0.500	15	0.511
0.667					16	0.536
0.688			11	0.542		
0.708					17	0.569
0.750	6	0.571	12	0.600	18	0.609
0.792					19	0.656
0.813			13	0.675		
0.833					20	0.710
0.875	7	0.750	14	0.767	21	0.772
0.917					22	0.841
0.938			15	0.875		
0.958					23	0.917
1.000	8	1.000	16	1.000	24	1.000

2.3 Group Size

Table 1 shows that the Probability of Agreement corresponding to a majority proportion of 0.75 varies from group to group. For a group of eight it is 0.571, for sixteen 0.600, and for twenty-four 0.609. Thus the PA measure is higher when the same majority proportion is achieved by a larger group. This accurately reflects the intuitive idea discussed in the previous module that it is harder for a large group to achieve the same high proportion in the majority.

We can use the same norm of behavior as we did for the Rice Index, to see whether this property works generally. The probability distribution for the different possible majority sizes will be the same as those calculated for Table 2 of Unit 271a and can be used to calculate the expected value of PA under this norm.

We can, however, derive a general expression for the expected value of the PA measure which is very revealing. Recall that the possible majority sizes for a group of n voters will range from [(n+1)/2] to n, so we may represent this expected value as follows:

$$E(PA_n) = \sum_{k=[(n+1)/2]}^{n} PA_n(k)\ P(M=k)$$

$$= \sum_{k=[(n+1)/2]}^{n} P[AG_2|M=k]\ P[M=k]\ .$$

The second expression uses our alternate formulation of the PA measure in terms of the conditional probability. But we observe that in this expression the summation runs over all possible values of M, so that we have the equivalent expression:

$$E(PA_n) = \sum_{k=[(n+1)/2]}^{n} P[AG_2 \text{ and } M=k] \qquad (3)$$

$$= P[AG_2]\ .$$

The last is the *a priori* probability, under our assumption about the behavior of individuals, that two randomly selected voters will agree on the vote. But using our norm this probability can be calculated directly. Since each voter is equally likely to vote yes or no, independent of the others, we have

$$P(AG_2) = P(\text{both vote yes}) + P(\text{both vote no})$$

$$= (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2$$

$$= \tfrac{1}{2}\ .$$

Under the neutral behavior which we assume for comparison, the expected value of the Probability of Agreement is *always* 0.5. Consequently, the size of the group is not a significant factor in the determination of PA, so PA may reasonably be used to compare the unity of different size groups.

In addition, the value 0.5 which is the midpoint of the range of the PA measure is also the natural result. Groups with a PA greater than 0.5 can be regarded as relatively united and groups with a PA less than 0.5 as disunited.

2.4 Exercises

1. Calculate the Probability of Agreement measure for all the majority size for a group with n = 5 voters.
2. Repeat exercise 1 for n = 7.
3. Repeat exercise 1 for n = 10.
4. Use the probability distributions calculated for the exercises in the previous module combined with the results of exercises 1, 2, and 3 to directly calculate the expected value of the Probability of Agreement for n = 5, 7, and 10.

3. OTHER PROBLEMS

3.1 The Range

Although the Probability of Agreement measure has solved one difficulty raised by the Rice Index, it has a different problem. Careful examination of the range of values of PA shown in Table 1 shows that this measure never has values near zero. Although the theoretical range of PA is zero to one, because it is defined as a probability, in practice its lowest values are never far below 0.5. Consequently, the use of the PA measure will make distinctions between groups with values in this range difficult.

3.2 A Normalization

We can apply a normalization method in order to correct the problem raised above. This approach was developed by Brams and O'Leary (1970). We use a method of normalization just like the one which we used to derive the Rice Index. However, since the actual range of the PA index varies with group size, we do the normalization for each size of group separately. We use the definition of the *Agreement Level* Index, AL, given by Brams and O'Leary:

$$(4) \qquad AL_n(k) = \frac{PA_n(k) - \min(PA_n)}{\max(PA_n) - \min(PA_n)}$$

In the numerator we subtract the smallest possible value of the PA measure for a group of size n (step 1) and then we divide by the length of the range of the PA measure (step 2). Consequently, the AL measure will range from zero to one and both extreme values are possible. Observe that $\max(PA_n) = 1$ for any n and $\min(PA_n) = PA_n[(n+1)/2]$. We may write

$$(5) \qquad AL_n(k) = \frac{PA_n(k) - PA_n([(n+1)/2])}{1 - PA_n([(n+1)/2])}\ .$$

Unfortunately, this normalization procedure destroys the simple interpretation of the Probability of Agreement measure.

3.3 Calculation of AL

In order to calculate AL for a group of size n, we must first calculate the smallest possible value of PA for that size group, then calculate the actual PA for the group and use formula (5).

Example 2: We calculate all AL_8 values for a group of 8 voters

$$\min(PA_8) = PA_8(4) = 0.428$$

from Example 1, so the AL formula is

$$AL_8(k) = \frac{PA_8(k) - 0.428}{1 - 0.428} = \frac{PA_8(k) - 0.428}{0.572}\ .$$

Using the values for PA_8 calculated in Example 1 we get the following results:

$$AL_8(4) = 0.000$$

$$AL_8(5) = 0.063$$

$$AL_8(6) = 0.250$$

$$AL_8(7) = 0.563$$

$$AL_8(8) = 1.000$$

The values of AL for groups of eight, sixteen, and twenty-four are given in Table 2.

TABLE 2

Agreement Level

	n = 8		n = 16		n = 24	
k/n	k,	AL	k,	AL	k,	AL
0.5	4	0.000	8	0.000	12	0.000
0.542					13	0.007
0.563			9	0.016		
0.583					14	0.028
0.625	5	0.063	10	0.063	15	0.063
0.667					16	0.111
0.688			11	0.141		
0.708					17	0.174
0.750	6	0.250	12	0.250	18	0.250
0.792					19	0.340
0.813			13	0.391		
0.833					20	0.444
0.875	7	0.563	14	0.563	21	0.563
0.917					22	0.694
0.938			15	0.766		
0.958					23	0.840
1.000	8	1.000	16	1.000	24	1.000

3.4 Nothing New

Table 2 shows that different size groups with the same majority proportion have the same agreement level measure. This property also holds for the Rice Index (since it is defined in terms of the majority proportion) and suggests that there is a systematic relationship between AL and RI. Careful examination of Table 2 and Table 1 from Unit 271a on the Rice Index reveals that for the case when the size of the group is even:

$$AL_n(k) = \frac{PA_n(k) - PA_n(n/2)}{1 - PA_n(n/2)}, \quad \text{by (5)}$$

$$= \frac{\{\binom{k}{2} + \binom{n-k}{2}\}/\binom{n}{2} - 2\binom{n/2}{2}/\binom{n}{2}}{1 - 2\binom{n/2}{2}/\binom{n}{2}}, \quad \text{by (2)}$$

$$= \frac{k(k-1) + (n-k)(n-k-1) - 2(n/2)(n/2-1)}{n(n-1) - 2(n/2)(n/2-1)}$$

$$= \frac{(k-(n-k))^2}{n^2}$$

(6) $$AL_n(k) = RI_n(k)^2.$$

When n is odd it can be shown that the same relation holds, except that there is a small error of the order $1/n^2$. This calculation is left as a problem.

The Agreement Level measure has no intrinsic meaning of its own and it is difficult to compute directly. Since it is essentially the same as the Rice Index, there is no point in using it as an index of unity. This is a good example of a natural effort to develop a measure which simply does not work out.

In the third module in this series, Unit 271c, we will develop another measure of unity which avoids the major problems of both the Rice Index and the Probability of Agreement and also has other desirable properties.

3.5 Exercises

5. Calculate the Agreement Level index for all majority sizes for a group of $n = 5$. How do these values compare with the values of the Rice Index computed in the exercises of the previous module?

6. Repeat exercise 5 for $n = 10$.

4. MORE THAN TWO ALTERNATIVES

4.1 A New Problem

In some situations, a political scientist may want to measure the unity of a group which has more than two alternatives on a vote. The United Nations often has votes where abstention is an important distinct alternative. In that case there are three alternatives: Yes, no abstain. Another situation which would have more than two alternatives is an election with more than two candidates.

The Probability of Agreement is a natural measure of unity in a situation with more than two alternatives. It has the same definition and natural meaning as for the two alternative case and it still has the desirable property that group size does not affect the expected value (which may not be 0.5, depending on what norm for behavior is used).

The Agreement Level Index was originally developed by Brams and O'Leary for a many alternative situation. In this case the close relation to the Rice Index is not as evident. However, the AL index again turns out to be essentially the same as a simpler measure, the extended Rice Index squared.

The Rice Index squared can be written as

(7) $$RI^2 = \frac{(M-m)^2}{n^2},$$

where M and m are the number in the majority and minority for a group of n voters. The advantage of squaring is that the formula may be expressed in terms of yes votes, Y, and no votes, N, without using absolute values:

(8) $$RI^2 = \frac{(Y-N)^2}{n^2}$$

If there are t voting options, we let m_1, m_2, ..., m_t be the number who vote for each of the alternatives. Then the natural extension of (8) is the average of the squared pairwise differences of the fractions, m_i. Consequently, we adopt the definitions:

$$RI^2(m_1, m_2, \ldots m_t) = \frac{1}{t-1} \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} \frac{(m_j - m_i)^2}{n^2}$$

Thomas Casstevens [1970] has also suggested this extension of the Rice Index. He called the square root of this expression his "general index of cohesion". The idea of this measure is that it is the average of the pairwise squared differences of the fractions. We divide by t - 1 rather than t for the average to compensate for the fact that when we know all but one of the m_i, the value of the last one is determined. In statistics we would say we have t - 1 degrees of freedom. The essential fact for us is that t - 1 is the correct factor to scale the measure to range from 0, for an even split, to 1 for a unanimous vote.

4.2 Calculation of PA and RI^2

We shall calculate the values of these two measures for a group of six voting on three options.

Example 3: Calculation of PA for a group of six with three options. The possible ways the group can vote are 6-0-0, 5-1-0, 4-2-0, 4-1-1, 3-3-0, 3-2-1, and 2-2-2.

$$PA_6(6,0,0) = \frac{\binom{6}{2}+\binom{0}{2}+\binom{0}{2}}{\binom{6}{2}} = 1$$

$$PA_6(5,1,0) = \frac{\binom{5}{2}+\binom{1}{2}+\binom{0}{2}}{\binom{6}{2}} = \frac{10}{15} = 0.667$$

$$PA_6(4,2,0) = \frac{\binom{4}{2}+\binom{2}{2}+\binom{0}{2}}{\binom{6}{2}} = \frac{7}{15} = 0.467$$

$$PA_6(4,1,1) = \frac{\binom{4}{2}+\binom{1}{2}+\binom{1}{2}}{\binom{6}{2}} = \frac{6}{15} = 0.400$$

$$PA_6(3,3,0) = \frac{\binom{3}{2}+\binom{3}{2}+\binom{0}{2}}{\binom{6}{2}} = \frac{6}{15} = 0.400$$

$$PA_6(3,2,1) = \frac{\binom{3}{2}+\binom{2}{2}+\binom{1}{2}}{\binom{6}{2}} = \frac{4}{15} = 0.267$$

$$PA_6(2,2,2) = \frac{\binom{2}{2}+\binom{2}{2}+\binom{2}{2}}{\binom{6}{2}} = \frac{3}{15} = 0.200$$

Example 4: We calculate the values of RI^2 under the same circumstances:

$$RI_6^2(6,0,0) = \frac{1}{2}\,\frac{(6-0)^2+(6-0)^2+(0-0)^2}{6^2} = 1.000$$

$$RI_6^2(5,1,0) = \frac{1}{2}\,\frac{(5-1)^2+(5-0)^2+(1-0)^2}{6^2} = 0.583$$

$$RI_6^2(4,2,0) = \frac{1}{2}\,\frac{(4-0)^2+(4-2)^2+(2-0)^2}{6^2} = 0.333$$

$$RI_6^2(4,1,1) = \frac{1}{2}\,\frac{(4-1)^2+(4-1)^2+(1-1)^2}{6^2} = 0.250$$

$$RI_6^2(3,3,0) = \frac{1}{2}\,\frac{(3-3)^2+(3-0)^2+(3-0)^2}{6^2} = 0.250$$

$$RI_6^2(3,2,1) = \frac{1}{2}\,\frac{(3-2)^2+(3-1)^2+(2-1)^2}{6^2} = 0.083$$

$$RI_6^2(2,2,2) = \frac{1}{2}\,\frac{(2-2)^2+(2-2)^2+(2-2)^2}{6^2} = 0.000$$

Each of these measures seems to work reasonably. The Rice Index squared will have the same dependence on group size as the original Rice Index whereas the Probability of Agreement will avoid this problem. The PA measure, on the other hand, does not really range from zero to one and the Rice Index squared does. Which measure is appropriate for a particular problem will depend on how the investigator balances these characteristics.

The third module in this series, Unit 271c, will develop another measure of unity which seems to combine the best features of the PA and RI indices.

4.3 Exercises

7. Calculate the Probability of Agreement for all possible ways in which a group of n = 7 can vote on 3 alternatives.

8. Calculate the Rice Index squared for all possible ways in which a group of n = 7 can vote on 3 alternatives.

5. THEORETICAL PROBLEMS

1. Show that if we assume that each voter votes yes with a probability p and no with the probability q = 1 - p, independent of the other voters, then

the expected value of the Probability of Agreement will be $p^2 + q^2$ independent of group size.

Prove that for n odd,

$$AL_n(k) = (RI_n(k))^2 + \frac{(RI_n(k))^2 - 1}{n^2 - 1}$$

3. (a) Give the definition of the Agreement Level measure for many voting options.

   (b) If there are t options with factions of $m_1, m_2, \ldots, m_t$, then show that

$$AL_n(m_1,m_2,\ldots,m_t) = RI_n^2(m_1,m_2,\ldots,m_t) + \varepsilon$$

   where $\varepsilon = (RI_n^2 - 1/\{n^2(\frac{t-1}{(t-r)r}) - 1\}$

   where r is the remainder of n/t and $\varepsilon = 0$ if $r = 0$.

4. Show that the minimum value for PA_n approaches 0.5 as the group size increases.

5. What is the minimum value of PA_n when there are t voting options? Does is approach a limit as n increases?

6. BIBLIOGRAPHY

Brams, Steven, and Michael O'Leary (1974), "An Axiomatic Model of Voting Bodies", American Political Science Review, v. 64.

Born, Richard, and Christopher Nevison (1974), "The 'Agreement Level' Measure, and the Rice Index of Cohesion Revisited", American Journal of Political Science, v. 18, 617-624.

________ (1975), "A Probabilistic Analysis of Roll Call Cohesion Measures", Political Methodology, v. 2, 131-149.

MEASURES OF VOTING UNITY III: THE α-INDEX

1. INTRODUCTION

In Units 271a and b we discussed the properties of two measures of voting unity, the *Rice Index* and the *Probability of Agreement.* Each of these measures has useful properties as well as disadvantages. The major disadvantage with the Rice Index is that it is dependent on the size of a group, so comparisons with the Rice Index of different size groups are questionable. The Probability of Agreement, on the other hand, has an expected value which is invariant with regard to group size. However, it does not really range from zero to one, so that comparisons of values of the Probability of Agreement near 0.5 are difficult. Efforts to normalize the Probability of Agreement lead to what we call the *Agreement Level* index which turns out to be, essentially, the square of the Rice Index.

In this module we introduce the *α-index* of voting unity which shares the better properties of the Rice Index and the Probability of Agreement and avoids other problems which we discuss in section 2. The disadvantages of the α-index are that it is harder to compute and it does not easily extend to many alternative voting situations.

2. THE CONTEXT OF A VOTE

2.1 External Circumstances

In our discussions of the Rice Index and the Probability of Agreement, following Rice, we established as a norm of behavior to serve as a neutral basis for comparison the following assumption: each individual

is equally likely to vote yes or no independent of the others. Under this norm we found that the expected value of the Rice Index is closely tied to the size of the group, whereas the expected value of the Probability of Agreement is 0.5, regardless of group size.

For some kinds of analysis the norm is not appropriate. Suppose we are studying the unity of small groups relative to a large context. For example we might ask how united party state delegations to Congress are, *relative to Congress as a whole.* If a delegation of 10 votes 7-3, when the Congress as a whole has voted 205-230 on one issue, but on another the delegation votes 9-1 when Congress votes 405-30, how are we to compare the unity of the group on the two issues? The measures of unity which we have studied so far would show that the group was more united on the second issue than on the first. However, Congress as a whole was very united on this issue as well, so there is some question as to which group is relatively more united.

Another circumstance where a relative degree of unity might be appropriate is when the external circumstance is a measure of public opinion or opinions of the appropriate large group, such as all Republicians, by techniques such as opinion polls. We might desire to measure the relative unity of a small group, such as a delegation to a party convention, in a context like this.

Neither the Rice Index nor the Probability of Agreement can be used directly to measure relative unity. We might be able to use statistical tools with these measures to assess the degree of unity of a small group due to the external circumstances and the residual unity which must be ascribed to the group itself. However, this technique can be quite difficult. Instead, we will introduce a direct measure of relative unity which we call the α-index.

2.2 The Expected Value of RI and PA

We can model the situation where we measure unity relative to the behavior of a larger group by using the behavior of the larger group as the norm for the behavior of the small group under study. We can then evaluate how the behavior of the large group affects a measure of unity by calculating the expected value of the measure under the norm.

There are two ways in which we can establish the norm based on external circumstances. If we have data, such as opinion polls, which indicate the probabilities that an individual voter will vote yes, p, or no, $q = 1 - p$, then we assume for the norm that each individual in the group votes with these probabilities independent of the others. This norm assumes that the group is randomly made of individuals from the observed population forming the context for the study.

A second method of establishing a norm could be used when the exact way the overall population has voted is known. It could be used, for example, to establish a norm for a group voting in Congress when the overall vote of Congress is known. In this case, rather than assume each individual in the small groups votes independently, we assume instead that the small group is randomly selected from the larger population which consists of a faction who voted yes and a faction who voted no. The probabilities for the possible majority sizes in the small group is based on this procedure of drawing without replacement from the larger population.

For our examples we will use the first method of establishing a norm.

Example 1: We calculate the expected value of the Rice Index for a group of eight for two norms. The first norm will assume a probability of a yes vote is $p = 0.6$ and the second $p = 0.7$. The possible values for RI are

$$RI_8(4) = 0,\ RI_8(5) = 0.25,\ RI_8(6) = 0.5,$$
$$RI_8(7) = 0.75,\ RI_8(8) = 1.0.$$

With either norm

$$P(RI_8 = 0) = P(4 \text{ yes}) = \binom{8}{4}P^4(1-P)^4,$$

$$P(RI_8 = 0.25) = P(5 \text{ yes}) + P(3 \text{ yes}) = \binom{8}{5}P^5(1-P)^3 + \binom{8}{3}P^3(1-P)^5,$$

$$P(RI_8 = 0.5) = P(6 \text{ yes}) + P(2 \text{ yes}) = \binom{8}{6}P^6(1-P)^2 + \binom{8}{2}P^2(1-P)^6,$$

$$P(RI_8 = 0.75) = P(7 \text{ yes}) + P(1 \text{ yes}) = \binom{8}{7}P^7(1-P) + \binom{8}{1}P(1-P)^7,$$

$$P(RI_8 = 1.0) = P(8 \text{ yes}) + P(0 \text{ yes}) = \binom{8}{8}P^8 + \binom{8}{0}(1-P)^8.$$

Table 1 gives the distribution for the two norms.

Table 1

RI_8	Probability, P = 0.6	Probability, P = 0.7
0	0.232	0.136
0.25	0.403	0.301
0.5	0.250	0.306
0.75	0.097	0.199
1.0	0.017	0.058

The expected value when P = 0.6 is $E(RI_8) = 0.316$ and when P = 0.7, $E(RI_8) = 0.436$. In Unit 271a in this series we calculated the expected value for P = 0.5, $E(RI_8) = 0.273$. We see that the expected value of the Rice Index is quite dependent on the external circumstances which are assumed to hold.

We can calculate the expected value for the Probability of Agreement independent of group size. The same argument which was used in the previous module, applies so that

$$E(PA) = P(AG_2),$$

where AG_2 denotes the event that two randomly selected voters in the group agree, and the probability is calculated according to the norm assumed. Again assuming that individual voters vote yes with probability p and no with probability q, independent of other voters, we get

$$E(PA) = P(\text{both yes}) + (P(\text{both no})$$

$$= p^2 + q^2.$$

When p = 0.6, $E(PA) = 0.6^2 + 0.4^2 = 0.52$ and when p = 0.7, $E(PA) \cdot 0.7^2 + 0.3^2 = 0.58$. Again we see that the probability of agreement measure depends on the external norm which is assumed to hold.

3. THE α-INDEX

3.1 Motivation

When we developed the Probability of Agreement measure we used the *a posteriori* probability of agreement as a means of making the expected value independent of the group size. Because the *a posteriori* probability of the agreement of two individuals in the group has no connection with the context in which the vote was taken, it could not reflect the external circumstances of the vote. Consequently, since there can be no built in compensation, the expected value of PA must change as the assumptions about external circumstances change.

In order to include some compensation for external circumstances in a measure based on a probability, that probability must be *a priori*. That is, it must be a probability based on the assumptions made about external circumstances. The α-index is just such a measure.

3.2 Definition

The basic idea of the α-index is to ask the question: Under whatever assumptions are made about the behavior of the group, what is the probability that the group voted with the observed majority, k, or a *smaller* majority. This probability will be 1.0 for the case of unanimity, since the group is sure to vote with a majority less than or equal to the size of the group. This probability will be near zero when the group is evenly split, since even for modest size groups under the Rice norm, the probability of achieving exactly an even split is fairly small.

The probability described above is simply the (cumulative) distribution function for the random variable, M, majority size, under whatever assumptions are made about behavior:

$$F_M(k) = P(M \leq k),$$

where the probability is calculated according to the assumed norm.

We make a small adjustment in the distribution function to establish the α-index. The need for this adjustment is evident if we reverse the range of the measure, using zero to indicate complete unanimity and one to indicate disunity. We can do this naturally in two ways. First, we take our measure of unity and subtract it from one. For the distribution function this yields $G(k) = 1 - F(k) = P(M > k)$. We get a value of exactly zero for unanimity, and a value near one for an even division.

The second way to approach the reversed measure is to measure the probability that the majority size M is at least equal to the observed value k, $P(M \geq k)$. This will give exactly one when the group is evenly split and nearly zero for unanimity. The two approaches give different results because the distribution function lacks symmetry. We overcome this by splitting the difference for M = k and we make this definition:

The α-index of unity for a group of n individuals with k in the majority is

$$\alpha_n(k) = P(M < k) + 1/2P(M = k),$$

where M is the random variable of the majority size and the probabilities are calculated under the appropriate norm for behavior.

The measure so defined can capture any external circumstances which can be reasonably expressed in terms of probabilities of behavior, and yields a measure of unity *relative* to those external circumstances. If no external circumstances are assumed, the neutral assumption that each voter is equally likely to vote yes or no independent of the others can be used. The values of the α-index will range from zero to one.

3.3 Calculation of α

In order to calculate the α-index we must calculate the probabilities of the different majority sizes under the appropriate norm. We shall do this for three norms. We assume that voters vote independently with the

probability of a yes vote of P, for P = 0.5, 0.6, and 0.7, for the three norms.

Example 2: We calculate the α-index for the norms given above for a group of n = 8. We have already calculated the probabilities of the different majority sizes, in example 1 for P = 0.6 and P = 0.7 and in Unit 271a for P = 0.5. These distributions are listed in Table 2

TABLE 2

Distribution of Majority size, M, When n = 8

M	P = 0.5 probability	P = 0.6 probability	P = 0.7 probability
4	0.273	0.232	0.136
5	0.438	0.403	0.301
6	0.219	0.250	0.306
7	0.063	0.097	0.199
8	0.008	0.017	0.058

In order to calculate $\alpha_n(k)$, we simply sum the values for the correct column for values of $M < k$ and add half the value for M = k. For example, when P = 0.5,

$$\alpha_8(4) = 1/2\ (0.273) = 0.137$$

$$\alpha_8(5) = 0.273 + 1/2\ (0.438) = 0.492$$

$$\alpha_8(6) = 0.273 + 0.438 + 1/2\ (0.219) = 0.821$$

Table 3 shows values of the α-index for a group of eight under each of these norms.

TABLE 3

Values of the α-index for a group of 8

k	P = 0.5 $\alpha_8(k)$	P = 0.6 $\alpha_8(k)$	P = 0.7 $\alpha_8(k)$
4	0.137	0.116	0.068
5	0.492	0.434	0.287
6	0.821	0.760	0.590
7	0.962	0.934	0.843
8	0.997	0.991	0.971

Tables 4 and 5 show the values of the α-index for groups of eight, sixteen, and twenty-four. Table 4 uses the norm P = 0.5 and Table 5, P = 0.7.

TABLE 4

The α-index, P = 0.5 = q

k/n	k	α_8	k	α_{16}	k	α_{24}
0.50	4	0.137	8	0.098	12	0.081
0.54					13	0.310
0.56			9	0.371		
0.58					14	0.576
0.63	5	0.492	10	0.668	15	0.770
0.67					16	0.892
0.69			11	0.857		
0.71					17	0.957
0.75	6	0.820	12	0.951	18	0.985
0.79					19	0.996
0.81			13	0.987		
0.83					20	0.999
0.88	7	0.961	14	0.998	21	1.000
0.92					22	1.000
0.94			15	1.000		
0.96					23	1.000
1.00	8	0.996	16	1.000	24	1.000

TABLE 5

The α-index, p = 0.7, q = 0.3

k/n	k	α_8	k	α_{16}	k	α_{24}
0.50	4	0.068	8	0.024	12	0.010
0.54					13	0.045
0.56			9	0.108		
0.58					14	0.111
0.63	5	0.287	10	0.253	15	0.213
0.67					16	0.355
0.69			11	0.444		
0.71					17	0.523
0.75	6	0.590	12	0.652	18	0.691
0.79					19	0.830
0.81			13	0.827		
0.83					20	0.923
0.88	7	0.843	14	0.937	21	0.973
0.92					22	0.993
0.94			15	0.985		
0.96					23	0.999
1.00	8	0.971	16	0.998	24	1.000

3.4 Exercises

1. Calculate the values of the α-index for all possible majorities for a group of n = 5 under the neutral norm, p = 0.5.
2. Repeat exercise 1 for n = 7.
3. Repeat exercise 1 for n = 10.
4. Repeat exercise 1 for the norm p = 0.7.
5. Repeat exercise 1 with n = 7, p = 0.7.
6. Repeat exercise 1 with n = 10, p = 0.7.
7. Repeat exercise 1 with n = 5, p = 1.0. What is the significance of this result?

4. PROPERTIES OF THE α-INDEX

4.1 Group Size

Table 4 shows that the α-index credits larger groups with a higher degree of unity for the same majority proportion. For example, for a majority proportion of 0.75, a group of eight receives an α of 0.820, a group of sixteen has α = 0.951, and a group of twenty-four has α = 0.985. This confirms our sense that it is more difficult for a large group to achieve the same high majority proportion. The α-index shares this desirable property with the PA measure.

In addition, however, we see from Tables 4 and 5 that the α-index extends over the full range of the interval 0-1. The lowest values are close to zero and the highest close to one. This is an improvement over the behavior of the PA measure.

4.2 Expected Value

The invariance of the α-index with respect to group size is confirmed by calculating its expected value. Indeed, we will find that the α-index has the same expected value, no matter what norm for behavior is used (as long as that norm is used for the calculation of α), and that value is 0.5.

We can write the α-index for a group of n with k in the majority as follows:

$$\alpha_n(k) = P(M < k) + 1/2\, P(M = k)$$

$$= \sum_{i=1}^{k-1} P_i + 1/2\, P_k,$$

where $P_i = P(M = i)$. But the expected value of α can be written

$$E(\alpha_n) = \sum_{k=1}^{n} \alpha_n(k)\, P_k$$

$$= \sum_{k=1}^{n} P_k\Big(\sum_{i=1}^{k-1} P_i + 1/2\, P_k\Big).$$

Consequently,

$$2E[\alpha_n] = 2\sum_{k=1}^{n} p_k\Big[\sum_{i=1}^{k-1} p_i + 1/2\, p_k\Big]$$

$$= \sum_{k=1}^{n} p_k\Big[\sum_{i=1}^{k-1} p_i + 1/2\, p_k\Big] + \sum_{i=1}^{n} p_i\Big[\sum_{k=1}^{i-1} p_k + 1/2\, p_i\Big] \tag{1}$$

$$= \sum_{k=1}^{n} p_k\Big[\sum_{i=1}^{k-1} p_i + 1/2\, p_k\Big] + \sum_{i=1}^{n} p_i\Big[\sum_{k=1}^{i-1} p_k\Big] + \sum_{k=1}^{n} 1/2\, p_k^2 \tag{2}$$

$$= \sum_{k=1}^{n} p_k\Big[\sum_{i=1}^{k-1} p_i + 1/2\, p_k\Big] + \sum_{k=1}^{n} p_k\Big[\sum_{i=k+1}^{n} p_i + 1/2\, p_k\Big] \tag{3}$$

$$= \sum_{k=1}^{n} p_k\Big[\sum_{i=1}^{n} p_i\Big]$$

$$= 1, \text{ since } \sum_{i=1}^{n} p_i = 1.$$

Therefore $E[\alpha_n] = 1/2$.

In the above sequence, (1) is simply a relabeling of subscripts, (2) a separation of the second summation into components, and (3) a change of the order of summation in the second sum. The fact demonstrated above is well known in probability theory.

We may conclude that for any circumstances reflected by the norm under which α is calculated, an α value of 0.5 for a group represents the degree of unity which would be expected for the group, higher values of α indicate a relatively united group, lower values of α indicate a relatively disunited group. Consequently, with α as a measure of unity it is reasonable to compare the unity for groups of different sizes and the relative unity of groups voting under different external circumstances. No other measure of voting unity has this property.

The α-index has the disadvantage, compared to the Rice Index or the Probability of Agreement, that it is somewhat more difficult to calculate. However the use of computers for data analysis ameliorates this objection somewhat.

A second disadvantage of the α-index is that there is no obvious way to extend it to a situation where there are more than two alternatives. In the previous module, Unit 271b, we saw how this could be done for either the Rice Index or the Probability of Agreement.

4.3 Exercises

8. Calculate directly the expected value of the α-index for a group of $n = 5$ with $p = 0.5$, using the results of exercise 1 and the probability distribution calculated in Unit 271a.

5. AN APPLICATION OF THE α-INDEX

As an example of how the α-index might be used, we shall make a comparison of the unity of state party delegations to Congress for a particular vote but under different external circumstances. This will enable us to draw conclusions about factors which influence the voting behavior of the members. The example we shall consider is taken from Born and Nevison (1975). The data is given in Table 6. It records the votes of fifteen eastern and mid-western Republican state delegations on the Teague Amendment to the Agricultural Act of 1973 (July 19, 1973). The unsuccessful amendment, which would have removed from the bill "escalator clause" provisions adjusting price support payments to farm production costs for wheat, feed grains, and cotton, was strongly favored by the Nixon administration as anti-inflationary.

The first column of the table records the vote of each delegation on the amendment. The second column records the α index for each delegation, calculated using the vote of the Republican Party as a whole for a norm. The third column records the α index for each delegation, calculated using the respective regional Republican totals as norms.

When the whole party vote is used as a norm, nearly all the states exceed the 0.5 level attributable to chance, with the exceptions being Minnesota, Nebraska, and Wisconsin. This would suggest that these states are more united than might be expected from the party as a whole.

However, when the regional party totals are used, the eastern states show levels of unity close to chance. This suggests that the factors promoting unity in the eastern states are regional rather than state factors. Indeed, since the east as a whole contained virtually

TABLE 6

Republican State Delegation α Values

On Teague Amendment to the Agriculture Act of 1973

State Party Delegation	Delegation Voting Split		α Values (Calculated Using Over-all Republican Totals)[a]		α Values (Calculated Using Respective Regional Party Totals)[b]
Midwest					
Illinois	14Y, 0N		0.993		0.998
Indiana	6Y, 1N		0.715		0.796
Iowa	0Y, 3N		0.783		0.816
Kansas	0Y, 4N		0.842		0.878
Michigan	11Y, 0N		0.981		0.993
Minnesota	1Y, 3N		0.448		0.509
Nebraska	1Y, 2N		0.283		0.316
Ohio	13Y, 2N		0.852		0.934
Wisconsin	2Y, 2N		0.106		0.132
		Ave.	0.667	Ave.	0.708
East					
Connecticut	3Y, 0N		0.783		0.531
Maryland	3Y, 0N		0.783		0.531
Massachusetts	3Y, 0N		0.783		0.531
New Jersey	7Y, 0N		0.936		0.573
New York	17Y, 0N		0.997		0.677
Pennsylvania	10Y, 1N		0.887		0.115
		Ave.	0.862	Ave.	0.493

[a] Overall Republican totals: 135Y, 45N

[b] Regional totals: Midwestern Republicans -- 48Y, 20N
Eastern Republicans -- 47Y, 1N

no districts directly affected by the amendment, an almost unanimous 47-1 majority felt free to support their Republican president.

In contrast, all the midwestern states increase their measure of unity when the regional norm is used. This suggests that the factors causing this higher-than-chance level of unity are not regional, but perhaps state, sub-regional, or cross-state factors.

This example demonstrates how the α-index may be used to analyze the factors which influence voting on certain issues.

6. THEORETICAL PROBLEMS

1. Suggest appropriate criteria for calling a group more or less united in the t-option voting situation, $t > 2$. How well do the extended Rice Index squared or Probability of Agreement conform to these criteria?

2. Suggest an extension of the α-index to measure unity in the case of t-options, $t > 2$. Assess your measure according to the criteria established in problem 1. Is the expected value of your measure invariant to group size or other factors? How does your measure compare to the extended Rice Index squared or the Probability of Agreement?

7. BIBLIOGRAPHY

Richard Born and Christopher Nevison (1975), "A Probabilistic Analysis of Roll Call Cohesion Measures", Political Methodology, v. 2, 131-149.

SOLUTIONS TO EXERCISES AND THEORETICAL PROBLEMS

1. Unit 271a, *The Rice Index*

1.1 Solutions to Exercises

The solutions to Exercises 1 - 6 are collected in the following tables.

Exercises 1, 4: $n = 5$

k	$RI_n(k)$	$P(M=K)$
3	0.2	0.625
4	0.6	0.313
5	1.0	0.063

Exercises 2, 5: $n = 7$

k	$RI_n(k)$	$P(M=K)$
4	0.143	0.547
5	0.428	0.328
6	0.714	0.109
7	1.000	0.016

Exercises 3, 6: $n = 10$

k	$RI_n(k)$	$P(M=K)$
5	0.0	0.244
6	0.2	0.401
7	0.4	0.234
8	0.6	0.088
9	0.8	0.019
10	1.0	0.002

Exercise 7: $E(RI_5) = 0.375$

Exercise 8: $E(RI_7) = 0.273$

Exercise 9: $E(RI_{10}) = 0.246$

Exercise 10: $\text{Average} = \frac{2\cdot(0.25) + 0.5 + 1.0}{4} = 0.5$

$E(RI_8) = 0.273$. Consequently, the group appears to be relatively united.

1.2 Solutions to Problems

1. y/n has a range from zero, when all vote no, to one, when all vote yes. In order to normalize this to range from -1 to 1, we first multiply by 2 in order to make the range 2 units long, then we subtract 1, in order to translate the lowest value to -1. The normalized measure would be

$$Z = 2y/n - 1.$$

2. Suppose $n = 2r$, even. Then

$$E(RI_{n+1}) = \frac{1}{2^n}\binom{n}{n/2}$$

$$= \frac{1}{2^{2r}}\binom{2r}{r}$$

$$= \frac{1}{2^{2r}}\frac{(2r)!}{r!\,r!}$$

$$= \frac{2r}{2\cdot r}\frac{1}{2^{2r-1}}\frac{(2r-1)!}{r!(r-1)!}$$

$$= \frac{1}{2^{n-1}}\binom{n-1}{[(n-1)/2]} = E(RI_n).$$

3. $$\sum_{k=0}^{n}\frac{n-2k}{n}\binom{n}{k}\left(\frac{1}{2}\right)^n = 2\sum_{k=0}^{[(n-1)/2]}\frac{n-2k}{n}\binom{n}{k}\left(\frac{1}{2}\right)^n$$

$$= \sum_{k=1}^{[n-1)/2]}\frac{n-2k}{n}\binom{n}{k}\left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^{n-1}$$

$$= \left(\frac{1}{2}\right)^{n-1}\left\{\sum_{k=1}^{[(n-1)/2]}\left[\frac{n-k}{n}\binom{n}{k} - \frac{k}{n}\binom{n}{k}\right] + 1\right\}$$

$$= \left(\frac{1}{2}\right)^{n-1}\left\{\sum_{k=1}^{[(n-1)/2]}\left[\binom{n-1}{k} - \binom{n-1}{k-1}\right] + 1\right\}$$

$$= \left(\frac{1}{2}\right)^{n-1}\left\{\binom{n-1}{[(n-1)/2]} - 1 + 1\right\} = \left(\frac{1}{2}\right)^{n-1}\binom{n-1}{[(n-1/2]}.$$

2. Unit 271b, *The Probability of Agreement Measure*

2.1 Solutions to Exercises

Solutions to exercises 1, 2, 3, 5, 6 are in the following tables.

Exercises 1, 5: $n = 5$

k	$PA_n(k)$	$AL_n(k)$
3	0.4	0.000
4	0.6	0.333
5	1.0	1.000

Exercise 2: $n = 7$

k	$PA_n(k)$	$AL_n(k)$
4	0.429	0.000
5	0.524	0.166
6	0.714	0.499
7	1.000	1.000

Exercises 3, 6: $n = 10$

k	$PA_n(k)$	$AL_n(k)$
5	0.444	0.000
6	0.467	0.041
7	0.533	0.160
8	0.644	0.360
9	0.800	0.640
10	1.000	1.000

4. $E(PA_5) = (0.4)(0.625)+(0.6)(0.313)+(1.0)(0.063) = 0.5008$

$E(PA_7) = (0.429)(0.547)+(0.524)(0.328)+(0.714)(0.109)+(1)(0.016) = 0.5014$

$E(PA_{10}) = (0.444)(0.244)+(0.467)(0.401)+(0.533)(0.239)+(0.644)(0.088)+(0.800)(0.019)+(1)(0.002) = 0.4922$

The deviations from 0.5 are due to round-off errors.

7,8. $n = 7$

m_1, m_2, m_3			$PA_7(m_1,m_2,m_3)$	$RI_7^2(m_1,m_2,m_3)$
3	2	2	0.238	0.020
3	3	1	0.286	0.082
4	2	1	0.333	0.133
4	3	0	0.429	0.265
5	1	1	0.476	0.326
5	2	0	0.524	0.377
6	1	0	0.714	0.633
7	0	0	1.000	1.000

2.2 Solutions to Problems

1. $E(PA_n) = P(AG_2)$, as established in section 2.3.

$$P(AG_2) = P(\text{both vote yes}) + P(\text{both vote no})$$
$$= p^2 + q^2,$$

since each voter acts independent of the others.

2. $$AL_n(k) = \frac{PA_n(k) - PA_n((n+1)/2)}{1 - PA_n((n+1)/2)}$$

$$= \frac{\{\binom{k}{n}+\binom{n-k}{2}\}/\binom{n}{2} - \{\binom{(n+1)/2}{2}+\binom{(n-1)/2}{2}\}/\binom{n}{2}}{\binom{n}{2}/\binom{n}{2} - \{\binom{(n+1)/2}{2}+\binom{(n-1)/2}{2}\}/\binom{n}{2}}$$

$$= \frac{k(k-1)+(n-k)(n-k-1)-\{((n+1)/2)((n-1)/2)+((n-1)/2)((n-3)/2)\}}{n(n-1)-\{((n+1)/2)((n-1)/2)+((n-1)/2)((n-3)/2)\}}$$

$$= \frac{4k^2 - 4nk + n^2 - 1}{n^2 - 1}$$

$$= \frac{(k-(n-k))^2-1}{n^2-1} = \frac{(k-(n-k))^2}{n^2} + \frac{\frac{(k-(n-k))^2}{n^2} - 1}{n^2-1}$$

$$= RI_n(k)^2 + \frac{RI_n(k)^2 - 1}{n^2}.$$

For $n = 5$, $\frac{RI_5(3)^2-1}{5^2-1} = -0.040$.

For $n = 7$, error $= -0.020$.

For $n = 9$, error $= -0.012$.

3. (a) $AL_n(m_1,m_2,\ldots,m_t) = \frac{PA_n(m_1,m_2,\ldots,m_t) - \min PA_n}{1 - \min PA_n}$,

where $PA_n(m_1,m_2,\ldots,m_t)$ is the probability of agreement of two voters from a group of n which voted $m_1, m_2, \ldots m_t$ for each of t alternatives. Min PA_n denotes the smallest possible value for this quantity, which will for the most even split.

(b) Assume $n = t\ell + r$. Then

$AL_n(m_1,m_2,\ldots,m_t) =$

$$\frac{\{\binom{m_1}{2}+\binom{m_2}{2}+\ldots+\binom{m_t}{2}\}/\binom{n}{2} - \{(t-r)\binom{\ell}{2}+r\binom{\ell+1}{2}\}/\binom{n}{2}}{1 - \{(t-r)\binom{\ell}{2}+r\binom{\ell+1}{2}\}/\binom{n}{2}}$$

$$= \frac{m_1(m_1-1)+m_2(m_2-1)+\ldots+m_t(m_t-1)-\{(t-r)\ell(\ell-1)+r(\ell+1)\ell\}}{n(n-1) - \{(t-r)\ell(\ell-1) + r(\ell+1)\ell\}}.$$

The expression which appears twice in brackets in the last formulation can be written as follows:

$$\{(t-r)\ell(\ell-1)+r(\ell+1)\} = t\ell^2 + 2r\ell - t\ell$$

$$= \frac{t^2\ell^2+2tr\ell+r^2-r^2-t^2\ell-tr+tr}{t}$$

$$= \frac{n^2}{t} - n + \frac{r(t-r)}{t}$$

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$$= \frac{m_1^2+m_2^2+\ldots+m_t^2-2m_1m_2-2m_1m_3-\ldots-2m_{t-1}m_t}{t}$$

$$-(m_1+m_2+\ldots+m_t) + \frac{r(t-r)}{r}$$

When we substitute this into the last expression for AL we get:

$$AL_n = \frac{((t-1)(m_1^2+m_2^2+\ldots+m_t^2)-2m_1m_2-2m_1m_3-\ldots-2m_{t-1}m_t-r(t-r)}{(t-1)n^2 - r(t-r)}$$

$$= \frac{\sum_{i\neq j}(m_i-m_j)^2 - r(t-r)}{(t-1)n^2 - r(t-r)}$$

Observe that $\frac{a-c}{b-c} = \frac{a}{b} + \frac{(a/b)-1}{(b/c)-1}$.

Letting $a = (m_i-m_j)^2$, $b = (t-1)n^2$, $c = r(t-r)$, this reduces AL_n to

$$= \frac{\sum(m_i-m_j)^2}{(t-1)\,n^2} + \frac{(\sum(m_i-m_j)^2/(t-1)n^2 - 1)}{(n^2(t-1)/r(t-r) - 1)}$$

$$= RI_n^2(m_1,m_2,\ldots,m_t) + \frac{(RI_n^2(m_1,m_2,\ldots m_t)-1)}{(n^2(t-1)/r(t-r)-1)}$$

as desired.

4. The minimum value of PA_n will occur when the group is evenly split. For $n > 2k$,

$$\min PA_n = PA_n(k) = 2\binom{k}{2}/\binom{2k}{2}$$

$$= \frac{2k(k-1)}{2k(2k-1)} = \frac{k-1}{2k-1}.$$

This last expression approaches 1/2 as k grows large.

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5. The minimum value of PA_n for t options will occur when voters are equally divided among the t options. Suppose $n = t\ell$, then

$$\min PA_n = PA_n(\ell,\ell,\ldots,\ell) = t\binom{\ell}{2}/\binom{t}{2}$$

$$= t\ell(\ell-1)/t\ell(t\ell-1).$$

This will approach $1/t$ as ℓ, and hence n, grows large.

3. Unit 271c, *The α-Index*

3.1 Solutions to Exercises

The solutions for exercises 1-6 are in the following tables;

Exercises 1,4: $n = 5$

k	$\alpha_n(k)$, p = 0.5	$\alpha_n(k)$, p = 0.7
3	0.313	0.221
4	0.778	0.635
5	0.969	0.915

Exercises 2,5: $n = 7$

k	$\alpha_n(k)$, p = 0.5	$\alpha_n(k)$, p = 0.7
4	0.274	0.162
5	0.711	0.496
6	0.930	0.793
7	0.992	0.959

Exercises 3,6: $n = 10$

k	$\alpha_n(k)$, p = 0.5	$\alpha_n(k)$, p = 0.7
5	0.122	0.052
6	0.444	0.222
7	0.762	0.478
8	0.923	0.734
9	0.990	0.915
10	0.999	0.986

Exercise 7: $n = 5$

k	$\alpha_n(k)$, p = 1
3	0.0
4	0.0
5	0.5

Exercise 8: $n = 5$

$$E(\alpha_5) = (0.313)(0.625) + (0.778)(0.313) + (0.969)(0.063) = 0.500086.$$

The divergence from 0.5 is due to round-off error.

STUDENT FORM 1

Request for Help

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Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ____________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ____________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

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Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)