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ABSTRACT

Vector analysis is viewed to play a key role in many branches of engineering and the physical sciences. This unit is geared towards deriving identities and establishing "machinery" to make derivations a routine task. It is noted that the module is not an applications unit, but has as its primary objective the goal of providing science, engineering, and mathematics students with powerful means of deriving vector and vector field identities. It is felt the skills a student would gain in studying the material should be very valuable in the practice of applied mathematics. Exercises and a model exam are provided, with answers included for both. (MP)

umap

UNIT 427

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

THE LEVI-CIVITA TENSOR AND IDENTITIES IN VECTOR ANALYSIS

by Chang-li Yiu and Carroll O. Wilde

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix}$$

VECTOR FIELD IDENTITIES

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THE LEVI-CIVITA TENSOR AND IDENTITIES IN VECTOR ANALYSIS

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Intermodular Description Sheet: *UMAP Unit 427*

Title: THE LEVI-CIVITA TENSOR AND IDENTITIES IN VECTOR ANALYSIS

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Classification: VECTOR FIELD IDENTITIES

Prerequisite Skills:

1. Prior or concurrent course in vector analysis.
2. Ability to multiply matrices and manipulate determinants.

Output Skills:

1. Ability to derive vector identities and vector field identities using the Levi-Civita tensor.

Other Related Units:

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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THE LEVI-CIVITA TENSOR AND IDENTITIES IN VECTOR ANALYSIS

1. INTRODUCTION

Vector analysis plays a key role in many branches of engineering and physical sciences. In electromagnetic theory and in fluid mechanics, for example, we often use vector identities such as

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C},$$

and vector field identities such as

$$\nabla\times(\nabla\times\vec{u}) = \nabla(\nabla\cdot\vec{u}) - \nabla^2\vec{u}.$$

In many textbooks on vector analysis and physics, proofs of identities are either very difficult or simply omitted. This situation is also encountered when we learn quantum field theory, where four-dimensional vectors appear. The problem is not that we are unable to learn the identity itself, because we can always accept the result without proof; the more serious consequence is an inability to derive a new result when the need arises.

In this unit we study a systematic way to derive these identities, and we establish "machinery" that makes such derivation a routine task. Three-dimensional vectors and vector fields are studied in detail, and a brief indication of the extension to four dimensions is also included.

We note that this is not an "applications unit"; the primary objective is to provide science, engineering and mathematics students with a powerful means of deriving vector and vector field identities. The skills that you gain should be valuable to you in the practice of applied mathematics.

2. THE KRONECKER-δ*

We consider three-dimensional vectors in the rectangular xyz-coordinate system. Let us denote the unit vectors along the coordinate axes by $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. (These vectors are often denoted by $\vec{i}$, $\vec{j}$, $\vec{k}$, respectively.) Then, any vector $\vec{A} = (A_1, A_2, A_3)$ may be expressed as

$$\vec{A} = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3 = \sum_{i=1}^{3} A_i\vec{e}_i. \tag{2.1}$$

The vectors $\vec{e}_i$, i = 1,2,3, all have length one, and they are pairwise orthogonal. These properties can be expressed in terms of the scalar (or dot) product, as follows: for i,j = 1,2,3 we have

$$\vec{e}_i\cdot\vec{e}_j = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases} \tag{2.2}$$

The right-hand side of Equation (2.2) can be written in more convenient form by means of the Kronecker-δ. This useful device is a function which is defined on two indices i,j by the formula

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases} \tag{2.3}$$

For example, $\delta_{13} = 0$ and $\delta_{11} = 1$.

Using the Kronecker-δ, we may rewrite Equation (2.2) in the shorter form

$$\vec{e}_i\cdot\vec{e}_j = \delta_{ij}. \tag{2.4}$$

Recall that for any two vectors

$$A = \sum_{i=1}^{3} A_i\vec{e}_i \quad\text{and}\quad \vec{B} = \sum_{i=1}^{3} B_i\vec{e}_i,$$

*Read "Kronecker-delta."

the scalar product is given by

$$(2.5) \qquad \vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A_i B_i .$$

We may also express $\vec{A}\cdot\vec{B}$ in terms of the Kronecker-δ: applying basic properties, we first find

$$\vec{A}\cdot\vec{B} = \left(\sum_{i=1}^{3} A_i \vec{e}_i\right) \cdot \left(\sum_{j=1}^{3} B_j \vec{e}_j\right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j (\vec{e}_i \cdot \vec{e}_j).$$

Then using Equation (2.4) we obtain

$$(2.6) \qquad \vec{A}\cdot\vec{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij} .$$

Exercises

1. Show that for any i,j we have

   $$(2.7) \qquad \delta_{ij} = \delta_{ji} .$$

   (This equation, although very easy to establish, is numbered for reference later on.)

2. a. Use the defining relation (2.3) to verify that for $i = 1,2,3$ we have $B_i = B_1\delta_{i1} + B_2\delta_{i2} + B_3\delta_{i3}$.

   b. Substitute the expression for B_i in part (a) into Equation (2.5) to obtain an alternate derivation of Equation (2.6).

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3. PERMUTATIONS

We recall briefly some basic concepts associated with permutations. These ideas will then be used to define the fundamental concept of this module, the Levi-Civita tensor.

A permutation of the integers 1,2,...,n is an arrangement (or ordering) of these numbers. A permutation may be regarded mathematically as a one-to-one function of the set {1,2,...,n} onto itself. In describing a permutation, we usually omit all commas separating the integers, and simply write them in the order of the arrangement. Thus, for n = 3 there are six possible permutations:

123, 231, 312, 213, 321, 132.*

The permutations 123, 231, 312 are called *even* permutations, while 213, 321, 132 are called *odd* permutations. These names are associated with the number of pairwise exchanges needed to obtain the given permutation from the natural ordering 123. For example, 231 can be obtained from 123 by two exchanges: first exchange 1 and 2 to obtain 213, then exchange 1 and 3 to obtain 231. These exchanges may be represented as follows:

$$1\ 2\ 3 \rightarrow 2\ 1\ 3 \rightarrow 2\ 3\ 1.$$

Since 231 can be obtained from 123 by two exchanges and two is an even number, we call 231 an even permutation.

In a similar way we may illustrate that 312 is an even permutation

$$1\ 2\ 3 \rightarrow 1\ 3\ 2 \rightarrow 3\ 1\ 2.$$

There are two interchanges involved, so 312 is even. The permutation 123 is even because the number of exchanges required to obtain 123 from 123 is zero, and zero is an even number.

The odd permutations can be described in a similar way:

---

*Read "one-two-three, two-three-one, three-one-two," etc.

1 2 3 → 1 3 2 (1 exchange)

1 2 3 → 2 1 3 (1 exchange)

1 2 3 → 2 1 3 → 2 3 1 → 3 1 2 (3 exchanges)

or,

1 2 3 → 3 2 1 (1 exchange)

An easy way to determine whether a given permutation is even or odd is to write out the permutation and then write the natural arrangement directly below it. Then connect corresponding numbers in these two arrangements with line segments, and count the number of intersections *between pairs* of these segments; if this number is even, then the given permutation is even; otherwise it is odd. (See Figure 1.) The reason this scheme is valid is that each pairwise exchange corresponds to an intersection of two of the lines.

a. Two intersections; 231 is even.

b. Three intersections; 321 is odd.

Figure 1. A geometric scheme for determining whether a given permutation is even or odd. (Be careful! The intersections in (b) could all occur at one point—remember we are counting intersections of pairs of lines.)

Exercises

3. Determine whether the permutation 4231 of the integers 1,2,3,4 is even or odd by:
   a. counting integers;
   b. using intersections of lines (the scheme depicted in Figure 1).

4. THE LEVI-CIVITA TENSOR

We are now in a position to present the central concept of this module, the Levi-Civita tensor. This tensor is a function of three indices i,j,k which is related to the vector (or cross) product in much the same way as the Kronecker-δ is a function of two indices i,j which is related to the scalar product.

The vector product relations among the basis vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are given by:

$$(4.1)\quad \begin{cases} \vec{e}_1\times\vec{e}_1 = 0, & \vec{e}_2\times\vec{e}_2 = 0, & \vec{e}_3\times\vec{e}_3 = 0, \\ \vec{e}_1\times\vec{e}_2 = \vec{e}_3, & \vec{e}_2\times\vec{e}_3 = \vec{e}_1, & \vec{e}_3\times\vec{e}_1 = \vec{e}_2, \\ \vec{e}_2\times\vec{e}_1 = -\vec{e}_3, & \vec{e}_3\times\vec{e}_2 = -\vec{e}_1, & \vec{e}_1\times\vec{e}_3 = -\vec{e}_2. \end{cases}$$

A careful examination of the relations in (4.1) reveals some patterns that turn out to be most useful. In the first line, the two indices that appear in any one of the three equations are identical. In the second line, the indices are arranged 123, 231, 312 in the three equations; in the third line they are arranged 213, 321, 132. These observations lead to a formulation of the Levi-Civita tensor,* which is the function of three indices i, j, k defined by

$$(4.2)\quad \varepsilon_{ijk} = \begin{cases} 0, \text{ if two or more of the indices i, j, k are equal;} \\ 1, \text{ if ijk is an even permutation of 123;} \\ -1, \text{ if ijk is an odd permutation of 123.} \end{cases}$$

For example, $\varepsilon_{113} = 0$, $\varepsilon_{333} = 0$, $\varepsilon_{123} = 1$, $\varepsilon_{231} = 1$, $\varepsilon_{213} = -1$, $\varepsilon_{132} = -1$.

*Strictly speaking, what we define here is a tensor component. The Levi-Civita tensor is a collection of these components, just as a vector is a collection of its components. However, we shall use the term "tensor" instead of "tensor component" for simplicity.

Using the Levi-Civita tensor, we may summarize all the relations in (4.1) in a single equation:

$$(4.3) \qquad \vec{e}_i \times \vec{e}_j = \sum_{k=1}^{3} \varepsilon_{ijk} \vec{e}_k \quad \text{for } i,j = 1,2,3.$$

Exercises

4. Substitute the values 1,2,3 for i,j in Equation (4.3) to obtain the nine relations in (4.1).

5. Show that for any given i,j,k we have

$$(4.4) \qquad \varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki};$$

$$(4.5) \qquad \varepsilon_{ikj} = \varepsilon_{kji} = \varepsilon_{jik} = -\varepsilon_{ijk}.$$

Recall that in Section 2 we obtained the expression

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij}$$

for the scalar product in terms of δ_{ij}. In a similar way, we can express the vector product in terms of ε_{ijk}: applying basic properties, we first find

$$\vec{A} \times \vec{B} = \left(\sum_{i=1}^{3} A_i \vec{e}_i \right) \times \left(\sum_{j=1}^{3} B_j \vec{e}_j \right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \vec{e}_i \times \vec{e}_j .$$

Then using Equation (4.3) we obtain

$$(4.6) \qquad \vec{A} \times \vec{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} A_i B_j \varepsilon_{ijk} \vec{e}_k .$$

Exercises

6. For the vectors $\vec{A} = 2\vec{e}_1 - \vec{e}_2 + 3\vec{e}_3$, $\vec{B} = 3\vec{e}_1 + 4\vec{e}_2 + \vec{e}_3$, substitute the appropriate values into Equation (4.6) and simplify. Check your result by using another method (such as representation by a determinant) to find $\vec{A} \times \vec{B}$.

5. A USEFUL NOTATIONAL CONVENTION

At this point we introduce a convention to simplify our mathematical writing. The reason for this device is to reduce the number of summation symbols we must write. (Just look at Equation (4.6)!)

We first note that in several equations above, for example, (2.6), (4.3) and (4.6), the indices over which we sum (i.e., the dummy indices) appear twice. We adopt the convention that whenever an index appears *exactly twice* in an expression, it will be a dummy index, and we must sum on this index over the appropriate values. For three-dimensional vectors the appropriate range of values is from 1 to 3.

Example 1. To interpret the notation $A_{ij}b_j$, we note that the subscript j appears *exactly twice*, so under our convention we have

$$A_{ij}b_j = A_{i1}b_1 + A_{i2}b_2 + A_{i3}b_3 .$$

That is, we sum on j over the range from 1 to 3. The notation $A_{ik}b_k$ has the same meaning:

$$A_{ik}b_k = A_{i1}b_1 + A_{i2}b_2 + A_{i3}b_3 ;$$

only the index of summation is changed, not the sum itself.

Example 2. We may rewrite several results from the text above in much shorter form: Equations (2.5), (2.6), (4.3) and (4.6) become, respectively,

$$(5.1) \qquad \vec{A} \cdot \vec{B} = A_i B_i ;$$

$$(5.2) \qquad \vec{A} \cdot \vec{B} = A_i B_j \delta_{ij} ;$$

$$(5.3) \qquad \vec{e}_i \times \vec{e}_j = \varepsilon_{ijk} \vec{e}_k ;$$

$$(5.4) \qquad \vec{A} \times \vec{B} = A_i B_j \varepsilon_{ijk} \vec{e}_k .$$

The final relation in Example 2 may also be given by specifying the kth component of $\vec{A} \times \vec{B}$:

$$(\vec{A} \times \vec{B})_k = A_i B_j \varepsilon_{ijk}. \tag{5.5}$$

The next exercises check skills and present results that are necessary in the remainder of this module. Make sure that you understand these exercises *clearly*, before going on; if necessary, refer to the solutions in Section 11.

Exercises

7. Verify the following relations:

(5.6) $A_i \delta_{ij} = A_j$;

(5.7) $B_j \delta_{ji} = B_i$;

(5.8) $\delta_{1j} \delta_{j1} = 1$;

(5.9) $\delta_{ij} \delta_{ji} = \delta_{ii} = 3$;

(5.10) $\delta_{ij} \delta_{j\ell} = \delta_{i\ell}$ (i and ℓ are arbitrary but fixed; in particular, they may have the same value).

8. Show that the ith component of $\vec{B} \times \vec{C}$ is:

(5.11) $(\vec{B} \times \vec{C})_i = B_j C_k \varepsilon_{ijk}$.

9. Prove the following relations:

a. for scalar triple products we have

(5.12) $\vec{A} \cdot (\vec{B} \times \vec{C}) = A_i B_j C_k \varepsilon_{ijk}$;

b. for vector triple products we have

(5.13) $[\vec{A} \times (\vec{B} \times \vec{C})]_i = A_j B_\ell C_m \varepsilon_{ijk} \varepsilon_{k\ell m}$.

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6. RELATION BETWEEN THE KRONECKER-δ AND THE LEVI-CIVITA TENSOR

By checking directly in the definition of the Levi-Civita tensor, Equation (4.2), we can establish that for i,j,k = 1,2,3 we have

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{k2} \\ \delta_{i3} & \delta_{j3} & \delta_{k3} \end{vmatrix}. \tag{6.1}$$

For example, if i = j, then $\varepsilon_{ijk} = 0$, and both determinants in Equation (6.1) also equal zero; the first because two rows are identical, the second because two columns are identical. If i = 1, j = 2, k = 3, then $\varepsilon_{ijk} = 1$, and both determinants in Equation (6.1) also equal one, since both reduce to

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \tag{6.2}$$

Finally, an even permutation of the indices corresponds to an even permutation of the rows (columns) of the determinant (6.2), so in this case both sides of (6.1) equal one; and an odd permutation of the indices corresponds to an odd permutation of the rows (columns) of (6.2), so in this case both sides of (6.1) equal -1. For example,

$$\varepsilon_{132} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix},$$

since both sides equal -1. Thus, we have established Equation (6.1).

Next, we derive a second, and very important, relation between the Kronecker-δ and the Levi-Civita tensor. For $j,k,\ell,m = 1,2,3$ we have

$$(6.3) \quad \varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}.$$

For example,

$$\varepsilon_{i23}\varepsilon_{i32} = \varepsilon_{123}\varepsilon_{132} + \varepsilon_{223}\varepsilon_{232} + \varepsilon_{323}\varepsilon_{332} = -1,$$

and

$$\delta_{23}\delta_{32} - \delta_{22}\delta_{33} = -1,$$

which verifies Equation (6.3) for the values $j = 2$, $k = 3$, $\ell = 3$, $m = 2$.

To establish Equation (6.3), apply Equations (6.1) and (2.7) to obtain

$$(6.4) \quad \varepsilon_{ijk}\varepsilon_{i\ell m} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \cdot \begin{vmatrix} \delta_{1i} & \delta_{1\ell} & \delta_{1m} \\ \delta_{2i} & \delta_{2\ell} & \delta_{2m} \\ \delta_{3i} & \delta_{3\ell} & \delta_{3m} \end{vmatrix}.$$

Since the determinant of the product of two square matrices equals the product of the determinants, we can find the product in Equation (6.4) by performing a matrix multiplication. Since

$$\delta_{i1}\delta_{1i} + \delta_{i2}\delta_{2i} + \delta_{i3}\delta_{3i} = 3$$

by Equation (5.9), and since we may apply Equation (5.10) to arbitrary but fixed indices, we have

$$(6.5) \quad \varepsilon_{ijk}\varepsilon_{i\ell m} = \begin{vmatrix} 3 & \delta_{i\ell} & \delta_{im} \\ \delta_{ji} & \delta_{j\ell} & \delta_{jm} \\ \delta_{ki} & \delta_{k\ell} & \delta_{km} \end{vmatrix}.$$

Expanding this determinant and simplifying (see Exercise 10), we obtain the desired Equation (6.3).

Exercises

10. Expand the determinant in Equation (6.5), and simplify using Equations (2.7) and (5.10), to obtain the right-hand side of Equation (6.3).

11. For the following sets of values of j,k,ℓ,m, verify Equation (6.3) by direct substitution:

 a. 1,1,2,3;

 b. 1,2,1,2;

 c. 1,3,2,1;

 d. 1,1,1,1.

12. Prove that for $k,m = 1,2,3$ we have $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$.

7. IDENTITIES IN VECTOR ALGEBRA

We first apply Equation (6.1) to prove a well-known identity for scalar triple products. Study the proof carefully! Remember, the use of the Levi-Civita tensor in proving identities is the main theme of this module. The identity we prove is:

$$(7.1) \quad \vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

Proof:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_iB_jC_k\varepsilon_{ijk} \qquad \text{(Equation (5.12))}$$

$$= A_iB_jC_k\begin{vmatrix}\delta_{i1} & \delta_{i2} & \delta_{i3}\\ \delta_{j1} & \delta_{j2} & \delta_{j3}\\ \delta_{k1} & \delta_{k2} & \delta_{k3}\end{vmatrix} \qquad \text{(Equation (6.1))}$$

$$= \begin{vmatrix}A_i\delta_{i1} & A_i\delta_{i2} & A_i\delta_{i3}\\ B_j\delta_{j1} & B_j\delta_{j2} & B_j\delta_{j3}\\ C_k\delta_{k1} & C_k\delta_{k2} & C_k\delta_{k3}\end{vmatrix} \qquad \text{(Multiplication of determinants by constants)}$$

$$= \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}. \qquad \text{(Equation (5.7), applied also to } A_i \text{ and } C_k\text{)}$$

As a corollary of the proof, we obtain a formula for a 3×3 determinant in terms of the Levi-Civita tensor:

$$\text{(7.2)} \qquad \begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix} = A_iB_jC_k\varepsilon_{ijk}.$$

Equation (6.3) is the key formula in proving many identities. We illustrate the utility of this equation by proving the following well-known identity for vector triple products:

$$\text{(7.3)} \qquad \vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}.$$

Proof:

We show that the ith components of both sides of Equation (7.3) agree:

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \varepsilon_{kij}\varepsilon_{k\ell m}A_jB_\ell C_m \qquad \text{(Equations (5.13) and (4.4))}$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})A_jB_\ell C_m \qquad \text{(Equation (6.3))}$$

$$= A_jC_jB_i - A_jB_jC_i \qquad \text{(Sum over } \ell \text{ and } m\text{)}$$

$$= (\vec{A}\cdot\vec{C})B_i - (\vec{A}\cdot\vec{B})C_i. \qquad \text{(Equation (2.5))}$$

Since the corresponding components agree, the vectors on both sides of Equation (7.3) must be equal.

Exercises

13. Prove the vector identity:

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}).$$

8. IDENTITIES IN VECTOR FIELDS

First, we shall introduce some useful notation. For coordinate variables we shall use x_1,x_2,x_3, instead of x,y,z. For example, a function u will be denoted by $u(x_1,x_2,x_3)$, and the partial derivative with respect to the first variable x_1 by $\partial u/\partial x_i$. In addition, the differential operator $\partial/\partial x_i$ will be denoted by ∂_i. For example, if we have three functions u_1,u_2,u_3, then we shall use $\partial_2 u_3$ to denote $\partial u_3/\partial x_2$. In this notational system, the curl of a vector field $\vec{u} = u_i\vec{e}_i$ is given by

$$\text{(8.1)} \qquad \nabla\times\vec{u} = \begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ \partial_1 & \partial_2 & \partial_3\\ u_1 & u_2 & u_3\end{vmatrix}$$

The ith component of $\nabla \times \vec{u}$ is given by

(8.2) $$(\nabla \times \vec{u})_i = \varepsilon_{ijk}\partial_j u_k.$$

(Compare with Equation (5.11)!)

Exercises

14. With ∇^2 defined, as usual, by $\nabla^2\phi = \nabla\cdot(\nabla\phi)$, show that $\nabla^2 = \partial_i\partial_i$.

We shall prove several vector field identities that are used extensively in applications. For vector fields $\vec{u},\vec{v}$ we have

(8.3) $$\nabla\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\nabla\times\vec{u}) - \vec{u}\cdot(\nabla\times\vec{v});$$

(8.4) $$\nabla\times(\nabla\times\vec{u}) = \nabla(\nabla\cdot\vec{u}) - \nabla^2\vec{u};$$

(8.5) $$\nabla\cdot(\nabla\times\vec{u}) = 0.$$

Study these proofs carefully—they are important in reaching our main objectives.

Proof of Equation (8.3):

$$\begin{aligned}
\nabla\cdot(\vec{u}\times\vec{v}) &= \partial_i(\vec{u}\times\vec{v})_i \\
&= \partial_i(\varepsilon_{ijk}u_j v_k) \\
&= \varepsilon_{ijk}(\partial_i u_j)v_k + \varepsilon_{ijk}u_j(\partial_i v_k) \\
&= \varepsilon_{ijk}(\partial_i u_j)v_k - \varepsilon_{jik}u_j(\partial_i v_k) \\
&= (\nabla\times\vec{u})_k v_k - u_j(\nabla\times\vec{v})_j \qquad \text{(Equation (8.2))} \\
&= (\nabla\times\vec{u})\cdot\vec{v} - \vec{u}\cdot(\nabla\times\vec{v}).
\end{aligned}$$

Proof of Equation (8.4):

$$\begin{aligned}
[\nabla\times(\nabla\times\vec{u})]_i &= \varepsilon_{ijk}\partial_j(\varepsilon_{k\ell m}\partial_\ell u_m) \qquad \text{(Equation (8.2) applied twice)} \\
&= \varepsilon_{ijk}\varepsilon_{k\ell m}(\partial_j\partial_\ell u_m) \\
&= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\partial_j\partial_\ell u_m \\
&= \partial_i(\partial_j u_j) - \partial_j\partial_j u_i \\
&= \partial_i(\nabla\cdot\vec{u}) - \nabla^2 u_i.
\end{aligned}$$

Equation (8.4) now follows, since corresponding components from both sides agree.

Proof of Equation (8.5):

$$\begin{aligned}
\nabla\cdot(\nabla\times\vec{u}) &= \partial_i(\varepsilon_{ijk}\partial_j u_k) \\
&= \varepsilon_{ijk}\partial_i\partial_j u_k \\
&= \varepsilon_{ijk}\partial_j\partial_i u_k \qquad (\partial_i\partial_j = \partial_j\partial_i) \\
&= \varepsilon_{jik}\partial_i\partial_j u_k \qquad \text{(Rename the dummy indices } i,j.) \\
&= -\varepsilon_{ijk}\partial_i\partial_j u_k \\
&= -\nabla\cdot(\nabla\times\vec{u}).
\end{aligned}$$

Hence,

$$\nabla\cdot(\nabla\times\vec{u}) = 0.$$

Exercises

15. Prove the vector field identity

$$\nabla\times(\phi\vec{u}) = \phi(\nabla\times\vec{u}) + (\nabla\phi)\times\vec{u}$$

where ϕ is a scalar function of x_1, x_2, x_3.

---

9. THE LEVI-CIVITA TENSOR IN FOUR DIMENSIONS

We conclude our study with a brief indication of the extension of the Levi-Civita tensor to four dimensions. The definition of the "four-dimensional Levi-Civita tensor" is straightforward: for $\alpha,\beta,\gamma,\tau = 1,2,3,4$, the tensor component is

$$(9.1)\qquad \epsilon_{\alpha\beta\gamma\tau} = \begin{cases} 0, & \text{if two or more of the indices are equal;} \\ 1, & \text{if } \alpha\beta\gamma\tau \text{ is an even permutation of } \{1,2,3,4\}; \\ -1, & \text{if } \alpha\beta\gamma\tau \text{ is an odd permutation of } \{1,2,3,4\}. \end{cases}$$

(It is customary to use Greek letters as indices for four-dimensional quantities.)

Exercises

16. Use Equation (9.1) to find:

a. ϵ_{2143};

b. ϵ_{3142};

c. ϵ_{4321}.

17. Show that the four-dimensional Levi-Civita tensor may be expressed in determinant form, as follows:

$$\epsilon_{\alpha\beta\gamma\tau} = \begin{vmatrix} \delta_{\alpha1} & \delta_{\alpha2} & \delta_{\alpha3} & \delta_{\alpha4} \\ \delta_{\beta1} & \delta_{\beta2} & \delta_{\beta3} & \delta_{\beta4} \\ \delta_{\gamma1} & \delta_{\gamma2} & \delta_{\gamma3} & \delta_{\gamma4} \\ \delta_{\tau1} & \delta_{\tau2} & \delta_{\tau3} & \delta_{\tau4} \end{vmatrix}$$

(You may wish to review the discussion immediately following Equation (6.1).)

18. Establish the following relations:

a. $\epsilon_{\kappa\mu\nu\lambda}\epsilon_{\alpha\beta\gamma\lambda} = (\delta_{\kappa\alpha}\delta_{\mu\beta}\delta_{\nu\gamma} + \delta_{\kappa\beta}\delta_{\mu\gamma}\delta_{\nu\alpha} + \delta_{\kappa\gamma}\delta_{\mu\alpha}\delta_{\nu\beta})$
$\qquad - (\delta_{\kappa\alpha}\delta_{\mu\gamma}\delta_{\nu\beta} + \delta_{\kappa\beta}\delta_{\mu\alpha}\delta_{\nu\gamma} + \delta_{\kappa\gamma}\delta_{\mu\beta}\delta_{\nu\alpha})$;

b. $\epsilon_{\kappa\mu\gamma\lambda}\epsilon_{\alpha\beta\gamma\lambda} = 2(\delta_{\kappa\alpha}\delta_{\mu\beta} - \delta_{\kappa\beta}\delta_{\mu\alpha})$;

c. $\epsilon_{\kappa\beta\gamma\lambda}\epsilon_{\alpha\beta\gamma\lambda} = 6\delta_{\kappa\alpha}$;

d. $\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = 24$.

We close with an indication of how to extend the definition of the vector product to four dimensions. For vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, the "vector product" is the vector $\vec{D}$ which is given by:

$$(9.2)\qquad \vec{D} = A_\alpha B_\beta C_\gamma \epsilon_{\alpha\beta\gamma\tau} \vec{f}_\tau,$$

where $\vec{f}_1$, $\vec{f}_2$, $\vec{f}_3$, $\vec{f}_4$ are the usual basis vectors, and the appropriate range of summation for repeated indices is from 1 to 4.

Exercises

19. Show that the vector $\vec{D}$ in Equation (9.2) is orthogonal to $\vec{A}$, i.e., show that $\vec{A}\cdot\vec{D} = 0$.

---

10. MODEL EXAM

Use the Levi-Civita tensor technique in solving the following problems.

1. Prove the vector identities:

a. $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$;

b. $(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$.

2. Prove the vector field identities:

a. $\nabla\times\nabla\phi = \vec{0}$;

b. $\nabla\times(\vec{u}\times\vec{v}) = (\nabla\cdot\vec{v})\vec{u} - (\nabla\cdot\vec{u})\vec{v} + (\vec{v}\cdot\nabla)\vec{u} - (\vec{u}\cdot\nabla)\vec{v}$.

11. ANSWERS AND SOLUTIONS TO EXERCISES

1. Use Equation (2.3):

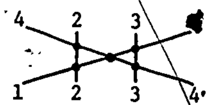
$$\delta_{ji} = \begin{Bmatrix} 0, & \text{if } j \neq i \\ 1, & \text{if } j = i \end{Bmatrix} = \begin{Bmatrix} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{Bmatrix} = \delta_{ij}$$

(i.e., the definition is symmetric in i and j).

2. a. $B_i\delta_{i1} + B_2\delta_{12} + B_3\delta_{13} = B_1 \cdot 1 + B_2 \cdot 0 + B_3 \cdot 0 = B_1$, and a similar calculation applies for $i = 2,3$.

   b. $\vec{A}\cdot\vec{B} = \sum_{i=1}^{3} A_i \sum_{j=1}^{3} B_j\delta_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_iB_j\delta_{ij}$.

3. a. Odd; 5 exchanges.

   b. 

   There are 5 intersections; this scheme also shows the permutation odd.

4. $\vec{e}_1 \times \vec{e}_1 = \sum_{k=1}^{3} \varepsilon_{11k}\vec{e}_k = 0\vec{e}_1 + 0\vec{e}_2 + 0\vec{e}_3 = \vec{0}$;

   $\vec{e}_1 \times \vec{e}_2 = \sum_{k=1}^{3} \varepsilon_{12k}\vec{e}_k = 0\vec{e}_1 + 0\vec{e}_2 + 1\vec{e}_3 = \vec{e}_3$;

   the remaining parts are similar.

5. If ijk is an even permutation of {1,2,3}, then kij and jki are also even, and ikj, kji and jik are all odd; hence in this case we have all terms equal to 1 in (4.4) and -1 in (4.5). If ijk is odd, then each term in (4.4) equals 1 and each in (4.5) equals -1.

6. $\vec{A}\times\vec{B} = -13\vec{e}_1 + 7\vec{e}_2 + 11\vec{e}_3$.

7. • $A_i\delta_{i1} = A_1\delta_{11} + A_2\delta_{21} + A_3\delta_{31} = A_1$, and the result is similar for $j = 2,3$.

   • Equation (5.7) can be obtained in a similar way.

   • $\delta_{1j}\delta_{j1} = \delta_{11}\delta_{11} + \delta_{12}\delta_{21} + \delta_{13}\delta_{31} = 1$.

   • $\delta_{ij}\delta_{ji} = \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = 3$ (all other terms equal zero).

   • $\delta_{1j}\delta_{j1} = 1$, as already shown;

   $\delta_{11}\delta_{12} + \delta_{12}\delta_{22} + \delta_{13}\delta_{32} = 0$, and $\delta_{12} = 0$;

   the remaining cases are similar.

8. By Equation (5.5), with A replaced by B, B by C, k by i, i by j, j by k, we have

   $(\vec{B}\times\vec{C})_i = B_jC_k\varepsilon_{jki}$.

   Now apply Equation (4.4).

9. a. Apply Equations (5.1) and (5.11).

   b. Apply Equation (5.5) twice (with appropriate replacements for $\vec{A},\vec{B}$ and the indices).

10. Expand the determinant to obtain

   $3(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}) - (\delta_{km}\delta_{ji}\delta_{i\ell} - \delta_{jm}\delta_{ki}\delta_{i\ell}) - (\delta_{j\ell}\delta_{ki}\delta_{im} - \delta_{k\ell}\delta_{ji}\delta_{im})$.

   Now apply Equation (5.10).

11. Both sides of Equation (6.3) reduce to the values:

   a. 0;

   b. 1;

   c. 0;

   d. 0.

12. Sum on the left side to obtain

$$\varepsilon_{ijk}\varepsilon_{ijm} = \varepsilon_{12k}\varepsilon_{12m} + \varepsilon_{13k}\varepsilon_{13m} + \varepsilon_{23k}\varepsilon_{23m} + \varepsilon_{21k}\varepsilon_{21m} + \varepsilon_{31k}\varepsilon_{31m} + \varepsilon_{32k}\varepsilon_{32m}$$

   (all other terms equal zero).

   If $k = m$, then exactly two of these terms equal one, and if $k \neq m$, then all terms equal zero. Thus, the sum equals δ_{km}.

13. $$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\times\vec{B})_i(\vec{C}\times\vec{D})_i$$

$$= A_j B_k \varepsilon_{ijk} C_\ell D_m \varepsilon_{i\ell m}$$

$$= A_j B_k C_\ell D_m (\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell})$$

$$= (A_j C_\ell \delta_{j\ell})(B_k D_m \delta_{km}) - (A_j D_m \delta_{jm})(B_k C_\ell \delta_{k\ell})$$

$$= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}).$$

14. $\nabla^2\phi = \nabla\cdot\nabla\phi = \partial_i(\nabla\phi)_i = \partial_i(\partial\phi)_i = (\partial_i\partial_i)\phi$, from which it follows that $\nabla^2 = \partial_i\partial_i$.

15. $$(\nabla\times\phi\vec{u})_i = \varepsilon_{ijk}\partial_j(\phi\vec{u})_k$$

$$= \varepsilon_{ijk}\partial_j(\phi u_k)$$

$$= \varepsilon_{ijk}[(\phi\partial_j u_k) + (u_k\partial_j\phi)]$$

$$= \phi(\varepsilon_{ijk}\partial_j u_k) + \varepsilon_{ijk}u_k\partial_j\phi$$

$$= [\phi(\nabla\times\vec{u})]_i + [(\nabla\phi)\times\vec{u}]_i$$

The desired result follows, since the ith coordinates of both sides agree.

16. a. 0;

b. -1;

c. 1.

17. If any two indices are equal, then two rows of the determinant are identical, hence in this case both sides equal zero. Next,

$$\varepsilon_{1234} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1,$$

and an even permitation of 1234 corresponds to an even permutation of the rows, which yields +1 on both sides in this case. Similarly, an odd permutation of 1234 leads to -1 on both sides. Hence equality is obtained in all cases.

18. a.

$$\varepsilon_{\kappa\mu\nu\lambda}\varepsilon_{\alpha\beta\gamma\lambda} = \begin{vmatrix} \delta_{\kappa 1} & \delta_{\kappa 2} & \delta_{\kappa 3} & \delta_{\kappa 4} \\ \delta_{\mu 1} & \delta_{\mu 2} & \delta_{\mu 3} & \delta_{\mu 4} \\ \delta_{\nu 1} & \delta_{\nu 2} & \delta_{\nu 3} & \delta_{\nu 4} \\ \delta_{\lambda 1} & \delta_{\lambda 2} & \delta_{\lambda 3} & \delta_{\lambda 4} \end{vmatrix} \cdot \begin{vmatrix} \delta_{1\alpha} & \delta_{1\beta} & \delta_{1\gamma} & \delta_{1\lambda} \\ \delta_{2\alpha} & \delta_{2\beta} & \delta_{2\gamma} & \delta_{2\lambda} \\ \delta_{3\alpha} & \delta_{3\beta} & \delta_{3\gamma} & \delta_{3\lambda} \\ \delta_{4\alpha} & \delta_{4\beta} & \delta_{4\gamma} & \delta_{4\lambda} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{\kappa\alpha} & \delta_{\kappa\beta} & \delta_{\kappa\gamma} & \delta_{\kappa\lambda} \\ \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\gamma} & \delta_{\mu\lambda} \\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\gamma} & \delta_{\nu\lambda} \\ \delta_{\lambda\alpha} & \delta_{\lambda\beta} & \delta_{\lambda\gamma} & 4 \end{vmatrix}$$

$$= 4\begin{vmatrix} \delta_{\kappa\alpha} & \delta_{\kappa\beta} & \delta_{\kappa\gamma} \\ \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\gamma} \\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\gamma} \end{vmatrix} - \delta_{\lambda\gamma}\begin{vmatrix} \delta_{\kappa\alpha} & \delta_{\kappa\beta} & \delta_{\kappa\lambda} \\ \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\lambda} \\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\lambda} \end{vmatrix}$$

$$+ \delta_{\lambda\beta}\begin{vmatrix} \delta_{\kappa\alpha} & \delta_{\kappa\gamma} & \delta_{\kappa\lambda} \\ \delta_{\mu\alpha} & \delta_{\mu\gamma} & \delta_{\mu\lambda} \\ \delta_{\nu\alpha} & \delta_{\nu\gamma} & \delta_{\nu\lambda} \end{vmatrix} - \delta_{\lambda\alpha}\begin{vmatrix} \delta_{\kappa\beta} & \delta_{\kappa\gamma} & \delta_{\kappa\lambda} \\ \delta_{\mu\beta} & \delta_{\mu\gamma} & \delta_{\mu\lambda} \\ \delta_{\nu\beta} & \delta_{\nu\gamma} & \delta_{\nu\lambda} \end{vmatrix}.$$

The second term expands to:

$$-[\delta_{\kappa\alpha}\delta_{\mu\beta}\delta_{\nu\gamma} + \delta_{\kappa\beta}\delta_{\mu\lambda}\delta_{\gamma\alpha} + \delta_{\kappa\lambda}\delta_{\mu\alpha}\delta_{\nu\beta}$$

$$- \delta_{\kappa\alpha}\delta_{\mu\lambda}\delta_{\nu\beta} - \delta_{\kappa\beta}\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\kappa\lambda}\delta_{\mu\beta}\delta_{\nu\alpha}]$$

$$= -\begin{vmatrix} \delta_{\kappa\alpha} & \delta_{\kappa\beta} & \delta_{\kappa\gamma} \\ \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\gamma} \\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\gamma} \end{vmatrix}$$

Thus, the first two terms of the expansion reduce to:

$$3\begin{vmatrix}\delta_{\kappa\alpha} & \delta_{\kappa\beta} & \delta_{\kappa\gamma}\\ \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\gamma}\\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\gamma}\end{vmatrix}.$$

The last two terms also reduce to the negative of the same determinant, and the indicated subtraction yields the desired result.

b. Use the result of part a, taking care to sum correctly, to obtain:

$$\begin{aligned}\varepsilon_{\kappa\mu\gamma\lambda}\varepsilon_{\alpha\beta\gamma\lambda} &= (4\delta_{\kappa\alpha}\delta_{\mu\beta} + \delta_{\kappa\beta}\delta_{\mu\alpha} + \delta_{\mu\alpha}\delta_{\kappa\beta})\\ &\quad - (\delta_{\kappa\alpha}\delta_{\mu\beta} + 4\delta_{\kappa\beta}\delta_{\mu\alpha} + \delta_{\mu\beta}\delta_{\kappa\alpha})\\ &= 2(\delta_{\kappa\alpha}\delta_{\mu\beta} - \delta_{\kappa\beta}\delta_{\mu\alpha}).\end{aligned}$$

c. Use the result of part b:

$$\varepsilon_{\kappa\beta\gamma\lambda}\varepsilon_{\alpha\beta\gamma\lambda} = 2(4\delta_{\kappa\alpha} - \delta_{\kappa\alpha}) = 6\delta_{\kappa\alpha}.$$

d. Use the result of part c:

$$\varepsilon_{\kappa\beta\gamma\lambda}\varepsilon_{\alpha\beta\gamma\lambda} = 6\cdot 4 = 24.$$

19. $\vec{A}\cdot\vec{D} = A_\tau D_\tau$

$$= A_\tau A_\alpha B_\beta C_\gamma \varepsilon_{\alpha\beta\gamma\tau}$$

This form can be recognized immediately as the expansion of the determinant

$$\begin{vmatrix}A_1 & A_2 & A_3 & A_4\\ A_1 & A_2 & A_3 & A_4\\ B_1 & B_2 & B_3 & B_4\\ C_1 & C_2 & C_3 & C_4\end{vmatrix}$$

along the second row. The determinant is zero, since two rows are equal.

12. ANSWERS TO MODEL EXAM

1. a. $\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i(\vec{B}\times\vec{C})_i = A_i(B_jC_k\varepsilon_{ijk})$

$$= (A_iB_j\varepsilon_{ijk})C_k = (\vec{A}\times\vec{B})_kC_k = \vec{A}\times\vec{B}\cdot\vec{C}.$$

b. $[(\vec{A}\times\vec{B})\times\vec{C}]_k = (\vec{A}\times\vec{B})_iC_j\varepsilon_{ijk}$

$$\begin{aligned}&= (A_\ell B_m\varepsilon_{i\ell m})C_j\varepsilon_{ijk}\\ &= A_\ell B_mC_j\varepsilon_{ijk}\varepsilon_{i\ell m}\\ &= A_\ell B_mC_j(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell})\\ &= (A_\ell C_j\delta_{j\ell})B_m\delta_{km} - (B_mC_j\delta_{jm})A_\ell\delta_{k\ell}\\ &= (\vec{A}\cdot\vec{C})B_k - (\vec{B}\cdot\vec{C})A_k\\ &= [(\vec{A}\cdot\vec{C})B - (\vec{B}\cdot\vec{C})A]_k\end{aligned}$$

Thus, $(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$.

2. a. $(\nabla\times\nabla\phi)_i = \varepsilon_{ijk}\partial_j(\nabla\phi)_k = \varepsilon_{ijk}\partial_j\partial_k\phi$. Since $\varepsilon_{ijk} = -\varepsilon_{ikj}$, and since $\partial_j\partial_k = \partial_k\partial_j$, we see that for every nonzero term in the sum, the negative of that term also appears. Hence the sum is zero.

b. $[\nabla\times(\vec{u}\times\vec{v})]_i = \varepsilon_{ijk}\partial_j(\varepsilon_{k\ell m}u_\ell v_m)$

$$\begin{aligned}&= \varepsilon_{kij}\varepsilon_{k\ell m}\partial_j(u_\ell v_m)\\ &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})(u_\ell\partial_jv_m + v_m\partial_ju_\ell)\\ &= \delta_{i\ell}\delta_{jm}u_\ell\partial_jv_m - \delta_{im}\delta_{j\ell}u_\ell\partial_jv_m\\ &\quad + \delta_{i\ell}\delta_{jm}v_m\partial_ju_\ell - \delta_{im}\delta_{j\ell}v_m\partial_ju_\ell\\ &= (\nabla\cdot\vec{v})u_i - (\nabla\cdot\vec{u})v_i + (\vec{v}\cdot\nabla)u_i - (\vec{u}\cdot\nabla)v_i,\end{aligned}$$

after the indicated sums are performed. Since corresponding coordinates agree, the two sides of the desired equation must be equal.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

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STUDENT FORM 2

Unit Questionnaire

Name____________________ Unit No.____________ Date____________

Institution____________________ Course No.____________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain____________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain____________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)