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ABSTRACT

One of the great strengths of mathematics is viewed as the fact that apparently diverse real-world questions translate into that same mathematical question. It is felt that studying a mathematical problem can often bring about a tool of surprisingly diverse usability. The module is geared to help users know how to use graph theory to model simple problems, and to support elementary understanding of vertex coloring problems for graphs. The module includes exercises. Answers to selected problems are provided. (MP)

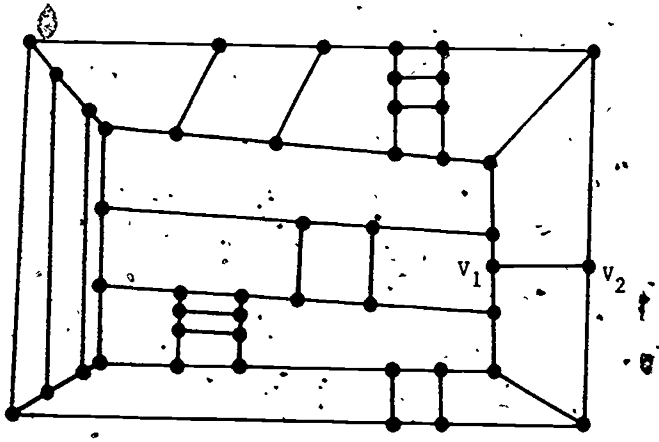
umap

UNIT 442

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

APPLICATIONS OF VERTEX COLORING PROBLEMS FOR GRAPHS

by Joseph Malkevitch



APPLICATIONS OF GRAPH THEORY

IN MODEL CONSTRUCTION

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APPLICATIONS OF VERTEX COLORING PROBLEMS FOR GRAPHS

by

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Intermodular Description Sheet: UMAP Unit 442

Title: APPLICATIONS OF VERTEX COLORING PROBLEMS FOR GRAPHS

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Review Stage/Date: III 10/79

Classification: APPL GRAPH THEORY/MODEL CONSTRUCTION

Prerequisite Skills: None

Output Skills:
1. Know how to use graph theory to model simple problems.
2. Elementary understanding of vertex coloring problems for graphs.

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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APPLICATIONS OF VERTEX COLORING PROBLEMS FOR GRAPHS

1. INTRODUCTION

One of the great strengths of mathematics lies in the fact that apparently diverse real world questions translate into the same mathematical question. Hence, by studying a mathematical problem, one often has a tool of surprisingly diverse usability.

2. SOME EXAMPLES

EXAMPLE 1

The registrar of a college is planning to schedule the final examinations for the summer session of the college. The courses offered meet at the following times:

Section:	A	M, W, F,	9	
	AB	M, W, 9 F,	10	(i.e., Monday and Wednesday at 9, Friday at 10)
	A*	T, TH,	9	
	$\overline{B}$	M, T, TH,	10	
	B	W, F,	10	
	BC	M, W, 10 F,	11	
	$\overline{C}$	M, T, TH,	11	
	C*	T, TH,	11	
	D	M, T, W,	12	
	X	M, 9,10 W,	9,10	
	Y	T, 9,10 TH,	9,10	

On any final examination day, exams are given from:
8 A.M. - 10 A.M. 12 Noon - 2 P.M. 3 P.M. - 5 P.M.

What is the minimum number of days required to complete the final examinations so no student has an examination conflict?

EXAMPLE 2

The pet shop of a new department store decides to stock tropical fish. It is planned to purchase tanks in which to store the fish. Unfortunately, not all the fish are compatible; that is, if placed in the same tank, some types of fish may fight with one another. The shop plans to stock the twelve most popular varieties of fish. The table below shows the compatibility of the types of fish. An X in the table means the fish types in the row and column with the X are incompatible.

TABLE I

Species	1	2	3	4	5	6	7	8	9	10	11	12
1		X	X			X			X		X	
2	X		X	X	X			X		X		X
3	X	X		X			X					X
4		X	X		X		X	X				
5		X		X		X			X	X		
6	X				X		X					X
7			X	X		X		X			X	
8		X		X			X		X			
9	X				X			X		X		X
10	X				X				X		X	
11	X						X			X		X
12		X	X			X			X		X	

Note that a blank entry corresponds to a pair of compatible species.

Question

What is the minimum number of tanks which must be purchased to store the fish such that the fish in any one of the tanks are compatible?

3. CONSTRUCTION OF A GRAPH MODEL

Although these two problems have a very different verbal flavor, their structure leads to the same mathematics. The essential element, common to both problems, is "compatibility." The second example is concerned with the compatibility of fish and the first example is concerned with the compatibility of the times at which courses can have their final examinations scheduled. Extracting, by abstraction the common thread of both problems, we might consider the following generic problem:

Generic Problem

Given a collection of objects $A_1, \ldots, A_k$ some of which may not be compatible. What is the minimum number of groups that the $A_1, \ldots, A_k$ can be put into so that members of the same group are compatible? (In Example 2, $A_1, \ldots, A_k$ are fish types and k is 12, while in Example 1, $A_1, \ldots, A_k$ are courses and k is 11.)

To model this generic problem mathematically, we make use of a *graph*. A *graph* is a collection of points called *vertices* joined by a collection of straight or curved line segments called *edges*, where at most one edge joins a pair of distinct points.* Some examples of graphs are shown in Figure 1, where the dark dots are vertices. The other crossings of the edges arise by "accident," because of the way the graph has been drawn on the flat paper. (Figure 1(c) has several "accidental" crossings.)

To construct a graph to assist in solving Example 2, we proceed as follows:

Represent each fish type by a vertex of a graph. Join two vertices together with an edge if the fish they represent will fight (i.e., are incompatible). The resulting graph (Figure 2) shows all of the compatibility information given in the original problem. Note that someone else modeling this problem might draw a seemingly different graph (Figure 3). However, it is easy to check that the graph in Figure 3 has the same structure as the one in Figure 2, in the sense that the labeling of the graph in Figure 3 will give the same incidence relations between the vertices and edges as in Figure 2. The fact that the graphs in Figure 2 and Figure 3 have the same structure (are *isomorphic*) can be seen by noting that the vertex labeled i in Figure 2 corresponds to the vertex labeled i' in Figure 3. (Edges in Figure 2 will correspond to edges in Figure 3.) The graph modeling Example 1 in the generic problem is constructed in the same way.

*Sometimes a graph is defined so as to allow  (loops) and  (parallel edges), but there is no advantage to us in doing this.

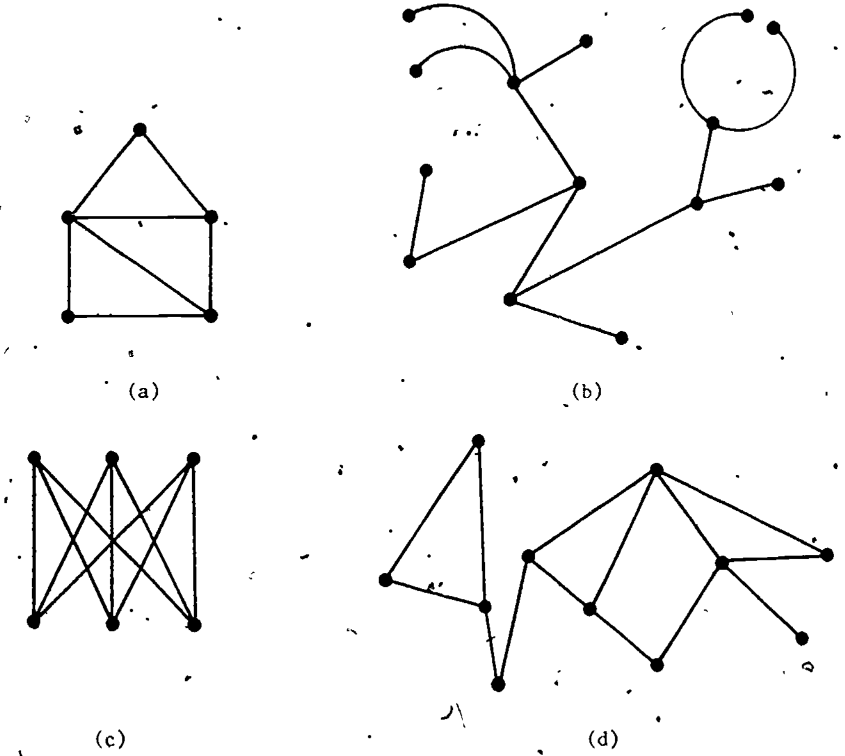

Figure 1.

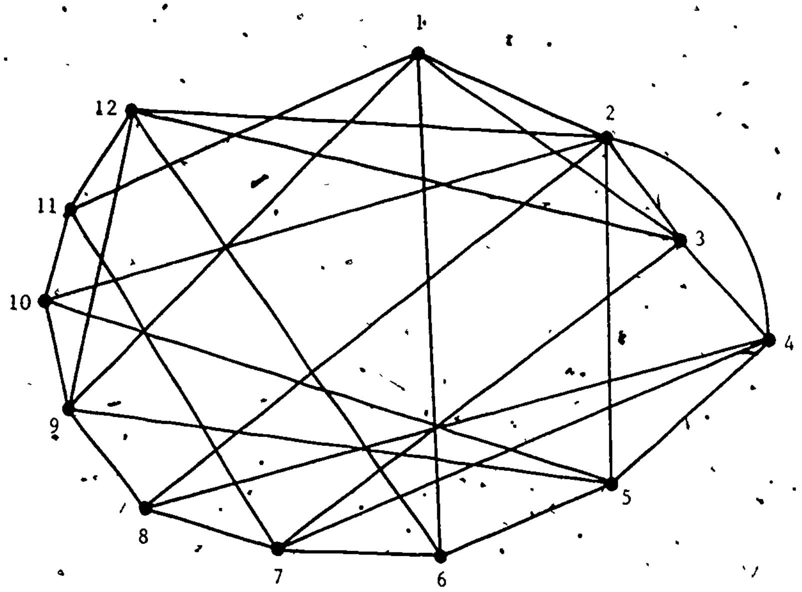

Figure 2.

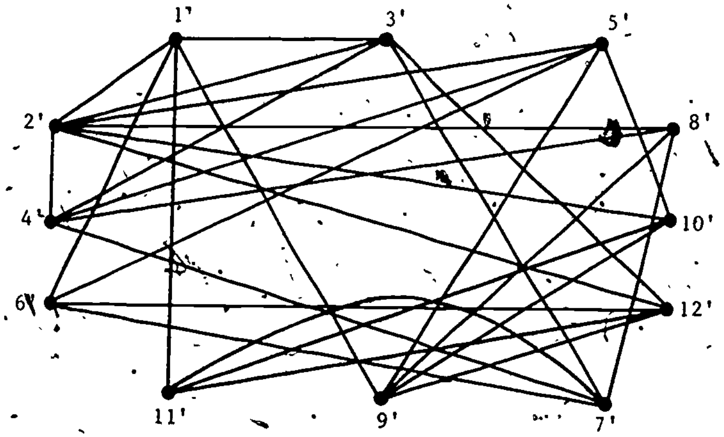

Figure 3.

Having represented the information in such problems by graphs, we are now faced with determining what problem in graph-theoretic language, will solve the original problem. This can easily be done. In Example 2, the fact that two types of fish, A_1 and A_2, are incompatible, and must be placed in different tanks is reflected by the fact that the vertices representing these fish are joined by an edge in the graph used as the model. We wish to assign a minimum number of labels to the vertices of the graph, the label being thought of as the tank the fish will be placed in, so that if two vertices are joined by an edge the vertices get different labels. (Thus, incompatible fish will be assigned to different tanks.) In graph theory it is customary to refer to labels assigned to vertices as *colors*. Thus, we have the *vertex coloring problem:*

> Given a graph G, the determination of the minimum number, $\chi(G)$, of colors (labels) necessary to color (label) the vertices of the graph so that no two vertices which are joined by an edge get the same label is called the *vertex coloring problem for G*. $\chi(G)$ is called the *chromatic number of G*.

Note that if G has m vertices, then the vertices of G can always be colored with m colors, so that no two vertices joined by an edge get the same color. However, $\chi(G)$ is usually less than m.

Figure 4 shows a graph G, labeled with four colors (denoted A, B, C, D) so that no two vertices joined by an edge get the same color. Thus, for this graph $\chi(G) \leq 4$.

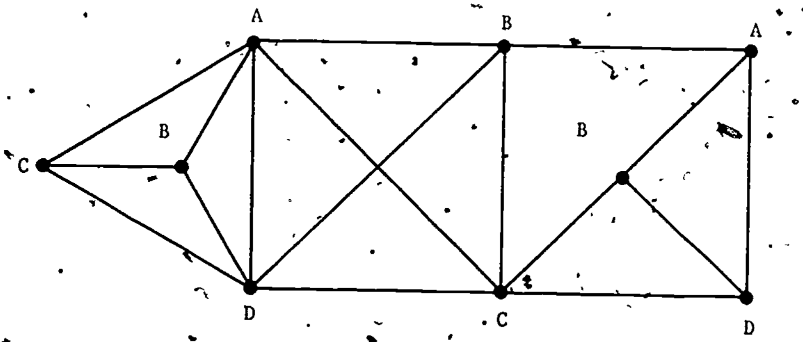

Figure 4.

It is not possible to color the vertices of this graph with three colors, as can be seen by examining the four leftmost vertices in Figure 4. Hence, for this graph $\chi(G) = 4$.

4. DETERMINING $\chi(G)$

How can $\chi(G)$ be determined for a given graph, for example, the graph in Figure 2? For any particular graph, such as the one in Figure 2, trial and error or systematic ennumeration is a feasible method. For graphs with relatively large numbers of vertices or for the determination of $\chi(G)$ for many graphs, trial and error is unattractive.

It is here that mathematical ideas perhaps can be of help. Unfortunately, no efficient algorithm (mechanical technique) is known for finding $\chi(G)$ for an arbitrary graph. In fact, it is known that the problem of determining $\chi(G)$ for a graph belongs to the class of problems called "NP - complete." This class of problems has the property that if one member of the class could be shown to require an exponential amount of time to solve the problem, as a function of the problem's size, then all the problems in the class would require exponential time. The current feeling is that all of these problems will require exponential amounts of time. The class "NP - complete" contains many of the most interesting applications oriented problems in graph theory and operations research. The breaking of this "log jam" would be a major contribution to the theory of graphs and the theory of algorithms.

Despite the above, many facts about $\chi(G)$ are known. For example:

Theorem. If k is the maximum number of edges at any vertex of G, then $\chi(G) \leq k+1$.

In what follows, it is convenient to use the following definition. If v is a vertex of graph G, the number of edges at v is called the *valence* of v. For example, the graph in Figure 5 has vertices of valence 0 and 1.

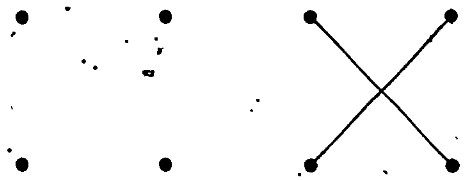

Figure 5.

Applying the theorem above to this graph, we would conclude that its vertices can be colored with 2 or fewer colors.

To prove the theorem, we proceed as follows. Let m denote the number of vertices of the graph in question. If m = 1, then the graph consists of one vertex and no edges. It has a maximum valence 0, and can be colored with one color. Thus $\chi(G) \leq 1$. Hence, the theorem holds for graphs with m = 1.

Now assume that any graph with m vertices and maximum valence k is colorable with $\leq (k+1)$ colors. Let G be any graph with (m+1) vertices and maximal valence k. Let v be any vertex of G. Form G' by deleting from G, v and the edges attached to v. Since G' has m vertices and maximal valence k, by the assumption above, G' can be colored with k+1 or fewer colors. Since v (in G) was attached to at most k vertices, even if all these k vertices are colored differently, there would still be a $(k+1)^{st}$ color to color v differently from any vertices to which it may be joined by an edge. Hence, G can be colored with k+1 or fewer colors. We may now conclude that the theorem holds for a graph with any number of vertices. This argument illustrates the technique of theorem proving called *mathematical induction*. Although the proof is much more complicated the theorem above can be improved to:

Theorem (R.L. Brooks, 1941). If G has maximum valence k ($k \geq 3$) then $\chi(G) \leq k$ unless G is the graph with (k+1) vertices where each vertex is joined to every other one with an edge, in which case $\chi(G) = k+1$.

It might seem that if a graph H has a large number of vertices, such as the one in Figure 6, then it might have a large chromatic number. However, H in Figure 6 requires no more than 3 colors to color its vertices, by Brook's theorem. In fact, $\chi(H) = 3$. Note that if vertices v_1 and v_2, and their connecting edge are deleted from H, to form H', then $\chi(H') = 2$. On the other hand, Brook's

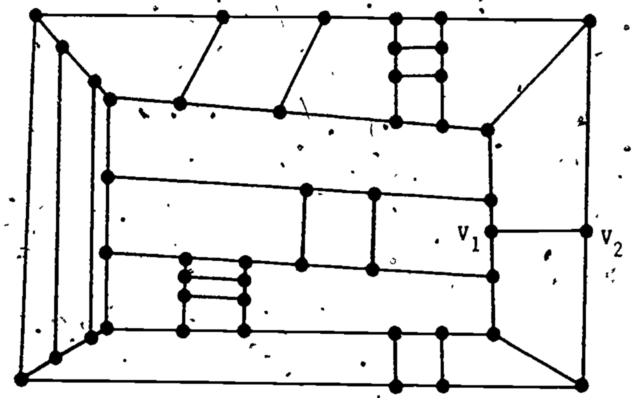

Figure 6.

theorem often gives bad estimates for $\chi(G)$. For the graph in Figure 7, applying Brook's theorem gives that $\chi(G) \leq 10$. In fact, $\chi(G) = 3$.

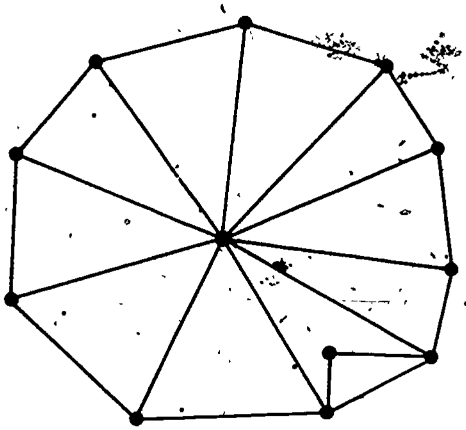

Figure 7.

Another approach to determining $\chi(G)$ is to determine a lower bound for $\chi(G)$. For example, if G contains a subgraph consisting of m vertices, each joined to every other, then $\chi(G) \geq m$. The graph in Figure 8 arises in attempting to solve Example 1 (Exam Scheduling). Since it has 6 vertices, each joined to every other by an edge, χ for this graph is ≥ 6. In fact, this graph has chromatic number 6.

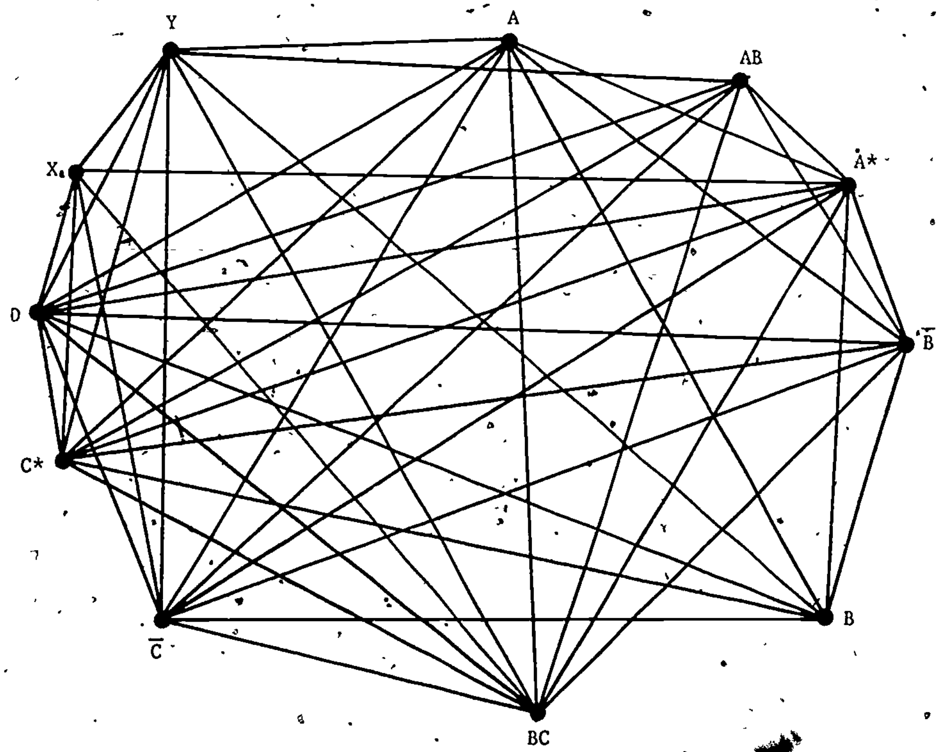

Figure 8.

Occasionally, it may be the case that a graph can be drawn on a piece of paper without accidental crossings. If this is true, then in fact Brook's theorem can be vastly improved. The theorem of K. Appel and W. Haken shows that:

Theorem. If graph G can be drawn on a sheet of paper without accidental crossings then $\chi(G) \leq 4$.

The conjecture which this theorem settles, was originally proposed in 1852 by Francis Gutherie, and was only proven in 1976. Unfortunately, from the practical point of view, graphs which arise in the "real world" usually have accidental crossings no matter how they are drawn on a sheet of paper.

5. CONCLUSION

This module has merely touched the surface of the difficult problem of determining the chromatic number of a graph. Additional results, and historical information about coloring problems can be found in the references. What we hope has been demonstrated here is that mathematical ideas help greatly in unifying our thinking about real world problems, and can greatly assist in solving these problems.

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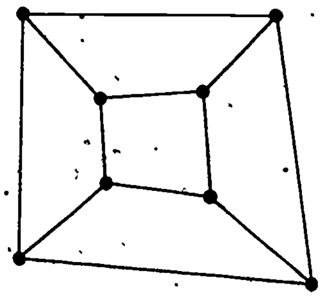
Saaty, T.L. and P. Kainen, "The Four Color Problem," McGraw Hill, New York, 1977.

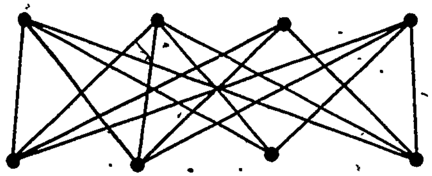
7. EXERCISES

1. For the graphs below determine the number of vertices and edges:

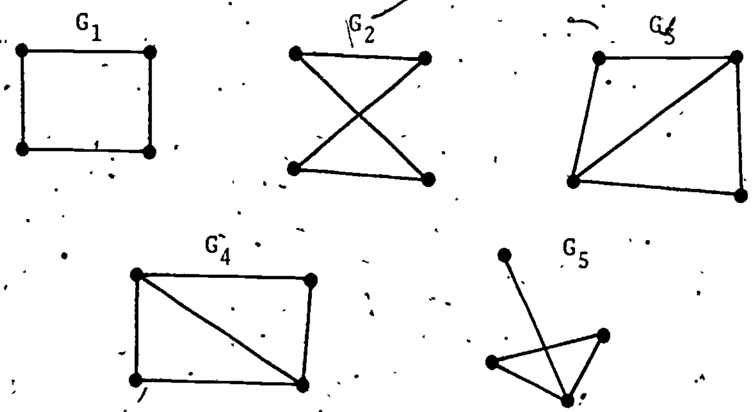
(a)



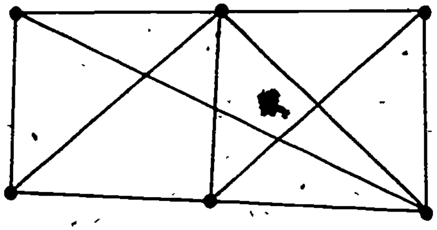
(b)



2. Determine which pairs of graphs below are isomorphic:

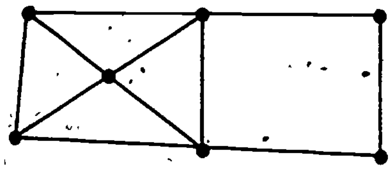


3. Draw a graph isomorphic to G below but without "accidental crossings."

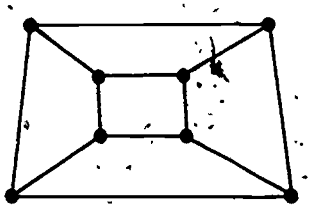


4. Use trial and error to compute $\chi(G)$ for each graph below and color the graph's vertices with $\chi(G)$ colors.

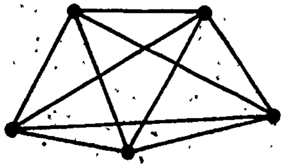
(a)



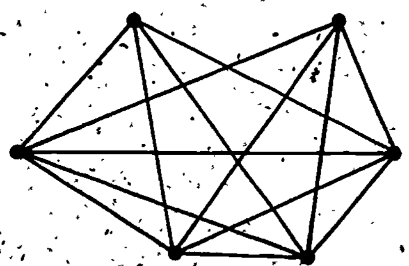
(b)



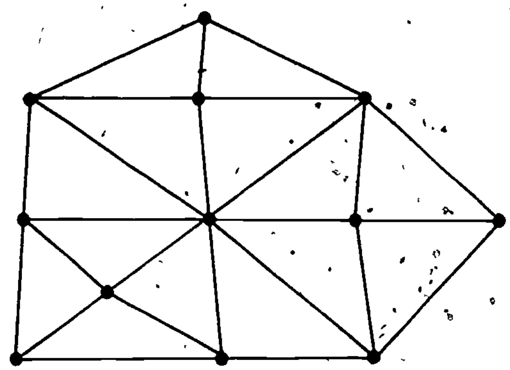
(c)



(d)



(e)



5. For each problem below, decide on an appropriate graph to draw so that the problem can be solved as a coloring problem for the graph.

(i) The committees of the legislature of State Y have members in common. The chart below shows committees that have at least one member in common.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(1) Legislative Affairs	-	X			X		X	X
(2) Education	X	-	X	X		X		
(3) Finance		X	-		X	X	X	
(4) Urban Affairs		X		-	X		X	X
(5) Rules	X		X	X	-		X	
(6) Policy		X	X			-	X	X
(7) Health	X		X	X	X	X	-	
(8) Higher Education	X			X		X		-

What is the minimum number of time slots in which meetings can be held so no conflicts result?

(ii) The curator of a zoo is designing a new area where animals will not have cages but roam free on "islands" separated from viewers by moats. Unfortunately, some of the animals will eat one another if they are on the same island. An X below indicates which animals will not eat one another. How few "islands" will accommodate all the animals?

				Species				
	1	2	3	4	5	6	7	8
Species 1	–	X	X			X		X
2	X	–		X	X		X	
3	X		–	X	X			
4		X	X	–		X	X	
5		X	X		–	X		X
6	X			X	X	–		
7		X		X			–	
8	X				X			

(iii) A group of scouts are driving to Lion Mountain in vans. The scoutmaster wishes to avoid "fights" in vans. An X in the chart below indicates scouts who often "fight." What is the minimum number of vans necessary to accommodate the scouts and avoid fights? If each van will accommodate only 4 scouts, how many vans are needed?

	1	2	3	4	5	6	7	8	9	10	11	12
1	–	X						X		X	X	X
2		–	X	X	X	X	X					
3		X	–				X	X	X	X		
4		X		–	X						X	X
5		X		X	–	X		X		X	X	
6		X			X	–			X			X
7		X	X				–			X	X	
8	X		X		X			–	X	X	X	X
9			X			X		X	–		X	
10	X		X		X		X	X		–		X
11	X			X	X		X	X	X		–	X
12	X			X		X		X		X	X	–

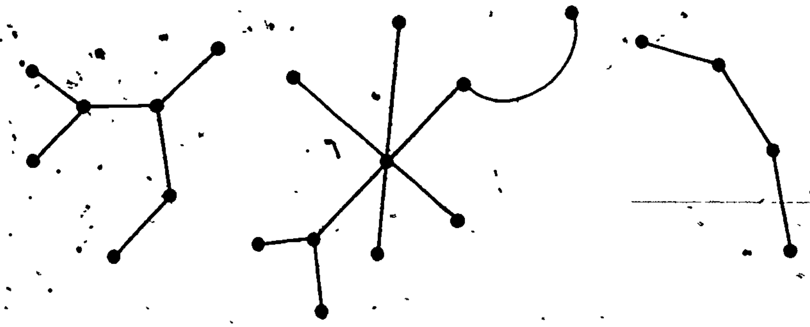
6. Formulate three problems which might occur in the real world, which can be modeled by a graph and which involve coloring the vertices of the graph to solve the original problem.

7. Draw a graph G which has no accidental crossings when drawn in the plane, and for which $\chi(G) = 4$.

8. Construct a family of graphs F_n, for which the graph F_n has n vertices and $\chi(F_n) = n$.

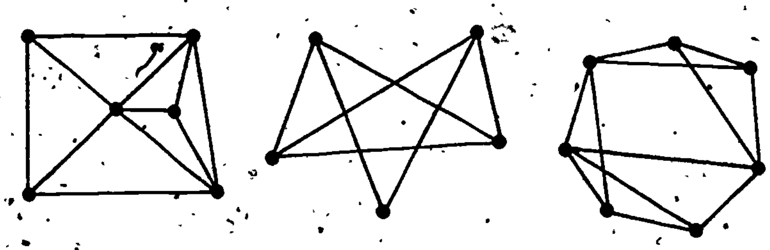
9. Construct an infinite family of graphs for which the chromatic number is 2.

10. Graphs such as the ones shown in the figure below are called trees:



(A tree is a graph which is in one piece and has no circuit.) Show that $\chi(\text{tree}) = 2$.

11. For each graph below:
   (a) Determine the valence of each vertex in the graph.
   (b) Determine the chromatic number of the graph.
   (c) Compare the actual value of the chromatic number with the estimated lower and upper bounds discussed in the text.



G_1 G_2 G_3

8. ANSWERS TO SOME EXERCISES

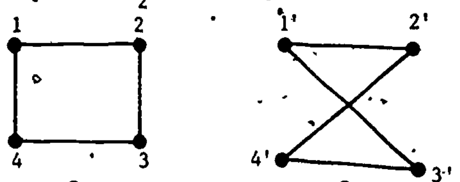
1. (a) 8 vertices (= dark dots)

   12 edges (= line segments).

   (b) 8 vertices (Note that crossings other than at dark dots are not vertices.)

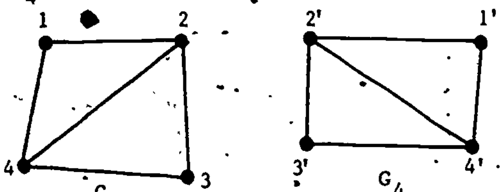
   16 edges.

2. G_1 is isomorphic to G_2 as seen by labeling below:



G_1 G_2

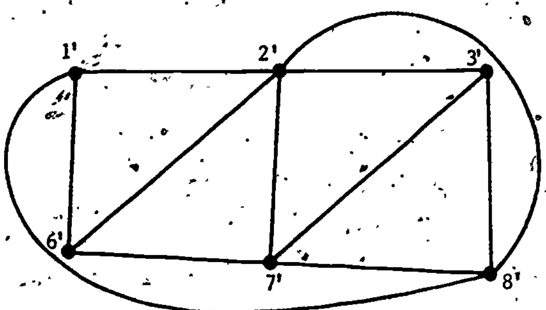
G_3 and G_4 are isomorphic as seen by the labeling below:



G_3 G_4

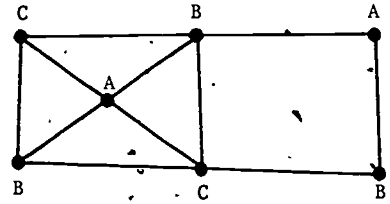
G_5 is not isomorphic to any of the others. G_5 has 4 edges. For two graphs to be isomorphic they must have the same number of edges (and vertices). Hence G_5 is not isomorphic to G_3 and G_4. Since G_5 has a vertex of valence 1, G_5 cannot be isomorphic to G_1 or G_2. If two graphs are isomorphic the number of vertices of valence 1 in each must be the same.

3.

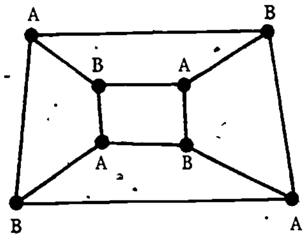


Note: The labeling shows the graphs are isomorphic.

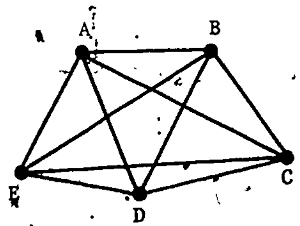
4. (a) $\chi(G) = 3$



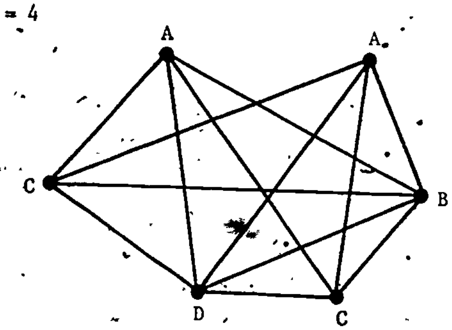
(b) $\chi(G) = 2$



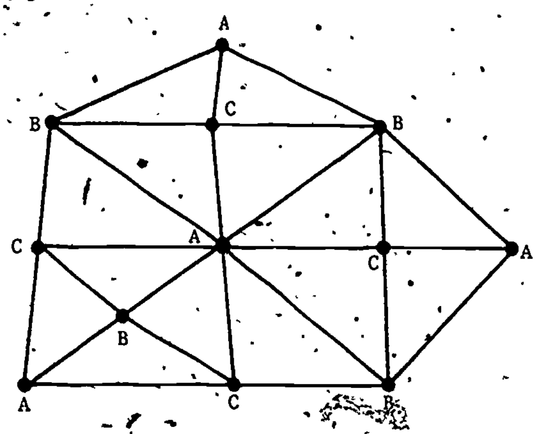
(c) $\chi(G) = 5$



(d) $\chi(G) = 4$



(e) $\chi(G) = 3$



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STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Name________________________ Unit No.____________ Date____________

Institution________________________ Course No.________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)