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AUTHOR Whitley, W. Thurmon
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ABSTRACT

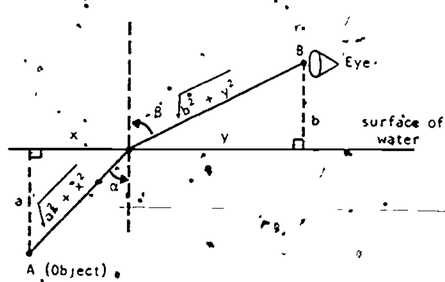
The emphasis is on so-called "best solution" problems to questions that frequently arise in practical situations, such as finding an answer for the least amount of time, greatest volume, least amount of work, maximum profit, and minimum cost. One of this module's purposes is to help users become acquainted with the types of calculations necessary to solve such real-world problems. The material looks at: A) Minimum Cost Problem in Industry; B) A Maximum Profit Problem; C) Snell's Law-Light Refraction; D) Surface Area of a Bee's Cell, and E) Arterial Branching. Exercises and a model exam are included, with an answer key to both provided. (MP)

UMAP MODULE 341

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

Five Applications of Max-Min Theory from Calculus

by W. Thurmon Whitley



Applications of Max-Min Theory

Birkhäuser Boston Inc.
380 Green Street
Cambridge, MA 02139

FIVE APPLICATIONS OF MAX-MIN THEORY FROM CALCULUS

by

W. Thurman Whitley
University of New Haven
West Haven, Connecticut 06516

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Intermodular Description Sheet: *UMAP Unit 341*

Title: FIVE APPLICATIONS OF MAX-MIN THEORY FROM CALCULUS

Author: W. Thurmon Whitley
University of New Haven
West Haven, Connecticut 06516

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Classification: APPL MAX-MIN THEORY

References: See Section 10 of text.

Prerequisite Skills:

1. Use of first and second derivative tests to optimize functions of a single variable.
2. Differentiation of and manipulation with trigonometric functions.

Output Skills:

1. Further knowledge of differentiation and optimization skills.
2. Apply calculus techniques to obtain information from mathematical models of real world situations.
3. Be able to construct mathematical models of simple physical situations.
4. Understand assumptions and refinements needed in the construction of mathematical models.

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FIVE APPLICATIONS OF MAX-MIN THEORY FROM CALCULUS

1. INTRODUCTION

A question that frequently arises in practical situations is, "What is the best solution to this problem?" For example, a scientist or a business analyst might require such information as least amount of time, greatest volume, least amount of work, maximum profit, minimum cost. (Can you think of other examples of "best solutions"?)

In many cases, answers to such "best solution" problems can be obtained, or at least approximated, by using derivatives to maximize or minimize single-variable functions. In this unit we consider several "real-world" problems in which "best solutions" are obtained in this way.

As we proceed through these examples you will frequently be asked to carry out manipulations on your own. In some cases these steps will be routine; in others they will involve more tedious calculations. In real-world situations, people must often carry out such tasks in order to solve a problem. Thus, one purpose of the exercises in this unit is to help you become acquainted with the type of calculations necessary to solve realistic problems.

2. A MINIMUM COST PROBLEM IN INDUSTRY

2.1 Outline of the Problem

Let us first consider a typical problem faced by manufacturers of commercial products. Suppose that a certain manufacturer wishes to minimize the total cost of producing a particular item. We consider two of the types of costs that are involved; the cost of actually manufacturing the product, and the cost of storing it. (In so doing we ignore other costs, such as marketing, shipping, and management.)

In our example we assume that the product units are manufactured in batches of equal size at equally spaced time intervals throughout the year, and that demand for the product is at a known constant rate. We further assume that the total number of units produced in the year is predetermined to be equal to the total demand. We wish to find the number of batches the company should produce annually to minimize the total production and storage cost.

While we cannot expect our assumptions to be satisfied exactly in real situations, the agreement may be close enough for our results to provide the manufacturer with a useful approximation.

2.2 Some Notation

We shall use the following notation:

X = the number of batches of the product produced annually;

k = the cost in dollars of storing one unit of the product for one year;

F = the fixed cost in dollars of setting up the factory to manufacture each single batch (usually includes insurance, cost of equipment, etc.);

v = the cost in dollars of manufacturing one unit of the product (called the variable cost);

T = the total number of units produced annually.

In our problem we assume that k, F, v and T are known constants.

2.3 Derivation of the Cost Equation

Let us first consider the manufacturing cost. Since there are T units produced annually in X batches of equal size, there are T/X units in each batch. Thus, the total production cost M(X) for X batches per year is

$$(2.1) \qquad M(X) = (F + \frac{vT}{X})X \text{ dollars.}$$

Next, we consider storage costs. We assume that each batch of T/X units is put into storage, with the supply depleted at a constant rate down to zero when the next batch is then completed and stored. Thus, the average number of units in storage at any given time is approximately (1/2)(T/X) = T/2X. Since the cost of storing one unit for a year is k dollars, the total annual storage cost is

$$(2.2) \qquad S(X) = \frac{kT}{2X} \text{ dollars.}$$

Combining Equations (2.1) and (2.2), we obtain the total cost of production and storage (in dollars):

$$(2.3) \qquad C(X) = (F + \frac{vT}{X})X + \frac{kT}{2X}$$

2.4 The Minimum Cost

At first glance the variable X in Equation (2.3) seems to be a discrete variable, since it represents the number of batches produced per year, and this representation suggests integer values. However, X may also assume rational values; for example, a production rate of 12.5 batches per year would be interpreted to mean a rate of 25 batches every two years. Since any real number can be approximated by rational numbers, we go one step further and regard X as a continuous real variable. This assumption permits us to apply calculus techniques to find the minimum cost.

Differentiating Equation (2.3), we obtain

$$(2.4) \qquad C'(X) = F - \frac{kT}{2X^2}$$

Setting $C'(X)$ equal to zero and solving for X we obtain as the only positive critical value

(2.5) $X_0 = \sqrt{kT/2F}$.

That the value of X_0 in (2.5) yields a minimum for $C(X)$ can be shown by the second derivative test. (See Exercise 1.)

Using (2.5) we find $C(X_0)$ to obtain the minimum cost:

(2.6) $C_{min} = vT + \sqrt{2kTF}$.

2.5 Some Final Observations

As noted in Section 2.1, we cannot expect real manufacturing situations to be described by our model exactly. In those situations in which agreement is good however, Equation (2.5) provides a reasonable approximation for the number of batches per year that should be made in order to minimize production and storage costs.

In those situations for which our assumptions are not reasonably accurate, adjustments must be made in the model. For example, if demand is not constant, then the expression T/2X may be inappropriate for the number of units in storage at any given time, and we would need a different expression, depending on the demand curve assumed.

Note that the result $X_0 = \sqrt{kT/2F}$ for the critical value of X is reasonable. As the storage cost k increases, so should the critical number of batches, as it does here since $\sqrt{k}$ appears in the numerator. Similarly, as production T rises, so should the number of batches, and this also agrees with the model. Finally, as the fixed cost per batch F goes up, the number of batches should go down; this behavior is consistent with $\sqrt{F}$ in the denominator.

2.6 Exercises

1a. Use the second derivative test to show that the value of X_0 in (2.5) yields a minimum for C(X).

b. Find $C(X_0)$ using the expression in (2.5) to show that C_{min} is as given in (2.6).

2. In Section 2.4 we noted that the only positive critical point of C(X) was the one given by (2.5).

a. Find the other critical point of C(X).

b. Explain briefly why the other critical point is disregarded in the solution of the problem.

c. Why is 0 not a critical point of C(X) even though C'(0) does not exist?

3. Assume that the average annual storage cost per unit is $2.00, that 10,000 units are to be produced per year, that the fixed cost per batch is $100.00, and that the variable cost is $3.00 per unit. Under the assumptions of Section 2.1, find:

a. the number of batches that will minimize the annual (production/storage) cost.

b. the minimum annual cost.

4. Using the values given in Exercise 3, sketch the graph of C(X), as given by Equation (2.3), for positive values of X.

5. Let us assume that all conditions are the same as those in Section 2.1, except that now X is held constant and T allowed to vary. Then the right side of Equation (2.3) becomes a function of T, rather than X; we denote this function by another symbol, say B instead of C, for correct use of functional notation:

$$B(T) = (F + vT/X)X + kT/2X .$$

Find the value of T that will minimize B(T).

3. A MAXIMUM PROFIT PROBLEM*

3.1 The Selling Price

Several years ago, the Boeing Aircraft Company was faced with the problem of determining the selling price for a new model jet airliner. The basic problem was to find the price per aircraft which would maximize the company's profit.

In this particular case, Boeing had one competitor, which had a similar plane. It was understood that the companies would charge the same price, since any price adjustment by one company would automatically be met by the other. Thus, the price would not affect the relative shares of the market. It could, however, have a significant impact on the total size of the market.

3.2 Factors To Be Considered

The following quantities were considered:

p = the selling price per airliner (in millions of dollars);

$N(p)$ = the total number of airliners that would be sold at price p by Boeing and its competitor;

$C(X)$ = the total cost (in millions of dollars) to Boeing of manufacturing X airliners;

h = the fraction of the market to be won by Boeing ($0 \le h \le 1$); thus, if Boeing produces X airliners, $h = X/N(p)$;

P = the total profit (in millions again) to Boeing.

The profit P is a function of the price p, and is the quantity that the company wished to maximize.

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* Based on Brigham, Georges, "Pricing, Investment, and Games of Strategy," in Management Sciences, Models and Techniques, edited by C.W. Churchman and M. Verhulst. v. 1 pp 271-87 (Pergamon Press, 1960).

3.3 The Profit Function

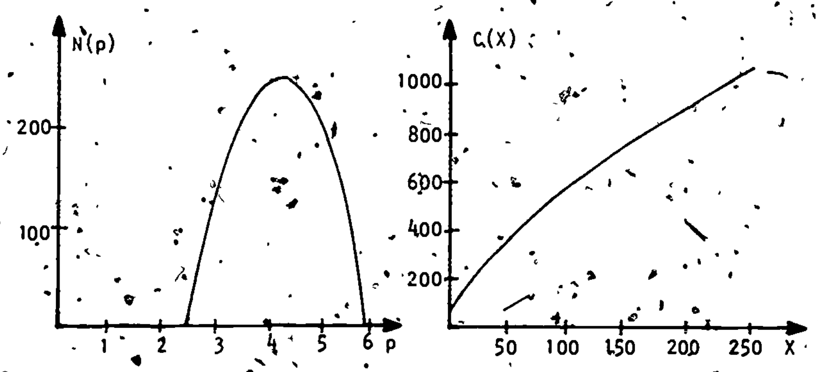
Our first objective is to express the profit P explicitly as a function of the price p. Since profit is total income minus total cost, we first need expressions for both of these quantities.

Analysts at Boeing made both predictions on the market and estimates of costs involved, and used their results to arrive at the expressions

(3.1) $N(p) = -78p^2 + 655p - 1125$

(3.2) $C(X) = 50 + 1.5X + 8X^{3/4}$

as estimates of the total market, N(p), at price p, and the total cost to Boeing, C(X), (as p, also in millions of dollars), of producing X airliners. Sketches of N(p) and C(X) are shown in Figure 3-1, with smooth curves drawn through actual discrete points. Routine calculations with Equation (3.1) (e.g., the quadratic formula) show that the



(a) Predicted number of airliners salable at price p, where p is in millions of dollars.

(b) The cost C(X), in millions of dollars, of producing X airliners.

Figure 3-1.

least value of p for which N(p) is nonnegative is slightly more than \$2.408 million. Therefore, p must be at least \$2.408 million (also see Exercise 6).

Since profit is total income minus total cost, we have

(3.3) $P = pX - C(X)$.

Since also $X = hN(p)$, we obtain an expression for P in terms of p by substituting hN(p) for X in (3.3):

(3.4) $P = phN(p) - C[hN(p)]$.

3.4 Determination of the Best Selling Price

We note first that p and X are both discrete variables, each assuming only integer values. Nevertheless, formulas (3.1) and (3.2) both determine functions of continuous real variables, as indicated in Figure 3-1, and these functions can be used to obtain approximations for the required integer values.

Equations (3.1) and (3.4) constitute a chain of functions that would yield an explicit formula for P if we carried out the actual substitution. However, this substitution would produce an unnecessarily complicated expression, and we avoid this difficulty by differentiating (3.4) with respect to p as it stands, using the chain rule:

(3.5) $P'(p) = phN'(p) + hN(p) - C'[hN(p)]hN'(p)$.

Next, if we set P'(p) equal to zero, we see that p must satisfy the equation

(3.6) $p + \frac{N(p)}{N'(p)} = C'(X)$ (if $N'(p) \neq 0$).

A word of caution is in order here. If $N'(p) = 0$, then the equation $P'(p) = 0$ reduces to the equation $hN(p) = 0$. You should show (Exercise 7) that the only value of p for which $N'(p) = 0$ is not a critical point of P, unless $h = 0$.

3.5 Numerical Value of the Best Selling Price

Using the relations $N'(p) = -156p + 655$; $C'(X) = 1.5 + 6X^{-1/4}$, and $X = hN(p) = h(-78p^2 + 655p - 1125)$, we could substitute into Equation (3.6) and solve the resulting equation for p. However, this approach would lead to difficult calculations, and so we try an alternative approach.

Let us suppose that the company will produce 70 airliners. Then since $X = 70$, Equation (3.6) reduces to

$$(3.7) \qquad p + \frac{-78p^2 + 655p - 1125}{-156p + 655} = 3.57432.$$

Equation (3.7) can be reduced easily to a quadratic equation, which can be solved by elementary techniques (a calculator will be most useful here). The roots of the equation are, to two places, 5.05 and 2.94. Using the second derivative test (see Exercise 9), we see that $P''(5.05) < 0$ and $P''(2.94) > 0$. Hence, if the company produces 70 airliners, it should charge approximately \$5.05 million to maximize its profit.

Similarly, if $X = 100$, then $C'(100) = 3.39737$. In this case, solution of (3.6) for p yields $p = 5.0$ and $p = 2.86$. Of these values, $p = 5$ yields the desired maximum.

3.6 A Look at the Cost Equation

From our calculations in Section 3.5, you might have noticed that a large change in X produced a relatively small change in p (when p is measured in millions). Let us see why this is so.

Recall that $C'(X) = 1.5 + 6X^{-1/4}$. Hence, $C''(X) = -(3/2)X^{-5/4}$. Thus, for large X the graph of $C'(X)$ has slope near zero. Thus, changes in $C'(X)$ will be relatively small for large X. (See Figure 3-2.) Such changes, in turn, will produce only small changes in the right side of Equation (3.6), and thus, small changes in the value of p.

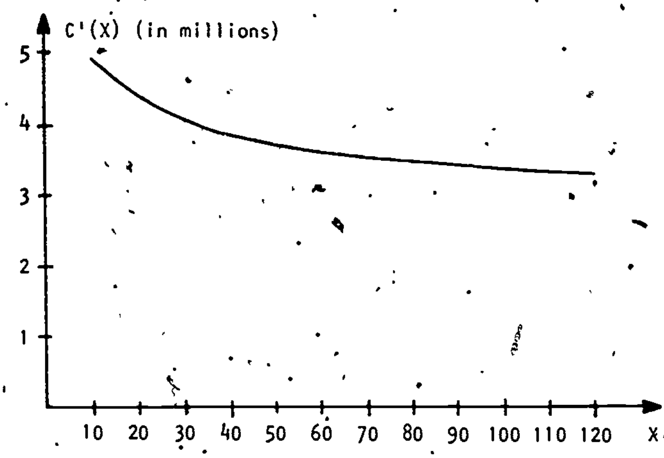


Figure 3-2. Graph of C'(X).

3.7 Exercises

6. Find the maximum value of the function N(p) given by (3.1). How many airliners should actually be sold at price p? (Recall that a company cannot sell a fraction of an airliner).

7. For N(p) given by Equation (3.1), show that if $N'(p) = 0$, then $P'(p) \neq 0$, unless $h = 0$.

8. Using (3.5), compute $P''(p)$.

9a. Suppose Boeing decides to produce 70 airliners. Recall that for X = 70, we showed in Section 3.5 that $P'(p) = 0$ for $p = 5.05$ and for $p = 2.94$. Now use the second derivative test to show that P attains its maximum for $p = 5.05$ and its minimum for $p = 2.94$. (Hint: Compute $C''(70)$ from (3.2). Then use the fact that $C'(X) - p = \frac{N(p)}{N'(p)}$ (from 3.6) to simplify your answer in Exercise 8. Finally use the fact that $0 \leq h \leq 1$ to show that $P''(5.05) < 0$ and $P''(2.94) > 0$. A calculator will help!).

b. For X = 70, find the maximum value of P (i.e. find P(5.05)).

10. If Boeing produces 70 airliners, compute n(5.05), the total number of airliners sold by Boeing and its competitor. Also compute h in this case.

4. SNELL'S LAW - LIGHT REFRACTION

4.1 How Water "Bends" Light

In this section we examine a very important property of light rays.

You may have noticed that when you see an object partially submerged in water, it appears to be bent. This phenomenon is known as refraction. Our understanding of refraction is based on a principle due to Pierre Fermat, a famous 17th century mathematician and physicist. According to Fermat's principle, when light travels through one or more homogeneous media, it follows the path that requires the least amount of total time. Thus, when there is only one medium, such as air, the path that will minimize distance is a straight line, since the rate is constant. For example, when light travels from water into air, it travels along one straight line to the surface and along another in the air. As a result, we see the "bending" effects. (See Figure 4-1.)

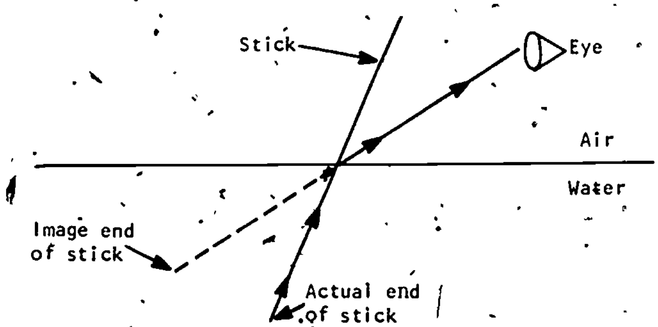


Figure 4-1. The image of a partially submerged stick being "bent" by water. The arrows indicate the path of the light rays to the eye.

4.2 Some Notation

Let us assume that the speed of light in water is v (in appropriate units) and in air w. Assume that the object is a units below the surface, the eye b above. Let x, y, α, β be as indicated in Figure 4-2, and put c = x + y.

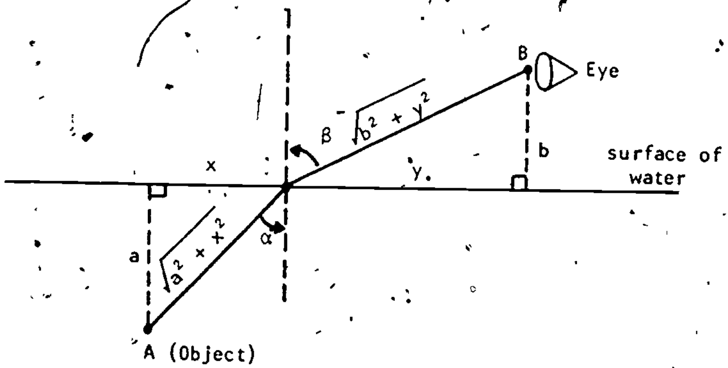


Figure 4-2. Light is diffracted (deflected from its path) as it crosses the boundary between transparent media that have different densities.

4.3 Minimizing the Travel Time of Light

We wish to minimize the total time light takes to travel from point A to point B. Using the formula time = distance ÷ rate, we obtain the total time as a function of x:

$$(4.1) \qquad T(x) = \frac{\sqrt{a^2 + x^2}}{v} + \frac{\sqrt{b^2 + (c - x)^2}}{w}$$

Then we have

$$(4.2) \qquad T'(x) = \frac{x}{v\sqrt{a^2 + x^2}} + \frac{x - c}{w\sqrt{b^2 + (c - x)^2}}$$

and

$$(4.3) \qquad T''(x) = \frac{a^2}{v(a^2 + x^2)^{3/2}} + \frac{b^2}{w(b^2 + (c - x)^2)^{3/2}}$$

Now, since v and w are both positive, T"(x) is always positive. Hence any critical value will yield a minimum. (See Exercise 12.) Setting T'(x) = 0, we obtain

$$(4.4) \qquad \frac{x}{v\sqrt{a^2 + x^2}} = \frac{c - x}{w\sqrt{b^2 + (c - x)^2}}$$

or

$$(4.5) \qquad \frac{v}{w} = \frac{x/\sqrt{a^2 + x^2}}{y/\sqrt{a^2 + y^2}} = \frac{\sin \alpha}{\sin \beta}.$$

The equation

$$(4.6) \qquad \frac{v}{w} = \frac{\sin \alpha}{\sin \beta}$$

is known as *Snell's Law*. Snell's Law states that, since v and w are known constants and since, by Fermat's Principle the travel time of light from point A to point B is minimized, the ratio of sin α to sin β is constant. (That is, changes in x, y, a or b will not change this ratio.) This constant, v/w, is called the index of refraction. Snell's Law is believed to have been first discovered by Willebrord Snell in 1621.

4.4 Some Concluding Remarks

Notice that the choice of air and water is not crucial to the derivation of Snell's Law. In fact, any two media through which light travels at a constant rate could have been used, with similar results.

You should note also that we derived Snell's Law without explicitly finding a critical value for T(x). The actual solution of the equation T'(x) = 0 (see (4.2)) would involve a cumbersome fourth degree polynomial. In addition, it is not at all important to have an explicit expression for a critical value. The ability to obtain useful results knowing only the existence of certain numbers (without knowing their values) is a phenomenon which occurs frequently in applied mathematics.

4.5 Exercises

11. Using equation (4.1), compute $T'(x)$ and $T''(x)$. Your answers should be equations (4.2) and (4.3), respectively.

12. (This exercise is for those who are familiar with the intermediate value theorem.) Using (4.2), show that there is a number x between 0 and c for which $T'(x) = 0$. (Hint: find $T'(0)$ and $T'(c)$, then . . . .)

13. Suppose that a light ray passes through a transparent plate (Figure 4-3). Prove that, with reference to that figure, $\alpha = \delta$. (Hint: let v be the speed of light in air, w the speed of light in the plate, and apply Snell's Law.)

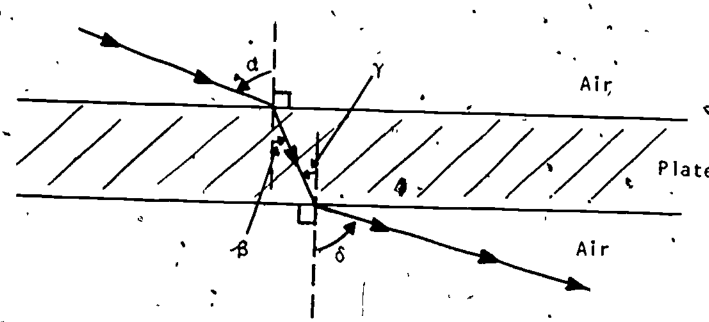


Figure 4-3. Light passing through a transparent plate. By Exercise 13, although the ray is displaced by the plate, its direction is unchanged.

5. SURFACE AREA OF A BEE'S CELL

5.1 The Shape of a Honey Bee's Cell

One of nature's most remarkable creatures is the honey bee--nature appears to have given him some amazing engineering abilities. (See Thompson.) In this section we study the construction of a honeycomb cell.

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The open face of a single cell in the comb approximates a regular hexagon. (See Figure 5-1a.) The horizontal (vertical in our diagram) portion of the cell is constructed geometrically as follows (see Figure 5-1b). Over the regular hexagon abcdef with sides of length s construct a right hexagonal prism of height h, with top vertices A, B, C, D, E and F, respectively. (Vertices D and F are not shown in Figure 5-1b.) The corners B, D and F

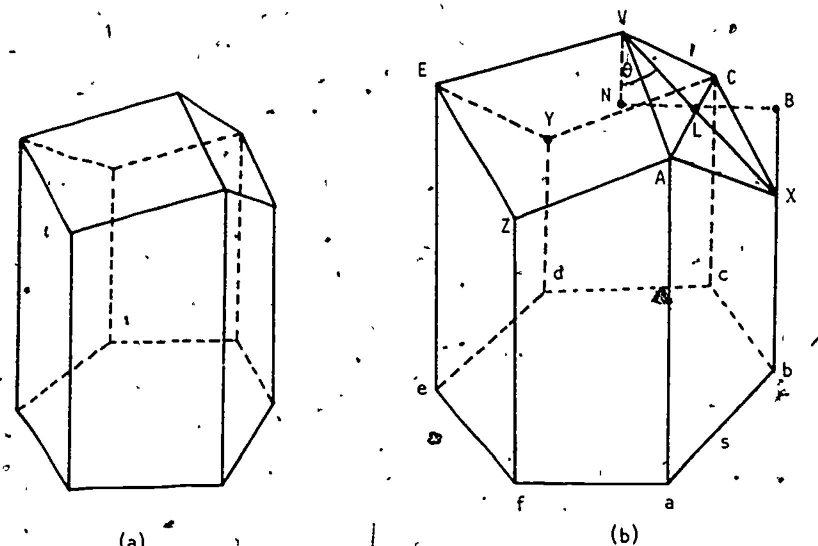

Figure 5-1. A Honey Bee's cell.

are cut off by planes passing through the lines AC, CE and EA, respectively, which meet at a point V on the axis VN of the prism. (The point N is the intersection of the axis of the prism with the line through B and L.) The "cut-off" pieces are then placed atop the cell so that the points that were at X, Y and Z will meet at V.

Observe that the new solid (the cell) has the same volume as the original prism. Note also that the lines AC, CE and EA are axes of rotation for the "cut-off" pieces, i.e., the tetrahedra ACXB, CEYD and EAZF, respectively.

5.2 Description of the Problem

The bees form their cells in such a way as to minimize surface area for a given volume. Thus, they use the least amount of wax in cell construction. The mathematical problem is to cut the hexagonal prism at an angle ($\theta = \angle NVL$ in Figure 5-1b) which minimizes the total surface area.

5.3 The Surface Area Equation

First, let us express the area of the top portion of the cell in terms of θ. Notice that, by the construction outlined in Section 5.1, the top surface consists of three rhombi, AXCV and two others that are congruent to it (Figure 5-2).

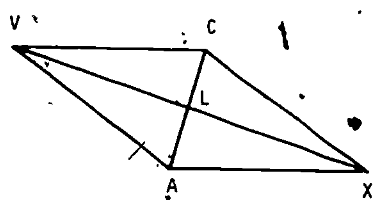

Figure 5-2. One of the three rhombi that form the upper surface of the cell.

Let L be the intersection of the segments VX and AC. Then, VNL and LBX (Figure 5-1b) are congruent right triangles (the point N lies in the plane of the vertices A, B, C, [illegible], F and F), so that NL = LB. From plane geometry we recall that in a regular hexagon of edge length s the distance from the center to each vertex is also s. Thus, NL = s/2 (since NB = s).

Now, BCN is an equilateral triangle with altitude CL, so

$$CL = s\sqrt{3}/2.$$

In addition, from Figure 5-1b, we see that

$$VL = NL \csc\theta = (s/2)\csc\theta,$$

and hence the area of each rhombus of the three that form the upper surface (see Figure 5.2) is

$(s^2\sqrt{3}/2) \csc\theta$.

Therefore the total area of the three rhombi is

(5.1) $(3s^2\sqrt{3}/2) \csc\theta$.

From triangle NLV we have VN = $(s/2)\cot\theta$, and since VN = BX, we have

$$\text{Area }(abXA) = s(h + (h - (s/2)\cot\theta))/2.$$

Hence, the total lateral surface area of the cell is

(5.2) $6(sh - (s^2/4)\cot\theta) = 6sh - (3s^2/2)\cot\theta$.

Finally, combining (5.1) and (5.2), we find the total surface area, $f(\theta)$, of the cell to be

(5.3) $f(\theta) = 6sh - (3/2)s^2\cot\theta + (3s^2\sqrt{3}/2)\csc\theta$.

5.4 Minimizing the Surface Area

To minimize $f(\theta)$, which is given by Equation (5.3), first note that for θ measured in radians, $0 < \theta < \pi/2$. Differentiating Equation (5.3) and simplifying (Exercise 14), we obtain

(5.4) $f'(\theta) = \frac{3s^2}{2}(\csc\theta)(\csc\theta - \sqrt{3}\cot\theta)$.

Setting $f'(\theta)$ equal to 0, we obtain $\cos\theta = 1/\sqrt{3}$. The only value of θ between 0 and $\pi/2$ that satisfies this equation is $\theta \approx 0.9553$ radians. (The symbol $\approx$ means approximately equal to.) In degrees, this angle is about 54°44'.

You should carry out the details to show that this value of θ does indeed minimize the total area of the cell, as given by Equation (5.3) (Exercise 15).

Note that the minimizing value of θ does not depend upon either s or h.

5.5 Do Bees Know Mathematics?

Several points are worth noting here. First, actual honeycomb cells are not perfectly hexagonal, as we assumed in deriving Equation (5.3). However, as you might have noticed from pictures or by actual observation, these cells are usually close to perfect hexagons.

As you might imagine, actual measurement of the angle θ in a beehive is difficult. However, such measurements can be made, and the measured angles seldom differ from our calculated value by more than two degrees. (See Fejes Tóth in the references.)

Finally, while hexagonal solids allow economy in the use of wax in cell construction, other polyhedra permit even greater economy (see Tóth). However, Tóth also points out that if bees used the most efficient design in terms of minimum surface area for a given volume, they would save less than two-fifths of one percent in wax. In addition, the actual construction described in Section 5 is considerably easier for bees to carry out than is the building for more efficient cells.

For more information on these fascinating creatures, see Batschelet, Thompson, and Tóth in the references.

5.6 Exercises

14. Carry out the details of deriving (5.4) from (5.3).

15a. Show that the value $\theta = \arccos(1/\sqrt{3})$ yields a minimum value for $f(\theta)$, as given in (5.3). (Hint: It is more efficient to analyze the sign of $f'(\theta)$ to determine values for which $f(\theta)$ is decreasing and those for which it is increasing than to apply the second derivative test here.)

b. Use (5.3) to find the value of the minimum surface area of the cell. (Hint: Use the exact result $\cos\theta = 1/\sqrt{3}$ to evaluate $\sin\theta$ and $\cos\theta$.)

16a. Find the surface area of a right, regular hexagonal prism with base edge of length s and altitude h, if the surface is closed at the top and open at the bottom (no corners cut off here).

b. Compare the result of 16a with that of Exercise 15b.

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6. ARTERIAL BRANCHING

6.1 A Surgeon's Dilemma

Consider the case of a surgeon who must attach a blood vessel to an artery AB, which will lead to a point C. (See Figure 6-1.) The surgeon wishes to attach the vessel

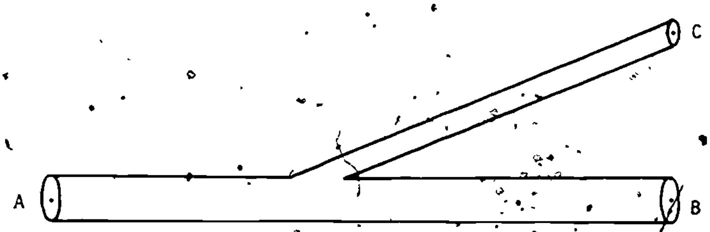


Figure 6-1. A blood vessel connecting artery AB to a point C.

in such a way as to minimize the resistance to the flow of blood from point A to point C, since this, in turn, will minimize the strain on the heart.

Now, according to Poiseuille's Law for laminar fluid flow in rigid pipes, resistance is directly proportional to the length of the pipe and inversely proportional to the fourth power of the radius of the pipe:

(6.1) $R = kd/r^4$

where k is the constant of proportionality, d the length and r the radius of the pipe.*. ("Laminar flow" means that all particles of the fluid pass through the tube along paths that are parallel to its wall, and that the rate of flow increases smoothly from the wall toward the center.)

* For a very readable discussion of Poiseuille's Law, see Philip Tuchinsky's Viscous Fluid Flow and the Integral Calculus, UMAP Unit # 210, Education Development Center, Newton, Massachusetts, 1978.

The proportionality constant is determined by the viscosity of the fluid, blood in this case. Of course, blood vessels are not actually rigid, but for short distances, as is usually the case in surgery, they are nearly rigid.

Now the surgeon's problem becomes apparent. If the vessel is attached closer to the point A, (Figure 6-1), the blood travels less total distance, but farther in the vessel, which has a smaller radius. On the other hand, if the vessel is attached closer to B, so that the blood flows farther in the tube of larger radius, then the total distance is increased. The problem is to find the point between A and B at which the vessel should be attached in order to minimize the resistance to the flow of blood.

6.2 A Resistance Equation

We shall use the notation in Figure 6-2. (Note that, without loss of generality, we may locate B so that ABC is

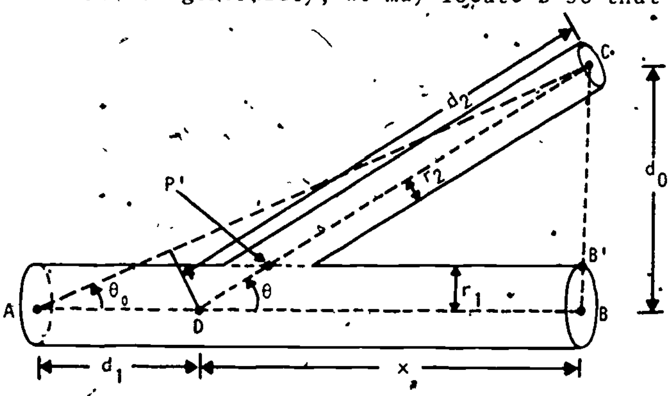


Figure 6-2. Notational diagram for the arterial branching problem.

a right angle.) The known quantities are d_0, r_1, r_2, θ_0 and $d_1 + x$, while the variables are θ, d_1, d_2, and x. The problem is to find the value of θ that minimizes the total resistance to the flow of blood from A to C.

We first express d_1, d_2, and x in terms of θ. From Figure 6-2 we see that $\csc \theta = d_2/d_0$, $\cot \theta_0 = (d_1 + x)/d_0$, and cot $\theta = x/d_0$; hence

$$d_2 = d_0 \csc \theta,$$

$$d_1 = -x + d_0 \cot \theta_0, \text{ and}$$

$$x = d_0 \cot \theta.$$

From the latter two expressions we obtain

$$d_1 = d_0(\cot \theta_0 - \cot \theta).$$

Let $R = R(\theta)$ denote the total resistance to the flow of blood. Then R is the sum of the resistance through the artery plus the resistance through the vessel. By (6.1), we have

$$R = \frac{kd_1}{r_1^4} + \frac{kd_2}{r_2^4}$$

$$= k\left(\frac{d_0 \cot \theta_0 - d_0 \cot \theta}{r_1^4} + \frac{d_0 \csc \theta}{r_2^4}\right).$$

If we put $K = kd_0/r_1^4 \cot \theta_0$, a constant, we obtain the following expression for the resistance:

(6.2) $$R(\theta) = K + kd_0\left(\frac{\csc \theta}{r_2^4} - \frac{\cot \theta}{r_1^4}\right).$$

6.3 Minimizing the Resistance

Without loss of generality, we may assume $0 < \theta < \pi$. Differentiating $R(\theta)$ using (6.2), we obtain

(6.3) $$R'(\theta) = kd_0\left(\frac{-\csc \theta \cot \theta}{r_2^4} + \frac{\csc^2 \theta}{r_1^4}\right).$$

(Be sure to carry out this differentiation in detail.) Setting $R'(\theta)$ equal to 0 and solving for θ, we find that the only critical value is

(6.4) $$\theta = \arccos(r_2^4/r_1^4).$$

(See Exercise 17.) Next, you should show that the value of θ given by Equation (6.4) yields a minimum for R(θ) (Exercise 18).

Notice that the critical value of θ depends only on the radii of the tubes. Therefore, the locations of the points A, B and C were not crucial to our work, except for the effect that the lengths involved may have on the assumption of rigidity.

You should note also that under the conditions of the problem, the surgeon is concerned chiefly with the fact that the resistance must be minimized, and not with the actual amount of the resistance. For this reason we do not find R_{min} in this example.

6.4 Some Concluding Observations

Experimental observations have shown that the angle given by Equation (6.4) is close to the actual angles at which blood vessels are attached to arteries in the body.

Finally, let us return to the surgeon, who is waiting patiently (!!??) for us to complete our calculations and observations. If we report the critical value of θ in Equation (6.4), he or she still won't know where to attach the vessel to the artery. Can you help out with a little right-angle trigonometry? (See Exercise 19.)

6.5 Exercises

17. With $R'(\theta)$ as given by (6.3), solve the equation $R'(\theta) = 0$ for θ to obtain (6.4).

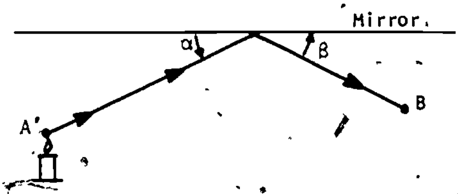
18a. Obtain $R''(\theta)$ from (6.3).

b. Show that the value of θ given by (6.4) satisfies the condition $R''(\theta) > 0$, and hence that this value minimizes $R(\theta)$.

19. With reference to Figure 6-2, for a known θ (such as the value in (6.4)), find the distance P'B' in terms of θ, d_0 and r_1.

7. MODEL EXAMINATION

1. Suppose that a light ray from a point A is reflected in a mirror to a point B, with the path of the ray completely in air and forming angles θ_1, θ_2 with the mirror, as shown in the diagram. Apply Fermat's Principle to prove that $\alpha = \beta$.



2. Suppose that analysts for a refrigerator manufacturing firm have determined that the total cost in dollars involved in producing x refrigerators of a certain kind is approximately

$$C(x) = \frac{1}{12^4 \times 10^5} x^4 - \frac{1}{7200} x^2 + 500,$$

and that at a price of p dollars per unit, a total of

$$N(p) = 120\sqrt{10p}$$

refrigerators could be sold. (Here C(x) includes all costs, such as material, manufacturing, storage, advertising, shipping, and management.)

a. Find an expression for the total profit P in terms of the selling price p.

b. Find the value of p that maximizes the profit $P = P(p)$.

8. ANSWERS TO EXERCISES

1a. $C''(X) = kTX^{-3}$. Since $X_0 > 0$ (see Equation 2.3), $C''(X_0) > 0$, and hence X_0 yields a minimum for $C(X)$.

b. $C(X_0) = (FX_0 + vT) + kT/2X_0$

$$= F\sqrt{kT/2F} + vT + kT/2\sqrt{kT/2F}$$

$$= vT + \sqrt{kFT/2} + \sqrt{kFT/2}$$

$$= vT + \sqrt{2kFT}.$$

2a. $\sqrt{kT/2F}$.

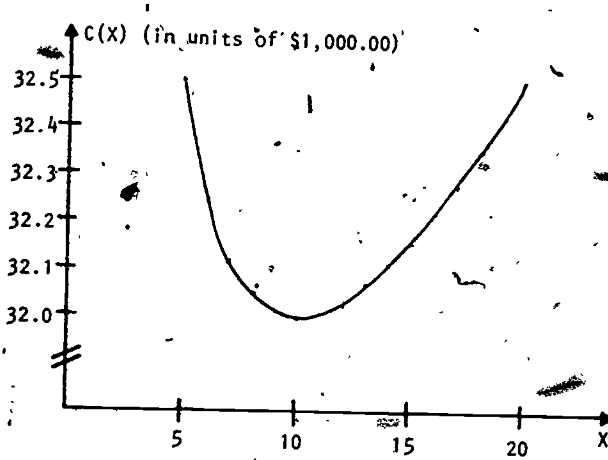
b. All the quantities in the cost equation must be positive.

c. Zero is not in the domain of the function $C(X)$.

3a. $X_0 = 10$.

b. $C_{min} = \$32,000.00$

4.



5. $T=0$. $B'(T)$ is never zero, but the domain of $B(T)$ is $\{T: T \geq 0\}$. Note that $B(T)$ is a linear function of T, with positive slope.

6. $N_{max} = 250.08$; $N = 250$.

7. If $N'(p) = 0$, then $P'(p) = hN(p)$: Since $N(p) \neq 0$, (otherwise the problem would be trivial) we have $P'(p) \neq 0$ unless $h = 0$.

8. $P''(p) = h(2N'(p) - C''(X)h(N'(p))^2 - (C'(X) - p)N''(p))$.

9. By Equation (3.6) the expression for $P''(p)$ in the answer to Exercise 8 reduces to:

$$P''(p) = h[2N'(p) - C''(X)h(N'(p))^2 - \frac{N(p)}{N'(p)}N''(p)].$$

Also note that $C''(70) = -0.0074$.

a. $P''(5.05) = h(130.65h - 492.97) < 0$, since $0 \leq h \leq 1$;

$P''(2.94) = h(285.64h + 493.22) > 0$, since $0 \leq h$.

b. By Equation (3.3), $P(5.05) = (5.05)(70) - C(70) =$ 4.90 million.

10. With $X = 70$ we use $p_0 = 5.05$ as the critical value that maximizes the profit. Then, $N(5.05) = 193.56$. Thus, a total of 194 airplanes will be produced, so the competition produces 124, and hence $h = .36$.

12. $T'(0) = -c/w\sqrt{b^2 + c^2} < 0$; $T'(c) = c/\sqrt{a^2 + c^2} > 0$. Since $T'(x)$ is continuous, there is a point x between 0 and c such that $T'(x) = 0$, by the intermediate value theorem.

13. By Snell's Law,

$$\frac{v}{w} = \frac{\sin \alpha}{\sin \beta}, \quad \text{and} \quad \frac{w}{v} = \frac{\sin \gamma}{\sin \delta}.$$

Since $\beta = \gamma$, we must have $\sin \alpha = \sin \delta$. Since $0 < \alpha, \delta < \pi/2$, we must have $\alpha = \delta$.

14. $f'(\theta) = -(3/2)s^2(-\csc^2 \theta) + (3s^2\sqrt{3}/2)(-\csc \theta \cot \theta)$

$$= \frac{3s^2}{2}\left(\frac{1}{\sin^2 \theta} - \sqrt{3}\frac{1}{\sin \theta}\frac{\cos \theta}{\sin \theta}\right)$$

$$= \frac{3s^2}{2}\left(\frac{1 - \sqrt{3}\cos \theta}{\sin^2 \theta}\right).$$

15a. Since $3s^2/2\sin\theta > 0$ for $0 < \theta < \pi/2$, the sign of $f'(\theta)$ is the same as that of the factor $1 - \sqrt{3}\cos\theta$. For $0 < \theta < \arccos\sqrt{3}$, $1 - \sqrt{3}\cos\theta < 0$; and for $\arccos\sqrt{3} < \theta < \pi/2$, $1 - \sqrt{3}\cos\theta > 0$. Thus, $f(\theta)$ is decreasing for $0 < \theta < \arccos\sqrt{3}$ and increasing for $\arccos\sqrt{3} < \theta < \pi/2$. Hence $f(\theta)$ has a minimum at $\theta = \arccos\sqrt{3}$.

b. For the critical value $\theta = \arccos\sqrt{3}$ we have $\sin\theta = \sqrt{2}/\sqrt{3}$, $\cot\theta = 1/\sqrt{2}$, and

$$f(\theta) = 6sh + 3s^2/\sqrt{2} = 6sh + 3s^2\sqrt{2}/2.$$

16a. $6hs + 3\sqrt{3}\, s^2/2$

b. $6hs + \frac{3\sqrt{2}}{2} s^2 < 6hs + \frac{3\sqrt{3}}{2} s^2$.

17. $R'(\theta) = 0 + kd_0\left[\frac{-\csc\theta\cot\theta}{r_2^4} - \frac{-\csc^2\theta}{r_1^4}\right]$;

the calculation requires only elementary differentiation formulas and the fact that K, k, d_0, r_1, r_2 are constants. The formula (6.3) follows by elementary algebra.

18a. $R''(\theta) = kd_0\left[\frac{\csc\theta\cot^2\theta + \csc^3\theta}{r_2^4} - \frac{2\csc^2\theta\cot\theta}{r_1^4}\right]$

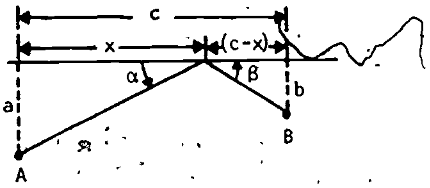
b. $\csc\theta = r_1^4/\sqrt{r_1^8 - r_2^8}$; $\cot\theta = r_2^4/\sqrt{r_1^8 - r_2^8}$;

$R''(\theta) = k\,d_0\, r_1^4/r_2^4\sqrt{r_1^8 - r_2^8} > 0$.

19. $P'B' = (d_0 - r_1)\cot\theta$.

9. ANSWERS FOR MODEL EXAMINATION

1. Since the path is completely in the air there is no change in the speed of light, and hence the travel time will be at a minimum if we minimize the distance.



The distance traveled by the light ray is

$$D(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2},$$

from which

$$D'(x) = \frac{x}{\sqrt{a^2 + x^2}} + \frac{-(c - x)}{\sqrt{b^2 + (c - x)^2}}.$$

The equation $D'(x) = 0$ yields $\sin \alpha = \sin \beta$, and since $0 < \alpha, \beta < \pi/2$, we must have $\alpha = \beta$.

2a. The profit can be found by subtracting the total cost from the total income from sales:

$$P(p) = (120\sqrt{10p})p - C(120\sqrt{10p})$$

$$= 120\sqrt{10}\, p^{3/2} - 10p^2 + 20p - 500.$$

b. $P'(p) = 180\sqrt{10}\, p^{1/2} - 20p + 20,$

$P''(p) = 90\sqrt{10}\, p^{-1/2} - 20.$

The equation $P'(p) = 0$ has two solutions, $p = 812.00$ and $p = .0012$. Direct calculation yields $P''(812) < 0$ and $P''(0.0012) > 0$. To maximize its profit, the firm should charge \$812.00 per refrigerator.

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