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ABSTRACT

This document is seen as a framework that presents suggestions for mathematics programs to fill the void left with the demise of "new math." The child is identified as the central figure in the educational scene. The material is in two major parts. The first part is a framework for mathematics as revised for 1975, and covers all grades. The second part is an addendum to this mathematics framework, as adopted in 1980. The document as a whole is intended to aid teachers through suggestions of goals, objectives and programs that would enable California school children to learn basic mathematics. (MP)

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Mathematics Framework and the 1980 Addendum

for California Public Schools

Kindergarten Through Grade Twelve

CALIFORNIA STATE DEPARTMENT OF EDUCATION
Wilson Riles—Superintendent of Public Instruction
Sacramento, 1982

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ADOPTION INFORMATION

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Contents

Addendum to the Mathematics Framework

Foreword

The "post-new math framework" might be an appropriate title for this document to signal its timing. With the demise of "new math," this framework presents improved mathematics to fill the void. The contents reflect the concerns of teachers rather than those of mathematicians.

The new framework identifies the child as the central figure in the educational scene, and that is as it should be. "The teacher assumes the role of a guide," say the writers of this document, a guide "who directs learners to explore, investigate, estimate, and solve everyday, realistic, pupil-oriented problems."

The "metric framework" might be another title ascribed to this document, because it establishes the International System of Units (SI) as the standard for measurement. With my endorsement and encouragement, the writers submitted and won this concession from the State Board of Education.

However, my preference for a title is the "basics framework," because the major concern of the writers is clearly the increased use of sound teaching techniques to enable California schoolchildren to learn basic mathematics. I wholeheartedly support this approach, and I hope for every teacher and student the excitement that comes with this way of teaching and learning.

Superintendent of Public Instruction

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Preface

In 1963 the first framework for mathematics was published by the California State Department of Education. It was commonly referred to as "The Strands Report" because Part One of the framework outlined eight fundamental concepts or strands which "tied" the mathematics curriculum together in kindergarten through grade eight. Also considered in the framework was the dynamic character of good mathematics instruction; that is, pupils should be encouraged to guess, to experiment, to hypothesize, and to understand through active participation in the teaching-learning process.

The Second Strands Report (*Mathematics Framework for California Public Schools. Kindergarten Through Grade Eight*) was accepted by the State Board of Education in 1968. In this second report, the network of strands was designed as an integrated whole, and a satisfactory instructional program was described as one that would provide a balanced emphasis upon each of the strands.

The Statewide Mathematics Advisory Committee (SMAC), 1967–1970, which prepared the *The Second Strands Report*, was charged by the Curriculum Development and Supplemental Materials Commission and the State Board of Education to consider a suitable extension of the strands concept through grade twelve. Under the direction of its chairman, John L. Kelley, the advisory committee sponsored a conference of 45 participants, including mathematicians, scientists, secondary teachers of mathematics and science, and persons using mathematics in industry and computer technology. The present Ad Hoc Mathematics Framework Committee, whose members were appointed by the Curriculum Development and Supplemental Materials Commission in October, 1973, is indebted to SMAC for the work it accomplished. The framework committee gathered information from agencies, teachers, professional organizations, and concerned individuals; and it conducted meetings throughout the state in an attempt to ensure that a variety of opinions would be heard.

Because the mathematics program for individual high school students varies according to their interests, skills, and career objectives, the strands for the high school level were designed to respond to the flexibility of school programs. In this framework is contained what a total mathematics program can and should provide for high school students. Although the Department of Education plays no direct role in the selection of materials for mathematics programs in grades nine through twelve, the Department believes that this framework provides information useful to those responsible for mathematics programs at every level.

Those responsible for the selection of materials for mathematics programs in kindergarten through grade eight should find that the screening criteria contained in the framework are quite useful. The updated criteria reflect a number of concerns about the acquisition of basic mathematics skills that prevailed at the time the framework was revised. It is anticipated that the forthcoming statewide adoption of materials in mathematics will reflect the impact of this publication.

In the development of a school mathematics program for California, it seems pointless to refer to contemporary mathematics programs as the "new math." Our concern should be to make the very best in mathematics curriculum and instructional practices available to our students and teachers. We are also interested in informing the public that high-quality preservice and inservice teacher education programs are needed to prepare people to teach mathematics with the knowledge, skill, and enthusiasm necessary to serve our pupils with excellence.

The Ad Hoc Mathematics Framework Committee is hopeful that this publication will provide a set of creative guidelines for teachers, authors, and publishers to employ in the development of instructional materials. Further, we expect that the report will be useful to administrators and teachers in the development of instructional materials and comprehensive mathematics programs which have objectives consonant with the needs of pupils and society.

This publication represents the combined efforts of many interested and concerned individuals, and we express our thanks to them, especially to the members of the Ad Hoc Mathematics Framework Committee, who are listed on page vi, and to the members of the Addendum Committee, who are listed on page 55.

DONALD R. McKINLEY
Chief Deputy Superintendent of Public Instruction

DAVIS W. CAMPBELL
Deputy Superintendent for Programs

Introduction

The California State Board of Education, recognizing the need for a reappraisal of *The Second Strands Report: Mathematics Framework for California Public Schools* published in 1972, mandated the development of a new and more extensive framework. This 1975 framework will encompass the mathematics program from the kindergarten level through grade twelve.

This is the third mathematics curriculum framework developed for use by California public schools. *The Second Strands Report* provided an excellent basis for mathematics curriculum development in the state, but a continuous assessment of a mathematics framework is needed because of expanding information and knowledge, shifting emphasis in subject areas, and changing organizational patterns for instruction.

The principal assumption which underlies the thinking of this ad hoc framework writing committee is that the school mathematics program should be designed to educate each child to the child's optimum potential in mathematics.

The recommendations of the revised framework for kindergarten through grade eight reflect the following changes in emphasis:

1. An *increased emphasis* on the application of mathematical concepts to physical objects familiar to children
2. An *increased emphasis* on computational skills along with the development of the structural aspects of mathematics
3. An *increased emphasis* on ways to improve children's attitudes toward mathematics
4. An *increased emphasis* on metric units known as the International System of Units, which will be the basis for standard measurement instruction
5. An *increased emphasis* on the total concept of decimal numbers
6. An *increased emphasis* on application and problem-solving skills
7. A *decreased emphasis* on numeration systems other than the familiar decimal-based system
8. A *decreased emphasis* on the computation of fractional numbers in kindergarten through grade six
9. A *decreased emphasis* on set theory

The purpose of a framework is to provide a base from which schools, school districts, and offices of county superintendents of schools can develop adequate goals and objectives for their programs. In addition, the mathematics framework provides the basis for the development of criteria for the evaluation of instructional materials to be considered for adoption by the state of California. This mathematics framework contains a description of the major components of the school mathematics program, kindergarten through grade twelve. These components are:

- *Broad goals and objectives*
  General goals for learners
  Content and topic goals
  Program objectives
- *General content guidelines*
  The elementary strands
  The secondary strands
- *Methods and materials*
  Classroom climate
- *Suggestions for program evaluation*
- *Criteria for screening instructional materials*

The Climate and Environment for Learning Mathematics

The most effective and efficient climate and environment for learning provides for the following:

- Experience with objects from which the learner can develop concepts
- A means of communication that the learners can understand
- Opportunities for learners to become involved in activities
- Opportunities for the teacher to study the learner's habits of work and thought
- Motivation for learners to improve continually their proficiencies in mathematical skills and concepts

The kindergarten through grade twelve mathematics program provides for the following:

- A rich variety of opportunities for the learning of mathematical concepts
- The application of these mathematical concepts to socially useful mathematical problems
- The accumulation of mathematical maturity and proficiency for use in other disciplines
- A climate for learning

Learning is a group experience in that group behavior affects the learning process, as pupils do learn from one another. Mathematics becomes a vibrant, vital subject when points of view are argued, and for this reason interaction among pupils should be encouraged. As pupils build mathematics together, they develop special pride in learning activities, and their work gains momentum. Manipulative materials provide effective means for facilitating learning. These materials are often simple and can be pupil-made or collected. Manipulating may mean handling an object, comparing objects, viewing objects represented in a pictorial mode, or engaging in paper-and-pencil activities. The materials should provide a smooth transition from concrete learning experiences to the abstract.

A significant feature of a mathematics learning environment is the spirit of free and open investigation. The learning of mathematics is many-faceted. Pupils and their teacher must feel free to express and explore those facets that have particular meaning for them. The classroom environment is an important but often overlooked facet. It should be organized and equipped to appear as a laboratory for learning and should relate learning to past experiences while providing new experiences as needed. Well-equipped and organized classrooms allow pupils to accept the responsibility for their own learning and progress.

The learning climate in the classroom should provide an atmosphere of open communication between pupil and teacher. The teacher should encourage questions and accept problems from the pupils. The mathematics' instructional materials should be relevant to the pupil's interest and needs and should provide for pupil experimentation.

The establishment of a classroom climate, under the direction of the teacher, should be pupil oriented, self-directed, and non-threatening. Using definable instructional objectives, the teacher assumes the role of a guide who directs learners to explore, investigate, estimate, and solve everyday, realistic, pupil-oriented problems.

The ideal classroom climate fosters the spirit of "discovery." It provides a variety of ways for pupils to direct their own learning under mature, patient guidance of an experienced, curiosity encouraging teacher. Self-directed learning requires pupil involvement in creative learning experiences that are both pupil motivated and teacher motivated. The classroom climate should encourage pupils to solve problems in a variety of different ways and accept solutions in many different forms. All pupils should express creative thinking, even when it differs from the pattern anticipated by the teacher or when it produces a different conclusion or result.

Instructional materials adopted by the state to implement the mathematics program should be sufficiently flexible to be able to be used with a variety of teaching methods and organizational plans. Whether or not ability grouping takes place, it is clear that in any classroom the rates of learning will vary, and the pacing of instruction should be planned accordingly. Perhaps of more significance, the pupils' modes of thinking will differ: some think best in concrete terms, others, in abstract formulations. The introduction of a new mathematical concept should be done in such a way as to appeal to each of these ways of thinking.

Mathematics Program Evaluation

Program evaluation is a sequence of activities that culminates in a judgment about the success or failure of a program. In evaluating a program at any level, one must provide a response to the question, "Did the program achieve its objective(s)?" The evaluation discussion which follows is designed to provide information useful at the classroom level.

Teachers conduct their classes so that pupils learn mathematics. In order to evaluate their own efforts, teachers use a variety of tools and techniques to assess the progress of each pupil. If pupils do not progress as expected, then the programs should be modified or expanded to accommodate the talents and the needs of those pupils. If the objectives of a mathematics program are reasonable and comprehensive, the quality of the program can be measured according to the accomplishment of those objectives.

Evaluation is a multipart process. Program objectives should be stated and based on an assessment of the needs of pupils. The target population then can be surveyed to ensure an accurate appraisal of its needs, and the program can be adjusted to reflect current conditions in the pupil population. When the instructional program is complete, the population can be assessed to determine the degree to which the program objectives were accomplished.

Matrix sampling is used in California. The procedure requires the development of a pool of test items that provide comprehensive coverage of the mathematics content. After a valid pool of items has been developed, the items are distributed randomly to a number of subtests so that subtests are of similar difficulty. The subtests are randomly assigned and administered to pupils in the examinee population. The underlying theoretical model permits the resultant data to be used to estimate the achievement characteristics of the examinee population as if every examinee in the population had responded to every test item on every subtest. At the school level, the examinee population might be all the sixth-grade pupils in the school. At the classroom level, the examinee population would include all the pupils in the class. This testing procedure has potential for use at the district level, at the school level, and even at the class level.

When the examinee population is small, it is necessary to administer a greater number of items to each examinee to preserve, at least partially, the integrity of the results. Models for the development of a matrix sampling plan are presently available. However, the procedure estimates group characteristics only and cannot be used to measure the achievement of individuals.

The progress of individual pupils in a class is vitally important at the class, and possibly the school, decision-making level. If one wishes to learn of the needs of a particular pupil, it is necessary to consult pupil personnel files, to conduct diagnostic testing and interviews, to observe the learning behavior of the pupil, to utilize achievement test results and teacher-made test results, and to consult parents regarding the status of their children. It is essential that teachers learn as much about their pupils as they can to better serve pupil needs. A needs assessment process sets up the pupil-level objectives which direct the teacher's behavior. Thus, teaching behavior can be directed both by total class achievement and by the achievement of individual pupils.

A cyclical evaluation process that teachers could employ is presented in Figure 1. The cycle is entered by making a preliminary review of the accomplishments and talents of the pupils in the class. That needs assessment gives rise to a tentative set of objectives and a corresponding mathematics program. While the program operates, the teacher uses various tools and techniques to gather data on the condition of pupils in the class. The teacher also seeks parental input regarding the status of the children with respect to school activities. This interim information-gathering activity provides feedback about the progress of pupils and provides a quasi-scientific basis for making program adjustments to better accommodate pupil strengths and weaknesses. That continual needs assessment activity is the link between program development and program relevance.

As the time allotted to the program runs out, the teacher should complete the development of the final evaluation system. The testing instruments for assessing the achievement of individual pupils should be selected or developed with both the stated program objectives and the pupil-level objectives well in mind. In developing an item pool for matrix sampling applications (usually accomplished with the cooperation of other teachers), the program objectives should be used to

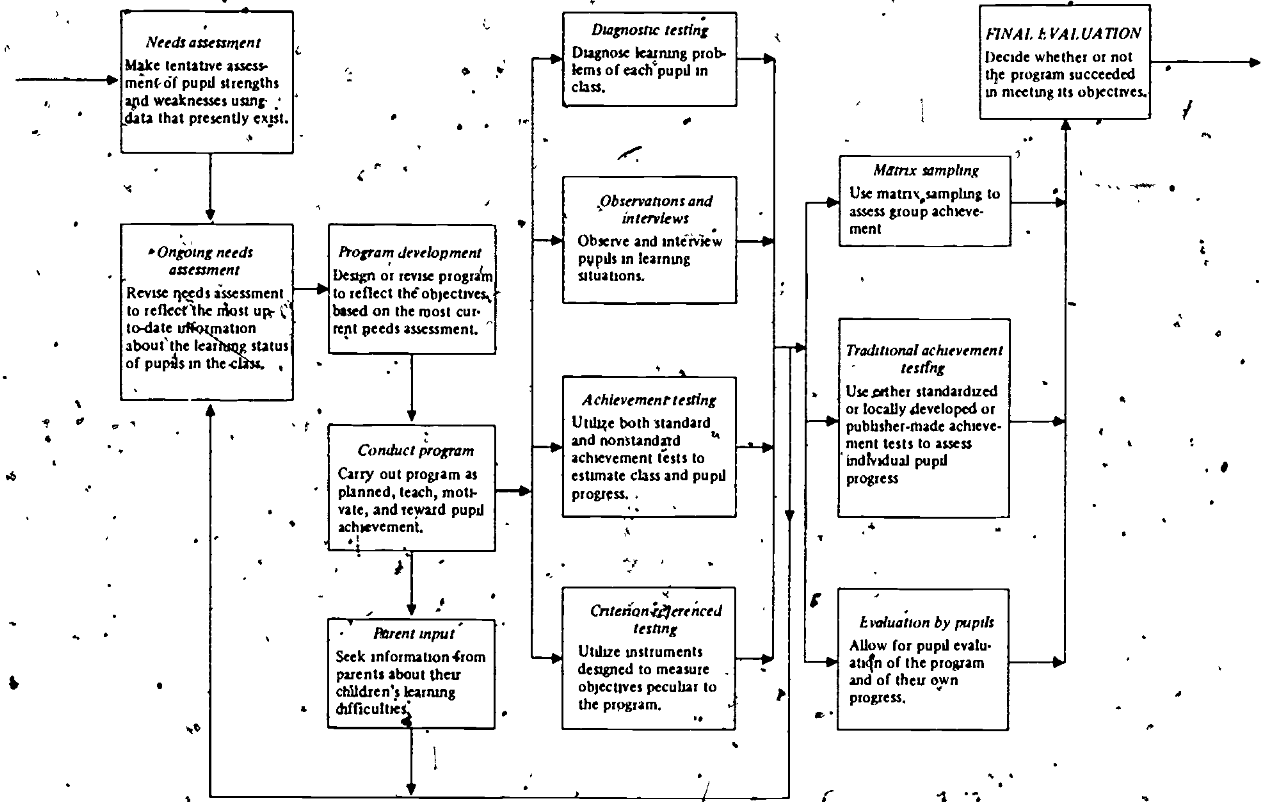


Fig. 1. The schematic of a plan for evaluating a mathematics program at the classroom level.

ensure comprehensive coverage of the content domain, both topically and by level of development or cognition. When the final evaluation activities are carried out and the data on the total program impact are in, then it is necessary to judge how successful the program has been. Did it accomplish its objectives? Should it continue or should it be changed or dropped?

The procedures outlined in this section should help to remind the reader of the things that might be done in evaluating a program. However, the philosophy which governs evaluation is eminently more important than the tools alluded to earlier. At the class level, the teacher must deliver a program to pupils which fits the needs of those pupils, a program that is designed to help pupils learn the mathematics they must know to enrich their lives. The teacher must keep the program abreast of pupil needs. And, finally, the teacher must realize that the program is successful only if it serves to diminish the learning difficulties of the pupils in the class.

The Mathematics Program in Kindergarten Through Grade Eight

Mathematics programs should respond to the needs of children and the needs of a career-oriented society in presenting the content and structure of mathematics. While the study of mathematics for its own sake is possible in such a response, it is not likely to be the prevailing reason for mathematics study by children. When children become aware of the fact that the study of mathematics tends to open up certain career options, their inclination or enthusiasm for mathematics increases. Some children acquire such awareness slowly. It is recommended that programs be designed to permit pupils to study and learn mathematics as long as they attend school, regardless of their level of attainment in the subject matter or the lateness of their decision to engage in further study.

Goals of Mathematics Instruction

As a result of mathematics study, pupils should learn to function smoothly in their everyday encounters with mathematical situations. Their studies should allow them to advance to further study commensurate with their ability and desire to so do. The study of mathematics should also acquaint pupils with the richness of the design of mathematics to allow that element of beauty to become a part of their knowledge. It is expected that most pupils will not become mathematicians; however, no program should prevent such a career outcome.

A number of other goals for programs have had a pervasive influence on the preparation of the framework. They are listed below:

- Mathematics programs should progress from concrete experiences to abstract experiences for all learners, with substantial emphasis on those elements of the environment which are familiar and likely to kindle interest.
- Mathematics programs, to be maximally effective, must be implemented by the efforts of a sensitive, knowledgeable, and skilled teacher.

- The program shall strive to have pupils learn to reason logically and independently and to develop a fondness and inclination for inquiry.
- The experiences of pupils in the mathematics program should equip them with the skill to think and communicate in mathematical terms.
- The program should result in continuous individual pupil growth in the skills of computation and measurement to ensure functional competency of pupils as citizens in a complex society.
- The experience of learners in a mathematics program should result in an understanding and appreciation for the fundamental concepts, structure, and usefulness of mathematics.
- Mathematics programs should be more activity oriented than theoretical, and mathematics programs should require pupils to engage in useful activities designed to generate enthusiastic learning and positive attitudes toward mathematics as a useful tool in their lives.
- The program design should be flexible and provide for a variety of teaching and learning styles. More specifically, the programs conducted should lead pupils to acquire the following:

1. A sound background in the concepts and skills of the real number system, including experiences with:
   a. Sets of numbers and basic operations defined on those sets
   b. Computational algorithms for the basic operations
   c. Properties of the basic operations defined on the sets of numbers
   d. Equality, inequality, and other relations
   e. Functions and other relations
   f. Mathematical sentences
   g. Decimal systems of numeration and place value
2. A background in the concepts of geometry, including experiences with:
   a. Simple geometric constructions
   b. Basic plane and solid geometric configurations
   c. Congruence and similarity
   d. Perpendicularity and parallelism
   e. Symmetry
   f. Circles and polygons
   g. Transformations
   h. Measurement of angles, perimeters, areas, and volumes
   i. Maps and scale drawings

j. Graphing and coordinate geometry in one and two dimensions

3. An appreciation of or an ability to apply mathematics, including experiences with:
   a. Measurement with standard units, including the use of decimals
   b. Estimation, comparison, and scientific notation
   c. Probability and statistics
   d. Discovery of mathematical relationships
   e. Simple deductive systems
   f. Telling time
   g. Strategies and tactics for problem solving
   h. Analysis of problems using mathematical models
   i. Methods of logical reasoning

The relative success of a program can be estimated in terms of how well it seems to meet the above goals. However, a formal evaluation will require assessment against a set of specific program objectives based on these goals and on the specific needs of the pupils and the community served. A program should have a formal evaluation component for judging strengths and weaknesses. With such a component, educators can make meaningful adjustments for program improvement and continuing student growth.

The Early Education Program

The interrelated ideas of mathematics become part of our human experiences at a preschool age. These early experiences provide intuitive background essential to the development of later mathematical content. Therefore, it is important that the instructional program in mathematics begin in the early education of the child.

In the beginning, the development of mathematical concepts for all children should be of an informal and exploratory nature; the goals and objectives set up in this framework provide for the establishment of a program of such a nature. For early mathematical experiences to be effective, guidelines need to be established which will provide a frame of reference for guiding adults involved in developing learning experiences for children. Incidental learning is a useful tool for developing mathematical concepts.

Activities should provide for the involvement of children with physical objects that are usually found in the environment. Activities may also be centered on materials designed to develop certain mathematical ideas. Children should be encouraged to compare, classify, and arrange objects according to shape, color, and size; to experiment with symmetry and balance; and to discover and create patterns. They should discover the comparative relations of "more," "fewer," and "as many as" through the activity of matching groups of objects. In these activities, the child develops understanding of mathematical concepts while learning to name and understand the number of a set, to count, and to develop positional relationships such as inside, outside, on, first, next, last, before, after, between, left, right, above, and below.

Throughout their activities, children should be encouraged to ask questions and talk about what they are doing, both with the teacher and among themselves. Children at this level are imitative and are interested in words. They are increasing their vocabulary rapidly. If the teacher introduces word and language patterns easily and naturally, children will begin to assimilate the words and patterns into their own speech and thoughts. The key words here are *easily* and *naturally*. Children's own ways of conveying their ideas must be accepted at the time; but, concurrently, they should have the opportunity to learn to express ideas with clarity and precision.

Strands for Kindergarten Through Grade Eight

The content of the mathematics programs for elementary levels, kindergarten through grade eight, is organized into seven major content areas (strands), while the secondary program has nine major content areas. For each strand in kindergarten through grade eight, the program objectives are listed by levels in Appendix D, as illustrated in Figure 2.

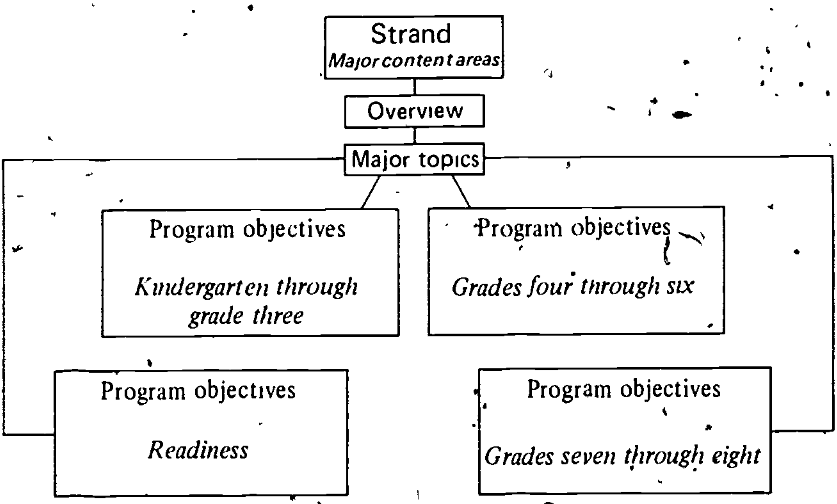


Fig. 2. Organization of mathematics program strands in kindergarten through grade eight

The program objectives identify skills and concepts to be presented by teachers in the following order of conceptualization:

1. Exploration
2. Identification
3. Recognition
4. Development
5. Acquisition/demonstration
6. Application/utilization
7. Maintenance

Arithmetic, Numbers, and Operations Strand

The development of the *arithmetic, numbers, and operations* strand constitutes the most important portion of a mathematics program in all levels in kindergarten through grade eight. This strand reflects the growing concerns of educators throughout California with regard to the urgent need for a reemphasis on children's learning and maintenance of computational skills. The study of the real number system begins with the counting numbers to be followed by the whole numbers, integers, rational numbers, and real numbers.

At the early childhood level, children are provided with exploratory counting and comparison experiences using concrete sets of objects. The basic addition and multiplication facts should be presented and mastered early in the mathematics program. Intuitive experiences with manipulative materials should be used to motivate the development of computational algorithms. At appropriate levels, these techniques should be tied to the ideas from which they derive their validity; namely, the properties of closure, commutativity, associativity, distributivity, identity elements, and inverse elements for each of the number systems studied. Place value is the fundamental principle for naming numbers. In this respect, decimal notation, as well as computational skills with decimal numbers, should receive early priority at the primary level, as described in the *measurement* strand.

Informal mathematical experiences are important at all levels of learning. Most learners need to relate the symbols of mathematics to objects and to images of events from their own experiences in order for the symbols to become meaningful. In developing the ability to work with abstract symbols, pupils should first work with physical models: (1) build with blocks and other materials; (2) handle objects of different shapes and sizes, noting characteristic features; (3) sort and classify objects; (4) fit objects inside others; (5) arrange objects in order of size; (6) experiment with a balance; (7) recognize positional relationships and symmetry; and (8) search for patterns.

Pupils may then progress through pictorial representations to the more abstract and symbolic representations of the same concepts.

A mathematics program should provide for the introduction of new terms and language patterns in close association with other learning activities. Assimilation of vocabulary and language patterns into a learner's own speech and thought is expected to develop gradually. When learners show evidence of familiarity with an idea, they should be provided with opportunities for many relevant activities. Gradually, attention can be given to new terms in reading vocabulary. Most learners should acquire an understanding of, and an ability to read, the standard terminology and language patterns. New terms and language patterns can take place through many opportunities in using the terminology rather than through memorization and parrotlike repetition.

The program-level objectives for the *arithmetic, numbers, and operations* strand are grouped under the following major topics:

1. Counting
2. Operations
3. Place value
4. Patterns
5. Nature of numbers
6. Properties

Activities that guide a learner to recognize and generalize the central unifying ideas in the real number system aid the learner in development of a methodology for systematic thinking. Continual diagnosis of a learner's growth, a planned program of maintenance, reinforcement of skills, and remedial instruction are essential ingredients of the *arithmetic, numbers, and operations* strand.

Counting. Activities in which learners can compare the number of objects in different sets, without resorting to counting, lead to counting concepts. In these activities, pupils compare the number of objects in two given sets by pairing the members and then discovering the size relationship. Counting requires matching the members of a set of objects on a one-to-one basis with the members of the set of counting numbers. More sophisticated experience in counting can be obtained by grouping sets into ones, tens, hundreds, and so on. Learners should have experiences with many types of counting activities, which include experiences with equal sets, equivalent sets, finite sets, and infinite sets.

Operations. The learners should be provided with experiences that will enable them to acquire proficiency in computational skills. The possession of such skills should help the pupil to develop self-confidence in the ability to deal with numbers, the basic operations of numbers, and the applications of numbers. The developmental stages for these concepts range from an intuitive development of the definitions of the basic operations by joining and separating sets to the higher-level ability of developing and using the algorithms for the operations on the systems of whole numbers, integers, and rational numbers in decimal and fractional form.

Place value. The study of place value develops an understanding of the decimal place-value notation. The two major principles of a place value of numeration are base and position. The learner's concept of the decimal place-value numeration system has its beginning in prekindergarten and kindergarten experience when the learner first names numbers. These concepts should be refined and extended throughout the mathematics curriculum. Some of these concepts developed as the learner gains understanding of decimal place-value notation are the following:

1. A decimal system requires ten symbols.
2. The order of numbers.
3. Positional notation to indicate value.

Consideration of systems of numeration that utilize the principle of nonplace-value system can lead to an appreciation of the advantages of a place-value system and an awareness of the historical development of numeration systems. However, numeration systems other than the decimal should be a minor part of any mathematics program.

Patterns. The study of patterns is valuable to the pupil in the study of number systems (their operations and properties). The study of patterns assists the learner in the discovery and development of generalizations, providing not only practice in using the basic facts but experience in working with large numbers. Mathematics has been described as the study of patterns. Important applications of mathematics are a result of the search for trends or patterns among data derived from experiments or from the solutions of problems. The discovery of new ideas through the study of numerical relationships that display unusual patterns should be a regular part of the school mathematics program.

Nature of numbers. The study of the nature of numbers leads to an understanding of the real number system. The nature of numbers encompasses the following:

1. The chief characteristics of numbers, such as whether they are prime or composite, whether they are even or odd, what their factors and multiples are, and what their relation is to other numbers (greater than, less than, relatively prime to)
2. How numbers are used in daily living, such as in counting, measuring, and computing; and how numbers appear in nature, such as in plants, flowers, and shells

Properties. The learner should develop an intuitive understanding and appreciation of the properties of the basic operations and of their applications to everyday problems. The study of properties of the basic operations should include some level of conceptual understanding of commutativity, associativity, identity elements, inverse elements, distributivity, and closure. The transitive property for equality and order relations should be presented. Number sentences are particularly useful in guiding learners to discover patterns for the properties of operations. The same properties are later applied to the solutions of mathematical equations and inequalities.

Geometry Strand

The mathematics program for kindergarten through the various levels of the curriculum should provide for the development of a strong, intuitive grasp of basic geometric concepts such as point, line, plane, and three-dimensional space. Experiences in geometry should relate to familiar objects, since so much of the world is of a geometric nature. Opportunities should be provided at all levels to use manipulative materials for investigation, exploration, and discovery; the opportunities should consist of a wide assortment of informal geometric experiences, including the use of instruments, models, and simple arguments. Thus, a geometry program should provide the foundation for later, formal study.

Teacher awareness of geometry in the environment will enhance the total curriculum and lead to opportunities to incorporate geometry with other disciplines. Examples can be drawn from art forms of all cultures, from industry, and from nature.

From the outset, deliberate effort should be made by the teacher to use appropriate and correct terminology in the development of the geometric concepts. Yet, language should not become a barrier or deterrent to the exploration of and experimentation with geometric ideas. Vocabulary building should include the language of sets in a natural way.

The program-level objectives for the geometry strand are grouped under the following major topics:

1. Geometric figures
2. Reasoning—logical thinking
3. Coordinate geometry
4. Measuring geometric figures

Geometric figures The identification of geometric figures should begin by handling physical models, such as triangular, rectangular, and circular objects. Sorting or grouping objects according to shape and size in the early learning level makes pupils aware of *similarities* and *differences* Later, classification of geometric figures should be refined by specifying additional properties. Pupils should be familiar with figures, such as triangles, rectangles, cubes, spheres, and pyramids.

Two basic concepts of geometry are similarity and congruence. Similarity can be thought of as a transformation which preserves shape but not necessarily size. Congruence is a transformation that preserves size as well as shape. While early experiences with similarity and congruence can be accomplished through sorting and matching, later experiences can include tracing and paper-folding activities and the use of measuring instruments. Many learning activities lead to identifying specific conditions that will ensure these relationships.

Pupil experiences should include experiences with the four transformations of reflection, rotation, translation, and dilation (i.e., scale drawing).

Reasoning–logical thinking. The elementary geometry program in kindergarten through grade eight is a program of "informal" geometry. The word informal refers not to casual manner of presentation or emphasis but to the absence of a formally developed subject using an axiomatic approach. A goal of the geometry program from kindergarten through grade eight should be to provide the foundation for later formal study. When appropriate, the teacher may present short deductive and inductive arguments.

Coordinate geometry. First experiences with concepts of coordinate geometry should be informal and preferably of a physical nature. Arrangements in rows and columns and movement in specified directions are appropriate activities. Children then can plot points in the first quadrant and can graph data recorded in experimental situations. Successive experiences should involve all four quadrants, leading to the ability to graph simple linear and quadratic equations.

Measuring geometric figures. Through realistic situations, the concepts of length, perimeter, area, volume, and angle measurement should be developed. Experiences such as pacing the perimeter of a rectangle (e.g., classroom, hallway, or school yard), tiling plane surfaces with regular shapes, and constructing rectangular solids with building blocks assist pupils to discover patterns leading to general statements or formulas.

"Measuring geometric figures" is an obvious intersection of the two strands *measurement* and *arithmetic, numbers, and operations* with the strand *geometry*. The application of the concepts of measurement provides a wealth of problem situations that frequently demonstrate the practical value of geometry.

Measurement Strand

Often one is unaware of the extensive use made of the process of measurement in daily living because measurement so permeates one's experience. In fact, measuring is a key process in many of the applications of mathematics and serves as a connecting link between mathematics and the evironment. Nonetheless, measurement skill is an acquired skill that is best learned through the *act* of measuring.

In most recent textbook series, only a single chapter or unit at each grade level has dealt specifically with measurement concepts. Additionally, incidental teaching of measurement occurred in problem-solving and application activities in other sections of the instructional materials. Such presentations of measurement often had little to contribute to the overall mathematical development of the learner, except for some possible computation practice and memorization of facts needed to convert within a measurement system.

With the introduction of a metric system, a stronger feeling for measurement can be developed easily because of the way metrics ties directly into our decimal system of numeration. A metric standard will foster a better understanding of measurement concepts because the decimal (tens) nature of metrics is related to the base ten place-value system.

The *measurement* strand is not merely an outline for transition to a different measurement standard but rather an outline for improving the presentation of measurement. Pupils must be given extensive opportunities to use measurement tools and to acquire skills useful in adult life. The transition to metrics provides a convenient opportunity to improve measurement instruction along with the adoption of a universal and less complex standard for measurement.

Earliest experiences should center on physical activities requiring arbitrary units (e.g., width of a hand, capacity of bottles, and clapping of hands) to develop concepts of measuring distance, capacity, or the passage of time. In learning to measure, pupils should begin to use simple measurement tools to measure quantitative attributes of familiar objects. As learning progresses, pupils should be provided experience in carrying out more complex measurements.

The *measurement* strand intersects with the strands of *geometry* and *arithmetic, numbers, and operations.* The use of rulers and protractors to measure geometric figures will lend added insight to the concepts in geometry. When the units used for measuring are metric units, this activity promotes an understanding of decimal notation. Computation involving measurements will provide practice in the operations of arithmetic. The *measurement* strand also provides opportunities for numerous interdisciplinary learning situations that relate mathematics to subjects such as social studies, geography, science, industrial arts, and home economics. Measurement also includes the topics of time and temperature.

The program-level objectives for measurement are grouped under the following topic headings:

1. Arbitrary units of measure
2. Standard units of measure
3. The approximate nature of measurement
4. Estimation in measurement

Arbitrary units of measure. In the introductory stage of learning measurement skills, pupils first become familiar with the properties of the objects to be measured. Next, they learn to make discriminations among those properties. They then learn to compare objects according to the quantitative properties the objects possess in such terms as "is equal to," "is less than," or "is greater than." At this point, arbitrary units of measure are selected or devised to allow the comparison of common properties of objects. Pupils should understand that the arbitrary units selected by others may differ. Experience with arbitrary units should lead pupils to discover the merits of selecting more widely accepted units of measure and to establish the need for standard units.

Standard units of measure. Measurements expressed in standard units result in measurement statements that can be universally understood. The International System of Units (SI) should be the system of standard units taught in the schools of California. Conversion, involving computation from metric to U.S. Customary units and from U.S. Customary units to metric units, must be avoided. However, informal comparisons of metric units with comparable U.S. Customary units may be profitably used during the transitional period and for historical discussions.

The techniques of measurement learned while using arbitrary units are the same as those used with standard units. The metric system, long in use by most of the countries of the world, is a decimal system of standard units. Thus, early instruction with numbers and operations using decimal notation should precede instruction in the use of metric units. The precision required and the complexity of the ideas represented dictate the level at which different units and associated measurement terminology are introduced into the instructional program.

The approximate nature of measurement. The exact measure of a line segment is called its length, just as the exact measure of a surface is called its area. The physical act of measuring a segment with a ruler or a tape measure produces, at best, an approximate measure, due to the limitations of the ability to read a measuring instrument and of the precision of the measuring instrument. The process of "rounding off," familiar to students as a part of operations with numbers, becomes meaningful when applied to the measuring procedure.

In general, pupils should learn to understand that in making and recording measurements, they are dealing in approximations. Ordinarily, one tries to obtain as accurate an approximation of the measurement as possible, although frequently a good estimate may serve the purpose.

Estimation in measurement. The ability to estimate effectively is a skill that has great practical value. Frequently, an offhand estimate will provide a ready check for the result of a calculation and will act as a deterrent to continued calculations with incorrect measurements. Because a good estimate is an "educated guess," skill at estimating develops through many measuring experiences.

In early grade levels, estimates with measurements should be encouraged through comparisons with already accepted measures, whether arbitrary or standard. Later, pupils should develop an intuitive grasp of and familiarity with the standard units so as to be able to make reasonable estimates through direct observation, using visual or other appropriate senses.

Problem Solving/Applications Strand[1]

One major goal of a mathematics program is for pupils to develop the ability to formulate and solve problems and the ability to apply these problem-solving skills in practical situations. In applying mathematics, we are concerned with situations that arise inside as well as outside the domain of mathematics. Application of mathematics requires one to (1) formulate problems that are suggested by

[1]See pages 59 through 74 for the changes in this strand made by the Mathematics Framework Addendum Committee.

given situations; (2) construct, if possible, adequate mathematical models of these formulated problems; (3) find the solution of these models; and (4) interpret these solutions back in the original situations. Problem solving requires one to select strategies for the analysis of a given problem and to use certain mathematical skills and techniques identified in the analysis to solve the problem. Clearly, the ideas of problem solving and mathematical applications are interrelated.

Concrete mathematical applications selected from a wide range of sources should be systematically included in a mathematics program, along with the development of problem-solving strategies and skills. Constant exposure to concrete mathematical applications enables pupils to use concepts, techniques, and skills they have already developed to attack and solve useful problems. This exposure to interesting and useful problems can motivate students to develop new and more significant mathematical skills and techniques.

The strand *problem solving/applications* should be consistently interwoven throughout the mathematics program. Each of the other six strands provides tools for the development of problem-solving strategies and skills. The other strands also contribute toward the development of techniques for expressing and relating mathematical concepts that arise both inside and outside the domain of mathematics.

The ideas of *problem solving/applications* are so important that a mathematics program should include periodic study of formulation techniques, problem-analysis strategies, and problem-solving techniques. However, the strategic principles of problem solving should not be presented as a specific format that must be followed nor as a step-by-step procedure to which all solutions must conform. The creative solution of a problem is more valuable than a burdensome routine. *Creative thought or insight should not be stifled by having to conform to unnecessary formalism.*

The program-level objectives for the *problem solving/applications* strand are grouped under the following major topics:

1. Problem formulation
2. Problem-analysis strategies and tactics
3. Constructing mathematical models of problems
4. Finding the solution
5. Interpreting the solution

Problem formulation. Problem formulation should be an outgrowth of pupil experiences that arise in the context of some interesting event—often a phenomenon arising in everyday life, in the social sciences, in the life sciences, in the physical sciences, in the humanities, or in mathematical recreations. The situations selected for study should be so meaningful that the pupil will honestly expect to experience the situation or will have some assurance that other people actually do experience the situation. As pupils work with concrete situations, they consciously or unconsciously pose problems that seem to need solution, or they ask questions, such as, "Why does this work?" The ability to formulate meaningful problems has as much value in the marketplace as does the ability to solve problems.

Each attempt on the part of a pupil to formulate a problem should be nurtured and encouraged. To this end, it is recommended that mathematics programs include a significant number of concrete situations that require pupils to explore, analyze, and investigate. Some of these situations should lead to problems that are open-ended in the sense that they invite conjectures.

Problem-analysis strategies and tactics. A mathematics program should systematically assist pupils in devising strategies for analyzing problems that lead to some success in solving the problems. The first step in any strategy is to make sure the problem is understood. Regardless of the origin of the problem, the solvers must understand the problem so well that they can restate it in their own words. The solver should be able to pinpoint the purpose of the problem, to indicate the unkowns, and to identify the given data. Several tactics are available to help the pupil at this point:

1. Guess some answers, try them out, and observe the results of the different guesses.
2. Construct a diagram, a graph, a table, a picture, or a geometric representation of the situation, and observe the relationship between the various parts of the problem.
3. Construct a physical model of the situation, or use physical materials to simulate the features of the problem.
4. Search for and identify underlying functional relationships in the problem.
5. Compare the problem or parts of the problem with similar or simpler problems that are more easily understood.

The development of problem-analysis strategies and tactics should start with the pupil's first mathematical experiences and accompany the development of basic mathematical concepts and skills. The use of a variety of analysis tactics should become the habitually accepted thing to do.

Constructing mathematical models of problems. Mathematics does not literally deal directly with the raw physical situation but only with a refined model of the situation. Mathematics does not divide ten apples by five children. Rather, mathematics provides an operation which divides the number ten by the number five; the answer, two, is interpreted as meaning that each child will have two apples. The distinction between the model and its origin is crucial, especially in more complex situations in which the model does not fit the situation so exactly. To illustrate this, one assumes in the preceding example that the ten apples are all at least edible.

The form of the mathematical model is usually one or more written sentences using mathematical symbols. Other models may, at times, be more appropriate; e.g., a picture of sets and a geometric figure. Basically, a mathematical model of a problem is any representation which permits manipulation by mathematical principles. A mathematical model tries to copy some of the characteristics of a given situation. To be successful, a model should accomplish the following:

1. Include as many of the main characteristics of the given situation as practical.
2. Be designed so that the included characteristics of the given situation are related in the model as they are in real life.
3. Be simple enough so that the mathematical problems that are suggested by the model can be solved readily.

A mathematics program should provide pupils with experience in discussing and constructing mathematical models of given situations. The reverse process is equally important: Given a mathematical model, the pupil will construct a real situation for it.

Finding the solution. The solution of a problem requires a wide variety of technical skills. Basic computational skills and an understanding of number properties are essential to finding solutions. Pupils also need the skills related to solving equations and inequalities, to graphing, to constructing geometric figures, and to analyzing tabular data. A mathematics program should include a substantial number of ready-to-solve problems that are designed specifically to develop and reinforce these technical skills and concepts.

In most problem situations, the results should be anticipated by estimating in advance. Estimating should be introduced early to all pupils as a standard operating procedure. Sometimes a solution when compared with an estimate may reflect a significant oversight, and then the major concerns should be: How did you go about it? Is the model adequate or valid? Was the solution process completed correctly? or Were the assumptions made too broad or restrictive?

Interpreting the solution. A mathematics program should systematically include experiences in the interpretation of the solutions obtained. The problem and its solution should be reviewed to judge the validity of the model and the accuracy of the mathematical manipulations. Discussion of a solution should resolve the following questions:

1. Was the problem solved?
2. Would another model work?
3. Can the model be improved?
4. Can the model be extended to solve related problems?

Probability and Statistics Strand

People today are overwhelmed with data from the mass media. They need to understand, interpret, and analyze these data in order to make decisions that affect everyday life. Therefore, experiences in collecting, organizing, and interpreting data should be included in a school mathematics program. These experiences should begin in kindergarten and should be a part of the instructional program at each succeeding level through the eighth grade and beyond.

Statistics is the art and science of collecting data, organizing data, interpreting data, and making inferences from data. One deals with some degree of uncertainty when trying to make these inferences. It is at the stage of decision making that one applies the concepts of probability so as to select alternative courses of action which are likely to produce desired results.

The program-level objectives for the *probability and statistics* strand are grouped under the following major topics:

1. Collection, organization, and representation of data
2. Interpretation of data
3. Counting techniques
4. Probability

Collection, organization, and representation of data. Collecting data should be the outgrowth of experiences involving observations by the pupil. The classroom, as well as the world, provides the pupil with an abundant source of data.

Organizing data is an art that the pupil must learn. The information in a table, graph, or chart must be presented in such a way that it fits the purpose for which the data were orginally gathered. In this topical area, the emphasis is on the construction and interpretation of the various graphs and tables needed to organize data.

Pupils working in the *probability and statistics* strand should have opportunities to see the relationship of mathematics applied to other areas of the curriculum.

Interpretation of data. Graphic devices are useful for offering a quick visual summary of a large collection of measurements or facts. If further interpretation or comparison of data is required, then measure of central tendency or scatter is needed (e.g., range, percentiles, mean, median, mode, and standard deviation).

Counting techniques. Before the more formal aspects of probability theory are presented, the pupils should develop a feeling for techniques of counting events. Counting procedures include tree diagrams, combinations, permutations, and sample space.

Probability. It is possible to introduce some of the beginning concepts of probability at the elementary level; however, most of the concepts in probability should be presented at the high school level.

Mathematical models of many scientific and economic problems exist within probability theory. The ability to assign numerical values to ideas that involve uncertainty is one of the concerns of probability. Probability theory is necessary to carry the interpretation of data to the point of making statistical inferences or "wise decisions" in the face of uncertainty.

The following ideas are considered appropriate for pupils in the elementary schools: sample space, definition of probability, probability of an event, independent events, probability of certainty, probability of nonoccurrence, P $(A \cap B)$, P $(A \cup B)$, and complementary events.[2]

Relations and Functions Strand

Mathematics offers a way of organizing and understanding most observations of the world about us, both in and out of school. One justification for including mathematics in the school curriculum seems to reside in the exploration of the notion of patterns and relationships. This approach to mathematics enables a child to discover and describe something of the shape and pattern of the universe. From the day children enter school, teachers should organize experiences that will encourage the children to think, seek, and discover ideas for themselves, to look for patterns and relationships, and to form generalizations. As these relationships are seen and discussed, concepts become clearer, and fundamental principles emerge that have value in unifying the study of mathematics to follow. Mathematics *is* the story of relationships.

[2] $P(A \cap B)$ represents the probability of A *and* B occurring. $P(A \cup B)$ represents the probability of A *or* B occurring.

The program-level objectives for the *relations and functions* strand are grouped under the following major topics:

1. Patterns
2. Relations
3. Functions
4. Graphs

Patterns. In preschool activities, the term "patterns" refers to painting, drawing, woodworking, making collages, and constructing models. The language used while engaged on a particular piece of work and the child's and the teacher's observations on various aspects of the work will help to heighten the awareness of size, shape, pattern, and relative position of objects.

At the primary level, patterns activities provide visual representations for discussing symmetry, repetition, counting, ordering, and pattern discovery. The child should develop an appreciation for the use of patterns to predict and make conjectures about future events. From experiences with patterns of two elements, the child should also become familiar with the notion of ordered pairs. In another application of patterns, the child can learn to recognize physical or pictorial representations of fractions.

Relations. Most of mathematics is concerned with relations. The young child learns early to relate certain objects or sounds with other objects. For example, a child associates other children with their parents. Intuitively, a child recognizes without formal articulation certain associations between pairs of objects or names of objects. Thus, the child learns to form ordered pairs, such as name and object, in the development of his or her language.

Sets of related pairs of objects are studied throughout the mathematics program. The process of forming pairs should be introduced early in the mathematics program. In beginning arithmetic, pupils learn to associate a set of objects with a number. First, they learn to count by pointing to the objects in sequence and pairing the objects with the set of ordinals. Then the pupils find that counting is a way of determining what number is to be associated with a certain collection of objects. Thus, counting determines certain related pairs; namely, *(set, number).*

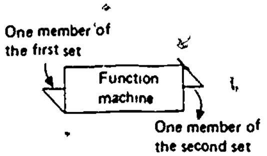
Another example of related pairs can be observed in the relationship "greater than," which may be thought of in the form *(number, greater number).* This is an example of a relation in which a single number is related to many other numbers.

As pupils develop skill in collecting empirical data, they should begin to search for meaningful relations in the data. For example, a pupil could discover the relation between the number of diagonals that can be drawn and the number of sides of a polygon.

Functions. One type of relation is of particular importance to science and mathematics: Each member of a set is related to one and only one member of a second set. For example, for each child there is just one natural mother. Such a relation is called a functional relation or, simply, a function. Thus, the pairs of related objects that form a functional relation have the property that just one related pair exists with a given first member. Some functional notation should be used systematically by the end of the eighth grade. Several notational schemes may be suggested, as shown in Figure 3, and different notational schemes should be used on occasion since different notations are suggestive of different aspects of the function concept.

The function concept includes mathematical operations. Elementary pupils encounter functions when learning the number facts. Multiplication by five, for example, identifies a function defined on the set of numbers.

Intuitive experiences will enable pupils to develop the concept of a function as a set of ordered pairs in which no two pairs have the same first element. Pupils should further realize that functions can be identified by statements, formulas or equations, tabulated data, and graphs. The pupils will then be on the way toward understanding a mathematical idea that has many applications. The mathematics program should offer the pupil familiarity with the functional notations given here and should enable the pupil to plot linear and quadratic functions, as well as functions with jumps, such as the greatest integer function.



INPUT	OUTPUT
3	6
1	2
5	10
0	0

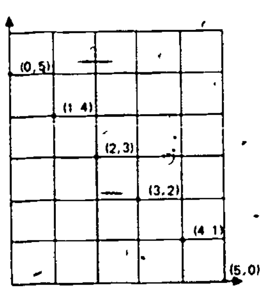


Fig. 3. Examples of notational schemes for functions

Graphs. Learning to present information in a graphical mode is essential. Graphs provide pupils with an organized method of recording and communicating their observations. Observations can be recorded as soon as a child has reached the collecting stage; it is while collecting and sorting data that the pupil begins to make comparisons and to form relations.

Use of a pictorial description of pairs of objects should begin early in the mathematics program. Plotting could be initiated with games such as tic-tac-toe; a class could graph the height of a plant on successive days of a month; or the class could record temperatures for each day in a month. The pupil's experience with such graphing reinforces the concept of the number plane, presents a picture for linear relations, and provides an excellent way of intuitively developing an understanding of the concept of grasping relations.

Graphing is also invaluable in the applications of mathematics. A class may record the length of a spring or rubber band as weights of successive sizes are suspended; or, at a more advanced stage, the period of a simple pendulum could be recorded while the length of the string is varied. In the study of measurement, a class may record the number of grams (or any standard unit) of water required to fill cylindrical jars of various diameters to a fixed depth. In the study of geometry, a class may plot circumference as related to diameter for different circles.

The mathematics program in kindergarten through grade eight should include the study of the coordinate plane. The association between each point in the plane with an ordered pair of numbers is basic to the mathematics that connects geometry and algebra. This association is also basic to the understanding of maps and, more generally, of scale drawings.

Logical Thinking Strand

The beginning approach to logical thinking is informal in grades one through three; beyond grade three, the requisites are more precise. Experiences with different kinds of sentences (using *and, or, not, if . . then, all,* and *some*) and with some fundamental patterns of reasoning should be provided. These sentences should be verbal sentences, as well as mathematical, so that the experiences with the sentences will aid the pupils in seeing the importance of logic in relation to patterns of thought in ordinary life as well as in mathematics.

The elementary mathematics program should help pupils learn to organize ideas and to understand what they learn. Though much of informal logical thinking and deductive reasoning is a matter of common sense, the use of standard logical techniques can help pupils to organize the thought processes involved. Children should be able to decide whether a particular mathematical construct fits a definition and to recognize a specific application of a general principle.

In fact, logical thinking is both a desired outcome of mathematics programs and a capability essential for learning mathematics. If pupils are to progress steadily in mathematics, they must learn to recognize patterns. To recognize patterns, they must think logically enough to make discriminations and to find order in those discriminations. Hence, logical thinking can be considered to embrace two topical divisions: (1) patterns in mathematics; and (2) formal and informal reasoning.

Patterns in mathematics. The close connection between the ability to recognize patterns and the ability to think logically should be utilized at all levels; it will prove invaluable to the pupil in the study of number systems and operations. Patterns exist in most life situations, in nature, in history, in music, and so forth. In mathematics the pupil can be taught to utilize not only these patterns but also those existing in numbers and geometric figures in order to gain an understanding and appreciation of the beauty, logic, and order of the world.

Formal and informal reasoning. Any program for kindergarten and the early grades should provide many opportunities for children to explore and manipulate concrete objects, to identify likenesses and differences, to classify and categorize objects by their characteristic features, and to state generalizations.

At the lower elementary level, the use of many definitions of mathematical terms should be nontechnical in nature. The pupils should become acquainted with terms such as *all, some, and, or, if . . . then,* and *not* in such a way that they will be able to understand the meaning of these terms in a mathematical context.

Venn diagrams and a variety of mathematical sentences should be used consistently throughout the program and in every strand where appropriate. Starting with kindergarten, the language of sets should be used as needed to gain clarity, precision, and conciseness in mathematical communication and to aid in the reasoning process.

The Mathematics Program in Grades Nine Through Twelve

One important ingredient for a successful mathematics program in grades nine through twelve is the establishment of confidence on the part of learners. A program with "built-in" success can improve the student's self-image and can assist the student in gaining confidence in mathematical capabilities. A positive and open classroom climate is essential in the attainment of this goal.

If students are involved in the learning process through activities that emphasize discovery, inquiry, or experimentation, then the students can be provided with meaningful success experiences. Successful learning experiences encourage the student's further involvement and can also set the stage for increased motivation to learn mathematics. Such an approach in grades nine through twelve can start a positive learning cycle for many students in mathematics.

Diagnostic techniques should be used with students who have learning difficulties to identify the areas of difficulty, and then prescriptive teaching can be designed to meet the needs of those students. One danger in this strategy is that this technique may not be motivating to many students. Pointing out to students their past failures and asking them to eliminate their shortcomings often result in an immediate "turnoff." Instead, the introduction of new techniques and materials that emphasize a more positive approach can capture the interest of students and promote a better learning environment.

Continuous and flexible in-service training programs must be organized and funded on state, regional, and local levels to develop and maintain the mathematics program described in this framework. Up-to-date and responsive preservice programs must also be under constant development and evaluation. The success of the mathematics program in grades nine through twelve depends in large part on the mathematical competence of the teachers.

The improved preparation of teachers is certainly an important prerequisite of any improved mathematics program. While many teachers may need updating in content areas, such as probability, statistics, transformational geometry, linear algebra, metric system, or computer mathematics, it appears equally important that teachers be prepared to teach average and below-average students who have career or school goals that require mathematics. Teachers need to be able to guide students in learning activities that emphasize discovery and inquiry. Relevant and systematically organized in-service training programs and realistic preservice programs in mathematics can provide the background and support that teachers need to implement a strong mathematics program for all students.

The magnitude of the role that mathematics teachers play in the counseling of students into mathematics classes must increase substantially. Additionally, to communicate clearly and correctly the many changes in the needs and requirements in job training and college programs, mathematics teachers must accept and be given greater responsibility in the placement of students in mathematics courses in grades nine through twelve. Mathematics teachers universally regard the placement of students in classes for which the students are emotionally, mentally, or technically unprepared as one of the major causes of deterioration of the learning evironment, not only for the misplaced students but also for their classmates. Mathematics teachers should be given the time, information, and support necessary to act as informed advisers to students in relation to the students' progress and selections of mathematics courses. Mathematics departments in grades nine through twelve should develop systematic couseling programs and procedures for all students enrolled in mathematics.

Finally, it is recognized that the development of new mathematics technical equipment and multimedia materials will continue to have an impact on the mathematics program in grades nine through twelve. Secondary schools should prepare for (1) a substantial increase in the use of computers and minicalculators in many mathematics classes; (2) the establishment of mathematics resource centers, laboratories, and media centers; and (3) the increased use of aides or paraprofessionals in their mathematics programs.

Goals of Mathematics Instruction

The goals of the mathematics program in grades nine through twelve are the following:

- Develop, commensurate with each student's ability, the mathematical competence that is necessary to function in society. This includes the ability to (1) recall or recognize mathematical facts, definitions, and symbols; (2) count, measure, and handle money; and (3) conceptualize spatial properties.
- Develop, commensurate with each student's ability, the skill of performing mathematical manipulations. This goal includes

(1) the ability to do straightforward computation; and (2) the ability to manipulate relations or to perform the computations required in a variety of mathematical models.

- Develop, commensurate with each student's ability, the understanding of mathematical concepts and processes. This goal includes the ability to transform or translate from one form of symbolism to another, such as from words to symbols, symbols to words, equation to graph, physical situation to formula.
- Develop, commensurate with each student's ability, the skill to select knowledge, information, and techniques that are needed to solve a particular problem—social, technical, or academic—and to apply these selections in the actual solution of a problem.
- Develop, commensurate with each student's ability, the capability of using mathematics and mathematical reasoning to analyze given situations, to define or formulate hypotheses, to make optimum decisions, and to verify the validity of results.
- Develop an appreciation of the importance and relevance of mathematics as a substantial part of the cultural heritage of the human race that permits people to invent and discover relationships that influence and order their environment.

General Objectives of the Mathematics Program

The mathematics program in grades nine through twelve should provide for the following:

1. Aquisition of the skills and concepts presented in the framework for kindergarten through grade eight
2. Development of courses and curriculum organizations to provide the opportunity and encouragement for all students to continue their study of mathematics to meet their specific career and educational goals
3. Development of a series of topical minicourses as an alternative for the traditional yearlong general mathematics course for the noncollege-bound student
4. Development of alternatives for the traditional one-year blocks of algebra and geometry (to serve one of the needs of the large middle majority of nontechnically oriented secondary school students)
5. Development of a remedial clinic program for mathematics students who are achieving below their expected level of achievement
6. Development of mathematics resource centers or mathematics laboratories to be used as an integral part of the instructional program of each mathematics class

7. Acquisition by all students of knowledge about the nature of a computer and the roles computers play in our society; and for some students the opportunity to acquire skills and concepts in computer science, including career training
8. Development of programs for talented students leading to the completion of one year of calculus or another advanced elective course by the end of the twelfth year

Concepts of the Framework at the Elementary Level

Through the mathematics program in grades nine through twelve, students should have adequate opportunities to acquire, as necessary, the skills and concepts presented in the framework for kindergarten through grade eight. The increased need for mathematical learning on the part of citizens in a modern society is recognized in the framework for kindergarten through grade eight, which is devoted to the development of mathematical skills and concepts that all citizens should know to function satisfactorily in our rapidly expanding technological society. Some students in grades nine through twelve, in spite of their best efforts, will need additional study in the content of the kindergarten through grade eight mathematics program.

The program in grades nine through twelve must give ample opportunity for learning basic computational skills and applications of mathematics at the level of the students' needs. However, new materials and strategies are required for the students in grades nine through twelve who need study in the content of the kindergarten through grade eight program; high school teachers should not continue teaching these students, using the same methods which have proved unsuccessful in the earlier grades. New mathematical concepts should be included in each instructional unit, incorporating new approaches and techniques and thus recapturing the interests of students and indirectly improving their performance. A reorganization of staff (such as differentiated staffing; use of specialist teachers, teacher assistants, or individualization of instruction; use of nongraded classroom organization; or different grouping patterns) may be necessary to achieve this objective.

Encouragement to Study Mathematics

The grade nine through twelve mathematics program should provide for the development of courses and curriculum organizations that would provide the opportunity and encouragement for all students to continue their study of mathematics to meet their specific career and educational goals. The program in grades nine through twelve must meet the needs of students aiming for various careers in technical fields, as well as the needs of college-bound students interested in social sciences, humanities, economics, or biological and physical sciences. For example, a school could offer courses specially designed to assist students to prepare for examinations for apprentice programs, for industrial positions, or for civil service by utilizing modern technology, equipment, and media appropriate to the particular fields of employment.

The result of implementing this objective would be in sharp contrast to present practice in which students who have arithmetic difficulties are often permanently shut out from all other mathematics courses. Courses could be designed and offered to cover the usual content at different rates, or new approaches could be offered, depending on the background, motivation, and ability of the student. Statewide in the 1960s and 1970s, a minority of students successfully completed first-year algebra. In fact, state reports indicate that a majority of California students are permitted to take only a general mathematics course. In some areas, students are required to take mathematics in grades nine through twelve and may spend two or more years in general mathematics courses that are essentially grade-six or grade-seven arithmetic, with no possibility of studying concepts of algebra, geometry, statistics, computers, and so forth. These limitations cannot remain if the needs of the students and society are to be met.

Minicourses in Lieu of General Mathematics

The grade nine through twelve mathematics program should provide for the development of a series of topical minicourses as an alternative for the traditional yearlong general mathematics course for the noncollege-bound student in grades nine through twelve. The topical minicourse approach to general mathematics provides a way to accommodate the large numbers of above-average, average, or below-average students with diverse goals and abilities who elect to take general mathematics in grades nine through twelve.

Topical minicourses could be packaged into nine-week quarter blocks, allowing students to select up to four different minicourses in place of the usual yearlong course in general mathematics. Schools, for example, that now offer two identical yearlong general mathematics classes could offer up to eight different minicourses. Additional minicourses could increase the length, adaptability, and flexibility of this recommended program.

Some of the minicourses would have prerequisites, but prerequisites should be kept to a minimum so that these elective minicourses can be taken in a variety of sequences. Diagnostic tests could be used to measure student need for the minicourses. For example, students who demonstrate a need to improve their basic computational skills could be required to enroll in minicourses, such as the following:

Mathematics clinic
Whole numbers, integers, and rational numbers
Ratios, properties, and percent
Geometry and measurement

Some other topical minicourses that could be offered are the following:

Calculating devices and minicalculators
Mathematics and living things
Measurement, measuring devices, and the metric system
Practical geometric constructions
Reading and using tables and graphs
Quality-control statistics
Flowcharts, computers, and programming
Credit and installment buying
Consumer economics
The mathematics laboratory
Mathematics and games

Alternatives to Algebra and Geometry

The mathematics program in grades nine through twelve should provide for the development of alternatives for the traditional one-year blocks of algebra and geometry to serve one of the needs of the large middle majority of nontechnically oriented secondary school students. Alternative courses should be true alternatives with equivalent college preparatory standing, not the conventional sequence of algebraic or geometric topics presented at a slower pace in "watered-down" courses. The usual grade placement of topics should be replaced by offering topics chosen from arithmetic, algebra, and geometry; the topics should be arranged in a logical sequence so that they provide mutual support. Topics such as functions, coordinate geometry, transformations, computer programming and flowcharts, and probability and statistics should be interwoven throughout. A strong effort should be made to make clear to the students the applications and relevance of mathematics to the real world.

For those students in the alternative mathematics courses who want a third or fourth year of mathematics, a third-year transition course should be offered that could prepare them to take such fourth-year courses as probability and statistics, computer programming, linear algebra, elementary functions, or a course to prepare for the AB Advanced Placement Examination in calculus. To provide another degree of flexibility, alternative courses could be organized into nine-week, semi-independent minicourses, thus allowing students a greater choice in the depth and direction that they could choose to follow.

Remedial Clinic Program

The mathematics program in grades nine through twelve should provide for the development of a remedial clinic program for mathematics students who are achieving below their expected level of achievement. A clinic program should be designed to provide individualized instruction aimed at meeting the needs of selected students whose mathematics achievement is significantly below their expected level of achievement. The program should be organized and planned to meet the identified needs of students at each school. A procedure for identification of students should include teacher/counselor recommendations and testing data. The program for each student should be planned individually and include diagnostic testing, pretesting, individualized instruction, and post-testing.

The clinic may operate as a "pullout" program or as a quarter or semester course. Aides should be provided to assist the teacher of the mathematics clinic. This aide(s) could be a paraprofessional, a parent, or a student assistant. A clinic should operate at a low pupil-teacher ratio (maximum 16:1) and should be adequately funded by state and local funds so as to provide for special materials and equipment and for optimum conditions for remediation and learning.

Mathematics Resource Centers

The mathematics program in grades nine through twelve should provide for the development of mathematics resource centers or mathematics laboratories that are used as an integral part of the instructional program of each mathematics class. Mathematics laboratories create the opportunity for and the encouragement of student research activities in applied mathematics. They also provide the environment for learning mathematical skills and concepts through the use of manipulative materials or the use of equipment and techniques that are a part of the daily procedures of business, industry, or science.

Sufficient funds should be provided to ensure that appropriate manipulative materials and up-to-date equipment are available and that adequate staffing is provided. In particular, teacher aides should be provided to maintain, organize, process, and control the use of materials and equipment. A professional staff member should also be designated as a director to train teachers in the use of the laboratory and to spearhead the development of new and innovative materials.

Computational aids, such as desk calculators, slide rules, tables, electronic programmable calculators, and computer terminals, should be an integral part of the laboratory learning system.

Knowledge of Computers

The mathematics program in grades nine through twelve should provide for acquisition by students of knowledge about the nature of a computer and the roles computers play in our society; and for some students, the opportunity to acquire skills and concepts in computer science, including career training. The average U.S. citizen has little idea of how computers work and how pervasive their influence actually is. The average citizen is, in short, culturally disadvantaged. It is essential that our educational system be expanded in such a way that every student becomes acquainted with the nature of computers because of the current and potential roles that computers play in our society. At a minimum, courses that include instruction in "computer literacy" should accomplish the following:

1. Give the student understanding about the way the computer works so that the student can understand what computers can and cannot do.
2. Include a broad sampling of the ways in which computers are used in our society, including nonnumeric as well as numeric applications. The impact of these various uses on the individual citizen should be made clear.
3. Introduce algorithms (and their representation by flowcharts). If time and equipment are available, computer programs representing the algorithms should be written and run on a computer, with printouts made available to the students.

Additional computer instruction should be designed to develop proficiency in the use of computers, particularly in the mathematical, physical, biological, and social sciences. Also, opportunities in vocational computer training should be more generally available. Currently, more than a million workers find employment in the computer industry, and this number will likely continue to increase.

Programs for Talented Students

The mathematics program in grades nine through twelve should provide for the development of programs for talented students leading to the completion of one year of calculus or another advanced elective course by the end of the twelfth year. The mathematics program in grades nine through twelve should include programs for talented students whose education or career plans require a strong mathematical background. It is estimated that at least 10 percent of the high school population, if interested, could complete a calculus course and receive college credit. However, it should be recognized that some capable students are not technically oriented and may not elect to take calculus. Alternative topics for these students could include linear programming, linear algebra, probability and statistical inference, introduction to logic, introduction to computers and computer programming, analytic geometry, or game theory.

Strands for Grades Nine Through Twelve

As in the program for kindergarten through grade eight, the mathematical content of the program for grades nine through twelve is classified by strands, seven strands for kindergarten through grade eight and nine strands for grades nine through twelve. The two additional strands in grades nine through twelve are *algebra* and *computers*. The name of one strand has been changed for grades nine through twelve to reflect a more sophisticated level of content; *arithmetic, numbers, and operations* becomes *the arithmetic of real numbers.*

Teachers in the program in grades nine through twelve are responsible to the students for instructional activities which reinforce and maintain all the concepts and skills identified in the kindergarten through grade eight program. Except for the two additional strands, each of the strands in the program in grades nine through twelve builds on the corresponding strands in the kindergarten through grade eight program. To comprehend fully the scope of each strand, the reader of this framework should first reread the corresponding strand and program-level objectives in the kindergarten through grade eight program.

The preparation of the grade nine through twelve program in this framework did not include the identification of the major topics in each strand nor the development of program-level objectives as was done in the kindergarten through grade eight program. Those tasks, as well as the development of specific instructional activities, await the reaction of the educational community to this framework. (See the addendum for an updating of the framework.)

The Arithmetic of Real Numbers Strand

The number concept in the program in kindergarten through grade eight is extended to the entire set of real numbers, rational and irrational. Through the discussion of roots, particulary of the irrationality of the square root of two, pupils intuitively understand that there are in any unit interval infinitely many points that correspond to irrational numbers. Such treatment of the number concept should not exclude practical applications of numbers at any ability level. It is important in this regard to continue the use of concrete applications, manipulative materials, multimedia materials, minicalculators, and exploratory laboratory activities.

Algebra Strand

Algebra is simultaneously tool and product—a prerequisite skill to studying mathematics and an end in itself. In both cases, algebraic concepts are an essential part of vocationally oriented mathematics courses. Learning the basic algebraic manipulations, as well as the logical system of algebra, is essential to all the strands in the program at the high school level.

The standard algebra course has undergone many changes that should be included in the program in grades nine through twelve. The concept of function has become the central theme throughout the algebra program. Inequalities and graphing now receive increased emphasis. New topics include linear algebra, linear transformations, matrices, and linear programming. Trigonometric functions are part of the current algebra program. The use of logical proof based on definitions, axioms, and postulates is now as essential to algebra as it once was to geometry (although care must be exercised by teachers to avoid overemphasis in the algebra program).

The algebra program should be allowed to continue to evolve, taking into account new ideas as they emerge.

Geometry Strand

The main functions of teaching geometry at the high school level are: organizing geometric facts into a more formal mathematical structure, extending and broadening the student's knowledge of mathematics, and applying geometric concepts in problem-solving situations.

Geometry has its roots deep in the historical development of mathematics. The evolvement of π, of irrational numbers, and of a postulational system as a model for logical reasoning are examples of mathematical ideas derived from the study of geometry. On the practical side, geometric ideas find wide application in such diverse career fields as clothing design, industrial design, architecture, construction skills, engineering, art, scientific research, advertising, and packaging. Geometry holds great potential for helping students gain understanding of and insight into arithmetic and algebra through a visual approach to learning number and algebra relationships. Not to be underestimated is the fact that for a large number of high school students, geometry provides a gateway to mathematics, creating an awareness of mathematics' breadth and depth and initiating many into logical reasoning.

The teaching of geometry in the high school can range from brief topical units to a full course along the lines of a traditional curriculum. Contemporary thinking about the nature of approaches to teaching formal geometry recognizes that there is no one best scheme. Pedagogically speaking, the degree of intuitive understanding desired relative to the amount of formal learning through logical reasoning depends on the ability and interest levels of students and teachers. Mathematically speaking, different approaches center on the choice of postulates that characterizes the system.

A majority of present-day geometry courses follow the postulational design of two- and three-dimensional Euclidean geometry modified to accommodate the natural relationship with the real numbers. Still other course designs emphasize coordinates, leading to applications in analytic geometry, calculus, and transformations and appealing both to manipulative experiences with concrete objects and to techniques of modern algebra. It is recommended that formal geometry courses be designed to maximize the mutual benefits in the understanding of both algebra and geometry; courses departing from this norm should be considered only where highly qualified teachers are available and where the total mathematics curriculum is able to accommodate different approaches.

Although no general agreement exists about what a geometry course should be, there is general recognition of some features that should characterize every geometry course. First of all, the course should be based on a set of postulates that is adequate, at a high school level of sophistication, to support proof of the theorems to be studied. The course should blend geometry of two and three dimensions and should contain substantial coverage of the topics of perpendicularity, parallelism, congruence, and similarity. The ability to use deductive methods to establish proofs of theorems is a desirable outcome. Mensuration theory, as applied to developing the usual formulas for measuring lengths, areas, and volumes, should be taught, and considerable practice in the use of these formulas should be given. Even in a course not emphasizing coordinate methods, an introduction to coordinates should provide the basis for an extended treatment in more advanced study.

Measurement Strand

Measurement skills and concepts are well-covered in the program in kindergarten through grade eight, but as pupils near the age of vocational preparation and consumer responsibility, measurement activities take on a more sophisticated nature. Some aspects of measurement (such as the approximate nature of measurement, precision, accuracy, and relative accuracy) then become worthy of consideration, because these aspects exhibit the strengths and limitations of measurement. Every learner should emerge from the study of measurement with an understanding of its approximate nature, as well as with an ability to select and use basic measuring instruments correctly and efficiently.

The SI system of measurement (the International System of Units) is gradually and steadily gaining acceptance as the standard units of measurement in the United States. A shift to such a system became inevitable for the United States when it became the only industrial power that was not utilizing a metric system as its measurement standard. To prepare learners for this transition, the mathematics program should use SI units as the basis for instruction and practice in measurement. Initially, references to the U.S. Customary system may be useful in making the transition to metrics. In most cases, conversions between systems should be avoided and discouraged. Conversions within the SI system may be profitable in establishing familiarity with the units and nomenclature of the SI system.

Instruction should be activity oriented. However, the activities should relate as much as possible to the experience of students and should serve to improve their consumer skills and their ability to obtain gainful employment.

Some of the activities that promote measurement skill can and should improve buying skills. The school experience can be enlarged to include experiences in unit pricing and in comparisons between items to determine best buys. Weather-forecasting activities can be used to enhance skill in measuring temperature, barometric pressure, quantity of precipitation, and wind velocity.

Problem Solving/Applications Strand[1]

The strand *problem solving/applications* is developed thoroughly in the program in kindergarten through grade eight and utilizes a five-step procedure which is appropriate also for the program in grades nine through twelve. At each of the five steps, important skills are identified, and activities are suggested that provide a systematic approach for solving all types of problems which arise in applied mathematics.

Problem-solving situations should be an outgrowth of student experiences involving phenomena that arise in the context of some

[1]See pages 59 through 74 for the changes in this strand made by the Mathematics Framework Addendum Committee.

natural event. A search for an "understanding of phenomena" has dominated human intellectual activity from the beginning of time. This pursuit of knowledge, of structure, and of causation is usually motivated by the desire for comfort, by fear of the unknown, or by curiosity. The search for understanding is usually accompanied by a strong desire for predictability; that is, within reasonable bounds the theoretical predictions will agree with known results of the actual situation. The consistent development in all students of creative problem-formulation skills is essential if the search for understanding is to be motivated and improved.

A grade nine through twelve mathematics program should systematically assist students in devising strategies or tactics for analyzing problems that lead to some success in solving them. Problems arise in many situations that seem complicated and difficult to understand. By increasing the number of ways that a student can organize information presented in a given problem, the chances of understanding the essential features of the problem are increased. Some problem-solving strategies for the program in grades nine through twelve are the following:

1. Using diagrams or drawings to organize and analyze information.
2. Using tables or graphs to organize and find new information
3. Using established forms of logical reasoning to discover characteristics of problems
4. Assigning numerical estimates to unknown quantities in a problem and using simple arithmetic to derive new information about the problem
5. Using simpler or similar problems to discover relationships in a given problem
6. Employing translation techniques

Attempts to teach specific applications of mathematics lead quickly to the identification of the following two major difficulties:

1. So many applications of mathematics exist that it is impossible to select any definite subset to represent all possible situations.
2. Nontrivial applications of mathematics usually require considerable teaching of nonmathematical topics.

To counteract these difficulties, the high school program should, in general, be concentrated on teaching the skills and techniques associated with the process of applying mathematics rather than on teaching specific applications of mathematics.

To provide students systematic, in-depth experiences in which to apply mathematics realistically, it is recommended that problem-solving blocks be designed that last from a few days to two weeks during which the student has the opportunity to immerse himself or herself in a problem-solving situation, exercise investigative and experimental skills, and apply mathematics knowledge. Such experiences would enable students to develop enough of an understanding of the given situation that they could then answer more substantive questions about it and could see their mathematical skills used in a realistic manner.

Probability and Statistics Strand

Societal needs demand that students at the secondary level become knowledgeable about fundamental concepts of probability and statistics as methods of analyzing data. Statistical findings and graphic presentations are used in all facets of daily life. Significant decisions and predictions are systematically made at various levels of business, industry, and government on the basis of statistical interpretations and inferences. Similarly, students throughout their lives will be confronted with statistical presentations of data and, hence, will be forced to make decisions based on the analysis and interpretation of those data.

Interest and enthusiasm for the study of probability and statistics should be easy to foster among students, because the applications of statistics have widespread significance in almost every discipline. A resourceful teacher can introduce problems and experiments that will be relevant to the spectrum of experience possessed by any given group of students. When possible, experiments should be planned and conducted by students to enhance the opportunities for student understanding and motivation for learning statistical concepts.

Through actual experimentation, the program can enable students to develop the nature of probability from an empirical viewpoint, leading to the development of theoretical probabilities based on methods of counting outcomes.

Students should acquire concepts and methodology for calculating measures of central tendency, dispersion, and skewness. The binomial distribution, based on repeated independent trials of events, is a probability function that should provide insight about discrete variables. The central role played by the normal distribution in the study of observed data needs to be carefully developed. Students should develop facility in using tables of the normal distribution for the solution of inferential problems.

Learning experiences in the methodology of analyzing data should provide students with opportunities for collecting, organizing, presenting, graphically representing, interpreting, and making inferences about data. If statistical experiences are to be meaningful to students, then data collection should be the outgrowth of measuring and observing experiences pertinent to the students' environment.

Students who are interested in continuing their studies of probability and statistics should have opportunities for further exploration of more complex probability distributions, simple correlations between paired variables, and curve fitting for distributions of data.

Relations and Functions Strand

The notions of relations and functions are among the most significant and useful ideas in mathematics. Functions and relations should be identified in many fields, such as economics, science, and education and in nature. The early introduction of the concepts of relations and functions makes it possible for students to unify parts of mathematics in a natural way and to apply mathematical techniques to many fields of study.

If students can learn to identify, represent, and use functions and relations, they gain the understanding and power to predict results from causes known or supposed. Furthermore, using the mathematical representations of functions and relations, they can literally reproduce a situation millions of times without having to perform experiments or use expensive equipment each time they want to acquire additional information.

The mathematics program in grades nine through twelve should provide for each student who studies mathematics the opportunity for a gentle but sustained development and use of these simple yet powerful unifying concepts.

As a minimum, the program in grades nine through twelve should provide the opportunity for each student to develop the following:

- A beginning understanding of relations and functions
- Some skill in representing relations and functions in word descriptions, in tabular form, in the use of formulas, in a diagram, in a graph, as a map, or as a set of ordered pairs
- Some skill in drawing graphs of simple functions and relations
- Some understanding of how functional relationships can be used to discover new information about a situation that may not have originally been apparent

As students progress in their study of mathematics, processes for constructing and manipulating functions should be identified, and the opportunity to study and use elementary functions (polynomial, rational, algebraic, exponential, logarithmic, and trigonometric) should be available. The notion of the inverse of a function should be emphasized, using the relationship between the graph of a function and the graph of its inverse. Functions and relations in the form of equalities and inequalities should be solved graphically as well as algebraically, and some introduction to the application of these relations should be given in the study of simple linear programming problems.

Logical Thinking Strand

The process of reasoning is basic to all mathematics, particularly to deductive reasoning. In fact, if anything typifies mathematics, it is the free spirit of making hypotheses and definitions, rather than a mere recognition of facts.

If logical thinking is to be divided into categories, the most natural are deductive reasoning and inductive reasoning. In instruction at the secondary level, there should be conscious experience with both types of reasoning in order that students can understand and make applications. In solving problems, the students should be helped to realize whether they are starting from general premises and seeking consequences or whether they are aiming at universal conclusions by examining particular instances. Programs in mathematics must help students (1) to understand the relation between assumptions and conclusions and thus to test the implications of ideas; (2) to develop the ability to judge the validity of reasoning that claims to establish proof; (3) to generalize both of the preceding skills; and, finally (4) to apply all of the preceding skills to situations arising in many fields of thought.

Inductive reasoning. Inductive reasoning characterizes an early stage in the process of growth and maturation that culminates in a mature understanding of both induction and deduction. Inductive reasoning is usually informal and intuitive, but it includes several different modes of thought.

1. *Simple enumeration.* If enough cases are collected, there will be some assurance that the conclusion drawn from the evidence is reasonably certain. Simple enumeration is extremely worthwhile in that it gives the student reasonable assurance of correctness. The danger present is that students will tend to think that mere numbers constitute proof.
2. *Method of analogy.* Some of the most influential hypotheses have their origins in a person's ability to use analogous reasoning. One must be certain, however, that the analogy fits before conclusions are drawn.
3. *Extension of a pattern of thought.* Many ideas are the result not merely of enumeration and analogy but of extrapolation—extending ideas beyond the observed instances. The process of extension approaches the formality of deductive inference, but

extension does not carry the authority nor the necessity of the implication.

4. *Hypothesis.* Guessing has a place in any area of mathematics, from the earliest elementary school experiences to courses in the high school curriculum. When students have many experiences in making reasonable conjectures, they soon come to see the value of hypothesizing—extending their perception beyond what is immediately evident.

Deductive reasoning. The process of deduction involves moving from assumptions or reasons to conclusions, such a process being called an inference. Students are exposed to simple inferences in elementary school, and their experiences in daily living provide them with a wealth of this type of reasoning.

Any mathematics program should incorporate general procedures for presenting students with the basic rules for forming valid inferences. At times this may be done within the context of courses already in the curriculum, such as in a course in algebra or one in geometry. Other programs may include a course in logical thinking that will encompass the basic principles of deduction.

Computers Strand[2]

The advent of the use of computers in high schools has raised questions which demand attention: How much should be taught about computers? To whom? By whom? What are the vocational responsibilities of the school? How can schools keep abreast of the rapid technological advances? Should the use of computers be applied to all areas of the curriculum? What are the effects of computer-assisted instruction on attitudes toward learning? Does the use of computers "dehumanize" society?

Regardless of the problems, the reality of computers in the educational environment has prompted the establishment of the following guidelines for the mathematics program in grades nine through twelve. When feasible, computers should be used in educational programs in the following three ways: (1) instruction about computers; (2) learning with the aid of computers; and (3) management of instruction.

Instruction about computers. Different student capabilities and interests will prescribe the scope of instruction about computers. A minimal level of computer literacy for all students includes:

1. Flowcharting

[2]See pages 75 through 83 for the changes in this strand made by the Mathematics Framework Addendum Committee.

2. The functions (storage, computation, and control) of the processing component of computers
3. The electronic method of coding (the binary system)
4. Input-output devices for communicating with computers (hands-on experiences)
5. The history and evolution of computers

For students with vocational interests in computers, instruction should be expanded to include the following:

1. Learning a compiler language (e.g., BASIC and FORTRAN)
2. Computer programming (e.g., alphabetizing and calculating)
3. Data processing (e.g., keypunching and sorting)
4. Business applications (e.g., inventory control and payroll)
5. On-site visits to computer installations

Finally, students who elect to pursue a career in computer science should be offered the following topics:

1. Advanced programming techniques
2. Additional skills in machine languages
3. Writing computer-assisted instruction programs
4. Writing simulation programs
5. Formulating and solving real problems in science, mathematics, social studies, economics, ecology, and so forth

Learning with the aid of computers. The diverse applications of computers to learning concepts and skills in mathematics preclude the establishment of a separate "computer department." Computers should be used in every mathematics class in the following ways:

1. Drill and practice with immediate correction
2. Remedial instruction with branching
3. Self-contained presentations of new material
4. Exploring interesting problems with repetitious calculations
5. Solving problems which arise in science, social studies, economics, ecology, and so forth
6. Simulation models (e.g., predictions and games)

Management of instruction. Computers should be used extensively to aid the teacher in the classroom. Some possible services that can be provided are the following:

1. Scheduling and evaluating resource materials
2. Cataloging topics by cross-reference
3. Recording individual student progress
4. Prescribing individual instruction
5. Selecting test items from a bank of questions
6. Scoring and analyzing test results

Suggestions for Mathematics Programs

In the objectives of the framework for grades nine through twelve, several different mathematics programs were suggested. This section will attempt to clarify some of those suggestions. The suggestions are not intended to be definitive but should be considered only as a point of departure for local program development.

College-Preparatory Program for Nontechnically Oriented Students

An alternative two-year college-preparatory program may be offered for nontechnically oriented college capable students. The objectives of the curriculum are as follows:

1. The curriculum should be devoted almost entirely to those mathematical concepts that *all* citizens should know in order to function satisfactorily in our society.
2. The traditional grade placement of topics should be ignored. Instead, topics from arithmetic, algebra, and geometry should be interwoven in such a way that they illuminate and support each other.
3. The basic ideas about certain new topics, such as computer mathematics, functions, coordinate geometry, transformations, probability, and statistics, should be made available to all students.
4. It is important to make clear to all students that mathematics is indeed useful; that it can help us in understanding the world we live in and in solving some of the problems that face us.

The content of an alternative college-preparatory program could be as follows:

Grade nine

1. Structuring space
2. Functions
3. Informal algorithms and flowcharts
4. Problem formulation
5. Number theory
6. The integers
7. The rational numbers
8. Congruence
9. Equations and inequalities
10. Decimal representation for rational numbers
11. Probability
12. Measurement
13. Perpendiculars and parallels (I)
14. Similarity

Grade ten

15. The real number system
16. Area, volume, and computation
17. Perpendiculars and parallels (II)
18. Coordinate geometry
19. Problem solving
20. Solution sets of mathematical sentences
21. Rigid motions and vectors
22. Computers and programs
23. Quadratic functions
24. Statistics
25. Systems of sentences in two variables
26. Exponents and logarithms
27. Logic
28. Applications of probability and statistics

A third-year program could include the following:

1. Organizing geometric knowledge
2. Concepts and skills in algebra
3. Formal geometry
4. Equations, inequalities, and radicals
5. Circles and spheres
6. The complex number system
7. Equations of the first and second degree in two variables
8. Systems of equations
9. Logarithms and exponents
10. Introduction to trigonometry
11. The system of vectors
12. Polar form of complex numbers
13. Sequences and series
14. Permutations, combinations, and the binomial theorem[3]

Flexible Minicourse Program

A flexible minicourse program can be offered for noncollege-preparatory students. The following is a brief outline of topics that could be made available in a nine-week minicourse format. Some topics in the outline have prerequisites, while others do not. (No attempt has been made to arrange the topics in any fixed sequence.) The instructional materials for the minicourses should be written for average or below-average achievers in grades nine through twelve. A school with four general mathematics classes could offer a one- to

[3]*Newsletter No. 36. Final Report on a New Curriculum Project.* Prepared by the School Mathematics Study Group. Pasadena, Calif.: A. C. Vroman, Inc., 1972, pp. 7–8.

four-year program with minicourses, such as the following. Students could select four courses per year, depending on their needs and interests:

1. Numerical Trigonometry and Introduction to Surveying
2. Geometric Constructions and Designs
3. Ratio, Proportion, and Variation
4. Problem Solving and Our Society
5. Practical Measurement and Measuring Devices
6. Mathematics and Nature
7. Mathematics and Games
8. Consumer Economics and Home Management Mathematics
9. Business Mathematics and Business Statistics
10. Flowcharts and Introduction to Programming
11. Credit and Installment Buying
12. Solutions of Equations and Inequalities and Applications
13. Desk Top and Minicalculators and Probability
14. Reading, Constructing, and Using Tables and Graphs
15. Practical Statistics
16. Coordinate Systems, Decision Making, and Solutions of Systems of Mathematics Sentences

Mathematics Clinic

A mathematics clinic program could be designed to provide individualized instruction aimed at meeting the needs of selected students whose math achievement is significantly below their expected level of achievement. The program could be organized and planned to meet the identified needs of students. A procedure for identification of students should include teacher-counselor recommendations and testing data. The program for each student is planned individually and includes diagnostic testing, pretesting, individualized instruction, and post-testing.

A clinic could operate in a pullout program (students are temporarily released from their regular classes) or as a quarter or semester course. It is desirable to provide aides to assist the teacher of the mathematics clinic. An aide might be a paraprofessional, a parent, or a student assistant.

Some examples of different students whose needs are met in clinic programs are as follows:

1. Students with an ability level within the average range who are achieving below their expected level in mathematics
2. Able students (including gifted) who are below their expectancy level of achievement in computational skills

3. Target students in mathematics as identified for the Elementary and Secondary Education Act, Title I. program

One example of a mathematics clinic program is a program designed for students with an average range of ability who are achieving below their expected level (see number 1 above). The program is designed to provide individualized instruction in basic concepts and computational skills suited to the needs of selected students. Offered as a nine-week quarter course, the clinic serves students whose test scores indicate deficiencies of approximately two years below expected grade level. The clinic is organized for a maximum number of 16 students per class period.

Some factors to be considered in recommending students to the clinic are the following:

1. *California Tests of Basic Skills* computational scores or some other appropriate test scores to determine a list of students from whom the clinic teacher will make recommendations to the counselors
2. Ability level of stanine three or above
3. Previous grades in math
4. Teacher recommendations

Recommended equipment for the clinic is as follows:

1. Four calculators that have floating decimal-point capability; the operations of addition, subtraction, multiplication, and division; and a percent key
2. Computational skills kit
3. Cassette players with tapes
4. Film-loop projector (Super-8) with tapes
5. Programmable calculators with a card reader and accompanying drill and practice instructional materials
6. Arithmetic facts kit with fact pacer

Recommended materials for the clinic are the following:

1. Programmed practice materials
2. Basic textbooks (or appropriate textbook for possible ungraded quarter courses) with accompanying workbooks

Another desirable equipment item for the math clinic is a videocassette player.

Math clinic activities are as follows:

1. Upon entering the mathematics clinic, a student is:
   a. Interviewed by the clinic teacher for interest in improving computational skills
   b. Given an orientation to the clinic procedures

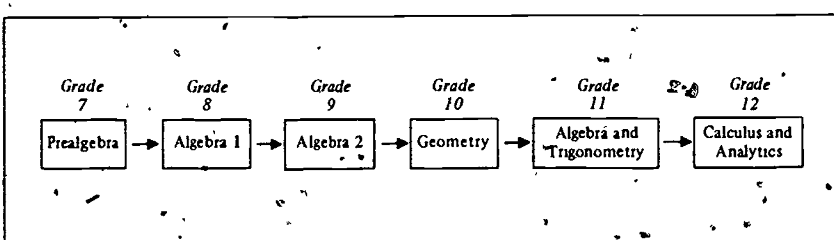
c. Given standardized diagnostic tests to verify specific areas of difficulty
d. Involved in determining individualized work based on the deficiencies indicated by the diagnostic test
e. Involved in developing the student's individual program

2. When a student completes the individual program, he or she:
a. Is given a post-test
b. Discusses the results of the test with the teacher
c. Is programmed into the workbook accompanying the basic textbook that is used in the regular classroom (This interface assignment is given to bridge the gap between the clinic effort and current levels of classroom assignments.)
d. Can cover in detail some of the main topics studied in the student's regular classroom

Since each student is involved in developing an individual plan of progress, the teacher must observe the student's motivation and efficiency pursuant to correcting the student's problems.

College-Preparatory Program for Technically Oriented Students

Alternative college-preparatory programs can be offered for technically oriented college capable students. Two alternatives for including calculus in college-preparatory mathematics programs are suggested below. The first alternative program is designed to provide an articulated program beginning in grade seven and leading to advanced college placement for mathematically talented students. The pattern of courses is as follows:



The course in grade twelve could be offered by a community college on the high school campus. While commercial textbooks are used at each level, other materials on topics not found in the books, such as mathematical systems, non-Euclidean geometry, source books of challenging problems, projects, and library references, need to be prepared.

Another alternative program is a highly sequential, integrated six-year program beginning in grade seven. Students may leave the program at any grade level. This program removes the traditional barriers between separate courses, such as between algebra and geometry, and it provides students with the equivalent of two full years of college-preparatory mathematics.

Addendum to the Mathematics Framework

Guidelines for

- Problem Solving/Applications
- Calculators/Computers
- Proficiency Standards
- Staff Development

Approved by
The California State Board of Education
on September 4, 1980

Acknowledgments for the Addendum

In June, 1979, the Curriculum Development and Supplemental Materials Commission of the State Board of Education established a *Mathematics Framework* Addendum Committee and nominated 15 persons to serve on that committee. The committee was charged with the responsibility of preparing a document that would update and supplement the *Mathematics Framework for California Public Schools,* which was adopted by the State Board of Education in 1974 and is not scheduled for revision until 1984.

The committee received suggestions and reactions from many knowledgeable teachers, consultants, other leaders in mathematics education, and curriculum commission members. The following persons were particularly helpful in adding their expertise to the deliberations of the committee: Jack D. Wilkinson, University of Northern Iowa; Twila Slesnick, University of California, Berkeley; Patrick Chladek, California State Department of Education; Robert Tardif, California State Department of Education; Gary G. Bitter, Arizona State University; and Warren Mott, Carlmont High School, Belmont.

The 15 members of the *Mathematics Framework* Addendum Committee were:

Carolyn Aho, District Resource Teacher, San Francisco Unified School District
Joan Aker, District Resource Teacher, Santee School District
Barbara Berg, Member, Lompoc School Board
Clyde Corcoran, Mathematics Department Chairman, California High School, Whittier Union High School District
Robert Hamada, District Mathematics Supervisor, Los Angeles Unified School District
Charles T. Hebert, District Mathematics Supervisor, Stockton Unified School District
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Susan Ostergard, Mathematics Education Lecturer, University of California, Davis
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Dale Seymour, Author of mathematics instructional materials, Los Altos

Linda Silvey, Teacher and Counselor, Sepulveda Junior High School, Los Angeles Unified School District

Anita Solza, Mathematics Resource Teacher, Miller Elementary School, Burbank Unified School District

Georgia Strecker, District Mathematics Supervisor (deceased), Berkeley Unified School District

Harold Taylor, Committee Chairperson; and Department Chairman, Mathematics, Aragon High School, San Mateo Union High School District

The committee's coordinators were:

Ruth Hadley, Mathematics Chairperson, Curriculum Development and Supplemental Materials Commission to the California State Board of Education; and Mathematics Resource Teacher, Vandenberg Middle School, Lompoc Unified School District

Joseph Hoffmann, Mathematics Consultant, State Department of Education

Introduction to the Addendum

This 1980 addendum to the *Mathematics Framework for California Public Schools* is designed to provide assistance to teachers, administrators, mathematics curriculum planners, and others who find themselves facing problems not addressed in the 1975 edition of the framework. Until a completely new *Mathematics Framework* is prepared by the Curriculum Development and Supplemental Materials Commission for the California State Board of Education, the reader should continue to use certain sections of the 1975 edition but modify or replace other sections with this addendum. For example, the problem solving/applications strand, which is described on pages 19—23 and 40—42, has been extensively rewritten and is presented in the addendum on pages 59—74 as the "umbrella" over all other skills and concepts in mathematics. The descriptions of the remaining strands in the framework are still valid; however, the term "strand" is replaced by "skill and concept area" in the addendum to emphasize the subordinate role these topics have under problem solving. Also, the computer stand, which is described on pages 45—46, has been expanded in the addendum on pages 75—83 to address the rapidly burgeoning role of calculators and computers in mathematics.

The three appendices of the 1975 *Mathematics Framework* are omitted in this edition. "Mathematics Program Strands and Objectives for Kindergarten Through Grade Eight," formerly Appendix A, has been extensively revised and is replaced by Appendix D of this edition. The "Time Line for Metrics," formerly Appendix B, set important goals for metric education that are generally fulfilled and no longer needed. The "Criteria for Evaluating Instructional Materials in Mathematics," designated as Appendix C in both the former and present editions, was revised by action of the State Board of Education on March 13, 1980.

The four topics addressed in this 1980 addendum were selected because of the critical need for statewide consensus and attention in school mathematics programs. Problem solving/applications demanded the highest priority, because this strand greatly influences the treatment given to all other topics in mathematics education. Closely related to this was the need for an up-to-date position on calculators and computers—a topic barely conceived at the time the 1975 framework was written (1973-1974). The third topic, proficiency standards

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and remediation in mathematics, arose in 1976 with new legislation and began creating confusion that demanded the attention of mathematics education specialists.

The final topic that the *Mathematics Framework* Addendum Committee addressed was one the committee felt would provide the key to improvement in school mathematics programs: how to plan, organize, and carry out a staff development program in mathematics.

Other topics were identified as critical and deserving of more than a mere mention here, but for the present they will have to be dealt with through other forums. One valuable source of assistance is available through the California Mathematics Council and its many regional affiliate organizations for teachers of mathematics, kindergarten through grade twelve and college/university. The many conferences and newsletters sponsored by those organizations will keep the concerned mathematics educator abreast of current issues and the latest strategies being tried by colleagues.

Some other critical problems which the *Mathematics Framework* Addendum Committee felt mathematics education is facing at present are these:

- Female aversion to the study of mathematics in high school
- Learning difficulties in mathematics resulting from low reading abilities
- The shortage of qualified mathematics teachers at all levels
- The lack of public support for comprehensive school mathematics programs
- Inadequacy of testing programs to measure achievement in mathematics at higher cognitive levels
- The need for a learning theory that will explain how mathematics is learned and how it should be taught

Problem Solving/Applications

The recent growth of the problem-solving emphasis in mathematics is the result of a number of converging influences and concerns. At the same time, many new concerns have arisen: clear definitions, measurable objectives, and learning strategies, as well as practical considerations of public acceptance of an emphasis on problem solving, in-service training of teachers, and development of instructional materials. Such concerns must be resolved as soon as possible. However, mathematics educators do not have the luxury of waiting for a smooth, unhindered transition to a curriculum that will meet the need to solve problems in an uncertain future as well as meet the needs that exist right now. Recognizing this, the members of the *Mathematics Framework* Addendum Committee developed this section of the addendum to serve as a framework that could be used by mathematics educators for exploring new ground while meeting the responsibilities of day-to-day classroom instruction.

Restructuring the Mathematics Framework

This addendum removes the former strand, problem solving/applications (pages 19—23, 40—42 of the framework), from its role as one of several equal parts of an instructional program in mathematics and positions it as an "umbrella" over all of the other strands.

The organization of the 1975 framework identified problem solving/applications as a separate strand, with the intent that it would receive equal emphasis in the curriculum of California public schools. However, in practice, this strand has received little or no additional attention in the instructional materials or in the organization of school mathematics programs. Since the primary reason to study mathematics should be imbedded in problem-solving/application skills, which now define the basic skill for mathematics programs in kindergarten through grade twelve in California public schools, the restructuring of the framework to implement this view was undertaken.

This restructuring means that throughout the instructional programs in mathematics, there should be a highly visible, consistent, organized set of components of problem-solving/applications skills. The program-level objectives for the strand, problem solving/applications, which were formerly separated from the program-level objectives of the remaining strands, have been redesigned and are now included in the appropriate program objectives for the remaining skill and concept areas. Program-level objectives for kindergarten through grade eight are listed in Appendix D.

The remaining strands in the 1975 *Mathematics Framework* are not to be deemphasized. The skills and concepts in arithmetic, geometry, measurement, probability and statistics, relations and functions, logical thinking, and, for grades nine through twelve, algebra and computers are still to be fully developed in the most learner-efficient manner and at the appropriate grade level. As Figure A-1 illustrates, all seven program objective categories should be developed through problem-solving techniques.

This restructuring of the problem solving/applications strand will give greater emphasis to the needs of the learner in the instructional programs in mathematics and will help students become functionally competent at whatever levels will meet their needs and expectations. They should not only be able to compute and demonstrate knowledge of mathematical skills and concepts but also be able to apply their skills and knowledge in an organized and effective way; and thus, they should be able to investigate, interpret, and solve problems

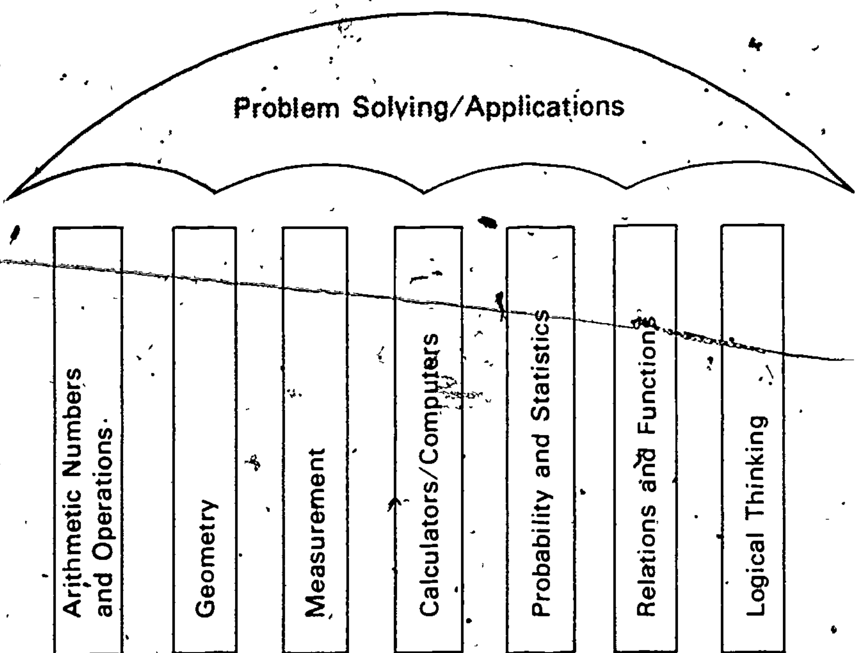

Fig. A-1. Problem solving/applications, the umbrella for all strands in mathematics related to a wide variety of practical situations at several levels of difficulty. The direct outcome of this new emphasis shall be for students at all levels to:

- Develop the skills necessary to apply their mathematical understandings to recognize, identify, and formulate problems from given practical situations outside and within the domain of mathematics.
- Develop skills in analyzing problem situations and selecting appropriate strategies and modeling techniques for solving the problems.
- Develop skills necessary to solve equations and simplify mathematical expressions.
- Develop skills necessary to interpret and evaluate the solution in the original situation or in a new situation or to generalize for a class of situations.

Rationale for Restructuring the Framework

Mathematics exists, both now and historically, because the human intellect poses problems. In many cases, mathematics aids in the solution of these problems by providing an abstract model capable of simpler analysis. The study of these abstract models *is* mathematics, and in many instances, models created within the "abstract world" make significant contributions to the "real world." Some problems, from both the real and abstract worlds, have profound implications on our lives, while others are merely riddles. Some problems have solutions, while others only lead to more problems.

Students have a natural curiosity about phenomena found in the physical world. They often unconsciously pose problems or ask questions, such as "Why does this work?" or "How much is that?" in dealing with these physical situations. Students need many experiences in applying mathematics to situations that arise outside the domain of mathematics and in the construction, analysis, and solution of the mathematical models that represent those situations. They also need to be exposed to contrived problem situations in order to build proficiencies in mathematics itself and to develop specific problem-solving/application skills.

The study of mathematics is an exploration of the loftiest endeavors of the human mind and entices those with a peculiar interest in logic and precision of thinking. Over the years that study has been structured in strands so that the learner will develop an appreciation for the beauty of mathematics.

The strands in mathematics are fragmented substructures which eventually unite, for the persistent learner, to form the total structure. "New math" represented an attempt to interweave those strands and reveal the structure as early in the learner's experience as possible. Though the focus on mathematical structure is no longer emphasized, the strands remain and benefit only those persistent learners who eventually may have an opportunity to see the total structure.

Moving from this strand-based mathematics curriculum to one which focuses on the needs of the learner requires a simple look at educational goals. If our goal is to transfer skills, knowledge, and values to each succeeding generation, then emphasis on the structure and beauty of mathematics is valid. But if our goal is to equip the learners to function with personal satisfaction in a rapidly changing society, then we want the educated person to learn the skills of exploring, studying, testing, reading, asking, and especially thinking critically in the face of challenging situations. Therefore, one must conclude that the value of mathematics lies in the way mathematics can be used in real situations to analyze, estimate, predict, save time, make decisions, and so on. These are the very reasons why mathematics came into being in the first place.

Many people find personal satisfaction in the beauty and structure of mathematics; these are the persistent learners who go beyond learning mathematical strategies to solve problems in the "real world" and begin to create and study models in the "abstract world." The eminent mathematician and author, George Polya, has eloquently demonstrated how the structure of mathematics can be derived from problem-solving skills. And helping students develop such skills is an important responsibility for teachers of mathematics at all levels, and it calls on them to recognize and give attention to those students who look for underlying mathematical principles and are eager to see how those principles interweave to form the total structure of mathematics. These explorations into the abstract world should be conducted in the problem-solving mode with Professor Polya's approach.

Components of Problem-Solving/Applications Skills

The four specific components of problem-solving/applications skills that should form the basis of an instructional program in mathematics, kindergarten through grade twelve, are:

1. Formulating the problem
2. Analyzing the problem
3. Finding the solution
4. Interpreting the solution

The components described should not be followed blindly in a step-by-step fashion. At any stage, a creative solution to a problem is more valuable than a burdensome routine. On the other hand, it is intended that sufficient instructional time be spent on each of these components so that the student is not confused or inhibited in problem-solving activities by failing to have available at least some explicit steps that can be taken to solve a problem. It is also intended that the mathematical skills and concepts brought to bear upon a situation be drawn from as broad a range as possible to reduce or eliminate the artificial effect of the partitioning of mathematics into fragmented skills and concepts. The emphasis upon problem solving applications should be to provide a vehicle that allows all of the mathematical skill and concept areas to interact in a normal way. Descriptions of each of the four components of problem-solving skills and procedures are presented in the paragraphs that follow.

Formulating the Problem

The ability to formulate meaningful problems is more useful in the marketplace than the ability to find a solution to a textbook "word" problem. Real problems do not always exist in neat, written textbook form. They often appear in very poorly defined, complex physical or abstract situations. The ability to ask questions or pose problems which clarify the relationships among the variables in a situation is indeed a valuable skill. It is proposed here that students be given considerable experience in formulating mathematical questions. To this end, it is expected that instructional programs in mathematics shall include a significant number of concrete, meaningful but perplexing situations which students can explore and discuss.

Instructional materials should provide detailed and varied suggestions to teachers for stimulating classroom discussions, and a variety of situations should be included to provide for differing interests and differing classroom situations.

This does not mean that students should spend all their time discussing perplexing situations and formulating problems. Sometimes people face problems that are already formulated, and these problems can provide students with practice in other problem-solving skills. However, the emphasis on problem-formulation skills is important because students often complete their schooling without ever formulating a problem. Problem formulation is to mathematics what writing skill is to language arts: both are closely related to organizational skills and divergent thinking, are subjective in nature, and depend extensively on the learner's experiences.

Analyzing the Problem

A mathematics program should systematically help students develop strategies and tactics for analyzing problems and devise appropriate mathematical models to represent the problems. A strategy here means a general plan of attack; while a tactic means a single technique which will help a part of the problem. The first step in developing a strategy is to identify those features which are significant to the central problem. This part may be referred to as analyzing the problem or "planning" and may involve selecting one or more of the following tactics:

1. Guess some answers, try them out, and observe the results of the different guesses. If a guess "works," the problem is solved!
2. Construct a diagram, a graph, a table, a picture, or a geometric representation of the situation, and observe the relationship between the various parts of the problem.
3. Construct a physical model of the situation, or use physical materials to simulate the features of the problem.
4. Identify extraneous or missing information in the problem.
5. Search for and identify underlying functional relationships in the problem.
6. Sort out similar elements in the problem and list the information.
7. Simplify the problem by breaking it into manageable parts or steps.
8. Translate the information into mathematical symbols and notations.

It is important to understand that the listing of these tactics does not suggest a formal step-by-step procedure or a specific order to which analyses of all problems must conform. The last tactic listed above, translating information into mathematical notations, is a crucial one, for in it the conditions of the problem must be made explicit and correct. This tactic should be the conclusion of careful analysis of the problem, not the first step in the analysis. Jumping directly from the statement of the problem to a formula or equation suggests that problem solving has been reduced to an algorithmic skill. Students should have many opportunities for deciding which tactics might be used and then use the one they feel is most appropriate for the situation.

The form of the mathematical model may be one or more written sentences or statements using mathematical symbols, a graph, a formula, a table, a chart, a diagram, or a geometric figure. Basically, a mathematical model of a problem is any representation which permits manipulation by mathematical principles. A mathematical model should represent some of the characteristics of the situation as it exists in the real world. To be successful, a model should accomplish the following:

- Include as many of the main characteristics of the given situation as practical.
- Be designed so that the included characteristics of the given situation are related in the model as they are in real life.

- Be simple enough so that the mathematics can be performed within the scope of the student's knowledge and skill.

It is important to note that the reverse process—given a mathematical model, construct a real situation for it—is pedagogically important. Such an experience gives the student significant insight into and understanding of problem analysis and of constructing a mathematical model.

The development of problem analysis strategies and mathematical models should start with the student's first mathematical experiences and accompany the development of basic mathematical concepts and skills. Some simple tactics appropriate to primary level mathematics are these:

1. Act out the problem with real people.
2. Represent the problem with real objects
3. Act out the problem with puppets or dolls.
4. Draw a picture of the situation.
5. Measure objects described in the problem.
6. Restate the problem in the student's own words.
7. Use manipulatives or paper folding to represent parts of the problem.
8. Make a pictorial representation of the problem; e g., sketches, arrays, tables, graphs, Venn diagrams, number lines, coordinate grids.
9. Use a calculator to check a guess.
10. Look for key words.

Finding the Solution

The solution of problems requires a wide variety of technical skills. Computational skills, and an understanding of operations and number properties are essential to solutions to many problems. Students need a wholesome attitude toward "risk taking" if they are to attack problems that are new to them. They should not become embarrassed if their first attempt does not lead to an answer. Students should be aware that some problems have more than one answer and that some have none. In addition to skills related to solving equations and inequalities, students need skills of graphing, constructing geometric figures, and analyzing tabular data. An instructional program in mathematics should include a substantial number of ready-to-solve problems that are designed specifically to develop and reinforce these technical skills and concepts.

Also, in most problems, the results should be anticipated by estimating in advance. Estimating should be introduced early to all students as a standard operating procedure. Sometimes a solution, when compared with an estimate, may reflect a significant oversight.

Interpreting the Solution

An instructional program in mathematics should systematically include experiences in the interpretation of the solution obtained. These experiences should be an explicit part of the instructional program and occur at all levels of instruction. Too often students accept answers that are found without regard to the original situation. The problem and its solution should be reviewed to judge both the validity of the model and the accuracy of the mathematical manipulations used to find the solution. The major concerns should be:

1. Did the answer match the estimate?
2. How was the analysis done?
3. Was the model adequate or valid?
4. Was some feature of the situation forgotten or ignored?
5. Was the solution process completed correctly?
6. Were the assumptions made too broad or restrictive?
7. Was the problem solved?
8. Can the model be improved?
9. Would another model work?
10. Was the mathematics performed correctly?
11. Were extraneous solutions introduced?
12. If you change some of the numbers in the problem, how does that change the answer?
13. Can the model be extended to solve related problems?

Skill in interpreting the solution may be developed by providing students with problems, tentative answers, and the discussion questions listed above.

A Definition of "Problem"

Throughout this discussion, the definition of "problem" has remained vague. Ordinarily, one thinks of a problem as a situation involving some knowns and unknowns which is not simple to resolve; that is, a situation that cannot be solved by a familiar rule or algorithm. For example, "adding with trading" might be a problem to a student meeting that situation for the first time, but after learning the algorithm it becomes a simple exercise. Textbooks may present a series of word problems which are solved in exactly the same way. Only the first exercise is a problem, and the rest can be solved by the same procedure once the learner notes that they meet the conditions for using that procedure.

It is useful to list some of the purposes for teaching problem solving and develop an operational definition by consideration of these purposes:

1. To develop an understanding of a broad range of strategies
2. To translate words into mathematical symbols

3. To form mathematical models from real-world situations
4. To develop deductive or convergent thinking skills
5. To improve and enhance decision-making skills
6. To motivate students to learn and appreciate certain mathematical skills
7. To develop mental and logical skills related to creative or divergent thinking

The following definition is broad enough for these purposes while excluding perfunctory drill that demeans the student as well as the mathematics:

> A mathematical problem is an understandable situation which provides some information and which asks for additional information not explicit in the situation. That is, the required information can only be derived by.
>
> 1. Understanding the elements in the situation and relationships among those elements;
> 2. Selecting a strategy for representing the elements and relationships symbolically; *and*
> 3. Manipulating the symbolic representation according to mathematical principles so as to reveal the required information.

Examples of Problems

Some examples of how problem solving can be taught are given here for various mathematical skills and concepts at the primary, intermediate, junior high, and senior high school levels. Although these examples are brief, suggestions for possible explorations and investigations will be hinted at by the questions suggested. See Appendix A for a list of most elementary level mathematical skills and concepts, with suggestions of the types of problems for which they may be used.

EXAMPLE AT PRIMARY LEVEL (KINDERGARTEN THROUGH GRADE THREE)

How should we decide whether to buy lunch at school or bring a lunch from home? (Number and operation)

Formulating the problem. Questions concerning the price of food could arise as an outgrowth of a nutrition unit or as part of a discussion following a graphing experience about which children buy their lunch at school and which children bring a lunch from home. Many primary-aged children may think that bringing a lunch from home is cheaper, because it is "free." They may not see the money that their parents spend for the food, nor do they realize that the value of the time someone spent making a lunch is worth money.

The problem formulation is more useful when children realize that the type of food "buyers" eat is different from what "bringers" eat. The students will need to decide what constitutes a "typical" sack lunch. Is it a sandwich (meat or peanut butter), fruit, or what? Should we look at nutritional value of what we say is typical, should we go by what tastes good, or should we consider only cost, picking the cheapest?

Analyzing the problem. After a "typical" lunch is agreed upon, students may guess at the cost of a lunch from home and record their results to compare them with what will be determined later. What suggestions can the students give for determining the price of a lunch brought from home? We may go to the store on a field trip and buy the ingredients for the lunch. Children at this age may not have the computational skills needed to find a solution nor understand the prices of food. The children should see and use the real objects. They need to make a lunch with real food and figure out the price using real money.

Finding the solution. Students at this age may want to break the problem down into parts to solve. For example, they may want to figure how many sandwiches can be made per loaf of bread. What is the price for bread for one sandwich? (This can be done by first grade children counting out the price of the loaf in pennies, physically separating the bread into two-slice piles, and then dividing the pennies among the piles.)

They could go through the same procedure with other sandwich fillings, with the fruit, and so forth. After the prices of the parts of the lunch are figured, they could then be added together to obtain the total cost.

Interpreting the solution. After the price of the "typical" lunch has been determined, the students need to discuss the results and interpret the solution. Discussion may include the following questions:

1. How did the solution compare with the guesses originally made from the cost of the lunch brought from home?
2. Which was the cheapest. the lunch bought at the school or the lunch brought from home?
3. How would the outcome be different if other decisions had been made about the components of a "typical" sack lunch?
4. What factors, other than cost, might influence whether a person buys or brings a lunch?

EXAMPLE AT INTERMEDIATE LEVEL (GRADES FOUR THROUGH SIX)

How typical are the heights of the students in the sixth grade class? (Measurement)

Formulating the problem. What information would you have to know? What unit of measure do you use to measure the students? When you have measured all of the students, what do you do with the information? Should students be measured with shoes or without? What do we want to know? Where can we get information about normal ranges of heights?

Analyzing the problem. Would charts or graphs help to solve the problem? How can we organize the data?

Finding the solution. Depending on the strategy chosen, computations should be made and answers labeled.

Interpreting the solution. What are the mean and median heights of the sixth graders in the class? Do the charts add new information? Does the mean or the median give a better solution?

EXAMPLE AT INTERMEDIATE LEVEL

Is the distribution of birthdays of members of the class typical or unusual? (Probability)

Formulating the problem. What do we want to find out? Do we need to get information from other classes? How many classes have at least two people with the same birthday?

Analyzing the problem How do we organize the data? What probability formulas apply? Can we devise a simpler problem (such as the sizes of families) and use the results to help us solve this problem? Are there patterns in the data?

Finding the solution. What are the answers to each of the strategies used in solving this problem?

Interpreting the solution. Are some of the answers better than others? Why? Are the results useful? Would the results change very much if several people were removed from the class?

EXAMPLE AT THE JUNIOR HIGH SCHOOL LEVEL (GRADES SEVEN AND EIGHT)

The student government at Cisco Junior High School wants to place a "Suggestion Box" in the main office for use by students and faculty. Your job is to prepare this box by covering a cardboard carton with fancy paper. (Geometry/Measurement)

Formulating the problem. What is the size of the box? Of the sheets of wrapping paper? Is cost a factor? Will the paper be chosen by you? By a group? How will the paper be attached? Is the bottom to be covered? Is it to be covered with one piece of paper or may it be pieced together?

Analyzing the problem. Measure the box. Make a plan (such as a detailed sketch or scale drawing). Cut a pattern according to the plan to fit the box and see if it fits the box. How well does it hold up?

Finding the solution. Use the pattern to cut the fancy paper and attach it to the box.

Interpreting the solution. Does it cover the box? How much paper did you use? If you were to do it over again, what would you do differently?

ANOTHER EXAMPLE AT THE JUNIOR HIGH LEVEL

The principal announces that if a student has a certain number of tardies, they will be worked off by spending so many days picking up paper. Is it a good rule? (Relations and functions)

Formulating the problem. Get a complete statement of the principal's tardy rule. What will determine if the rule is helpful? Will it cause more problems?

Analyzing the problem. Try to develop a formula or chart for the rule. Get data on the present rate of tardies, number of students involved, and so forth.

Finding the solution. Poll the students and the faculty. Make a table. Chart the tardies on a daily basis.

Interpreting the solution. Are the results useful? Are all significant factors included? Is your conclusion convincing?

EXAMPLE FOR SENIOR HIGH SCHOOL LEVEL (GRADES NINE THROUGH TWELVE)

For this section, a series of problem situations is presented and discussed, in turn, for each component of problem solving/applications.

Formulating the problem. Often the first contact with an actual problem situation outside a textbook reveals a very "fuzzy" picture of that situation. To sharpen the focus on such situations, it is helpful to ask questions, screen out extraneous information, and collect information that appears necessary to understand the situation. Many times this process of "sharpening the focus" involves a more or less precise formulation of mathematical problems that are amenable to solution. It is important that high school students be systematically involved in this type of activity as they progress through their mathematics program. Here are some examples of situations that can be used to illustrate the process:

1. "You are given a job of determining if the elevators ~~in an office~~ building are overloaded." State some questions that will help sharpen the focus on this situation. What information do you need to answer these questions? In gathering the information, are any simplifications made that might lead to an error in the solution?

   The following are some questions that could arise in a class discussion of this situation and which could lead to the more precise formulation of mathematical problems related to the situation. The solutions of these problems, to be useful, should shed some light on the situation.

   a. "How many passengers can the elevator hold?"
   b. "How many passengers can the elevator safely carry?"
      (1) "How powerful is the motor?"
      (2) "How much weight can the cable hold?"
      (3) "What is the weight of each passenger?" ("What is an average adult weight?")
      (4) "What is the weight of the passenger car?"
      (5) "What modifications must be made in the calculations to ensure the safety of the passengers? (A safety factor)"

2. An owner of a supermarket needs to redesign the parking lot
   a. Questions or comments could lead students to identify simple questions that place the given situation in a mathematical context involving linear and area measurements, ratio and proportion, and time and money.
   b. Other comments could lead students to identify questions that lead to geometric concepts, such as parallel and perpendicular relationships between lines, similarity and congruence, and other properties of geometric figures.

3. A new basketball coach organizes a group of nonplaying students to help improve the performance of the team. Class discussions could lead to these questions:

a. "How will we decide if the plan helps the team?"
b. "What data do we want to collect?"
c. "How do we organize and display the data?"
d. "What information can we find using the collected data? range? mean? median?"

4. A supermarket owner wants to open an "express lane" for customers with no more than a certain number of items. Class discussions could lead to these questions:
a. "What have you observed about the number of items required in supermarkets that feature this kind of express lane?"
b. "How many packages should be allowed?"
c. "How do customers feel about express lines?"
d. "On what basis is the waiting time of the customer to be averaged and minimized?"
e. "If a customer has purchases that must be weighed and priced (such as produce), is he or she to be allowed to use the express lane?"
f. "What data should be gathered on the number of packages that a customer has when he or she gets to the check-out counter?"

5. Your parents tell you that you can paint your bedroom. Class discussions could lead to these questions
a. "How much paint is needed?"
b. "What kind of paint is to be used?"
c. "What is the size of the room?"
d. "What is the shape of the room?"
e. "How much wood trim is in the room?"
f. "How many windows and doors are there?"
g. "How much area does one can of paint cover?"
h. "Is it more economical to buy large cans of paint, with some not used, or to buy just enough paint in large cans and small cans?"
i. "How much does paint cost?"
j. "What is the cost of paint for the bedroom?"

6. In an experiment to determine the effectiveness of a germicide, it is placed in the center of a colony of bacteria growing in a circular dish Class discussions could lead to these questions:
a. "How fast does the bacteria colony grow in a 24-hour period of time?"
b. "What is to be assumed is the shape of the bacteria colony at any given time? How can the size be calculated?"
c. "How fast does the germicide spread?"
d. "Does the germicide kill the bacteria immediately upon contact?"
e. "What is assumed to be the shape of the spread of the germicide at any given time? How can its size be calculated?"
f. "When will the colony of bacteria be wiped out?"

7. A patient requires radiation treatment of 27,000 roentgen units Class discussion could lead to these questions:
a. "What is the length of the treatment?"
b. "How many treatments are there to be?"
c. "Does the dosage limit depend on the size of the patient?"
d. "Does the dosage [illegible] decrease with increasing exposure?"
e. "How many roent[illegible]nits can be safely administered at a given time?"
f. "If the dosage limi[illegible]eases, is this decrease a linear function?"
g. "Is there a recomme[illegible] optimum interval between successive exposures to radiation?"

Analyzing the problem. In problem formulation the focus was on methods of taking a vague situation and developing some specific problems whose solutions might produce some useful information about the given situation. Here the problem-solving process is continued, with examples of possible strategies and possible mathematical models that could be developed for a few of the problems that were previously formulated. It is pedagogically important that students experience some sense of completion in the problem-solving process. Thus, these various strategies should not necessarily be presented all at one time, but probably should be presented and used one at a time, using the skills appropriate to the level of mathematical development of the student:

1. Consider the situation involving the elevator above and the problem, "How many passengers can the elevator safely carry?" The model of this problem is essentially arithmetic. To solve this problem, a person would need certain information and would need to make certain assumptions, as follows.
   a. Assume that, by looking at the cable, it is determined that the cross-section of the cable is a circle whose area is one square unit.
   b. Assume that the cable is steel and that it is known that a steel cable with a cross-sectional area of one square unit can hold up to a maximum of 60,000 kilograms.
   c. Assume that the weight of the passenger car is 1,000 kilograms.
   d. Assume that the average weight of an adult is 60 kilograms.
   e. Assume that the safety factor is ten, that is, the maximum allowable weight that can be carried is one-tenth of the possible weight.
   f. How can the maximum allowable weight to be carried be calculated?
   g. If the passenger car is 1,000 kilograms, then how much carrying capacity is left for the passengers?
   h. How many passengers can the elevator safely carry?

2. Consider the situation involving the painting of the bedroom. This is a list of some of the information and some of the geometric and arithmetic models needed to solve the problem, "What is the cost of the paint for the bedroom?".
   a. The shape of the room (a rectangular box?)
   b. The shape of each wall and the ceiling (rectangles?)
   c. The size (length and width) of each wall and the ceiling
   d. The size, shape, and number of doors and windows (one rectangular door and window?)
   e. The number of coats of paint required (two?)
   f. The fact that the enamel will be used on the wood and latex-based paint on the walls
   g. How much area a can of enamel will cover, how much a can of latex-based paint will cover
   h. The cost per can of each kind of paint, and the cost per can of enamel
   i. The total area of the wood trim, including the door
   j. The number of cans of latex-based paint needed and the number of cans of enamel needed (The areas must be doubled if two coats are required.)
   Number of cans of paint =

$$\frac{\text{Total wall area}}{\text{Area covered per can}}$$

   k. The total cost = (number of cans of latex)(cost per can of latex) + (number of cans of enamel)(cost per can of enamel)

3. Consider the bacteria colony and germicide situation. Here are some of the mathematical models and a list of some of the information needed to solve the problem, "When will the bacteria colony be wiped out?":
   a. Assume that the shape of the bacteria colony is circular.
   b Assume the circle of bacteria grows by 7 mm in radius each 24 hours.
   c Assume that the shape of the spread of the germicide is circular.
   d. Assume that the germicide spreads by 10 mm in radius each 24 hours.
   e. The germicide is put in the dish 36 hours after the bacteria starts to grow and kills immediately upon contact.
   f. The area of the bacteria colony at t days (24 hours) is $(7t)^2\pi$.
   g. The area of the germicide at time t days is $[10\ (t\text{-}36/24)]^2\pi$
   h The solution can be found by solving $(7t)^2\pi = [10\ (t\text{-}36/24)]^2\pi$ *for t.*

Finding the solution. The mathematical techniques available for finding solutions to problems in high school are many and include, for example, a wide variety of graphical techniques, geometric techniques, equation-solving techniques, and reasoning techniques. Usually, one finds that a combination of two or more techniques is needed in the solutions of problems arising from real situations. Most of these techniques are sufficiently well developed in existing secondary school mathematics programs. However, more emphasis could be given to the use of estimation of solutions as part of the solution process. Estimations not only serve as a valuable check upon the reasonableness of a solution found using mathematical procedures, but they also become a necessary first step in the techniques required to solve these problems:

1. Find the $\sqrt{197}$ by the estimate, divide, and average technique.
2. Find the value to three decimal places of any irrational roots of $x^3 + x^2 + x - 2 = 0$ using Newton's method
3. Find an approximate solution of $e^x = \sin x$ to three decimal places by drawing the graphs of two functions, e^x and $\sin x$ to find the approximate value of the x-coordinate of one of their points of intersection, and then use the interval bisection method to refine the approximation.

Interpreting the solution. Students too often accept answers without question if they were found using standard mathematical techniques. Consider these examples that illustrate at least one need for interpreting the solutions back in the original situation:

1 $d = 9.8t^2$ This is a common formula for the distance, in metres, an object falls from rest in t seconds This formula assumes that
   a. The object is a point.
   b The earth is a plane.
   c. There is no air resistance.
   d The paths of two falling objects are parallel.
   e The height from which the object falls has no effect upon the distance it falls in one second.

In fact every one of these assumptions is wrong. The earth is roughly spherical (oblate spheroid). Objects fall toward the center, and the paths of two objects will not be parallel The force of gravity does decrease with the distance from the center of the earth, and since the earth is not uniformly dense, the force of gravity is not independent of the location. Air resistance is not always negligible (a wad of dry notebook paper falls more slowly than a ping pong ball). However, for most simple situations, these assumptions will produce negligible differences but not in all situations.

2. $d = rt$. This is a common formula for finding the distance traveled in kilometers at r km/h for t hours. Consider this problem: "An architect flies in to the airport in Santa Barbara and rents a car to get to an appointment with a client in San Luis Obispo. If it is 160 km from Santa Barbara to San Luis Obispo, the architect estimates that he can drive 80 km/h, and he has allowed two hours for the drive, can he get to his appointment with his client on time?

$d = rt$, $160 = (80)t$, $2 = t$

Answer: Yes; it takes two hours for the architect to get to his appointment with the client in San Luis Obispo.

What does this solution mean? Is this solution correct? Consider these questions:

a. Is the exact distance between the airport and the client 160 km?
b. Is the airport likely to be in downtown Santa Barbara?
c. Can the architect step off the plane, pick up his luggage, rent a car, and instantly start driving 80 km/h?
d. Does the highway from Santa Barbara to San Luis Obispo go right beside the airport?
e. Does the client live in the center of San Luis Obispo or on the outskirts?

The mathematical model, $d = rt$, gave us a solution. However, if one looks more closely, one can see that the solution really does not answer the question. However, we are usually quite willing to accept this solution as an approximation to the solution of the physical situation. In problem solving many incorrect assumptions are often made. Therefore, it is important to decide whether or not these errors can be pushed aside as inconsequential, or whether the solution must be changed to take these errors into account in the final solution. Basically, the question should be asked: "Is the mathematical model adequate for finding the answer to the problem? If not, under what conditions would it be adequate?"

Impact of Technological Developments on the Mathematics Curriculum

As the result of the back-to-basics movement, there is a tendency by some educators to allocate too much time in mathematics classes to working with drill and factual recall. With the advent of low-cost, high performance microprocessors and calculators, it becomes possible for computation to be done more accurately and in less time than in the past. This allows more time for problem solving, the major focus of the mathematics curriculum.

The data explosion during World War II created the need for incredibly faster methods of organizing, processing, and retrieving data. As the technology improved, the need increased to support mathematical research and other disciplines and, indeed, to support the very technology which developed the computer itself. Concomitant with an increase in the need, the necessary computer technology was being developed so that the first generation of computers began to emerge in the 1940s.

Because of the recent rapid developments in all disciplines, the need for a more numerate society has intensified. The number of computers from 1955 to 1975 increased from 1,000 to 220,000. Projecting at the same rate, which is probably conservative, 48 million computers will be in use in the United States by 1995. The number of people using computers will increase so that probably more than one of every five people in the U.S. will be using computers by 1995.

Computers are but one dimension of this technological expansion. There are many other exciting new developments, including: (1) Nav-Star location satellites, with their related plane, ship, and personal locators capable of locating any position on the globe within a 40 metre square; (2) home communication systems linking telephone, television, and large, centrally located, main-frame placements capable of storing and retrieving masses of data and information heretofore unimaginable; and (3) the development of simulation games and game generators on computers and small calculators, which allow for more interaction between individuals while analyzing numerical data.

Calculators and Computers in the Curriculum

The impact of technological advances on the school curriculum is staggering. It becomes imperative that school curricula should provide enough flexibility for educators to respond to technological advances in our society. Specifically, the mathematics curriculum should continue to build on new ideas as it has through the centuries to enhance our abilities to cope with the environment.

One of the technological advances, the small, hand-held calculator, will allow most of the population to make use of more mathematical concepts, such as ratios and proportions. As recently as 1975, only 44 percent of the nation's seventeen-year-olds could handle problems of proportionality. Furthermore, research shows that computating skills are not decreased when calculators are used extensively in the classroom.

Because of computers the use of mathematics in other disciplines is expanding. For example, the environmental sciences use a sampling technique which can be more easily understood with the use of improved computers and calculators. In nutrition education the coordination of video tape, the television monitor, and the computer will make it possible for a student to take diet information and project a person's life span through simple numerical analysis. Simulation techniques of the same type are also available in social science; for example, with computer simulation topics, such as election results and the impact of food production on world hunger, can be made more meaningful.

Three primary considerations with which the math educators of the 1980s must concern themselves are: (1) the use of the available hardware and software to develop problem-solving skills, work with simulations, and user-interactive materials; (2) the development of programming ability so that students may construct their own software packages to make full use of the technological developments available; and (3) the sequencing of fractions, decimals, and percents in the mathematics curriculum, taking into account the decimal characteristics of calculators and computers.

The increasing gap between available hardware and software pinpoints the need for people with the ability to unite the power of technology with the needs of the users. Computer literacy includes both areas, and educators should put more emphasis on both areas through the development of programming abilities.

Computer-Assisted Instruction

Computer-assisted instruction (CAI) will become increasingly important to educators as technology expands and the costs of computers diminish.

An issue that needs to be addressed for any product being placed in public school classrooms is one of quality control. And the quality of CAI products should be as good as conventional teaching tools. Of course, the value of CAI products, as with other products, is determined by their effect on the learner. And CAI products will have a better chance of fulfilling their function as teaching resources if the needs of students are identified and it is determined that the products are appropriate for meeting those needs.

The calculator should not be overlooked as a teaching tool, because it can be used in classrooms to enhance the learning of mathematics in many ways, such as the following:

- *Motivation*—The calculator generates interest in learning mathematics because computation can be simplifed, therefore providing more time to expand and reinforce the concepts. The student becomes more inquisitive, creative, and independent as he or she experiments with the calculator and mathematical ideas. The recreational aspects of mathematics become more challenging at all levels.
- *Discovery*—The calculator can be a unique aid in investigating concepts and problems that are new to students. Using the calculator, students discover relationships and develop concepts that involve tedious or difficult computations.
- *Drill and Practice*—The calculator is an excellent instructional aid for varying the routine nature of drill and practice. The essential skills of estimating and checking answers for their reasonableness can be practiced with the calculator.
- *Enrichment*—The calculator is an extremely important aid for exploration and extension of the curriculum, because it provides nonstructured enrichment opportunities through games and problem-solving activities.
- *Application*—Because calculators are widely used in government, science, business, and industry, students using them in the classroom can apply mathematics to out-of-school situations. Other applications outside the classroom include balancing checkbooks and family budgets, calculating income tax, computing gasoline mileage, converting U.S. dollars to foreign currencies, and inventing and playing calculator games.
- *Problem Solving*—The calculator can be used effectively in problem-solving techniques; e.g., "trial and error," or "guess-then-check." This latter technique is frequently neglected, because the computations in the "check" are often laborious. A calculator makes "trials" or "guesses" much simpler, and using the calculator in this way becomes a very efficient way to work out a

problem. Also, by reducing the computation time with the use of a calculator, students can spend more time on other problem-solving skills or on solving more problems than they could without the calculator.

- *Reinforcement*—A calculator can provide immediate reinforcement of the learning of definitions, functions, and basic properties. It provides immediate feedback for accuracy in computation; it can be used in place of flash cards and other drill and practice materials. Also, locating errors in complex computations becomes routine with a calculator.
- *Conceptualization*—Calculators are providing more opportunities for students to learn mathematical concepts than they had without the devices. The calculator is also used to learn the number systems concepts, such as place-value, decimals, and powers of ten.

A New Skill and Concept Area for Mathematics

The inclusion of technological developments in the mathematics curriculum has been rapid and, therefore, difficult to plan. Also, the impact of the technology of calculators and computers on the curriculum is much broader in scope than most educational planners had anticipated. In the broad sense, the technology is based on the concept of an organization of machine functions to accomplish complex or repetitive tasks.

The study of calculators and computers is possible at two levels: (1) actual involvement with the machines; and (2) simulation of the machines' inner workings. Both levels will enhance computer literacy for students. The first is preferable, but for those schools without calculators and computers, a simulation model is suggested. By examining the basic functions of input, output, processing, and memory, students will simulate machine operation and in the process may learn many basic skills; e.g., the study of algorithms will differ from textbook presentations and should serve to reinforce understanding. This simulation model will also provide students with another view of the everyday use for mathematics.

The examples of how problem solving can be taught, as shown on pages 67—74, clearly show that computing is subordinate to problem-solving skills. A calculator may be used to enhance the learning of these skills, and in some cases using a computer might be even more helpful. Two more examples that are presented in Figures A-2 and A-3 show how especially good use can be made of calculators or computers. Figure A-2 describes a budget problem at four grade levels, and Figure A-3 presents a business problem involving ways of saving money.

Grade level	Problem-solving skills: Formulating the problem (FP)	Analyzing the problem (AP)	Finding the solution (FS)	Interpreting the solution (IS)
Kindergarten through grade three	How can I keep track of my allowance?	Keep a record. What operations do I need?	Set up a journal, add income, and subtract out-go.	Did it balance? Did it satisfy my parents? Is there an easier way?
Grades four through six	How can I keep track of my earnings?	Plan an earnings record.	Establish the record.	Does my record balance, and is it complete?
Grades seven through nine	How should a school club manage money?	What is involved? Check with the Student Activities Director	Set up the financial records for a club.	Do the club members understand the reports? Do I?
Grades ten through twelve	How can I plan a budget as an adult?	What types of income and expenditures do adults have?	Develop a tentative budget with categories.	Discuss the budget with at least three adults and revise the budget.

Fig. A-2. Example of a Personal Budget Problem for Which a Calculator Can Be Used

Grade level	Problem-solving skills			
	Formulating the problem (FP)	Analyzing the problem (AP)	Finding the solution (FS)	Interpreting the solution (IS)
Kindergarten through grade three	Where can I keep money given to me as a savings gift?	What does the bank do to keep a record of my money? How does my money earn interest?	Calculate savings account as deposits are made. Calculate interest on savings at regular intervals	How much money do I now have? How much money did the bank pay me in interest? Could I have earned more money by going to a different bank?
Grades four through six	How can I make sure I have enough money for Christmas? How much money do I want to spend on Christmas?	What amount will I need to deposit each month to attain my goal? What recording procedures will I use?	Calculate monthly payments plus accrued interest.	How much money did I finally have? What would have happened if I had missed a month? Could I have earned this much in another way?
Grades seven through nine	How can I regulate the income and out-go of my paper route collections and my monthly bill payment?	How does a checking account work? Which checking account is best for me?	Check to make sure bank statement is accurate. Enter deposits or bank slips. Write checks for business expenditures	What were my profits? How can I maximize my net income? Do bank statements vary among banks? Which is best for me?

Grade level	Problem-solving skills: Formulating the problem (FP)	Analyzing the problem (AP)	Finding the solution (FS)	Interpreting the solution (IS)
Grades ten through twelve	How do I qualify to get an auto loan at a bank?	What kind of auto can I afford? What will the monthly payments be? Will my interest on the loan be simple or compound? Does the amount of time the loan runs matter? How much is insurance? What will maintenance expenses be?	Fill out loan forms. Figure out amount of interest paid to bank for loan of original money. Figure principal and interest amounts per month.	Would it have been possible to obtain a loan that required me to pay less interest? Are auto and house loans calculated similarly? How do bank loans compare to auto dealership loans?

Fig. A-3. Example of an Interest Problem for Which a Computer Can Be Used

Other areas which lend themselves well to the use of calculators/computers are:

Business	Sales Predictions Development costs
Personal	Games Budget Shopping
Government	Taxes Census Inventory
Applications to schools	Elections (social science) Food chains—simulation (science) Pattern generation (art) Simulation of slope variation (algebra)

See Appendix B for additional curriculum considerations related to calculators and for tips on purchasing calculators.

Suggested List of Objectives for Grades Nine Through Twelve

A suggested list of objectives for instruction in the use of calculators and computers in grades nine through twelve follows (See Appendix D for objectives for kindergarten through grade eight.):

1. Estimation
   a. Explore the skills necessary to estimate quantity-related information.
   b. Develop skills necessary to estimate quantity-related information.
   c. Acquire the skills necessary to estimate quantity-related information.
   d. Apply the skills necessary to estimate quantity-related information.
2. Operations
   a. Acquire the skills required to calculate addition, subtraction, multiplication, and division using real numbers.
3. Relations and Patterns
   a. Develop the ability to analyze number relationships and order priorities.
   b. Acquire the ability to analyze number relationships and order priorities.
   c. Apply the ability to analyze number relationships and order priorities to other situations.
   d. Develop the skills to analyze number patterns.
   e. Apply the skills necessary to analyze number patterns to other situations.
4. Applications
   a. Explore the use of motivation and enrichment activities.

b. Explore extended application problems to concepts that are normally obscured by tedious computation.
c. Develop extended application problems to concepts that are normally obscured by tedious computation.
d. Apply knowledge of extended application problems to concepts that are normally obscured by tedious computation.
e. Develop integrated curriculum opportunities.
f. Apply integrated curriculum opportunities.
g. Apply skills with computer simulations.

5. Computer Literacy
   a. Develop working knowledge of functions, logic, and mechanics.
   b Develop the historical perspective of improved technology.
   c. Develop programming skills.
   d. Apply programming skills.
   e. Explore the use of technology in today's society.

Proficiency Testing and Remediation in Mathematics

When the first proficiency standards law in California was enacted in 1976, the public was expressing much concern about the reading, writing, and computation skills of students in California schools. It was found that too many high school graduates lacked basic skills, and the Legislature initiated a policy that required each school district to set proficiency standards for its graduates. The law prohibits the award of a high school diploma to any graduate who fails to meet those standards; the law also requires the district to maintain a program for students to remedy deficiencies detected in the formal assessment process after grade three.

The intent of the Legislature in enacting the proficiency standards law was the establishment of minimum standards for those skills one might reasonably expect of any high school graduate. Meeting those standards is a necessary but not a sufficient condition for the award of a diploma. Each district must require its students to complete a course of study in addition to meeting adopted minimum standards prior to awarding diplomas. To allow meeting minimum standards to become a sufficient condition for receiving diplomas may contribute to schools abandoning more ambitious programs for meager instructional outcomes. The law also specifies a course of study for students that includes content in mathematics beyond a functional minimum and, thus, makes it clear that minimum programs are neither desired nor allowed. (See Education Code sections 51210[b] and 51220[f].)

Involving parents and students along with professional educators in developing proficiency standards for a school district will minimize the problems inherent in this change process. It is also important to make certain the standards at the elementary level are in concert with those at the secondary level. Since the notion of standards was appealing to educators and the public alike before the law was passed, these standards can be developed to reflect a consensus of local beliefs about the purpose of education. The State Department of Education's publication, the *Technical Assistance Guide for Proficiency Assessment,* makes useful suggestions for developing proficiency standards, assessing student learning, designing school programs relative to standards, and identifying how to avoid difficulty in implementing a proficiency standards policy.

Suggestions for Setting Proficiency Standards

Proficiency standards should not be set in concrete but should be reviewed regularly by local committees whose membership includes parents, students, teachers, administrators, counselors, and other members of the community. The law was not intended to prevent students from graduating from high school; rather, it was designed to make certain that the awarding of diplomas indicated that those receiving them had achieved at least minimum functional skills and that those students who found it difficult to acquire such skills would be accommodated by appropriate remedial instruction.

While much of the rhetoric about the law has centered on assessment and testing, it is most important for educators to notice that the law represents the Legislature's desire for having changes made in school programs more than it does for regulating such programs. It is an opportunity to make school programs more successful than they have been in the past in imparting knowledge and skill to less able learners.

The emphasis in this framework addendum is on improving mathematics learning, but in this section the focus is on how computation relates to the proficiency standards and the rest of the mathematics curriculum. Most districts have a mathematics continuum of learning objectives, and computation and other identified minimum proficiency objectives should be a part of that master continuum. The following statements are intended to provide a helpful perspective:

1. Computation is an important skill, and students profit by knowing how to compute and which operation to use in a given situation.
2. Mathematics is more than computation, and to limit any student's mathematics learning to the acquisition of computation skill is not desirable.
3. Mathematics instruction presents many opportunities for learning reading, writing, and computing; and regardless of the subject, those opportunities should be used to help students meet proficiency standards for graduation.
4. The mathematics program for every student should include instruction in every skill specified in the proficiency standards adopted by that student's school district. Failure to do so diminishes the content validity of the assessment process; i.e., tests should not be designed to examine skills which are not taught.
5. Remedial programs established for students who fail to meet proficiency standards should be designed to help students meet those minimum standards, thus, those responsible for remedial programs should teach to the objectives in the standards, not to the test items. On the other hand, those responsible for other remedial programs that are designed for students who can meet the minimum standards but have

difficulty with other mathematical skills should teach to whatever objectives have been set for those programs.

6. Since remedial education presents some of the most difficult responsibilities schools and teachers face, it is appropriate that staff assigned to remedial classes be highly qualified and that teachers receive in-service training in teaching less able learners.
7. Those planning remedial programs should give adequate attention to such issues as student-teacher ratios, scheduling, motivation of student learning, choice of material appropriate to slow learners, and alternative modes of instruction.
8. Proficiency standards should be coordinated between high schools and their feeder schools through rigorous articulation efforts.

Policies for proficiency standards have been firmly implemented in most school districts, and these policies should provide for a continual review of standards and objectives as well as ongoing attention to improving regular school programs and remedial instruction. School improvement, as a goal, is a never-ending process, and mathematics educators have an important role to play in that process.

Staff Development

Staff development is the most important activity for ensuring the implementation of the concepts embodied in this addendum to the *Mathematics Framework*. Staff development efforts in California should be designed to help teachers while focusing on the new mathematical orientation toward problem solving. To achieve this goal, those responsible for staff development must have a workable process, persons with expertise in certain subjects, and a plan for formative evaluation.

A workable process needs both a structure within which to function and the personnel to accomplish the task. Both teachers and administrators working on a staff development program will need or must secure persons who have expertise in certain areas in order to provide a curriculum consistent with this addendum.

In order to be successful, staff development programs need to include the following components:

1. Coordination of all relevant staff development programs
2. Coordination with district, county, and state offices and with professional mathematics educational groups
3. A closed loop to ensure adequate and effective planning, as shown in Figure A-4

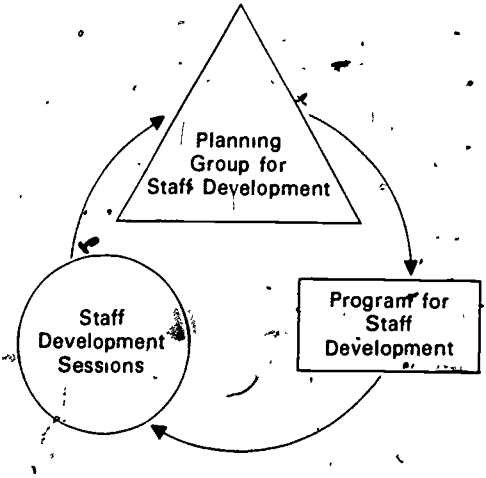

Fig. A-4. The Closed Loop in Staff Development Planning

4. The content of the addendum to the *Mathematics Framework*
5. Evaluation, which is basic to the implementation of high quality staff development programs

The interrelationships of the components of staff development are shown in Figure A-5. The following is an example of the manner in which a school staff might implement the paradigm shown in Figure A-5.

A school staff development planning group is given the charge of developing a plan for a mathematics in-service training program. The group begins by drawing upon both external and internal groups or sources, including the *Mathematics Framework* addendum, for planning a staff development program. The plan then goes to the total staff for evaluation and suggestions. As the revised program is implemented, continual evaluation of both the process and the outcome of the sessions will improve future staff development efforts.

Specific recommendations for planning and carrying out a staff development program follow:

1. Program planning needs to encompass all skill levels of staff members.
2. Specific participant objectives need to be identified for each session, and the degree to which the objectives were attained needs to be determined at the conclusion of the training.
3. Coordination of training among schools is necessary.
4. To facilitate continued teaching improvement, a need exists for one individual to provide liaison between the staff development program and the classroom teacher.
5. At regular intervals time needs to be set aside for the target population to discuss the implementation of the program.

The following topics in mathematics are suggested as subjects of a staff development program:

1. Teaching for Competency Mastery—Alternative Approach to Remediation
2. Hand-held Calculators in the Classroom
3. Computer Literacy
4. A Problem-Solving Curriculum
5. The Role of Estimation in Problem Solving
6. The Complete Mathematics Program—More Than the Textbook
7. Mathematics as It Relates to Career Choices
8. Reading and Its Impact on Mathematics
9. Mathematics Centers—Uses and Abuses
10. Techniques for Grouping Within Secondary Classrooms
11. Mathematics Labs—Organizing, Alternative Functions, and Efficient Recordkeeping Systems
12. Use of the Computer to Simulate Conditions Found in Social Science, Science, and Other Subject Areas
13. The Relationship of Mathematics with Other Disciplines

APPENDIX A

A List of Uses of Mathematical Skills and Concepts in Applications of Arithmetic

This outline is adapted from an April, 1980, report of the Arithmetic and Its Applications Project, which was codirected by Zalman Usiskin and Max Bell, with some revisions that incorporate the work of that project as late as June, 1980. This work was partially supported by National Science Foundation grant number 79-19065; however, the opinions expressed are those of the codirectors, not necessarily those of the National Science Foundation.

The outline adheres to the following hierarchy:

I. (Capital Roman numerals)—Steps of the modeling process
 A. (Capital letters)—Curricular topics, usually standard
  1. (Numerals)—Use categories (For what reason do we use . . .?)
   a. (Small letters)—Use subcategories (listed occasionally)

I. Confronting Numbers and Labels

- A. Single numbers
  1. Counts
  2. Measures
  3. Locations in reference frames
     a. Rank orders
     b. Time and space frames
     c. Scales (linear, normalized, exponential, and so forth)
  4. Comparisons
  5. Conversion factors and constants
  6. Codes
     a. Numerical codes
     b. Alphanumeric codes
- B. N-tuples (ordered pairs, triples, and so forth) of numbers
  1. Storing counts or measures
  2. Locations
  3. Comparisons
  4. Codes
- C. Labels
  1. Counting units

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I. (Capital Roman numerals)—Steps of the modeling process
  A. (Capital letters)—Curricular topics, usually standard
    1. (Numerals)—Use categories (For what reason do we use . . .?)
      a. (Small letters)—Use subcategories (listed occasionally)

I. Confronting Numbers and Labels
  A. Single numbers
    1. Counts
    2. Measures
    3. Locations in reference frames
      a. Rank orders
      b. Time and space frames
      c. Scales (linear, normalized, exponential, and so forth)
    4. Comparisons
    5. Conversion factors and constants
    6. Codes
      a. Numerical codes
      b. Alphanumeric codes
  B. N-tuples (ordered pairs, triples, and so forth) of numbers
    1. Storing counts or measures
    2. Locations
    3. Comparisons
    4. Codes
  C. Labels
    1. Counting units

2. Measure units
   a. Standardized
      (1) Base
      (2) Multiples of base
      (3) Derived
   b. Quasistandardized (teaspoon, pace, and so forth)
   c. Arbitrary
3. Monetary units
4. Scale labels
   a. Discrete scales
   b. Linear scales
   c. Normalized scales
   d. Exponential scales
   e. Other scales
5. Informational labels
   a. Directional labels
   b. Descriptive labels

II. Adjusting Concepts in Part I for Study

A. Adjustments of notation (fractions to decimals, and so forth)
   1. For facility in applying algorithms
   2. For easier comparison
   3. For consistency of information
   4. For saving space for easier reading

B. Microadjustments of value (estimates, rounding, and so forth)
   1. For safety tolerance
   2. For necessity
      a. Exact computation is impossible (e.g., vacation cost).
      b. Situation has built-in variability (e.g., typing speed).
      c. An exact value would mean ability to predict future.
      d. An exact value is impossible in real world terms.
      e. An exact value is unavailable.
   3. See reasons under A.

C. Macroadjustments of value (scaling, shifting, and so forth)
   1. See reasons under A.

D. Adjustments of mode of representation (prose to photographs)
   1. For increasing information being conveyed
   2. For clarity

III. Translation of Situational Questions into Mathematical Relationships

A. Addition
   1. Totaling counts
   2. Totaling measures
   3. Changing by fixed quantity
   4. Combining changes

B. Subtraction
   1. Take-away with counts
   2. Take-away with measures
   3. Filling up (finishing, completion)
   4. Comparison

C. Multiplication
   1. Measure product
      a. Cartesian product (permutation without repetition)
      b. Area
      c. Other derived measures
   2. Rate factor
      a. Set of sets
      b. Continuous rate factor
   3. Conversion factor
      a. Within system
      b. Between systems
   4. Scale factor
      a. Part of, multiple of
      b. Size changing

D. Division
   1. Measure quotient (from measure product)
   2. Rate
      a. Calculation
      b. Rate divisor (from rate factor)
   3. Ratio
      a. Calculation
      b. Scale divisor (from scale factor)
   4. Conversion
      a. Calculation of conversion rate
      b. Conversion divisor

E. Powering
   1. Permutation with repetition
   2. Volume
   3. Change of unit in dimension
   4. Growth
   5. Acceleration, and so forth

F. Absolute valuing—undirecting distance

G. Combinations of operations—many applications utilize more than one of the above operations. The schema cannot give more than a few examples of these. Our categorization here is not by intent but by nature of the relationship of the combination to what is given above.
   1. Use categories involving two separable operations
      a. Linear combinations
      b. Rate of change (slope and so forth)
   2. Use categories involving more than two separable operations
      a. Weighted average

b. Linear conversion
c. Formula for hypotenuse of right triangle given two legs, and so forth

3. Where separation seems difficult (use categories based upon mathematical properties of the operations)
a. "Best fit" functions
b. Trigonometric formulas, and so forth

H. Operations on sets (max, min, mode, median, gcd, lcf)
1. Measuring central tendency
2. Measuring extrema
3. Measuring fit

APPENDIX B

Curriculum Considerations for Use of the Hand-Held Calculator

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Excerpts from *The Teacher*, February, 1977

As the calculator is used more and more, the mathematics curriculum, being quite hierarchical and definitive, is open to some considerations. Teachers, researchers, and curriculum experts need to examine some of the calculator implications on the curriculum. These considerations are the purpose of this article.

Different Logic Systems

Many types of calculators perform in different ways. Calculators have various capability features that work to the advantage as well as to the disadvantage of teaching mathematical concepts. The first hurdle is to be able to cope with a specific calculator that a student brings into the classroom and to be flexible in pointing out that certain problems may not work out a certain way on the machine. The algebraic logic (*4* ⊟ *3* ⊟), arithmetic logic (*4* ⊞ *3* ☐), and Reverse Polish notation (*4* [*enter*] *3* ☐) logic systems are ways in which different calculators solve algorithms. The most common is the algebraic system which seems to correlate with the algorithms taught to children.

Rounding

Consumer computations involving money usually have answers of more than two decimal places. Most students are not aware of this, as they only see money answers in dollars and cents (two place decimals). Therefore, careful development of rounding needs to be in the curriculum at earlier grade levels (than presently developed) for students to relate to our monetary system.

Fractions to Decimals

The ability to convert fractions into decimals is essential. The formula for the area of a triangle $1/2\ bh$ *for volume or* $4/3\ \pi\ r^3$ of a sphere can be confusing even if a student knows how to represent the fraction 1/2 or 4/3 in decimal form. Therefore, to change a fraction to a decimal is a necessary skill which the student must know in order to relate to the fraction in its new form.

Decimal Representation of a Fraction

Many calculators represent fractions such as 1/6 as 0.166666. If someone is doing a chain of operations with the fraction 1/6, leaving it as 0.166666 gives a more accurate answer. But our math books show that 0.166666 can be rounded to 0.16667 or 0.167. This affects computation results and can be confusing in relation to our present math curriculum. (Another common notation for infinite decimals is shown here: $0.1\overline{6}$)

Overflow

1,234,567 ⊠ *123,456* is too large (overflow) a product for most calculators to display since they usually have displays of eight digits or less. Although many calculators indicate that the answer is too large, techniques need to be developed to solve these "large" problems. Some calculators give the first eight digits—therefore, estimation of *1,234,567* ⊠ *123,456* or work with powers can enable the student to have an approximate answer to such situations.

Underflow

Answer to problems such as *0.0001* ⊠ *0.000056* ⊟ *0* (underflow) may also be quite confusing to students. In addition, many calculators have no means to indicate that the answer is too small to have meaning on the calculator. Calculators need to be developed so students can solve the problem using scientific notation. In the curriculum, students should be taught about "smallness" just as they are about "bigness."

Truncation

The calculator's manipulation of numbers involving truncation needs to be explored. For example: *1* ⊟ *15* ⊟ *15* ⊟ *15* ⊟ *0.0002962*. Multiplying 0.0002962 by 15 repeatedly should get back the original number 1, instead, the answer is 0.999675. Parts of the product have been truncated making it impossible to regain the original number. The idea of truncation needs to be discussed, and real life truncation problems need to be explored. For example, banking continually faces the truncation dilemma. In addition, we need to consider expressions of the decimal form with irrational numbers as fractions.

Order of Operations

Order of operations rules need to be emphasized for use on most calculators. The problem 4⊞20⊟4⊞6 solved from left to right with most calculators gives the answer 12. The correct answer is 15 since you divide and multiply before you add and subtract. (My Dear Aunt Sally rule.) This procedure is overlooked in most calculator applications and often accounts for wrong answers. Although the use of parentheses is emphasized to help solve this type of problem, the "order of operations" rules need to be included in the math curriculum. It is included in upper grades, such as eighth grade math and in algebra. It should be in lower grades also.

"Add-on" Percent Key

The understanding of the "add-on" percent key is essential for consumer use. To realize that it does a lot of arithmetic in a simple calculator operation is not common knowledge to most people since it does not relate directly to percent. It does not show that 5 percent is 0.05 in a separate step.

Constant Key

The constant key (automatic repetition of a constant addend, multiple, etc.) allows computation by a number without entering the constant (function) each time. For example, divide each number, 18, 16, 24, and 27 by 9.

18 ⊟ *9* ⊟ *2*
16 ⊟ *1.7777777*
24 ⊟ *2.6666666*
27 ⊟ *3*

Calculator Arithmetic Using Internal Counting

Some interesting nonmathematical "arithmetic" problem solving may occur. For example: *37 + 37 + 37*. Calculator solution. *37* ⊞ ⊟ ⊟ ⊟ *111*. or some calculators may only require *37* ⊞ ⊟ ⊟ *111*. Problem. Using only four 4s, make the calculator equal 3:

4 ⊠ *4* ⊞ *4* ⊞ *4* ⊞ ⊟ ⊟ ⊟

Of course this ignores order of operations as well as writing a real mathematics formula.

Division by Zero

Some calculators show *5* ⊟ *0* ⊟ *0*. Many calculators show *5* ⊟ *0* ⊟ *0* — (flashing zero). Students need to have an understanding that this solution is incorrect and any number divided by 0 is undefined.

The Negative Sign

Displays on many calculators print out one less digit if the answer is negative. Knowing that the negative sign takes up a digit place can be misleading for eight digit answers. Students expecting an eight digit solution will only get seven digits with a negative sign. Lots of calculators have a "minus" operation key ⊟ and a "change sign" key [+/-] or [CS].

Consecutive Operations

Some calculators perform consecutive operations which do not follow the algebraic logic and rules of mathematics. For example, compounding interest on some calculators can be done as follows: $200 at 6 percent annual interest compounded quarterly:

200 ⊠ *1.5* [%] ⊟ [%] ⊟ [%] ⊟ *206.045*

The use of consecutive operations is then unique to some calculators, and explanations are not dependent on mathematics utilizing a binary operation.

These are a few of the considerations which the calculator presents to the curriculum. Either the curriculum needs to include exploration of these considerations or the teacher who uses a specific calculator needs to be aware of its possible functions (or capabilities).

If the present variations in calculators continues, teachers need to be aware of problems and questions which may occur. Different companies will certainly have different models with different internal functions— so teachers must have an open mind to different types of calculators. Also, many of the basic considerations are already in many basal math texts, but they must be emphasized more. Curriculum development needs to include such topics as underflow, overflow, estimation, scientific notation, powers of ten, rounding, and truncation to make it possible for students to successfully take full advantage of the calculator and its capabilities.

Tips on Purchasing a Calculator

Trying to select the right pocket calculator for use in the classroom can be confusing. Many different brands, types, prices, and features are available. The following guidelines may help you choose the type of calculator best suited to your needs:

- A four-function calculator is sufficient for students through the eighth grade. Memory, square root, change sign, and particularly percent keys would also be beneficial for sixth, seventh, and eighth graders.
- The casing of the calculator should be rugged and preferably made of hard, durable plastic. Any bolts or screws should be recessed or concealed to make the case, as well as the battery storage compartment, difficult to open.
- The keys should be large and well-spaced. Keys that make a sound when pressed are especially useful, because they tell the children that the keys have been activated. This also prevents students from pressing the keys unnecessarily hard.
- The read-out display should be clearly legible and visible in bright light. It is more convenient to have the minus or negative sign next to the number than on the far left side or indicated by a light. When the solution to a computation is too long to fit on the display, a symbol denoting "overflow" should be clearly visible on the display.
- The algorithmic computation approach (algebraic logic) is important. In other words, entering *3* ☐ *2* ⊟ into the calculator should produce the solution—one. Fortunately, few calculators compute otherwise. Some nonalgorithmic (nonalgebraic logic) calculators would print the solution as minus one. In order to compute this problem on such a calculator you would enter *3* ⊞ *2* ⊟⊟ to get the answer—one.
- A printout display that accommodates a minimum of six digits is essential, eight digits are desirable. A "floating" decimal point is more useful than a two- or four-place fixed decimal point display. For example, using a floating point, the solution to the problem *1* ⊟ *8* would read *0.125* instead of *0.12* with a fixed decimal format.

- Calculators with keys that show two functions, such as [+ =] or [- =], are not beneficial because they may cause confusion.
- A calculator that "counts" can provide many activities for primary children. By entering the number and the plus sign, such as *1* [+], and then continually pushing the equal key, the calculator should count by ones, such as *1, 2, 3, 4*, ... Or, enter *5* [+] [=] [=] . . . to get *5, 10, 15*. . . .
- Calculators with long-life batteries are now available and relatively inexpensive compared to the cost of buying and replacing short-life batteries. These calculators often have an automatic shut-off feature which extends the battery life even longer.
- A dependable no-risk warranty is also important.

Not every calculator has all the above features, but these should be taken into consideration when making selections for children's use. The greater the number of functions and degree of sophistication the more expensive will be the machine.

APPENDIX C

Criteria for Evaluating Instructional Materials in Mathematics (K—8)

Adopted by the California State Board of Education
March 13, 1980

The classroom teacher continues to be the gatekeeper in curriculum—the one who stages the learning opportunities. The teacher is most responsible for knowing and providing the basic mathematics program as described in the *Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve* (1975), and the addendum to that framework (1980), the mathematics curriculum mandated by the California State Board of Education. The following criteria are written in conformance with the *Mathematics Framework* and its addendum and constitute a mandate by the State Board to the publishers who submitted their instructional materials for consideration on the 1982 adoption list.

Several substantial changes have been made in the criteria from the previous version:

- The problem solving/application strand has become the all encompassing theme of mathematics instruction and is no longer a separate topic.
- Part II of the criteria, instructional methods, has been expanded to include process skills of mathematical thinking necessary for solving problems.
- Mathematical skills and concepts related to calculators and computers are now elevated to the same level as other traditional concepts.
- The directive for metric instruction has been clarified and strengthened in view of progress being made toward a metric America.

The criteria are organized to facilitate the efforts of the Instructional Materials Evaluation Panels that are responsible for screening all submitted materials and recommending to the Curriculum Frameworks and Supplemental Materials Commission of the State Board those which should be included in the adoption list.

I. Students and Their Needs

A well-rounded mathematics program will appeal to many types of learners. Hence, the instructional materials should provide learning opportunities premised on the following student characteristics:

A. Achievement Patterns

1. Superior achievement patterns of pacing, depth of learning, recall, and application in mathematics
2. Average achievement patterns of pacing, depth of learning, recall, and application in mathematics
3. Limited achievement patterns

B. Learning Modalities

4. Visual learning
5. Sensory or tactile learning
6. Symbolic learning

C. Reading Skills

7. Average or above
8. Below average

D. Affective Qualities

9. Highly motivated, self-directive
10. Unmotivated, nonself-directive
11. Variable, but typically average

E. Special Needs

12. Language disabilities and restrictions
13. Educationally handicapped
14. Able learners

II. Instructional Needs

The instructional materials shall facilitate:

A. The Problem-Solving Perspective of Mathematics

The major goal of the instructional program in mathematics, kindergarten through grade eight, is the development of the ability to solve real problems and to understand the basic concepts involved in the problem-solving process. Concepts and skills related to problem solving should be reviewed and emphasized frequently throughout the mathematics program so that students understand that mathematics is problem solving.

Every topic in mathematics should derive its justification for existence on the basis of its value as an aid in problem solving. Problem solving is no longer an end-of-chapter application section designed to provide practice in using just-learned mathematical concepts and skills. Concepts and skills should be organized in the instructional program around the problem-solving processes which are outlined briefly here as follows:

1. How to formulate problems from situations with which students are familiar
2. How to analyze problems using a wide variety of mathematical strategies and models
3. How to solve problems by mathematical simplification
4. How to interpret the solution, verify its accuracy, and explore the implications

B. A Variety of Instructional Techniques

1. Student groupings in the classroom
2. A variety of classroom strategies for instruction

Classroom strategies	Student groupings		
	Large	Small	Individualized
Laboratory			
Audiovisual (films, loops)			
Expository teaching			
Team teaching			

C. Consistent and Valid Metric Instructions

1. The first *standard* units of measure presented for a given measurement task should be metric and thoroughly understood before making any reference to the present U.S. system of units for that task.
2. Suggestions for instructional examples for the teacher to use in explaining mathematical concepts should be metric if reference to standard units of measure are involved.
3. Measurement tasks should be presented to develop skills of estimation with metric units.
4. The present U.S. system of units should be presented only to the extent necessary for simple measurement tasks pupils encounter in out-of-school applications.
5. Measurement tools for classroom use should be replaced as rapidly as possible with metric-only tools.
6. In each set of application exercises, metric units should be used in a majority of those problems which involve measurement units.
7. Only decimal notation should be used to report measurements in metric units—no fractions.
8. All metric symbols and terms should conform to the International Standard of Units (SI).
9. *All* graphs, pictures, and illustrations involving measurement units should clearly portray metric units as the dominant, acceptable, and standard units of measurement.
10. No dual-dimensional measurements should be used.

III. Mathematics Content

The criteria for content coverage is organized around *major* mathematical concepts and skills. Material should be presented to students in a problem-solving context, as outlined elsewhere in the addendum to the *Mathematics Framework*.

The materials shall provide activities leading to the development of the following concepts and skills:

A. *Arithmetic, Numbers, and Operations*

1. One-to-one correspondence, number, counting, and order
2. The number line and the coordinate plane
3. Positive and negative numbers
4. Decimal notation and computation with decimals (prior to formal computation with numbers in fraction form)
5. Memorization and use of basic arithmetic facts of addition and multiplication
6. Addition in the development of the operation of subtraction
7. Subtraction and multiplication in developing the operation of division
8. Equality and order relations
9. Properties of operations in the development of computation skills
10. Elementary number theory concepts
11. Skills of computation with positive and negative numbers
12. Selection of the appropriate operations for given situations
13. Mental arithmetic
14. Place value in the decimal numeration system
15. Exponential and scientific notation
16. The real number system
17. Ratio, proportion, and percent
18. Rounding off numbers and estimation skills

B. *Geometry*

19. Intuitive, informal geometry, utilizing environmental models
20. Similarity and congruence
21. Parallelism, perpendicularity, and skewness
22. Classification of geometric shapes
23. Use of geometric instruments
24. Construction of three-dimensional models
25. Length, circumference, perimeter, area, volume, and angle measures of simple geometric figures
26. Indirect measurement and the Pythagorean formula
27. Elementary coordinate geometry

C *Measurement*

28. Measuring familiar objects through "hands-on" experience
29. Arbitrary units for measuring (preceding instruction in standard units)
30. Standard units as a uniform way of reporting measurements
31. Understanding the structure of and using the metric system of units (SI)
32. Convenient references for metric units without computational conversions between the U.S. customary units and SI units
33. Practice with the numerical values of the metric prefixes

34. Estimating distance, area, volume, mass, and temperature in metric units
35. Reading simple measuring instruments and the approximate nature of measurement
36. Scale drawings and maps
37. Formulas for determining perimeter, area, and volume

D. Calculators and Computers

38. Estimation
39. Calculating +, −, ×, ÷
40. Immediate feedback
41. Reinforcement of number relationships
42. Exploring number patterns
43. Motivation and enrichment
44. Extended application problems to develop concepts that are normally obscured by tedious computations
45. Providing integrated curriculum opportunities
46. Working knowledge of the functions, logic, and mechanics
47. Historical perspective
48. Order properties
49. Programming and flow charting
50. Examples of use in society

E. Probability and Statistics

51. Collecting, organizing, and representing data derived from real-life situations
52. Tree diagramming and counting procedures for sample spaces
53. Permutations (arrangements) and combinations (selections)
54. Making guesses about patterns or trends in data
55. Statistical inferences
56. Various measures of central tendency and dispersion
57. Elementary concepts of probability
58. Reliability of statistical inference

F. Relations and Functions

59. Constructing and interpreting tables, graphs, and schedules
60. Mappings, correspondences, ordered pairs, and "rules" leading to the concept of a mathematical relation
61. The function concept and function notation
62. A function in mathematical applications
63. Patterns and relationships and forming generalizations

G. Logical Thinking

64. Manipulatives, games, problems, and puzzles which stimulate and develop elementary reasoning patterns
65. Trial and error strategies
66. Applying reasoning patterns to nonmathematical situations, such as advertising
67. Direct and indirect reasoning patterns

68. Inductive and deductive reasoning patterns
69. Sentences using: *and, or, not, if . . . then, all,* and *some*

IV. Teacher Materials

The materials should assist the teacher by providing:

A. Organization

1. Clear identification of intended student audience
2. Scope and sequence
3. List of student or program objectives
4. Integrated ancillary materials

B. Strategies

5. Innovative alternatives for classroom activities (transparencies, cassettes, hand-held calculators, and so forth)
6. Assistance in overcoming reading difficulties, slow learning rates, and special student needs
7. Suggestions for applying mathematics to career education
8. Suggestions for utilizing mathematical ideas in reference to environmental needs
9. Ideas for developing students' investigative skills
10. Suggestions for reinforcing, reviewing, and applying previously learned skills and concepts
11. Techniques and materials for encouraging home assistance
12. Suggestions for making a smooth transition from concrete learning to abstract learning
13. Integration of mathematics with other curriculum areas

C. Evaluation Instruments

14. Alternatives for assessment of student achievement of instructional objectives
15. Suggestions for and samples of student progress records
16. Assessment of overall achievement of a program's objectives
17. Reports and forms established for student achievement
18. Techniques for student self-assessment, including some answers in student materials
19. Solutions to all exercises

D. Management Systems

20. Suggestions for appropriate placement of students
21. Suggestions for a variety of groupings
22. Alternative schedules for directing students through daily mathematical activities
23. Recommended use of teacher-support personnel (instructional aides, parent volunteers, and so forth) and suggestions for handling student papers

E. Appreciation

24. Presentation of mathematics beauty, history, and value
25. Stimulation of favorable attitudes toward learning mathematics

APPENDIX D

Mathematics Program Objectives for Kindergarten Through Grade Eight

Arithmetic, Numbers, and Operations

Counting

Counting: Readiness

Explore and identify the number of objects in a set.
Explore the concept of *how many.*
Explore and identify equal and unequal sets of physical objects.
Acquire the skill of counting concrete objects.

Counting: Kindergarten—Grade Three

Acquire the skill of counting pictorial representations of objects.
Develop the skill of identifying a sequence of whole numbers.
Develop the skill of counting by multiples of numbers.
Explore the conservation of number.
Using the number line, explore the order relation for whole numbers.
Develop the order relation for the whole numbers.
Using sets of physical objects, explore the concepts of equality and inequality.
Using one-to-one correspondence, explore the concepts of more than, less than, and equality.
Acquire the skill of reading and writing numerals and number names written in words.
Explore reading graphs that use pictorial representations of objects.
Maintain counting skills through review and practice.

Counting: Grades Four—Six

Acquire the skill of reading and writing non-negative rational numbers in decimal and fraction form.
Acquire the skill of counting by multiples.
Develop the number line concept to include rational numbers.
Develop the order relation for the integers.

Counting: Grades Seven-Eight

Acquire the skill of determining the order relation for rational numbers.
Acquire the skill of reading and writing rational numbers.

Operations

Operations: Readiness

Using concrete objects, explore the process of joining and separating sets.
Using sets of concrete objects, explore addition and subtraction.

Operations: Kindergarten—Grade Three

Explore the skill of adding and subtracting whole numbers.
Develop the appropriate use of the operational symbols (+, −, ×, ÷) between numbers.
Develop addition and subtraction facts.
Acquire addition and subtraction facts.
Acquire the skill of adding or subtracting with and without renaming.
Acquire the skill of adding and subtracting amounts of money.
Acquire the skill of using the terms *addend, sum, factor,* and *product.*
Maintain addition and subtraction skills for whole numbers through review and practice.
Explore the concept of multiplication.
Develop basic multiplication facts.
Acquire the basic multiplication facts.
Acquire an understanding of, and use of, the multiplication algorithm.
Develop the skill of adding and subtracting of decimal fractions (with and without renaming).
Acquire the skill of using number sentences.
Explore the meaning of division.
Develop basic division facts.
Acquire basic division facts.
Maintain multiplication and division skills for whole numbers through review and practice.

Operations: Grades Four—Six

Maintain the addition and subtraction facts.
Acquire an understanding of, and skill in using, the operational symbols (+, −, ×, ÷).
Develop mastery and accuracy in using addition and subtraction algorithms.
Maintain addition and subtraction skill for whole numbers through review and practice.
Maintain the basic multiplication and division facts.
Develop the division algorithm.
Acquire skill in using the division algorithm.
Develop skill in estimating sum, difference, product, and quotient of whole numbers.
Explore the use of positive and negative integers in everyday situations.
Develop the skill of adding and subtracting positive and negative integers.
Maintain skill in working with number sentences.
Maintain multiplication and division skill for whole numbers through review and practice.
Acquire computational skill for addition and subtraction of decimal fractions.
Acquire computational skill for multiplication and division of decimal fractions.
Develop the skill of converting a rational number into its equivalent fraction, decimal fraction, or percent forms.
Explore computational skill for the addition and subtraction of common fractions.

Explore computational skill for multiplying and dividing common fractions.
Explore percents through related work with fractions.
Develop the use of whole numbers as exponents.
Explore the skill of multiplying and dividing positive and negative integers.

Operations: Grades Seven-Eight

Develop the skill of using the standard order of operations in computations.
Explore shortcuts in the basic algorithm operations.
Develop the skill of multiplying and dividing positive and negative integers.
Acquire the skill of adding and subtracting rational numbers in fraction and mixed number forms.
Acquire multiplication and division skill for fractions.
Maintain computational skill for addition and subtraction of whole numbers and decimal fractions.
Maintain computational skill for the multiplication and division of whole numbers and decimal fractions.
Develop skill in estimating the sum, difference, product, and quotient of rational numbers.
Develop skill in rounding off.
Acquire the skill of using integers as exponents.
Acquire the skill of using numbers written in percent form.
Develop the skill of simplifying complex fractions.
Acquire the skill of solving simple equations by using the operations of addition, subtraction, multiplication, or division.
Acquire the skill of squaring a rational number.
Develop the skill of estimating square root.

Place Values

Place Values: Readiness

Explore the process of grouping and counting concrete objects.
Develop the process of grouping and counting concrete objects.
Explore grouping and counting concrete objects by tens and ones.
Develop grouping and counting concrete objects by tens and ones.

Place Values: Kindergarten—Grade Three

Develop the writing of the digits zero through nine.
Explore the role of decimal numbers (e.g., money, metric measurement).
Develop skill in counting concrete objects and recording tens and ones.
Develop the concept of place value by grouping objects.
Explore place value of digits in numerals.
Develop place value of digits in a three-digit numeral.
Explore the skill of writing a number in expanded notation form.
Develop skills of counting by ones, twos, fives, tens, and hundreds.

Place Values: Grades Four—Six

Acquire the skill of using a decimal point in place value notation.
Acquire the skill of identifying the place value for any digit in a numeral.
Develop the skill of representing a number in expanded notation form and in exponential form.
Explore scientific notation of numbers.

Place Values: Grades Seven-Eight

Develop the use of integers as powers of ten in place value notation.
Develop scientific notation of numbers.

Patterns

Patterns: Readiness

Explore simple patterns made with objects.
Develop the skill of recognizing patterns made with objects and the extension of those patterns.
Explore even and odd whole numbers.

Patterns: Kindergarten—Grade Three

Using pictures and drawings, develop the skill of recognizing simple patterns.
Explore pattern recognition in sequences of numbers.
Explore sequences of even and odd numbers and their properties.
Using multiples of numbers, explore methods of counting.
Develop recognition of even and odd numbers.

Patterns: Grades Four—Six

Develop the skill of pattern recognition in number sequences.
Explore number patterns of the real number system.

Patterns: Grades Seven-Eight

Develop recognition of number patterns of the real number system.
Explore the patterns in sets of ordered pairs.
Explore the use of variables to represent number patterns.

Nature of Numbers

Nature of Numbers: Readiness

Explore the skill of reading numerals.
Explore the use of numbers and numerals in daily life.
Develop the skill of recognizing half of an object and half of a set of objects.

Nature of Numbers: Kindergarten—Grade Three

Develop the order relation for whole numbers and decimal fractions.
Develop the skill of selecting relational symbols to make true statements ($=$, $\neq$, $>$, $<$).
Explore situations involving negative numbers.
Explore fractional parts of concrete objects.
Develop the skill of recognizing fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$.
Acquire the skill of identifying fractional parts $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$.
Explore the skill of recognizing decimal equivalents of fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{10}$.
Explore the skill of ordering fractions.

Nature of Numbers: Grades Four—Six

Explore the property of betweenness for numbers.
Develop fraction notation and the properties of fractions.
Develop the skill of selecting a number in fraction form that corresponds to part of a concrete object.

Acquire the skill of changing fractions to equivalent fractions.
Develop an understanding of least common multiple and greatest common factor.
Explore the technique of finding common multiples.
Acquire the skill of computing the least common multiple of two or more whole numbers.
Develop an understanding of prime and composite numbers.
Develop the skill of identifying composite numbers.
Explore techniques to determine if a given number is prime.
Develop the skill of finding the prime factors of a composite number.
Explore the concept of relatively prime numbers.
Acquire the skill of ordering decimal fractions.
Develop the skill of converting percents to decimals to fractions and vice versa.

Nature of Numbers: Grades Seven-Eight

Acquire the skill of ordering rational numbers.
Acquire the skill of finding factors in terms of divisibility.
Develop techniques to determine if a given number is prime.
Explore some of the simple properties of primes.
Develop rules for divisibility.
Acquire a technique to determine the prime factors of a number.
Acquire the skill of converting percents to decimals to fractions and vice versa.
Explore the periodicity of the decimal form of rational numbers.

Number Properties

Number Properties: Readiness

Explore joining the empty set to a set of objects.

Number Properties: Kindergarten—Grade Three

Explore commutative and associative properties of addition and multiplication for whole numbers.
Explore the distributive property of whole numbers.
Explore the use of parentheses in grouping.
Explore the special properties of zero and one.

Number Properties: Grades Four—Six

Develop the distributive property.
Develop the use of parentheses to illustrate the associative and distributive properties
Explore the closure property for addition and multiplication on the set of whole numbers.
Develop the role of identity elements in addition and multiplication.
Explore the role of inverse elements in addition and multiplication.
Explore the use of inverse elements in operations with integers and fractions.
Develop subtraction and division as inverse operations of addition and multiplication, respectively.
Explore the use of properties of number systems in the development of algorithms for basic operations.

Number Properties: Grades Seven-Eight

Acquire the skill of using the identity elements for the operations of addition and multiplication with rational numbers.

Acquire the skill of using the inverse elements for the operations of addition and multiplication with rational numbers.

Develop an understanding of absolute value.

Explore the presence or absence of the property of denseness in different subsets of real numbers.

Explore the use of the field properties in the development of algorithms for basic operations with integers and rational numbers.

Explore the concept of one-to-one correspondence between real numbers and points on the number line.

Explore the concept of irrational numbers.

Geometry

Geometric Figures

Geometric Figures: Readiness

Explore familiar physical objects representing two-dimensional and three-dimensional geometric shapes.

Explore, through design, building with three-dimensional materials, including pattern formation.

Explore recognition of geometric shapes according to their properties.

Explore recognition of special properties; e.g., inside, outside, flat, curved, straight, round, square, tall, or deep.

Explore simple patterns involving symmetry in the environment; e.g., blot patterns or leaves.

Geometric Figures: Kindergarten—Grade Three

Identify two-dimensional and three-dimensional geometric shapes.

Identify properties of geometric shapes, using precise language, written and oral.

Identify perpendicular and parallel lines in two-dimensional and three-dimensional geometric models and figures.

Identify open and closed curves.

Develop classification skills for plane geometric figures.

Develop the ideas of similarity and congruence.

Develop pattern designs and tessellations.

Acquire the ability to demonstrate properties of geometric shapes.

Explore simple patterns of picture symmetry; e.g., paper folding.

Identify symmetry in the natural world.

Geometric Figures: Grades Four—Six

Explore transformations with geometric shapes.

Develop precise language and appropriate symbols to use with closed and open curves, closed and open surfaces, cubes, spheres, and triangular prisms.

Develop an understanding of lines, planes, and space as sets of points.

Acquire definition of perpendicular and parallel relationships of lines.
Develop skill of construction of models and patterns to illustrate geometric ideas using appropriate instruments.
Develop ability to classify and name geometric figures.
Identify patterns of symmetry on the plane.

Geometric Figures: Grades Seven-Eight
Develop the use of transformations, such as reflections, rotations, translations, and dilations on geometric figures.
Acquire the ability to demonstrate concepts of congruence, similarity, perpendicularity, and parallelism.
Develop patterns of symmetry on figures and models.

Reasoning

Reasoning: Readiness
Explore short chains of reasoning through the sorting of two-dimensional and three-dimensional shapes.

Reasoning: Kindergarten—Grade Three
Explore logical verification of similarity and congruence relationships with physical objects.

Reasoning: Grades Four—Six
Develop ways of verifying similarity and congruence.
Explore inductive reasoning, leading to generalizations.
Identify ways to verify congruence and similarity.
Identify short chains of deductive reasoning.

Reasoning: Grades Seven-Eight
Acquire ways to verify similarity and congruence.
Develop simple deductive and inductive reasoning processes, especially those related to similarity, congruence, and sum of angle measures for angles of a triangle.
Acquire the ability to perform short, sequential reasoning exercises.
Identify logic errors in faulty reasoning.

Coordinate Geometry

Coordinate Geometry: Readiness
Explore two-dimensional patterns as a foundation for coordinates, e.g., tile floor or checkerboard.

Coordinate Geometry: Kindergarten—Grade Three
Explore the use of coordinates to find locations in the first quadrant.
Identify points on a line corresponding to positive and negative numbers.

Coordinate Geometry: Grades Four—Six
Explore graphing solution sets on the number line.
Develop concepts related to the coordinate plane, such as map reading and graphing.
Identify points on the coordinate plane with appropriate language and symbols in all four quadrants.

Coordinate Geometry: Grades Seven-Eight

Acquire the concepts related to the coordinate plane.

Acquire skill of graphing solution sets of number sentences on the number line and the coordinate plane.

Explore the concepts related to the three-dimensional coordinate space.

Acquire the skill of graphing, using coordinates and symmetry.

Measurement of Geometric Figures

Measurement of Geometric Figures: Readiness

Explore the comparison of size of geometric figures and models.

Measurement of Geometric Figures: Kindergarten—Grade Three

Explore concepts of length and perimeter, using arbitrary units; (e.g., pacing).

Identify the concepts of length, width, perimeter, and circumference.

Using standard units, develop the measurement of length, width, perimeter, and circumference.

Measurement of Geometric Figures: Grades Four—Six

Develop concepts of area, volume, and measurement of angles.

Using computation skills, develop geometric measurement concepts.

Measurement of Geometric Figures: Grades Seven-Eight

Develop an understanding of the derivation of π (pi).

Develop informal derivations of geometric formulas for areas of circles and triangles.

Develop informal derivations of geometric formulas for volumes of cones, pyramids, cylinders, and spheres.

Measurement

Arbitrary Units of Measurement

Arbitrary Units of Measurement: Readiness

Explore the attributes of measurement together with the development of appropriate vocabulary.

Explore the use of arbitrary units of measure.

Explore the reasons for measuring objects.

Arbitrary Units of Measurement: Kindergarten—Grade Three

Explore measurement, using a variety of arbitrary units of measure.

Develop useful vocabulary for measurement.

Develop and organize techniques related to measuring, using selected arbitrary units of measure.

Arbitrary Units of Measurement: Grades Four—Six

Explore a variety of measuring experiences within the child's environment.

Acquire appropriate vocabulary of measurement.

Acquire skills in measuring with common arbitrary units of measure.

Arbitrary Units of Measurement: Grades Seven-Eight

Maintain skills in using arbitrary units for measurement situations arising in daily living.

Maintain proficiency in using correct language for measurement.

Standard Units of Measurement

Standard Units of Measurement: Readiness

Explore comparative measuring with unmarked objects of standard metric unit size.

Standard Units of Measurement: Kindergarten—Grade Three

Develop a familiarity with informal methods of measuring.
Explore comparative measures within the child's environment, using objects marked with standard metric units.
Explore the selection of appropriate units for measuring concrete objects.
Explore measuring to the nearest whole unit.
Develop appropriate vocabulary of metric units of measure.
Develop measuring techniques, using standard metric units of measure, including temperature and time.
Develop skills using simple measuring instruments.
Explore activities that involve the expression of measurements in decimal notation.

Standard Units of Measurement: Grades Four—Six

Explore concepts related to scale drawings and interpretation of maps.
Acquire an understanding of measuring to the nearest unit.
Develop the ability to choose the appropriate unit for measuring objects.
Explore conversion between units within the metric system.
Develop the ability to convert units within the metric system.
Develop correct vocabulary of measurement with metric units.
Develop skills in representing measurements in decimal notation.

Standard Units of Measurement: Grades Seven—Eight

Acquire correct vocabulary.
Develop informal comparisons between metric units and U.S. customary units.

Approximate Nature of Measurement

Approximate Nature of Measurement: Kindergarten—Grade Three

Explore the approximate nature of measurement.
Acquire skill in reading approximate time on a clock.

Approximate Nature of Measurement: Grades Four—Six

Explore the approximate nature of measurement.
Explore the relation between the size of the unit and the "error" in the measurement.
Develop an understanding of the approximate nature of measurement.
Develop the relationship between the approximate nature of measurement and rounding off.
Acquire skill in choosing the appropriate unit of measure.

Approximate Nature of Measurement: Grades Seven-Eight

Acquire an understanding of the approximate nature of measurement.
Acquire an understanding of "error" in measurement.
Maintain skill in choosing appropriate units of measure.

Estimation

Estimation: Readiness

Explore estimation of the size of objects compared to familiar objects within the range of a child's environment.

Estimation: Kindergarten—Grade Three

Develop the skill of estimating the size of objects compared to familiar objects within the range of a child's environment.
Develop the skill of estimating by guessing and measuring.
Develop the skill of choosing the correct unit of measure for estimates.
Develop techniques of estimation using standard metric units of measure.

Estimation: Grades Four—Six

Acquire skill of estimation with metric units.
Acquire skill of choosing the correct unit of measure for estimates.

Estimation: Grades Seven-Eight

Apply techniques for the refinement of an estimate.
Apply estimation skill in situations found inside and outside the classroom.

Calculators and Computers

Calculators and Computers: Readiness

Explore the skill necessary to estimate quantified data.
Explore the skill required to calculate addition, subtraction, multiplication, and division, using whole numbers.
Explore the use of motivation and enrichment activities.
Explore the use of technology in today's society.

Calculators and Computers: Kindergarten—Grade Three

Explore the skill necessary to estimate quantified data.
Develop the skills required to calculate addition, subtraction, multiplication, and division, using whole numbers.
Explore number relationships and order.
Explore number patterns through the use of calculators.
Explore the use of calculators for recreational activities.
Explore integrated curriculum opportunities.
Explore the mechanics of the calculator and its functions and logic.
Explore the use of computer simulations.
Develop the historical perspective of computer technology.
Explore the use of technology in today's society.

Calculators and Computers: Grades Four—Six

Acquire the skill necessary to estimate quantified data.
Acquire the skill required to calculate addition, subtraction, multiplication, and division, using decimal numbers.
Develop the ability to analyze number relationships and order.
Develop the skill of using calculators to change fractions to decimals to percents, and vice versa.
Develop the skill to analyze number patterns.

Develop the use of calculators for recreational activities.
Explore extended application of calculators to concepts that are normally obscured by tedious computation.
Develop integrated curriculum opportunities.
Develop a working knowledge of calculator functions, logic, and mechanics.
Develop the historical perspective of computer technology.
Develop computer simulations.
Explore programming skills.
Develop awareness of the use of technology in today's society.

Calculators and Computers: Grades Seven-Eight
Maintain the skill necessary to estimate quantified data.
Maintain the skill required to calculate addition, subtraction, multiplication, and division, using decimal numbers.
Acquire the ability to analyze number relationships and order.
Acquire the skill of using calculators to change fractions to decimals to percents, and vice versa.
Develop the skill to analyze number patterns.
Develop extended application of calculators to concepts that are normally obscured by tedious computation.
Apply integrated curriculum opportunities.
Acquire the historical perspective of computer technology.
Acquire skill using computer simulations.
Develop programming skills.
Acquire awareness of the use of technology in today's society.

Probability and Statistics

Counting Techniques

Counting Techniques: Readiness
Explore techniques of collecting data for establishing one-to-one correspondence.
Explore sorting and grouping of data.

Counting Techniques: Kindergarten—Grade Three
Develop techniques of collecting data for establishing one-to-one correspondence.
Develop sorting and grouping of data.
Explore the use of tally markers and other symbols to record data.
Explore activities involving the generation of organized lists of data.

Counting Techniques: Grades Four—Six
Explore techniques of data-counting, using manipulative materials.
Develop the ability to create organized lists.
Explore ways of finding the number of arrangements of objects.
Explore ways of finding the number of subsets of a set.

Counting Techniques: Grades Seven-Eight
Explore the concept of a sample space for a particular event.
Explore the permutations formula of n things taken r at a time.

Apply the fundamental counting procedure to determine the sample space of an event.

Collection, Organization, and Representation of Data

Collection, Organization, and Representation of Data: Readiness
Explore ways of grouping physical objects.
Explore simple inferences drawn from collected data.

Collection, Organization, and Representation of Data: Kindergarten—Grade Three
Explore ways students can generate data.
Explore construction and interpretation of simple bar graphs and line graphs.
Develop skill in construction and interpretation of circle graphs.
Explore ways of drawing inferences from simple graphs.

Collection, Organization, and Representation of Data: Grades Four—Six
Develop ways students can generate data.
Develop construction and interpretation of graphs.
Explore techniques for drawing inferences from collected data.
Explore techniques for developing tables for the organization of data.

Collection, Organization, and Representation of Data: Grades Seven-Eight
Acquire the ability to generate data.
Develop techniques of drawing inferences from data collected.
Explore the technique of random sampling.
Explore techniques for construction of frequency tables and histograms.

Interpretation of Data

Interpretation of Data: Readiness
Explore the interpretation of student-collected data.

Interpretation of Data: Kindergarten—Grade Three
Explore the range of a set of data.
Explore techniques for drawing inferences from a set of data.

Interpretation of Data: Grades Four—Six
Develop the ability to calculate the arithmetic mean (average) and range for a set of data.
Develop the ability to calculate the median and mode for a given set of data.
Develop techniques for drawing inferences from data.

Interpretation of Data: Grades Seven-Eight
Acquire an understanding of arithmetic mean and range for a set of data.
Acquire an understanding of the median and mode for a set of data.

Probability

Probability: Readiness
Explore techniques of guessing, hypothesizing, and making predictions, followed by experimentation and discussion.

Probability: Kindergarten—Grade Three
Develop experiences in guessing, hypothesizing, and making predictions.

Probability: Grades Four—Six

Explore the definition of probability.

Explore the definition of odds.

Develop the concept of a probabilistic event.

Probability: Grades Seven-Eight

Develop an understanding of the probability of an event that is certain to occur.

Develop an understanding of the probability of an event that is certain not to occur.

Relations and Functions

Patterns

Patterns: Readiness

Explore construction of pictorial representations of patterns.

Explore identification of relationships among, and properties of, objects.

Patterns: Kindergarten—Grade Three

Explore the recognition of patterns of symmetry and repetition in geometric objects or drawings.

Explore the recognition of patterns in simple numerical sequences.

Develop skill in recognizing relationships among, and properties of, objects.

Develop skill in identifying missing terms in numerical sequences.

Explore a vocabulary of comparison.

Patterns: Grades Four—Six

Explore recognition of, and use of, specific mathematical patterns.

Develop a vocabulary of comparison.

Explore generalizations of patterns of data.

Patterns: Grades Seven-Eight

Develop skill in identifying patterns in numerical sequences.

Develop skill in using variables in the representations of mathematical patterns.

Acquire a vocabulary of comparison.

Develop skills to generalize patterns of data.

Relations

Relations: Readiness

Explore the concept of a set of ordered pairs.

Explore the concept of pairing names with objects.

Explore examples of relations.

Relations: Kindergarten—Grade Three

Explore construction of sets of ordered pairs through pictorial representation.

Explore comparison of sets through matching.

Develop a simple mathematical language for sets of ordered pairs.

Develop skill in recognizing equivalent sets.

Develop skill in the rules for finding the second number or ordered pairs.

Develop skill in using the comparison relationships between sets.

Relations: Grades Four—Six
Develop graphing ordered pairs of numbers.
Develop skills in expressing a given relation in a mathematical sentence.
Explore the transitivity property of the ordering relation.

Relations: Grades Seven-Eight
Acquire skills in expressing a given relation in a mathematical sentence.
Develop skill in defining one-to-one correspondence between sets, both finite and infinite.
Develop the domain, range, and rule of a relation.
Develop the three properties of the equivalence relation.

Functions

Functions: Readiness
Explore simple sets of ordered pairs.
Explore pictorial representations of functions.

Functions: Kindergarten—Grade Three
Explore the interpretation of graphs of functions.
Explore addition and subtraction in *function machine* language.
Explore the many functional relations existing in nature.
Develop the properties of graphs of functions.

Functions: Grades Four—Six
Explore finding rules for given *function machines.*
Develop skill for recognizing functions.
Develop skill for defining functions.
Develop the four operations in *function machines.*

Functions: Grades Seven-Eight
Develop the domain and range and inverse property of functions.
Acquire the definition of relations and functions.
Develop the ability to determine whether or not formulas, statements, graphs, or tabulated data represent functions.

Graphs

Graphs: Readiness
Explore graphing related to physical objects.
Explore the use of simple charts for reference, comparison, and recordkeeping.
Develop the recognition of patterns through the pictorial representaton of relations.

Graphs: Kindergarten—Grade Three
Explore recognition and construction of various kinds of graphs.
Develop interpretation of graphs and tables.
Explore representation of number pairs in tabular and graphical forms.

Graphs: Grades Four—Six
Explore interpreting and graphing of data given as sets of ordered pairs.
Explore construction of the Cartesian product for any two sets of whole numbers.

Develop skill for generating sets of ordered pairs from tables, graphs, and formulas.
Develop skill for determining a rule for the graph of a relation.
Acquire skill for identifying, interpreting, and constructing points on the coordinate plane.
Acquire skill for recognizing graphs of functions.
Acquire skill for graphing linear functions, using whole numbers, rationals, and integers.
Develop skill for constructing and interpreting line and circle graphs.

Graphs: Grades Seven-Eight
Acquire skill in recognition, construction, interpretation, and demonstration of various kinds of graphs.
Develop skill for construction and use of points on the coordinate plane.
Develop skill in construction of graphs of inequalities.
Acquire skill in plotting of linear and quadratic functions, step functions, and constant functions.

Logical Thinking

Logical Thinking: Readiness.
Explore methods of sorting and matching objects, using appropriate vocabulary.
Develop ability to make comparisons.
Develop concepts of more, fewer, and same number as, using sets of objects.

Logical Thinking: Kindergarten—Grade Three
Explore short chains of logical reasoning, using manipulatives.
Explore the vocabulary of logic in simple mathematical sentences.
Explore patterns for logical reasoning.
Develop the concepts of between, before, and after.
Explore the use of logical reasoning in situations with one or two conditions.
Develop a vocabulary of logic terms; e.g., *and, or, not, if...then.*

Logical Thinking: Grades Four—Six
Explore the logical meaning of *all, some, each,* and *every.*
Develop the logical meaning of *and, or, not, if . . . then.*
Explore the use of flowcharts to show steps in operations.

Logical Thinking: Grades Seven-Eight
Explore inductive arguments.
Explore puzzles and games to extend concepts of logical thinking.
Develop simple deductive arguments.
Develop precise statements in logical reasoning procedures.

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†Also available in Spanish, at the price indicated
*Developed for implementation of School Improvement.