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ABSTRACT

Three of the modules deal with applications of calculus to other mathematics. These are: 240-Pi is Irrational; 241-The Wallis Approximation of Pi; and 242-Buffon's Needle Experiment. The first of these units focuses on a proof of the irrational nature of pi, and provides exercises and answers. The second module of this group reviews the history of attempts to calculate pi, and leads to a discussion of sequences in the Wallis formula, as originally discovered by the English mathematician John Wallis in about 1650. The third unit covers a "fun and games" method of approximating pi using needle tosses. The module gives a brief review of aspects of probability prior to discussing details of the experiment. A fourth unit in this document set focuses on applications of calculus to physiology and psychology: 251-A Strange Result in Visual Perception. The module describes an experiment on the process by which the eye sees bright light, and reviews the physiological background of eyes. A model is shown to predict seemingly paradoxical results. Exercises are presented at the conclusion to encourage further thought. Answers to these problems are included. (MP)

UMAP MODULE 240

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

π is Irrational

by Brindell Horelick and Sinan Koont

$$\pi \notin Q$$

Applications of Calculus to Other Mathematics

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π IS IRRATIONAL

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Intermodular Description Sheet: *UMAP Unit 240*

Title: π IS IRRATIONAL

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References:

Eves, Howard. 1969. An Introduction to the History of Mathematics (Third Edition), Holt, Rinehart and Winston, New York.
Niven, I. "A Simple Proof of the Irrationality of π," Bulletin of the American Mathematical Society, Vol. 53, 1947, p. 509.

Prerequisite Skills:

1. Know the Fundamental Theorem of Calculus.
2. Be able to work with factorials.
3. Know the product rule of differentiation.
4. Be able to differentiate and antidifferentiate sin and cos.
5. Be able to compute higher derivatives.
6. Know the chain rule of differentiation.
7. Know the binomial theorem.

*8. Be able to perform integration by parts.
*9. Understand mathematical induction.

*For those without prerequisites 8 and 9, Exercise 3 can be used to replace Section 2.3.

Output Skills:

1. To be able to prove that π is irrational, and to discuss the fact briefly in the context of algebraic and transcendental numbers.

Other Related Units:

The Wallis Approximation of π *(U241)*
Buffon's Needle Experiment *(U242)*

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1. INTRODUCTION

Certainly you have met the number π frequently in your study of mathematics. You have learned various approximations to it. Perhaps the most familiar are 22/7 and 3.14.* Through the centuries there have been many other approximations of π. In 1967 a computer program calculated π to 500,000 decimal places. But no one has ever found an "exact" value of π, and no one ever will, for a very good reason. The title of this unit states that reason, and in this unit we shall prove the statement.

1.1 Rational and Irrational Numbers

Recall that a real number is said to be rational if and only if it can be expressed in the form p/q (i.e., as a ratio) where p and q are integers. A real number which is not rational is said to be irrational. On the face of it, it is not clear that there are any irrational numbers. In fact, the ancient Greeks believed that all numbers were rational. We can hardly blame them, since the Greeks thought geometrically, and geometric "common sense" seems to confirm their belief. After all, we can draw a straight line (in modern terms, think of the x-axis) and mark off upon it equally spaced points corresponding to the integers. Then, by a straightforward technique of Euclidean geometry, we can divide each unit interval on the line into q equal intervals, where q is any positive integer we wish. If we do this, for example, to the interval from 0 to 1 we get points whose distances from 0 are $\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}$. If we do this for all possible q and for all unit intervals, it is hard for us (and was

*Many people, apparently overawed by decimals, assume without calculation that 3.14 must be the better approximation, presumably because it is in decimal form. But notice that $22/7 \approx 3.1429$, which is closer to π (≈ 3.1416) than is 3.14.

hard for the Greeks) to believe that we do not get all the points on the line (in effect, all real numbers).

But we don't. It is very easy to construct a line segment whose length is $\sqrt{2}$; for example, the hypotenuse of an isosceles right triangle with legs of unit length. If we then place this segment on the line of the preceding paragraph with its left end at 0, its right end will be at $\sqrt{2}$. And $\sqrt{2}$ is irrational, as the Greeks discovered to their consternation,* and as you may have seen proved elsewhere. Therefore it is not one of the points marked off in the construction of the preceding paragraph.

1.2 Decimal Representation

Mathematicians now know that there are very many irrational numbers. In fact, as you may already know, a real number expressed in decimal form is rational if and only if it eventually becomes repeating. (A terminating decimal can be regarded as a decimal that repeats zeros.) Some examples are:

$$0.25000\ldots = \frac{25}{100} = \frac{1}{4}$$

$$0.5183000\ldots = \frac{5183}{10{,}000}$$

$$0.3333\ldots = \frac{1}{3}$$

$$0.142857142857\ldots = \frac{1}{7}$$

$$0.111\ldots = \frac{1}{9}$$

$$0.1333\ldots = 0.333\ldots - 0.2 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$$

This is not the same as saying that the decimal expansions of irrational numbers never have a pattern. There are some patterns which are not repeating. One example is 0.101100111000111100000....

*The Greek's consternation was based on much more than having their "common sense" jarred. Some of their mathematical theory and even of their philosophy was based upon the erroneous belief that the ratio of any two line segment lengths was rational.

Exercise 1

Each of the following decimals can be obtained from the repeating decimals listed above by simple algebra. Use this fact to express each of them in the form $\frac{p}{q}$, where p and q are integers.

(a) 0.003333...

(b) 0.766666...

(c) 0.892857142857142857...

(d) 0.253968253968...

---

2. THE PROOF

In 1767 Johann Heinrich Lambert (1728-1777), an Alsatian philosopher, scientist, and mathematician, proved that π is irrational (see Exercise 10). It follows that we can never write down a fraction or terminating decimal which equals π; we must always settle for an approximation.

Since then there have been many different proofs that π is irrational. The proof presented in this unit was originally discovered by Ivan Niven in 1947. It has the virtue that a first-year calculus student can read it, but it does contain, unfortunately, a considerable amount of computational detail. It will be much easier to follow if, before getting into that detail, we present an outline.

2.1 Outline

It is much easier to work computationally with the notion of rational number than with that of irrational number. After all, an irrational number is defined by what it is not (not rational) rather than what it is. So most proof that any specific number is irrational are indirect; they begin by assuming it is rational, and then arrive at a contradiction. Our proof is no exception.

If we assume π is rational, then so is π^2, and we can write $\pi^2 = p/q$ where p and q are integers. The heart of our proof will consist of an intensive study of the expression

$$(1) \qquad K = \pi p^n \int_0^1 F(x) \sin \pi x \, dx$$

where

$$(2) \qquad F(x) = \frac{x^n (1 - x)^n}{n!}$$

and where n is a fixed positive integer. Notice that K depends upon n (but not upon x).

First, we shall make an estimate of the size of the integrand and, using this estimate, shall show that *for very large integers* n, *K < 1.*

Second, we shall perform the indicated integration, obtaining an expression for K in terms of F(0), F(1), and higher derivatives of F evaluated at x = 0 and x = 1.

Third, we shall derive certain properties of F(x) and its derivatives and apply them to this expression to show that *for all positive integers* n, *K is a positive integer.*

The two italicized statements contradict each other. Since our only assumption along the way is that π is rational, this assumption must be false.

2.2 Part One (K < 1)

The function we must integrate in computing K is $\frac{x^n (1 - x)^n \sin \pi x}{n!}$. We are integrating from x = 0 to x = 1. In this interval the numerator is the product of three factors, each of which is between 0 and 1, and so it must also be between 0 and 1.

$$0 \leq x^n(1 - x)^n \sin \pi x \leq 1$$

$$0 \leq \frac{x^n(1 - x)^n \sin \pi x}{n!} \leq \frac{1}{n!}$$

$$\int_0^1 0 \, dx \leq \int_0^1 \frac{x^n(1 - x)^n \sin \pi x}{n!} \, dx \leq \int_0^1 \frac{1}{n!} \, dx$$

(3) $$0 < \int_0^1 \frac{x^n(1 - x)^n \sin \pi x}{n!} \, dx < \frac{1}{n!}.$$

We have introduced strict inequality signs in the last line, since the only way we could have equality would be for the integrand to be identically 0 or identically $\frac{1}{n!}$ for $0 \leq x \leq 1$. Clearly this is not so.

Multiplying by πp^n we get

$$0 < K < \frac{\pi p^n}{n!}.$$

This part of the proof will be complete if we can show that $\frac{\pi p^n}{n!} < 1$ for very large n. To show this we let

$$r = 2p.$$

There is some positive integer m for which

$$2^m > \frac{\pi p^r}{r!}. \text{ (See Exercise 2).}$$

Then

$$\frac{1}{2^m} < \frac{r!}{\pi p^r}$$

and we have

$$\frac{\pi p^{r+m}}{(r+m)!} = \frac{\pi \, p^r \, p^m}{r!(r+1)(r+2)\ldots(r+m)}$$

$$= \frac{\pi p^r}{r!} \cdot \frac{p}{r+1} \cdot \frac{p}{r+2} \cdot \ldots \frac{p}{r+m}$$

$$< \frac{\pi p^r}{r!} \cdot \underbrace{\frac{p}{r} \cdot \ldots \cdot \frac{p}{r}}_{m \text{ times}} = \frac{\pi p^r}{r!} \left(\tfrac{1}{2}\right)^m < 2^m \cdot \frac{1}{2^m} = 1.$$

Exercise 2

Assuming p and r are given positive integers, find positive integer m so that $2^m > \frac{\pi p^r}{r!}$. Try your answer out for specific values of p and r.

2.3 Part Two (Integration)

The second part of our proof consists of performing the integration indicated in (1). The most direct way to go about this requires integration by parts and mathematical induction. If you have never studied these topics you will find an alternate approach in Exercise 3 at the end of this section. Although it is a little awkward it is perfectly correct.

Let f(x) be any differentiable function, and, for any positive integer k, let $f^{(k)}(x)$ denote the k^{th} derivative of f(x). We shall use mathematical induction on n to prove that, for all non-negative integers n,

$$(4) \qquad \int_0^1 f(x)\sin \pi x\, dx = \frac{f(1)+f(0)}{\pi} - \frac{f''(1)+f''(0)}{\pi^3} + \cdots$$

$$\pm \frac{f^{(2n)}(1)+f^{(2n)}(0)}{\pi^{2n+1}} \mp \frac{1}{\pi^{2n+2}}\int_0^1 f^{(2n+2)}(x)\sin \pi x\, dx$$

where the upper signs apply if n is even, and the lower signs if n is odd. The proof, like all proofs involving mathematical induction, requires two steps: (A) Proving (4) for an initial value of n (in this case, n=0); (B) Proving that if (4) holds for n=k then it holds for n = k+1.

(A) The first step involves two integrations by parts. Replacing n by 0 in (4), we see that we must prove

(5) $\int_0^1 f(x)\sin \pi x\,dx = \frac{f(1)+f(0)}{\pi} - \frac{1}{\pi^2}\int_0^1 f''(x)\sin \pi x\,dx$

The formula for integration by parts tell us

$$\int_0^1 u\,dv = uv\Big|_0^1 - \int_0^1 v\,du.$$

We can apply this to the left side of (5), using

$u = f(x)$ $\qquad dv = \sin \pi x\,dx$

$du = f'(x)dx$ $\qquad v = -\frac{1}{\pi}\cos \pi x\,dx.$

We get

(6) $\int_0^1 f(x)\sin \pi x = -\frac{f(x)}{\pi}\cos \pi x\Big|_0^1 + \frac{1}{\pi}\int_0^1 f'(x)\cos \pi x\,dx$

$$= \frac{f(1)+f(0)}{\pi} + \frac{1}{\pi}\int_0^1 f'(x)\cos \pi x\,dx.$$

Applying integration by parts to the integral on the right, with

$u = f'(x)$ $\qquad dv = \cos \pi x\,dx$

$du = f''(x)dx$ $\qquad v = \frac{1}{\pi}\sin \pi x$

we get

$$\int_0^1 f'(x)\cos \pi x\,dx = \frac{f'(x)}{\pi}\sin \pi x\Big|_0^1 - \frac{1}{\pi}\int_0^1 f''(x)\sin \pi x\,dx$$

$$= -\frac{1}{\pi}\int_0^1 f''(x)\sin \pi x\,dx.$$

Substituting this result into (6) gives us (5) and completes the first step in our mathematical induction.

(B) Now we assume (4) holds for n=k. In other words, we assume

(7) $$\int_0^1 f(x)\sin \pi x\,dx = \frac{f(1)+f(0)}{\pi} - \frac{f''(1)+f''(0)}{\pi^3} + \cdots \pm \frac{f^{(2k)}(1)+f^{(2k)}(0)}{\pi^{2k+1}} \mp \frac{1}{\pi^{2k+2}}\int_0^1 f^{(2k+2)}(x)\sin \pi x\,dx.$$

Equation (5), which we have just proved, applies to any differentiable function f. If we apply it to $f^{(2k+2)}$ (replacing f(x) by $f^{(2k+2)}(x)$ throughout), we get

$$\int_0^1 f^{(2k+2)}(x)\sin \pi x\,dx = \frac{f^{(2k+2)}(1)+f^{(2k+2)}(0)}{\pi} - \frac{1}{\pi^2}\int_0^1 f^{(2k+4)}(x)\sin \pi x\,dx.$$

Multiplying this by $\mp \frac{1}{\pi^{2k+2}}$ gives us

$$\mp \frac{1}{\pi^{2k+2}}\int_0^1 f^{(2k+2)}(x)\sin \pi x\,dx$$

$$= \mp \frac{f^{(2k+2)}(1)+f^{(2k+2)}(0)}{\pi^{2k+3}} \pm \frac{1}{\pi^{2k+4}}\int_0^1 f^{(2k+4)}(x)\sin \pi x\,dx.$$

Making this substitution on the right side of (7) yields

(8) $$\int_0^1 f(x)\sin \pi x\,dx = \frac{f(1)+f(0)}{\pi} - \frac{f''(1)+f''(0)}{\pi^3} + \cdots \pm \frac{f^{(2k)}(1)+f^{(2k)}(0)}{\pi^{2k+1}} \mp \frac{f^{(2k+2)}(1)+f^{(2k+2)}(0)}{\pi^{2k+3}} \pm \frac{1}{\pi^{2k+4}}\int_0^1 f^{(2k+4)}(x)\sin \pi x\,dx$$

which is precisely (4) with n = k+1. We have completed the proof of (4).

Now let us apply this result to (1), setting f(x) = F(x). Since F(x) is a polynomial of degree 2n,

$F^{(2n+2)}(x)$ is identically zero, and the integral on the right side of (8) drops out. Therefore

$$(9)\quad \int_0^1 F(x)\sin \pi x\, dx = \frac{F(1)+F(0)}{\pi} - \frac{F''(1)+F''(0)}{\pi^3} + \cdots \mp \frac{F^{(2n)}(1)+F^{(2n)}(0)}{\pi^{2n+1}}.$$

Multiplying by π and using the fact that $\pi^{2k} = \frac{p^k}{q^k}$ for any k,

$$(10)\quad \pi \int_0^1 F(x)\sin \pi x\, dx = F(1)+F(0) - \frac{F''(1)+F''(0)}{p} q + \frac{F^{iv}(1)+F^{iv}(0)}{p^2} q^2 - \cdots \mp \frac{F^{(2n)}(1)+F^{(2n)}(0)}{p^n} q^n .$$

Finally, multiplying (10) by p^n,

$$(11)\quad K = p^n(F(1)+F(0)) - p^{n-1}q(F''(1)+F''(0)) + \cdots \mp q^n(F^{(2n)}(1)+F^{(2n)}(0)).$$

This is the expression for K promised in phase two of our outline.

Exercise 3

Here is a way to obtain (9) without using integration by parts or mathematical induction.

Consider the function

$$g(x) = -\frac{1}{\pi}F(x)\cos \pi x + \frac{1}{\pi^2}F'(x)\sin \pi x + \frac{1}{\pi^3}F''(x)\cos \pi x - \frac{1}{\pi^4}F'''(x)\sin \pi x - \frac{1}{\pi^5}F^{iv}(x)\cos \pi x + \frac{1}{\pi^6}F^{v}(x)\sin \pi x + \cdots$$

$$\mp \frac{1}{\pi^{2n+1}} F^{(2n)}(x) \cos \pi x \pm \frac{1}{\pi^{2n+2}} F^{(2n+1)}(x) \sin \pi x$$

where, for example, $F^{(2n)}(x)$ denotes the $(2n)^{th}$ derivative of $F(x)$. (The signs change after every odd-numbered term: -++--++-... The last sign will be + if n is even and - if n is odd.)

(a) Show that $g'(x) = F(x) \sin \pi x$.

(b) Use (a) to deduce Equation (9).

Exercise 4

Compute the following integrals.

(a) $\int_0^1 x^2 \sin \pi x \, dx$

(b) $\int_0^1 x^9 \sin \pi x \, dx$ [Do not multiply all the numbers in your answer.]

Exercise 5

Compute the following integrals. In each case you will have to make a change of variable (substitution).

(a) $\int_0^\pi y^9 \sin y \, dy$ [Do not multiply all the numbers in your answer.]

(b) $\int_0^1 y \sin (\pi y^2) \, dy$

(c) $\int_0^1 y^5 \sin (\pi y^2) \, dy$.

2.4 Part Three (K is an Integer)

What remains to be done is to show that the right side of (11) equals a positive integer. It is made up of sums and products of p, q, and terms of the form $F^{(k)}(0)$ and $F^{(k)}(1)$ for various integers $k \geq 0$. Therefore it will be more than enough to show that F(x) and all its derivatives take on integer values at x = 0 and x = 1. This will show that K is an integer. That K > 0 follows immediately from (1) and (3).

Let us first consider $x = 0$. We have already remarked that F(x) is really a polynomial of degree

2n. So for any $k \leq 2n$ its k^{th} derivative is a polynomial of degree $2n - k$ (for $k > 2n$ the k^{th} derivative is of course identically zero). Further, the value of any polynomial at $x = 0$ equals its constant term. And, since differentiation reduces the exponent of each non-constant term by one, the constant term of $F^{(k)}(x)$ is determined solely by the "x^k" term of $F(x)$.

With these background facts in mind we can go to work. To begin with we can expand $(1 - x)^n$:

$$(1 - x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$$

where $C_0, \ldots, C_n$ are integers. (It is easy to compute them explicitly [see Exercise 6], but it will not be necessary for this proof.) Then

$$x^n(1-x^n) = C_0x^n + C_1x^{n+1} + C_2x^{n+2} + \ldots + C_nx^{2n}$$

$$F(x) = \frac{C_0}{n!}x^n + \frac{C_1}{n!}x^{n+1} + \frac{C_2}{n!}x^{n+2} + \ldots + \frac{C_n}{n!}x^{2n}.$$

If $0 \leq k < n$ there is no "x^k" term in $F(x)$, and therefore $F^{(k)}(0) = 0$, which is certainly an integer. If $n \leq k \leq 2n$ then the "x^k" term is $\frac{C_{k-n}}{n!}x^k$. Its successive derivatives are

$$\frac{kC_{k-n}}{n!}x^{k-1}, \quad \frac{k(k-1)C_{k-n}}{n!}x^{k-2}, \quad \frac{k(k-1)(k-2)C_{k-n}}{n!}x^{k-3},$$

etc. Each differentiation introduces another factor in front of the C_{k-n}. By the time we differentiate k times there are k such factors, and so we have

$$F^{(k)}(0) = \frac{k(k-1)(k-2)\ldots 2\cdot 1\ C_{k-n}}{n!} = \frac{k!}{n!}C_{k-n}$$

Since $k \geq n$, $\frac{k!}{n!}$ is an integer, and so is $F^{(k)}(0)$.

To show $F^{(k)}(1)$ is an integer for each k, first observe that

(12) $F(x) = F(1 - x)$

for all x. We can differentiate both sides of (12) with respect to x, applying the chain rule

$$[f(g(x))]' = f'(g(x))g'(x)$$

to the right side with g(x) = 1-x and f ≡ F. We get

$$F'(x) = -F'(1 - x).$$

Differentiating again (and again using the chain rule),

$$F''(x) = F''(1 - x).$$

Doing this k times we see

$$F^{(k)}(x) = \pm F^{(k)}(1 - x)$$

with the sign depending on whether k is even or odd. Putting x = 1 we get

$$F^{(k)}(1) = \pm F^{(k)}(0)$$

and since we already know $F^{(k)}(0)$ is an integer we are done.

Exercise 6

Compute the coefficients $C_1, \ldots, C_n$ if $(1-x)^n = 1 + C_1x + C_2x^2 + \ldots C_nx^n$.

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3. AN EXTENSION OF THE RESULT

As mentioned in Section 2, the result that π is irrational is in a sense a negative result. It tells us what kind of number π is *not*. Another way of putting this is that, if we start with the integers, no amount of addition, subtraction, multiplication, or division can produce π exactly. But there are other operations we can perform on integers; for example, we can take square roots, cube roots, etc. Is it possible that one of these will produce π exactly?

The answer is "No." This, and much more, follows from a result first proved by F. Lindemann (1852-1939) in 1882. In these concluding sections we shall discuss Lindemann's results briefly.

3.1 Algebraic and Transcendental Numbers

First we need a couple of definitions. We call a number algebraic if it is a root of some polynomial equation with rational coefficients. Otherwise we call it transcendental. For example, $\frac{3}{5}$ is algebraic since it is a root of $5x - 3 = 0$, and $\sqrt{2}$ is algebraic since it is a root of $x^2 - 2 = 0$. In fact, every rational number is algebraic (see Exercise 7), and every number of the form $\sqrt[m]{n}$ is algebraic, where m and n are positive integers (see Exercise 8). Further, there are lots of polynomial equations which we have no idea how to solve. Nonetheless, we can assert that their roots, whatever they may be, are algebraic. For example, the equation

$$\frac{15}{43}x^5 - 2x^4 + \frac{73}{74}x^3 + \frac{10}{17}x^2 - \frac{92}{13}x + \frac{41}{3} = 0$$

has at least one real root, and that root is algebraic.

Exercise 7

Prove that every rational number is algebraic.

Exercise 8

If m and n are positive integers, prove that $\sqrt[m]{n}$ is algebraic.

Exercise 9

Some writers replace the word "rational" with "integer" in the definition of algebraic. Prove that the two forms of the definition are equivalent. That is, prove both of the following:

(a) if x is the root of a polynomial equation with rational coefficients, then it is the root of a polynomial equation with integer coefficients;

(b) if x is the root of a polynomial equation with integer coefficients, then it is the root of a polynomial equation with rational coefficients.

3.2 Lindemann's Result

Lindemann proved in 1882 that π is trancendental. So Exercise 8 immediately tells us that, as already asserted, we cannot produce π exactly by taking roots of integers. But it also tells us what kind of number π is *not* (i.e., π is not algebraic) and extends the list of procedures (i.e., solving polynomial equations) which can *not* give us π exactly. Incidentally, the ancient Greeks could not have conceived of this result because of the strictly geometrical way in which they worked.

Exercise 10

Lambert's original proof that π is irrational is based on the following, which he proved: if x is any nonzero rational number, then tan x is irrational.

Assuming this is true, explain how it follows that π is irrational.

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4. HINTS FOR SELECTED EXERCISES

3. (b) Use the Fundamental Theorem of Calculus.

4. Use (4) with $f(x)$ = (a) x^2 (b) x^9. All derivatives of f after the (a) second (b) ninth are identically zero.

5. (a) $x = y/\pi$. (b) $x = y^2$. (c) $x = y^2$. Then in each case use (4).

6. Use the binomial theorem.

10. Consider $x = \pi$.

5. ANSWERS TO EXERCISES

1. (a) $\frac{1}{100} \times \frac{1}{3} = \frac{1}{300}$ (b) $\frac{1}{10} \times \frac{2}{3} = \frac{23}{30}$

   (c) $\frac{3}{4} + \frac{1}{7} = \frac{25}{28}$ (d) $\frac{1}{9} + \frac{1}{7} = \frac{16}{63}$

2. Any integer greater than $\dfrac{\ln \frac{\pi p^r}{r!}}{\ln 2}$.

4. (a) $\frac{1}{\pi} - \frac{4}{\pi^3}$ (b) $\frac{1}{\pi} - \frac{9\cdot 8}{\pi^3} + \frac{9\cdot 8\cdot 7\cdot 6}{\pi^5} - \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{\pi^7} + \frac{9!}{\pi^9}$

5. (a) $\pi^{10}\left(\frac{1}{\pi} - \frac{9\cdot 8}{\pi^3} + \frac{9\cdot 8\cdot 7\cdot 6}{\pi^5} - \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{\pi^7} + \frac{9!}{\pi^9}\right)$

   (b) $\frac{1}{\pi}$

   (c) $\frac{1}{2\pi} - \frac{2}{\pi^3}$

6. $c_i = (-1)^i \binom{n}{i} = \frac{(-1)^i n!}{i!(n-i)!}$

7. $\frac{p}{q}$ is a solution of $x - \frac{p}{q} = 0$.

8. $\sqrt[m]{n}$ is a solution of $x^m - n = 0$.

9. (a) Given a polynomial equation with rational coefficients, multiply it by the least common multiple of the denominators. The new equation has integer coefficients and the same roots.

   (b) Any polynomial with integer coefficients is already a polynomial with rational coefficients.

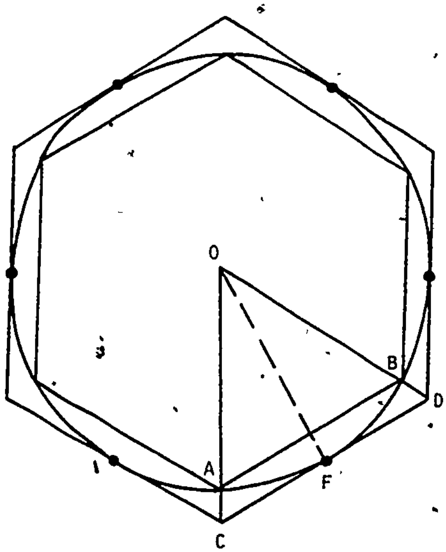
10. Take $x = \pi$. If π were rational, then tan x = 0 would be irrational.

UMAP MODULE 241

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

The Wallis Approximation of π

by Brindell Horelick and Sinan Koont



Applications of Calculus to Other Mathematics

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THE WALLIS APPROXIMATION OF π

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Intermodular Description Sheet: *UMAP Unit 241*

Title: THE WALLIS APPROXIMATION OF π

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References:

Beckman, P. (1971). A History of π. (Second Edition). The Golem Press, Boulder, Colorado.
Eves, H. (1969). An Introduction to the History of Mathematics. (Third Edition). Holt, Rinehart and Winston, New York.
Courant, R. and H. Robbins. (1941). What is Mathematics? Oxford University Press, London.
Kitchen, J.W., Jr. (1968). Calculus of One Variable. Addison-Wesley, Reading, Massachusetts.
Thomas, G.B., Jr. (1960). Calculus and Analytic Geometry. (Third Edition). Addison-Wesley, Reading, Massachusetts.

Prerequisite Skills:

1. Know the definition of rational number.
2. Possess at least an informal understanding of the notion of limit of a sequence.
3. Be able to integrate powers and products of sin and cos.
4. Be able to integrate by parts.
5. Understand and be able to manipulate factorial notation.

Output Skills:

1. Be able to sketch some history of the approximation of π.
2. Be able to state and derive the Wallis formula for approximating π.

Other Related Units:

π is Irrational *(Unit 240)*
Buffon's Needle Experiment *(Unit 242)*

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1. HISTORICAL BACKGROUND

1.1 Introduction

The number π is defined as the ratio of the circumference to the diameter of a circle. It is of central importance in innumerable mathematical and scientific results, including many which do not appear to have anything to do with circles.

For thousands of years mathematicians and others have been fascinated by the challenge of determining the value of π as precisely as possible, and now computers have gotten into the act. In 1967 a computer in Paris determined π to 500,000 decimal places.

It is undeniably true that computers, because of the phenomenal speed with which they can do arithmetic, have enabled us to obtain many more digits of π than ever before. In 1948 (just before computers) the record was 808 places. But all the machines can do is use formulas supplied to them by us slowpoke humans, who are much more intelligent -- no matter what you may have heard to the contrary. It has been said of the computer that "the most intelligent thing it is capable of doing without the help of its programmers is to go on strike when required to work without air conditioning."*

Well, then, where do these formulas come from? How do "mere" humans find the value of π, even without the aid of computers? After all, 808 places isn't peanuts. Certainly no one could measure the circumference of any circle, however large, with that accuracy.

---

* Beckmann, page 102.

1.2 Early Circle Measurers

About 4000 years ago the Babylonians, who knew nothing about π except its definition, probably did actually measure circles. They may have, for example, drawn a large circle on the ground and marked off its diameter on a rope. They could have then observed that laying this rope segment along the circle three times almost, but not quite, took them around the circle. Perhaps they then measured the excess with another rope and determined that this shorter segment went into the original segment about eight times (we now know seven would have been more accurate). At any rate, they came up with the estimate $\pi \approx 3\frac{1}{8}$.

1.3 Archimedes and other Polygon Measurers

In about 240 BC the Greek mathematician Archimedes (287-212 BC) became involved in our tale. Archimedes is universally regarded as one of the two or three greatest men in the history of Western mathematics. He was apparently the first person who attempted to estimate π in any way other than literally measuring the circumference of a circle. Instead he said, in effect: "Let me start with a circle of diameter one. Its circumference will of course be π. Instead of trying to measure the circumference, I'll inscribe a polygon and circumscribe another polygon. Clearly the circumference π is between the two perimeters. If I choose the polygons wisely, I should be able to use the geometry I know to compute these perimeters."

For an illustration of Archimedes' thoughts look at Figure 1, which shows a circle of diameter one with inscribed and circumscribed regular hexagons. Each of the hexagons consists of six equilateral triangles, so the calculations are straightforward.

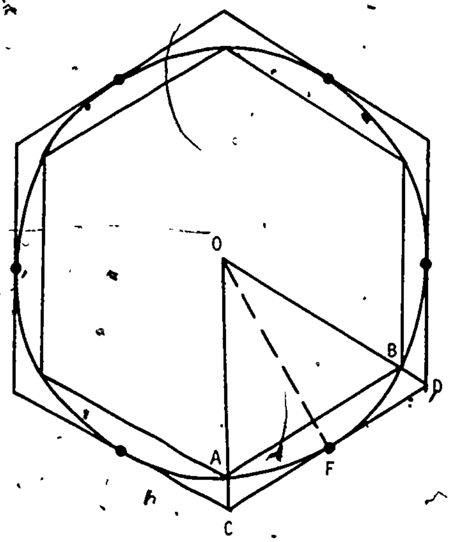

Figure 1. Archimedes' calculation of π.
The circle has diameter one.

For the inscribed hexagon $\overline{AB} = \overline{OA} = \frac{1}{2}$, so the perimeter $p = 6 \times \overline{AB} = 6 \times \frac{1}{2} = 3$. For the circumscribed hexagon we can observe that $\overline{OF} = \frac{1}{2}$, so

$$\overline{OF}^2 = \frac{1}{4} .$$

But also,

$$\overline{OF}^2 = \overline{OC}^2 - \overline{CF}^2 = \overline{OC}^2 - (\tfrac{1}{2}\,\overline{CD})^2$$
$$= \overline{OC}^2 - (\tfrac{1}{2}\,\overline{OC})^2 = \tfrac{3}{4}\,\overline{OC}^2 .$$

Equating these two expressions for $\overline{OF}^2$ we get

$$\frac{3}{4}\,\overline{OC}^2 = \frac{1}{4},$$

so

$$\overline{OC}^2 = \frac{1}{3}$$

and

$$OC = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} .$$

Then the perimeter

$$P = 6 \times \overline{CD} = 6 \times \overline{OC} = 2\sqrt{3} .$$

We could have used trigonometric functions, since we know the value of sin 60°, but they were introduced long after Archimedes, and we wanted to show that he did not really need them.

We've shown

$$3 < \pi < 2\sqrt{3}$$

or

$$3.00 < \pi < 3.47 .$$

This result in itself would not be worth the trouble, but Archimedes used some rather clever geometry to compute the perimeters of inscribed and circumscribed regular polygons with 96 sides. He got

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

or, in modern decimal notation,

$$3.140 < \pi < 3.143.$$

As a modern mathematician would put it, he determined π to two decimal places.

Although Archimedes broke away from direct measurement of circumferences, his method of estimating π was still driectly related to the fact that π is the circumference of a circle with diameter one. Until the 17th century all attempts to compute π were based upon this fact, or upon the closely related fact that such a circle has area $\pi/4$. As of 1630 the record for digits of π was apparently 35, set by an otherwise obscure Dutch mathematician named Ludolph von Ceulen (1539 - 1610) who is supposed to have based his work on polygons having 2^{62} ($\approx 4.6 \times 10^{18}$) sides and to have devoted most of his life to this task.

1.4 Analytic Attempts

In the 17^{th} century the search for digits of π began to take a fundamentally different tack for the first time since Archimedes. Mathematicians began to turn away from circles, polygons, and other geometric considerations. The mathematical concepts which were to become "Calculus" under Isaac Newton (1642 - 1727) and Gottfried Wilhelm Leibniz (1646 -1716) were in their infancy then, and mathematicians were just beginning to understand the notion of "sequence of rational numbers." From the 17^{th} century on, all attempts to approximate π, with or without a computer, have amounted to finding sequences of rational numbers whose limit is π. It was this approach which made it possible to extend the record from 35 digits in 1630 to 527 digits in 1874 (before desk calculators) and 808 digits in 1948 (before computers).

In this unit we shall derive one of the earliest of these sequences. It was originally discovered by the English mathematician John Wallis (1616 - 1703) in about 1650.

2. THE WALLIS FORMULA

2.1 Outline; Definition of I_n

Our derivation of the Wallis formula is based upon a study of the definite integral

$$\int_0^{\pi/2} \sin^n x \, dx ,$$

where n can be any positive integer, or zero. Since we will be talking about this integral quite a lot, let's give it a name:

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

We'll proceed in five steps.

1) We'll use integration by parts to obtain a formula expressing I_n in terms of I_{n-2} for $n \geq 2$.

2) We'll use this formula to compute I_n.

3) We'll then compute I_{n+1}/I_n. It will turn out to be $\frac{2}{\pi}$ times a certain rational number r_n which depends on n.

4) We'll go back to the definition of I_n and show directly from it, without using (1), (2) or (3), that

$$\lim_{n\to\infty} \frac{I_{n+1}}{I_n} = 1.$$

5) Combining (3) and (4), we'll observe that $\frac{2}{\pi} \times r_n \to 1$ and therefore $2r_n \to \pi$. The sequence $\{2r_n\} = \{2r_1, 2r_2, 2r_3, \ldots\}$ is the Wallis sequence.

2.2 Reduction Formula for I_n

We'll begin by attempting to compute I_n using integration by parts. Using the formula.

$$\int u \, dv = uv - \int v \, du$$

with

$$u = \sin^{n-1} x \qquad dv = \sin x \, dx$$

$$du = (n-1)\sin^{n-2} x \cos x \, dx \qquad v = -\cos x$$

we get

$$\int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx.$$

Replacing $\cos^2 x$ by $1 - \sin^2 x$ in the last term:

(1) $\int \sin^n x\,dx$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,dx$$

$$- (n-1)\int \sin^n x\,dx.$$

Now evaluating from $x = 0$ to $x = \frac{\pi}{2}$:

$$I_n = -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n = (n-1)\ I_{n-2} - (n-1)\ I_n$$

$$n I_n = (n-1)\ I_{n-2}$$

(2) $$I_n = \frac{n-1}{n} I_{n-2}.$$

Exercise 1. From (1) obtain an iteration formula for $\int \sin^n x\,dx$, and use it to determine:
(a) $\int \sin^5 x\,dx$; (b) $\int \sin^4 x\,dx$.

2.3 Computation of I_n

If you have never seen a derivation like the one leading to Equation (2) you may think we have failed in our attempt to compute I_n using integration by parts. After all, we have "merely" expressed one unknown integral in terms of another. But I_{n-2} involves a lower power of $\sin x$ than does I_n. Formula (2) is called a *reduction* formula. If we start with the direct computation

$$I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} 1\ dx = x \Big|_0^{\pi/2} = \frac{\pi}{2}$$

we can use (2) again and again to work our way up to I_n for any even n:

$$n = 2: \quad I_2 = \frac{1}{2} \cdot I_0 = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$n = 4: \quad I_4 = \frac{3}{4} \cdot I_2 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$n = 6: \quad I_6 = \frac{5}{6} \cdot I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

and in general

(3) $$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

for any even n. (This is called an *iterative* process, from the verb iterate to repeat).

If we start with

$$I_1 = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 0 - (-1) = 1$$

we obtain:

$$n = 3: \quad I_3 = \frac{2}{3} \cdot I_1 = \frac{2}{3} \cdot 1$$

$$n = 5: \quad I_5 = \frac{4}{5} \cdot I_3 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$n = 7: \quad I_7 = \frac{6}{7} \cdot I_5 = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

and in general

(4) $$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

for any odd n.

2.4 Computation of I_{n+1}/I_n

Now we are ready to consider the ratio I_{n+1}/I_n. The formula for this ratio depends on whether n is even or odd. Let us suppose n is even. (It will turn out that this is the only case we'll have to consider.) Then (3) gives us the demoninator I_n. Since n is even, n+1 is odd, and we can obtain the numerator I_{n+1} from (4) if we replace n by n+1 in that formula:

(5) $$I_{n+1} = \frac{n}{n+1} \cdot \frac{n-2}{n-1} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

for any *even* n.

Dividing (5) by (3):

$$\frac{I_{n+1}}{I_n} = \frac{\frac{n}{n+1} \cdot \frac{n-2}{n-1} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1}{\frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}} .$$

Taking the factors alternately from the numerator and denominator, and remembering to invert those from the denominator,

$$\frac{I_{n+1}}{I_n} = \frac{n}{n+1} \cdot \frac{n}{n-1} \cdot \frac{n-2}{n-1} \cdot \frac{n-2}{n-3} \cdots \frac{6}{7} \cdot \frac{6}{5} \cdot \frac{4}{5} \cdot \frac{4}{3} \cdot \frac{2}{3} \cdot \frac{2}{1} \cdot 1 \cdot \frac{2}{\pi}$$

for any even n. It is conventional to write these factors in the opposite order:

(6) $$\frac{I_{n+1}}{I_n} = \frac{2}{\pi} \cdot 1 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{n}{n-1} \cdot \frac{n}{n+1} .$$

2.5 The Limit of I_{n+1}/I_n

Let's calculate the right side of (6) for the first few even values of n.

$n = 2$: $\frac{I_3}{I_2} = \frac{2}{\pi} \cdot 1 \cdot \frac{2}{1} \cdot \frac{2}{3} = \frac{8}{3\pi} \approx 0.8488$

$n = 4$: $\frac{I_5}{I_4} = \frac{2}{\pi} \cdot 1 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} = \frac{128}{45\pi} \approx 0.9054$

$n = 6$: $\frac{I_7}{I_6} = \frac{2}{\pi} \cdot 1 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} = \frac{4608}{1575\pi} \approx 0.9313$

$n = 8$: $\frac{I_9}{I_8} = \cdots \approx 0.9461$

$n = 10$: $\frac{I_{11}}{I_{10}} = \cdots \approx 0.9556$

$n = 12$: $\frac{I_{13}}{I_{12}} = \cdots \approx 0.9623$

It looks like this sequence may be approaching one. Can we prove this?

We can prove even more. As stated in Section 2.1, we can go right back to the definition of I_n and show that

$$\lim_{n\to\infty} \frac{I_{n+1}}{I_n} = 1.$$

In other words, the sequence I_2/I_1, I_3/I_2, I_4/I_3, ... approaches one. Since our sequence I_3/I_2, I_5/I_4, I_7/I_6, ... consists of every other term in this sequence, it must also approach one.

Remember that all of the I_n are integrals from $x = 0$ to $x = \frac{\pi}{2}$. For $0 \leq x \leq \frac{\pi}{2}$ we know that $0 \leq \sin x \leq 1$. Since $0^2 = 0$, $1^2 = 1$, and $y^2 < y$ for all y between 0 and 1 we have

(7) $$0 \leq \sin^2 x \leq \sin x \leq 1.$$

Since $\sin x \geq 0$, $\sin^n x \geq 0$ for all integers n. We can multiply (7) by $\sin^n x$, getting

$$0 \leq \sin^{n+2} x \leq \sin^{n+1} x \leq \sin^n x$$

and therefore

$$0 \leq I_{n+2} \leq I_{n+1} \leq I_n.$$

(If $0 \leq f(x) \leq g(x)$ throughout an interval [a, b], then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

since these integrals represent areas, one of which is contained in the other. See Figure 2.)

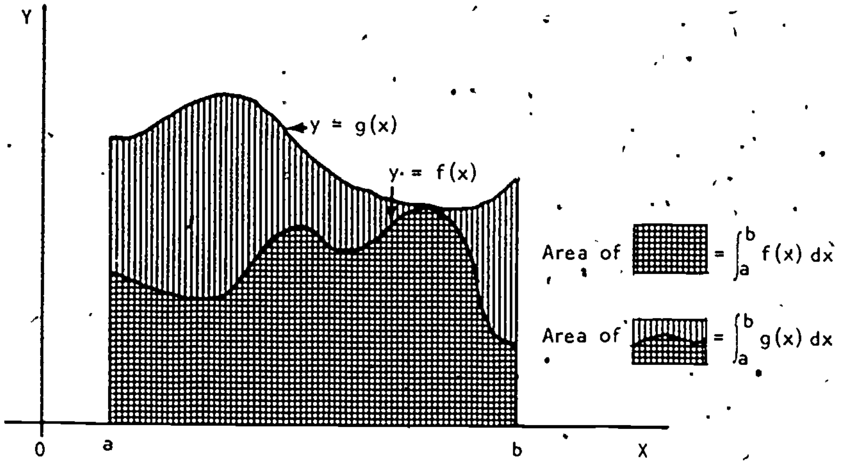

Figure 2. Graphical proof that $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$ when $0 \le f(x) \le g(x)$.

Since $I_n > 0$ we can divide by it:

$$0 \le \frac{I_{n+2}}{I_n} \le \frac{I_{n+1}}{I_n} \le 1$$

or, using (2), with n replaced by n+2,

$$\frac{n+1}{n+2} \le \frac{I_{n+1}}{I_n} \le 1.$$

Now suppose $n \to \infty$. Then certainly $\frac{n+1}{n+2} \to 1$. The middle expression is trapped between two expressions which are near one when n is large (one of them actually equals one), and so it too is near one when n is large. That is, it approaches one as n approaches infinity:

$$\lim_{n\to\infty} \frac{I_{n+1}}{I_n} = 1.$$

(This is sometimes called the *squeeze principle*. See Figure 3.)

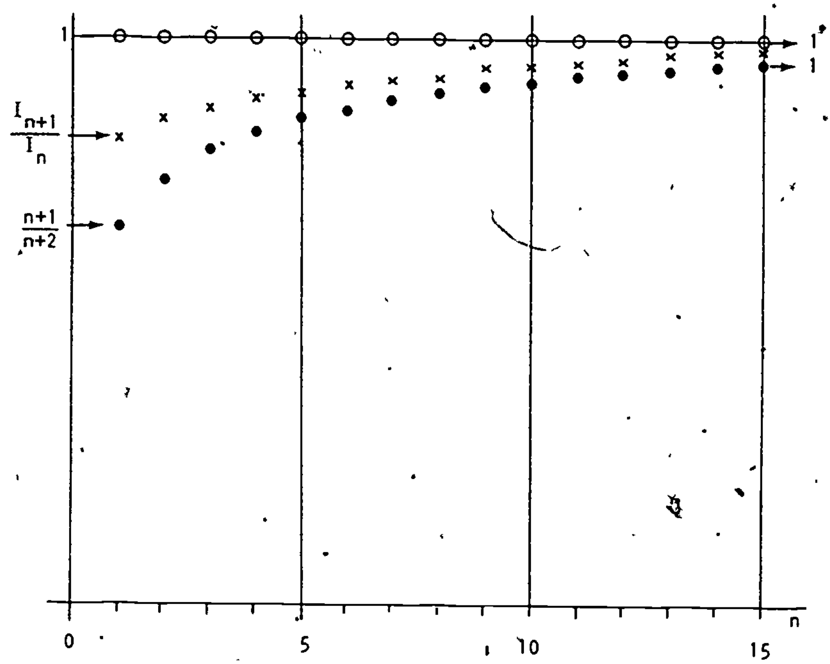

Figure 3. The squeeze principle for $\{\frac{n+1}{n+2}\}$, $\{I_{n+1}/I_n\}$, and $\{1\} = \{1,1,1,\ldots\}$.

2.6 The Wallis Formula for π

We have now proved what we guessed might be true earlier in Section 2.5; that the sequence whose first six terms we calculated there does indeed approach one. Therefore its terms must eventually be near one. A bit more precisely,

$$(8) \qquad \frac{2}{\pi} \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{n}{n-1} \cdot \frac{n}{n+1} \approx 1,$$

and the left side can be made as near to one as we like by taking n large enough (and even).

Multiplying (8) by π we get

$$2\left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{n}{n-1} \cdot \frac{n}{n+1}\right) \approx \pi$$

for large even n. This is the Wallis formula.

3. EXERCISES

2. (a) Show that $2\cdot4\cdot6\cdots(2n) = 2^n n!$ for any positive integer n, where n! denotes $1\cdot2\cdot3\cdots n$.

   (b) Show that

$$\frac{2\cdot2\cdot4\cdot4\cdots(2n)\cdot(2n)}{1\cdot3\cdot3\cdot5\cdots(2n-1)\cdot(2n+1)} = \frac{(n!)^4\ 2^{4n}}{[(2n)!]^2(2n+1)}$$

   for any positive integer n.

   (c) Show that

$$\lim_{n\to\infty} \frac{(n!)^2\ 2^{2n}}{(2n)!\ \sqrt{n}} = \sqrt{\pi}.$$

3. This problem presents a derivation of another formula for π. The formula is credited to Leibniz.

   (a) Show that

$$1 - x^2 + x^4 - x^6 + \dots + x^{2n} = \frac{1}{1+x^2} + \frac{x^{2n+2}}{1+x^2}$$

   for any real number x and any even positive integer n.

   (b) From (a) deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{1}{2n+1} = \frac{\pi}{4} + \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx.$$

   (c) Show that

$$0 \le \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx \le \int_0^1 x^{2n+2}\,dx = \frac{1}{2n+3}.$$

   (d) Finally, deduce that

$$\lim_{n\to\infty} 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots + \frac{1}{2n+1}\right) = \pi.$$

   This is the Leibniz formula for π.

4\. (a) Integrate the formula in 3(a) from 0 to u (where $0 < u < 1$) and then, by steps similar to those in problem 3, show that

(9) $$\tan^{-1}u \approx u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \ldots + \frac{u^{2n+1}}{2n+1}$$

where n is a large even positive integer.

(b) What value of u should give $\frac{\pi}{6}$ on the left side of (9)?

(c) Use this value of u to obtain another formula for π.

The Project would like to thank Charles Votaw of Fort Hays State University and Solomon Garfunkel of the University of Connecticut for their reviews, and all others who assisted in the production of this unit.

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4. HINTS AND SOLUTIONS TO EXERCISES

1. (a) $\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

Starting with $\int \sin x\,dx = -\cos x$.

$n = 3$: $\int \sin^3 x\,dx = \frac{-\sin^2 x \cos x}{3} - \frac{2}{3}\cos x$

$n = 5$: $\int \sin^5 x\,dx = \frac{-\sin^4 x \cos x}{5} + \frac{4}{5}\int \sin^3 x\,dx$

$= \ldots = -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x$

$- \frac{8}{15}\cos x.$

(b) $n = 2$: $\int \sin^2 x\,dx = \frac{-\sin x \cos x}{2} + \frac{1}{2}\int 1\,dx$

$= -\frac{1}{2}\sin x \cos x + \frac{1}{2}$

$n = 4$: $\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx.$

$= \ldots = -\frac{1}{4}\sin^3 x \cos x$

$-\frac{3}{8}\sin x \cos x + \frac{3}{8}x.$

2. (a) $2\cdot4\cdot6\cdots(2n) = (2\cdot1)(2\cdot2)(2\cdot3)\ldots(2\cdot n)$

$= \underbrace{(2\cdot2\cdot2\cdots2)}_{n}(1\cdot2\cdot3\cdots n) = 2^n n!.$

(b) Multiply left side by $\frac{2\cdot2\cdot4\cdot4\cdots(2n)(2n)}{2\cdot2\cdot4\cdot4\ldots(2n)(2n)}$.
Show that the new numerator is $(2\cdot4\cdot6\cdots(2n))^4$ and use part (a).

(c) The Wallis formula together with part (b) says

$$\pi \approx 2 \frac{(n!)^4\ 2^{4n}}{[(2n)!]^2(2n+1)}.$$

Taking square roots we get

$$\sqrt{\pi} \approx \sqrt{2}\ \frac{(n!)^2\ 2^{2n}}{(2n)!\ \sqrt{2n+1}}$$

or

$$\lim_{n\to\infty} \frac{(n!)^2\ 2^{2n}\sqrt{2}}{(2n)!\ \sqrt{2n+1}} = \sqrt{\pi}.$$

Multiply by $\sqrt{n}/\sqrt{n}$ and observe that

$$\frac{\sqrt{2n}}{\sqrt{2n+1}} = \sqrt{\frac{2n}{2n+1}} \to 1 \text{ as } n\to\infty.$$

3. (a) Use the fact that

$x^{2n+2} = (1+x^2)(1-x^2+x^4-x^6+\ldots+x^{2n})$,

which can be confirmed by direct multiplication.

(b) Integrate the equation in 3(a) from $x = 0$ to $x = 1$:

$$\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{x^{2n+1}}{2n+1}\right)\Bigg|_0^1$$

$$= \tan^{-1} x \Bigg|_0^1 + \int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx.$$

(c) Since $x^2 \geq 0$, $1+x^2 \geq 1$, and therefore $\frac{x^{2n+2}}{1+x^2} \leq x^{2n+2}$. Recall that if $f(x) \leq g(x)$ for all x between 0 and 1, then $\int_0^1 f(x)dx \leq \int_0^1 g(x)dx$.

(d) $\frac{1}{2n+3} \to 0$ as $n \to \infty$. Therefore, using the result in 3(c) and the squeeze principle,

$$\int_0^1 \frac{x^{2n+2}}{1+x^2}\,dx \to 0.$$

Applying this to the equation in 3(b) we get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{1}{2n+1} \approx \frac{\pi}{4}.$$

4. Taking $u = \frac{1}{\sqrt{3}}$:

$$\frac{\pi}{6} \approx \frac{1}{\sqrt{3}} - \frac{1}{3\cdot 3\sqrt{3}} + \frac{1}{5\cdot 3^2\sqrt{3}} - \frac{1}{7\cdot 3^3\sqrt{3}} + \ldots + \frac{1}{(2n+1)3^n\sqrt{3}}$$

or

$$\pi \approx \frac{6}{\sqrt{3}}\left[1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \ldots + \frac{1}{(2n+1)3^n}\right]$$

$$\approx 2\sqrt{3}\left[1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \ldots + \frac{1}{(2n+1)3^n}\right].$$

umap

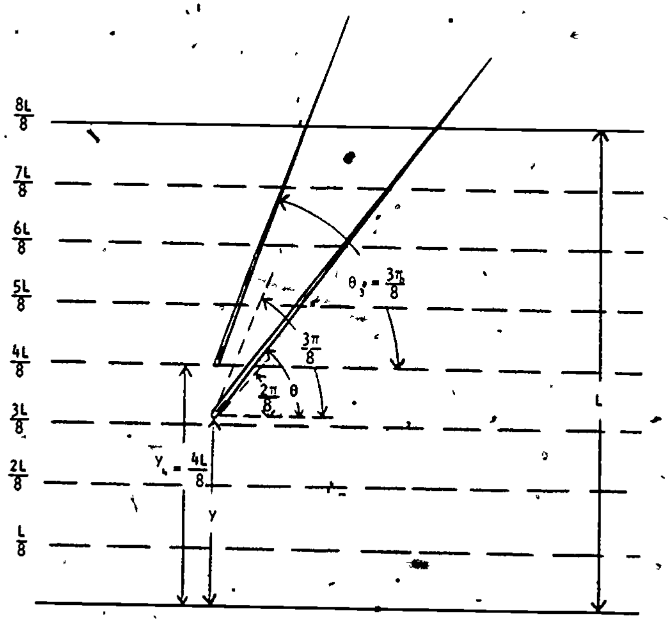
UNIT 242

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

BUFFON'S NEEDLE EXPERIMENT

by

Brindell Horelick and Sinan Koont



APPLICATIONS OF CALCULUS TO OTHER MATHEMATICS

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BUFFON'S NEEDLE EXPERIMENT

by

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TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 242*

Title: BUFFON'S NEEDLE EXPERIMENT

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Review Stage/Date: IV 6/27/79

Suggested Support Material: A hand calculator with the sine function.

References:

Chover, J. 1972. The Green Book of Calculus. W.A. Benjamin, Menlo Park, California.

Eves, H.W. 1969. In Mathematical Circles. Prindle, Weber & Schmidt, Inc., Boston, Massachusetts.

Prerequisite Skills:

1. Know the definition of a definite integral (Riemann integral).
2. Knowledge of

$$\int_b^a \sin\theta \, d\theta.$$

3. For Section 6 and Exercise 15: computer programming ability and access to a computer.

Output Skills:

1. Understand the meaning of the statement "The probability of an event is x."
2. Know and be able to replicate the Buffon needle experiment for approximating π.
3. Approximate the probability underlying the Buffon needle experiment by an appropriate Riemann sum.
4. Express the limit of this Riemann sum as a definite integral, and evaluate this integral.

Other Related Units:

The Irrationality of π *(U240)*
The Wallis Approximation of π *(Unit 241)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

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1. INTRODUCTION

1.1 The Experiment

Certainly you have met the number π ($\approx$ 3.141592) very frequently in your study of mathematics. It is defined as the ratio of the circumference of a circle to its diameter, but it arises in many places which appear to have nothing whatever to do with circles. In this unit we shall describe an experiment where π does indeed arise unexpectedly. It provides a "fun and games" method of approximating π.

You can easily set up and perform this experiment yourself. All you need is a large flat tabletop, enough paper to cover it, and some thin object a few inches long, such as a toothpick. As the unit title suggests, Comte de Buffon (French; 1707-1788) referred to a needle when he first discussed this experiment. Out of respect for the Count, we shall refer to the object as a needle throughout this unit. A toothpick, however, is probably much more convenient and somewhat less dangerous.

Cover the tabletop with paper. On this paper draw a bunch of parallel lines. Make sure the distance from each line to the next is exactly equal to the length of the needle.

Now pick up the needle and toss it onto the table. When it comes to rest it will either cross one of the lines you have drawn, or it will not. Let us call this toss a "success" if the needle crosses a line. Pick up the needle and toss it again. Continue doing this, keeping track of the total number S of successes and the total number T of tosses.

1.2 A Sample Run

Let us consider the ratio S/T; that is, the fraction of tosses which are successes. A running calculation of S/T might look like this:

TABLE I

T	S	S/T	T	S	S/T
1	1	1.00000	102	65	0.63725
2	1	0.50000	103	66	0.64078
3	1	0.33333	104	66	0.63462
4	2	0.50000	105	67	0.63810
5	3	0.60000		. . . .	
6	4	0.66667			
7	5	0.57143	9995	6368	0.63712
8	5	0.62500	9996	6369	0.63715
. . . .			9997	6370	0.63719
			9998	6370	0.63713
100	64	0.64000	9999	6370	0.63706
101	64	0.63366	10000	6371	0.63710

In practice it would be unreasonable to toss the needle 10,000 times (although in Section 6 we discuss how a computer could easily simulate this, and much more). But we want to make a point. Namely, the values of S/T at first fluctuate wildly, but after a very large number of tosses they tend to settle down. We shall use calculus to show that in a certain sense they are most likely to settle down to a value near $\frac{2}{\pi}$. To understand exactly what that means, we must make a brief digression into probability.

Exercise 1

Perform the experiment described. Toss the needle 100 times. Then write $\frac{S}{100} \approx \frac{2}{\pi}$, where S is the number of successes you have recorded, and compute an estimate of π. Finally, compute the percentage error in this estimate.

---

2. PROBABILITY

2.1 Equally Likely Events

If we toss a coin, it may come down heads or tails. Assuming the coin is honest, there is every reason in the world to think these two possibilities are equally likely. We say that the probability of heads is $\frac{1}{2}$ and the probability of tails is $\frac{1}{2}$.

If we throw a die, it may come up 1, 2, 3, 4, 5, or 6. Again, these possibilities are equally likely. We say that the probability of each of them is $\frac{1}{6}$.

Generally, if there are t possible results, of which exactly one will happen, *and if these results are equally likely*, we say that each of the results has probability $\frac{1}{t}$.

Exercise 2

If you pick one card from a well-shuffled deck of playing cards, find the probability it will be the Jack of Hearts.

Exercise 3

Suppose you toss a coin with your left hand while throwing a die with your right hand, and then record the combined result; for example, Heads - 6 or Tails - 3. Find the probability you will get Heads on the coin and 4 on the die.

Exercise 4

Suppose you throw one die with your left hand while throwing another die with your right hand, and then record the combined result. For example, you may get Left - 3, Right - 5, which you could abbreviate (3,5). Or Left - 5, Right - 3, abbreviated (5,3). Find:

(a) the probability you will get 4 on the left die and 6 on the right die,

(b) the probability you will get "boxcars" (6 on both dice).

2.2 "In The Long Run"

There is another way of interpreting probability. Getting back to the die as an example, suppose we were to throw it 10,000 times. Since the six possibilities are equally likely, we would expect that each of them would come up about the same number of times -- about $\frac{1}{6}$ of the 10,000 tosses would give 1, about $\frac{1}{6}$ of them would give 2, etc. We can say that the probability $\frac{1}{6}$ is then a prediction of about what fraction of the tosses will give a certain result.

Sometimes probability is defined in terms of this kind of prediction of what will happen "in the long run", rather than how likely an event is on a "one-shot" basis. We must be careful, though. This "long run" view of probability is not, and cannot be, an assertion of exactly what will happen. For one thing, $\frac{1}{6}$ of 10,000 is $1666\frac{2}{3}$, which is not an integer. So we certainly cannot get exactly that number of, say, 5's. But it would also be wrong to interpret the probability as an assertion that exactly 1666 or 1667 tosses will result in a 5. The die has no memory and can't keep count. What it does on each toss is not determined by the previous tosses. If it comes up 5 on one toss, it cannot say, "Well, I'd better lay off 5 for the next six tosses." If it fails to come up 5 for five or six consecutive tosses, it cannot say, "Hey, I'm overdue. Better make it a 5 this time." And so it may come up 5 a bit more or less than predicted. It may even run a string of ten consecutive 5's, although this is extremely unlikely.

To impress this upon your memory, think again of tossing a coin. The probability of getting heads is $\frac{1}{2}$. In the long run, about $\frac{1}{2}$ of all tosses will be heads. But this does not mean that if we toss a coin twice, exactly one of the tosses must result in heads. The most likely number of heads is one, but we would not be the least bit surprised to see it land heads both times. If it happened to land heads the first time, we would certainly not say it was guaranteed to land tails the next time.

2.3 Compound Events

Now suppose we toss a die once, and are interested in how likely it is to come up greater than 4 (that is, 5 or 6). It should come up 5 about $\frac{1}{6}$ of the time, and 6 about $\frac{1}{6}$ of the time. This adds up to $\frac{2}{6}$, or $\frac{1}{3}$, of the time it comes up greater than 4. We say the probability of this occurring is $\frac{2}{6}$ (or $\frac{1}{3}$).

Generally, if there are t possible *equally likely* results, of which exactly one will occur, and if s of these satisfy a certain condition, the probability that this condition will be satisfied is $\frac{s}{t}$.

Exercise 5

If you throw one die, find the probability it will be even.

Exercise 6

In the experiment of Exercise 2, find the probability of each of the following events:

(a) the card is a Jack

(b) the card is a Heart

(c) the card is a picture card

(d) the card is an even-numbered card.

Exercise 7

In the experiment of Exercise 4, find the probability that:

(a) the sum of the two numbers will equal ten,

(b) the number on the right die will exceed the number on the left die

(c) the two numbers will have an odd sum and an odd product.

---

3. THE THEORETICAL RESULT OF THE EXPERIMENT

3.1 Statement of The Result

Now we can get back to that needle on the tabletop and state precisely what we are going to prove, and how it is helpful in approximating π. We'll prove that *the probability the needle will cross a line is* $\frac{2}{\pi}$. That is, in a very large number of tosses, about $\frac{2}{\pi}$ ($\approx$0.63662) of them will be "successes." The result is independent of the length of the needle.

3.2 Application of the Result

To apply this result, toss the needle t times and count the number s of successes. The fraction $\frac{s}{t}$ should be close to $\frac{2}{\pi}$, although our discussion in Section 2.2 suggests that you shouldn't be too optimistic about the degree of accuracy. Then write $\frac{s}{t} \approx \frac{2}{\pi}$ and "solve for" π.

In the example of Table I, t = 10,000, s = 6371, and $\frac{s}{t}$ = 0.6371. This is less than 0.1% off the true value of $\frac{2}{\pi}$, which, under the circumstances, is pretty good. Writing $\frac{2}{\pi} \approx 0.6371$ and solving for π gives us $\pi \approx 3.13922$ (again within 0.1%).

Exercise 8

Show that with t = 10,000 it is impossible to get better than three decimal place accuracy in estimating π with this experiment.

---

4. PROOF OF THE RESULT

4.1 Locating the Needle Numerically

Although the application of our result depends upon the "long run" interpretation of probability, it is more convenient to use the "equally likely events" interpretation in proving the result. The events will be the various positions in which the needle may land. To base any calculations upon the needle's landing place we must first decide upon a scheme for describing the landing place numerically.

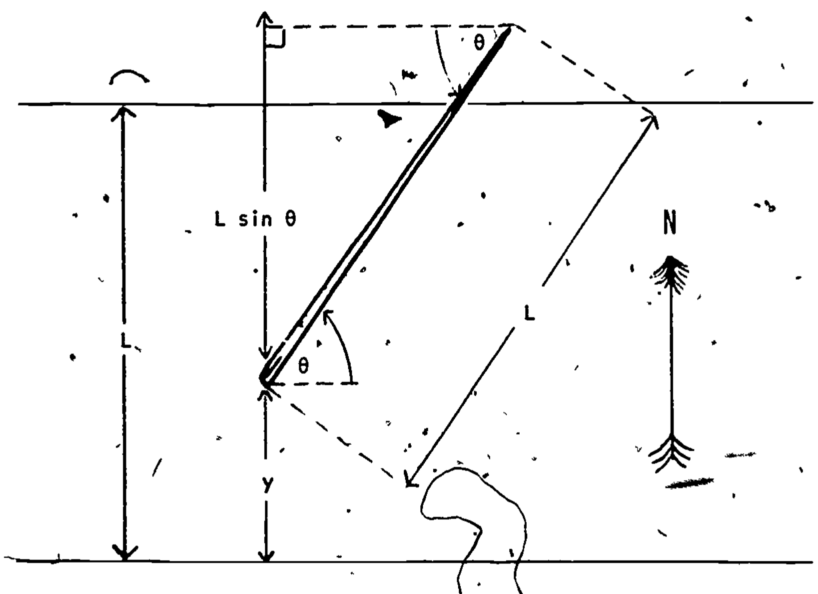


Figure 1. Typical position of needle.

Let us assume the lines on the paper run east-west, and that the distance between consecutive lines is L. A little thought should convince you that two numbers will tell you all you need to know to decide whether or not the needle is on a line (see Figure 1). The first of these (call it y) is the distance of the southern end of the needle from the nearest line to the south. (If the needle's southern end should happen to be on a line, set y = 0. If the needle should happen to lie east-west, think of its western end as the "southern" end.) The second number (call it θ is the angle the needle makes with a ray running eastward from its southern end. So we have an ordered pair (y,θ) with $0 \le y < L$ and $0 \le \theta < \pi$.

Notice that we have simplified matters by restricting our attention narrowly to what concerns us. The ordered pair does not really tell us where the needle is (how far east is it? which line is it straddling?) but, as we have said, it does tell us whether the needle is on a line. In fact, you can see from Figure 1 that this happens if and only if

(1) $$y + L \sin \theta > L.$$

4.2 Equally Likely Positions

A problem arises when we attempt to list the "equally likely events" -- the possible positions of the needle. There is no difficulty finding them; the trouble is that there are too many of them. The southern end of the needle is just as likely to be anywhere as anywhere else. The needle is just as likely to be oriented in any direction as in any other. In other words, all possible pairs (y,θ), with $0 \le y < L$ and $0 \le \theta < \pi$, are equally likely. But there are infinitely many of them, and an infinite subset of these satisfy (1). Our definition in Section 2.3

applies only to finite situations. We can make nothing of the ratio $\frac{\infty}{\infty}$.

4.3 A Finite Approximation

Let us replace our problem by one which involves a large but finite number of equally likely events, and whose answer will be a good approximation to that of the original problem. Let us pick a large positive integer n and partition the interval $0 \leq y \leq L$ into n intervals:

(2) $y_1 = \frac{L}{n},\ y_2 = \frac{2L}{n},\ \ldots,\ y_i = \frac{iL}{n},\ \ldots,\ y_n = \frac{nL}{n} = L.$

Similarly, we shall partition the interval $0 \leq \theta \leq \pi$ into n equal intervals:

(3) $\theta_1 = \frac{\pi}{n},\ \theta_2 = \frac{2\pi}{n},\ \ldots,\ \theta_j = \frac{j\pi}{n},\ \ldots,\ \theta_n = \frac{n\pi}{n} = \pi.$

$0 \quad \pi/n \quad 2\pi/n \quad 3\pi/n \quad \ldots \quad \frac{(n-1)\pi}{n} \quad \pi$

$0 \quad L/n \quad 2L/n \quad \ldots \quad \frac{(n-1)L}{n} \quad L$

There are now n^2 pairs of the form (y_i, θ_j) where i and j are each integers from 1 to n inclusive (it may happen that $i = j$).

Now let us imagine that when we throw the needle, instead of recording its actual position (y, θ), we record (y_i, θ_j), where y_i and θ_j are the smallest numbers in (2) and (3) greater than or equal to y and θ respectively. This amounts to pretending the needle is slightly north of its actual position, and rotated slightly counterclockwise.

In Figure 2 we have illustrated this for the case $n = 8$. The needle has fallen in the position (y, θ) where $\frac{3L}{8} < y < \frac{4L}{8}$ and $\frac{2\pi}{8} < \theta < \frac{3\pi}{8}$. Therefore we replace y by $y_4 = \frac{4L}{8}$ and θ by $\theta_3 = \frac{3\pi}{8}$, recording (y_4, θ_3) as the approximate position of the needle.

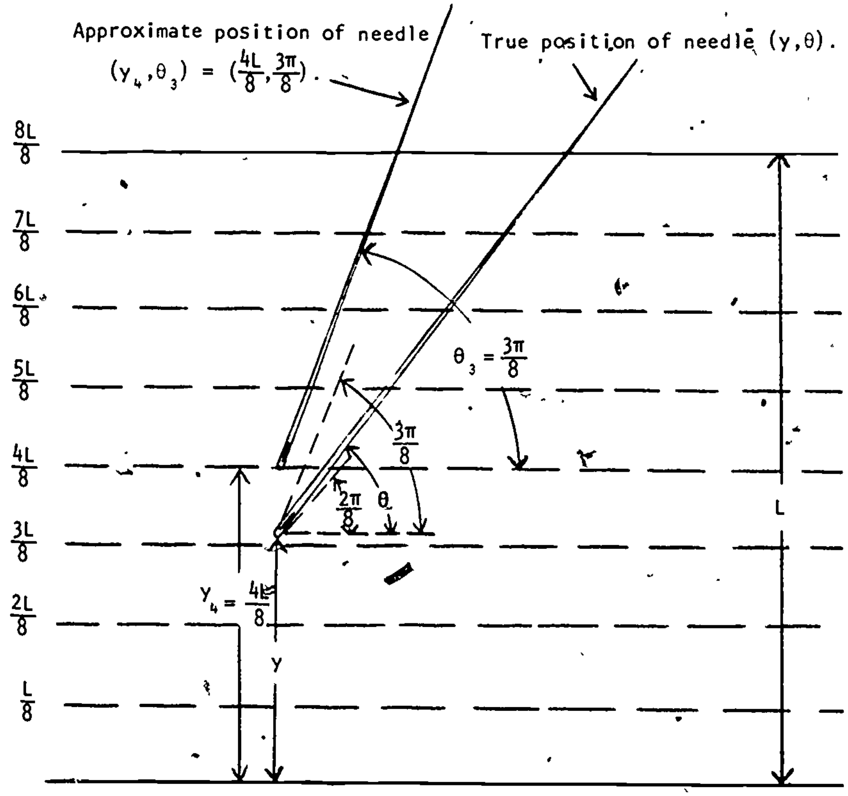


Figure 2. Typical approximation of needle position (n = 8).

We shall define p_n to be the probability the needle crosses a line after it has been moved in this way. This amounts to the probability that

(4) $$y_i + L \sin \theta_j > L.$$

Since, for any actual position (y, θ) of the needle, $y_i \approx y$ and $\theta_j \approx \theta$, and since these approximations become better and better as $n \to \infty$, it follows that p_n approaches the true probability which we seek. The result stated in Section 3.1 can thus be expressed

$$\lim_{n\to\infty} p_n = \frac{2}{\pi}$$

4.4 Counting Successes in the Finite Case

Since the intervals $0 \le y \le L$ and $0 \le \theta \le \pi$ are each partitioned into equal subintervals, each of the n^2 ordered pairs (y_i, θ_j) is equally likely. Therefore, according to Section 2.3, all that is necessary to determine p_n is to count the number of ordered pairs which satisfy (4). If this number is m_n, then $p_n = \frac{m_n}{n^2}$.

To determine m_n we begin by looking at any one particular θ_j. There are n paris involving this θ_j. Of these, the ones satisfying (4) are those for which

$$y_i > L\,(1 - \sin \theta_j)$$

$$\frac{iL}{n} > L\,(1 - \sin \frac{j\pi}{n})$$

(5) $$i > n\,(1 - \sin \frac{j\pi}{n}).$$

The right side of (5) is between 0 and n. Let k_j be the integer part of the right side; that is, the unique integer such that

(6) $$k_j \le n(1 - \sin \frac{j\pi}{n}) < k_j + 1.$$

If $k_j = 0$ then $n(1 - \sin \frac{j\pi}{n})$ is between 0 and 1 and therefore less than any positive integer. So (5) is *true* for all permissible values of i $(i = 1, 2, \ldots, n)$. If $k_j \ne 0$ then it is a positive integer, and (5) is *false* for all values of i up to k_j $(i = 1, 2, \ldots, k_j)$ and *true* for all values of i after that $(i = k_j+1, k_j+2, \ldots, n)$. In either case k_j counts the number of ordered pairs with this particular θ_j for which (5) is *false*, and therefore $n - k_j$ counts the number for which it is true. So

(7) $$m_n = \sum_{j=1}^{n} (n - k_j) = n^2 - \sum_{j=1}^{n} k_j .$$

Figure 3 and Table 2 illustrate the determination of k_j and of m_n for $n = 7$. The figure shows clearly that, for each j, k_j denotes the largest i for which the needle fails to cross a line.

Exercise 9

Find all the k_j's $(j = 1, 2, 3, \ldots, n)$, m_n, and p_n:

(a) for $n = 10$

(b) for $n = 100$ (this is easiest if you write a little computer program)

Some of your computations will be helpful in Exercise 10.

4.5 Approximating the Count

The result in (7) easily provides the value of p_n for any particular n, but not in a form convenient for computing the limit. To achieve that end, we begin by noting that (6) tells us

$$k_j \approx n\,(1 - \sin \frac{j\pi}{n})$$

with an error less than one. This approximation leads to an error of less than $\frac{1}{n}$ in the value of p_n, as you should be able to verify (see Exercise 11). It follows that

$$n - k_j \approx n - n(1 - \sin \frac{j\pi}{n})$$

$$n - k_j \approx n \sin \frac{j\pi}{n}$$

and this is approximately the number of ordered pairs, with this particular θ_j, for which (5) is true.

If we make the same approximation for each θ_j and then add the results, we obtain

$$m_n \approx \sum_{j=1}^{\infty} n \sin \frac{j\pi}{n} .$$

TABLE II

Calculation of m_n (Section 4.4) and of the approximation of m_n (Section 4.5) for $n=7$

j	$\frac{j\pi}{n}$	$\sin\frac{j\pi}{n}$	$n(1-\sin\frac{j\pi}{n})$	k_j	$n-k_j$	$n\sin\frac{j\pi}{n}$
1	$\frac{\pi}{7}$	0.4339	3.9628	3	4	3.0372
2	$\frac{2\pi}{7}$	0.7818	1.5272	1	6	5.4728
3	$\frac{3\pi}{7}$	0.9749	0.1755	0	7	6.8245
4	$\frac{4\pi}{7}$	0.9749	0.1755	0	7	6.8245
5	$\frac{5\pi}{7}$	0.7818	1.5272	1	6	5.4728
6	$\frac{6\pi}{7}$	0.4339	3.9628	3	4	3.0372
7	π	0.0000	7.0000	7	0	0.0000
					$m_n = 34$	$m_n \approx 30.6990$

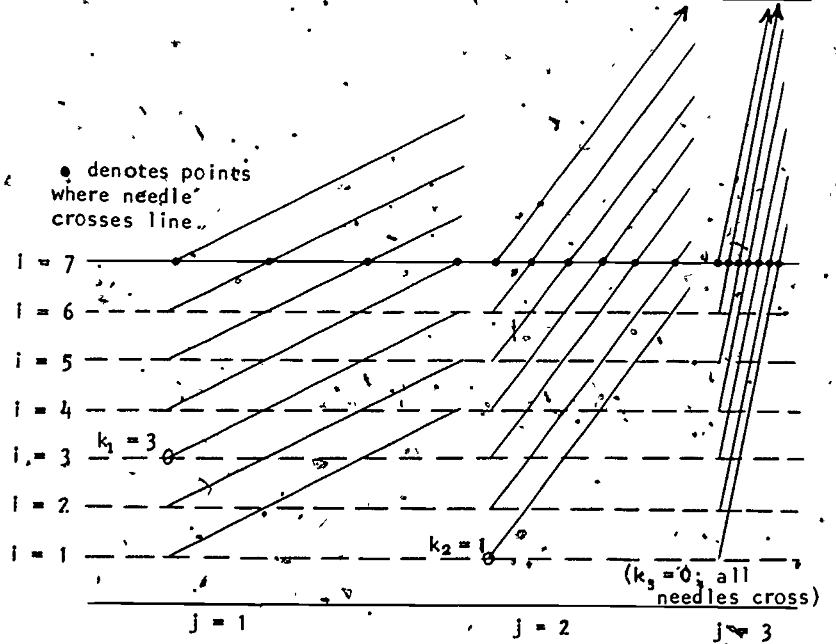


Figure 3. Graphical Illustration of k_1, k_2, k_3 for $n = 7$.

This computation is illustrated, for $n = 7$, in the last column of Table II.

Remembering that there are n^2 pairs altogether, we obtain

(8) $$p_n \approx \frac{\sum_{j=1}^{n} n \sin\frac{j\pi}{n}}{n^2}$$

Exercise 10

Compute the approximation of p_n given in (8):

(a) for $n = 10$

(b) for $n = 100$.

Exercise 11

(a) Show that the error in estimating p_n by (8) is less than $\frac{1}{n}$.

(b) Show that the estimated value of p_n given by (8) is less than the true value.

4.6 Taking the Limit

Let us call the right side of (8) q_n. We shall show that $\lim_{n\to\infty} q_n = \frac{2}{\pi}$. Since, as you have shown in Exercise 11(a), $|q_n - p_n| < \frac{1}{n}$, it will follow that $\lim_{n\to\infty} p_n = \frac{2}{\pi}$, and we shall be done.

We shall begin with a little algebra. Starting from the definition of q_n:

(9) $$q_n = \frac{\sum_{j=1}^{n} n \sin\frac{j\pi}{n}}{n^2} = \sum_{j=1}^{n} \frac{1}{n} \sin\frac{j\pi}{n}$$

$$= \sum_{j=1}^{n} \frac{1}{\pi}\,\frac{\pi}{n} \sin\frac{j\pi}{n} = \sum_{j=1}^{n} \left[\frac{1}{\pi}\left(\sin\frac{j\pi}{n}\right)\frac{\pi}{n}\right].$$

Now look again at (3) where we partitioned the interval $0 \le \theta \le \pi$. The right hand endpoints of the subintervals are π/n, $2\pi/n$, etc. The numbers $\frac{1}{\pi}\sin\frac{j\pi}{n}$ are the values of the function $f(\theta) = \frac{1}{\pi}\sin\theta$ at these endpoints. And $\frac{\pi}{n}$ is the length of each subinterval. In other words (see Figure 4), this is a Riemann sum for the integral

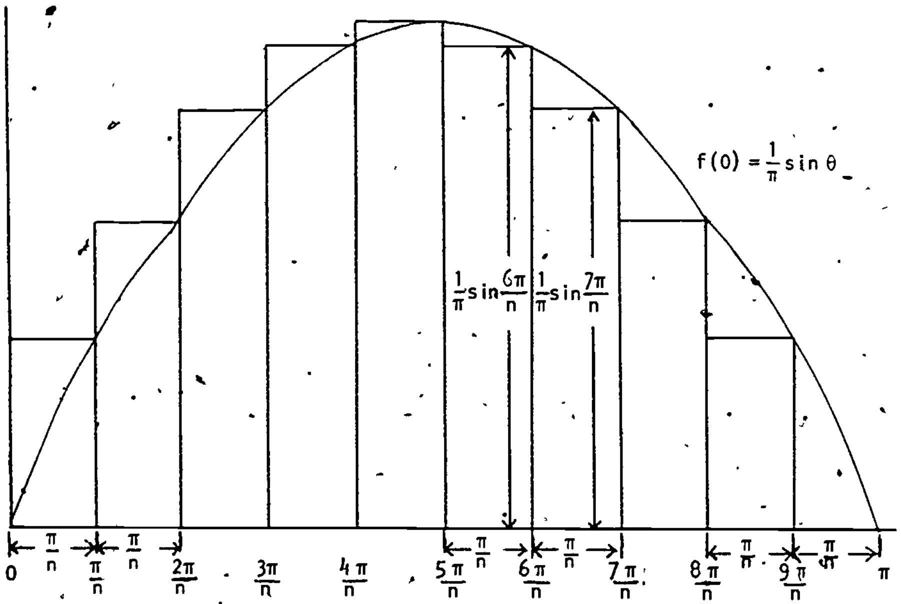


Figure 4. Relationship between $\sum_{j=1}^{n}\left(\frac{1}{\pi}\sin\frac{j\pi}{n}\right)\frac{\pi}{n}$ (rectangular areas) and $\int_0^\pi \frac{1}{\pi}\sin\theta\, d\theta$ ($n = 10$).

$\int_0^\pi \frac{1}{\pi}\sin\theta\, d\theta$. Thus,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{\pi}\left(\sin\frac{j\pi}{n}\right)\frac{\pi}{n} = \int_0^\pi \frac{1}{\pi}\sin\theta\, d\theta.$$

From (9) we then get

$$(10) \qquad \lim_{n\to\infty} q_n = \int_0^\pi \frac{1}{\pi}\sin\theta\, d\theta = \frac{1}{\pi}(-\cos\theta)\Big|_0^\pi = \frac{2}{\pi}.$$

Exercise 12:

Suppose in setting up our calculation we measured the angle ϕ which the needle makes with the northward direction, instead of θ.

(a) What inequality would replace (1)?

(b) Over what values would ϕ range?

(c) What integral would replace the one in (10)?

(d) Show that this integral equals $\frac{2}{\pi}$.

---

5. VARYING THE NEEDLE LENGTH

We have already remarked that our main result is independent of the length L of the needle. But it is very much dependent upon the fact that the distance between consecutive lines equals the needle length. Certainly if we were to switch to a needle, say, half as long, while continuing to use the same ruled paper, the needle would be less likely to cross a line.

In practice it is unrealistic to insist upon this equality. It is very likely that in setting up this experiment you will have available some ruled paper whose lines are a distance D apart, and a needle of length L, where $L \neq D$. It turns out that, as long as L is less than D, the probability that the needle will cross a line is $\frac{2}{\pi} \times \frac{L}{D}$. In fact, it's not at all hard to modify our proof, starting with (1), to get this result. We'll leave it to you (see Exercise 13).

Exercise 13

Suppose the needle has length L but the parallel lines are D units apart, where $L < D$.

(a) What inequality replaces (1)?

(b) What integral replaces the one in (10)?

(c) Show that this integral equals $\frac{2}{\pi} \times \frac{L}{D}$.

Exercise 14

The result stated in this section clearly cannot be true if $L > D$, since then $\frac{2}{\pi} \times \frac{L}{D}$ might be greater than one, and could not possibly be the *fraction* of tosses which are successes. Exactly where does our proof break down if we try to modify it as in Exercise 12?

---

6. GETTING THE COMPUTER TO HELP

Whatever the merit of this approach to approximation of π, in view of Exercise 8, no one can claim it to be a realistic way of getting a good estimate. Remember, not only does Exercise 8 show that the accuracy after 10,000 tosses cannot be better than three places, but the discussion in Section 2.2 says there is no reason to presume it will even be that good.

It is possible for a computer, figuratively at least, to toss the needle for us. In our proof we described the approximate position of the needle by an ordered pair (y,θ) where $0 \leq y < L$ and $0 \leq \theta < \pi$. Since the result does not depend on L anyway we can take L = 1 for convenience. We can ask the computer to pick a number at random between 0 and 1, and call it y; pick another number at random between 0 and $4 \tan^{-1} 1$ (note that $4 \tan^{-1} 1 = \pi$), and call it θ; and then determine if

$$y + \sin \theta > 1$$

is true.

The machine can easily count the number of "tosses" T and of "successes" S. We can program it to perform any predetermined number of "tosses" and then to compute an estimate of π just as in Section 3.2

In an actual computer run,* we asked the computer to "toss the needle" 100,000 times. It reported 63,449

* Computer work for this unit supported by the University of Maryland Computer Center.

"successes," giving 0.63449 as an estimate of $\frac{2}{\pi}$, and thus 3.15214 as an estimate of π, an error of about $\frac{1}{3}$ of one percent. The BASIC program we used contained only 10 lines. If you know BASIC or any other computer language, you should be able to try this yourself.

Exercise 15

(a) If you know a computer language, write a computer program to simulate Buffon's needle experiment for 100,000 "tosses."

(b) If you have access to a computer, run this program, compute the estimate of π resulting from this run, and compute the percentage error in this estimate.

The Project would like to thank Charles Votaw of Fort Hays State University, Hays, Kansas, and Solomon Garfunkel of the University of Connecticut, Storrs, Connecticut for their reviews, and all others who assisted in the production of this unit.

This unit was field-tested and/or student reviewed in preliminary form at Malone College, Canton, Ohio, University of Louisville, Louisville, Kentucky, Roberts Wesleyan College, Rochester, New York, St. John's University, Collegeville, Minnesota, and The Wheatley School, Old Westbury, New York, and has been revised on the basis of data received from these sites.

7. ANSWERS TO EXERCISES

2. $\frac{1}{52}$

3. $\frac{1}{12}$

4. (a) $\frac{1}{36}$ (b) $\frac{1}{36}$

5. $\frac{1}{2}$

6. (a) $\frac{1}{13}$ (b) $\frac{1}{4}$ (c) $\frac{3}{13}$ (d) $\frac{5}{13}$

7. (a) $\frac{1}{12}$ (b) $\frac{5}{12}$ (c) 0

8. The best possible result is s = 6366, giving $\frac{2}{\pi} = 0.6366$ and $\pi \approx 3.1417$.

9. (a)

j	1	2	3	4	5	6	7	8	9	10
k_j	6	4	1	0	0	0	1	4	6	10

$m_{10} = 68$; $p_{10} = 0.68$.

(b)

j	1	2	3 ...	43	44	45	46 ...	50
k_j	96	93	90 ...	2	1	1	0 ...	0

For $50 < j < 100$, $k_j = k_{100} - j$; $k_{100} = 100$;
$m_{100} = 6412$; $p_{100} = 0.6412$.

10. (a) $p_{10} \approx 0.6314$ (b) $p_{100} \approx 0.6366$

11. You are looking for an inequality of the form

$$0 \le p_n - \frac{\Sigma n \sin \frac{j\pi}{n}}{n^2} < \frac{1}{n}.$$

To get it, remember the exact value of p_n is

$$p_n = \frac{n^2 - \sum_{j=1}^{n} k_j}{n^2}$$

Since the formula for p_n involves k_j, write down the defining inequality for k_j;

$$k_j \le n\,(1 - \sin \frac{j\pi}{n}) < k_j + 1.$$

Now take the following steps.

Subtract k_j from all three qualities in the inequality

$$0 \le n(1 - \sin \frac{j\pi}{n}) - k_j < 1.$$

Sum from $j = 1$ to n, and divide by n^2.

$$0 \le \frac{\sum_{j=1}^{n} n(1 - \sin \frac{j\pi}{n}) - \sum_{j=1}^{n} k_j}{n^2} < \frac{n}{n^2}.$$

Split the sum of the left, and simplify the far right

$$0 \le \frac{\sum_{j=1}^{n} n - \sum_{j=1}^{n} n \sin \frac{j\pi}{n} - \sum_{j=1}^{n} k_j}{n^2} < \frac{1}{n}.$$

Then split the fraction in the middle this way,

$$0 \le \frac{n^2 - \sum_{j=1}^{n} k_j}{n^2} - \frac{\Sigma n \sin \frac{j\pi}{n}}{n^2} < \frac{1}{n},$$

to get the inequality you seek:

$$0 \le p_n - \frac{\Sigma n \sin \frac{j\pi}{n}}{n^2} < \frac{1}{n}.$$

12. (a) $y + L \cos \phi > L$.
(b) $-\frac{\pi}{2} \le \phi < \frac{\pi}{2}$.
(c) $\int_{-\pi/2}^{\pi/2} \frac{1}{\pi} \cos \phi \, d\phi$.

13. (a) $y + L \sin \theta > D$.
(b) $\int_0^{\pi} \frac{1}{\pi} \frac{L}{D} \sin \theta \, d\theta$.

14. The k_j defined in (6) is not an accurate count of those values of i from 1 to n for which (5) is false. For certain j, $1 - \sin \frac{j\pi}{n} < 0$, so (5) is true for all i and false for 0 i. But $k_j < 0$.

15. (a) Here is one BASIC program which should work. Depending on the computer you are using, it might require minor modifications.

```
100   RANDOMIZE
200   FOR T = 1 TO 100000
300   Y = RND
400   A = 4 * RND * ATN(1)
500   IF Y + SIN(A) < 1 THEN 700
600   S = S + 1
700   NEXT T
800   R = S/100000
900   PRINT 'PI =  '; 2/R
1000  END
```

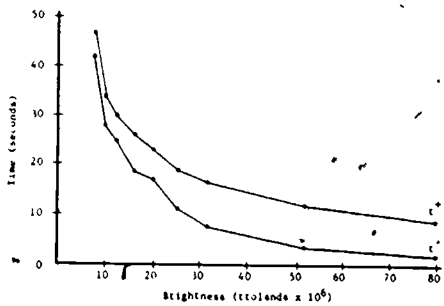
UMAP

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

MODULE 251

A Strange Result in Visual Perception

by Brindell Horelick
and Sinan Koont



Applications of Calculus to Physiology and Psychology

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Intermodular Description Sheet: *UMAP Unit 251*

Title: A STRANGE RESULT IN VISUAL PERCEPTION

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Review Stage/Date: III 6/28/79

Classification: APPL CALC/PHYSIO & PSYCH

Prerequisite Skills:

1. Algebraic manipulation of logarithmic and exponential functions.
2. Differentiation of exponential functions.
3. Solution of $z' = kz$.
4. Knowledge of $\lim_{z\to\infty} e^z$ and $\lim_{z\to\infty} e^{-z}$.
5. Use of $f'(t)$, $f''(t)$, and $\lim_{z\to\infty} f(t)$ in graphing $f(t)$.
6. Knowledge of $\lim_{z\to\infty} f(z)$, where $f(z)$ is a polynomial.

Output Skills:

1. Understand how elementary calculus may be used to explain the results of T.N. Cornsweet's experiment as described in the module.

The Project would like to thank Michael J. Kallaher of Washington State University, Pullman, Washington, Mitchell Lazarus of Education Development Center, Newton, Massachusetts, and Joel Greenstein of New York City Community College (CUNY), New York, for their reviews, and all others who assisted in the production of this unit.

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A STRANGE RESULT IN VISUAL PERCEPTION

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TABLE OF CONTENTS

*These sections are included in the instructional unit but omitted in the UMAP Journal version for the sake of brevity.

A STRANGE RESULT IN VISUAL PERCEPTION

1. INTRODUCTION

1.1 The Modeling Problem

In 1962 Tom N. Cornsweet reported on the experimental verification of some seemingly paradoxical results.* He (and others) had predicted these results by constructing a remarkably simple mathematical model of the process by which the eye "sees" bright light, based in turn on some very simple physiological assumptions. We shall describe the experiment and then present his model.

1.2 The Experiment

Essentially, subjects whose eyes had had a chance to adapt to darkness fixated upon a point in a brightly lighted region, across which was a non-opaque bar which filtered out a fixed amount of the light. At a certain time t^- the bar rather abruptly appeared much brighter than the background even though it was less brightly illuminated! Then, the apparent brightnesses gradually became equal, and finally at time t^+ the background became and remained slightly brighter than the bar. The subjects were asked to press buttons at the times t^- and t^+ as accurately as possible. This experiment was rerun for different background brightnesses. The details of the experiment can be found in Cornsweet's article (pages 261-263). In Figure 1 average empirical values of t^- (lower curve) and t^+ (upper curve) are plotted against a measure of background brightness.

*Tom N. Cornsweet, "Changes in the Appearance of Stimuli of Very High Luminance," *Psychological Review* 69(1962): 257-273.

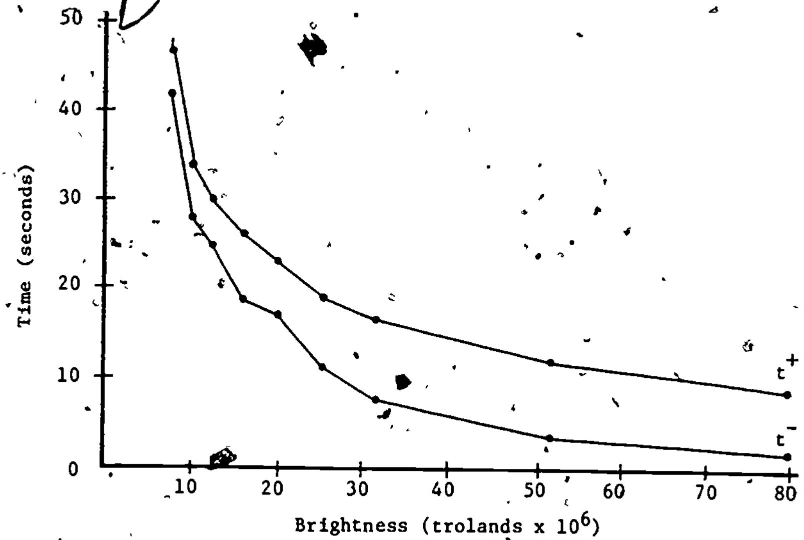

Figure 1. Times at which apparent brightnesses of bar and background were equal, plotted against background brightness. (Source: Cornsweet, p. 261)

2. PHYSIOLOGICAL BACKGROUND

Before constructing a mathematical model, we must have some idea what we are modeling. In this section we shall describe briefly what we need to know about the physiological process by which the eye "sees."

2.1 Activation

In the retina of one's eye there are certain cells, call "receptor" cells. Each of these contains molecules which are capable of being "activated" by light. This activation results in the discharge of a certain chemical. Accumulation of a certain minimal amount of this chemical triggers a nerve impulse which results in one "seeing" the light.

For various reasons, the chemical tends to be destroyed very quickly after its discharge. This means that, if there is to be any hope that the required minimal amount accumulates before it is destroyed, this hope will depend upon very many molecules being activated in a short period of time. That is, it will depend on a high *rate* of activation.

2.2 Regeneration

When one of these molecules has just been activated it is not capable of "immediately" responding to further stimulation. It must take a small, but not negligible, amount of time to recover, or be "regenerated." So at any moment only a certain fraction of the molecules are in this regenerated state.

3. THE MODEL

3.1 Assumptions of the Model

There are four mathematical assumptions we must make in order to construct our model. We hope each of these assumptions will seem reasonable to you. As for whether they are "true," we cannot say for sure, but we remind you that a very striking and seemingly paradoxical result predicted by the model has been confirmed experimentally.

(1) How much light one "sees" (one's perception) is directly proportional to the rate of activation.

(2) The rate of activation is directly proportional to the amount of light (brightness) falling upon the receptor cells.

(3) At any time t, the rate of activation is also directly proportional to the fraction of the molecules which are in the regenerated state at that time.

(4) At any time t, activated molecules are being regenerated at a rate directly proportional to the fraction of the molecules which are activated at that time.

3.2 The Assumptions Rewritten Mathematically

Let $x = x(t)$ stand for the fraction of molecules which are regenerated at time t. Then the number of regenerated molecules is mx, where m is the total number of molecules. Assumptions (2) and (3) together say that an amount of light q shining on the retina activates molecules (decreasing mx) at the rate cqx, where c is a positive constant of proportionality. Assumption (4) says that at the same time other molecules are being regenerated (increasing mx) at the rate $k(1-x)$, where k is another positive proportionality constant. Notice that while the rate of activation of regenerated molecules depends upon q, the rate of regeneration of active molecules does not.

All told, then:

(1) $$(mx)' = k(1-x) - cqx.$$

Finally, assumption (1) says that our perception of light is proportional to cqx. Since we are studying perception, it is this quantity we are interested in.

3.3 Summary of the Notation

A lot of notation is beginning to pile up, and there will be more. There is no need to remember all the notational details. Just keep in mind:

q (the amount of light entering the eye) is a positive constant controlled by the experimenter.
c, k, and m are positive constants determined by the physiology of the eye, and are not controllable.
x is a function of the time t.
cqx is proportional to perception, and is what we are interested in.

4. A FORMULA FOR PERCEPTION

4.1 Solving the Equation of the Model

Since m is a constant, Equation (1) can be written

$$mx' = k(1-x) - cqx$$

$$mx' = (-k-cq)x + k$$

(2) $$x' = \frac{-k-cq}{m}x + \frac{k}{m}.$$

We are getting buried by notation. At the price of introducing still more letters, let's simplify it. In effect, Equation (2) just says x' is a linear function of x:

(3) $$x' = rx + s$$

where r and s are constants.

Here is a very useful trick for solving *any* equation like Equation (3). You may want to remember it. Just make the substitution

$$z = rx + s.$$

Then $z' = rx' = r(rx+s) = rz$. And of course the solution of $z' = rz$ is $z = Ce^{rt}$. Going back to the x-notation, this is $rx + s = Ce^{rt}$; or

(4) $$x = C^*e^{rt} - \frac{s}{r}$$

where $C^* = \frac{C}{r}$.

4.2 Determining the Constant

To determine C* we must know the value of x for any one particular t. Remember, in the experiment the subject's eye was allowed to adapt to darkness before being exposed to a bright light. In other words, almost all the molecules in the receptor cells were in the regenerated state just before the initial exposure, so at that moment

$x = 1$. If we label the moment of first exposure $t = 0$, then $x(0) = 1$.

Putting $t = 0$ and $x = 1$ in Equation (4) yields

$$C^* = 1 + \frac{s}{r} = \frac{r + s}{r},$$

and so

$$(5) \qquad x = \frac{r + s}{r} e^{rt} - \frac{s}{r}.$$

4.3 Determining the Activation Rate

To interpret Equation (5) we must return to our original notation. Comparing Equation (3) with Equation (2) we see that

$$r = \frac{-k-cq}{m}$$

and $s = \frac{k}{m}$, so

$$x = \frac{cq}{k + cq} e^{-(k+cq)t/m} + \frac{k}{k + cq}$$

and the rate of activation is

$$(6) \qquad cqx = \frac{c^2q^2}{k + cq} e^{-(k+cq)t/m} + \frac{cqk}{k + cq}.$$

The right side of Equation (6) tells us, if our model has any validity, how bright a light will appear to one as a function of time t, assuming it is turned on at time $t = 0$, and that one's eye was in darkness before then. Let's give this function a name: $f(t)$, and let's investigate its properties.

5. ANALYSIS OF THE FORMULA

5.1 Simplifying the Notation

Again, let's simplify the notation to avoid being buried by it. Let's write

$$M = \frac{c^2q^2}{k + cq}, \quad N = \frac{k + cq}{m}, \quad P = \frac{cqk}{k + cq}.$$

Then we have

(7) $$f(t) = Me^{-Nt} + P.$$

There is no point in trying to interpret M, N, and P physiologically. What we should remember is the material in the box in Section 3.3, from which it follows that M, N, and P are positive constants.

5.2 The Shape of the Perception Graph

It is easy to confirm (Exercise 1) that $f(0) = M + P$, $\lim_{t\to\infty} f(t) = P$, $f'(t) < 0$ for all t, and $f''(t) > 0$ for all t. Therefore the graph of f(t) for positive t looks like Figure 2. The horizontal asymptote at P represents an equilibrium position. It corresponds to that value of x (the fraction of molecules which are regenerated) for which regenerated molecules are being activated and activated molecules are being regenerated at the same rate.

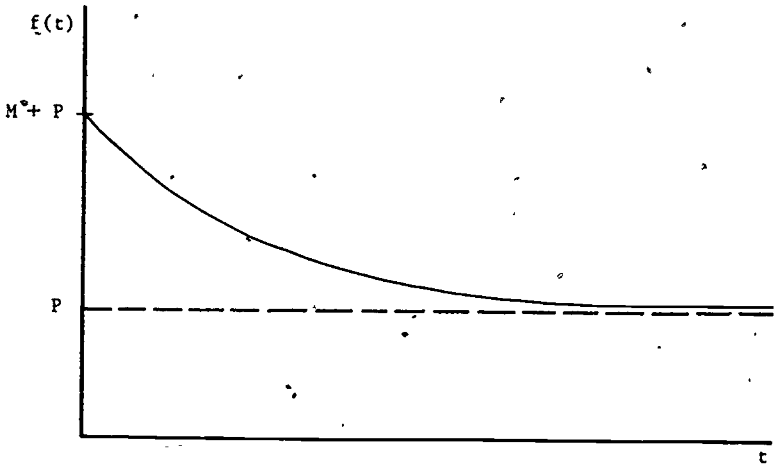

Figure 2. A typical graph of the activation rate as a function of time.

5.3 The Effect of Changing the Brightness

If we change q, then we also change M, N and P. But they are still positive constants. So we get a new f(t) whose graph has the same shape as that in Figure 2, but with a different intercept and a different horizontal asymptote.

6. RELATION OF THE FORMULA TO THE EXPERIMENT

6.1 Explaining the Experimental Result

Now let's look at the experiment. What happened was that the subject's eye received two amounts of light: q_1 from the background, and $q_2 < q_1$ through the bar. So there were two graphs like the one in Figure 2:

$$f_1(t) = M_1e^{-N_1t} + P_1$$

with intercept $M_1 + P_1$ and asymptote at P_1, representing the subject's perception of the background, and

$$f_2(t) = M_2e^{-N_2t} + P_2$$

with intercept $M_2 + P_2$ and asymptote at P_2, representing the subject's perception of the bar.

In Figure 3 we have drawn two curves of the right shape on the same pair of coordinate axes. We have not considered the specific values of the constants. We just want to show that it is at least believable that the two curves might intersect twice in the region $t > 0$, in which case there would be two reversals as to which curve was the higher of the two. *If* this actually happens, then for $t^- < t < t^+$ the subject will perceive the less brightly illuminated bar to be brighter than the background, and there will be two reversals of apparent relative brightness of bar and background. Further, Figure 3 suggests that the first reversal (at $t = t^-$) will be rather abrupt, while the second (at $t = t^+$) will be much more gradual. This is just what happened in the experiment!

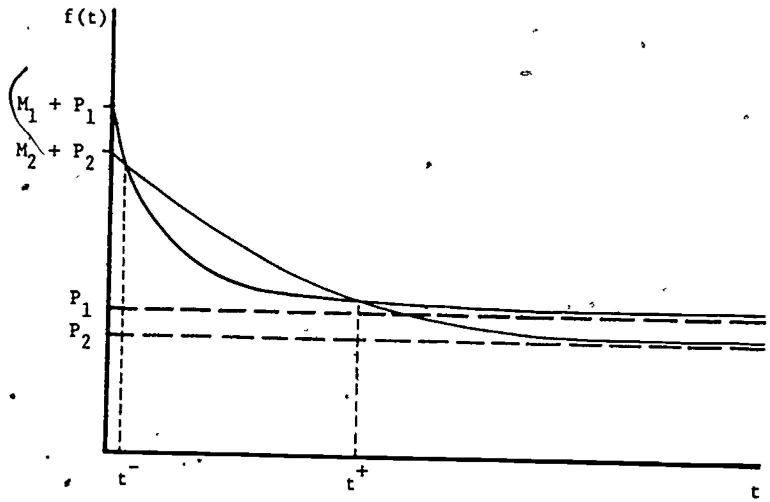

Figure 3. Two typical graphs of the activation rate (same subject, different brightnesses).

6.2 Predicting the Experimental Result

Now for that all-important "If." Could we have actually calculated that $f_1(t)$ and $f_2(t)$ intersect (and thus have *predicted* the experimental result), rather than settle for "it is at least believable . . ." and (after the fact) "This is . . . what happened"?

This amounts to solving $f_1(t) = f_2(t)$, or

$$M_1 e^{-N_1 t} + P_1 = M_2 e^{-N_2 t} + P_2 \tag{8}$$

for t. There is usually no easy way to do this. But for certain carefully selected values of q_1 and q_2 it can be done. (Remember, q is the only thing we can control). Let's select q_1 and q_2 so that $N_1 = 2N_2$. (This *can* be done. In Exercise 7 you are asked to show that $q_1 = Rq_2$ will do it, where

$$R = \frac{k + 2cq_2}{cq_2} = \frac{k}{cq_2} + 2.)$$

Then Equation (8) becomes

$$M_1e^{-2N_2t} + P_1 = M_2e^{-N_2t} + P_2$$

(9) $$M_1e^{-2N_2t} - M_2e^{-N_2t} + (P_1 - P_2) = 0.$$

Putting $y = e^{-N_2t}$ we get

(10) $$M_1y^2 - M_2y + (P_1 - P_2) = 0$$

which is a plain old quadratic equation. (This is the payoff for making $N_1 = 2N_2$.) Its solutions are of course

(11) $$y = e^{-N_2t} = \frac{M_2 \pm \sqrt{M_2^{\ 2} - 4M_1(P_1 - P_2)}}{2M_1}.$$

There will be two positive values of t satisfying Equation (9) if there are two values of y between 0 and 1 satisfying Equation (10). So we must show that the right side of Equation (11) is (a) real, (b) greater than zero, and (c) less than one.

The sticky part of this is (a). Some fairly messy algebra is required, which we have put in the Appendix. It turns out that the right side of Equation (11) is real if q_2 is large enough compared with k/c. After that, (b) and (c) are relatively easy, and we'll leave them as exercises (see Exercises 5 and 6).

So if q_2 is sufficiently large (see the Appendix), and if q_1 is sufficiently larger than q_2 (Exercise 7), then $f_1(t)$ and $f_2(t)$ will intersect at two positive values of t, producing the paradoxical effect confirmed by the experiment.

APPENDIX

This appendix is devoted to showing that the solutions to Equation (10) are real provided q_2 is large enough. As stated in Section 6.2, these solutions are

$$y = \frac{M_2 \pm \sqrt{M_2^2 - 4M_1(P_1 - P_2)}}{2M_1}.$$

They will be real (and distinct) if

$$M_2^2 - 4M_1(P_1 - P_2) > 0.$$

In working with this inequality, we shall use

(a) the definitions:

$$M_1 = \frac{c^2 q_1^2}{k + cq_1} \qquad M_2 = \frac{c^2 q_2^2}{k + cq_2}, \qquad P_1 = \frac{cq_1 k}{k + cq_1} \qquad P_2 = \frac{cq_2 k}{k + cq_2};$$

(b) the fact that $q_1 = Rq_2$ where

$$R = \frac{k + 2cq_2}{cq_2} = \frac{k}{cq_2} + 2;$$

(c) the fact that $k + cRq_2 = 2(k + cq_2)$, which you are asked to prove as an exercise (see Exercise 4).

We have

$$M_2^2 - 4M_1 (P_1 - P_2)$$

(by (a)) $$= \frac{c^4 q_2^4}{(k+cq_2)^2} - \frac{4c^2 q_1^2}{k+cq_1}\left(\frac{cq_1 k}{k+cq_1} - \frac{cq_2 k}{k+cq_2}\right)$$

(by (b)) $$= \frac{c^4 q_2^4}{(k+cq_2)^2} - \frac{4c^2 R^2 q_2^2}{k+cRq_2}\left(\frac{cRq_2 k}{k+cRq_2} - \frac{cq_2 k}{k+cq_2}\right)$$

(by (c)) $$= \frac{c^4 q_2^4}{(k+cq_2)^2} - \frac{2c^2 R^2 q_2^2}{k+cq_2}\left(\frac{cRq_2 k}{2(k+cq_2)} - \frac{cq_2 k}{k+cq_2}\right)$$

$$= \frac{c^4q_2^4 - c^3R^3q_2^3k + 2c^3R^2q_2^3k}{(k+cq_2)^2}$$

$$= \frac{c^3q_2^3}{(k+cq_2)^2}\left(cq_2 - R^3k + 2R^2k\right).$$

For this expression to be positive, the part in parentheses must be positive. Using the definition of R in (b):

$$cq_2 - R^3k + 2R^2k$$

$$= cq_2 - \left(\frac{k+2cq_2}{cq_2}\right)^3 k + 2\left(\frac{k+2cq_2}{cq_2}\right)^2 k$$

$$= cq_2 - \frac{k^4+6k^3cq_2+12k^2c^2q_2^2+8kc^3q_2^3}{c^3q_2^3} + \frac{2k^3+8k^2cq_2+8kc^2q_2^2}{c^2q_2^2}$$

$$= \frac{c^4q_2^4-k^4-6k^3cq_2-12k^2c^2q_2^2-8kc^3q_2^3+2k^3cq_2+8k^2c^2q_2^2+8kc^3q_2^3}{c^3q_2^3}$$

$$= \frac{c^4q_2^4-k^4-4k^3cq_2-4k^2c^2q_2^2}{c^3q_2^3}.$$

For this fraction to be positive, the numerator must be positive. That is,

$$c^4q_2^4 - k^4 - 4k^3cq_2 - 4k^2c^2q_2^2 > 0.$$

Since it is q_2 which we can control, let's rearrange terms:

$$c^4q_2^4 - 4k^2c^2q_2^2 - 4k^3cq_2 > k^4.$$

The left side of this inequality is a fourth degree polynomial in q_2. Since the leading coefficient c^4 is positive, the left side approaches $+\infty$ as q_2 approaches

$+\infty$. And so the left side must be greater than k^4 for q_2 large enough.

Actually, you can easily check that if $q_2 = 3k/c$, then the left side equals $33k^4$, which is certainly greater than k^4.

EXERCISES

1. Starting with Equation (7):

   (a) compute $f'(t)$ and $f''(t)$;

   (b) show that $f'(t) < 0$ for all t;

   (c) show that $f''(t) > 0$ for all t;

   (d) show that $f(0) = M + P$;

   (e) show that $\lim_{t \to \infty} f(t) = P$.

2. Find the time t between t^- and t^+ on Figure 3 at which the second light is perceived to exceed the first light in brightness by the greatest amount.

3. In Equation (2), find the value of x for which we have equilibrium, and show that this is consistent with the remarks in Section 5.2 about the equilibrium position.

4. Show that if

$$R = \frac{k + 2cq_2}{cq_2} = \frac{k}{cq_2} + 2,$$

   then

$$k + cRq_2 = 2(k + cq_2).$$

5. Show that the right side of Equation (11) is positive, assuming it is real.

6. Show that the right side of Equation (11) is less than one, assuming it is real.

7. Show that if $q_1 = Rq_2$, where

$$R = \frac{k + 2cq_2}{cq_2} = \frac{k}{cq_2} + 2,$$

then

$$N_1 = 2N_2.$$

International Journal for Physical, Biological and Human Geosciences and their Application in Environmental Planning and Ecology

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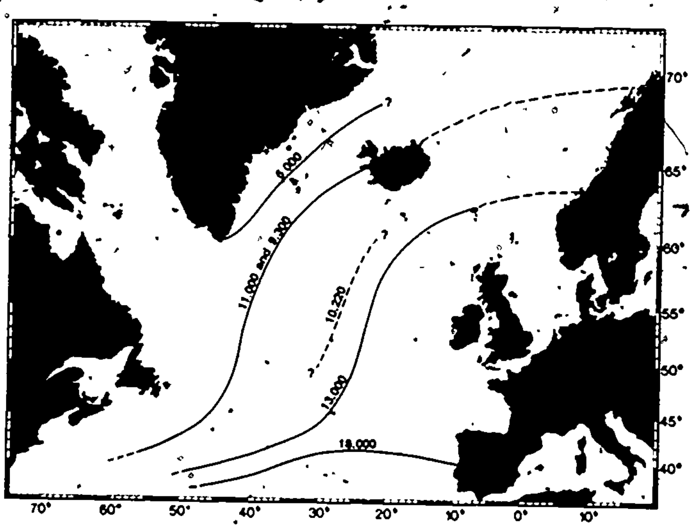

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ANSWERS TO EXERCISES

1. (a) $f'(t) = -MNe^{-Nt}$; $f''(t) = MN^2e^{-Nt}$.

   (b) $e^z > 0$ for all real z, $M > 0$, and $N > 0$.

   (c) same as (b).

   (d) $f(0) = Me^0 + P = M + P$.

   (e) $\lim_{t\to\infty} (Me^{-Nt} + P) = M(\lim_{t\to\infty} e^{-Nt}) + P = 0$

   since $e^{-Nt} = \frac{1}{e^{Nt}} \to 0$.

2. $t = \dfrac{\log M_1N_1 - \log M_2N_2}{N_1-N_2}$.

   (Maximize the function $g(t) = f_2(t) - f_1(t)$ by setting $g'(t) = 0$ and solving for t.)

3. $x = \dfrac{k}{k + cq}$. (Set $x' = 0$ and solve for x.)

   For this x, the rate of activation would be

   $$cqx = \frac{cqk}{k + cq} = P,$$

   as stated in Section 5.2.

4. $$\frac{k + cRq_2}{k + cq_2} = \frac{k + cq_2\left(\frac{k + 2cq_2}{cq_2}\right)}{k + cq_2} = \frac{k + (k + 2cq_2)}{k + cq_2} = 2.$$

5. The solution with the plus sign is certainly positive. The other will be positive if

   $$M_2 - \sqrt{M_2^2 - 4M_1(P_1 - P_2)} > 0,$$

   $$M_2 > \sqrt{M_2^2 - 4M_1(P_1 - P_2)},$$

   $$M_2^2 > M_2^2 - 4M_1(P_1 - P_2).$$

For this to be true we require $P_1 - P_2 > 0$. To see that this is so, we write

$$P_1-P_2 = \frac{cq_1k}{k+cq_1} - \frac{cq_2k}{k+cq_2};$$

$$= \frac{k^2cq_1+c^2q_1q_2k-k^2cq_2-c^2q_1q_2k}{(k+cq_1)(k+cq_2)}$$

$$= \frac{k^2c}{(k+cq_1)(k+cq_2)} \cdot (q_1-q_2),$$

which is positive since $q_1 > q_2$.

6. We must show

$$M_2 + \sqrt{M_2^2 - 4M_1(P_1-P_2)} < 2M_1,$$

$$\sqrt{M_2^2 - 4M_1(P_1-P_2)} < 2M_1 - M_2,$$

$$M_2^2 - 4M_1(P_1-P_2) < 4M_1^2 - 4M_1M_2 + M_2^2,$$

$$P_1 - P_2 > M_2 - M_1,$$

$$M_1 + P_1 > M_2 + P_2.$$

To show this, note that

$$M_1 + P_1 = \frac{c^2q_1^2+cq_1k}{k+cq_1} = cq_1,$$

and similarly that

$$M_2 + P_2 = cq_2.$$

Remember that $q_1 > q_2$.

7. $\frac{N_1}{N_2} = \frac{k+cq_1}{k+cq_2} = \frac{k+cRq_2}{k+cq_2} = 2$ (using Exercise 4).

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