DOCUMENT RESUME

ED 214 785 SE 036 476

AUTHOR	Cannon, Raymond J.
TITLE	UMAP Modules-Units 84-88.
INSTITUTION	Education Development Center, Inc., Newton, Mass.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	78
GRANT	SED76-19615
NOTE	139p.; Contains occasional light type.
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	*College Mathematics; Functions (Mathematics); Higher Education; Instructional Materials; *Learning Modules; *Mathematical Applications; Mathematical Concepts; *Problem Solving; Secondary Education; Secondary School Mathematics; Supplementary Reading Materials; Textbooks; Undergraduate Study
IDENTIFIERS	Exponentiation (Mathematics); Heuristics

ABSTRACT

This document is designed to help the user recognize problems which can be solved by use of the exponential function, to show a wide variety of such problems, and to teach how to actually solve them. The material is divided into five individual units, numbered and labeled as follows: 84-Recognition of Problems Solved by Exponential Functions; 85-Exponential Growth and Decay; 86-Development of the Function y equals the quantity A times e to the power of C times x; 87-Numerical Approximations to y equals e to the x; and 88-How to Solve Problems Involving Exponential Functions. Each unit includes a series of exercises, an answer key, a model exam, and answers to the exam. (MP)

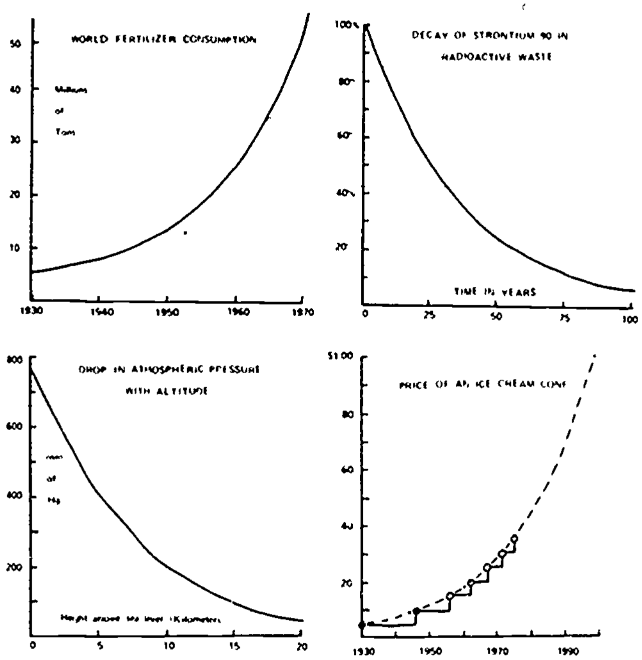
umap

UNIT 84

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

RECOGNITION OF PROBLEMS SOLVED BY EXPONENTIAL FUNCTIONS

by Raymond J. Cannon



INTRODUCTION TO EXPONENTIAL FUNCTIONS

UNITS 84-88

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RECOGNITION OF PROBLEMS SOLVED BY EXPONENTIAL FUNCTIONS

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9/6/77

TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 84*

Title: RECOGNITION OF PROBLEMS SOLVED BY EXPONENTIAL FUNCTIONS

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Review Stage/Date: IV 9/6/77

Classification: EXPN FNCTN/RECOG EXPN PROBS (U 84)

Suggested Support Material:

References:

Riggs, D. S. The Mathematical Approach to Physiological Problems. The MIT Press, Cambridge, Massachusetts, 1963. Chapter 6.

Simmons, G. F. Differential Equations with Applications and Historical Notes. McGraw-Hill Book Company, New York, 1972. Chapter 1.

Prerequisite Skills:
1. Know that the derivative is the instantaneous rate of change of the function.
2. Be able to draw a line tangent to a curve.
3. Know that the derivative is slope of the tangent.

Output Skills:
1. Name the function that satisfies the equation $y' = ky$.
2. Given a small portion of a graph, be able to decide whether the graph satisfies $y' = ky$.
3. Given a word problem, be able to decide whether the solution to the problem involves the exponential function.

Other Related Units:
Exponential Growth and Decay *(Unit 85)*
Recognizing Exponential Functions *(Units 63, 64, 65)*
Development of Function $y = Ae^{cx}$ *(Unit 86)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

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The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is one of many projects of Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank Lois Heflin, Ruth Hailperin and William T. Fishback for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED 76-19615. Recommendations expressed are those of the authors and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

Text: UNIT 84

RECOGNITION OF PROBLEMS SOLVED BY EXPONENTIAL FUNCTIONS

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1. INTRODUCTION

The purpose of this unit is to help you recognize problems which can be solved by use of the exponential function, and to show you some of the wide variety of such problems. You will learn how to actually solve such problems in Units 85-88.

2. THE EQUATION $y' = ky$

Problems whose solutions involve the exponential function all have one thing in common: the rate of change of the quantity being measured is proportional to the quantity itself. Knowing the mathematical language will enable us to develop the mathematical machinery needed to solve these problems. Let y stand for the quantity being measured. You are familiar with the concept of rate of change expressed in mathematical terms; rate of change is given by the derivative, in this case written y'. The phrase "proportional to" means "is a constant multiple of." We let k stand for the constant.

(The rate of change of the quantity being measured)	is proportional to	(The quantity being measured)
y'	$= k$	y

With this translation, we can say now that each problem whose solution involves the exponential function satisfies the mathematical equation $y' = ky$.

3. GRAPHICAL PROBLEMS

3.1 Geometric Meaning of $y' = ky$

If you are given the graph of a function of x, you can use a straight edge to draw lines tangent to the graph at various points and then find the slope of the tangent line as you would for any straight line (the change in y divided by the change in x). Now recall the geometric significance of the derivative: it is the slope of the tangent line. Thus, the slope of the tangent line is also the value of y'.

If the curve is the graph of an exponential function, then it satisfies the equation $y' = ky$, or if $y \neq 0$, $y'/y = k$. We may use the method above to check whether or not a given curve is the graph of an exponential function by evaluating y'/y at various places.

3.2 Example of a Graph that Satisfies $y' = ky$

(See next page.)

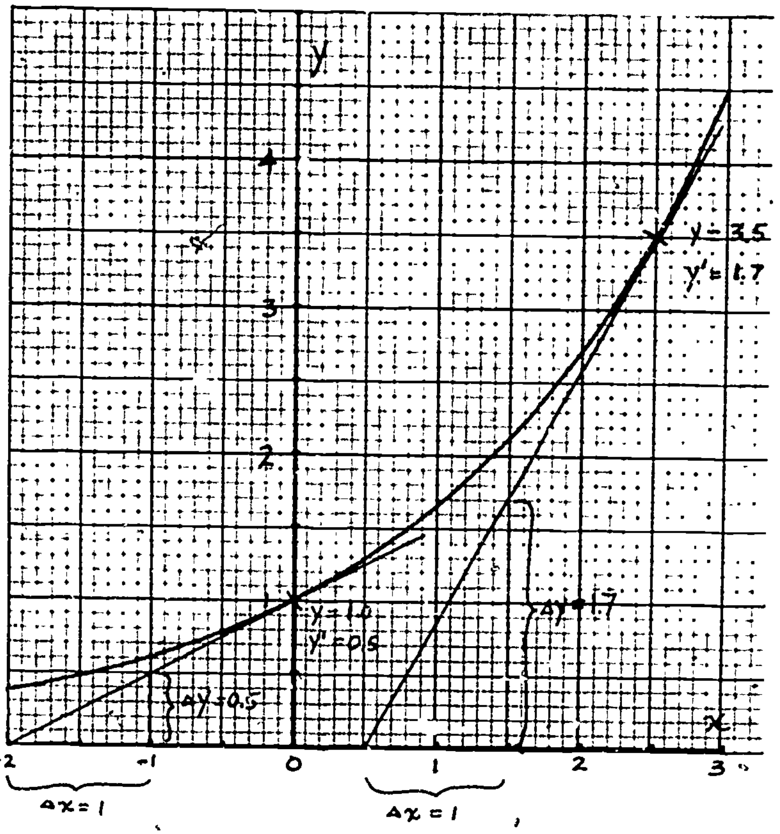

Figure 1. Example of a graph that satisfies y' = ky.

We have drawn a line tangent to the graph at (2.5, 3.5) and computed its slope by extending the tangent line down to the line where y = 0; from this point we move one unit to the right, and then go back up to the tangent line. The point where we meet the line again has 1.7 as its y-coordinate. Since we started with y = 0, our change in y is 1.7, which remains unchanged when divided by our change in x which is 1; the slope of this tangent line is 1.7. The quotient y'/y is 1.7/3.5 or approximately .5.

The curve also goes through the point (0, 1), and the tangent line has been drawn at this point. The slope of this line is approximately .5 which gives us the value of y'; the value of y at the point of tangency is 1 and y'/y is .5/1; or .5. You probably now suspect that this graph satisfies the equation y' = .5y. You may wish to confirm this suspicion by trying the same process at some other point on the graph. Do not be upset if your calculations give y'/y as approximately .55 or .45; these are only approximations. We will assume the curve is exponential if the values of k agree to within ten percent of each other. You should not get numbers like 2.3 or .1 as approximations. You also need not feel you have to extend the tangent line down to the x-axis to compute the slope. But it is easier if you always let the change in x be 1 or 2 or 10, etc., because the slope is then easy to figure from the change in y.

3.3 Graphical Exercises

Choose at least three points on each of the following graphs and draw the tangent line at each of these points. Use the slope of these lines to estimate y' and fill in the table accompanying each graph.

1.

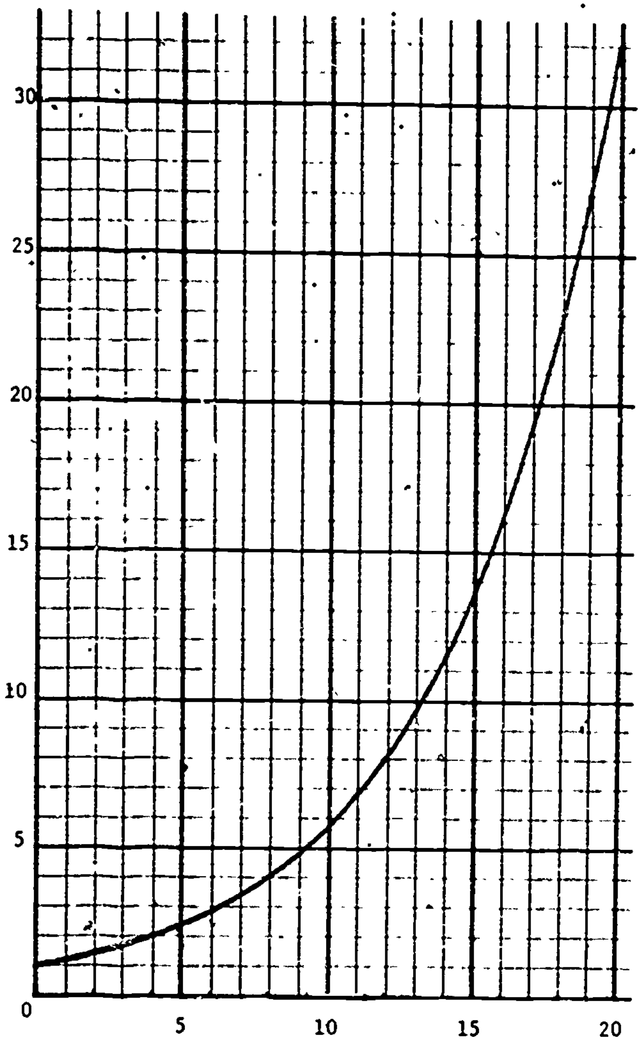

Figure 2. Is this the graph of an exponential function?

TABLE I

y				
y'				
y'/y				

a) Are the numbers in the last row approximately the same?

b) Is Figure 2 the graph of an exponential function?

2.

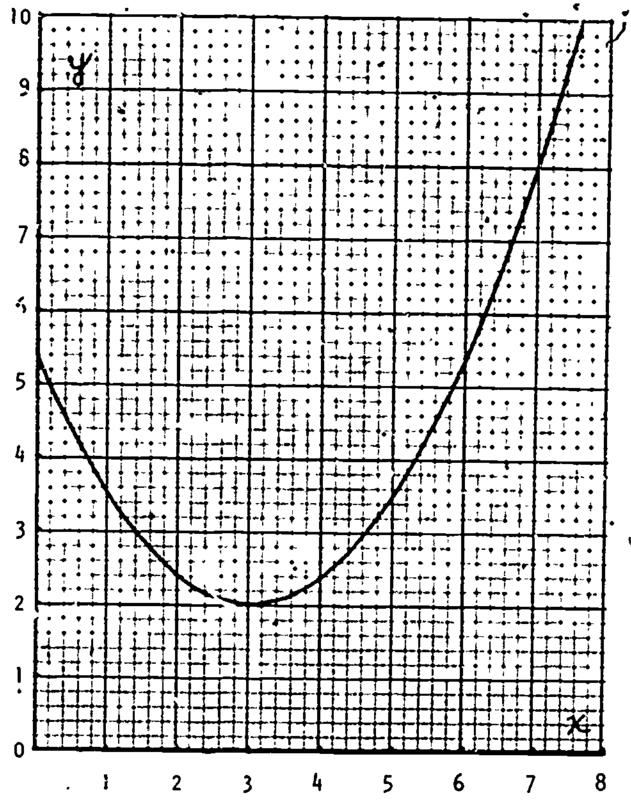

Figure 3. Is this the graph of an exponential function?

TABLE II

y				
y'				
y'/y				

a) Are the numbers in the last row approximately the same?

b) Is Figure 3 the graph of an exponential function?

3.

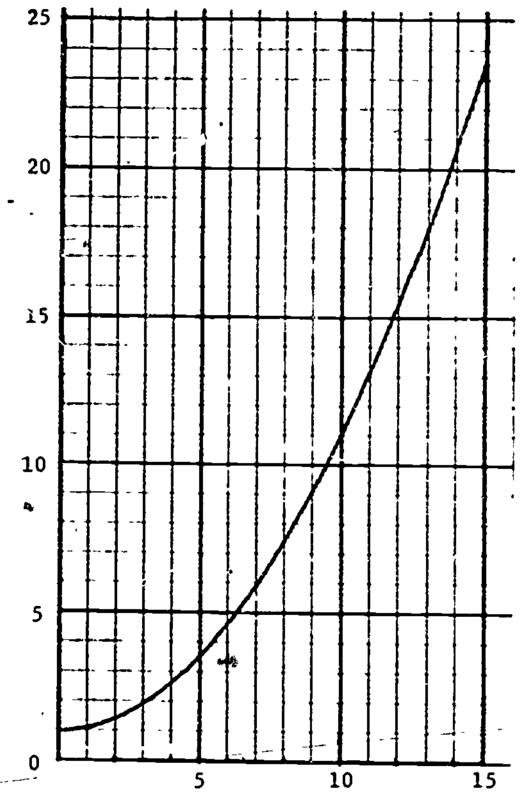

Figure 4. Is this the graph of an exponential function?

TABLE III

y				
y'				
y'/y				

a) Are the numbers in the last row approximately the same?

b) Is Figure 4 the graph of an exponential function?

4.

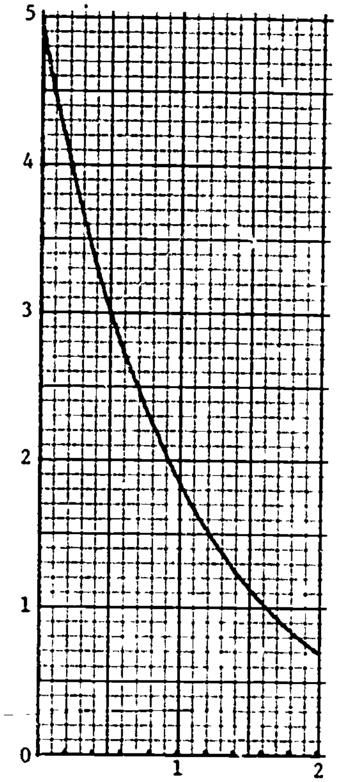

Figure 5. Is this the graph of an exponential function?

TABLE IV

y				
y'				
y'/y				

a) Are the numbers in the last row approximately the same?

b) Is this a graph of an exponential function?

4. WORD PROBLEMS

4.1 Some Key Phrases

People often have difficulty with "word" or "story" problems because they do not know how to get started. This section is to help you get started on some word problems by being able to recognize key words or phrases which indicate that the solution of the problem will involve the exponential function. We saw the general phrasing of such problems in Section 2 of this unit, and now we look at some specific instances.

Sometimes you are lucky enough to see a phrase that is very close to the general phrase given in Section 2, such as "the rate of change of the cost is proportional to the cost." Sometimes the phrase is slightly disguised, as in "the acceleration is proportional to the velocity"; here you must recognize that acceleration is the rate of change of the velocity. Another way of saying the rate of change is proportional to the amount is to express this proportion as a percentage, as in "increasing at the rate of six percent per year." This phrase would translate to the mathematical equation $y' = .06y$. It is important to distinguish between a growth rate that is constant, say $6 per year and, as we have here, a growth rate that is a constant percentage, six percent per year. In other cases, the proportionate rate of growth is not given as a yearly rate, but in terms of how long it takes the quantity to increase or decrease by a given factor. You will recognize this in phrases such as "doubles every eight years", "increases by a factor of three every two years", "has a half-life of thirty minutes", or "decreases by a factor of 1/5 every twenty years." The concepts of half-life and doubling-period will be dealt with in greater detail in Unit 85. For now, try to develop recognition

of such phrases; your computational skill will be developed later.

4.2 Example of Word Problems that Involve $y' = ky$

Example 1

The Navy is testing a new torpedo, and launches one from a submarine. Because it contains enough air to offset the weight of the metal, it is weightless in the water and therefore stays at a constant depth. The torpedo misses its target and heads toward open ocean. Two miles from the launch its motor stops. At this instant it is traveling 80 miles per hour, but the water resistance slows the torpedo at a rate proportional to its velocity. Three miles from launch it is traveling at 40 miles per hour. Since the torpedo is a threat to navigation, it must be recovered. It can be picked out of the water if it is going less than one mile per hour. How soon can the Navy recover the torpedo, and how far down range should they go to make the recovery?

Discussion of Example 1

One of the difficulties of word problems is that there tends to be more information given than is really necessary to solve the problem. Another is that the information needed to completely solve the problem is not needed to begin the solution. The first task in solving this problem is to write the correct equation governing the change in velocity of the torpedo. This equation is given by the "water resistance slows the torpedo at a rate proportional to its velocity." This contains one of our key phrases, and if v is the velocity, we have $v' = kv$. The solution of this problem involves the exponential function.

4.3 Another Word Problem

Example 2

A fossil is found in a cave, and taken to a laboratory to be analyzed. It is found to emit about seven rays from carbon-14 per gram per hour. A living body radiates at a rate of 918 rays per gram, per hour, and radioactive carbon-14 has a half-life of about 5,600 years. Approximately how old is the fossil?

Discussion of Example 2

The word half-life was given earlier as a term that indicated a rate of change proportional to the amount of a quantity present. The term half-life can be expressed as a percentage rate of change. In this case the percentage is given as 50 percent per 5,600 years, instead of an annual rate of percentage. If we let $C(t)$ be amount of carbon-14 at time t, then the solution to the problem involves the solution to the equation $C'(t) = kC(t)$. Again, how to determine k, and how to solve this equation will be discussed in Unit 85.

4.4 Exercises

A question is posed in each problem to make the problem seem more like those you will encounter later; you are not expected to be able to solve these problems now. You are expected only to answer questions (a) and (b) following each problem statement.

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5. The Security Union Bank advertises that it pays five percent interest on saving accounts, and the interest is compounded continuously. If you opened a \$1,000 savings account with this bank today, how much money would be in the account a year from today if you make no withdrawals or deposits?

   a) Is the rate at which your account is growing a constant percent?

   b) If $a(t)$ is the amount in your account at time t, does $a(t)$ satisfy the equation $a'(t) = ka(t)$ for some constant k?

6. Roger wants to go scuba diving for lobsters. He must be able to dive to a depth of 100 feet. It is a cloudy day, and on his light

meter at home, Roger notes that the amount of light at the surface of the water is 400. Roger does not have an underwater lamp and is dependent on natural light. He knows from previous experience that every 20 feet of water will reduce the amount of light by one-half. Roger needs a reading of 40 on the bottom to see the lobsters. Can Roger plan on lobster for dinner, or should he defrost the hamburger?

   a) If $L(d)$ represents the amount of light at depth d, is $L'(d)/L(d)$ a constant?

   b) Does the solution to this problem involve the exponential function?

7. A rock is dropped from a cliff towards the ocean. The velocity at time t is proportional to t, and the constant of proportionality is 32. If the cliff is 50 feet above the ocean, when does the rock hit the water?

   a) If $p(t)$ represents the distance the rock has fallen, does $p'(t) = kp(t)$?

   b) Does the solution to this problem involve the exponential function?

8. Anne has just spent \$5,000 for an automobile. She knows that it will depreciate at a constant rate of 17 percent per year. For how much will she be able to sell the car in six years?

   a) If $p(t)$ is the price for which she can sell the car in t years, is $p'(t)/p(t)$ a constant?

   b) Does the solution to this problem involve an exponential function?

9. A curve in the plane satisfies the following geometric condition: the slope of the line tangent to the curve at any point is three times the x-coordinate of that point. The curve passes through the point (1, 4). What is the y-coordinate of the point on the curve when $x = 2$?

   a) Does this curve satisfy $y' = ky$?

   b) Does the solution to this problem involve the exponential function?

10. A piece of pottery is taken out of a kiln where it has been baking. Its temperature when removed from the kiln is $2300^{\circ}F$, and it is placed in a room where the temperature is kept at $75^{\circ}F$. Newton's law of cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and the surrounding room. (Note that the rate of cooling is equal to the rate at which the difference in temperature between the body and the room decreases.) After one hour, the pottery's temperature is $2000^{\circ}F$. When will it be safe to touch the pottery with bare hands?

   a) Let F(t) be the difference between the temperature of pottery and the temperature of the room. Is $F'(t) = kF(t)$?

   b) Does the solution to this problem involve the exponential function?

11. A certain factory has been dumping its chemical wastes into a river which flows into a lake. The chemical wastes of the factory cause a rash on the skin when their concentration in the water is 30 parts per million; they irritate the eyes at a concentration of five parts per million. The factory stopped dumping its waste into the river a month ago, and the concentration in the lake was then at 75 parts per million. The clean water of the river entering the lake mixes with the polluted water of the lake; then, as the river flows out of the lake, some of the polluting materials are carried off. The flow of the river is constant; together with our mixing assumptions, this means that the rate at which the waste material is being carried off is proportional to the amount of waste in the lake. The chemical waste now in the lake amounts to 70 parts per million. How long will it be before people can swim in the water without getting a rash? Without their eyes burning?

   a) If c(t) is the amount of chemical waste at time t, is $c'(t)$ proportional to c(t)?

   b) Does the solution to this problem involve the exponential function?

12. A drug is injected into the bloodstream of a patient. Enough of the drug is given so that its concentration in the bloodstream is three times its effective level. The rate at which the drug is eliminated is proportional to the amount in the bloodstream. After ten minutes, a sample of the patient's blood shows that the level of the drug is 2.7 times its effective level. How long will the level of the drug remain above its effective level?

   a) If a(t) is the amount of drug in the bloodstream at time t, is $a'(t) = ka(t)$?

   b) Does the solution to this problem involve the exponential function?

5. ANSWERS TO EXERCISES

1. The entries in your first two rows will depend on the choices of y you made, but your entries in the last row should all be about 0.17.
   a) yes, b) yes

2. The entries in the last row should differ considerably. For instance $y' < 0$ when $y = 4$ at one point, and $y' > 0$ when $y = 4$ at another point.
   a) no, b) no

3. This is harder than No. 2, but again the values in the third row should differ. y' is near 0 when $y = 1$ and is near 2 when $y = 12$. This makes it more difficult to answer but the answers are
   a) no, b) no

4. Each entry in the last row should be about -1.0.
   a) yes, b) yes

5. a) yes, b) yes

6. a) yes, b) yes

7. a) no; $p'(t) = 32t$, not $32p(t)$, b) no

8. a) yes, b) yes

9. a) no; y' = 3x, not y' = 3y, b) no

10. a) yes, b) yes

11. a) yes, b) yes

12. a) yes, b) yes

6. MODEL EXAM

1. The Security Union Bank advertises that it pays five percent interest on saving accounts, and the interest is compounded continuously. If you opened a $1,000 savings account with this bank today, how much money would be in the account five years from today if you make no withdrawals or deposits?

   a) Does solution of this problem involve an exponential function?

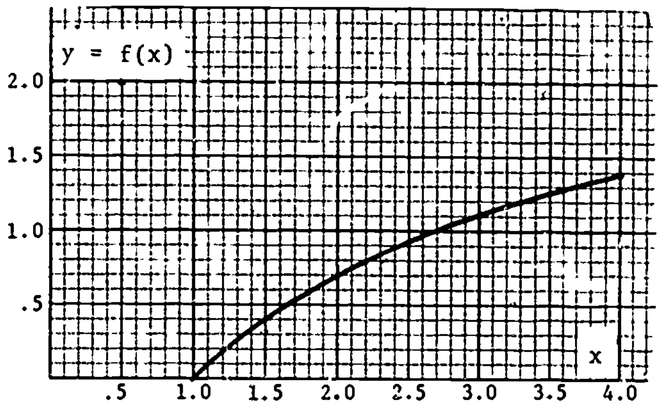
   b) Why or why not?

2. A jet is traveling at 1,300 miles per hour. The jet's engine burns out, and the plane is being slowed by the resistance of of the air. The deceleration rate is proportional to the square of the velocity of the airplane. After five minutes, the plane is traveling at 1,100 miles per hour. How long is it before the plane is traveling at 800 miles per hour?

   a) If v(t) represents velocity of airplane at time t, is $v'(t) = kv(t)$?

   b) Does solution of this problem involve an exponential function? Why or why not?

3.



Fill in the following table for various values of y.

y			
y'			
y'/y			

Is this the graph of an exponential function? Why or why not?

7. ANSWERS TO MODEL EXAM

1. Yes, because the bank adds to the account at a rate proportional to the total amount in the account at any time, i.e. it adds at the rate of five percent of the account balance per year.

2. a) No, the deceleration is proportional to the square of the velocity. The rate of change equation should read

   $$v'(t) = k[v(t)]^2$$

   b) No, because the rate of change of velocity is not proportional simply to the velocity but to the square of the velocity.

3. No, because y'/y is not approximately the same for all points.

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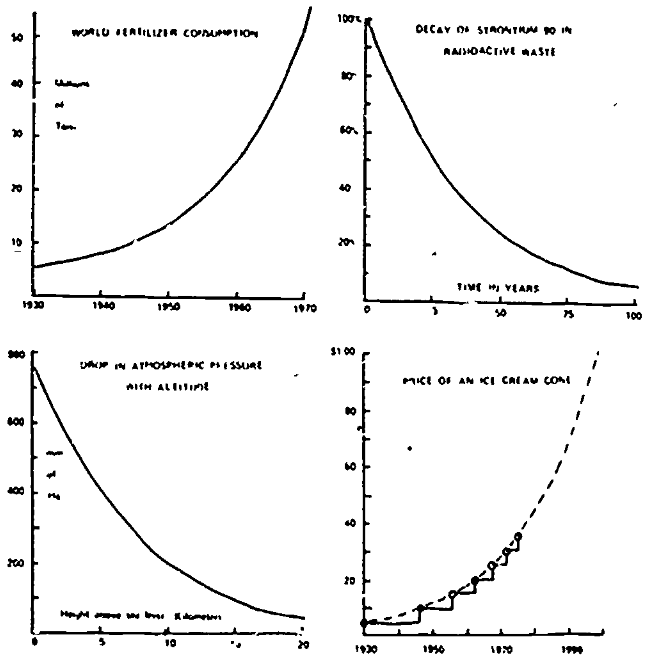
UNIT 85

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

EXPONENTIAL GROWTH AND DECAY

by Raymond J. Cannon



INTRODUCTION TO EXPONENTIAL FUNCTIONS

UNITS 84-88

edc/umap/55chapel st./newton, mass. 02160

EXPONENTIAL GROWTH AND DECAY

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9/6/77

TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 85*

Title: EXPONENTIAL GROWTH AND DECAY

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Review Stage/Date: IV 9/6/77;

Classification: EXPN FNCTN/GROWTH & DECAY (U85)

Suggested Support Material: Pocket Calculator

Prerequisite Skills:
1. Be able to use rules of exponents.
2. Be able to read graphs and tables of data.

Output Skills:
1. Be able to describe in words the meaning of each letter in the formulas $A(t) = A_0 2^{t/k}$ and $A(t) = A_0(\frac{1}{2})^{t/k}$.
2. Given a table of data or a graph showing the doubling period (half-life) of a quantity, be able to:
   a) State the length of the doubling period or half-life.
   b) Use the concept of half-life or doubling period to fill in missing points.
   c) Recognize a given function as an exponential by its constant doubling period or half-life.
   d) Write a formula that fits the data using an exponential function to the base 2 or to the base $\frac{1}{2}$.
   e) Use the formula to calculate intermediate points given by any rational exponent.

Other Related Units:
Recognition of Problems Solved by Exponential Functions *(Unit 84)*
Development of the Function $y = Ae^{cx}$ *(Unit 86)*
$e^x \rightarrow \ln x$ *(Unit 86B)*
Numerical Approximations to $y = e^x$ *(Unit 87)*
How to Solve Problems Involving Exponential Functions *(Unit 88)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
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The Project would like to thank Sister Eileen R. Benton, Lois Heflin, and Marjorie Fitting for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615. Recommendations expressed are those of the authors and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

EXPONENTIAL GROWTH AND DECAY

by

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9/6/77

1. INTRODUCTION

The words half-life and doubling period were emphasized in Unit 84 as key words (or concepts) in recognizing problems solved by exponential functions. This unit takes a closer look at these concepts, and develops formulas that express the amount of the quantity as a function of time.

2. REVIEW OF EXPONENTS

Since we will have need of the expression $2^{t/k}$, we want to review what the exponent means when t and k are integers; what it means when t and k are not integers will be dealt with later. We will use the letter b to represent an arbitrary positive number. In the expression $b^{t/k}$, b is called the *base*, and t/k, the *exponent*.

If t is a positive integer, then b^t is b multiplied by itself t times. For example, $2^3 = (2)(2)(2) = 8$. What is 2^5? ________ = ________.

If t is a negative integer, then $b^{-t} = (\frac{1}{b})^t = 1/b^t$. Thus, $2^{-5} = (\frac{1}{2})^5 = (\frac{1}{2})(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{32}$. What is 2^{-3}? 2^{-3} = ________ = ________ = ________.

If k is a positive integer, then $b^{1/k}$ is the number a such that $a^k = b$, and $b^{1/k}$ is called the k^{th} root of b. Thus, $8^{1/3} = 2$ since $2^3 = 8$. What is $32^{1/5}$? $32^{1/5}$ = _____, since $2^5 = 32$. What is $64^{1/3}$? _____. What is $64^{2/3}$? _____.

3. POPULATION GROWTH AND DOUBLING TIMES

3.1 Computation of a Doubling Period Given Annual Percentage Growth

Let us suppose that country A has population that is growing at the rate of three percent per year. Do you think of this as rapid growth or slow growth? _____. If the country had 10,000,000 people in 1975, how long will it take the population to double and reach the 20,000,000 level? Take what you think is a reasonable guess. _____. We can work out the answer and see how good your guess was.

That the population grows at a rate of three percent per year means the population in a given year is 1.03 times the population of the previous year. Thus, the population in 1976 will be 10,300,000 = (1.03)(10,000,000). The population in 1977 will be (1.03)(10,300,000) = 10,609,000. To compare this to the 1975 population, write 10,609,000 as (1.03)(1.03)(10,000,000) which is $(1.03)^2(10,000,000)$. The population in 1978 will be $(1.03)(10,609,000) = (1.03)(1.03)^2(10,000,000) = (1.03)^3(10,000,000)$.

This pattern is made clearer when we look at the data in the form of a table as in Table I.

Exercise

1. Fill in Table I. Use a calculator to do the multiplications.

(See next page.)

TABLE I

Year	Population
1975	$(1.03)^0$ (10,000,000) = 10,000,000
1976	$(1.03)^1$ (10,000,000) = 10,300,000
1977	$(1.03)^2$ (10,000,000) = 10,609,000
1978	$(1.03)^3$ (10,000,000) = 10,927,270
1979	$(1.03)^{\square}$(10,000,000) =
1980	$(1.03)^{\square}$(10,000,000) =
1990	$(1.03)^{\square}$(10,000,000) =
2000	$(1.03)^{\square}$(10,000,000) =

Careful, watch the jump.

How close was your guess to the value 20,937,779? In 25 years, the population has more than doubled! In fact, the population doubled in less than 23 $\frac{1}{2}$ years. (How to compute the actual doubling time will be shown in a later unit.) What will the population be in the year 2025? To answer this question, we want to compute $(1.03)^{50} \cdot (10,000,000)$. As a short cut in this computation, note that $(1.03)^{25}(10,000,000) \approx 20,937,779$ means that $(1.03)^{25} \approx 2.0937779$. This means $(1.03)^{50} = [(1.03^{25(2)}] = [(1.03)^{25}]^2 \approx (2.0937779)^2$.

In 25 years from 1975 to 2000 the population increased by a factor of 2.0937779, and in the next 25 years from 2000 to 2025, it increases by the same factor. We may express the population in 2025 in the following equivalent ways: $(1.03)^{50}(10,000,000) = (2.0937779)^2(10,000,000) = (2.0937779)(20,937,779) = 43,839,059$.

Exercises

2. Find the doubling period of each quantity from the following tables.

TABLE II
GROWTH OF BACTERIA

Time	Number of Bacteria
9 a.m.	1,400
10 a.m.	1,764
11 a.m.	2,222
12 noon	2,800
1 p.m.	3,528
2 p.m.	4,444
3 p.m.	5,600

a) Doubling period is ______________.

TABLE III
INCREASING COST OF AUTOMOBILE

Year	Cost of Car (dollars)
1971	2,500
1974	3,149
1977	3,967
1980	5,000
1983	6,298
1986	7,934

b) Doubling period is ______________.

3. Use the estimate that the population doubles every 24 years in country A to complete Table IV.

TABLE IV

Year	Population
1975	10,000,000
1999	20,000,000
2023	
2047	

Using the same doubling period, we see that the population must have gone from $\frac{1}{2}$(10,000,000) to 10,000,000 in 24 years. Thus, the population in 1951 was ______________. In 1927 the population was ______________. If we know the doubling period, we can say what did happen, as well as what will happen.

3.2 Computation of Doubling Period from Graph of an Exponential Function

The graph of an exponential growth function is given in Figure 1 (page 5). Notice the value of the function is 8 when x = 6. To find the doubling period, we must see what x is when y = 16; when the graph crosses the line

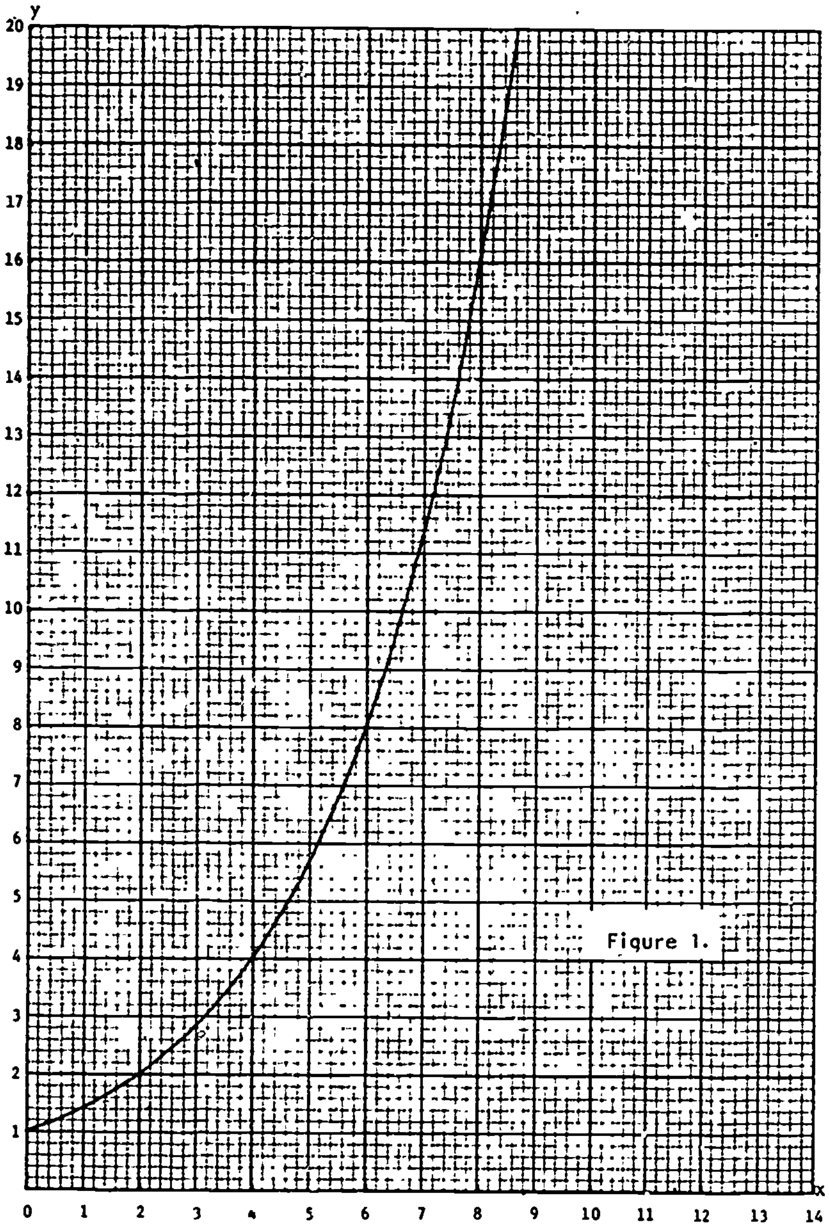


Figure 1.

y = 16, the value of x is 8. The value of y doubled when x increased by 2. Similarly, when y = 6, we see that x is about 5.1, and so we expect that when y = 12, x will be 5.1 + 2 = 7.1. Verify that this is so.

Notice that if x increases by 2, the value of y doubles; if x decreases by 2, the value of y halves. We can use this information to give certain values for the function that are not plotted. For instance, the graph goes through the point (1, 1.4) and if we were to extend to negative numbers, we would see it goes through (-1, .7) since -1 = 1 - 2 and $.7 = \frac{1}{2}(1.4)$.

Exercise

4. Fill in the blanks without looking at the graph. Then look at the graph in Figure 1 to verify answers when possible.

   a) The graph passes through (4.6, 5) and (___, 10).

   b) The graph passes through (1, 1.4) and (3, ___).

   c) The graph passes through (8, 16), (10, ___) and (12, ___).

   d) The graph passes through (0, 1) and (-2, ___).

   e) The graph passes through (-1, .7) and (___, .35).

---

4. FORMULA FOR EXPONENTIAL GROWTH

4.1 Formula for Growth of a Population

We want to find a formula for the population of country A so we will know what it is in an arbitrary year. Remember, we are assuming the population doubles every 24 years. Use this assumption and 1975 as our starting point in time to fill in Table V.

TABLE V

A Year	B Year Minus 1975	C Population
1927		2,500,000
1951	-24	
1975	0	10,000,000
1999	24	20,000,000
2023	48	
2047		80,000,000

Let t represent the entry in column B and verify that the corresponding entry in column C is $2^{t/24}(10,000,000)$ by completing Table VI.

TABLE VI

Year	t	$2^{(t/24)}$	$2^{(t/24)}(10,000,000)$
1927	-48	$2^{-48/24} = 2^{-2} = \frac{1}{4}$	
1951		$= 2^{-1} =$	
1975	0	$= 2^0 = 1$	
1999			
2023			
2047	72	$2^{72/24} = 2^3$	

Is the population in each year as computed by you in Table VI the same as the population given in Table V? _____ It should be.

The last column in Table VI gives us our formula for the population in any year:

$$P = (10,000,000)2^{t/24}$$

where t is the number of years measured from 1975.

Assuming the population doubles every 24 years, we can say the population in 1995 will be $(10,000,000)2^{20/24}$ and that the population in 1960 was $(10,000,000)2^{-15/24}$.

4.2 Formula for Growth of a Bank Account

Tom's father put $100 in the bank for Tom. The bank paid interest on the savings account so that the amount doubled every 12 years. In 1960 Tom had $400. When did Tom's father start the account? You can answer this question without a formula such as the one we worked out in Section 4.1, but let's use a formula anyway. Then we can use it to answer a question about the amount at a time when $\frac{t}{k}$ is not an integer.

Since we know how much was in the bank in 1960, we use 1960 as our starting point. The doubling period is 12, so we have k = 12 and the amount in the bank is given by $400(2^{t/12})$. We want the value of t so that

$$(400)(2^{t/12}) = 100.$$

That means $2^{t/12} = \frac{1}{4}$. Now, $\frac{1}{4} = (\frac{1}{2})(\frac{1}{2}) = 2^{-2}$. Thus, we have $\frac{t}{12} = -2$, so t = -24. The account had $100 in (1960 - 24) or in 1936.

When did the account have $200 in it? _____

But how much did Tom have in 1966? There are several ways to find out. If we continue to use 1960 as our starting date, we want to compute

$$(400)(2^{(66-60)/12}) = (400)(2^{1/2}) =$$

$$(400)(1.4142) = \$565.68.$$

We could have used the actual starting date of the account, 1936, and computed $(100)(2^{(66-36)/12}) = (100)(2^{5/2})$. Use the fact there was $200 in 1948 to do the computation.

From these calculations we see that the amount can be expressed as $A_o 2^{t/12}$ where A_o is the amount in a given year, and t is computed starting at that year. We call the year we start with *initial time*, and A_o is the *initial amount*.

Exercises

In these exercises use $2^{1/2} \approx 1.4142$.

5. Compute the amount in Tom's bank account by using a different initial time and amount. Fill in Table VII by using 1972 as the initial time. You have already computed A_o to be $800.

TABLE VII

Year	t	$A_o 2^{t/12}$
1954		
1960		
1972		
1978		
1984		

6. A utility company has discovered that the use of electricity in Central City is doubling every eight years. Fill in Table VIII.

(See next page.)

TABLE VIII

Year	Kilowatt Hours
1962	
1966	
1970	2,000,000
1974	
1978	
1982	

Can you compute the usage in 1971?

Hint: $2^{1/8} = \sqrt{\sqrt{\sqrt{2}}} \approx 1.0905$.

Usage in 1971 was about ________ kilowatt hours.

7. The graph of a function was given by Figure 1 in Section 3.2. We saw that the doubling period of that function is 2 and that the graph passes through (6, 8). If we let our initial time equal 6, initial amount equal 8, and the doubling period k equal 2, we have a formula for the function: $y = 8(2)^{(x-6)/2}$.

   a) Use the fact that the graph goes through (4, 4) to obtain another formula for the function. ________

   b) Use the fact that the graph goes through (0, 1) to obtain another formula for the function. ________

---

5. HALF-LIVES

5.1 Half-Life of the Charge in a Capacitor

Some quantities, instead of increasing exponentially, decrease exponentially, and instead of a doubling period we speak of a halving-period or more commonly a *half-life*.

Peter walks in on Fred who is conducting an experiment in the Physics Lab. Fred has charged a capacitor with a

90 volt battery, and then made a circuit, so the voltage in the capacitor is dropping. Peter is an observant person, and notices the needle on the voltmeter is slowing down as the voltage gets smaller. He decides to make the following table, which he starts with the voltmeter reading 50 volts.

Seconds Elapsed	Voltage
0	50
59	45
126	40
201	35
288	30
390	25
516	20
678	15
	10
	5

But after Peter had filled in the table this far, the needle was dropping very slowly, and Peter wanted to go to supper. He said, "Gosh, Fred, you don't have to sit here and watch that thing any more, I can tell you how to fill in the table."

Fred did not believe him, and challenged him to predict the entries. Can you do it?

Seconds Elapsed	Voltage
	10
	5

Peter had written

Seconds Elapsed	Voltage
906	10
1,296	5

Fred asked Peter how he was able to predict, and Peter told him, "Look, Fred, it dropped from 50 to 25 in 390 seconds, from 40 to 20 in 390 seconds, from 30 to 15 in 390 seconds, so it will drop from 20 to 10 in 390 seconds, and from 10 to 5 in 390 seconds. The charge has a half-life of 390 seconds." Fred was impressed, but not convinced. Peter said "Fred, if I can tell you how long you had been conducting this experiment before I came in here, will you believe me?"

Fred said he would. Peter gave Fred a time, and they both went to supper immediately. How many seconds had Fred been doing the experiment before Peter walked in? ________.

5.2 Half-Life of Radioactive Carbon

One of the most common uses of the term "half-life" is to describe the rate at which a radioactive element emits particles and thus changes into another element. Anthropologists use the amount of Carbon-14 found in fossils, together with the knowledge that the half-life of Carbon-14 is about 5,600 years, to estimate the age of the fossil.

Another radioactive isotope is Carbon-11, which decays into boron roughly at the rate of 3 1/2 percent per minute. Table IX contains two entries showing the amount of Carbon-11 in a given mass of material. Fill in the missing entries, and then answer the questions.

Exercise

8. TABLE IX

Time	Amount of C^{11}
3:00	
3:20	
3:40	24,000
4:00	12,000
4:20	
4:40	

What is the half-life of C^{11}? ________.

Can you tell how much there was at 3:50? ________.

To answer this last question, we use a formula like the one for growth; however, now we have a halving period and $A(t) = (24,000) \cdot (\frac{1}{2})^{t/20}$ where t is the number of minutes from 3:40. At 3:50 the amount is

$$(24,000)(\tfrac{1}{2})^{10/20} = (24,000)(\tfrac{1}{2})^{1/2} \approx (24,000)(.707) = 16,968.$$

5.3 Formula for Expressing Exponential Decay

When a quantity is doubling in a period of length k, we saw we could use $2^{t/k}$ to help express how much there is at arbitrary times. When a quantity is halving in a period of length k instead of doubling, we use $\frac{1}{2}$ instead of 2.

Using the data from Table IX, we can express the amount of C^{11} as $(24,000)\cdot(\frac{1}{2})^{t/20}$ where t is the number of minutes from 3·40.

The amount at 3:30 is $(24,000)(\frac{1}{2})^{-10/20} = (24,000)(\frac{1}{2})^{-1/2} = (24,000)(2^{1/2}) = (24,000)(1.414) = 33,936.$

Exercises

9. Use the data from Table IX to answer the following questions.

   a) How much C^{11} was there at 3:10? ________

   b) How much C^{11} was there at 4:10? ________

10. Use the graph in Figure 2 (see page 14) to answer these questions.

   a) The graph passes through the point (-3, ___) and (0, ___).

   b) If x increases by ___, then y decreases by a factor of $\frac{1}{2}$.

   c) The graph passes through (6, .25). If extended, it would pass through (9, ____).

   d) If extended in the other direction, it would pass through (-9, ___).

   e) Use the facts that the graph passes through (0, 5) and that k = 3 to give a formula for the function.

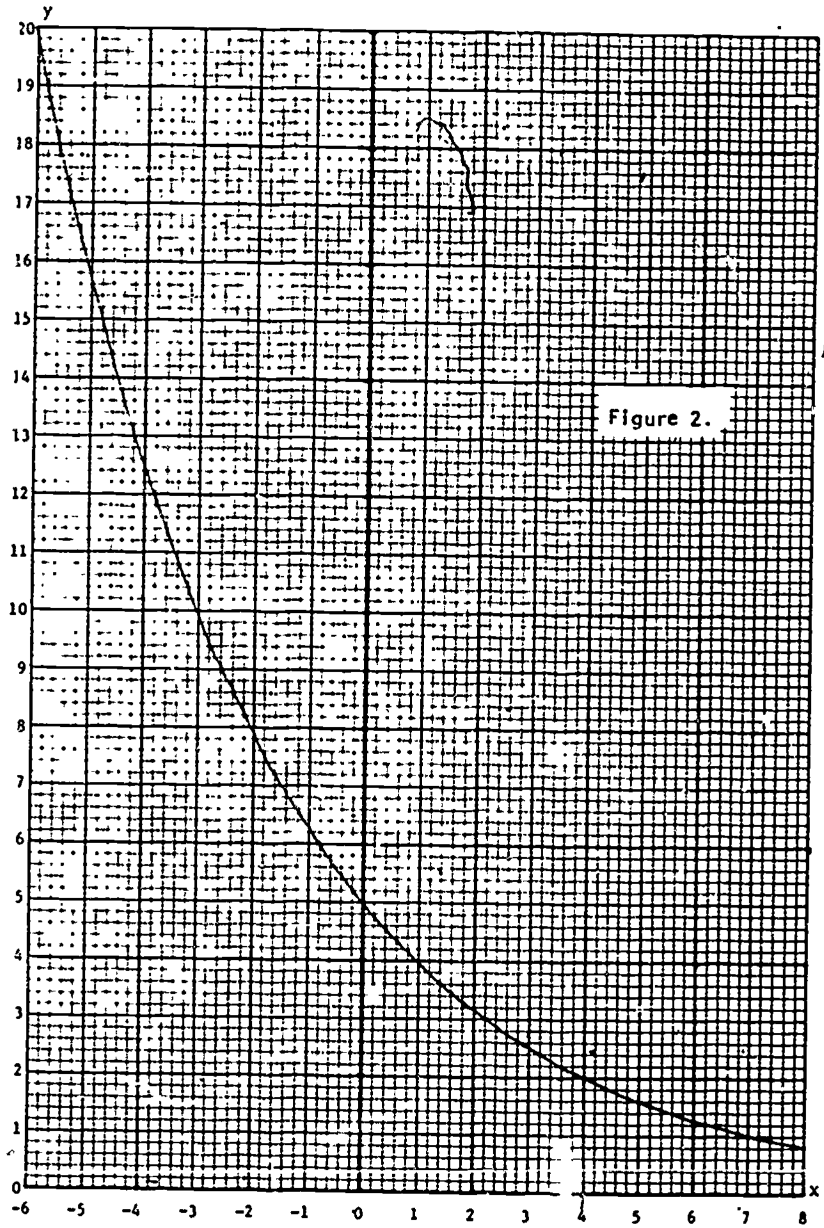


Figure 2.

6. ANSWERS TO EXERCISES

1.

TABLE I

Year	Population
1979	$(1.03)^4$ (10,000,000) = 11,255,088
1980	$(1.03)^5$ (10,000,000) = 11,592,740
1990	$(1.03)^{15}$(10,000,000) = 15,579,674
2000	$(1.03)^{25}$(10,000,000) = 20,937,779

2. a) Three hours, b) Nine years.

3.

TABLE IV

Year	Population
1975	10,000,000
1999	20,000,000
2023	40,000,000
2047	80,000,000

4. a) (6.6, 10), b) (3, 2.8), c) (10, 32) and (12, 64), d) (-2, .5), e) (-3, .35).

40

Section 4.1:

TABLE V

A Year	B Year Minus 1975	C Population
1927	-48	2,500,000
1951	-24	5,000,000
1975	0	10,000,000
1999	24	20,000,000
2023	48	40,000,000
2047	72	80,000,000

TABLE VI

Year	t	$2^{t/24}$	$2^{t/24}$(10,000,000)
1927	-48	$2^{-48/24} = 2^{-2} = \frac{1}{4}$	$\frac{1}{4}$(10,000,000)
1951	-24	$2^{-24/24} = 2^{-1} = \frac{1}{2}$	$\frac{1}{2}$(10,000,000)
1975	0	$2^{0/24} = 2^0 = 1$	1(10,000,000)
1999	24	$2^{24/24} = 2^1 = 2$	2(10,000,000)
2023	48	$2^{48/24} = 2^2 = 4$	4(10,000,000)
2047	72	$2^{72/24} = 2^3 = 8$	8(10,000,000)

41

Section 4.2: Account had \$200 in 1948.

5.

TABLE VII

Year	t	$A_o 2^{t/12}$
1954	-18	$(800)2^{-18/12} = (800)(\frac{1}{2})^{3/2} = \$282.84$
1960	-12	$(800)2^{-12/12} = (400)$
1972	0	800
1978	6	1,131.37
1984	12	1,600

6.

TABLE VIII

Year	Kilowatt Hours
1962	1,000,000
1966	1,414,214
1970	2,000,000
1974	2,828,427
1978	4,000,000
1982	5,656,865

2,181,000 kilowatt hours in 1971.

7. a) $y = 4 \cdot 2^{x-4/2}$, b) $y = 2^{x/2}$

Section 5.1: Fred had been working 331 seconds.

8.

TABLE IX

Time	Amount of C^{11}
3:00	96,000
3:20	48,000
3:40	24,000
4:00	12,000
4:20	6,000
4:40	3,000

Half-life of 20 minutes.

9. a) At 3:10, there was approximately $(24,000)(\frac{1}{2})^{-30/20} = (24,000)(2^{3/2}) \approx 67,882$.

b) At 4:10, there was approximately $(24,000)(\frac{1}{2})^{3/2} \approx (24,000) \cdot (.3536) \approx 8,485$.

10. a) (-3, 10) and (0, 5), b) 3 c) (9, .625), d) (-9, 40), e) $y = 5(\frac{1}{2})^{x/3}$.

7. MODEL EXAM

1. If the following table gives data for number of bacteria, and this number is growing exponentially,

a) Fill in the table, and answer the questions that follow.

Number of Bacteria	Time
	1:40
	1:45
	1:50
	1:55
4,000	2:00
	2:05
8,000	2:10
	2:15
	2:20

b) The number of bacteria doubles every _____ minutes.

c) A formula for the number of bacteria is $N(t)$ = __________. The initial time is __________. And t is measured in ______ starting at _____.

d) Write a formula for the number of bacteria where the initial time is 1:50.

2. The half-life of a radioactive isotope is four minutes.

a) Fill in the following table and answer the questions b and c.

Amount	Time
	9:20
	9:22
100	9:24
	9:26
	9:28
	9:30
	9:32

b) Write a formula for the amount of isotope with 9:24 as an initial time, and use it to compute the amount at 9:30.

A(t) =

A(9:30) =

c) Write formula with 9:22 as the starting time, and use it to compute the amount at 9:30.

A(t) =

A(9:30) =

44

8. ANSWERS TO MODEL EXAM

1. a)

Number of Bacteria	Time
1,000	1:40
1,414	1:45
2,000	1:50
2,828	1:55
4,000	2:00
5,657	2:05
8,000	2:10
11,314	2:15
16,000	2:20

(last place accuracy ±1)

b) Bacteria are doubling every ten minutes.

c) $A(t) = 4{,}000(2^{t/10})$, initial time is 2:00 and t is measured in minutes starting at 2:00 (or similar answer).

d) $A(t) = 2{,}000(2^{t/10})$.

2. a)

Amount	Time
200	9:20
142	9:22
100	9:24
71	9:26
50	9:28
35	9:30
25	9:32

(last place accuracy ±2)

b) $A(t) = 100(\frac{1}{2})^{t/4}$

$A(9{:}30) = 100(\frac{1}{2})^{6/4} =$

$100(\frac{1}{2})^{3/2} = 35.$

c) $A(t) = 142(\frac{1}{2})^{t/4}$;

$A(t) = 142(\frac{1}{2})^{8/4} =$

$(142)(\frac{1}{2})^2 = 35.$

45

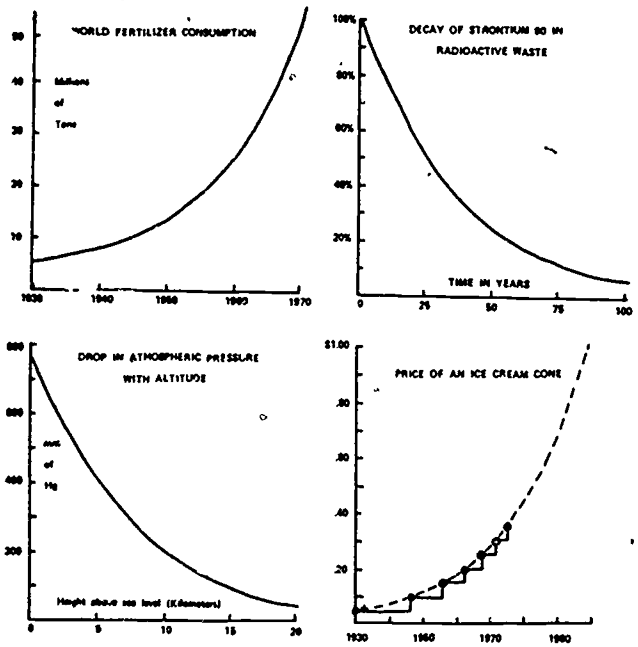
umap

UNIT 86

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

DEVELOPMENT OF THE FUNCTION $y = Ae^{cx}$

by Raymond J. Cannon



INTRODUCTION TO EXPONENTIAL FUNCTIONS

UNITS 84-88

edc/umap/55chapel st./newton, mass. 02160

DEVELOPMENT OF THE FUNCTION $y = Ae^{cx}$

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TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 86*

Title: DEVELOPMENT OF THE FUNCTION $y = Ae^{cx}$

Author: Raymond J. Cannon
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Review Stage/Date: IV 6/12/78

Classification: EXPN FNCTN/DVLP OF EXPN FNCTN

Suggested Support Materials: A calculator capable of taking square roots. (Optional.)

Prerequisite Skills:

1. Be able to use rules of exponents.
2. Know that the derivative of a function $y = f(x)$ at the point x is defined to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

when this limit exists.

3. Know the meaning of the symbol $\int f(x)\,dx$.

Output Skills:

1. Be able to define the number e.
2. Be able to estimate

$$\lim_{h\to 0} \frac{b^h - 1}{h}$$

by computing the value of the quotient for selected values of h.

3. Estimate coordinates of points on the graph of $y = e^x$.
4. Integrate and differentiate $y = Ae^{cx}$.
5. Integrate and differentiate $y = b^x$.

Other Related Units:

Recognition of Problems Solved by Exponential Functions *(Unit 84)*
Exponential Growth and Decay *(Unit 85)*
Numerical Approximations to $y = e^x$ *(Unit 87)*
How to Solve Problems Involving Exponential Functions *(Unit 88)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank W.T. Fishback, L.M. Larsen, Rosemary Bray, Thomas M. Lamm, and David E. Moxness for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

DEVELOPMENT OF THE FUNCTION $y = Ae^{cx}$

by

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6/12/78

1. INTRODUCTION

There are several physical quantities that have "doubling periods" or "half-lives". Unit 85 of this module shows that the amount of such a quantity with a doubling period of length k is given by the formula $A_0 2^{t/k}$ where A_0 is the amount at a starting time, and t is the time measured from that starting time. The corresponding formula for quantities with half-lives is $A_0(\frac{1}{2})^{t/k}$. These formulas have many applications and it is worth some time learning how to evaluate them for every value of t (to give meaning to $2^{\sqrt{2}}$ for example), and how to compute their derivatives (to enable us to talk about instantaneous rate of change as well as doubling rate).

2. SIMPLIFICATION OF THE EXPONENTIAL FORMULAS

2.1 Establishment of Formula $y = b^t$

Before we get into the computations involved in finding a formula for the derivative of an exponential function, we want to simplify the notation used. The first step in the simplification is to assume the initial amount A_0 is 1; we will remove this assumption later. The next simplification is one of notation only. We rewrite $2^{t/k}$ as $2^{(1/k)t}$ which we again can rewrite, this time as $(2^{1/k})^t$. Since k is a constant the number $2^{1/k}$ is also a constant. If we let b stand for this constant we may rewrite the expression $2^{t/k}$ as simply b^t.

We may go through a similar argument to also rewrite $(\frac{1}{2})^{t/k}$ in the form b^t; this time b is playing the role of the constant $(\frac{1}{2})^{1/k}$.

Thus, functions of the form $y = b^t$ include functions that have a doubling period as well as functions that have a half-life. We will now investigate functions of the form $y = b^t$ where the exponent t is the variable. It is for this reason that $y = b^t$ is called an *exponential function*, and b is called the *base*.

2.2 The Base Must Be Positive

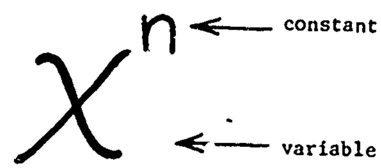
Because numbers of the form $2^{1/k}$ and $(\frac{1}{2})^{1/k}$ are positive, and because we want b^t to be defined for all possible values of t, we make the restriction that <u>in everything that follows b > 0.</u> Note that if we allowed b to be equal to -9 for example, then b^t would not be defined when $t = \frac{1}{2}$; no real number is the square root of -9.

3. THE DERIVATIVE OF $A(t) = b^t$

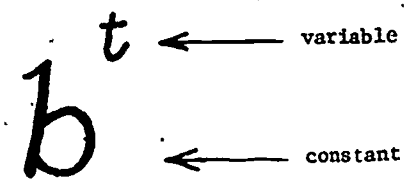
3.1 The Distinction Between x^n and b^t

Now that we have simplified the notation, we try to compute the derivative. We have no formula that we can use because the variable is in the *exponent*. The rule that worked for x^n, where the exponent is constant and the variable is in the base, does not apply; x^n and b^t are completely different kinds of functions:

Previous Functions (Polynomials)



New Functions (Exponentials)



3.2 Finding the Derivative of b^t

We will not prove that an exponential function does have a derivative; we take that as an unproven fact and seek a formula for the derivative. We will calculate all the following limits assuming they do indeed exist.

To find the derivative of b^t, we must go back to the definition and develop a new formula. If $A(t) = b^t$, we use the three step method (CALC, Module II, Unit D-6). Remember:

$$A'(t) = \lim_{h \to 0} \frac{A(t + h) - A(t)}{h} = \lim_{h \to 0} \frac{\Delta A}{h}$$

1. Find a formula for $\frac{\Delta A}{h}$.

$\Delta A = A(t + h) - A(t) = b^{t+h} - b^t$; by definition.

$\frac{\Delta A}{h} = \frac{b^{t+h} - b^t}{h}$; by dividing both sides by h.

2. Simplify algebraically.

$\frac{\Delta A}{h} = \frac{b^{t+h} - b^t}{h}$; from step 1

$\frac{\Delta A}{h} = \frac{b^t b^h - b^t}{h}$; because $b^{t+h} = b^t b^h$

$\frac{\Delta A}{h} = b^t(\frac{b^h - 1}{h})$; because $b^t b^h - b^t = b^t(b^h - 1)$

3. Let h approach zero.

$A'(t) = \lim_{h \to 0} \frac{\Delta A}{h}$; by definition

$A'(t) = \lim_{h \to 0} b^t(\frac{b^h - 1}{h})$; from last expression in step 2

$A'(t) = b^t \lim_{h \to 0} (\frac{b^h - 1}{h})$; because b^t does not depend on h.

3.3 An Important Property of the Exponential Function

Now we pause and examine what we have. The expression

$$\lim_{h \to 0} \frac{t^h - 1}{h}$$

does not involve t; it is a number that depends only on the base b. We have discovered that if

$$A(t) = b^t$$

then

$$A'(t) = A(t)(\lim_{h \to 0} \frac{b^h - 1}{h})$$

where

$$\lim_{h \to 0} \frac{b^h - 1}{h}$$

is a constant. More simply, if $A(t) = b^t$, there is a constant c such that $A'(t) = cb^t = cA(t)$. That is, an exponential function is proportional to its own derivative. This would be a very easy formula to remember and use if we could find a base b with

$$c = \lim_{h \to 0} \frac{b^h - 1}{h} = 1.$$

If we can find such a base, we will have $A'(t) = A(t)$. That is, $A(t)$ will be an exponential function which is equal to its own derivative. Is there such a base? We can try different bases, and see if we can find one.

4. VALUES OF THE FUNCTION $L(b) = \lim_{h\to 0} \frac{b^h - 1}{h}$

4.1 Introduction to the Function L

Since we are interested in the value

$$c = \lim_{h\to 0} \frac{b^h - 1}{h}$$

for various values of b, we are looking at a *function* of b. We can give it the name L, and write $L(b) = c$ where

$$L(b) = \lim_{h\to 0} \frac{b^h - 1}{h}.$$

4.2 Computation of L(b) for Various Values of b

4.2.1 b = 1

This is an easy limit to evaluate since $1^h = 1$ for every value of h. Fill in Table I.

TABLE I

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\frac{1^h - 1}{h}$	0	0	0					

We have just discovered $L(1) = 0$. This means if $A(t) = 1^t$, then $A'(t) = L(1)A(t) = 0 \cdot A(t) = 0$. This verifies what you already knew for if $A(t) = 1^t = 1$, then $A(t)$ is a constant function so its derivative is 0.

4.2.2 b = 2, 3, 4

We try to estimate

$$L(2) = \lim_{h\to 0} \frac{2^h - 1}{h}$$

by evaluating

$$\frac{2^h - 1}{h}$$

for values of h near 0. If we look at

$$h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \text{ etc.},$$

then 2^h can be calculated easily on hand calculators that have a square root key. Rounding off in the fourth decimal place, we have:

$$2^{1/2} = \sqrt{2} \approx 1.4142, \text{ and}$$

$$2^{1/4} = 2^{(1/2)(1/2)} = \sqrt{2^{1/2}} = \sqrt{\sqrt{2}} \approx 1.1892,$$

which is obtained by entering 2 and pushing the square root key twice.

Next,

$$2^{1/8} = 2^{(1/4)(1/2)} = \sqrt{2^{1/4}} = \sqrt{\sqrt{\sqrt{2}}}.$$

To calculate this, enter 2 and push the square root key three times, getting $2^{1/8} \approx 1.0905$. Now to compute

$$\frac{2^{1/8} - 1}{1/8},$$

first compute $2^{1/8}$, then subtract 1, and finally divide by $\frac{1}{8}$. Of course, dividing by $\frac{1}{8}$ is the same as multiplying by 8, and it may be easier to do it that way on your calculator.

Compute

$$\frac{2^h - 1}{h}$$

for $h = \frac{1}{16}$. Enter 2, push the square root key four times, subtract 1, and multiply by 16. Does your answer agree with the one in Table II? (Do not worry about accuracy in the last place.) If not, try it again, this time keeping track of each intermediate value computed.

Exercises

1. Check the next entry in Table II, and then fill in the missing entries.

TABLE II

Table for b = 2

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\frac{2^h - 1}{h}$	.8284	.7568	.7241	.7084	.7007			

2. To estimate L(3), check the entries given in Table III and fill in the missing entries using four-place accuracy.

TABLE III

Table for b = 3

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\frac{3^h - 1}{h}$	1.4641	1.2643	1.1776			1.1081		

3. To estimate L(4), check and complete Table IV.

TABLE IV

Table for b = 4

h	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$
$\frac{4^h - 1}{h}$	1.6569	1.4481		

We look now at the completed tables and use them to say: L(2) is some number near .69 and L(3) is approximately 1.10. What would you guess as an approximate value of L(4)? ________. Actually, L(4) ≈ 1.3863, correct to 4 places.

5. DEFINITION OF THE NUMBER e

5.1 A Graph of the Function L(b)

We have seen the following:

b	1	2	3	4
L(b)	0	about .69	about 1.10	about 1.39

It seems that as b gets bigger, L(b) also gets bigger. We substantiate this impression by estimating L(b) where b = 1.5, 2.5, and 3.5. We will then draw a graph of the function c = L(b).

Exercises

4. Use $h = \frac{1}{256}$ to fill in Table V.

TABLE V

b	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Approximation to L(b)	0.00		.69		1.10		1.39

5. Use the data from Table V to draw a graph of the function c = L(b), $1 \le b \le 4$. (See Figure 1 on the next page.)

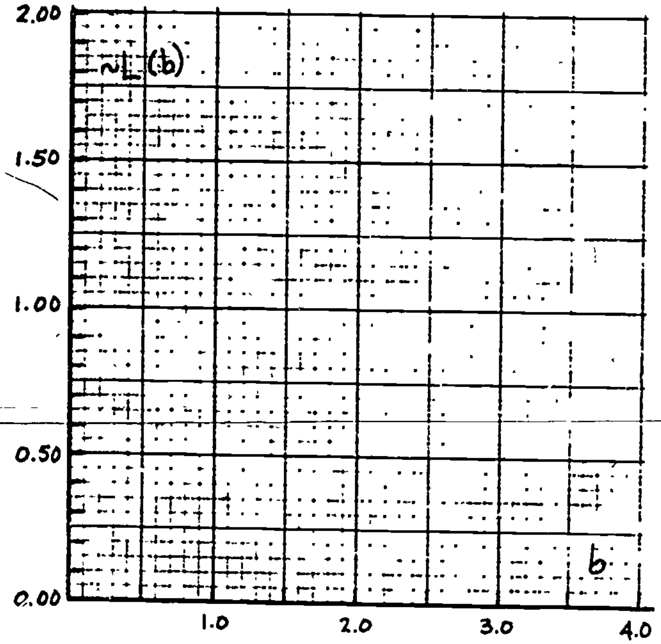


Figure 1.

5.2 The Value of b for Which L(b) = 1

We see from Exercises 4 and 5 that there is a number b between 2.5 and 3.0 for which L(b) = 1. This number is denoted by the letter e following the notation used by the Swiss mathematician Leonhard Euler (pronounced "oiler") (1707-1783). Thus, L(e) = 1 where e is a number between 2.5 and 3.0. Using this notation, we can write $A(t) = e^t$, and we have the formula

$$A'(t) = A(t) \cdot L(e)$$

and since L(e) = 1 we have

$$A'(t) = A(t).$$

Thus, e is the base such that $A(t) = e^t$ implies that $A'(t) = A(t)$. (Of course, this result doesn't depend on what letters we use for the variables; for example, if $y = e^x$, then $y' = e^x = y$.)

Exercise

6. Optional

Write a computer program that enables you to fill in Table VI.

TABLE VI

b	Approximate value of L(b)
2.50	
2.55	
2.60	
2.65	
2.70	
2.75	
2.80	
2.85	
2.90	
2.95	
3.00	

6. DEVELOPMENT OF THE FUNCTION $y = e^{cx}$

6.1 Writing $y = b^x$ As $y = e^{cx}$

The function $y = e^x$ is very special because of the nice formula for its derivative, but what about all the other bases and their exponential functions $y = b^x$? In this section we will see that if b is any positive base, then there is a number c with $b = e^c$, and the function $y = b^x$ can be written as $y = (e^c)^x = e^{cx}$.

When an exponential function is written in the form $y = e^{cx}$ the derivative is easy to compute. First we show, given b, how to find c so that $b^x = e^{cx}$.

6.2 Solving the Equation $e^t = b$

Given a number $b > 0$, we want to find a number c so that $e^c = b$. We divide the search for such a number into three cases depending on the value of b: $b > 1$, $b = 1$ or $b < 1$.

6.2.1 $b > 1$

We know e is somewhere between 2 and 3 and so we can write $e > 2$. Now given $b > 1$, find some number n so that $2^n > b$. Because $e > 2$ it follows that $e^n > 2^n$. Combining the statements $e^n > 2^n$ and $2^n > b$, we can say $e^n > b$. Now look at the section of the graph $y = e^x$ that lies above the interval $(0, n)$ on the x-axis.

(See graph on next page.)

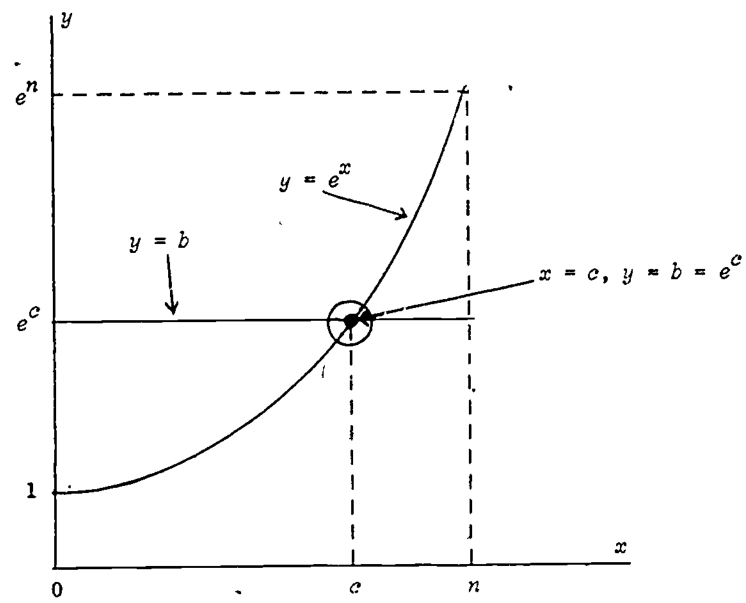


Figure 2.

The graph starts at the point $(0, 1)$ on the left and goes up as it moves to the right, ending at the point (n, e^n). Since $e^n > b > 1$, the graph begins at height 1 below the line $y = b$ and ends at height e^n above the line $y = b$. Somewhere, the graph must have crossed the line $y = b$.

What are the coordinates of that point of intersection? Let c denote the x-coordinate. Since the point is on the graph $y = e^x$, the y-coordinate must be e^c. But the point is also on the line $y = b$. so its y-coordinate must be b. But the point can only have one y-coordinate, and we must have $b = e^c$. The x-coordinate of the point on both the graphs $y = e^x$ and $y = b$ is the value c we were looking for.

Exercises

7. Use Figure 3 (page 14), the graph of $y = e^x$,* to solve the equation $e^c = b$ for different values of b. Remember, what you want is the first coordinate of the point where the graph $y = e^x$ crosses the line $y = b$. Figure 3 shows how to estimate the value of c when $b = 1.5$, and when $b = 3.0$. Now complete Table VIII.

TABLE VIII

b	1.0	1.5	2.0	2.5	3.0	3.5	4.0
c	0	.4			1.1		

8. Compare the values of c you computed in Exercise 7 with the values of L(b) computed in Exercise 4. Can you guess what the relationship is between c and $L(b) = L(e^c)$? (We return to this relationship in Section 7 of this unit.)

6.2.2 b = 1

This case is easy. If b = 1, then we want the value of c such that $e^c = b = 1$. But $e^0 = 1$ (in fact any number raised to the power 0 equals 1). So here we have $c = 0$.

6.2.3 b < 1

We could go through the same kind of argument as in Section 6.2.1, using this time $\frac{1}{e} < \frac{1}{2}$, finding a number n with $\frac{1}{e^n} < \frac{1}{2^n} < b$, and drawing the graph and solving for c. Instead of presenting this argument in detail, we use the result of Section 6.2.1. If $0 < b < 1$, then $\frac{1}{b} > 1$, and we know from Section 6.2.1 that there is a number, which we call d this time, with $e^d = \frac{1}{b}$. Taking reciprocals of both sides we have $b = \frac{1}{e^d}$; we may rewrite $\frac{1}{e^d}$ as e^{-d} and we have

*How these values of e^c were computed will be explained in Unit 87 of this module.

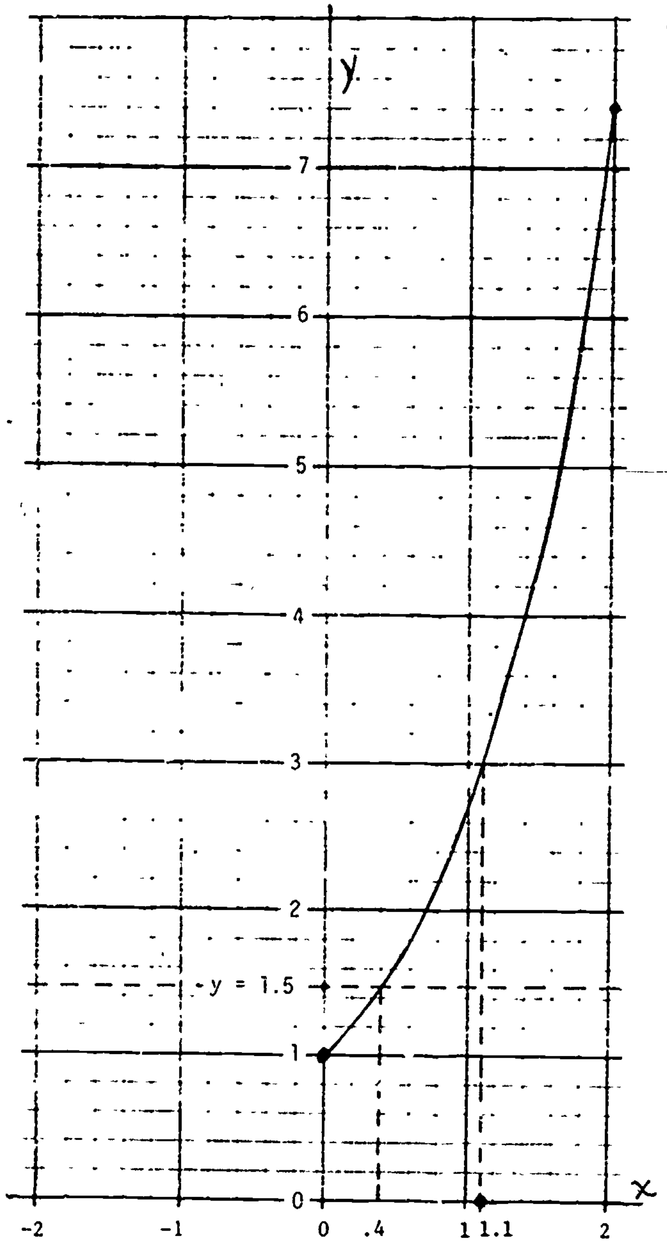


Figure 3.

now the equation $b = e^{-d}$. The number we are searching for is -d; if we write c = -d, we have $b = e^c$.

For example, we let $b = \frac{1}{2}$ and solve for c in the equation $e^c = \frac{1}{2}$. If $b = \frac{1}{2}$, then $\frac{1}{b} = 2$; we know $e^{.7} \approx 2$, so $(e^{.7})^{-1} \approx 2^{-1}$ and $e^{-.7} \approx 2^{-1} = \frac{1}{2}$. The value of c is $\approx -.7$.

Exercises

9. What is the value of c so $e^c = \frac{1}{3}$? $(e^{1.1} \approx 3)$ c ≈ ________.

10. What is the value of c so $e^c = .4$? c ≈ ________.

11. Use Figure 3 to sketch the curve $y = e^x$ for $-2 \le x \le 0$.

---

7. FORMULA FOR DERIVATIVE of $y = b^x = e^{cx}$

7.1 $L(e^c) = c$

In Section 4 we defined a new function, which we called L. This function presented itself when we were computing the derivative of $A(t) = b^t$ and we discovered that $A'(t) = b^t \cdot L(b)$. If we substitute e^c for b in the expressions for both the function and its derivative we have $A(t) = (e^c)^t = e^{ct}$ and $A'(t) = e^{ct}L(e^c)$. The purpose of this section is to establish that $L(e^c) = c$, a fact that you may have guessed at when you did Exercise 8. We give two proofs that $L(e^c) = c$. The first proof uses the definition of the function L and properties of derivatives. The second proof is easier but uses the chain rule for derivatives. If you are not familiar with the chain rule you may skip Section 7.1.2. If you have worked with exponentials and logarithms before you may recognize that the formula $L(e^c) = c$ means that the function we have called L is the same as the natural logarithm; you may have seen L(b) written as ln(b) or $\log_e b$. Please see the appropriate unit on logarithms for further discussion of this function and its derivative.

7.1.1 First proof that $L(e^c) = c$

This proof is a bit sneaky; it uses one algebraic and two notational tricks. We start with the definition of $L(e^c)$:

$$L(e^c) = \lim_{h\to 0} \frac{e^{ch} - 1}{h}.$$

We now perform our algebraic trick, which is to multiply numerator and denominator by c:

$$L(e^c) = \lim_{h\to 0} \frac{c(e^{ch} - 1)}{c \cdot h} = \lim_{h\to 0} c\left(\frac{e^{ch} - 1}{c \cdot h}\right).$$

Next, we perform our first notational trick and write c·h as d:

$$L(e^c) = \lim_{h\to 0} c\left(\frac{e^d - 1}{d}\right).$$

The next step is not tricky, but uses a property of limits: we factor the constant c out of the limit to get

$$L(e^c) = c \lim_{h\to 0} \frac{e^d - 1}{d}.$$

This is "legal" because c does not depend on h. Next, we use a second notational trick: since c is fixed, then as $h \to 0$, it is also true that $ch \to 0$. But ch = d, so we have $d \to 0$ as $h \to 0$. This means we can replace $h \to 0$ by $d \to 0$, and write

$$L(e^c) = c \lim_{d\to 0} \frac{e^d - 1}{d}.$$

Lastly, e was chosen precisely so this limit is 1: that is, e is that number with

$$\lim_{d\to 0} \frac{e^d - 1}{d} = 1.$$

We put 1 in for the limit and have

$$L(e^c) = c \cdot 1 = c$$

and have established the fundamental relationship $\underline{L(e^c) = c}$.

7.1.2 Second proof that $L(e^c) = c$

We start with $A(t) = e^{ct}$, and let $u(t) = c\cdot t$. Recall that in Section 5 e was defined so that if $w = e^u$ then $\frac{dw}{du} = e^u$. Now $\frac{du}{dt} = c$ so if $w = e^u = e^{ct}$, then $\frac{dw}{dt} = \frac{dw}{du}\cdot\frac{du}{dt} = e^u\cdot c = e^{c\cdot t}\cdot c$. In other notation, with $w = A(t)$ we have $A'(t) = e^{c\cdot t}\cdot c$. We also know from the definition of the function L that $A'(t) = e^{ct}L(e^c)$. Equating these two expressions for $A'(t)$ gives us $e^{ct}\cdot c = e^{ct}L(e^c)$; divide both sides by the (non-zero) quantity e^{ct} and obtain $c = L(e^c)$.

7.2 The Derivative of $y = e^{cx}$ is $y' = ce^{cx}$

In Section 3 we discovered the following formula:

if $\quad A(t) = b^t$

then $\quad A'(t) = b^t \lim_{h\to 0} \left(\frac{b^h - 1}{h}\right)$.

In Section 4 we let L(b) stand for this limit so we could write $A'(t) = b^tL(b)$, and then calculated some values of L(b). In Section 5 we defined e to be the number with $L(e) = 1$, so that if $A(t) = e^t$, then $A'(t) = e^t$. In Section 6 we rewrote b as e^c, and we were able to say that if $A(t) = e^{ct}$, then $A'(t) = e^{ct}L(e^c)$. In Section 7.1 we established the formula $L(e^c) = c$, and we are now able to say:

if $\quad A(t) = e^{ct}$,

then $\quad A'(t) = ce^{ct}$.

Changing the notation we may write:

if $\quad y = e^{cx}$,

then $\quad y' = e^{cx}L(e^c) = e^{cx}(c)$.

If we put the c factor in front we have

$$y = e^{cx} \Longrightarrow y' = ce^{cx}.$$

Examples:

i. If $y = e^{3x}$, then $y' = 3e^{3x}$.

ii. If $y = e^{x/2}$, then $y' = \frac{1}{2}e^{x/2}$.

iii. If $y = e^{x/4}$, then $y' = \frac{1}{4}e^{x/4}$.

iv. If $y = e^{-2x}$, then $y' = -2e^{-2x}$.

v. If $y = e^{-x}$, then $y' = (-1)e^{-x} = -e^{-x}$.

Exercise

12. Find the following derivatives:

i. If $y = e^{10x}$, then $y' =$ __________.

ii. If $y = e^{x/7}$, then $y' =$ __________.

iii. If $y = e^{-3x}$, then $y' =$ __________.

iv. If $y = e^{-x/2}$, then $y' =$ __________.

v. If $y = e^{-8x/9}$, then $y' =$ __________.

7.3 Other Forms of the Derivative

7.3.1 Review of the derivative $y = b^x$

Recall that in Section 3 we found that if $y = b^x$, then $y' = L(b)\cdot b^x$. Let us use this formula directly, and then find another form for writing the derivative formula.

Examples:

a) If $y = 2^x$, then $y' = L(2)2^x$. In Section 4 we used Table II to see that $L(2) \approx .69$; we use that information again to write the derivative of $y = 2^x$ as $y' \approx (.69)2^x$.

b) If $y = 5^x$, then $y' = L(5)5^x$. We did not approximate L(5) in Section 4, but we may use $h = \frac{1}{256}$ to do so now.

$$L(5) \approx \frac{5^{1/256} - 1}{1/256} = 256\,(5^{1/256} - 1)$$
$$\approx 1.61.$$

This enables us to say the derivative of $y = 5^x$ is $y' \approx (1.61)5^x$.

Exercise

13. Use either the information tabulated in Section 4, or $h = \frac{1}{256}$ to complete these problems.

a) If $y = 3^x$, then $y' = L(\_\_\_)3^x \approx \_\_\_ 3^x$.

b) If $y = 4^x$, then $y' = \_\_\_\_\_\_ \approx \_\_\_\_$.

c) If $y = 6^x$, then $y' = \_\_\_\_\_\_ \approx \_\_\_\_$.

d) If $y = (\frac{1}{2})^x$, then $y' = \_\_\_\_\_\_ \approx \_\_\_\_$.

7.3.2 $e^{L(b)} = b;\ b > 0$

This is the fundamental relationship we will be using to convert an arbitrary exponential function to one whose base is e. This equation says that the number c with $e^c = b$ is the same as the number L(b). Because this is so important we state it again. The number c that we found in Section 6 with $e^c = b$ is the same number as the number L(b) we encountered in Section 4; $c = L(b)$.

To establish this relationship we let $b > 0$ and c be the number so that

$$(*)\ e^c = b.$$

Apply the function L to both sides of (*) to obtain equation $(**) L(e^c) = L(b)$. Use the result of Section 7.1, $L(e^c) = c$, to rewrite the left hand side of (**) and obtain

$$(***) c = L(b).$$

We can substitute this expression for c into equation (*) to arrive at the result $e^{L(b)} = b$.

Examples:

a) If $y = 2^x$, we can rewrite $y = e^{L(2)x}$ and now use formula from Section 7.2 to write $y' = e^{L(2)x} \cdot L(2)$.

b) If $y = 5^x$, then $y = e^{L(5)\cdot x}$, and $y' = L(5)e^{L(5)x}$.

Exercise

13.1 Rewrite answers to Exercise 13 using base e.

7.4 The Derivative of $y = b^{kx}$

If you are given a function in the form $y = b^{kx}$, you may think of this as $y = (b^k)^x$, so that $y' = L(b^k)b^{kx}$. For example, if $y = 2^{3x}$, we may rewrite this as $y = (2^3)^x = 8^x$ and $y' = L(8)8^x = L(2^3)2^{3x}$.

As an alternative, you can rewrite $y = b^x$ as $y = e^{L(b)kx}$. Now $y' = L(b)k\ e^{L(b)\cdot kx}$ since $L(b)\cdot k$ is a constant. For example, if $y = 2^{3x}$, then $y = e^{L(2)3x}$ and $y' = L(2)\cdot 3\ e^{L(2)3x} = 3L(2)e^{L(2)3x} = 3L(2)2^{3x}$.

Exercise

14. Write each of the following in three ways:

a) $y = 3^{2x} \Longrightarrow y' = \_\_\_ L(\_\_\_)3^{2x} = L(\_\_\_)3^{2x} = L(\_\_\_)e$—.

b) $y = 5^{3x} \Longrightarrow y' = \_\_\_\_\_\_ = \_\_\_\_\_\_ = \_\_\_\_\_\_$.

c) $y = 5^{x/2} \Longrightarrow y' = \_\_\_\_\_\_ = \_\_\_\_\_\_ = \_\_\_\_\_\_$.

d) $y = (\frac{1}{2})^x = 2^{-1x} = 2^{-x} \Longrightarrow y' = \_\_\_\_\_\_ = \_\_\_\_\_\_ = \_\_\_\_\_\_$.

8. FORMULA FOR DERIVATIVE OF $y = Ae^{cx}$

We now remove the assumption we made in Section 2 that $A_0 = 1$. Let A be any constant and remember that $(Ay)' = A \cdot y'$. Then if $y = Ae^{cx}$, we have $y' = (Ae^{cx})' = A(e^{cx})' = A(ce^{cx}) = Ace^{cx}$.

Examples

i. If $y = 2e^{3x}$, then $y' = 2 \cdot 3e^{3x} = 6e^{3x}$.

ii. If $y = -2e^{x/3}$, then $y' = -2 \cdot \frac{1}{3}e^{x/3} = \frac{-2}{3}e^{x/3}$.

iii. If $y = \frac{1}{4}e^{4x}$, then $y' = (\frac{1}{4}) \cdot 4e^{4x} = e^{4x}$.

Exercise

15. Compute the following derivatives:

i. If $y = -5e^{6x}$, then $y' =$ ________.

ii. If $y = \frac{1}{4}e^{8x}$, then $y' =$ ________.

iii. If $y = 5e^{x/5}$, then $y' =$ ________.

iv. If $y = \frac{1}{2}e^{2x}$, then $y' =$ ________.

v. If $y = -\frac{1}{3}e^{-3x}$, then $y' =$ ________.

9. ANTIDERIVATIVES OF EXPONENTIAL FUNCTIONS

9.1 Formula for Antiderivative of $y = e^{cx}$

We have seen that the derivative of $y = Ae^{cx}$ is $y' = Ace^{cx}$; if we want to find a function whose derivative is e^{cx}, we should start with one of form Ae^{cx} where $A \cdot c = 1$. From this we get $A = \frac{1}{c}$ which means that an antiderivative of e^{cx} is $\frac{1}{c}e^{cx}$. Thus,

$$\text{ANTIDER } [e^{cx}] = \frac{1}{c}e^{cx} + k$$

or

$$\int e^{cx}dx = \frac{1}{c}e^{cx} + k$$

where k is an arbitrary constant. To check this result, notice that

$$\begin{aligned} [\tfrac{1}{c}e^{cx} + k]' &= [\tfrac{1}{c}e^{cx}]' + k' \\ &= \tfrac{1}{c}(e^{cx})' + 0 \\ &= \tfrac{1}{c}(ce^{cx}) \\ &= e^{cx}. \end{aligned}$$

Examples:

i. The antiderivative of e^{7x} is $\frac{1}{7}e^{7x} + k$.

ii. $\text{ANTIDER } [e^{1/3x}] = \frac{1}{1/3}e^{x/3} + k = 3e^{x/3} + k$

iii. $\int e^{-5x}dx = -\frac{1}{5}e^{-5x} + k$

iv. $\int e^{-x/2}dx = -2e^{-x/2} + k$

Exercise

16. Compute the following antiderivatives, and check your work by taking the derivative of your answer.

i. The antiderivative of e^{4x} is ________.

ii. $\text{ANTIDER } [e^{-2x}] =$ ________.

iii. $\int e^{-x/5}dx =$ ________.

iv. $\int e^{x/6}dx =$ ________.

v. $\int e^{-2x}dx =$ ________.

9.2 Formula for Antiderivative of $y = Ae^{cx}$

As with derivatives, the constant A causes little trouble, since $\int Aydx = A\int ydx$. This gives us the formula

$$\int Ae^{cx}dx = A\frac{1}{c}e^{cx} + k.$$

Examples:

i. $\int 6e^{3x}dx = \frac{6}{3}e^{3x} + k = 2e^{3x} + k$.

ii. $\int \frac{1}{3}e^{4x}dx = \frac{1}{3}\cdot\frac{1}{4}e^{4x} + k = \frac{1}{12}e^{4x} + k$.

iii. $\int 6e^{-2x}dx = 6(-\frac{1}{2})e^{-2x} + k = -3e^{-2x} + k$.

iv. $\int -5e^{x/3}dx = (-5)(\frac{1}{1/3})e^{x/3} + k = -15e^{x/3} + k$.

v. $\int 4e^{-x/6}dx = (4)(\frac{1}{-1/6})e^{-x/6} + k = -24e^{-x/6} + k$.

vi. $\int -3e^{-2x}dx = (-3)(-\frac{1}{2})e^{-2x} + k = \frac{3}{2}e^{-2x} + k$.

Exercise

17. Find the following antiderivatives.

i. $\int -3e^{6x}dx =$ ________.

ii. $\int 8e^{4x}dx =$ ________.

iii. $\int \frac{1}{2}e^{-6x}dx =$ ________.

iv. $\int -2e^{x/6}dx =$ ________.

v. $\int -\frac{1}{2}e^{-x/6}dx =$ ________.

vi. $\int 2e^{-6x}dx =$ ________.

9.3 Formula for Antiderivative of $y = Ab^x$

Simply rewrite $y = Ab^x$ as $y = A\,e^{L(b)x}$ and apply so the formula from Section 9.2.

$$\int Ab^x dx = \int Ae^{L(b)x}dx = A \cdot \frac{1}{L(b)}\, e^{L(b)x} + k$$

$$= \frac{A}{L(b)}\, b^x + k.$$

Examples:

i. $\int 2^x dx = \frac{1}{L(2)}\, 2^x + k$

ii. $\int 5 \cdot 3^x dx = \frac{5}{L(3)} 3^x + k.$

Exercise

18. Find the following antiderivatives.

i. $\int 3 \cdot 4^x dx =$ ________.

ii. $\int -2 \cdot 5^x dx =$ ________.

iii. $\int 5(\frac{1}{2})^x dx =$ ________.

9.4 Formula for Antiderivative of $y = Ab^{mx}$

With $c = L(b)$ we have

$$\int Ab^{mx}dx = \int Ae^{cmx}dx = \frac{A}{cm}\, e^{cmx} + k$$

from Section 9.2; translating back to a form that does not involve c we have

$$\frac{A}{cm}\, e^{cmx} + k = \frac{A}{L(b) \cdot m}\, b^{mx} + k.$$

Examples:

i. $\int 5 \cdot 3^{4x}dx = \int 5(3^4)^x dx = \int 5 \cdot 81^x dx = \frac{5}{L(81)} 81^x + k$

ii. $\int 5 \cdot 3^{4x}dx = \int 5 \cdot e^{L(3)4x}dx = \frac{5}{L(3) \cdot 4} e^{L(3)4x} + k$

$= \frac{5}{L(3) \cdot 4}\, 3^{4x} + k.$

Exercise

19. Find the following antiderivatives.

i. $\int 2^{3x} dx =$ ________.

ii. $\int 4 \cdot 3^{2x} dx =$ ________.

iii. $\int 3 \cdot 5^{-2x} dx =$ ________.

10. SUMMARY

We have found a number e so that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

We have calculated $2.5 < e < 3.0$.

Given a number b we were able to find a number c such that $e^c = b$ and to identify c as $L(b)$. If $y = b^x$ we have $y = e^{L(b)x}$ and $y' = L(b)b^x$. Finally, we found $\int e^{cx}dx = (\frac{1}{c})e^{cx} + k$ and that $\int b^x dx = \int e^{L(b)x} = (\frac{1}{L}(b))\, e^{L(b)x} + k = (\frac{1}{L}(b))b^x + k.$

We were able to find c two ways:

1. $c = \lim_{h \to 0} \frac{b^h - 1}{h} = L(b)$.

2. c is the x-coordinate of the point on graph $y = e^x$ (Figure 4) whose y-coordinate is b.

11. ANSWERS FOR UNIT 86

TABLE I

Table for b = 1

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\frac{1^h - 1}{h}$	0	0	0	0	0	0	0	0

Exercise 1

TABLE II

Table for b = 2

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\frac{2^h - 1}{h}$	.8284	.7568	.7241	.7084	.7007	.6969	.6950	.6941

Exercise 2

TABLE III

Table for b = 3

h	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\frac{3^h - 1}{h}$	1.4641	1.2643	1.1776	1.1372	1.1177	1.1081	1.1033	1.1010

Exercise 3

TABLE IV

Table for b = 4

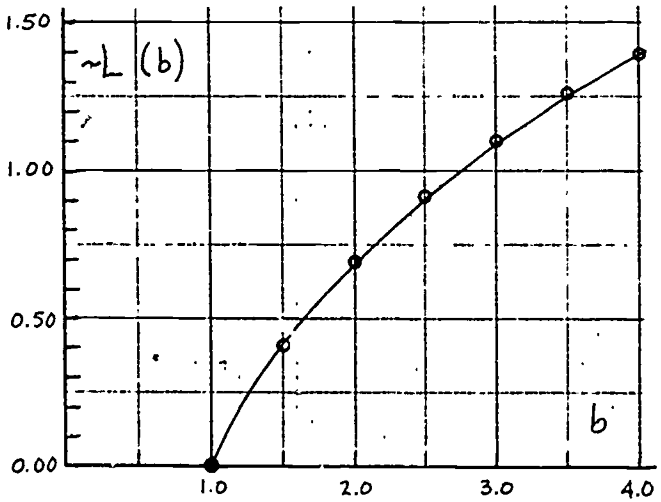
h	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$
$\frac{4^h - 1}{h}$	1.6569	1.4481	1.4014	1.3901

Exercise 4

TABLE V

b	1.0	1.5	2.0	2.5	3.0	3.5	4.0
≈ L(b)	0.00	.41	.69	.92	1.10	1.26	1.39

Exercise 5



Exercise 6

(Using BASIC)

```
*
* List
0010 FOR B=2.5 TO 3 STEP .05
0020 LET Y=256*(B↑(1/256-1)
0030 PRINT B,Y
0040 NEXT B
0050 END

* RUN
 2.5            .917969
 2.55           .937744
 2.6            .957275
 2.65           .976318
 2.7            .995117
 2.75           1.01367
 2.8            1.03174
 2.85           1.04956
 2.9            1.06689
 2.95           1.08398
 3.0            1.10097
END AT 0050
*
```

Exercise 7

TABLE VIII

b	1.0	1.5	2.0	2.5	3.0	3.5	4.0
c	0	.4	.7	.9	1.1	1.3	1.4

Exercise 8

$L(e^c) = c$

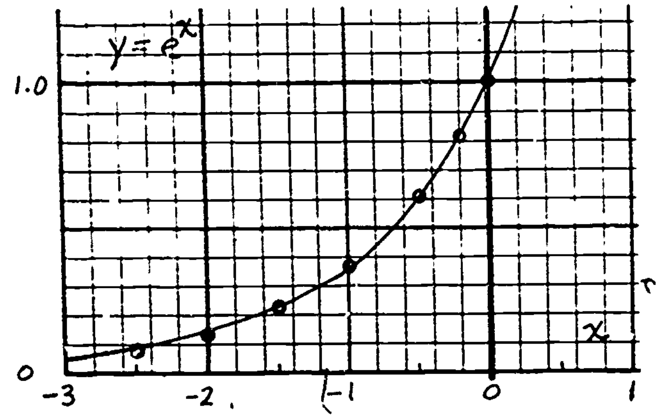
Exercise 9

$c \approx -1.1$

Exercise 10

$c \approx -.9 (e^{.9} \approx 2.5$ and $.4 = \frac{1}{2.5})$

Exercise 11



$e^{-x} = \frac{1}{e}x$

Exercise 12

i. $y' = 10e^{10x}$

ii. $y' = \frac{1}{7}e^{x/7}$

iii. $y' = -3e^{-3x}$

iv. $y' = -\frac{1}{2}e^{-x/2}$

v. $y' = -\frac{8}{9}e^{-8x/9}$

Exercise 13

a) $y' = L(3)3^x \approx 1.10\ 3^x$

b) $y' = L(4)4^x \approx 1.39\ 4^x$

c) $y' = L(6)6^x \approx 1.80\ 6^x$

d) $y' = L(\frac{1}{2})(\frac{1}{2})^x \approx -.69(\frac{1}{2})^x$

Exercise 13.1

a) $y' = L(3)\ e^{L(3)x} \approx 1.10e^{1.10x}$

b) $y' = L(4)\ e^{L(4)x} \approx 1.39e^{1.39x}$

c) $y' = L(6)\ e^{L(6)x} \approx 1.80e^{1.80x}$

d) $y' = L(\frac{1}{2})\ e^{L(\frac{1}{2})x} \approx -.69e^{-.69x}$

Exercise 14 (three of each of the following)

a) $y' = 2L(3)\ 3^{2x} = L(9)3^{2x} = 2L(3)\ e^{2L(3)x} = L(9)\ e^{2L(3)x}$

b) $y' = 3L(5)\ 5^{3x} = L(125)5^{3x} = 3L(5)\ e^{3 \cdot L(5)x} = L(125)e^{3L(5)x}$

c) $y' = \frac{1}{2}L(5)\ 5^{(\frac{1}{2})x} = L(\sqrt{5})5^{(\frac{1}{2})x} = \frac{1}{2}L(5)e^{(\frac{1}{2})L(5)x}$
$= L(\sqrt{5})\ e^{L(5)\cdot(\frac{1}{2})x}$

d) $y' = -L(2)2^{-x} = L(\frac{1}{2})(\frac{1}{2})^x = -L(2)e^{-L(2)x} = L(\frac{1}{2})e^{L(\frac{1}{2})x}$

Exercise 15

i. $y' = -30e^{6x}$

ii. $y' = 2e^{8x}$

iii. $y' = e^{x/5}$

iv. $y' = e^{2x}$

v. $y' = e^{-3x}$

Exercise 16

i. $\frac{1}{4}e^{4x} + k$

ii. $-\frac{1}{2}e^{-2x} + k$

iii. $-5e^{-x/5} + k$

iv. $6e^{x/6} + k$

v. $-\frac{1}{2}e^{-2x} + k$

Exercise 17

i. $-\frac{1}{2}e^{6x} + k$

ii. $2e^{4x} + k$

iii. $-\frac{1}{12} e^{-6x} + k$

iv. $-12e^{x/6} + k$

v. $3e^{-x/6} + k$

vi. $-\frac{1}{3}e^{-6x} + k$

Exercise 18

i. $\frac{3}{L(4)} 4^x + k$

ii. $\frac{-2}{L(5)} 5^x + k$

iii. $\frac{5}{L(1/2)}(\frac{1}{2})^x + k$ or $\frac{5}{-L(2)}(\frac{1}{2})^x + k$

Exercise 19

i. $\frac{1}{L(8)}2^{3x} + k$ or $\frac{1}{3L(2)}2^{3x} + k$

ii. $\frac{4}{L(9)} 3^{2x} + k$ or $\frac{4}{2L(3)} 3^{2x} + k$

iii. $\frac{3}{L(1/25)} 5^{-2x} + k$ or $\frac{3}{-2L(5)}5^{-2x} + k$ or $\frac{3}{-L(25)}5^{-2x} + k$

12. MODEL EXAM

1. Use $h = \frac{1}{256}$ to estimate $\lim_{h \to 0} \frac{5^h - 1}{h}$.

2. Define the number e.

3. Use the following table to give an interval of b of length .2 that contains e.

b	2.0	2.2	2.4	2.6	2.8	3.0
$\lim_{h \to 0} \frac{b^h - 1}{h}$	≈.69	≈.79	≈.88	≈.96	≈1.03	≈1.10

4. If $y = e^{5x}$, then $y' =$

5. If $y = 7e^{-2x}$ then $y' =$

6. If $y = 3e^{-x/8}$, then $y' =$

7. $\int e^{3x}dx =$

8. $\int \frac{1}{4}e^{2x}dx =$

9. $\int \frac{1}{5}e^{-4x}dx =$

10. If $e^{1.6} \approx 5$, then for what x does $e^x \approx .2$?

11. If $e^{2.3} \approx 10$, then $e^{-2.3} \approx$ ____________.

12. Since $e^{1.6} \approx 5$, if $y = 5^x$, then $y' = (\_\_\_\_\_\_)5^x$.

13. Since $e^{1.6} \approx 5$, then $5^x \approx e^{\_\_\_\_\_\_ x}$.

14. Since $e^{1.6} \approx 5$, then $\int 5^x dx =$ ________________.

13. ANSWERS TO MODEL EXAM

1. 1.61

2. i. e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

   ii. e is the number such that if $y = e^x$, then $y' = e^x$.

3. $2.6 < e < 2.8$.

4. $y' = 5e^{5x}$

5. $y' = -14e^{2x}$

6. $y' = -\frac{3}{8}e^{-x/8}$

7. $\int e^{3x}dx = \frac{1}{3}e^{3x} + K$

8. $\int \frac{1}{4}e^{2x}dx = \frac{1}{8}e^{2x} + K$

9. $\int \frac{1}{5}e^{-4x}dx = \frac{1}{20}e^{-4x} + K$

10. $x = -1.6$

11. .1 or $\frac{1}{10}$

12. $y' = (1.6)5^x$

13. $5^x \approx e^{1.6x}$

14. $\int 5^x dx \approx \frac{1}{1.6}5^x + K$

80

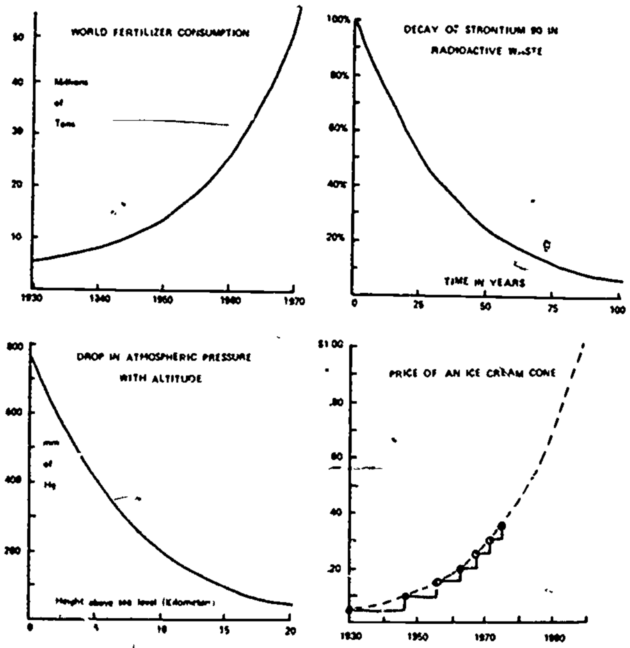
umap

UNIT 8

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

NUMERICAL APPROXIMATIONS TO $y = e^x$

by Raymond J. Cannon



INTRODUCTION TO EXPONENTIAL FUNCTIONS

UNITS 84-88

edc/umap /55 chapel st./newton, mass. 02160

NUMERICAL APPROXIMATIONS TO $y = e^x$

by

Raymond J. Cannon
Department of Mathematics
Stetson University
DeLand, FL 32720

12/22/77

TABLE OF CONTENTS

Intermodular Description Sheet: UMAP Unit 87

Title: NUMERICAL APPROXIMATIONS TO $y = e^x$

Author: Raymond J. Cannon
Stetson University
DeLand, FL 32720

Review Stage/Date: IV 12/22/77

Classification: EXPN FNCTN/NUM APRX TO EX (U 87)

Suggested Support Material: A hand calculator.

Prerequsite Skills:

1. Know that $\lim_{h\to 0} \frac{b^h - 1}{h} = 1$ implies $b = e$.
2. Know that $c = \lim_{h\to 0} \frac{e^{ch} - 1}{h}$.
3. Know that $y = e^x$ is a solution to $y' = y$ and that y' is slope of line tangent to curve at (x,v).
4. Take derivatives of all orders.
5. Plot points and draw graphs of polynomials.

Output Skills:

1. State the effects of "round-off error" in a calculator.
2. State that $e \approx 2.71818$.
3. Give the defining properties for the Taylor polynomial of degree n centered at $x = 0$ that approximates $y = e^x$.
4. Give a qualitative description of $P_n(x)$ as an approximation to $y = e^x$ when a) n is fixed, and x varies, b) x is fixed, and n varies.
5. Recognize that e^x can be approximated by $(1 + \frac{x}{n})^n$ for large values of n.
6. Define n!
7. Given a value of x, be able to compute $(1 + \frac{x}{n})^n$ as an approximation to e^x for various values of n.
8. Given a value of $\Delta x \geq 1$, be able to sketch a graph that approximates $y = e^x$ using the Euler Method, but without use of a table of va s.
9. Given any value of $\Delta x \geq 0$, be able to construct a table of values of a function that approximates $y = e^x$, using the Euler Method.
10. Given a polynomial, be able to state whether it is the Taylor Polynomial of degree n centered at $x = 0$ for $y = e^x$.
11. For any value of n, be able to give the Taylor Polynomial of degree n centered at 0 for $y = e^x$.
12. Be able to use three methods for approximating e.

Other Related Units:

Recognizing Exponential Functions *(Units 63,64,65)*
Recognition of Problems Solved by Exponential Functions *(Unit 84)*
Exponential Growth and Decay *(Unit 85)*
Development of the Function $y = Ae^{cx}$ *(Unit 86*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank Lois Helfin, W.T. Fishback, Ruth Hailperin, and Roland F. Smith for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615. Recommendations expressed are those of the authors and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

1. INTRODUCTION

We have seen that exponential functions have many applications; that an exponential function with base b, $y = b^x$, can be written as $y = e^{cx}$ for the right choice of c, and that this way of writing the function gives us simple differentiation and integration formulas. In working a problem involving an exponential function, we may want an answer expressed in decimal form rather than, for example, in the form $e^3 - \sqrt{e}$. The purpose of this unit is to show several methods for computing decimal approximations to e^x for any value of x with particular emphasis on the value $x = 1$.

It is not necessary to cover all of these methods at one time. You may prefer to save Section 4 for a more general discussion of differential equations. Similarly, Section 6 may be used as a specific application in a more general discussion of polynomial approximations.

2. METHOD 1: APPROXIMATION OF e USING ITS DEFINITION

The number e can be defined as the number that makes

$$\lim_{h \to 0} \frac{e^h - 1}{h}$$

equal to one. More generally, the function L(b) is defined by the equation

$$L(b) = \lim_{h \to 0} \frac{b^h - 1}{h},$$

and we can compute approximate values of L(b) for various values of b. We find that L(b) increases as b increases, and we can use this property to approximate e in Table I

below. From the approximations of L(b) in the table we conclude that $2.5 < e < 3.0$ because $0.92 < 1 < 1.10$. We can then take the midpoint of the interval from 2.5 to 3.0 to be our approximation to e, and say $e \approx 2.75$.

TABLE I

Approximations to L(b) for Assorted Values of b from b=1 to b=3.5

b	1.0	1.5	2.0	2.5		3.0	3.5
L(b)	0.00	0.41	0.69	0.92	1	1.10	1.25

To refine our approximation for e, we can make another set of approximations to L(b) for values of b near 2.75 (Table II). We conclude from these that $2.70 < e < 2.75$. If we again take the midpoint of the interval to be our approximation of e, we conclude that $e \approx 2.725$. We could continue in this vein by using smaller values of h, but this method is somewhat indirect and yields only an approximation for e rather than a general formula for e^x. A slight modification of this idea, however, results in a direct method for approximating e^c for any value of c.

TABLE II

Approximations to L(b) for Assorted Values of b from b=2.55 to b=2.80

b	2.60	2.65	2.70		2.75	2.80
L(b)	0.96	0.97	0.99	1	1.01	1.03

3. METHOD 2: $e^c \approx (1 + \frac{c}{n})^n$

3.1 Development of the Formula

Since

$$c = L(e^c) = \lim \frac{e^{ch} - 1}{h},$$

it follows that

$$c \approx \frac{e^{ch} - 1}{h}$$

when h is near 0.

This approximation is our starting point for Method 2 which, step-by-step, goes as follows:

Multiplying both sides by h gives $ch \approx e^{ch} - 1$.

Adding 1 to both sides results in $ch + 1 \approx e^{ch}$.

Letting h be of the form $\frac{1}{n}$ gives $c\frac{1}{n} + 1 \approx e^{c(1/n)}$.

Raising both sides to the n^{th} power yields

$$(c\frac{1}{n} + 1)^n \approx (e^{c(1/n)})^n.$$

Rewriting this last expression, we have the formula

$$e^c \approx (1 + \frac{c}{n})^n$$

In particular, for c = 1 we have $e \approx (1 + \frac{1}{n})^n$.

The closer $h = \frac{1}{n}$ is to 0, the better the approximation. Put another way, the larger the value of n, the better the approximation.

We will do some computations to test this method.

3.2 Calculator Hints

We can compute numbers like $3^{1/256}$ on a hand calculator by entering 3 and then pressing the square root key eight times since $(\frac{1}{2})^8 = \frac{1}{256}$. Similarly, we can

compute x^{256} by entering x and then pressing the squaring key eight times. This makes it particularly easy to compute $(1 + \frac{c}{n})^n$ when n is a power of two. This computation is made even easier by noting that $1 + \frac{c}{n} = \frac{n + c}{n}$. Using n = 16, and c = 1, we can approximate e by entering 16 + 1 = 17, dividing by 16, and squaring four times:

$$(\frac{17}{16})^{16} = ((((\frac{17}{16})^2)^2)^2)^2 \approx 2.6379.$$

Exercises

1. Use the method just described to complete Table III, rounding off to four decimal places.

2. Construct a table similar to Table III for higher powers of 2 and fill it in. The entries you obtain should obey the following two rules: (1) The numbers $(\frac{n + 1}{n})^n$ are all smaller than e, and (2) they increase as n increases. Does your extended table follow this pattern? If not, the reason may be a round-off error in your calculator. The more accurate your calculator, the higher value of n you can enter but eventually, because of the repeated multiplications, round-off error will begin creating trouble. For what power of 2 does your calculator start to show round-off error? ________.*

TABLE III

n	$(\frac{n+1}{n})^n \approx e$
2	2.25
4	
8	
16	2.6379
32	
64	
128	2.7077
256	
512	2.7156
1,024	

---

* If that number is small, such as 2^{14} = 16,384, Section 3.3 will be of special interest to you, but be sure to do Exercise 3 first.

3. Here we approximate $\frac{1}{e} = e^{-1}$ using our formula $e^c = \left(\frac{n + c}{n}\right)^n$. If we set $c = -1$, we have $e^{-1} \approx \left(\frac{n - 1}{n}\right)^n$. Fill in Table IV rounding off to four decimal places.

TABLE IV

n	$16 = 2^4$	$64 = 2^6$	$256 = 2^8$	$1{,}024 = 2^{10}$
$\frac{1}{e} \approx \left(\frac{n - 1}{n}\right)^n$				

3.3 Various Uses of the Formula to Obtain Better Approximations

Your last entry in Table IV should have been 0.3677. Since this is approximately $\frac{1}{e}$, its reciprocal is approximately e. Compute $\frac{1}{0.3677}$ to four places: $\frac{1}{0.3677} \approx$ ______. Using n = 1.024 in Exercise 1, $e \approx 2.7170$. If we average these two approximations, $e \approx \frac{1}{2}(2.7196 + 2.7170) =$ ______. This, in fact, is the correct value of e, rounded off to the fourth place. One approximation is too big, the other is too small, and when we average them, the errors tend to cancel each other. Of course, we are not getting an exact answer, only a generally better approximation.

The computation here involves a "sneaky trick," but sometimes one has to be devious to avoid round-off error. Since $e^{-1} \approx \left(\frac{n - 1}{n}\right)^n$, by taking reciprocals we have $e \approx \left(\frac{n}{n - 1}\right)^n$. We already have the formula

$$e \approx \left(\frac{n + 1}{n}\right)^n$$

and using each formula once, we have

$$e^2 = e \cdot e \approx \left(\frac{n}{n - 1}\right)^n \left(\frac{n + 1}{n}\right)^n = \left(\frac{n + 1}{n - 1}\right).$$

Now take the square root of both sides:

$$e = (e \cdot e)^{1/2} \approx \left[\left(\frac{n + 1}{n - 1}\right)^n\right]^{1/2} = \left(\frac{n + 1}{n - 1}\right)^{n/2}$$

which is another approximate formula for e.

$$e \approx \left(\frac{n + 1}{n - 1}\right)^{n/2}.$$

Exercises

4. Use this formula to complete Table V. Keep six-place accuracy.

TABLE V

(For n = 256, divide 257 by 255 and square 7 times.)				
n	$256 = 2^8$	$512 = 2^9$	$1{,}024 = 2^{10}$	$2{,}048 = 2^{11}$
$e \approx \left(\frac{n + 1}{n - 1}\right)^{n/2}$				

If your answers agree with those on the answer sheet, page 24, then the last entry is the correct value of e to six places. Even if your calculator has round-off error and your answers did not agree to six places, compare your answers to four places against the best approximation you were able to get in Exercise 1. This formula gives much better accuracy than did the previous one.

5. In the beginning of Section 3.3 we wrote e as $\frac{1}{1/e}$, and later in Section 3.3 we wrote e as $\sqrt{e^2}$. Think of another devious way to write e and use your way to approximate e.

4. METHOD 3: THE EULER METHOD

4.1 Description of the Method

This method uses a set of straight line segments to approximate the graph of $y = e^x$. We can improve the approximation by changing the length of the straight line segments: the shorter the segments, the better the approximation. The method which produces such a curve is called the Euler Method.

The property of the curve $y = e^x$ we use depends upon the special nature of e: $y = e^x$ is the function for which $y' = y$. Geometrically this means at each point of the curve $y = e^x$, the slope of the tangent line (which is y') is equal to the distance of the point from the x-axis (which is y).

We will use the Euler Method to approximate $y = e^x$ only on the interval $0 \leq x \leq 1$, producing curves which will have different "left-hand" and "right-hand" slopes at a finite number of special points. However, the "right-hand" slope at each such point will be the same as the y-coordinate of that point. These ideas will be clearer after an example.

4.2 First Approximation

Our first curve will have two special points, corresponding to $x = 0$, and $x = 0.5$. We know the curve $y = e^x$ goes through the point (0,1) since $e^0 = 1$. Because the second coordinate of this point is 1, the tangent line to the curve also has slope 1 (remember $y' = y$). We draw this tangent line from $x = 0$ to $x = 0.5$.

The y coordinate at the right end of this line segment, (that is, at $x = 0.5$) is $y = 1.5$. At the point (0.5, 1.5) we start a new line segment whose slope is equal to the y-coordinate, namely 1.5; this new line segment will extend from $x = 0.5$ to $x = 1$. We want to know the value of y when $x = 1$. Recall a property of straight lines: if m is the slope of

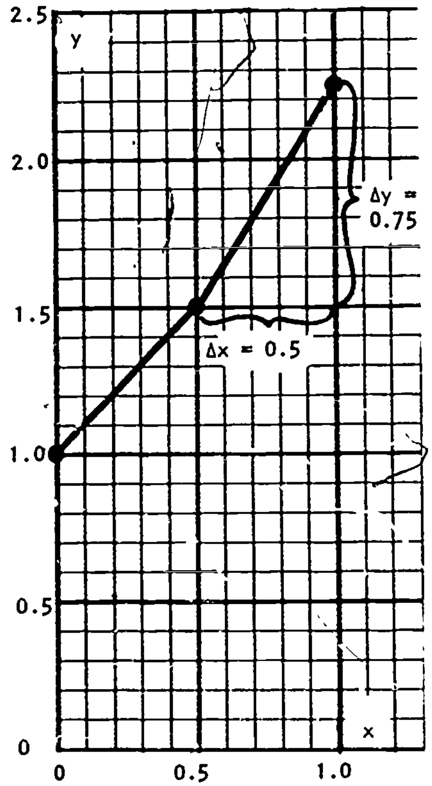


Figure 1.

the line, then $\Delta y = m(\Delta x)$. We use this to compute the y-coordinate of this segment when $x = 1$: we have $\Delta x = 0.5$ and $m = 1.5$ which gives $\Delta y = 0.75$. The final height is the height we started at (1.5) plus the change in height ($\Delta y = 0.75$), and $1.5 + 0.75 = 2.25$. Thus, the "curve" in Figure 1, which consists of two line segments, has the value 2.25 at $x = 1$ and thus yields the approximation $e \approx 2.25$.

4.3 Second Approximation

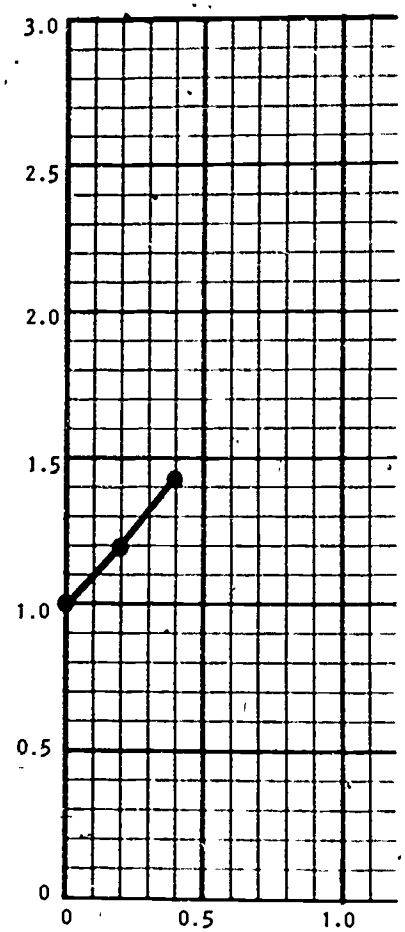


Figure 2.

For our next approximating function, we chop the interval from x = 0 to x = 1 into five equal sub-divisions so that $\Delta x = 0.2$ for each. As before, the graph will start with slope equal to 1, but will change slope at x = 0.2, 0.4, 0.6, and 0.8.

Exercises

6. Complete Table VI and then sketch the graph, using Figure 2.

TABLE VI

x-coordinate	y-coordinate	Slope of line leaving the point
0.0	1	1.0
0.2	1 + (1)(0.2) = 1.2	1.2
0.4	1.2 + (1.2)(0.2) = 1.44	1.44
0.6	1.44 + (1.44)(0.2) = 1.728	
0.8	1.728 + (1.728)(0.2) = 2.48832	
1.0		

The graph $y = e^x$ always lies above our approximation. As these approximations approach the exponential curve from below, the y-coordinate corresponding to x = 1 will approach $y = e^1 = e$. Thus, our first approximation to e using $\Delta x = 0.5$ was 2.25. Our second approximation using $\Delta x = 0.2$ was 2.48832.

7. Instead of making all the computations involved in Exercise 6, you can sketch the graph quickly using the following method: A straight line through (x,y) with slope y must also pass through the point (x-1, 0). Hold a ruler so that its edge goes through the point (x,y); now rotate it [keeping (x,y) on its edge] so that it also goes through the point (x-1, 0). In this position, the ruler's edge gives a straight line passing through (x,y) with slope y. Use this method to draw the approximate graph of $y = e^x$ on Figure 3 for the five segment case ($\Delta x = 0.2$) and compare it with the numerical computation in Exercise 6. (The first three segments have been done.) What approximate value for e do you obtain by this graphical procedure?

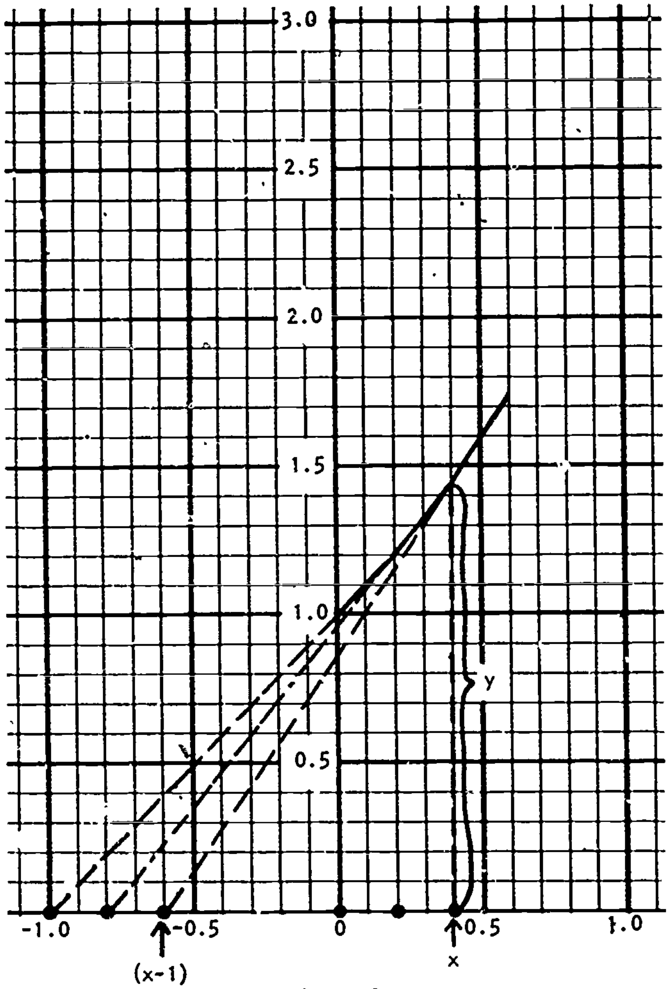


Figure 3.

8. Approximate e by completing Table VII using $\Delta x = 0.1$. Notice the constant ratio property: Each y value is $\frac{11}{10}$ of the previous y value. New y = Old y + (Δx)(old y) = $(1+\Delta x)$(old y). Keep four places of accuracy for your value of y.

TABLE VII

x	y	Δx y
0.0	1.0	0.1
0.1	1.1	0.11
0.2	1.21	0.121
0.3	1.331	0.1331
0.4		
0.5		
0.6		
0.7		
0.8		
0.9		
1.0		

What approximate value for e do you obtain in this case?

9. Use the method of Exercise 7 to sketch on Figure 4 the approximating curve, with $\Delta x = 0.1$. Use the results in Table VII to plot points and sketch curve in Figure 5. Compare the approximations.

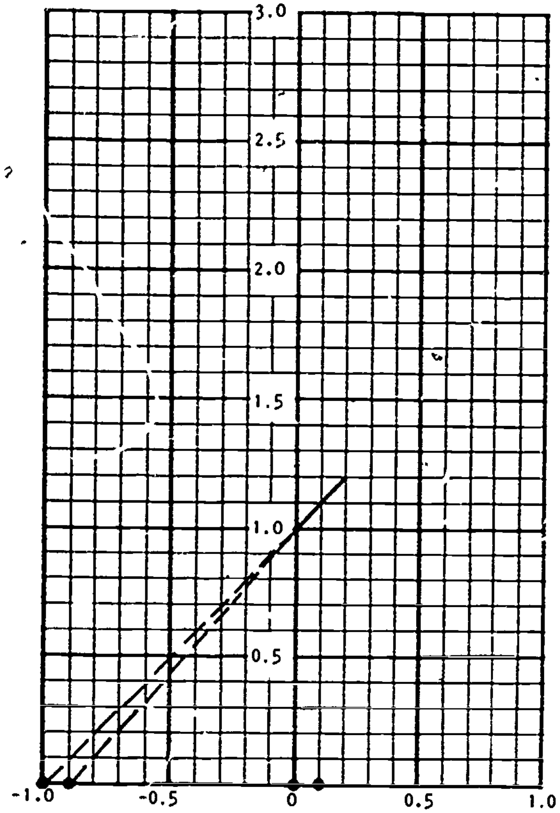


Figure 4.

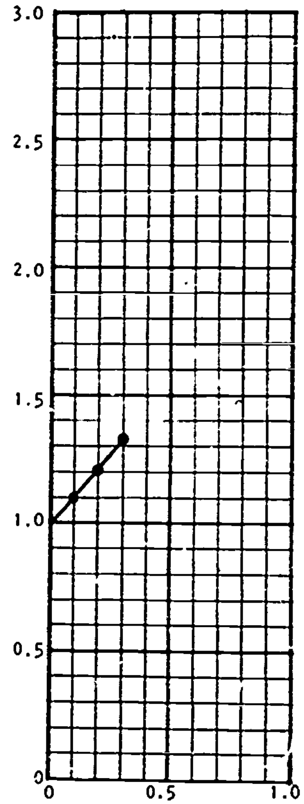


Figure 5.

5. COMPARISON OF METHOD 3 WITH METHOD 2

The approximations in Method 3 are not very accurate, although we do get some feel for the shape of the curve $y = e^x$ by sketching the graphs of the approximations. In fact, Methods 2 and 3 yield the same approximation for e when the n in Method 2 and the Δx in Method 3 are related by the equation $\Delta x = \frac{1}{n}$.

Exercises

10. a) Compare the estimate of e with $\Delta x = .5$ obtained in Section 4.2 with the estimate in Exercise 1 with $n = 2$.

b) Compute $(1 + 0.2)^5$ and compare with the estimate $\Delta x = 0.2$ in Exercise 6.

c) Compute $\left(1 + \frac{1}{10}\right)^{10}$ and compare with the estimate for e obtained in Exercise 8.

11. a) Compute $\left(1 + \frac{1}{5}\right)^0$, $\left(1 + \frac{1}{5}\right)^1$, $\left(1 + \frac{1}{5}\right)^2$, $\left(1 + \frac{1}{5}\right)^3$, $\left(1 + \frac{1}{5}\right)^4$, and $\left(1 + \frac{1}{5}\right)^5$. Compare these values to the values of the y-coordinates in Figure 2.

b) Compute $(1.1)^0$, $(1.1)^1$, $(1.1)^2$, $(1.1)^3$, . . . , $(1.1)^{10}$ and compare with y-coordinates computed in Exercise 8 and plotted in Figure 5.

c) Letting $\Delta x = \frac{1}{n}$, can you find a pattern that enables you to fill in Table VIII with the same values you would get if you used the Euler Method?

TABLE VIII

x	0			$\frac{3}{n}$	$\frac{4}{n}$	. . .	$\frac{k}{n}$	. . .	$\frac{n}{n} = 1$
y	1	$\left(1 + \frac{1}{n}\right)$	$\left(1 + \frac{1}{n}\right)^2$						

6. METHOD 4: TAYLOR POLYNOMIALS

6.1 Description of Taylor Polynomials

Here is another way to produce functions approximating e^x. The approximating functions are polynomials and the higher the degree of the polynomial, the better the approximation. This method is named after the English mathematician Brook Taylor (1685 -1731), although it was discovered by James Gregory and published in 1688 when Taylor was three years old. It is the most powerful of the four methods in this unit.

The only value of the function $y = e^x$ which we are able to write in exact decimal form at this stage in our work corresponds to $x = 0$; that is

$$e^0 = 1.$$

The first derivative has the same value as the function because $y' = y$. Consequently, we also know that

$$y'(0) = 1.$$

But we can get even more information about this function by taking derivatives of both sides of the equation $y' = y$. This gives $y'' = y'$ so that

$$y''(0) = y'(0) = 1.$$

If we continue this process, we discover, letting $y^{(n)}$ stand for the nth derivative of y, that $y^{(n)}(0) = 1$ for every n. The polynomial of degree n that we are about to construct (using this information) is called the Taylor polynomial of degree n for $y = e^x$ centered at $x = 0$.

We know that we can get a straight line approximation to $y = e^x$ by looking at the line tangent to curve at (0,1). The equation of this straight line is $y = x + 1$. $x + 1$ is a polynomial of degree 1, and for reasons that will

be obvious later, we write $P_1(x) = x + 1$. We will always mean $y = e^x$ whenever we use y. $P_1(x)$ has two essential features: $P_1(0) = y(0) = 1$ and $P_1'(0) = y'(0) = 1$. We should get a better approximation if we ask for a polynomial which has the same value as $y = e^x$, the same first derivative, *and the same second derivative* at $x = 0$. We look for the polynomial of lowest degree that satisfies these three conditions. Therefore let us write the general form of a polynomial of degree two

$$P_2(x) = a + bx + cx^2.$$

Now

$$P_2(0) = a = 1, \text{ since } y(0) \text{ must } = 1.$$

Also

$$P_2'(x) = b + 2cx,$$

so

$$P_2'(0) = b = 1, \text{ since } y'(0) \text{ must } = 1.$$

Finally

$$P_2''(x) = 2c,$$

so

$$P_2''(0) = 2c = 1, \text{ since } y''(0) \text{ must } = 1.$$

This gives us $c = \frac{1}{2}$. Thus our second order polynomial approximation to $y = e^x$ is

$$y = P_2(x) = 1 + x + \frac{x^2}{2}.$$

Exercises

12. Using Figure 6, graph the polynomial $P_2(x)$ for $-2 \leq x \leq 2$, and show that $P_2(1) = 2.5$. We call this the second degree approximation of e.

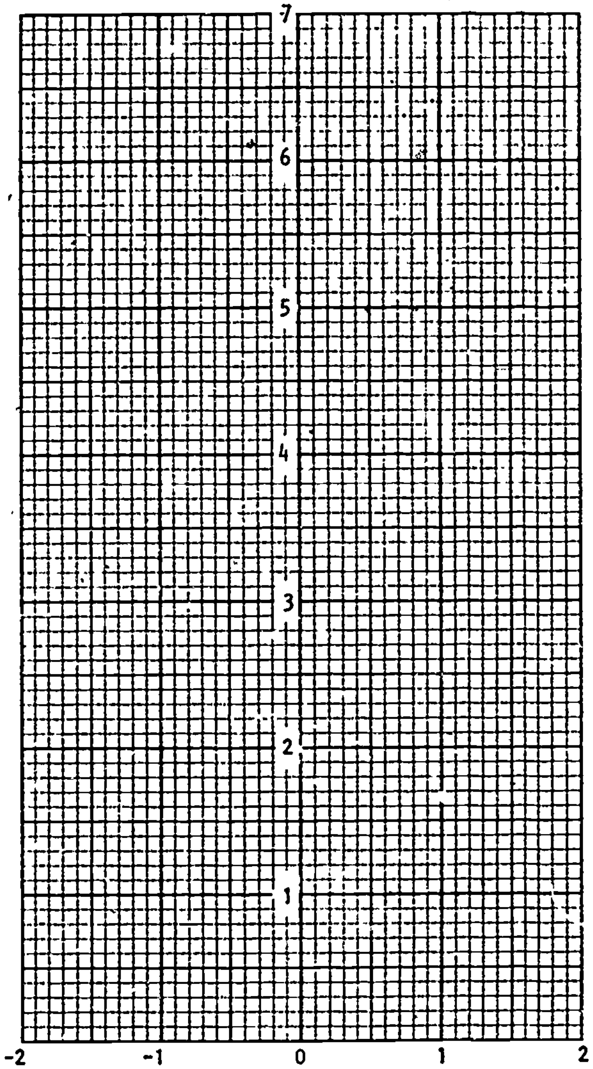


Figure 6.

13. Fill in the following computations which will give us the third degree approximation of e. We start with an arbitrary polynomial of degree three and then solve for the coefficients

by evaluating, at $x = 0$, successive derivatives of the arbitrary polynomials and equating each of them to 1.

We start with

$P_3(x) = a + bx + cx^2 + dx^3$; $1 = P_3(0) = a$, so $a = 1$.

$P_3'(x) = \underline{b + 2cx + 3dx^2}$; $1 = P_3'(0) = \underline{b}$, so $b =$ ______

$P_3''(x) =$ __________; $1 = P_3''(0) = 2c$, so $c =$ ______

$P_3'''(x) =$ __________; $1 = P_3'''(0) =$ __, so $d =$ ______

$P_3(x) =$ ____ + ____x + ____x^2 + ____x^3.

Now you know what $P_3(x)$ is. What is $P_3(1)$? ____. This is the third degree approximation to e.

14. Compute $P_4(x)$ and $P_4(1)$.

6.2 Factorial Notation

The symbol ! has special meaning to mathematicians when it follows a positive whole number: n! is read "n factorial" and it is defined to be the product of all positive integers less than or equal to n. In symbols $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$. As examples, $3! = 1 \cdot 2 \cdot 3 = 6$; $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$; $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 =$ ____; $6! =$ ____________.

Using this notation, $P_2(x) = 1 + x + \frac{1}{2!}x^2$ and $P_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$. Use this notation to express $P_4(x)$. $P_4(x) =$ ____________.

Exercises

15. Can you guess what $P_5(x)$ is? $P_5(x) =$ ______________.
Verify your guess by taking the first 5 derivatives of your guess and checking that they all equal 1 when $x = 0$. Use this guess to estimate e:

$e \approx P_5(1) =$ ____________.

6.3 Discussion of Accuracy of These Approximations

A proof of how accurate these approximationg are will have to wait, but $P_5(1)$ is within .00162 of e, and is too small. Next,

$$P_8(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8$$

and $P_8(1) = 2.7182788$ which again is too small, but is within .0000031 of e.

Note $P_9(1) = P_8(1) + \frac{1}{9!}$. What is $P_9(1)$? ______

We do not have to restrict our evaluations to the case $x = 1$, but can use these polynomials to approximate e^x for every x. Again, however, we must postpone a rigorous discussion of how good the approximation is.

Exercises

16. Compute $P_4\left(\frac{1}{2}\right)$ ______________. This is approximately $e^{1/2} = \sqrt{e}$, so square the approximation. $\left[P_4\left(\frac{1}{2}\right)\right]^2 =$ ________. Compare this to $P_4(1)$.

    What is a better approximation to e, $P_4(1)$ or $\left[P_4\left(\frac{1}{2}\right)\right]^2$? (Use the value of e given in Section 6.3 above.)

17. Compute $P_4\left(\frac{1}{8}\right)$ ______________. This is an approximation to $e^{1/8}$. What is $\left[P_4\left(\frac{1}{8}\right)\right]^8$? __________. Compare this to $P_4(1)$, and to $\left[P_2\left(\frac{1}{2}\right)\right]^2$? ______________. Which of the three is the best approximation to e?

6.4 General Behavior of These Polynomials

The approximating polynomials were picked to behave like e^x for $x = 0$. It would seem therefore that we can draw a conclusion about the behavior of these polynomials: the closer x is to 0, the better the approximation.

This behavior is shown by the results of Exercises 16 and 17 where we found that $P_4\left(\frac{1}{8}\right)$ is closer to $e^{1/8}$ than

$P_4\left(\frac{1}{2}\right)$ is to $e^{1/2}$, and that $P_4\left(\frac{1}{2}\right)$ is closer to $e^{1/2}$ than $P_4(1)$ is to $e^1 = e$. We can make a further test of this behavior by computing $P_4(2)$. Compute $P_4(2)$: $P_4(2) =$ ______________. We know $2.718 < e < 2.719$; consequently $(2.718)^2 < e^2 < (2.719)^2$, or

$$7.387524 < e^2 < 7.392961.$$

Your computations should show that $P_4(2)$ is not a very good approximation to e^2.

Exercise

18. Compute the following numbers:

    $P_5(2) =$ ______________

    $P_6(2) =$ ______________

    $P_7(2) =$ ______________

    $P_8(2) =$ ______________

From the results of Exercise 18 as well as previous exercises, we can draw another conclusion about these polynomials: the higher the degree of the polynomial, the better the approximation. This goes with our previous conclusion: the closer x is to zero, the better the approximation. Writing these more formally, we have:

1. If x is fixed, and $n > k$, $P_n(x)$ is closer to e^x than is $P_k(x)$.
2. If n is fixed, and $0 < a < b$, then $P_n(a)$ is closer to e^a than $P_n(b)$ is to e^b.

Exercise

19. Use Figure 7 to graph $P_4(x)$ on the interval $-2 \le x \le 2$. Compare this with the graph in Exercise 12 and then compare each to the graph of e^x drawn in Figure 8. See how these graphs support the two conclusions given above.

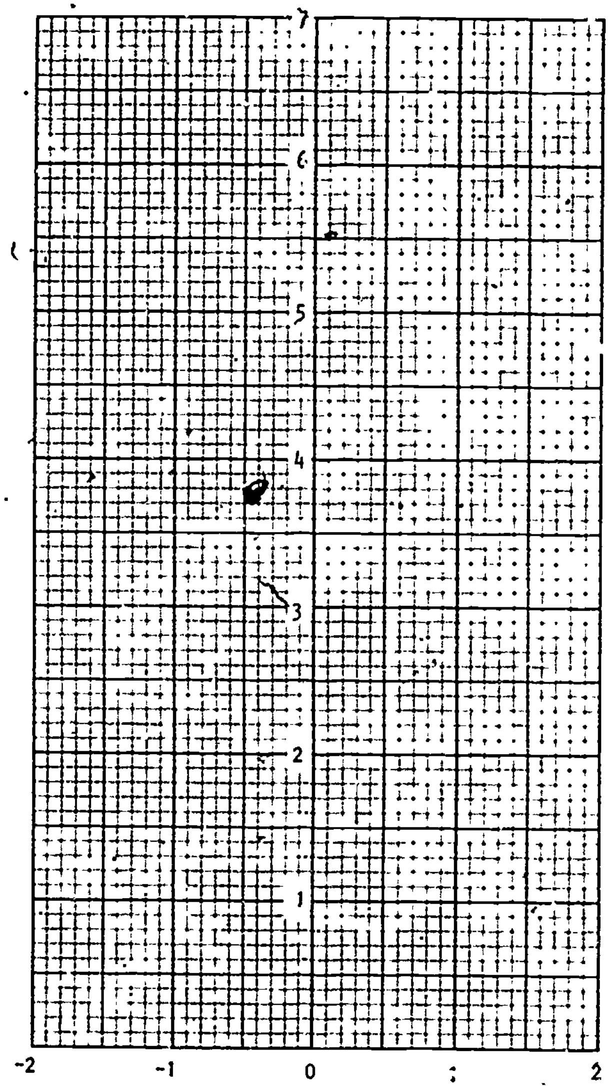


Figure 7.

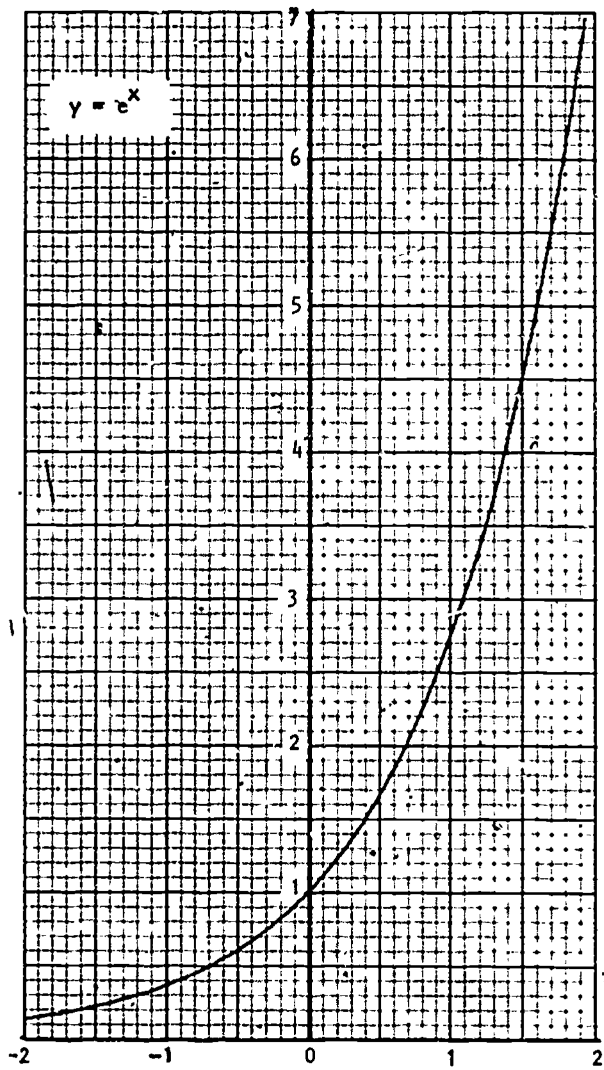


Figure 8.

7. SUMMARY

Of the four methods, the last is by far the most efficient. In fact, by computing $P_{12}(1)$ we have $e \approx 2.718281828$ and this approximation is accurate to the ninth decimal place. We should note that in spite of appearances, the decimal expansion of e does not have a repeating block. The number e is irrational.

8. ANSWERS TO EXERCISES

1.

TABLE III

n	$\left(\frac{n+1}{n}\right)^n \approx e$
2	2.25
4	2.4414
8	2.5658
16	2.6379
32	2.6770
64	2.6973
128	2.7077
256	2.7130
512	2.7156
1,024	2.7170

3.

TABLE IV

n	$16 = 2^4$	$64 = 2^6$	$256 = 2^8$	$1{,}024 = 2^{10}$
$\frac{1}{e} \approx \left(\frac{n-1}{n}\right)^n$	0.3561	0.3650	0.3672	0.3677

Questions in Section 3.3: $\frac{1}{0.3677} \approx 2.7196$; $e \approx 2.7183$.

4.

TABLE V

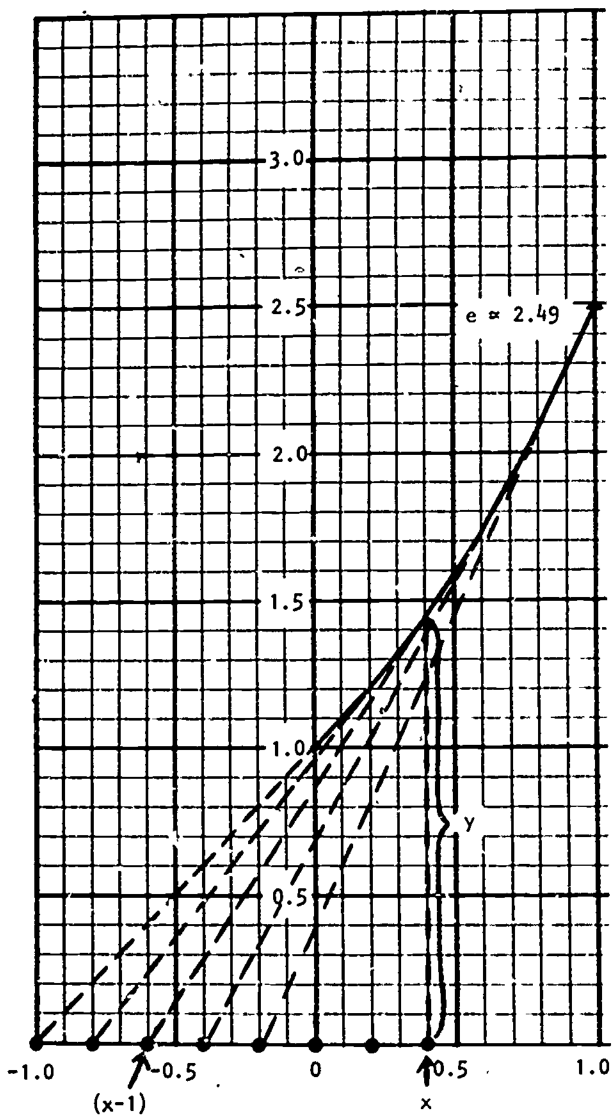
(For n = 256, divide 257 by 255 and square 7 times.)				
n	$256 = 2^8$	$512 = 2^9$	$1{,}024 = 2^{10}$	$2{,}048 = 2^{11}$
$e \approx \left(\frac{n+1}{n-1}\right)^{n/2}$	2.718296	2.718285	2.718283	2.718282

6.

TABLE VI

x-coordinate	y-coordinate	Slope of line leaving the point
0.0	1	1.0
0.2	1 + (1)(0.2) = 1.2	1.2
0.4	1.2 + (1.2)(0.2) = 1.44	1.44
0.6	1.44 + (1.44)(0.2) = 1.728	1.728
0.8	1.728 + (1.728)(0.2) = 2.0736	2.0736
1.0	2.0736 + (2.0736)(0.2) = 2.48832	2.48832

7.



8.

TABLE VII

x	y	Δx y
0.0	1	0.1
0.1	1.1	0.11
0.2	1.21	0.121
0.3	1.331	0.1331
0.4	1.4641	0.14641
0.5	1.6105	0.16105
0.6	1.7716	0.17716
0.7	1.9487	0.19487
0.8	2.1436	0.21436
0.9	2.3579	0.23579
1.0	2.5937	0.25937

9. See page 27 for graph for Exercise 9.

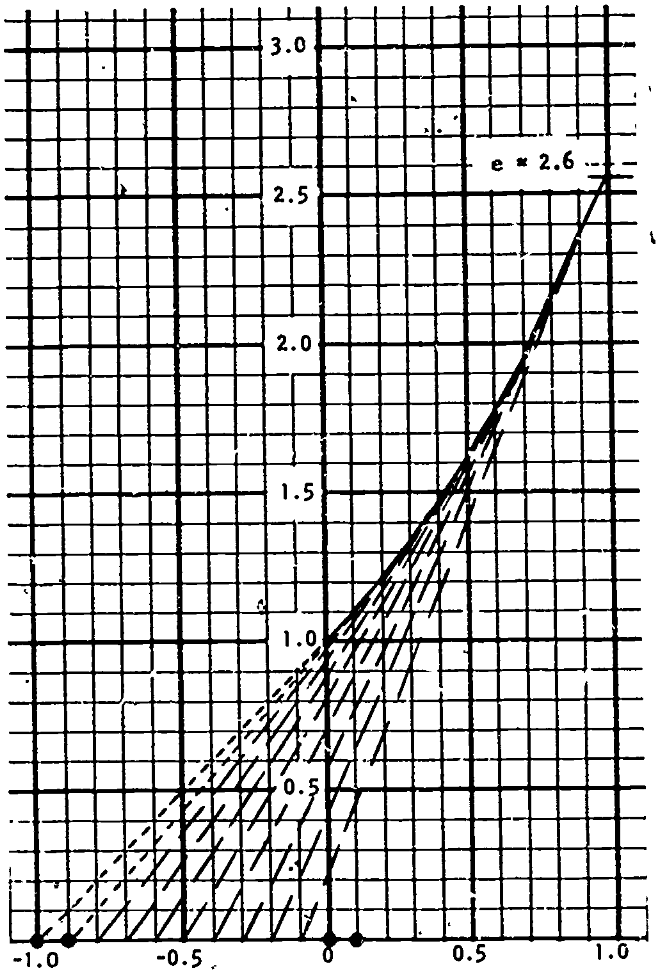
10. a) They are the same, 2.25.
    b) They are the same, 2.48832.
    c) They are the same, 2.5937.

11. a) They are the same.
    b) They are the same.
    c)

TABLE VIII

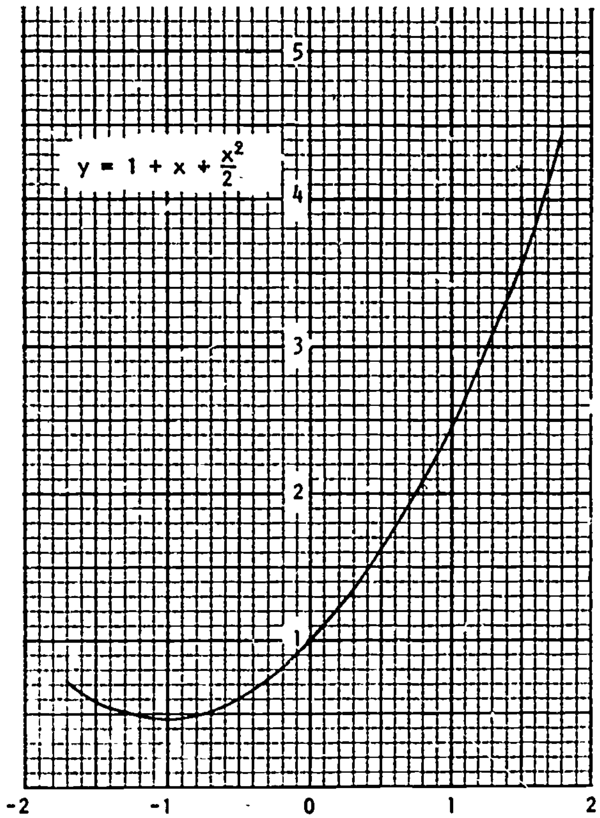
x	0	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{3}{n}$	$\frac{4}{n}$	....	$\frac{k}{n}$	....	$\frac{n}{n} = 1$
y	1	$\left(1+\frac{1}{n}\right)$	$\left(1+\frac{1}{n}\right)^2$	$\left(1+\frac{1}{n}\right)^3$	$\left(1+\frac{1}{n}\right)^4$		$\left(1+\frac{1}{n}\right)^k$		$\left(1+\frac{1}{n}\right)^n$

9. Answer from previous page.



111

12.



112

13. $P_3'(x) = b + 2cx + 3dx^2$; $1 = P_3'(0) = b$, so $b = 1$.

$P_3''(x) = 2c + 6dx$; $1 = P_3''(0) = 2c$, so $c = \frac{1}{2}$

$P_3'''(x) = 6d$; $1 = P_3'''(0) = 6d$, so $d = \frac{1}{6}$

$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

$P_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2\frac{2}{3} = 2.6666....$

14. $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

$P_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2\frac{17}{24} = 2.70833....$

Examples in Section 6.2

$5! = 120$; $6! = 720$

$P_4(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$

15. $P_5(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5$

$P_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2\frac{86}{120} \approx 2.71666....$

Example in Section 6.3

$P_9(1) \approx 2.7182816$ or 2.7182815.

16. $P_4\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{1}{24}\left(\frac{1}{2}\right)^4 \approx 1.6485475$

$\left[P_4\left(\frac{1}{2}\right)\right]^2 \approx 2.717346$

$P_4(1) \approx 2.708333...$

$\left[P_4\left(\frac{1}{2}\right)\right]^2$ is the better approximation.

17. $P_4\left(\frac{1}{8}\right) = 1 + \frac{1}{8} + \frac{1}{2}\left(\frac{1}{8}\right)^2 + \frac{1}{6}\left(\frac{1}{8}\right)^3 + \frac{1}{24}\left(\frac{1}{8}\right)^4 \approx 1.1331482$

$\left[P_4\left(\frac{1}{8}\right)\right]^8 \approx 2.7182768$

$\left[P_4\left(\frac{1}{8}\right)\right]^8$ is the best approximation to e.

Example in Section 6.4

$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

$P_4(2) = 1 + 2 + \frac{1}{2}(4) + \frac{1}{6}(8) + \frac{1}{24}(16) = 7.$

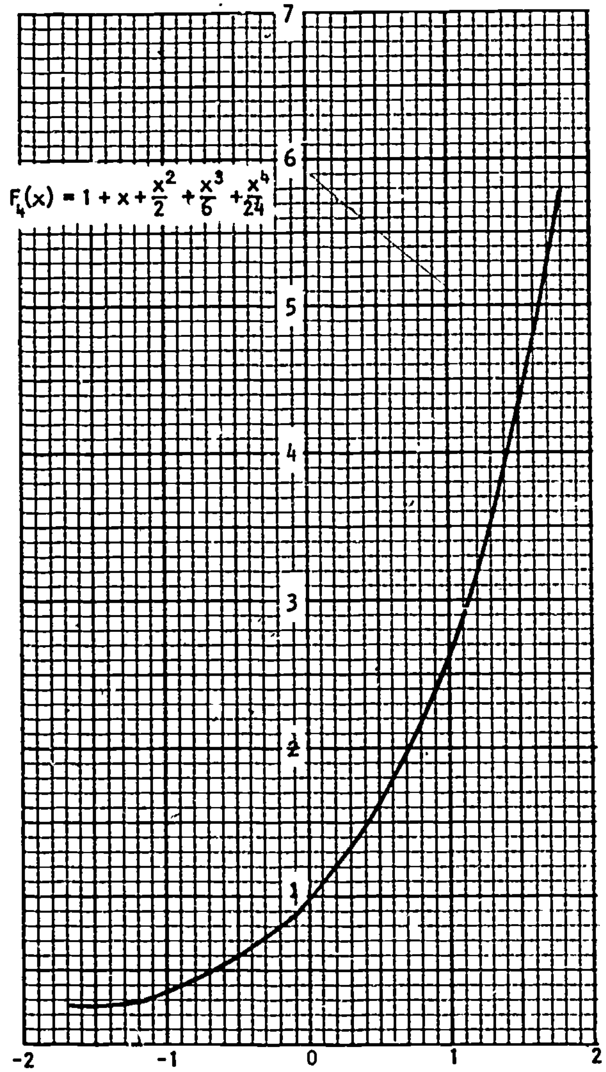
18. $P_5(2) = 7 + \frac{1}{120}(2)^5 = 7.2666....$

$P_6(2) = P_5(2) + \frac{1}{720}(2)^6 = 7.35555....$

$P_7(2) = P_6(2) + \frac{1}{5040}(2)^7 = 7.3809524$

$P_8(2) = P_7(2) + \frac{1}{40320}(2)^8 = 7.3873016.$

19.



9. MODEL EXAM

Part I

1. Complete this expression: $e^2 \approx \left(1 + \frac{\square}{100}\right)^{100}$.
2. Use a hand calculator with a squaring key to compute $\left(\frac{1025}{1024}\right)^{1024}$.
3. The number $\left(\frac{1025}{1024}\right)^{1024}$ is an approximation to $e^{\square}$.
4. If may be difficult to get a good approximation to e, using the formula $\left(\frac{n+1}{n}\right)^n$, because of __________ in the calculator being used.
5. What is the missing exponent? $\left(\frac{n+1}{n-1}\right)^{\square} \approx e$.

Part II

1. Use the Euler Method to sketch a curve that approximates $y = e^c$ for $0 \le x \le 1$ with $\Delta x = 0.25$.
2. Use the Euler Method to fill in the following table which gives an approximation to $y = e^x$. $0 \le x \le 2$ with $\Delta x = 0.4$.

x	0	0.4	0.8	1.2	1.6	2.0
y	1					

Part III

Let $P_n(x)$ denote the Taylor polynomial of degree n centered at $x = 0$ that approximates $y = e^x$.

1. What is $P_2(x)$?
2. Is $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3$ the same as $P_3(x)$? To answer, either show that it satisfies the defining properties of $P_3(x)$, or that it does not.
3. Use $P_4(x)$ to approximate e.
4. If we hold the value of x fixed, then how does $P_n(x)$ change as n changes?
5. The general reason $\left[P_6\left(\frac{1}{10}\right)\right]^{10}$ is a better approximation to e than is $\left[P_6(2)\right]^{1/2}$ is that __________________.

10. ANSWERS TO MODEL EXAM

Part I

1. $e^2 \approx \left(1 + \frac{2}{100}\right)^{100}$
2. $\left(\frac{1025}{1024}\right)^{1024} \approx 2.7170$
3. $\left(\frac{1025}{1024}\right)^{1024} \approx e^1$
4. round-off error
5. $\left(\frac{n+1}{n-1}\right)^{n/2} \approx e$

Part II

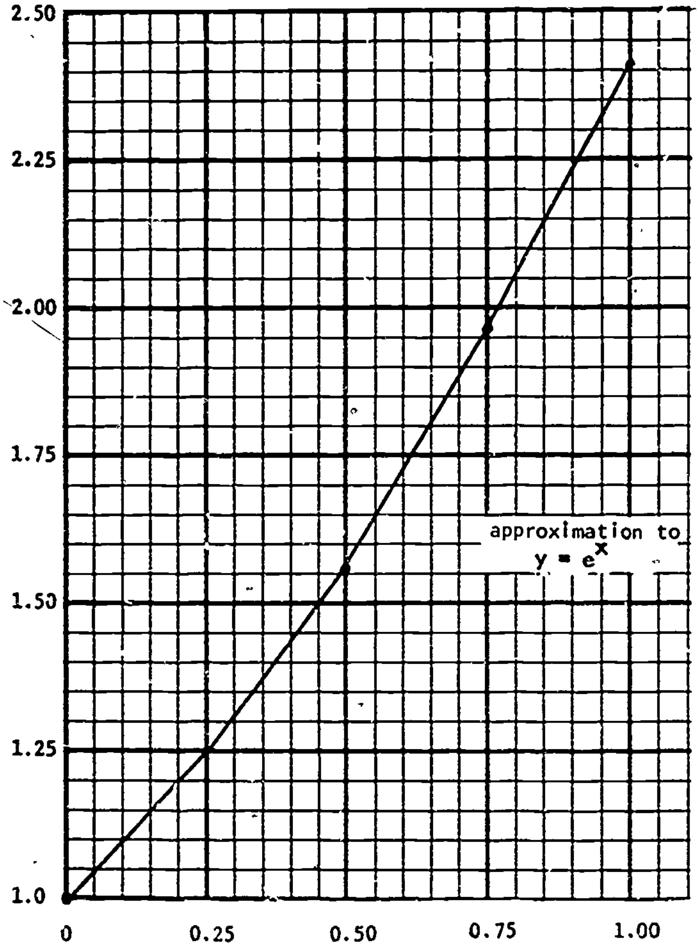
1. See answer graph on next page.
2.

x	0	0.4	0.8	1.2	1.6	2.0
y	1	1.4	1.96	2.744	3.8416	5.37824

Part III

1. $P_2(x) = 1 + x + \frac{1}{2}x^2$
2. $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \ne P_3(x)$, by looking at its third derivative. If $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3$, then $f'''(x) = 2$, while, by definition, $P_3'''(x) = 1$.
3. $P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$, so $e \approx P_4(1) = 2.708333...$
4. As n gets larger, $P_n(x)$ gets closer to e^x.
5. The smaller the value of x, the closer $P_n(x)$ is to e^x.

Answer to Model Exam Part II, Question 1.



117

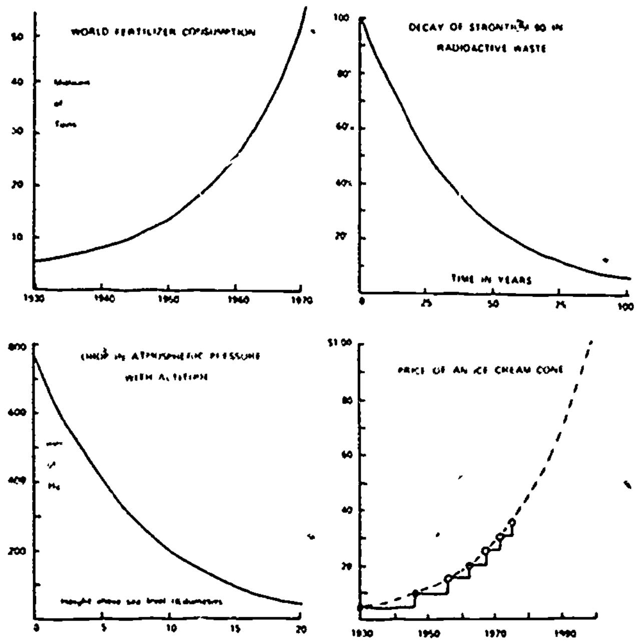
umap

UNIT 88

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

HOW TO SOLVE PROBLEMS INVOLVING EXPONENTIAL FUNCTIONS

by Raymond J, Cannon



INTRODUCTION TO EXPONENTIAL FUNCTIONS

UNITS 84–88

edc/umap/55chapel st./newton, mass. 02160

HOW TO SOLVE PROBLEMS INVOLVING EXPONENTIAL FUNCTIONS

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TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 88*

Title: HOW TO SOLVE PROBLEMS INVOLVING EXPONENTIAL FUNCTIONS

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Review Stage/Date: IV 6/12/78

Classification: EXPN FNCTN/SOLVING EXP PRDBS

Suggested Support Material:

References:

Riggs, Douglas Shepard, The Mathematical Approach to Physiological Problems, The MIT Press, Cambridge, Massachusetts, 1963, Chapter 6.

Simmons, George F., Differential Equations, with Applications and Historical Notes, McGraw-Hill Book Company, New York, 1972, Chapter 1.

Prerequisite Skills:

1. Be able to approximate values of e^x (Unit 87).
2. Be able to recognize problems that may be solved with exponential functions (Unit 84).
3. Be familiar with the terms half-life and doubling period (Unit 85).
4. (Recommended, but not essential) Be familiar with the limit

$$L(b) = \lim_{h \to 0} \frac{b^h - 1}{h} \quad \text{(Unit 86)}.$$

Output Skills:

1. Given a point on the graph of $y = Ae^{kx}$, and the slope of the graph at that point, be able to find A and k.
2. Given two points on the graph of $y = Ae^{kx}$, be able to find A and k.
3. Be able to solve elementary problems whose solutions involve exponential functions.

Other Related Units:

Recognition of Problems Solved by Exponential Functions *(Unit 84)*
Exponential Growth and Decay *(Unit 85)*
Development of the Function $y = Ae^{cx}$ *(Unit 86)*
Numerical Approximations to $y = e^x$ *(Unit 87)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank W.T. Fishback, Ruth R. Hailpern, L.M. Larsen, and Douglas F. Hale for their reviews and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

HOW TO SOLVE PROBLEMS INVOLVING EXPONENTIAL FUNCTIONS

by

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6/12/78

1. INTRODUCTION

The general form of an exponential function is $y = A_o e^{kx}$; this formula contains two arbitrary constants: A_o and k. In order to specify a particular exponential function, therefore, two pieces of information will be required. A given problem will generally provide the information by either 1) giving a point on the graph of the function and the slope of the graph at that point, or 2) giving two points on the graph. Our choice of e as the base makes the first case particularly easy, and we show how to solve both types of problems in this unit.

Before presenting these techniques in detail, however, we pause to present you with the following as indicative of the type of problems you will be able to solve.

1.1 Challenge Problem

A warm body cools at a rate proportional to the difference between it and the surrounding medium. Suppose you are in the cafeteria and you must leave in ten minutes for your next class. You have a cup of hot coffee and a small container of cold milk. You should begin drinking your coffee in five minutes. Will your coffee be colder if you pour the milk in now, or wait five minutes before adding the milk?

2. HOW TO SOLVE FOR y GIVEN A POINT AND SLOPE AT THAT POINT

Example 1:

An exponential function passes through the point x = 2, y = 4 with slope 3. What is the value of y at x = 1?

Solution to Example 1:

To answer this question, we want to know the particular exponential function whose general form is $y = A_o e^{kx}$. We must use the information we are given to solve for A_o and k. Because (2,4) is on its graph, we know that when x = 2, y = 4 and substituting these values in the expression $y = A_o e^{kx}$ we get our first equation:

$$4 = A_o e^{k \cdot 2}$$

The slope is also given: this is information about the derivative. If $y = A_o e^{kx}$, then $y' = k(A_o e^{kx})$, and $y' = ky$; we can solve for k immediately: $k = \frac{y'}{y}$. At the point (2,4) the graph has slope 3 so we use the values y = 4 and y' = 3 to obtain $k = \frac{y'}{y} = \frac{3}{4}$. We substitute this value of k in Equation 1 to produce the equation $4 = A_o e^{(3/4)2} = A_o e^{3/2}$; solving for A_o, we have

$$A_o = 4e^{-3/2}.$$

We now have $A_o = 4e^{-3/2}$ and $k = \frac{3}{4}$; substitute these values in $y = A_o e^{kx}$ to obtain

$$\begin{aligned} y &= 4e^{-3/2}e^{3x/4} \\ &= 4e^{3x/4 - 3/2} \\ &= 4e^{(3x-6)/4}. \end{aligned}$$

As an alternative way of writing this function, we could use one of the methods developed in Unit 87 to approximate $A_o = 4e^{-3/2}$. If we use e ≈ 2.72, then $e^{1/2} \approx$ (1.65); $(1.65)^{-3} = 1/(1.65)^3 = 1/4.49 = .22$, and 4(.22) = .88. We could say $A_o \approx .88$ and

$$y \approx .88e^{3x/4}.$$

Here is another example using this method.

Example 2:

What exponential function goes through the point (3, 4) with slope $\frac{1}{2}$.

Solution to Example 2:

Set $y = A_o e^{kx}$. Since $k = \frac{y'}{y}$, we have $k = \frac{1/2}{4} = \frac{1}{8}$. Now let x = 3 and y = 4 in the equation $y = A_o e^{kx}$ and obtain $4 = A_o e^{k3}$; substitute $k = \frac{1}{8}$ in this last equation and you have $4 = A_o e^{3/8}$; solving for A_o we have $A_o = 4e^{-3/8}$.

We may now use values $k = \frac{1}{8}$, $A_o = 4e^{-3/8}$ in the equation $y = A_o e^{kx}$ to write our answer as

$$y = 4e^{-3/8}e^{x/8} = 4e^{(x-3)/8}.$$

Alternatively, we may approximate $A_o = 4e^{-3/8} \approx 4(.69) \approx 2.76$ and write

$$y \approx 2.76e^{x/8}.$$

Exercises

1. Find the exponential function that goes through (0, 8) with slope 4.

   k = ______. A_o = ______. y = ______.

2. Find the exponential function that goes through (2, 12) with slope 3.

   i) k = ______. A_o = ______. y = ______.

   ii) Use some approximation techniques to write A_o in decimal form.

   $A_o \approx$ ______. $y \approx$ ______

---

3. HOW TO SOLVE FOR y GIVEN TWO POINTS ON THE GRAPH

3.1 A Typical Problem and Its Solution

We have to work a little harder in this case, since the value of k isn't so easy to find. What we must do is to solve two simultaneous equations involving k and A_o.

Example 3:

Find the exponential function that passes through (0, 4) and (2, 6).

Solution to Example 3:

Let $y = A_o e^{kx}$. Using x = 0, y = 4 we have $4 = A_o e^{k\cdot 0} = A_o e^0 = A_o \cdot 1 = A_o$. (In general notice that A_o is the value of y when x equals 0.) Using x = 2, y = 6, we have $6 = A_o e^{k\cdot 2}$. Since the first equation gives us $A_o = 4$, we substitute this value in the second equation and get $6 = 4e^{2k}$, or $e^{2k} = \frac{6}{4}$. The solution of this equation involves a function introduced in Unit 86 of this module, where it was called simply L. We pause in our solution of Example 3 for a brief review of this function. (The module on the logarithm function gives a more detailed treatment of this function, and different ways to evaluate it.)

3.1.1 How to Solve for k

The function L(b) is defined by the equation

$$L(b) = \lim_{h \to 0} \frac{b^h - 1}{h},$$

and this equation can be used to obtain a decimal approximation to L(b). There are two important properties that we will use in solving these problems. We label them for future reference:

(*) $L(e^c) = c$, and

(**) $e^{L(b)} = b$.

These formulas were developed in Unit 86.

Example 4:

Solve for k if $e^k = 5.4$.

Solution to Example 4:

Since $e^k = 5.4$, it follows that $L(e^k) = L(5.4)$. Now use (*) to write $L(e^k) = k$, and we have $k = L(5.4)$. We can leave the answer in this exact form, or use a method developed in Unit 86 to obtain a decimal approximation. If we let $h = \frac{1}{1{,}024}$ (a "small" number), we have

$$L(5.4) \approx 1{,}024(5.4^{1/1{,}024} - 1) \approx 1.69,$$

and so

$$k = L(5.4) \approx 1.69.$$

Exercises

In each of the following exercises, solve for k using the function L and then obtain a decimal approximation using $h = \frac{1}{256}$.

3. If $e^k = 3.3$, then k = ________ ≈ ________.
4. If $e^{3k} = 1.8$, then k = ________ ≈ ________.
5. If $e^k = .5$, then k = ________ ≈ ________.

We return to the solution of Example 3 which we left when confronted by the equation $e^{2k} = 1.5$. We can solve this now using the L function. Thus using (*), $L(1.5) = L(e^{2k}) = 2k$ and so $K = \frac{1}{2}L(1.5)$. We already have $A_o = 4$, and so our function is

$$y = 4e^{[L(1.5)/2]x}.$$

We can use (**) to rewrite this as $y = 4(1.5)^{x/2}$.

Example 5:

This is the most difficult example. Neither A_o nor k is readily apparent. Find the exponential function that passes through the points (2, 15) and (6, 135).

Solution to Example 5:

Set $y = A_oe^{kx}$. Use x = 2, y = 15 to get

(1) $15 = A_oe^{k\cdot 2}$.

Use x = 6, y = 135 to get

(2) $135 = A_oe^{k\cdot 6}$.

We get one equation involving only k by dividing Equation 2 by Equation 1. (More specifically by dividing the left-hand side of Equation 2 by the left-hand side of Equation 1, and also by dividing the right-hand side of Equation 2 by the right-hand side of Equation 1.)

$$\frac{135}{15} = \frac{A_oe^{k\cdot 6}}{A_oe^{k\cdot 2}}$$

The result is

(3) $9 = e^{6k - 2k} = e^{4k}$.

Thus using (*)

$$L(9) = L(e^{4k}) = 4k,$$

we find

$$k = \frac{L(9)}{4}.$$

We substitute this value of k in Equation 1 to solve for A_o:

$$15 = A_oe^{(\frac{L(9)}{4} \cdot 2)} = A_o(e^{L(9)})^{1/2};$$

by (**)

$$e^{L(9)} = 9$$

and we have

$$15 = A_o 9^{1/2} = 3A_o$$

and so

$$A_o = 5.$$

Thus, our function is $y = 5e^{[L(9)/4]x}$, which we can rewrite as

$$y = 5 \cdot 9^{x/4}.$$

3.1.2 How to Solve for A_o Given k

In the above example we used (**) to write $e^{L(9)} = 9$. We emphasize use of this formula with an example and some exercises.

Example 6:

Solve for A if $Ae^{L(16)/4} = 12$.

Solution to Example 6:

$12 = Ae^{L(16)/4} = A(e^{L(16)})^{1/4}$ which by using (**) we can say $= A(16)^{1/4} = A \cdot 2$. Thus, $12 = 2A$ and $A = 6$.

Example 7:

Find the exponential function whose graph goes through the points (2,27) and (4,9).

Solution to Example 7:

Let $y = A_o e^{kx}$. Set $x = 2$ and $y = 27$ to obtain

(1) $27 = A_o e^{2k}$.

Let $x = 4$ and $y = 9$ to obtain

(2) $9 = A_o e^{4k}$.

Divide Equation 2 by Equation 1, getting

$$\frac{9}{27} = \frac{A_o e^{4k}}{A_o e^{2k}}$$

or

$$\frac{1}{3} = \frac{e^{4k}}{e^{2k}} = e^{4k-2k} = e^{2k}.$$

Applying the function L to both sides we have

$$L(\tfrac{1}{3}) = L(e^{2k}) \quad 2k$$

so that

$$k = \tfrac{1}{2}L(\tfrac{1}{3}).$$

Now substitute this value of k back into either (1) or (2) and solve for A_o. Using (1) we have

$$27 = A_o e^{2(1/2)L(1/3)} = A_o e^{L(1/3)}$$

$$= A_o \cdot \tfrac{1}{3} \text{ and } A_o = 81.$$

Exercises

6. Solve for A if $Ae^{L(8)/3} = 10$. A = ________.

7. Solve for A if

$$Ae^{\frac{L(1/8)}{3}} = 10.$$

A = ________.

8. What exponential function goes through the points (3, 10) and (6, 50)?

---

4. WORD PROBLEMS

You learned in Unit 84 how to recognize word problems solved by using exponential functions. After recognizing such a problem, you must then analyze the way the data are given in the problem. Are two points given, or one point and the slope?

Example 8 :

The Surety Savings and Loan Company pays 5.25 percent interest compounded continuously. If a savings account contains $700 right now, how much will be there in six months?

Solution to Example 8 :

Let y(t) be the amount in account at time t. Obviously only one point is given: right now the account has $700. Let t = 0 correspond to now, so $A_o = 700$ and our function is $y = 700e^{kt}$. The rate of 5.25 percent is a yearly rate, so t is in units of years. The percent increase of 5.25 means $\frac{y'}{y}$ = 5.25 percent = .0525, and so k = .0525. Our function is thus $y = (700)e^{.0525t}$, where t is measured in years. Since six months is one-half of a year, the amount in six months will be $(700)e^{.0525/2}$. Using a calculator and $P_4(x)$, $x = \frac{.0525}{2} = .02625$, we have $P_4(.02625) = 1.0266$. Thus, the amount in the bank will be (700)(1.0266) = $718.62.

Example 9:

A biologist is studying a certain species of bacterium. At 1 p.m. she starts with 1000 bacteria. The temperature is kept constant and when she returns at 3 p.m. there are 8000 bacteria. What formula gives the number of bacteria under these conditions?

Solution to Example 9:

Discussion:

We will measure time in *minutes* with 1 p.m. as our starting point. Let A(t) be the number of bacteria t minutes after 1 p.m.; we know that the function has the form $A(t) = A_o e^{kt}$. Since 1 p.m. corresponds to t = 0 and there were 1000 at that time, we know the point (0,1000) is on the graph. Furthermore 3 p.m. corresponds to

t = 120 and the point (120,8000) is on the graph. We now proceed as in Example 7 and find the exponential curve through the points (0,1000) and (120,8000).

Solution:

Let A(0) = 1000 and t = 0 to find

(1) $1000 = A_o e^{k\cdot 0} = A_o$.

Next let A(120) = 8000 and t = 120 to find

(2) $8000 = A_o e^{k\cdot 120} = A_o e^{120k}$.

Note that (1) gives us A_o directly and we substitute the value $A_o = 1000$ into Equation (2) to obtain $8000 = 1000e^{120k}$ or

$$8 = e^{120k}.$$

Apply the function L to both sides and we have

$$L(8) = L(e^{120k}) = 120k.$$

Solving, k = L(8)/120 thus $A_o = 1000$ and k = L(8)/120; our function is $A(t) = 1000e^{(L(8)/120)t} = 1000\cdot 8^{t/120}$.

Remark on Example 9:

If we wish to find the doubling period for this species we would want to know for which t is $A(t) = 2\cdot A_o$; this gives us the equation

$$1000e^{(L(8)/120)t} = 2000$$

or

$$e^{(L(8)/120)t} = 2.$$

Applying the L function to both sides we have

$$(L(8)/120)\cdot t = L(2).$$

Multiply both sides by 120, $L(8)\cdot t = 120\cdot L(2)$ and

$$t = \frac{120\cdot L(2)}{L(8)}.$$

If you are familiar with logarithmic function you know we can write $L(8) = L(2^3) = 3L(2)$. (This was also derived in Unit 86, Section 7.4.) The doubling period for the species (at this temperature) is found to be

$$\frac{120\ L(2)}{3L(2)} = \frac{120}{3} = 40 \text{ minutes.}$$

Exercise

9. A country is growing at a rate of three percent per year. If it had 10,000,000 people in 1975, what will the population be in the year 2000?

Example 10:

We solve Exercise 11 of Unit 84 which is repeated here:

A certain factory has been dumping its chemical wastes into a river which flows into a lake. The chemical wastes of the factory cause a rash on the skin when their concentration in the water is 30 parts per million; they irritate the eyes at a concentration of five parts per million. The factory stopped dumping its waste into the river a month ago, and the concentration in the lake was then at 75 parts per million. The clean water of the river entering the lake mixes with the polluted water of the lake; then, as the river flows out of the lake, some of the polluting aterials are carried off. The flow of the river is constant; together with our mixing assumptions, this means that the rate at which the waste material is being carried off is proportional to the amount of waste in the lake. The chemical waste now in the lake amounts to 70 parts per million. How long will it be before people can swim in the water without getting a rash? Without their eyes burning?

Solution to Example 10:

Let c(t) be chemical concentration at time t, with t measured in months. Take the time when the factory

stopped dumping as initial time (t = 0) so we have c(0) = 75. A month later corresponds to t = 1, and c(1) = 70. So the curve passes through (0, 75) and (1, 70). Using (0, 75) we see $A_o = 75$. Thus $c(t) = 75e^{kt}$. Now using (1, 70) we get $70 = 75e^{k\cdot 1}$, so $e^k = \frac{70}{75}$ and $k = L(\frac{70}{75})$.

The function is $c(t) = 75e^{L(70/75)\cdot t}$ or

$$c(t) = 75(\frac{70}{75})^t.$$

We want to know when c(t) < 30; we have to solve for t in the inequality

$$75e^{L(70/75)t} < 30,$$

which is equivalent to

$$e^{L(70/75)t} < \frac{30}{75}.$$

By (*), this is the same as $L(\frac{70}{75})t < L(\frac{30}{75})$.

Since $\frac{70}{75} < 1$, $L(\frac{70}{75})$ is negative and dividing by a negative number reverses the inequality, so we want

$$t > \left[L(\frac{30}{75}) \div L(\frac{70}{75})\right].$$

Using $h = \frac{1}{256}$, we have

$$L(\frac{30}{75}) \approx -.915$$

$$L(\frac{70}{75}) \approx -.069.$$

Thus, $L(\frac{30}{75}) \div L(\frac{70}{75}) \approx 13.3$.

It will take over 13 months from the time the company stops dumping its waste into the river for the pollutant level to drop below 30 parts per million.

Exercises

10. Use this method to solve $75e^{L(70/75)t} < 5$.

    Answer: $t >$ ______

11. Use (**) to write $e^{L(70/75)}$ as $\frac{70}{75}$, and write $c(t) = 75(\frac{70}{75})^t$. Now take successive values of t = 2, 3, 4, . . ., and see when c(t) < 30. For what t is c(t) < 5? Compare these results with previous answers.

12. If a bank pays five percent interest compounded continuously, how long does it take for a saving's account to double in size?

13. Go back to Unit 84 and find the exponential function that solves Exercises 1, 4, 6, 8, and 12.

5. ANSWERS TO EXERCISES

1. $k = \frac{1}{2}$, $A_o = 8$, $y = 8e^{x/2}$

2. i) $k = \frac{1}{4}$, $A_o = 12e^{-1/2}$, $y = 12e^{(x-2)/4}$

   ii) $A_o \simeq 7.28$, $y \simeq 7.28e^{x/4}$

3. $k = L(3.3) \simeq 1.197$

4. $k = \frac{L(1.8)}{3} \simeq .196$

5. $k = L(.5) \simeq -.6922$

6. A = 5

7. A = 20

8. $y = 2e^{\frac{L(5)}{3}x}$ or $y = 2(5)^{x/3}$

9. $y = 10{,}000{,}000e^{.03t}$, t = 0 in 1975 so population is $10{,}000{,}000 \cdot e^{(.03)25} = (10{,}000{,}000)e^{.75}$. Using $e = 2.718$, $e^{3/4} = (e^{1/4})^3 \simeq (1.284)^3 \simeq 2.117$ and population $\simeq$ 21,170,000.

10. t > 39

11.

```
* RUN

t = 1        c(t) = 70
    2               65.3333
    3               60.9778
    4               56.9126
    5               53.1184
    6               49.5772
    7               46.272
    8               43.1872
    9               40.3081
    10              37.6209
    11              35.1128
    12              32.772
    13              30.5872
    14              28.548
    15              26.6448
    16              24.8685
    17              23.2106
    18              21.6632
    19              20.219
    20              18.8711
    21              17.613
    22              16.4388
    23              15.3429
    24              14.32
    25              13.3654
    26              12.4743
    27              11.6427
    28              10.8665
    29              10.1421
    30              9.46595
    31              8.83489
    32              8.2459
    33              7.69617
    34              7.18309
    35              6.70422
    36              6.25727
    37              5.84012
    38              5.45078
    39              5.08739
    40              4.74823
    41              4.43168
```

136

12. To solve for t in $e^{.05t} > 2$ or $.05t > L(2)$, $t > 20L(2)$: Using $h = \frac{1}{256}$, $L(2) \approx .694$, so $t > 13.88$ years, or about 14 years.

13. Repeat of the following:

Exercise 1: $k \approx .17$, and goes through (0, 1), $y \approx e^{.17x}$

Exercise 4: $k \approx -.5$, $A_o = 5$, $y \approx 5e^{-x/2}$

Exercise 6: $L(d) = (400)(\frac{1}{2})^{d/20}$

$L(100) = (400)(\frac{1}{2})^5 = \frac{400}{32} = 12.5$ and Roger should start defrosting.

Exercise 8: $P(t) = \$5,000e^{-.17t}$

$P(6) = \$5,000e^{-1.02} \approx \$1,803$

Exercise 12: The curve goes through (0, 3) and (10, 2.7); the problem is asking for t when y = 1. $y \approx 3e^{-.01x}$. Set y = 1 and solve for x; $x \approx 110$ minutes.

137

6. MODEL EXAM

1. Given an exponential function that passes through the point (1,8) with slope $\frac{1}{2}$, write the exponential function in the form

$$y = Ae^{kx}.$$

2. Given an exponential function that passes through (1,4) and (3,6), write the function in the form

$$y = Ae^{kx}.$$

3. A fossil is found in a cave, and taken to a laboratory to be analyzed. It is found to emit about seven rays from carbon-14 per gram per hour. A living body radiates at a rate of 918 rays per gram, and radioactive carbon-14 has a half-life of about 5,600 years. Approximately how old is the fossil?

7. ANSWERS TO MODEL EXAM

1. $y = (8e^{-1/16})e^{x/16} \approx 7.5e^{x/16}$

2. $y = 4(\frac{2}{3})^{1/2}e^{\frac{L(3/2)}{2}x}$.

3. The fossil is between 39,000 and 40,000 years old.