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ABSTRACT

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Children appear to interpret the "equals" sign as an operator ("adds up to") not a relational ("the same as") symbol--e.g., viewing equations like 13 = 7 + 6 or 8 = 8 as senseless. This study, a natural experiment, examined the effects of long-term instruction emphasizing a relational definition of "equals." In a partially standardized clinical interview, first- through third-graders evaluated a variety of familiar and unfamiliar equation forms. The curriculum seemed effective in inducing a relational view of "equals." An operator interpretation was also clearly evident, but attributed to the cognitive factor of assimilation--not to relatively immutable (age-related) cognitive limitations.

The Effects of Instruction on Children's Understanding of the "Equals" Sign

A basic concept in formal arithmetic is the equivalence relationship denoted by the "=" symbol. Various researchers have noted that children do not tend to view the "equals" sign as "the same as," i.e., as a "relational" symbol (e.g., Behr, Erlwanger, & Nichols 1976, 1980; Van de Walle 1980). Instead, primary school children appear to interpret it in terms of action performed--e.g., "adds up to" or "produces" (Ginsburg 1977). In other words, children appear to view "equals" as an "operator" symbol (a "write something symbol"). As a first-grader put it: "It means it would add up to and whatever the answer was you'd put down." Children, it appears, expect written (horizontal) equations to take a particular form: An arithmetic problem consisting of two (or perhaps more) terms on the left, the result on the right, and in between, a connecting ("equals") symbol (e.g., 3 + 2 = 5). Children tend to reject equations such as 13 = 7 + 6, 6 + 4 = 3 + 7, and 8 = 8 that do not adhere to the typical form and easily lend themselves to an operator interpretation of "equals" (e.g., Behr et al.; Ginsburg; Nichols 1976). Weaver (1971, 1973) found, moreover, that children had greater difficulty solving for a missing element in an equation when the arithmetic operation (problem) was on the right (e.g., □ = 5 + 8, 13 = 5 + □, 13 = □ + 8) than when it was on the left (e.g., 5 + 8 = □, 5 + □ = 13, □ + 8 = 13.

Research typically finds that viewing "equals" as an operator sign persists through elementary school (e.g., Behr et al. 1976, 1980). Moreover, a restricted understanding of equals may continue into high school and college and may affect math learning at these levels (e.g., Byers & Herscovics 1977; Frazer 1976). For instance, if "equals" is not viewed as a relational sign--as a bridge between numerically equivalent expressions--algebra solution strategies (such as adding identical elements to each side of an equation to simplify the expression on one side) may not be meaningful and may simply be learned by rote (Byers & Herscovics).

Why then do children seem to view "equals" as an operator rather than a relational symbol? One view is that it is an artifact of their early arithmetic training (Renwick 1932). Children are usually introduced to "equals" in the context of adding and in the format: 1 + 1 = ____. Workbook and ditto exercises reinforce this format, and the child becomes accustomed to "equals" implying "adds up to" (cf. Van de Walle 1980). Indeed Denmark, Barco, and Voran (1976) surveyed 10 elementary texts and found that "equals" as a relational symbol was generally not developed. In brief, the "equals" sign may be assimilated into notions propagated by instruction. Children may reject or have problems with atypical forms because they are generally unfamiliar with them (cf. Weaver 1973).

A second view is that children's limited conception of the "equals" sign is due to their cognitive limitations. These cognitive limitations are deep-seated in that they are tied to age--i.e., stage or maturational level. For example, a relational view of "equals" might depend upon consolidation of concrete operational thinking or the advent of more abstract formal operational thought. Kieran (1980b) notes that 13 years is a period of transition between the children's requiring an answer after the "equals" sign and their accepting it as a symbol of equivalence. Subjects between 12 and 14 years initially took an operator view of "equals," but after training, generally took a relational view (e.g., justifying equivalence statements by an "equal values argument"--that both sides have the same value) (Herscovics & Kieran 1980; Kieran 1980a). Similarly, Collis (1974) suggests that it is not until after 13 that children can deal with equations flexibly. He argues that children from 6 to 10 years of age are not capable of accepting a lack of closure, and as a result, equations such as 4 + 5 = 3 + 6 or 4 + 5 = 3 x 3 are incomprehensible. Closure of an operation on two elements depends on actually <u>seeing</u> the results <u>replace</u> the original elements. Thus 4 + 5 = ____ is meaningful only when a child sees 9 written on the right hand side of the equation. Approximately 10 to 13 years is a transitional period in which meaning still requires a unique result (such as 9 for 4 + 5) but not its physical expression.

The two views have very different educational implications. According to the first (instruction-related) view, we need only change the nature of mathematics instruction to promote a relational view of "equals." According to the second view, the conceptual inadequacy of equals is tied to deep-seated cognitive limitations, and hence, changing the nature of young children's instruction should not have much impact.

What empirical evidence is there for each view? Anderson (1976) undertook a training study in which second graders in the experimental group were taught to treat "equals" as a relational symbol. Indeed, children who received this training were more likely than control children to accept atypical equation forms. Denmark et al. (1976) undertook a training study which examined children's interpretations of "equals" as well as acceptance of atypical forms. Their subjects included first-graders who had not yet been introduced to the "+" and "-" signs in school. Over two months time, the children were exposed--via activities with a balance--to different equation forms such as 6 = 4 + 2. As a result, children were less inclined to consider equations of different forms as incorrect, but the "equals" sign was still primarily interpreted as an operator. Denmark et al. concluded that intellectual development as well as instruction was a contributing factor to viewing "eqals" as an operator symbol.

Are there indeed deep-seated--relatively immutable--cognitive factors such as developmental stage limitations or an inability to accept a lack of closure, which interfere with or inhibit an understanding of "equals" as a relational symbol? Or, is the primary issue one of instruction, in which case, intervention might be successful in promoting an understand of "equals" as a relation? Perhaps current instruction--as well as training efforts like those of Denmark et al. (1976)--have been insufficient in promoting a relational view.

We had the good fortune of finding a natural experiment which addressed this issue. In the course of visiting classrooms of a Rochester suburban school as part of another study, we discovered that the math curriculum defined equals as "the same as" and gave primary students experience in seeing a variety of equation sequence forms. This provided an opportunity to check the effects of a long term and systematic effort to teach a relational meaning of equals to young children before they reached the proposed transitional age 13. As a result of instruction, will six-to nine-year-olds accept as valid atypical forms which they had <u>and</u> had not been introduced to (e.g., 13 = 7 + 6 and 7 + 6 = XIII, respectively). Will they justify such forms in terms of each side having the same value, which has been taken to indicate a relational rather than operator view of "equals" (cf. Kieran 1980b)? To better assess the children's conception of "equals," we attempted to distinguish between an equation's form and its meaning. That is, we measured separately children's perception of the acceptability of or familiarity with an equation's form and their judgement of its validity. Thus, we could deal with a situation in which a child might say that 8 + 1 = 3 x 3 looks strange (its form is unusual or foreign) but is nevertheless correct.

One might expect that children receiving instruction emphasizing a relational view of "equals" will accept both the form and the validity of typical (e.g., 7 + 6 = 13) and atypical (e.g., 7 + 6 = 4 + 9) equations they had been exposed to; will accept the validity, but perhaps not the form of atypical equations they had not been exposed to; and will reject incorrect statements.

Method

The Wynroth Math Curriculum

The math curriculum we investigated, developed by Wynroth (1975), is individualized and consists of a sequence of games. Learning and using the rules of a game teaches the child one or two concepts. The first concepts taught are counting, same number, more than, less than, order (the number just before or after another in the count sequence) and (recognition of) the written numerals 0-9. No written work accompanies this first phase of the curriculum. After this the program branches off into three subsequences which are taught simultaneously: operations (addition, multiplication, subtraction and division in that order), missing number (in conjunction with addition and multiplication), and base ten. These concepts are also taught through game activities. Only after each game (concept) in these subsequences is mastered are children introduced to corresponding written representations of the concepts (worksheets).

The Wynroth teacher guide points out that the teacher must take care how concepts are defined. Indeed, the guide insists--rather dogmatically--that teachers should use and permit only curriculum specified definitions of terms. According to the curriculum guide, "equals" is defined as the "same number" in order to avoid the initial learning of "equals" as "the answer is."

Moreover, the term "equals" is not introduced in the context of addition, but in a manner which emphasizes a relational meaning. Dice games are used to teach the concept of addition. The child rolls the dice and the curriculum manual instructs the teacher to say, for example, "How much is 3 plus 4." If necessary, the teacher might add, "What number did you get when you counted them 3 [point] plus 4 [point]?" No mention of "equals" is supposed to be made in this context.

Children first see written equations in the missing number subsequence in such games as "Supposed to Be." In "Supposed to Be," each player picks a number of squares on which there is printed a one digit number 0 to 9. The first player then draws a card on which there is an equation with a missing element (e.g., 3 + 2 = □). If the first player has a number square that would correctly fill in the missing element, s/he may keep the equation, and draw a replacement square. The second player then would draw a new equation card and play would continue. If the first player did not have a number square which would correctly fit into the equation sentence, then s/he would pass the equation card to the second player who would then see if any of his/her squares fit the equation sentence, etc. The first player, meanwhile, would have the option of trading in one of his/her number squares and picking a new one. The player who collected the most completed equation sentences wins. It is important to note that this game immediately introduces children to a variety of equation forms (e.g., $1 + \square = 3 + 2$, $\square = 1 + 1$, etc.) Therefore, the child first sees the "equals" sign in a variety of contexts in an attempt to discourage an operator view of "equals."

Finally, the first written work (worksheets) involving equations are in the form of $4 = 4$, $4 \square 4$, $4 = \square$, $5 \neq 3$, etc. Moreover, written addition is introduced in the form $2 \overset{3}{+} 1$, where the child writes the answer above the "plus sign." The "equals" sign is first introduced with addition in the form $3 + 1 = 2 + 2$, $3 + 1 \neq 4 + 2$, or $3 + 1 < 4 + 3$ where the child would be asked to fill in a missing addend or relations sign (e.g., $3 + \square = 2 + 2$, or $3 + 1 \square 4 + 2$. Thus, the curriculum makes a concerted effort to encourage a relational rather than operator view of "equals."

<u>Participants</u>

Fifteen children from each of a first grade (6-3 to 7-3 years, *M* = 6-9), second grade (7-3 to 8-3 years, *M* = 7-9) and third grade (8-3 to 9-6 years, *M* = 8-9) class from a school serving a middle- to upper-class community participated in the study. Repeaters were not included. An additional first-, second-, and third-grader were not included in the study because of incomplete data. All participating children had parent or guardian permission.

Testing took place in April, 1981. Thus all first graders had been in the program for seven months. Except for four children in each grade who had been exposed to the curriculum the year before, second and third-graders had also been in the program since September, 1980.

The first author tested eight children at each grade level, and a research assistant seven. On the basis of age and sex, the children were paired and randomly assigned to the experimenters.

Procedure

In the first (familiarization) session, the experimenters played several math games, which lasted 15 - 20 minutes, with their assigned children. In the second session, the first equality sentences task (Task 1) was administered after an estimation task designed for another study. About a week later, the second equality sentences task (Task 2) was administered after another estimation task.

Task 1. The experimenter explained: "Cookie Monster [a hand muppet] did some math homework last night. He wrote out some math sentences, and was wondering if you would be his teacher and correct them for him." The child was then shown the following equality sentences (each printed on a separate 3 x 5 card) in random order:

$7 + 6 = 13$

$13 = 7 + 6$

$7 + 6 = 6 + 7$

$7 + 6 = 4 + 9$

$7 + 6 = 6 + 6 + 1$

$7 + 6 = 14 - 1$

7 + 6 = (//////////)
///

7 + 6 = XIII

$7 + 6 = 6$

$7 + 6 = 0.$

For each trial the child was asked (Qo) "What does that say?" (Q1) "Did Cookie Monster write that correctly--like you would in math class?" If the child responded yes to Q1, the experimenter asked (Q2) "Is it correct -- does it make sense?" If the child responded no to Q1, Q2 took the form: "Is there any way in which it is correct -- does it make any sense?" Finally the child was asked Q3, "Should we put this in the right or wrong pile?" If the child was unsure about what pile to put it in, or concluded that it belong in the "middle" or "in-between" pile, the experimenter commented: "If you had to choose, which pile would you put it in?" If necessary, the experimenter commented: "If it makes sense, put it in the right pile; if not, put it in the wrong pile." Q1 as well as Q2 were asked in order to distinguish between the child's judgment on the equation's form and her opinion of its validity. Q3 tested the consistency of a child's belief in the correctness of the form. Addition questions were asked if interesting or ambiguous responses were made. In brief, the experimenters conducted a partially standardized clinical interview.

The interviews were tape recorded and transcribed for scoring.

Q1 was scored:

2 = response implies that the written form was proper or familiar

1 = unsure

0 = not proper or familiar

Q2 was scored:

2 = response implies that the equation made sense (was true or valid) or, the child induces the correct answer (the child initially said the problem was wrong [senseless], but after computing the answer or deciphering the

symbolism [such as the Roman numerals] indicated the equality was valid).

1 = inconsistent response (e.g., hedges or changes answers with the exception described above).

0 = form did not make sense to the child.

Q3 was scored:

2 = child indicates the "right" pile.

1 = ambivalent (e.g., "may-be pile" or "both piles").

0 = "wrong" pile.

Interrater agreement on 8 subjects per grade level was 85% for Q1 and Q2 and 96% for Q3.

Task 2. The experimenter explained: "Cookie Monster did some more math sentences last night and would like you to check them again. Put a C if his math sentence is right -- makes sense. Put an X if it is wrong -- does not make sense." The experimenter presented the following problems in one of four random lists:

$$7 = 5 + 2$$

$$4 + 3 = 3 + 4$$

$$6 + 4 = 5 + 5$$

$$6 + 3 = 4 + 4 + 1$$

$$5 + 1 = 7 - 1$$

$$2 + 4 = 3 \times 2$$

$$8 = 8$$

5 + 3 = ~~||||~~ |||

3 + 2 = V

$$2 + 2 = 2$$

$$4 + 2 = 42$$

$$3 + 1 = 1 + 1 + 1$$

For each equation, a "C" was scored as 2 and an "X" as 0.

Categories of Equations

Category 1 includes equation forms to which a subject was actually exposed in the course of instruction, including the typical form (e.g., 7 + 6 = 13) and atypical forms (e.g., 13 = 7 + 6). These varied by grade and to some extent by individual subject. For example, only a few first-graders had seen equations of the forms 7 + 6 = 6 + 6 + 1; none had seen forms like 7 + 6 = 14 - 1.

Category 2 includes unexposed atypical forms which are either relatively conventional (e.g., 7 + 6 = 14 - 1 or 6 + 3 = 4 + 4 + 1) or unconventional (e.g., 7 + 6 = XIII or 5 + 3 = ~~||||~~ |||).

Category 3 designates incorrect forms (e.g., 7 + 6 = 0, 4 + 2 = 42).

Results

Judgment of Form

In terms of their known progress in the math curriculum, most children considered proper or familiar category 1 forms (those they had been exposed to) but not category 2 (unexposed) or category 3 (incorrect) forms. The participants were generally accurate, then, in identifying forms they had been exposed to and accepting them even if they were atypical. There were a few exceptions. While the curriculum provided considerable exposure to such equation forms as 13 = 7 + 6, only 47% of the first-graders and 20% of the second-graders considered such a form correct or familiar. Also, just less than half the second-graders (6 of 13) and third-graders (7 of 15) indicated acceptance or familiarity with the equation $7 + 6 = 6 + 6 + 1$--a variation of the form $5 + 4 = 2 + 6 + \square$, to which they had received some exposure.

<u>Judgment of Forms' Sense</u>

In general, children in all three primary grades tended to consider sensible equation forms which they had been introduced to in their written practice--including atypical forms such as $8 = 8$, $7 = 5 + 2$, and $7 + 6 = 4 + 9$. An average of 87% of the first-graders, 58% of the second-graders, and 88% of the third-graders considered category 1 equations as sensible.

Insert Table 1 about here

Moreover, about half the participants considered sensible unexposed equation forms. Category 2 equations averaged 56%, 47%, and 60% acceptance by first-, second-, and third-graders, respectively. For instance, Roman numerals had not been taught yet in any of the classes. Nevertheless, a number of participants said $7 + 6 =$ XIII made sense <u>if</u> XIII meant 13. Margaret's (third-grade) response illustrates this inferential process: "I don't know what this means. This is a 3 [pointing to III]; this -- oh! This is 13 [indicating XIII] and 7 + 6 is 13, so that's right!" Another unexposed form was 7 + 6 = (IIIIIIIIII) III. Mandy, a first-grader, had a common reaction. She counted the marks and concluded: "This one [7 + 6] has 13 and this one [the marks] has 13. But you'd never see this in math class." Asked if the equation was written correctly, Mandy indicated no. Asked if it made sense, she responded, "Well they're both equal, they're both umh, 13.... It makes sense." She then indicated that the problem should go in the correct pile. One final example was provided by another first-grader

Sharon in response to 7 + 6 = 6 + 6 + 1, which she had not been exposed to. Asked if the math sentence was written correctly, she commented, "13 = 6 + 6 + 1 [laughs]. I think it goes in the wrong pile."

I (interviewer): Is it written correctly?

S (Sharon): Yes.

I: Have you seen math sentences like this in math class?

S: I've seen stuff like that [7 + 6], but like that [6 + 6 + 1]?

I: Nothing with three numbers?

S: No.

I: What does 6 + 6 + 1 make?

S: That's 66.

I: I think this is 6, +, 6, + 1.

S: That's wrong. That's right!

I: It's right?

S: Ahuh. 6 + 6 is 12, another 1 is 13.

I: So, did Cookie Monster get this right or wrong?

S: Right. [claps]. That's good.

Repeatedly, then, children distinguished between unfamiliarity with a form and whether it made sense. Repeatedly, they used their existing knowledge to infer that unfamiliar equation forms made sense.

A response bias can be discounted as participants rarely considered incorrect equations as sensible. Only an average of 8% of the first-graders and 2% of second- and third-graders considered category 3 forms sensible. In fact, nearly all the inappropriate responses occurred with 4 + 2 = 42 and might be due to unfamiliarity with written two-digit numerals.

Especially interesting were the definitions of the "equals" sign given as well as the justifications for accepting equality statements. Three first-graders defined "equals" directly as the "same as," and nine (60%) used an equal values justification to explain why an equation made sense. For example, Jane responded to the question "Why is 13 = 7 + 6 correct?" by commenting, "Because 7 + 6 is 13...so this is 13 and that's 13." After inquiries by the experimenter, Jane answered that "=" meant "equals" and that "equals" meant "the same." Seven (47%) of the second-graders and five (33%) of the third-graders indicated that "equals" meant the "same as," and 4 (27%) and 10 (67%), respectively, gave equal value justifications on at least one trial. Many children, then, defined "equals" in relational terms, and a majority of the sample used an equal values justification, which has been taken as one indication of a relational view of equals in older children.

Relational vs. Operator View

While there was considerable evidence of a relational view of "equals," this view often seemed to conflict with an operator concept. Most (44%) participants were inconsistent in evaluating exposed, atypical equation forms as sensible (see Table 2). Few children, however, seemed to have a view of "equals" that tended to exclude a relational concept entirely. For example, only 9% of the sample considered invalid more than one-half of the atypical forms.

Insert Table 2 about here

The first-graders appeared to be the most consistent in judging exposed, atypical forms as sensible, while the second-graders were the least so. The difference between the first- and second-graders reached statistical significance (p = .035, Fisher Exact 2 x 2 when columns 1 and 2 vs. 3 and 4 of Table 2 are considered). The first-graders might have had an advantage over both the second- and third-graders in that all their formal math instruction has been through the Wynroth curriculum. Eleven of the second-graders had a traditional math curriculum (Holt School Mathematics, 1974), which did not emphasize a relational meaning of the "equals" sign. For example, in this program children first see the "equals" sign exclusively in the context of typical equation forms such as 3 + 1 = 4 (re: Units 3-12). Eleven of the third-graders were exposed to this traditional math curriculum for the first two years and four for one year.

The evidence from children's remarks is consistent with the pattern described above. Only four first-graders of eleven who made remarks (36%) indicated an operator view of "equals" (e.g., "He wrote [13 = 7 + 6] backwards" or "[Equals] means to put another number that they would add up to"). However, two of these participants did so in a unique context (8 = 8) and, otherwise, made remarks which suggested a relational notion of "equals." One of these, for example, who was otherwise correct and who defined "equals" as "the same as," concluded that 8 = 8 was incorrect because there "were no pluses." The other, who was correct on 20 of 22 trials and gave an equal values justification for three, also rejected 8 = 8 because: "He forgot to write something here [after the first 8]."

The pattern of results among the third-graders was similar, including a minority who seem to treat 8 = 8 as unique. Again, only two (13%) of the third-graders exclusively defined "equals" as an operator symbol. Jean, for example, noted that 13 = 7 + 6 should be the other way around, but they did have problems like that in school. Asked what "=" meant, she responded: "Equals.... It means that's how much it is. [It does not mean anything else] as far as I know. That's how much it makes." Eight (53%) made remarks suggesting both operator and relational views of "equals." For example, Bill read 13 = 7 + 6 as "13 equals the same as 7 + 6, yah," indicated it was correct and that some of his math sheets had "stuff like that." Asked if it made sense as it was written, Bill replied, "Hmhmm. Beca e in our math program, see, this doesn't mean equals, it's the same as...." Asked what it means to say equals, he responded, "Like it adds up." Nevertheless, Bill, who was otherwise consistently correct, argued that 8 = 8 did not make sense. Likewise, Dan, who was otherwise consistently correct, defined equals as "the same as," and provided an equal values explanation for 7 + 6 = 4 + 9, rejected 8 = 8: "It doesn't make sense. It already tells you the answer." It may be that these two third-graders and the two first-graders described above believe the "equals" sign can stand for a relationship between two statements, but that one of the statements must involve an arithmetic operation. In other words, this error may represent an attempt by these children to assimilate the curriculum's definition of "equals" to their own operator view.

The remarks of the second-graders reflected this group's more tentative grasp of a relational view. Four (27%) made remarks which were indicative ofan operator view only, six (40%) of both views, four (27%) of a relational view only, and one made no scorable remarks. For example, nine (60%) noted that 13 = 7 + 6 was written backwards. The following transcript of the interview with Dick illustrates the difficulty many of the second-graders had in coming to terms with the two views of equals:

I: What does that say?

D: 13 [laughs] = 7 + 6.

I: Did Cookie Monster write that correctly -- like in math class?

D: You're supposed to start with this [7 + 6].

I: Is it correct--does it make sense?

D: Yes, sometimes our teacher writes it backwards on the math worksheets.

I: Should we put it in the right or wrong pile?

D: Both

I: If we can only put it in one. Does this make sense--to say 13 = 7 + 6?

D: No.

I: Right or wrong pile?

D: Correct.

Conclusions

In conclusion the Wynroth program seems fairly successful in cultivating a relational concept of "equals." For example, it promoted acceptance of the sensibleness of atypical forms--both taught and untaught--as well as justifications such as "equal values" well before the onset of adolescence--the transition point implied by some (e.g., Kieran 1980). Moreover, an inability to accept a lack of closure did not seem to be an issue (cf. Collis 1974). Most children considered sensible equations such as 7 + 6 = 4 + 9, 2 + 4 = 3 x 2, 7 + 6 = 14 - 1 without actually seeing a written result (sum). These results, then, do not support the view that stage- or maturation-related cognitive limitations prevent the development of a relational view of "equals." They are consistent with the training studies (Anderson 1976; Denmark et al. 1976) which suggest that changing the nature of math instruction can promote such a view of the "equals" sign.

We disagree, therefore, with Kieran's (1980) conclusion that the "name for a number" approach advocated by the School Mathematics Study Group (SMSG) is based on unwarranted psychological assumptions. The SMSG approach argues that there are various names for a number--e.g., "7" is but one name for a number which can also be named 6 + 1, 9 - 2, etc. Thus, children should be introduced to expressions such as 4 + 5 = 3 + 6, since 4 + 5 and 3 + 6 are other names for the number also called "9." This approach is intended to help children develop a mathematically more accurate view of number and the "equals" sign (equivalent relationships). Kieran suggests that research indicates that young children cannot assimilate a relational view of the "equals" symbol and hence the SMSG approach is misguided. The results from this study suggest a less pessimistic conclusion.

While the Wynroth curriculum was fairly successful in promoting a relational concept of "equals," this view often conflicted with or was subordinate to an operator view. Thus there was a cognitive barrier to viewing "equals" as a relational symbol. The first-grade teacher did note, for example that problems in the form of 6 + 6 = □ seem to be the easiest for her children. Problems in the form of 6 + 3 = 4 + □ were hard, and problems such as □ = 6 + 6 were the hardest. We suspect that the cognitive factor militating against a relational view of the "equals" sign is the process of assimilation rather than age-related cognitive or developmental factors. The operations such as adding are familiar processes which make sense to the child even before entering school (Gelman & Gallistel 1978; Ginsburg 1977). For instance, children are accustomed to putting together two sets and counting the total, but relatively unaccustomed to separating a counted set and counting its components (Allardice 1981). It may be that when "equals" is introduced in school, it is assimilated into the child's familiar procedures--the operation of adding by counting. Moreover, assimilation of the "equals" sign to a child's informal knowledge of arithmetic is often reinforced by the child's formal instruction. Teachers, math texts and math workbooks in school, and parents, siblings and TV at home may emphasize an operator rather than relational view of "equals." Indeed, one reviewer noted that children's use of hand-held calculators would promote an operator view of equals: the arithmetic problem is punched in first and then the "equals" sign key is hit to produce the answer.

This may help to explain why the first-graders, overall, appeared to be more comfortable with "equals" as a relational symbol than the older children. The formal instruction of this youngest group--from the beginning--emphasized a relational view and minimized reinforcing an operator view. The earlier (traditional) formal instruction of the older children may have reinforced their informal basis for interpreting "equals" as an operator symbol. Thus, the youngest group--while not completely free of an operator view because of their informal experiences--may have had less cognitive resistance to learning a (new) relational view of "equals." The implication for educators is that if a relational view of "equals" is desired, it may be easier to teach an appreciation of this view if it is taught from the beginning of formal instruction.

Moreover, while a teacher may not be able to prevent assimilation of the "equals" sign to a child's informal knowledge or prevent an operator view from being reinforced outside the classroom, a teacher can minimize reinforcement of the operator view in the classroom--at least until the relational view of "equals" is secure. Therefore, it may be necessary to follow Wynroth's example, or indeed, go to greater lengths than allowed by that curriculum. As in the Wynroth curriculum, the "equals" sign might be introduced to show that the two sets have the same number. However, the "equals" sign might initially be used with objects rather than numbers (e.g., ⁛ = ..., ⁛ ≠ ⁙, etc.), then with numerals and objects (e.g., ⁛ = 3, ... = 4), then finally as in the Wynroth program (e.g., 8 = 8, 6 ≠ 7, etc.). A kindergarten teacher might spend the first half of the year with this system before introducing the "equals" sign with operations. This would be a much more thorough introduction to

A relational basis of "equals" than Wynroth now provides. If addition were introduced during this familiarization phase, it could be done as Wynroth suggests--putting the answer above the plus sign.

Follow-up studies which track children's understanding of equals as they proceed through curricula such as the Wynroth program, which emphasize a relational view, are needed to get a better appreciation of the influence of instruction of this important concept.

Note

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Table 1: Summary of the Subjects' Ratings of Correctness for Tasks 1 and 2 Equality Sentences.

Category	Sentence	1st Grade "Makes Sense"	1st Grade Unsure	1st Grade "Makes no Sense"	2nd Grade "Makes Sense"	2nd Grade Unsure	2nd Grade "Makes no Sense"	3rd Grade "Makes Sense"	3rd Grade Unsure	3rd Grade "Makes no Sense"
Category 1	8 = 8	73% (11)	-	27% (4)	47% (7)	-	53% (8)	60% (9)	-	40% (6)
	7 + 6 = 13	100% (15)	0% (0)	0% (0)	93% (14)	0% (0)	7% (1)	93% (12)	0% (3)	7% (1)
	13 = 7 + 6	73% (11)	20% (3)	7% (1)	33% (5)	47% (7)	20% (3)	80% (12)	20% (3)	0% (0)
	7 = 5 + 2	100% (15)	-	0% (0)	60% (9)	-	40% (6)	100% (15)	-	0% (0)
	7 + 6 = 6 + 7	87% (13)	0% (0)	13% (2)	73% (11)	7% (1)	20% (3)	87% (13)	13% (2)	0% (0)
	4 + 3 = 3 + 4	87% (13)	-	13% (2)	73% (11)	-	27% (4)	93% (13)	-	7% (1)
	7 + 6 = 4 + 9	73% (11)	13% (2)	13% (2)	60% (9)	20% (3)	20% (3)	93% (14)	0% (0)	7% (1)
	6 + 4 = 5 + 5	100% (15)	-	0% (0)	73% (11)	-	27% (4)	93% (14)	-	7% (1)
Categories 1 & 2	7 + 6 = 6 + 6 + 1 (1)	73% (11)	7% (1)	20% (3)	67% (10)	20% (3)	13% (2)	73% (11)	13% (2)	13% (2)
	6 + 3 = 4 + 4 + 1 (1)	93% (14)	-	7% (1)	67% (10)	-	33% (5)	93% (14)	-	7% (1)
	2 x 4 = 3 x 2 (2)	40% (6)	-	60% (9)	60% (9)	-	40% (6)	93% (14)	-	7% (1)
Category 2	7 + 6 = 14 - 1	73% (11)	13% (2)	13% (2)	67% (10)	20% (3)	13% (2)	87% (13)	13% (2)	0% (0)
	5 + 1 = 7 - 1	53% (8)	-	47% (7)	53% (8)	-	47% (7)	93% (14)	-	7% (1)
	7 + 6 = IIIIIIIIII III	87% (13)	0% (0)	13% (2)	80% (12)	0% (0)	20% (3)	80% (12)	7% (1)	13% (2)
	5 + 3 = IIIII III	47% (7)	-	53% (8)	33% (5)	-	67% (10)	40% (6)	-	60% (9)
	7 + 6 = XIII	27% (4)	13% (2)	60% (9)	20% (3)	0% (0)	80% (12)	73% (11)	7% (1)	20% (3)
	3 + 2 = V	47% (7)	-	53% (8)	27% (4)	-	73% (11)	47% (7)	-	53% (8)
Category 3	7 + 6 = 6	7% (1)	0% (0)	93% (14)	0% (0)	0% (0)	100% (15)	0% (0)	0% (0)	100% (15)
	2 + 2 = 2	0% (0)	-	100% (15)	0% (0)	-	100% (15)	7% (1)	-	93% (14)
	7 + 6 = 0	0% (0)	7% (1)	93% (14)	0% (0)	0% (0)	100% (15)	0% (0)	0% (0)	100% (15)
	4 + 2 = 42	27% (4)	-	73% (11)	7% (1)	-	93% (14)	0% (0)	-	100% (15)

(1) Only five first-graders had done worksheets involving this form; all but two second-graders had done worksheets involving this form.
(2) Only three first-graders had done worksheets involving this form; eight second graders had done worksheets involving this form.

Table 2: Summary of Response Consistency by Grade in Judging as Making Sense Exposed, Atypical Forms

Grade	Consistency				Total
	Completely Consistent	Highly Consistent (only one trial judged incorrectly)	Inconsistent (Two to One-half of the trials judged as making no sense	Non-acceptance of More than One-half of the trials	
1st	33% (5)	33% (5)	27% (4)	7% (1)	100% (15)
2nd	27% (4)	0% (0)	53% (8)	20% (3)	100% (15)
3rd	47% (7)	0% (0)	53% (8)	0% (0)	100% (15)
Total	37% (16)	11% (5)	44% (20)	9% (4)	100% (45)