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ABSTRACT

This document consists of eleven bulletins which present answers to questions about research on the teaching and content of elementary school mathematics, K-8. The bulletins are revisions of a set originally published in 1970 and revised in 1975. Specific research findings on eleven topics are cited with selected references. Titles are: (1) Attitudes and Anxiety; (2) Organizing the School Program for Instruction; (3) Promoting Effective Learning; (4) Differentiating Instruction; (5) Instructional Materials and Media; (6) Addition and Subtraction with Whole Numbers; (7) Multiplication and Division with Whole Numbers; (8) Rational Numbers: Fractions and Decimals; (9) Measurement, Geometry, and Other Topics; (10) Verbal Problem Solving; and (11) Planning for Research in Schools. The material is indexed by the questions answered in the bulletins as an aid to reference. (MP)

MATHEMATICS EDUCATION REPORTS

Using Research
A Key to Elementary School Mathematics

1981 Revision

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Using Research:

A Key to Elementary School Mathematics

1981 Revision

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Using Research:

A Key to Elementary School Mathematics

1981 Revision

This booklet consists of eleven bulletins which present answers to some questions about research on the teaching and content of elementary-school mathematics (including, in this case, kindergarten through grade 8). These are revisions of the bulletins which were originally prepared in 1970 as one facet of the "Interpretative Study of Research and Development in Elementary School Mathematics" and revised in 1975[1]. The questions are derived from ones frequently asked by teachers about the teaching and learning of mathematics. The bulletins are organized by topic, and specific research findings are cited, with lists of the selected references included for those who wish to explore a topic further. The intent is to provide a concise summary of specific findings which may be applicable in a classroom.[2]

In 1980, the National Council of Teachers of Mathematics issued An Agenda for Action: Recommendations for School Mathematics of the 1980s. At various points among the recommendations are references to research that is needed. What we already have learned from research is reflected in another way: in many cases, they formed a basis for a specific recommendation. These bulletins may help to clarify some questions you have about that foundation, and about the status of research on myriad other points.

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[1] The original study was funded by the Research Utilization Branch, Bureau of Research, U.S. Office of Education (Grant No. OEG-0-9-480586-1352-010), and was conducted at The Pennsylvania State University. The revision was funded by the ERIC Clearinghouse for Science, Mathematics and Environmental Education at The Ohio State University, under a contract from the National Institute of Education.

[2] For a set of bulletins which expand on specific ideas and activities suggested by research, see Driscoll, Mark J. Research Within Reach: Elementary School Mathematics. St. Louis, Missouri: CEMREL, Inc., 1980.

The first five bulletins consider research findings which may apply across various age levels. The first bulletin involves the affective factors of "Attitudes and Anxiety". The second bulletin, "Organizing the School Program for Instruction", cites research findings on ways of organizing the school and the curriculum, while the third, "Promoting Effective Learning", pertains to facets of learning which the teacher may directly control. "Differentiating Instruction" is the focus of the fourth bulletin, while some research on "Instructional Materials and Media" is included in the fifth.

Bulletins 6 through 10 cite research findings on the content of elementary-school mathematics: "Addition and Subtraction with Whole Numbers", "Multiplication and Division with Whole Numbers", "Rational Numbers: Fractions and Decimals", "Measurement, Geometry, and Other Topics," and "Verbal Problem Solving". The final bulletin, "Planning for Research in Schools", is designed to aid those teachers who want to become involved in doing research in their own schools. It can also aid readers of research reports, as it indicates important factors to consider.

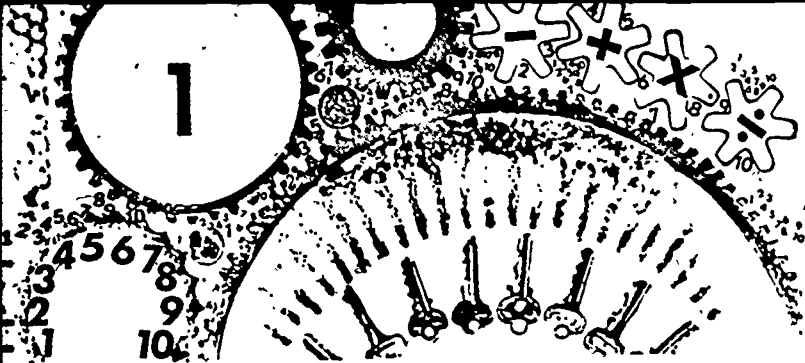
Following the last bulletin is an index of the questions posed in each bulletin, to aid in locating results of most interest.

In this revision, some sections from previous versions of the bulletins have been rewritten to reflect recent findings and other sections have been added to reflect recent concerns. Some questions have been reworded, some bulletins have been resequenced, and some topics have been deleted. There is still some research from past decades cited, for it is important to recognize the contribution that such research has made to our present state of knowledge about the teaching and learning of mathematics. That more of the studies from previous versions have not been included is largely a function of space: their findings may still be of interest.[3]

As with previous versions, studies have been selected by taking into consideration the quality of the research. Care was taken that the selection process would not distort what research may have to say about a particular question. In most cases, however, there are more findings from a study than are reported in these bulletins, as well as other studies which could have been cited to affirm a point. It must also be recognized that there are times when a point has been generalized, and occasionally editorial comments have been made, especially when no research evidence is available, or to make a particular point.

Any of the materials may be reproduced to meet your local instructional needs.

[3]Readers wishing to check the previous versions of this publication will find them in ERIC. The 1970 bulletins are separately listed in the August and September editions of Resources in Education; the 1975 version is listed as ERIC Document No. ED 120 013.



ATTITUDES AND ANXIETY

Attitudes are <u>affective</u> concerns, having to do with feelings. Increasingly, attitudes have come to be recognized as multi-dimensional, having a variety of aspects or facets. These range from awareness of the structural beauty of mathematics and of the usefulness of mathematics, to feelings about the difficulty and challenge of learning mathematics, to interest in a particular type of mathematics or particular methods of being taught mathematics. Attitudes are believed to exert a dynamic, directive influence on an individual's responses; thus, they may be related to the teaching and learning of mathematics.

In recent years, concern has accelerated about what has been termed mathematics anxiety -- that is, fears related to doing mathematics. The effect of such anxiety has also been studied in terms of its effect on teaching and learning.

Attitudes and anxiety frequently have been investigated by the use of scales on which one indicates the degree of agreement or disagreement with statements about mathematics or mathematics-related activities. Occasionally, various school subjects have been ranked by order of preference, or likes and dislikes have been indicated. Each method relies on the sincerity of the individual in expressing true feelings. Clearly, it is difficult to measure affective factors and to assess changes in affective development.

How do elementary school pupils feel about mathematics?

It is widely believed that most children, at all age levels, dislike mathematics. The evidence does not support this belief; many children do like mathematics. Generally, in fact, attitudes toward mathematics tend to be more positive than negative in the elementary school.

In studies conducted in the 1960s, approximately 20% of the pupils surveyed rated mathematics as the least-liked of their school subjects -- and about the same proportion cited mathematics as the best-liked subject (e.g., Curry, 1963; Greenblatt, 1962; Inskeep and Rowland, 1965). Data from Ernest (1976) correspond closely. In a study in which students in grades 2 through 12 were asked to rank four subjects, he found that mathematics was liked best by 30% of the boys and 29% of the girls. It was liked least by 27% of the boys and 29% of the girls.

Other studies indicated that attitudes were even more favorable. Levine (1972) found that pupils in grades 3, 4, and 6 ranked mathematics highest with respect to importance and enjoyment; it was thought to be the best subject and the subject the teacher taught best. About half of the students in grades 6 and 7 surveyed by Johnson (1977) reported favorable attitudes toward mathematics, while only 11% reported negative attitudes. Callahan (1971) reported that 62% of the eighth-grade students he surveyed said they liked mathematics, while only 20% expressed a dislike for it.

Data from the 1977-78 mathematics assessment of the National Assessment of Educational Progress (Carpenter et al., 1980) also showed that, for students at age 9, "mathematics did not necessarily evoke strong negative feelings. In fact, mathematics was liked and enjoyed by a majority of respondents" (p. 34). Physical education was liked best, while mathematics was next best liked -- by 65% at age 9 and 69% at age 13. It was perceived as "easy" by 42% at age 9 and 56% at age 13, with almost equal percentages rating it "in between". Students generally indicated they liked mathematics activities, and most said they wanted to do well in mathematics and were willing to work hard enough to achieve their goal.

Most of the 15 topics presented by Corbitt (1980) were rated as important or very important by a large majority of the eighth graders queried, although fewer topics were rated as liked or liked a lot by a majority. Kish (1980) similarly found that students in grades 7 and 8 thought that mathematics was useful and worthwhile, and expressed little fear of mathematics.

Despite such accrued evidence, one can still find statements such as that by Yamamoto et al. (1969): "Rather to our surprise, mathematics fared quite well in students' ratings" (p. 204). (Note the first four words.)

Only Beck (1977), using data from over 13 000 pupils in grades 1 through 8, found that mathematics was consistently ranked lower than reading/language, science or social studies. Using the same data, Hogan (1977) expanded on the mathematics findings. The average "liking" figure for all items was 58%, indicating a generally favorable attitude toward mathematics in the lower grades. In the upper grades, slightly under 40% liked mathematics, but average responses were on the liking side. Considerable variation was found in reactions to particular topics; thus, the students were favorable toward computation-related items at the intermediate level and toward measurement-related items at the primary level, but tended to dislike geometry-related items.

Generally, attitudes toward mathematics tend to become increasingly less positive as students progress through school. This decline may begin as early as grade 4 (Beck, 1977). Roland (1979), however, indicated that there was a trend for scores to be higher as grade level increased from grades 2 to 6. Most researchers have concluded that attitude remains high until grade 6 (e.g., Evans, 1971; Malcolm 1971; Muzeroll, 1976). Thus, Crosswhite (1972) reported that student attitude toward mathematics seemed to "peak" near the beginning of junior high school. Meece (1981), in a study with students in grades 5-10, found that as students advanced in grade level, they rated their mathematical abilities and performances as lower and viewed mathematics as more difficult and as less useful and valuable.

Do boys and girls differ in their attitudes toward mathematics?

In studies conducted 20 or more years ago, boys seemed to prefer mathematics slightly more than did girls, especially in the upper elementary grades (e.g., Chase and Wilson, 1958; Dutton, 1956; Stright, 1960). Research evidence from recent years indicates that there are few differences in the attitudes toward mathematics of girls and boys in the elementary school. (In the high school years, however, it has been found that the attitudes of girls increasingly become less positive.)

In a study with 1 320 students in grades 6 to 8, Fennema and Sherman (1978) found sex-related differences for only two of eight affective variables. Boys were significantly more confident of their ability to learn mathematics than were girls, and boys stereotyped mathematics as a male domain at higher levels than did girls. No differences were found on scales assessing attitudes toward success in mathematics; perceived attitudes of mother, father, and teacher toward one as a learner of mathematics; effectance motivation in mathematics; or the usefulness of mathematics. Similarly, Nelson (1979) found sex differences in favor of boys on only three of nine attitude scales used with fifth-grade Afro-American students, and Yamamoto et al. (1969) found sex differences on some but not all aspects of attitude in grades 6 through 9.

On the 1977-78 National Assessment (Carpenter et al., 1980), students' perceptions of mathematics as a male domain or female domain were assessed. Two thirds of boys and of girls at age 9 disagreed that "mathematics is more for boys than for girls" or that "mathematics is more for girls than for boys." At age 13, over 80% of the boys and over 90% of the girls disagreed with each statement.

Purdy (1976) found that attitudes toward the value of mathematics were related to sixth graders' views on masculinity and feminity. By grade 8, social desirability played a strong role in influencing students' attitudes --

that is, those who valued mathematics did so partly in order to achieve the approval of others. Boys had positive attitudes toward the value of mathematics, while girls had more positive attitudes toward the enjoyment of mathematics.

For students in grades 2 through 12, Ernest (1976) found that mathematics was the only subject in which no sex difference in preferences was observed.

Looking only at negative attitudes, Fluellen (1975) found no factors that contributed more to the development of negative attitudes toward mathematics of boys than of girls, although a few factors were identified that may have contributed either to boys' or to girls' negative attitudes in grade 6.

Beck (1977) is one of the few researchers who found that girls were significantly more positive toward mathematics than were boys in the primary grades. In the intermediate grades, however, differences were not significant.

Is a more favorable attitude related to higher achievement?

Most people believe that the affective component of learning is important: if children are interested in and enjoy mathematics, they will learn it better. However, there is no consistent body of research evidence to support the popular belief that there is a significant positive relationship between pupil attitudes toward mathematics and pupil achievement in mathematics.

When significant correlations are found between attitude and achievement, they generally range between .20 and .40; that is, no more than 4% to 16% of the variance is accounted for. There is, however, a rough balance between studies in which no significant differences are reported and those in which any significant correlation was found. Thus, no significant relationship between attitudes toward mathematics and achievement in mathematics was found in about half the studies (e.g., Abrego, 1966; Carey, 1978; Deighan, 1971; Johnson, 1977; Keane, 1969), while low positive correlations were reported by the other half (e.g., Antonnen, 1968; Beck, 1977; Burbank, 1970; Caezza, 1970; Quinn, 1978). Mastantuono (1971) analyzed data from four attitude scales and found that a significant correlation with achievement was obtained for only two of the four, thus indicating that the findings may be related to the type of measure used.

Whether boys and girls differ on this factor was considered by many of the researchers. Greenblatt (1962) reported a significant relationship between relative preference for mathematics and mathematical achievement level on the part of girls in grades 3 through 5, but no such significant relationship existed for boys. At the sixth-grade level, Neale et al. (1970) found attitudes and achievement to be significantly correlated for boys but not for girls, while Beattie et al. (1973) found differences in the relationship of attitude to achievement for boys and for girls in grade 4. However, in other studies, no significant differences between girls and boys were found (e.g., Abrego, 1966; Francies, 1971).

What is the relationship of teacher attitudes to pupil attitudes?

Teachers are often viewed as being prime determiners of a student's attitude and performance. There is some evidence to support this. Smith (1974), for instance, reported that students' perceptions of teachers were significantly correlated with mathematical growth in grades 4 through 6, and French (1979) reaffirmed this with students in grade 7 and higher. Rosenbloom et al. (1966) found that teaching effectiveness contributed significantly to the attitude and perceptions of pupils concerning their teachers and the methods they used, the school, text materials, and the class as a group.

However, Kester (1969) found that seventh graders' attitudes were not significantly affected by teacher expectations. Perhaps that is good, considering that Ernest (1976) found that, of a small sample of teachers (24 women and 3 men), 41% felt that boys did better in mathematics, while no one felt that girls did better.

As Aiken (1976) stated, "The belief that teachers' attitudes affect students' attitudes toward mathematics has not been as easy to confirm as might be supposed" (p. 299). Some studies have shown a strong agreement between teacher and pupil attitudes; thus, Chase and Wilson (1958) reported that when teachers preferred mathematics, a majority of their pupils preferred it. However, many researchers have found only small or non-significant correlations between teacher attitudes and pupil attitudes or achievement (e.g., Caezza, 1970; Deighan, 1971; Keane, 1969; Van de Walle, 1973; Wess, 1970), and some have found no relationship (e.g., Purdy, 1976).

Phillips (1970) reported evidence that the effect of teachers' attitudes may be cumulative. At the seventh-grade level, he found a significant relation between attitudes of students and the attitudes of their teachers in the sixth grade. He also observed that the type of teacher attitude encountered by students for two and for three of their past three years was related significantly to their present attitude and achievement. However, Blevins (1979) found no significant relationship between student attitudes and the attitudes of their former teachers.

It is also believed that parents determine the child's initial attitudes and affect their child's achievement. Poffenberger and Norton (1959) stated that attitude toward mathematics is a cumulative phenomenon caused by one experience building on another. Attitudes are developed in the home and carried to the school; self-concepts in regard to mathematical ability are well-established in the early school years, and it is difficult for even the best teacher to change them. Parents influence the child by their expectancy level, by their degree of encouragement, and by their own attitudes toward mathematics.

Significant relationships between students' attitudes toward mathematics and parents' attitudes have been found (e.g., Blevins, 1979; Burbank, 1970; Straman, 1979). Kahl (1979) found that active encouragement by parents had more effect than passive role modeling.

Some studies have shown that the attitudes of children tend to be more closely related to the attitudes of mothers than to those of fathers (Burbank, 1970; Hill, 1967; Levine, 1972). Ernest (1976) reported that mothers help children in mathematics more than fathers do in the elementary grades, but beginning with grade 6 the fathers help more.

What is the relationship of self-concept and achievement?

How children feel about themselves and their concept of themselves while doing mathematics are important components of the affective domain. Self-concept can influence what children expect of themselves, and thus would seem to have a bearing on what they can achieve.

In some studies with students in grades 4, 5, and 6, low positive correlations have been found between self-concept and mathematics achievement (e.g., Gaskill, 1979; Graham, 1975; Koch, 1972; Moore, 1972). Rubin (1978) found that self-esteem was more highly correlated with achievement as children grew older (from 9 to 15).

In at least one study, a significant effect was found between self-esteem and achievement for girls but not for boys in grades 5 and 6 (Primavera et al., 1974). It was suggested that the school plays a greater role in the affective development of a girl's self-esteem because it is a major source of approval and praise for her, whereas boys can seek approval through athletics and other activities.

In a study with fourth graders, Messer (1972) found that children who perceived their academic performance as contingent on their own effort and abilities had higher grades and achievement test scores than children who viewed their school performance as due to luck or the whims of others.

In other studies, no significant relationship was found between self-concept and mathematics achievement, even though a treatment to increase self-concept was tried in some of them (e.g., Devane, 1973; Hunter, 1974; Zander, 1973).

Is anxiety related to mathematics achievement?

Is anxiety helpful in learning mathematics? In general, research has shown that some anxiety facilitates achievement, but a high level of anxiety can be debilitating and negatively affect achievement. Thus, low achievement in mathematics has been associated with high anxiety; furthermore, low anxiety has been correlated with a high degree of confidence.

Most of the research to date on mathematics anxiety has been focused at the secondary and college levels. At the elementary school level, a few studies have considered test anxiety. Forhetz (1971) found that pupils in grades 4 and 6 who ranked mathematics as difficult showed more test anxiety before a mathematics test than before a test in easy-ranked spelling. Jonsson (1966) found a significant interaction between level of test anxiety of sixth-grade girls and version of mathematics test (easy versus difficult). The high-anxious students who took the difficult version had the poorest performance.

In another study with sixth graders, Logiudice (1970) reported that test anxiety and self-esteem were related; when self-esteem was satisfactory, students were more likely to score higher on mathematics tests. For students in grade 8, Degnan (1967) found that a group of achievers was more anxious than a group of underachievers. The achievers also had a much more positive attitude and ranked mathematics significantly higher.

What affects the development of attitudes?

Attitudes toward mathematics are probably formed and modified by many forces. The influence of other people could be named as one source: parents and other non-school-related adults, classmates and other children, and teachers in each of the grades. From a review of 124 dissertations, McMillan (1976) concluded that teacher attitudes and enthusiasm toward the subject and student-related variables such as previous attitudes, parents, and self-concept may have a greater impact on attitude formation than do instructional variables.

The way in which the teacher teaches nevertheless seems to be of importance -- the methods and materials he or she uses, as well as his or her manner, probably affect pupils' attitudes. Teacher enthusiasm and attitude toward mathematics may be the most important factors affecting attitude formation.

The subject itself undoubtedly has an influence on a child's attitude: the precision of mathematics when compared with many other subjects; the need for thorough learning of facts and algorithms; the "building block" characteristic where in many topics are built and often dependent on previous knowledge. Indeed, mathematics has traditionally been considered difficult, and its use as a means of disciplining the mental faculties is still touted by some persons and underlies the reasons many give for including mathematics in the school curriculum.

The learning style of the child is also an important factor to consider. The orderliness which discourages some is the very aspect which attracts others. Furthermore, Futterman (1981) found that ability has a causal role in the attitudinal process, directly affecting self-concept of ability and the value of mathematics to the individual.

For some children the practical value and usefulness of mathematics in out-of-class situations contribute to the development of more positive attitudes toward mathematics. Based on a survey of more than 1 000 pupils, Stright (1960) reported that 95% felt that mathematics would help them in their daily lives, while 86% classified mathematics as the most useful subject. Callahan (1971) reported that eighth-grade students gave the need for mathematics in life most frequently as the reason for liking it; not being good in mathematics was cited most often as the reason for disliking it.

How can attitudes be improved?

Among the reasons which children frequently give for disliking mathematics are lack of understanding, high level of difficulty, poor achievement, and lack of interest in certain aspects of mathematics.

On the other hand, children like mathematics primarily because they find it useful, interesting, challenging, and fun.

Certainly there are clues, in the reasons given above, for what to do in attempting to improve students' attitudes toward mathematics. And we have good reason to believe that attitudes can be improved if:

(1) the teacher likes mathematics and makes this evident to pupils;

(2) mathematics is an enjoyable experience, so that children develop a positive perception of mathematics and a positive perception of themselves in relation to mathematics;

(3) mathematics is shown to be useful, both in careers and in everyday life;

(4) instruction is adapted to students' interests;

(5) realistic, short-term goals are established -- goals which pupils have a reasonable chance of attaining;

(6) pupils are made aware of success and can sense progress toward these recognized goals;

(7) provision is made for success experiences and the avoidance of repeated failure (diagnosis and immediate remedial help are imperative); and

(8) mathematics is shown to be understandable, through the use of meaningful methods of teaching.

List of Selected References

Abrego, Mildred Brown. Children's Attitudes Toward Arithmetic. Arithmetic Teacher 13: 206-208; March 1966.

Aiken, Lewis R., Jr. Update on Attitudes and Other Affective Variables in Learning Mathematics. Review of Educational Research 46: 293-311; Spring 1976.

Anttonen, Ralph George. An Examination into the Stability of Mathematics Attitude and Its Relationship to Mathematics Achievement from Elementary to Secondary School Level. (University of Minnesota, 1967). Dissertation Abstracts 28A: 3011-3012; February 1968.

Beattie, Ian D. et al. The Relationship of Achievement and Attitudes Towards Mathematics in the Elementary School: A Longitudinal Study. February 1973. ERIC: ED 076 424.

Beck, Michael D. What Are Pupils' Attitudes Toward the School Curriculum? Elementary School Journal 78: 73-78; September 1977.

Blevins, Sandra Lee Cowan. The Interrelationships of Parent, Teacher, and Student Attitudes Toward Mathematics and Student Achievement in Mathematics. (East Tennessee State University, 1979.) Dissertation Abstracts International 40A: 1323; September 1979.

Burbank, Irvin Kimball. Relationships Between Parental Attitude Toward Mathematics and Student Attitude Toward Mathematics, and Between Student Attitude Toward Mathematics and Student Achievement in Mathematics. (Utah State University, 1968.) Dissertation Abstracts International 30A: 3359-3360; February 1970.

Caezza, John Francis. A Study of Teacher Experience, Knowledge of and Attitude Toward Mathematics and the Relationship of These Variables to Elementary School Pupils' Attitudes Toward and Achievement in Mathematics. (Syracuse University, 1969.) Dissertation Abstracts International 31A: 921-922; September 1970.

Callahan, Walter J. Adolescent Attitudes Toward Mathematics. Mathematics Teacher 64: 751-755; December 1971.

Carey, John F. O. The Relationship Between Attitude Toward School, Sex, Intelligence, and Academic Achievement. (The University of Rochester, 1978.) Dissertation Abstracts International 39A: 2824-2825; November 1978.

Carpenter, Thomas P.; Corbitt, Mary Kay; Kepner, Henry S., Jr.; Lindquist, Mary Montgomery; and Reys, Robert E. Students' Affective Responses to Mathematics: Results and Implications from National Assessment. Arithmetic Teacher 28: 34-37, 52-53; October 1980.

Chase, W. Lindwood and Wilson, Gilbert M. Preference Studies in Elementary School Social Studies. Journal of Education 140: 2-8; April 1958.

Corbitt, Mary Kay. Validation of Two Constructs of Attitude Toward School Mathematics. (University of Georgia, 1979.) Dissertation Abstracts International 40A: 3844; January 1980.

Crosswhite, F. Joe. Correlates of Attitudes Toward Mathematics. NLSMA Report No. 20. Stanford, California: School Mathematics Study Group, 1972. ERIC: ED 084 122.

Curry, Robert L. Subject Preferences of Fifth-Grade Children. Peabody Journal of Education 41: 21-27; July 1963.

Degnan, Joseph Arthur. General Anxiety and Attitudes Toward Mathematics in Achievers and Underachievers in Mathematics. Graduate Research in Education and Related Disciplines 3: 49-62; April 1967.

Deighan, William Patrick. An Examination of the Relationship Between Teachers' Attitudes Toward Arithmetic and the Attitudes of Their Students Toward Arithmetic. (Case Western Reserve University, 1970.) Dissertation Abstracts International 31A: 3333; January 1971.

Devane, James Ralph. An Exploratory Study of the Relationship Between Factors of Self-Concept and Over-Under Achievement in Arithmetic. (University of Minnesota, 1968.) Dissertation Abstracts International 33A: 4932-4933; March 1973.

Dutton, Wilbur H. Attitudes of Junior High School Pupils Toward Arithmetic. School Review 64: 18-22; January 1956.

Ernest, John. Mathematics and Sex. American Mathematical Monthly 83: 595-614; October 1976. See also ERIC: ED 107 535.

Evans, Robert Franklin. A Study of the Reliabilities of Four Arithmetic Attitude Scales and an Investigation of Component Mathematics Attitudes. (Case Western Reserve University, 1971.) Dissertation Abstracts International 32A: 3086; December 1971.

Fennema, Elizabeth H. and Sherman, Julia A. Sex-Related Differences in Mathematics Achievement and Related Factors: A Further Study. Journal for Research in Mathematics Education 9: 189-203; May 1978.

Fluellen, Alexander Hamilton. A Study of the Development of Negative Attitudes Toward Mathematics. (Indiana University, 1975.) Dissertation Abstracts International 36A: 2748-2749; November 1975.

Forhetz, John Elbert. An Investigation of Test Anxiety as Measured by the TASC in Content Areas Ranked Difficult and Easy with Fourth and Sixth Grade Students. (Southern Illinois University, 1970.) Dissertation Abstracts International 31A: 5124; April 1971.

Francies, Hallie Davis. Arithmetic Attitudes and Arithmetic Achievement of Fourth and Sixth Grade Students in Urban Poverty-Area Elementary Schools. (Case Western Reserve University, 1971.) Dissertation Abstracts International 32A: 1333; September 1971.

French, Yvonne Marie. A Study of the Relationships Between Student Achievement and Student Perception of Teacher Effectiveness. (The Louisiana State University and Agricultural and Mechanical College, 1979.) Dissertation Abstracts International 40A: 3016; December 1979.

Futterman, Robert. A Causal Analysis of Expectancies and Values Concerning Mathematics. (The University of Michigan, 1980.) Dissertation Abstracts International 41B: 3628; March 1981.

Gaskill, James Leslie. The Emotional Block in Mathematics: A Multivariate Study. (The University of British Columbia (Canada), 1979.) Dissertation Abstracts International 40A: 1931; October 1979.

Graham, Jean McClung. The Relationship of the Developmental Self-Concept to the Academic Achievement of Sixth-Grade Elementary Students. (The University of Oklahoma, 1974.) Dissertation Abstracts International 36A: 117; July 1975.

Greenblatt, E. L. An Analysis of School Subject Preferences of Elementary School Children of the Middle Grades. Journal of Educational Research 55: 554-560; August 1962.

Hill, John P. Similarity and Accordance Between Parents and Sons in Attitudes Toward Mathematics. Child Development 38: 777-791; September 1967.

Hogan, Thomas P. Students' Interests in Particular Mathematics Topics. Journal for Research in Mathematics Education 8: 115-122; March 1977.

Hunter, Marie L. Group Effect on Self-Concept and Math Performance. (California School of Professional Psychology, Los Angeles, 1973.) Dissertation Abstracts International 34B: 5169; April 1974.

Inskeep, James and Rowland, Monroe. An Analysis of School Subject Preferences of Elementary School Children of the Middle Grades: Another Look. Journal of Educational Research 58: 225-228; January 1965.

Johnson, Guy W. A Study of Cognitive and Attitudinal Interactions in Seventh Grade Mathematics. (The Louisiana State University and Agricultural and Mechanical College, 1967.) Dissertation Abstracts International 37A: 7008-7009; May 1977.

Jonsson, Harold Alfred. Interaction of Test Anxiety and Test Difficulty in Mathematics Problem-Solving Performance. (University of California, Berkeley, 1965.) Dissertation Abstracts 26: 3757-3758; January 1966.

Kahl, Stuart Reid. Selected Parental and Affective Factors and Their Relationships to the Intended and Actual Mathematics Course Enrollment of Eighth and Eleventh Grade Males and Females. (University of Colorado at Boulder, 1979.) Dissertation Abstracts International 40A: 1933-1934; October 1979.

Keane, Daniel Ford. Relationships Among Teacher Attitude, Student Attitude, and Student Achievement in Elementary School Arithmetic. (The University of Florida, 1968.) Dissertation Abstracts International 30A: 64-65; July 1969.

Kester, Scott Woodrow. The Communication of Teacher Expectations and Their Effects on the Achievement and Attitudes of Secondary School Pupils. (University of Oklahoma, 1969.) Dissertation Abstracts International 30A: 1434-1435; October 1969.

Kish, John Paul, Jr. Application of the Multitrait-Multimethod Matrix and Factor Analysis Techniques to Develop a Construct Valid Mathematics Attitude Scale for Junior High School Pupils. (The University of Toledo, 1980.) Dissertation Abstracts International 41B: 1395; October 1980.

Koch, Dale Roy. Concept of Self and Mathematics Achievement. (Auburn University, 1972.) Dissertation Abstracts International 33A: 1081; September 1972.

Levine, George. Attitudes of Elementary School Pupils and Their Parents Toward Mathematics and Other Subjects of Instruction. Journal for Research in Mathematics Education 3: 51-58; January 1972.

Logiudice, Joseph F. The Relationship of Self-Esteem, Test Anxiety and Sixth Grade Students' Arithmetical Problem Solving Efficiency Under Variant Test Instructions. (St. John's University, 1980.) Dissertation Abstracts International 31B: 2995-2996; November 1970.

Malcolm, Susan Vanderwal. A Longitudinal Study of Attitudes Toward Arithmetic in Grades Four, Six and Seven. (Case Western Reserve University, 1971.) Dissertation Abstracts International 32A: 1194; September 1971.

Mastantuono, Albert Kenneth. An Examination of Four Arithmetic Attitude Scales. (Case Western Reserve University, 1970.) Dissertation Abstracts International 32A: 248; July 1971.

McMillan, James H. Factors Affecting the Development of Pupil Attitudes Toward School Subjects. Psychology in the Schools 13: 322-325; July 1976.

Meece, Judith Lynne. Individual Differences in the Affective Reactions of Middle and High School Students to Mathematics: A Social Cognitive Perspective. (The University of Michigan, 1981.) Dissertation Abstracts International 42A: 2035-2036; November 1981.

Messer, Stanley B. The Relation of Internal-External Control to Academic Performance. Child Development 43: 1456-1462; December 1972.

Moore, Bobbie Dean. The Relationship of Fifth-Grade Students' Self-Concepts and Attitudes Toward Mathematics to Academic Achievement in Arithmetical Computation, Concepts, and Application. (North Texas State University, 1971.) Dissertation Abstracts International 32A: 4426; February 1972.

Muzeroll, Peter Arthur. Attitudes and Achievement in Mathematics in Student Choice and Non-Choice Learning Environments. (The University of Connecticut, 1975.) Dissertation Abstracts International 36A: 4233; January 1976.

Neale, Daniel C.; Gill, Noel; and Tismer, Werner. Relationship Between Attitudes Toward School Subjects and School Achievement. Journal of Educational Research 63: 232-237; January 1970.

Nelson, Roberta Emery. Sex Differences in Mathematics Attitudes and Related Factors Among Afro-American Students. (The University of Tennessee, 1978.) Dissertation Abstracts International 39A: 4793-4794; February 1979.

Phillips, Robert Bass, Jr. Teacher Attitude as Related to Student Attitude and Achievement in Elementary School Mathematics. (University of Virginia, 1969.) Dissertation Abstracts International 30A: 4316-4317; April 1970.

Poffenberger, Thomas and Norton, Donald A. Factors in the Formation of Attitudes Toward Mathematics. Journal of Educational Research 52: 171-176; January 1959.

Primavera, Louis H.; Simon, William E.; and Primavera, Anne M. The Relationship Between Self-Esteem and Academic Achievement: An Investigation of Sex Differences. Psychology in the Schools 11: 213-216; April 1974.

Purdy, Colleen Bridget. Attitudes Toward the Enjoyment and Value of Mathematics in Relation to Sex-Role Attitudes at the Sixth and Eighth Grade Levels. (Boston University School of Education, 1976.) Dissertation Abstracts International 37A: 1500; September 1976.

Quinn, D. William. The Causal Relationship Between Mathematics Achievement and Attitude in Grades 3 to 6: A Cross-Lagged Panel Analysis. Western Michigan University, 1978.) Dissertation Abstracts International 39A: 3433; December 1978.

Roland, Leon Harvey. Use of a Multidimensional Attitude Scale to Measure Grade and Sex Differences in Attitude Toward Mathematics in Second Through Sixth Grade Students. (University of Washington, 1979.) Dissertation Abstracts International 40A: 3174; December 1979.

Rosenbloom, Paul C. et al. Characteristics of Mathematics Teachers that Affect Students' Learning. Final Report, USOE project, September 1966. ERIC: ED 021 707.

Rubin, Rosalyn A. Stability of Self-Esteem Ratings and Their Relation to Academic Achievement: A Longitudinal Study. Psychology in the Schools 15: 430-433; July 1978.

Smith, Diane Savage. The Relationship Between Classroom Means of Students' Perceptions of Teachers as Related to Classroom Racial Composition and Grade Level, and the Effects on Classroom Means of Academic Growth. (Western Michigan University, 1974.) Dissertation Abstracts International 35A: 3309; December 1974.

Straman, Minerva Constantine. Expressed Parental Attitudes Toward Child Rearing in Relation to Study Habits, Study Attitudes and Study Skills Achievement in Early Adolescence. (Andrews University, 1979.) Dissertation Abstracts International 39A: 7253; June 1979.

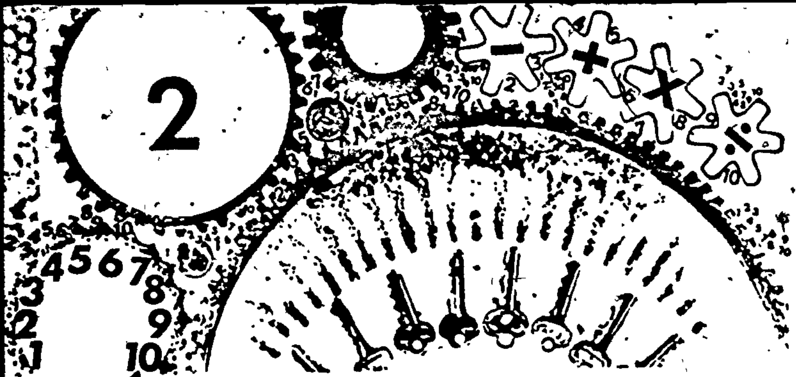
Stright, Virginia M. A Study of the Attitudes Toward Arithmetic of Students and Teachers in the Third, Fourth, and Sixth Grades. Arithmetic Teacher 7: 280-286; October 1960.

Van de Walle, John Arthur. Attitudes and Perceptions of Elementary Mathematics Possessed by Third and Sixth Grade Teachers as Related to Student Attitude and Achievement in Mathematics. (Ohio State University, 1972.) Dissertation Abstracts International 33A: 4254-4255; February 1973.

Wess, Roger George. An Analysis of the Relationship of Teachers' Attitudes as Compared to Pupils' Attitudes and Achievement in Mathematics. (University of South Dakota, 1969.) Dissertation Abstracts International 30A: 3845; March 1970.

Yamamoto, Kaoru; Thomas, Elizabeth C.; and Karns, Edward A. School-Related Attitudes in Middle-School Age Students. American Educational Research Journal 6: 191-206; March 1969.

Zander, Betty Jo Jackson. Junior High Students View Themselves as Learners: A Comparison Among Eighth Grade Students. (University of Minnesota, 1973.) Dissertation Abstracts International 34A: 2254-2255; November 1973.



ORGANIZING THE SCHOOL PROGRAM FOR INSTRUCTION

What is the best way to organize schools and classrooms for mathematics instruction?

Educators have long searched for the "perfect" organizational pattern to meet individual pupil needs and increase achievement. A vast number of studies have been conducted in the attempt to ascertain the efficacy or the superiority of departmentalization, team teaching, multi-grading, non-grading, or self-contained classrooms; other studies have considered such patterns of organization as "open education" or the middle school.

Attempting to isolate and measure the effects of any organizational pattern is extremely difficult, since factors such as curriculum and teacher behaviors interact with the pattern. The definitions of the various patterns also tend to overlap -- for instance, what one person labels team teaching another may define as departmentalization; or "middle school" may mean grades 5-8 in one system and grades 4-9 in another.

It is apparent from a continuing review of the research that no general conclusion can be drawn regarding the relative efficiency of any one pattern of organization for mathematics instruction. There appears to be no one pattern which, per se, will increase pupil achievement in mathematics. Proponents of any pattern can find studies that verify their stand. Achievement differences are affected more by other variables, such as the amount of time devoted to mathematics instruction, than by the organizational pattern. Perhaps the most important implication of the various studies is that good teachers can be effective regardless of the way schools and classrooms are organized. Moreover, it would appear that if a teacher or a group of teachers has a strong belief in the effectiveness of a particular pattern, then that pattern has a strong likelihood of being successful for them.

How should the content be sequenced in the curriculum?

For years the work of Washburne (1928) and the Committee of Seven strongly influenced the sequencing of topics in the elementary school curriculum. This group of superintendents and principals in the midwest surveyed pupils to find when topics were mastered, and then suggested the order and mental age or grade level in which each should be taught.

With the curriculum reform movement which began in the 1950s, much reorganization of content was suggested. Various topics were "tried out" to see if they could be taught at a proposed level: research reflects many such trials, most of which indicated that the topic could be taught at the proposed level.

Some researchers worked on the development of hierarchies of learning tasks. In one study, Gagné and Bassler (1963) structured a heirarchy of "subordinate knowledge" which led to the development of a concept. It was found that, in general, sixth-grade pupils learned a concept when it was developed according to such a hierarchy. Although they did not retain all of the subordinate knowledge, they did continue to achieve well on the final task.

In a study of fifth and sixth graders, Buchanan (1972) examined instructional sequences to determine how prior experiences with subordinate tasks affected mastery of a superordinate task, and the efficiency of performance within a sequence. The amount of prior experience with the introductory task had a significant effect on mastery of the superordinate task.

Phillips (1972) developed and evaluated procedures for validating a learning hierarchy from tests. Data from fourth-grade pupils indicated that sequence, even if random, seemed to have little effect on immediate achievement and transfer to a similar task. However, longer-term retention seemed quite susceptible to sequence manipulation.

Aside from studies on heirarchies, no major recent investigations have been conducted to determine the sequence of topics in the curriculum. There have been studies on the scope of the curriculum, however. One of these was the Priorities in School Mathematics survey (PRISM, 1981). Both educators and lay persons were queried on topics that should receive less, continued, or greater attention during the 1980s. While there was a tendency (not unexpectedly) for responses to favor the status quo, openness to change on many points was evident.

We know that the scope and sequence of the curriculum is constantly, but slowly, changing. Value judgments, reflected in such forces as the minimal competency movement, have had an impact on what textbooks present and teachers teach (Kasten, 1981). Many of the topics inserted into the curriculum during the curriculum reform movement of the 1950s have again been deleted (e.g., work with non-decimal numeration systems) or moved again to higher grade levels (e.g., work with negative numbers).

In addition, technology is beginning to force some changes on the curriculum. For instance, the use of calculators means that pupils encounter decimals at a far earlier age than they did when they did not use calculators, and the curriculum is beginning to reflect this need for a change in sequence.

How important is early mathematics instruction?

Today, with few exceptions, there is general agreement that we will begin to teach mathematics systematically in grade 1, if not in kindergarten. Fifty years ago, however, this was a matter of great debate. It was argued that formal study should be deferred "until the child could understand more and had a need for using mathematics." Therefore, until at least the third grade, mathematics should be learned "incidentally", through informal contacts with number.

Opponents argued that such delay was a waste of time. Data to support this were collected; for instance, Washburne (1928) found that pupils who began mathematics in either grade 1 or 2 made better mathematics scores in grade 6 than did pupils who began mathematics in grade 3. Clear evidence on the amount of mathematics that children could learn in grades 1 and 2 was provided by Brownell (1941).

A few studies (e.g., Sax and Ottina, 1958) have explored the question of delaying instruction until fifth grade or even later. While they found that achievement was apparently about as high in later grades despite the lack of formal instruction, the question of when to begin systematic instruction has not seriously been reopened.

A question asked more frequently today is how much should be taught in the nursery school or kindergarten program. There is concern that the interest of children becomes jaded by having a topic introduced at an early age so that, when it is presented in a later grade, children feel they've already had it and have less interest in it. Both interest and achievement in mathematics is also influenced by television programs and by the use of calculators, computers, and electronic games.

A question that recurs every few years pertains to the effects of pre-first-grade mathematical experiences, and in particular kindergarten experiences. About half of the studies of the effect of having or not having kindergarten have supported the conclusion that children with kindergarten experience had higher achievement in later grades than did children who had not been to kindergarten (e.g., Keen, 1976; Kristjansdottir, 1972). In the other half of the studies, no significant difference between the two groups was found (e.g., Clayton, 1981; Ricketts, 1976; Traywick, 1972).

Similarly, the research on pre-school experience (e.g., Fort, 1980; Yonally, 1972) indicates no clear-cut answer. One study suggested that mathematics skill teaching should be postponed until kindergarten for girls, but not for boys (Wintergalen, 1977). Another found that a home-based pre-school program was better than a school-based one; that is, parents could be highly successful at teaching children mathematical ideas (Washburn, 1977).

The effect of pre-school experience appears to be highly school-related; that is, the experiences that have been provided to build on the foundation provided in the pre-school or kindergarten (in addition to the pre-school or kindergarten experience itself) have a vital effect on how much children achieve.

How should the time available for mathematics instruction be used?

One body of research evidence concerns the distribution of time for mathematics instruction. To determine how the use of class time affects achievement, Shipp and Deer (1960) compared four groups, in which 75%, 60%, 40%, or 25% of class time was spent on group developmental work while the remainder was spent on individual practice. Higher achievement in computation, problem solving, and mathematical concepts was obtained when more than half of the time was spent on developmental activities.

In replications of this experiment, Shuster and Pigge (1965) and Zahn (1966) used other time allocations. They confirmed the finding that, when the greater proportion of time is spent on developmental activities, achievement is higher. Hopkins (1966) also compared two groups (in grade 5) which spent 50% time on meaningful activities and 50% time either on practice or in informal investigations of more advanced concepts. No significant differences between computation scores for the two groups were found, but significant differences on measures of understanding occurred. Hopkins concluded that the amount of time spent on practice "can be reduced substantially and still retain equivalent proficiency in arithmetic computation." If activities are carefully selected, understanding can be increased.

Another body of research evidence concerns the actual use of time by pupils -- the amount of time they spend "on task". There is rather definitive evidence (e.g., Denham and Lieberman, 1980; Jacobson, 1981) that time-on-task is highly related to achievement: the more time spent actually working on mathematics tasks, the higher the achievement that can be expected.

Is homework helpful?

No studies have shown that homework has a negative effect on achievement or attitudes of elementary school pupils. A few studies have reported an achievement gain when homework was used at the intermediate level (Doane, 1973; Maertens and Johnston, 1972). Others have found no significant differences between groups assigned or not assigned homework, either at the primary level (Harding, 1980) or intermediate level (Grant, 1971; Gray and Allison, 1971; Maertens, 1969).

Pressman (1980) found that homework constitutes a significant portion of a student's total opportunity to learn. Drill on work taught in class is the most frequently assigned homework activity. Teacher expectations about the importance of homework are reflected in the time and effort pupils devote to it.

Is tutoring helpful to elementary school pupils?

Tutoring is one way of providing remedial help to children who need it. Comparatively few studies have looked at adults in this role; this appears to be considered a natural aspect of teaching for teachers and other adults. It has been pointed out, however, that senior citizens could be effective tutors (Duffy, 1980), as could parents (Johnson, 1981).

Having children tutor other children, either at earlier or the same grade levels and either at lower or the same achievement levels, has been studied. As Jones (1979) pointed out, students helping students was as helpful as teachers helping students. Evidence from a number of students indicates, however, that peer or cross-age tutoring may not result in significant achievement gains for those being tutored over those not tutored (e.g., Carlson, 1973; Geer, 1978; Lee, 1980; Swenson, 1976). A few studies indicate that those tutored do achieve better (e.g., Ackerman, 1970; Guarnaccia, 1973; Levine, 1976). Hartley (1978) found in a meta-analysis of 153 studies that tutoring resulted in higher achievement than computer-assisted instruction, individual learning packets, or programmed instruction. Peer tutoring was as effective as were paid adult tutors, while cross-age tutoring was slightly better than the other two types of tutoring.

Some studies have look at a related question: is there an effect on those doing the tutoring? Some studies have found that tutoring did not result in better achievement for the tutors (e.g., Carlson, 1973; Kenemuth, 1975; Levine, 1976; Swenson, 1976). A number of researchers have found indications, however, that tutoring helped the tutors. Rosen et al. (1978) found that perceived achievement and satisfaction of tutors were greater on becoming the tutor rather than the tutee, particularly if one was the more competent partner. Hulse (1980) found that tutoring produced higher scores than being tutored did in grades 7 and 8 and Geer (1978) reported that tutoring improved the self-concept of sixth graders tutoring pupils in grades 1-4. Guarnaccia (1973) found that tutors learned at least as well as tutees in grade 3 and 4.

Schultz (1972) pointed out the need to consider the compatibility of tutors and those being tutored. Each of 20 tutors was assigned to one student who (according to a test) appeared most compatible to him or her, and one least-compatible student. Gains in achievement and in self-concept of arithmetic ability did not appear to be related to the degree of compatibility, but students rated the relationship with tutor as more facilitative when they were compatible.

List of Selected References

Ackerman, Arthur Peter. The Effect of Pupil Tutors on Arithmetic Achievement of Third-Grade Students. (University of Oregon, 1969.) Dissertation Abstracts International 31A: 918; September 1970.

Brownell, William A. Arithmetic in Grades I and II. Durham, North Carolina: Studies in Education, Duke University Press, No. 6, 1941.

Buchanan, Aaron Dean. An Experimental Study of Relationships Between Mastery of a Superordinate Mathematical Task and Prior Experience with a Special Case. (University of Washington, 1971.) Dissertation Abstracts International 32A: 6091; May 1972.

Carlson, Richard Tepper. An Investigation into the Effects Student Tutoring Has on Self-Concept and Arithmetic Computation Achievement of Tutors and Tutees. (Northern Illinois University, 1973.) Dissertation Abstracts International 34A: 2265; November 1973.

Clayton, Constance Elaine. An Analysis of the Persistence of High Achievement Among Low Income Fourth Graders with Prekindergarten Experience. (University of Pennsylvania, 1981.) Dissertation Abstracts International 42A: 923; September 1981.

Denham, Carolyn and Lieberman, Ann (Eds.). Time to Learn: A Review of the Beginning Teacher Evaluation Study. California: Commission for Teacher Preparation and Licensing, 1980.

Doane, Bradford Sayles. The Effects of Homework and Locus of Control on Arithmetic Skills Achievement in Fourth Grade Students. (New York University, 1972.) Dissertation Abstracts International 33A: 5548; April 1973.

Duffy, Charles Peter. The Demonstrated Effect of Senior Citizens in a Mathematical Tutorial Program with Clearly Defined Behavioral Objectives. (Columbia University Teachers College, 1980.) Dissertation Abstracts International 41A: 1303; October 1980.

Fort, Ientha Josephine. A Study of the Effects of a Prekindergarten Program on First and Second Grade Achievement. (Saint Louis University, 1979.) Dissertation Abstracts International 40A: 5665; May 1980.

Gagné, Robert and Bassler, Otto. Study of Retention of Some Topics of Elementary Nonmetric Geometry. Journal of Educational Psychology 54: 123-131; June 1963.

Geer, Charles Paul. The Effects of Cross-Age Tutoring on the Achievement and Self-Concept of Low-Achieving Students. (Arizona State University, 1977.) Dissertation Abstracts International 38A: 5909; April 1978.

Grant, Eugene Edward. An Experimental Study of the Effects of Compulsory Arithmetic Homework Assignments on the Arithmetic Achievement of Fifth-Grade Pupils. (University of the Pacific, 1971.) Dissertation Abstracts International 32A: 2401; November 1971.

Gray, Roland F. and Allison, Donald E. An Experimental Study of the Relationship of Homework to Pupil Success in Computation with Fractions. School Science and Mathematics 71: 339-346; April 1971.

Guarnaccia, Vincent Joseph. Pupil-Tutoring in Elementary Arithmetic Instruction. (Columbia University, 1970.) Dissertation Abstracts International 33B: 3283-3284; January 1973.

Harding, Robert Charles. The Relationship of Teacher Attitudes Toward Homework and the Academic Achievement of Primary Grade Students. (Lehigh University, 1979.) Dissertation Abstracts International 40A: 5668-5669; May 1980.

Hartley, Susan Strait. Meta-Analysis of the Effects of Individually Paced Instruction in Mathematics. (University of Colorado at Boulder, 1977.) Dissertation Abstracts International 38A: 4003-4004; January 1978.

Hopkins, Charles David. An Experiment on Use of Arithmetic Time in the Fifth Grade. (Indiana University, 1965.) Dissertation Abstracts 26: 5291-5292; March 1966.

Hulse, Dwight Lyman. A Study of Critical Factors Relating to Research on Cross-Age Tutoring. (State University of New York at Albany, 1980.) Dissertation Abstracts International 41A: 1554; October 1980.

Jacobson, Kerry Ray. The Relationship of Individual Student Time Allocation to Reading and Mathematics Achievement. (The University of Wisconsin - Madison, 1980.) Dissertation Abstracts International 42A: 35-36; July 1981.

Johnson, John, II. Effectiveness of a Parental Tutorial Program on Mathematics Achievement and Self-Concept Scores of Black Inner-City Students. (University of Miami, 1981.) Dissertation Abstracts International 42A: 1462; October 1981.

Jones, Robert Kenneth. An Evaluation of an Intra-Class Peer-Tutoring Instructional Strategy, Department of Defense Dependents Schools, Europe. (University of Southern California, 1979.) Dissertation Abstracts International 39A: 7135; June 1979.

Kasten, Margaret Bledsue. Minimum Competency Testing in Mathematics: A Chronicle of an Educational Movement of the 70s. (The Ohio State University, 1981.) Dissertation Abstracts International 42A: 2007; November 1981.

Keen, Susan Aberle. A Comparison of the Effect of Kindergarten Experience on First Grade Achievement. (Memphis State University, 1976.) Dissertation Abstracts International 37A: 3480-3481; December 1976.

Kenemuth, Gail L. An Experimental Study of the Effects on Achievement and Self-Concept of Sixth Grade Pupils as a Result of Tutoring Younger Elementary Pupils in Selected Activities. (The Pennsylvania State University, 1974.) Dissertation Abstracts International 35A: 7043-7044; May 1975.

Kristjansdottir, Thuridur Johanna. The Relationship of Preschool Experience to Achievement and Selected Aspects of Adjustment in Grade One. (University of Illinois at Urbana-Champaign, 1971.) Dissertation Abstracts International 32A: 5612-5613; April 1972.

Lee, Courtland Christian. The Academic and Affective Effects of Academic-Social Enrichment upon Elementary School Underachievers in an Urban School District. (Michigan State University, 1979.) Dissertation Abstracts International 40A: 5323; April 1980.

Levine, Kenneth Paul. The Effect of an Elementary Mathematics Tutoring Program upon the Arithmetic Achievement, Attitude Toward Mathematics, and the Self-Concept of Fifth Grade Low Achievers When Placed in the Role of Student Tutors. (Temple University, 1975.) Dissertation Abstracts International 37A: 2039; October 1976.

Maertens, Norbert. An Analysis of the Effects of Arithmetic Homework upon the Arithmetic Achievement of Third-Grade Pupils. Arithmetic Teacher 16: 383-389; May 1969.

Maertens, Norbert and Johnston, James. Effects of Arithmetic Homework upon the Attitudes and Achievement of Fourth, Fifth and Sixth Grade Pupils. School Science and Mathematics 72: 117-126; February 1972.

Phillips, Ernest Ray. Validating Learning Hierarchies for Sequencing Mathematical Tasks. (Purdue University, 1971.) Dissertation Abstracts International 32A: 4249; February 1972.

Pressman, Ellen Gloria. Allocation of School-Related Student Time. (Columbia University Teachers College, 1980.) Dissertation Abstracts International 41A: 1327-1328; October 1980.

PRISM: Priorities In School Mathematics: Executive Summary of the PRISM Project. Reston, Virginia: National Council of Teachers of Mathematics, 1981.

Ricketts, Jean Napier. The Effects of Prescribed, Sequential Programs of Reading and Mathematics on the Readiness of Middle Class Kindergarten Children and on Their Subsequent Achievement in Reading and Mathematics. (Temple University, 1976.) Dissertation Abstracts International 37A: 1979-1980; October 1976.

Rosen, Sidney; Powell, Evan R.; Schubot, David B.; and Rollins, Patricia. Competence and Tutorial Role as Status Variables Affecting Peer-Tutoring Outcomes in Public School Settings. Journal of Educational Psychology 70: 602-612; August 1978.

Sax, Gilbert and Ottina, John R. The Arithmetic Achievement of Pupils Differing in School Experience. California Journal of Educational Research 9: 15-19; January 1958.

Schultz, Edward W. The Influence of Teacher Behavior and Dyad Compatibility on Clinical Gains in Arithmetic Tutoring. Journal for Research in Mathematics Education 3: 33-41; January 1972.

Shipp, Donald E. and Deer, George H. The Use of Class Time in Arithmetic. Arithmetic Teacher 7: 117-121; March 1960.

Shuster, Albert and Pigge, Fred. Retention Efficiency of Meaningful Teaching. Arithmetic Teacher 12: 24-31; January 1965.

Swenson, Stuart Herbert. Effects of Peer Tutoring in Regular Elementary Classrooms on Sociometric Status, Self-Concept and Arithmetic Achievement of Slow Learning Tutors and Learners in a Special Education Resource Program. (Indiana University, 1975.) Dissertation Abstracts International 36A: 6003-6004; March 1976.

Traywick, Laura Ann McClendon Symank. Kindergarten Experience and Achievement of Elementary School Children. (The University of Texas at Austin, 1971.) Dissertation Abstracts International 32A: 5481; April 1972.

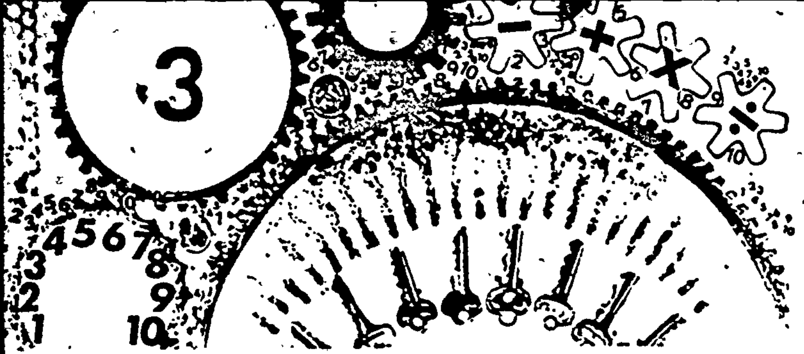
Washburne, Carlton. When Should We Teach Arithmetic? -- A Committee of Seven Investigation. Elementary School Journal 28: 659-666; May 1928.

Washburn, Jon Michael. The Effects of Two Different Pre-school Intervention Programs. (Wayne State University, 1976.) Dissertation Abstracts International 27A: 7027; May 1977.

Wintergalen, Patricia Ann. Achievement and Selected Related Variables in Relation to Participation in a Program of Early Childhood Education. (University of Denver, 1976.) Dissertation Abstracts International 37A: 4219; January 1977.

Yonally, James Lewis. Effect of a Preschool Experience on the Academic Achievement and Social Adjustment of Kindergarten and Second-Grade Children. (University of Kansas, 1972.) Dissertation Abstracts International 33A: 2679-2680; December 1972.

Zahn, Karl G. Use of Class Time in Eighth-Grade Arithmetic. Arithmetic Teacher 13: 113-120; February 1966.



PROMOTING EFFECTIVE LEARNING

Research to guide us in determining how children learn and how we can teach effectively encompasses far more than one curriculum area. We have selected that research which is based on some phase of the elementary school mathematics curriculum and provides specific suggestions to teachers of elementary school mathematics. We have noted some ideas from learning theory; considered the dependent variables of retention, transfer, and reinforcement; and included three prevalent methods of delivering instruction.

What do we know about how children learn mathematics?

In actuality, we don't know exactly how children learn -- but there are several theories which account for some of the characteristics of learning. Each recognizes the importance of the concrete level and describes movement toward the abstract. One that has been considered extensively during the past several decades was proposed by Piaget. In particular, he cited evidence, largely from interviewing children using carefully formulated tasks, that children at the elementary school level go through certain stages of cognitive development: the preoperational stage, in which the child makes judgments on the basis of perceptions rather than reason or logic, and the concrete operational stage, in which the child uses logic and reasoning in terms of concrete objects. At approximately age 12, many children move into the formal operational stage, using abstract thought and no longer needing to rely on the concrete.

A few of the studies on Piaget's theory have focused on its meaning for teaching and learning mathematics in the elementary school. For instance, it appears that conservation of number (awareness that the number of objects in a set remains unchanged in spite of changes in the arrangement of the objects) may be related to work with counting, addition and subtraction, and other mathematical ideas (e.g., LeBlanc, 1968; Smith, 1975; Souviney, 1980; Steffe, 1967). However, in other studies, it appears that children are able to do some mathematical tasks that would seem to be above their developmental level (e.g., Almy et al., 1970; Robertson, 1979). Thus, Baroody (1979) and Saxe (1979) reported that children appeared to develop counting strategies before number conservation concepts, while Mpiangu and Gentile (1975) and Pennington et al. (1980) concluded that conservation of number was not a prerequisite for work with numbers.

A similar set of stages was proposed by Bruner: children move from the enactive stage, where the child interacts directly with real objects; to the iconic stage, where they do representational thinking based on, for instance, pictures; to the symbolic stage, where they can manipulate symbols without regard to objects or pictures. (These stages correspond closely to those long-used in mathematics education: concrete, semi-concrete, and abstract.) A sizeable amount of research has been conducted to ascertain their importance (e.g., Baker, 1977; Beardslee, 1973; Gau, 1973).

Besides the developmental theories proposed by Piaget and Bruner, which propose that learning proceeds through stages as the child matures, there are behaviorism theories. These have their roots in stimulus-response theory and conditioned learning: behavior is shaped by rewards and punishments. Gagne is one researcher who has promulgated this theory by, for instance, stressing the need for sequences and hierarchies.

More recently, information processing, in which computer techniques are used as a method of analysis and comparison, has captured attention. Errors that are made as algorithms are attempted (e.g., Brown and Burton, 1978) is one type of a series of investigations.

In general, it seems that children acquire knowledge and understanding progressively, in a series of steps or stages. This learning is shaped by rewards and punishments, and characterized by errors that need to be corrected as information is assimilated by the child.

What helps pupils retain what they have learned?

One of the major goals of instruction is to have children retain what we are teaching and they are learning. There is much research to show that when something has meaning to learners and is understood by them, they will be more likely to remember. Furthermore, Shuster and Pigge (1965) state that retention is better when at least 50% of class time is spent on meaningful, developmental activities. Klausmeier and Check (1962) reported that when pupils solved problems at their own level of difficulty, retention was good regardless of IQ level.

Intensive, specific review will facilitate retention, according to Burns (1960). He prepared lessons which included not only practice exercises, but also review study questions which directed pupils' attention to relevant things to consider. Meddleton (1956) pointed out that such review should be systematic.

The mode of review does not have to be similar to the mode of initial instruction (Thomas, 1974). It can be administered individually or in small groups (Pence, 1974), or via a computer (Vinsonhaler and Boss, 1972). Games have been widely and successfully used to aid students in retaining knowledge (Bright et al., 1980).

Mastery learning has been proposed as a way of helping children learn and retain. A high level of mastery must be reached (often at least 80% of a set of performance objectives must be attained, or students must score at least 80% on a test) before they can go on to new topics. The evidence on mastery learning from studies at the elementary school level is equivocal -- some indicate that it is effective in teaching mathematics (e.g., Kingston, 1979), while others indicate no significant differences in achievement from "traditional" instruction (e.g., Gifford, 1980).

Many teachers have noted that children fail to retain well over the summer vacation. The amount of loss varies with the child's ability and age, but how long before the vacation material was presented is important. Practice during the summer and review concentrated on materials presented in the spring have been shown to be especially helpful in reducing retention loss. Grenier (1976) investigated whether seventh-grade pupils showed a significant loss in arithmetic over the summer, and tried to determine the length of time required to return to the pre-summer level of achievement when mean losses did occur. She found that the students had a significant loss on the computation subtest; some gain was found after two weeks in school. On some tests of concepts and applications, gains were found between spring and fall testings.

How can transfer be facilitated?

Transfer infers that something learned from one experience can be applied to another experience. Many years ago, Olander (1931) found that pupils who studied 110 addition and subtraction combinations could give correct answers to the 90 untaught combinations. What facilitated this transfer best was instruction in generalizing, in teaching children to see patterns. Transfer increases as the similarity of problems and experiences increases. Much research has shown that meaningful instruction aids in transfer of learning and the transfer is facilitated by discovery-oriented instruction (Weimer, 1975).

In general, the older the child and the higher the ability level, the better the child can transfer. However, Klausmeier and Check (1962) found that children of various IQ levels transferred problem-solving skills to new situations when work was at the child's own level of difficulty. Burton et al. (1975) reported that the performance of low-ability pupils in grade 5 on transfer tasks increased as the variety of methods which they were taught increased.

Learning a second method can interfere with the first method taught (Barszcz and Gentile, 1976). Other things that have aided transfer seem to be use of introductory materials and giving rules (Wenzelburger, 1975).

In one set of studies, Sawada (1972) studied a strategy for organizing a curriculum and instructional sequences with explicit provision for transfer from lower- to higher-order objectives, using content characterized by composition and reversibility. It was found that performance on an objective had little relationship with performance on the inverse objective. Pupils on their own apparently did not pick up the strategy of forming composites. In other words, pupils did not seem aware of reversibility inherent in the materials, nor of composition objectives. The need for explicit teaching, rather than expecting transfer to occur as a by-product, is indicated.

In most studies is this implication that transfer is facilitated when teachers plan and teach for transfer: we must teach children how to transfer.

What is effective for reinforcing instruction?

One of the best ways of reinforcing learning is to give the child "knowledge of results" -- by providing scores or by providing correct answers. Paige (1966) found that immediate reinforcement after a testing situation resulted in significantly higher achievement scores later. Having the student respond and then giving confirmation is more effective than prompting with the correct answer before giving a chance to respond (McNeil, 1965).

The use of token reinforcements -- plastic tokens which may be traded for candy, toys, or other desired items -- has been reported to result in achievement gains in other curricular areas. Hillman (1970) reported that fifth graders given per-item knowledge of results, whether with or without candy reinforcement, scored significantly higher in achievement with decimals than pupils given knowledge of results 24 hours later. He suggested that low achievers may profit more than high achievers. Heitzman (1970) studied pupils aged 6 to 9 in a summer arithmetic program. Those who were rewarded by tokens achieved significantly higher scores on a skills test than those who did not receive tokens (and also who may not have received knowledge of results). Immediate knowledge of results, rather than token reinforcement, may be the determining factor.

Masek (1970) reported significant increases in arithmetic performance and level of task orientation of underachieving first and second graders during periods when teachers emphasized reinforcement such as verbal praise, physical contact, and facial expression.

Praise has seemed to be an especially effective way to reinforce as well as to motivate learning. However, concerns about how it has been used were expressed by Brophy (1981) on the basis of a review of research (including but not limited to mathematics instruction). He enumerated some guidelines for effective praise, including: making it contingent rather than unsystematic; specifying the particulars of an accomplishment; showing spontaneity and variety and for credibility; rewarding the attainment of specified criteria; and attributing success to effort and ability.

Is there research which identifies outcomes of meaningful instruction?

Earlier this century, it was doubted that children needed to understand what they learned: it was sufficient for them to develop high degrees of skill. To take time to give explanations and develop understanding was deemed wasteful, besides being perplexing to the learners.

Then came the realization that certain things were to be gained if content made sense to the learner. When mathematics is taught according to the mathematical aim, learning becomes meaningful; when it is taught according to the social aim, it becomes significant. Children do not necessarily acquire meanings when they engage in real-life social activities involving mathematics. Significant mathematical experiences need to be supplemented by meaningful mathematical experiences.

In a summary of studies concerned with various aspects of meaningful instruction, Dawson and Ruddell (1955) concluded that meaningful teaching generally leads to greater retention, greater transfer, and increased ability to solve problems independently. They suggested that teachers should (1) use more materials, (2) spend more class time on development and discussion, and (3) provide short, specific practice periods. More recent studies have supported these findings, as another review (Weaver and Suydam, 1972) and continuing perusal of the research indicate.

How much guidance should be given to learners?

The question of the amount of guidance that students need has been viewed by many researchers in terms of comparing expository instruction with "guided discovery" approaches which encourage the child to discover mathematical ideas more or less independently. There is no clear evidence that guided discovery approaches are more effective than expository approaches -- but there are some studies in which expository instruction is clearly better than a guided discovery approach, others in which guided discovery is better, and still others in which there are no significant differences. It would seem that the method is dependent on the teacher's effectiveness and the particular content being taught.

The equivocal or inconclusive nature of research on "discovery" and its role in instruction may stem in part from the fact that the "discovery" label has been attached to methods or procedures that differ markedly in their distinguishing characteristics.

Robertson (1971) concluded that "it would appear that no one treatment or mode of instruction can be considered that best approach. The teacher who learns as many instructional modes as possible, identifies and diagnoses pupil needs and abilities, and uses this knowledge to individualize instruction may very well get the best results."

In research with fourth-grade classes, Good and Grouws (1979) found one pattern of lesson was characteristically used by teachers whose classes had high achievement. It consisted of daily review, development of new content, seatwork, and homework assignment, with special reviews as needed. Perhaps the strength of this "direct instruction" pattern lies in its emphasis on clarity, which has long been associated (in other research studies) with effective teaching.

How effective are activity-oriented approaches to instruction?

In a survey of research on activity-based learning in elementary school mathematics, Suydam and Higgins (1977) concluded that about half of the studies they reviewed reported achievement differences favoring the use of activities, while the other half reported no significant differences. Therefore, students using activity-oriented programs or units can be expected to achieve as well or better than students using programs not emphasizing activities.

Some research has focused on the effectiveness of games for teaching elementary school mathematics topics. Thus, as noted earlier, Bright et al. (1980) reported that using games was an effective way to help students in grades 5 and 6 maintain skills with basic multiplication facts.

Denoting a wide variety of procedures, the mathematics laboratory usually (but not always) involves use of manipulative materials plus a variety of other activities. At least equivalent achievement can be expected when mathematics laboratories are used (Suydam and Higgins, 1977). Almost all studies reported no significant differences in attitudes. Vance and Kieren (1971) summarized the results of the studies on mathematics laboratories which they reviewed by stating that the research indicates that students can learn mathematical ideas from laboratory settings. However, other meaningful instruction appears to work as well if not better.

Is it effective to teach mental computation skills and estimation skills?

Relatively little research as been directed toward these increasingly important skills. Austin (1970) found that eighth-grade pupils who spent one period a week on mental computation scored significantly higher on standardized tests than students not given such instruction. Fourth-grade pupils who were instructed in mental computation also made a significant increase in arithmetic achievement and were better able to solve problems mentally than were pupils for whom mental computation was not stressed (Grumbling, 1971). In another study, fifth graders were exposed to short, frequent periods of oral practice administered in various modes (Schall, 1973). It was found that the exercises resulted in increased ability to compute mentally and in a gain in attitude scores, although no significant differences were found between groups who used televised lessons, lessons on audio-tape, or programmed materials.

Rea and French (1972) reported on a small-scale research study with a class of sixth graders. One group used mental computation exercises; the other was given enrichment activities using the same content. For 24 days, both groups received their regular mathematics instruction plus 15 minutes daily of the special activities.

In both groups were individuals whose scores increased only slightly, and scores even decreased for a few. However, in both groups, the majority of the students gained rather dramatically; the average gain for the enrichment group on the achievement test was one full year, and for the mental computation group was eight months. There can be little doubt that the results were influenced by factors such as the halo effect, which often accompanies enthusiastic experimentation.

But why not capitalize on this in the classroom? Children do like variety -- and children enjoy experimenting and being part of an experiment. Research can be a way of motivating children.

Working with pupils in grades 4, 5, and 6, Schoen et al. (1981) compared estimation taught through meaningful instruction or with a computer-assisted drill-and-practice program. Both groups learned to estimate; the group taught meaningfully retained and transferred this skill.

A variety of different estimation strategies were demonstrated by the students in grades 7 and up interviewed by Reys et al. (1980). Several general processes were observed with regularity and seemed closely associated with good estimation skills: the ability to translate, to reformulate, and to use compensation.

List of Selected References

Almy, Millie and Associates. Logical Thinking in Second Grade. New York: Teachers College Press, 1970.

Austin, John C. Mental Mathematics Counts. Arithmetic Teacher 17: 337-338; April 1970.

Baker, Beverly Elizabeth. The Effect of Sequencing Levels of Representation on the Learning and Retention of Place Value Skills. (University of South Florida, 1977.) Dissertation Abstracts International 38A: 2623; November 1977.

Baroody, Arthur James. The Relationships Among the Development of Counting, Number Conservation and Basic Arithmetic Abilities. (Cornell University, 1979.) Dissertation Abstracts International 39A: 6640; May 1979.

Barszcz, Edward L. and Gentile, J. Ronald. Retroactive Interferences of Similar Methods to Teach Translation of Base Systems in Mathematics. Journal for Research in Mathematics Education 7: 176-182; May 1976.

Beardslee, Edward Clarke. Toward a Theory of Sequencing: Study 1-7: An Exploration of the Effect of Instructional Sequences Involving Enactive and Iconic Embodiments on the Ability to Generalize. (Pennsylvania State University, 1972.) Dissertation Abstracts International 33A: 6721; June 1973.

Bright, George W.; Harvey, John G.; and Wheeler, Margariete Montague. Using Games to Maintain Multiplication Basic Facts. Journal for Research in Mathematics Education 11: 379-385; November 1980.

Brophy, Jere. Teacher Praise: A Functional Analysis. Review of Educational 51: 5-32: Spring 1981.

Brown, John Seely and Burton, Richard R. Diagnostic Models for Procedural Bugs in Basic Mathematical Skills. Cognitive Science 2: 155-192; 1978.

Burns, Paul C. Arithmetic Books for Elementary Schools. Arithmetic Teacher 7: 147-149; March 1960.

Burton, John K.; Lemke, Elmer A.; and Williams, Jeral R. Transfer Effects as a Function of Variety of Method and Ability Level in Elmentary Mathematics. Journal for Research in Mathematics Education 6: 228-231; November 1975.

Dawson, Dan T. and Ruddell, Arden K. The Case for the Meaning Theory in Teaching Arithmetic. Elementary School Journal 5: 393-399; March 1955.

Gau, Gerald Elmer. Toward a Theory of Sequencing: Study 1-6: An Exploration of the Effect of Instructional Sequences Involving Enactive and Iconic Embodiments on the Attainment of Concepts Embodied Symbolically.

(Pennsylvania State University, 1972.) Dissertation Abstracts International 33A: 6728; June 1973.

Gifford, Robert Norman. The Rate of Mastery Learning of CAI Lessons in Basic Reading and Mathematics. (University of Southern California, 1980.) Dissertation Abstracts International 41A: 1553-1554; October 1980.

Good, Thomas L. and Grouws, Douglas A. Teaching and Mathematics Learning. Educational Leadership 37: 39-45; October 1979.

Grenier, Marie-Anne Cecile, C.S.C. An Investigation of Summer Mathematics Achievement Loss and the Related Fall Recovery Time. (University of Georgia, 1975.) Dissertation Abstracts International 36A: 5896; March 1976.

Grumbling, Betty Lou Nowlin. An Experimental Study of the Effectiveness of Instruction in Mental Computation in Grade IV. (University of Northern Colorado, 1970.) Dissertation Abstracts International 31A: 3775-3776; February 1971.

Heitzman, Andrew J. Effects of a Token Reinforcement System on the Reading and Arithmetic Skills Learnings of Migrant Primary School Pupils. Journal of Educational Research 63: 455-458; July/August 1970.

Hillman, Bill W. The Effect of Knowledge of Results and Token Reinforcement on the Arithmetic Achievement of Elementary School Chidren. Arithmetic Teacher 17: 676-682; December 1970.

Kingston, Neldon DeVere. Reality Based Mastery Intervention Procedures: An Assessment. (Utah State University, 1978.) Dissertation Abstracts International 40A: 793; August 1979.

Klausmeier, Herbert J. and Check, John. Retention and Transfer in Children of Low, Average, and High Intelligence. Journal of Educational Research 55: 319-322; April 1962.

LeBlanc, John Francis. The Performance of First Grade Children in Four Levels of Conservation of Numerousness and Three I.Q. Groups when Solving Arithmetic Subtraction Problems. (University of Wisconsin, 1968.) Dissertation Abstracts 29A: 67; July 1968.

Masek, Richard Martin. The Effects of Teacher Applied Social Reinforcement on Arithmetic Performance and Task-Orientation. (Utah State University, 1970.) Dissertation Abstracts International 30A: 5345-5246; June 1970.

McNeil, John D. Prompting Versus Intermittent Confirmation in the Learning of a Mathematical Task. Arithmetic Teacher 12: 533-536; November 1965.

Meddleton, Ivor G. An Experimental Investigation into the Systematic Teaching of Number Combinations in Arithmetic. British Journal of Educational Psychology 26: 117-127; June 1956.

Mpiangu, Benayame Dinzau and Gentile, J. Ronald. Is Conservation of Number a Necessary Condition for Mathematical Understanding? Journal for Research in Mathematics Education 6: 179-192; May 1975.

Olander, Herbert T. Transfer of Learning in Simple Addition and Subtraction. Elementary School Journal 31: 358-369; January 1931.

Paige, Donald D. Learning While Testing. Journal of Educational Research 59: 276-277; February 1966.

Pence, Barbara May Johnson. Small Group Review of Mathematics: A Function of the Review Organization, Structure, and Task Format. (Stanford University, 1974.) Dissertation Abstracts International 35B: 2894; December 1974.

Pennington, Bruce F.; Wallach, Lise; and Wallach, Michael A. Nonconservers' Use and Understanding of Number and Arithmetic. Genetic Psychology Monographs 101: 231-243; May 1980.

Rea, Robert E. and French, James. Payoff in Increased Instructional Time and Enrichment Activities. Arithmetic Teacher 19: 663-668; December 1972.

Reys, Robert E.; Bestgen, Barbara J.; Rybolt, James F.; and Wyatt, J. Wendell. Identification and Characterization of Computational Estimation Processes Used by Inschool Pupils and Out-of-School Adults. NIE Grant No. 79-0088, November 1980. Final Report. ERIC: ED 187 963.

Robertson, Howard Charles. The Effects of the Discovery and Expository Approach of Presenting and Teaching Selected Mathematical Principles and Relationships to Fourth Grade Pupils. (University of Pittsburgh, 1970.) Dissertation Abstracts International 31A: 5278-5279; April 1971.

Robertson, Joan H. Samson. The Effectiveness of Piagetian Conservation Tasks in the Prediction of Arithmetic Achievement of Second Grade Students. (Northeast Louisiana University, 1979.) Dissertation Abstracts International 40A: 2462; November 1979.

Sawada, Daiyo. Toward a Theory of Sequencing: Study 3-1: Curriculum Hierarchies and the Structure of Intelligence: A Strategy of Organizing Instructional Objectives into Mathematical Systems Employing Basic Piagetian Constructs. (The Pennsylvania State University, 1971.) Dissertation Abstracts International 32A: 6221-6222; May 1972.

Saxe, Geoffrey B. Developmental Relations Between Notational Counting and Number Conservation. Child Development 50: 180-187; March 1979.

Schall, William E. Comparing Mental Arithmetic Modes of Presentation in Elementary School Mathematics. School Science and Mathematics 73: 359-366; May 1973.

Schoen, Harold L.; Friesen, Charles D.; Jarrett, Joscelyn A.; and Urbatsch, Tonya D. Instruction in Estimating Solutions of Whole Number Computations. Journal for Research in Mathematics Education 12: 165-178; May 1981.

Shuster, Albert and Pigge, Fred. Retention Efficiency of Meaningful Teaching. Arithmetic Teacher 12: 24-31; January 1965.

Smith, James Edward. The Effect of Positioning of Manipulatives in Addition with Children at Two Levels of Conservation on Retention and Achievement. (University of Houston, 1974.) Dissertation Abstracts International 35A: 7144; May 1975.

Souviney, Randall J. Cognitive Competence and Mathematical Development. Journal for Research in Mathematics Education 11: 215-224; May 1980.

Steffe, Leslie Philip. The Performance of First Grade Children in Four Levels of Conservation of Numerousness and Three I.Q. Groups When Solving Arithmetic Addition Problems. (University of Wisconsin, 1966.) Dissertation Abstracts 28A: 885-886; August 1967.

Suydam, Marilyn N. and Higgins, Jon L. Activity-Based Learning in Elementary School Mathematics: Recommendations from Research. Columbus: ERIC/SMEAC, September 1977. ERIC: ED 144 840.

Thomas, Sally Halstead. A Comparison of Two Strategies for Reviewing Mathematics. (Stanford University, 1974.) Dissertation Abstracts International 35A: 1434-1435; September 1974.

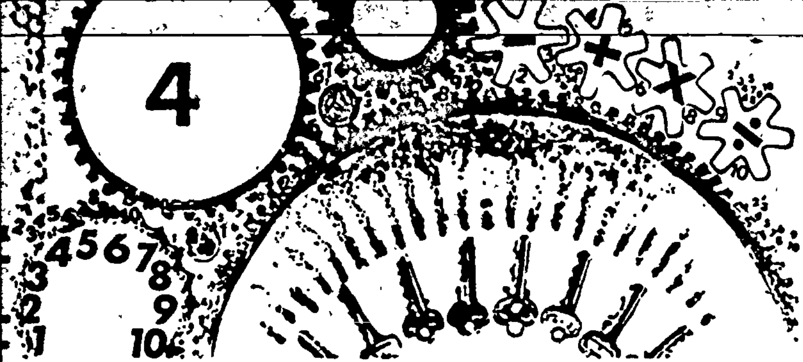
Vance, James H. and Kieren, Thomas E. Laboratory Settings in Mathematics: What Does Research Say to the Teacher? Arithmetic Teacher 18: 585-589; December 1971.

Vinsonhaler, John F. and Boss, Ronald K. A Summary of Ten Major Studies on CAI Drill and Practice. Educational Technology 12: 29-32; July 1972.

Weaver, J. Fred and Suydam, Marilyn N. Meaningful Instruction in Mathematics Education. Columbus: ERIC/SMEAC, June 1972. ERIC: ED 068 329.

Weimer, Richard Charles. A Critical Analysis of the Discovery Versus Expository Research Studies Investigating Retention or Transfer Within the Areas of Science, Mathematics, Vocational Education, Language, and Geography from 1908 to the Present. (University of Illinois at Urbana-Champaign, 1974.) Dissertation Abstracts International 35A: 7185-7186; May 1975.

Wenzelburger, Elfriede. Verbal Mediators in Mathematics for Transfer of Learning. (The University of Texas at Austin, 1974.) Dissertation Abstracts International 35A: 5142; February 1975.



Using Research: A Key to Elementary School Mathematics

DIFFERENTIATING INSTRUCTION

By differentiating instruction we mean attempts to organize mathematics programs and instruction in relation to the unique needs and abilities of individual children. This includes, but is not restricted to, plans in which individual pupils work more or less completely independently. It seems apparent that there is no one plan which is best. Provision for differentiating is conditioned in part by school organization, in part by particular teachers and pupils. Teachers must identify various factors related to pupils' achievement and interest in mathematics, and then decide on appropriate variations in content, materials, method, and time.

What factors are important to consider when differentiating instruction in mathematics?

Children in a given grade or class exhibit a wide range of ability and differ on a variety of factors; generally the range increases as grade level increases. Children also exhibit considerable variation in their patterns of behaviors. A variety of factors on which children differ must be considered as a teacher plans instruction; among them are:

Mathematical ability. The capability of the child to learn, transfer, and retain information and reason with ideas is certainly obvious to teachers. Mathematics achievement, like other aspects of school learning, is highly related to intelligence. Much research has shown that intelligence is highly related to ability to learn mathematical ideas. The factors of mathematical reasoning, verbal meaning, numerical ability, and spatial visualization are related to mathematics achievement (e.g., Westbrook, 1966).

Nature of the task. The difficulty of the content is obviously going to affect the way in which the content is taught.

Sex. Based on a review of 38 studies, Fennema (1974) concluded that during the early elementary grades, pupils' sex was not a factor that influenced mathematics achievement. In the upper grades, any observed achievement differences were apt to be in favor of boys on higher-level cognitive tasks and girls on lower-level ones. There appears to be no firm basis, however, for suggesting that mathematics instruction should be different for boys and for girls, but there is some evidence that teachers treat boys and girls differently, favoring boys (e.g., Gore, 1981). Moreover, girls may need more encouragement to pursue mathematics than boys do.

Language. It is readily apparent that, if a child does not speak English, it is difficult (if not impossible) to teach him or her in English. Beyond that, however, are concerns about the child who is bilingual.

Socioeconomic level. There is evidence from research that many children from low socioeconomic groups have less mathematical background when they enter school than do children from middle socioeconomic groups, and continue to achieve less well than those from higher socioeconomic groups through the elementary school years (e.g., Passy, 1964; Unkel, 1966).

Learning or cognitive style. We know that some children learn best from reading or seeing, while others learn best by listening, and still others by touching. Teachers usually try to cope with these differing modalities by varying instructional modes.

In other cases, it has been suggested that the most feasible way of coping with individual differences might be to match instructional methods with the cognitive style of the learner. Researchers have, for instance, identified children who learn better when taught inductively and others who achieve better when taught deductively (e.g., King et al., 1969). However, it has been much more difficult to use this identification to promote achievement. For instance, at the eighth-grade level, Gawronski (1972) found no significant achievement differences for inductive or deductive instructional approaches matched to students who had inductive or deductive learning styles. However, Branch (1974) did find that sixth graders with low-analytic cognitive styles were able to transfer better when taught inductively. Other researchers (e.g., Herrington, 1980; Hollis, 1975) also found a significant correlation between the cognitive style of individual students and the strategies used in instructional materials. However, Keane (1980) reported that level of ability rather than cognitive style accounted for most of the variance in the achievement of the sixth graders he studied.

Children differ in terms of being reflective or impulsive; thus, pupils who have a reflective cognitive style may take longer to consider their responses (and may achieve better) than pupils with an impulsive cognitive style (e.g., Cathcart and Liedtke, 1969; Radatz, 1979).

Children who are field independent (that is, they approach learning analytically, putting together parts to make a whole) are usually found to achieve better in mathematics than those who are field dependent (that is, they approach learning globally, seeing each situation as a whole) (e.g., Threadgill, 1979; Vaidya and Chansky, 1980). Cohen (1980) reported a significant interaction between field dependence/independence and degree of teacher guidance, and in another study with fifth graders, pupils who were field independent and reflective had better retention when they were taught deductively, while pupils who were field dependent and impulsive retained better when taught inductively (Koback, 1975).

Personality factors. Other studies have considered the effect of various personality factors. Peer acceptance and acceptability were each significantly related to mathematics achievement in grades 5, 7, 9, and 11 (Lawton, 1971). Poggio (1973) reported that grouping on the basis of personality characteristics appeared feasible for the sixth graders he studied, but the pertinent factors differed for boys and girls.

Is grouping for mathematics instruction effective?

Grouping to facilitate the individualization of reading instruction is a common practice in the elementary school. Evidence on the effectiveness of grouping for mathematics instruction is conflicting. Ability and achievement have each formed the basis for grouping, although it is possible to form groups on the basis of a combination of the two, or randomly.

From a review of studies on grouping, Begle (1979) concluded that "the evidence is quite clear that the most able students should be grouped together, separate from the rest of the student population. When this is done, these high ability students learn more mathematics than they would otherwise and do not develop any undesirable attitudes. ... On the other hand ... it seems to make little difference whether (other students) are grouped homogeneously or not". He also suggested that grouping should be on the basis of previous mathematics achievement rather than general ability.

When more recent studies are also considered, however, the results have been equivocal, whether grouping was by ability or by achievement: a few studies favored heterogeneous grouping; a few, homogeneous grouping; the remainder, found no significant differences. Thus, as is true across classroom organization, no clear conclusion about the achievement outcomes of grouping can be reached.

Another concern about grouping was addressed in a study with seventh graders grouped by ability (Brassell et al., 1980). They found some affective problems. Low-ranked students in groups at each of three levels of ability scored higher in anxiety, lower in self-concept, and had less enjoyment of mathematics than did other students; low-ranked students in the middle group were most anxious.

It may be that flexible grouping will best aid students in terms of both achievement and attitudes. Teachers can regroup at the beginning of each new topic, depending on pupil needs at that point. Holmes and Harvey (1956), however, found no significant differences in achievement, attitude, or social structure within the classroom whether pupils were grouped permanently or flexibly (with the topic introduced to all, followed by grouping for further work).

Second-grade children who spent more time in small groups with much interaction with the teacher were more likely to spend more of their time engaged in mathematical tasks than were students who spent most of their time on independent seatwork with little interaction with the teacher (BTES, 1978). Since it has been found that time-on-task is related to achievement, this has implications for classroom practice.

What is the effect of an individualized instruction program in which each child works alone or at his or her own pace?

For over a decade beginning in the mid-1960s, programs of individualized instruction were promoted. Many such programs individualized by (presumably) allowing children to move at their own pace through the same sequence of materials, often programmed to some extent.

Schoen (1976) reviewed 36 studies in which elementary school children in a self-paced mathematics program were compared to traditionally taught children. He found that only five of the 18 studies for kindergarten through grade 4 favored a self-paced program, five favored the traditional program, and no significant differences were found in 8. In grades 5 through 8, three studies favored self-pacing, 12 favored the traditional program, and no significant differences were found in three. He concluded that, considering the additional cost and time needed when using a self-paced program, as well as the achievement data, "a teacher or principal should not feel he or she is necessarily failing to allow for individual differences [by deciding] not to implement a self-paced instructional program."

Hartley (1978) also concluded, on the basis of a meta-analysis of studies, that little or no achievement gain resulted from the use of individual learning packets over traditional instruction. Analysis of 19 studies conducted between 1975 and 1980 which compared individualized with traditional programs at grade levels from 1 through 8 indicated similar findings. In 16, no significant differences were found; in two, the traditional program was favored over the individualized one; and in one, the individualized program was favored over the traditional one.

On the other hand, Horak (1981) concluded from her review that individualized approaches offered positive results on many specific instances, while Drayton (1979) reported that "the odds favor those in individualized programs gaining ¼ year over those in traditional programs" in grades 7 and 8.

Various other types of procedures to differentiate instruction have also been studied. Broussard (1971) found that fourth-grade students in inner-city schools given individually prescribed work through independent study, small-group discussion, large-group activities, and teacher-led discussions achieved significantly higher in skills and concepts than those taught by whole-class method using a textbook. Bierden (1970) found that, for seventh graders, a plan using group instruction followed by independent work on individualized objectives resulted in significant gains in computational skills, concept knowledge, and attitude, with a reduction in anxiety.

In another variation, Snyder (1967) found no significant differences in achievement between seventh and eighth graders who were allowed to select the mathematical topics they would study and those who could choose from a three-level assignment option. Both groups gained more on reasoning tests and less on skill tests than a third group receiving regular instruction. Powell (1976) asserted also that students should be given "preference options" in using self-directed study in middle schools.

Several studies indicated that learning cooperatively with others was more effective than learning individually (e.g., Madden, 1980; Prielipp, 1976). While Johnson et al. (1978) found that the achievement of fifth and sixth graders was higher with individual learning, attitudes and self-concept were better with cooperative learning.

How does diagnosis aid in differentiating instruction?

The purpose of diagnosis is to identify strengths as well as weaknesses, and, in the case of weakness, to identify the cause and provide appropriate remediation. As part of the process, there have been many studies which ascertained the errors pupils make. For instance, Cox (1975) reported on the systematic errors which children in grades 2 through 6 made on examples with each of the four operations with whole numbers. Roberts (1968) suggested that teachers must carefully analyze the child's method and give specific remedial help.

Gray (1966), in reporting on the development of an inventory on multiplication, called attention to the individualinterview technique pioneered by Brownell: "facing a child with a problem, letting him find a solution, then challenging him to elicit his highest level of understanding." The technique of skillful questioning and observing of pupils as they work can help to lead to devising ways of teaching by better methods.

Small-group instruction with an approach using diagnostic, prescriptive, goal-reference strategies for individual students (Fennell, 1973) and programmed materials to meet individual, diagnosed needs (Scott, 1970) are among the diagnostic procedures tried and found, to some degree, successful.

Is acceleration helpful?

A student who is accelerated in mathematics proceeds through the curriculum more rapidly than the average student -- by covering the regular curriculum faster or, most commonly in elementary school, by skipping part of the curriculum or a grade. In general, acceleration has been reported to be effective for some children. Thus, Klausmeier (1963) reported no unfavorable academic, social, emotion, or physical correlates of acceleration in fifth graders who had been "skipped" from second to fourth grade. Ivey (1965) found that fifth graders who were given an accelerated and enriched program in grade 4 gained significantly more than those receiving regular mathematics instruction. Jacobs et al. (1965) reported that seventh graders who were in an accelerated program for either three or four years did significantly better on concepts tests than those who had been accelerated for only one year. There were no significant differences on problem-solving tests.

Begle (1979) concluded from his review of research that, while acceleration is often appropriate for talented students and is probably more advantageous for them than enrichment, the grade level at which acceleration should start is open to question.

List of Selected References

Begle, E. G. Critical Variables in Mathematics Education: Findings from a Survey of the Empirical Literature. Washington, D.C: Mathematical Association of America; National Council of Teachers of Mathematics, 1979. ERIC: ED 171 515.

Bierden, James E. Behavioral Objectives and Flexible Grouping in Seventh Grade Mathematics. Journal for Research in Mathematics Education 1: 207-217; November 1970.

Branch, Robert Charles. The Interaction of Cognitive Style with the Instructional Variables of Sequencing and Manipulation to Effect Achievement of Elementary Mathematics. (University of Washington, 1973.) Dissertation Abstracts International 34A: 4857; February 1974.

Brassell, Anne; Petry, Susan; and Brooks, Douglas M. Ability Grouping, Mathematics Achievement, and Pupil Attitudes Toward Mathematics. Journal for Research in Mathematics Education 11: 22-28; January 1980.

Broussard, Vernon. The Effect of an Individualized Instructional Approach on the Academic Achievement in Mathematics on Inner-City Children. (Michigan State University, 1971.) Dissertation Abstracts International 32A: 2999-3000; December 1971.

BTES. Final Report: Beginning Teacher Evaluation Study. San Francisco: Far West Laboratory for Educational Research and Development, 1978.

Cathcart, W. George and Liedtke, Werner. Reflectiveness/Impulsiveness and Mathematics Achievement. Arithmetic Teacher 16: 563-567; November 1969.

Cohen, Jean R. The Interaction of Teacher Guidance and Sequence of Instruction on Mathematics Achievement in Field Dependent and Field Independent Students. (University of Maryland, 1979.) Dissertation Abstracts International 41A: 160; July 1980.

Cox, L. S. Systematic Errors in the Four Vertical Algorithms in Normal and Handicapped Populations. Journal for Research in Mathematics Education 6: 202-220; November 1975.

Drayton, Mary Ann Chatfield. A Bayesian Meta-Analytic Demonstration: Mathematics Achievement in Seventh and Eighth Grade. (Northern Illinois University, 1978.) Dissertation Abstracts International 39A: 4791; February 1979.

Fennell, Francis Michael. The Effect of Diagnostic Mathematics Instruction on the Achievement and Attitude of Low Achieving Sixth Grade Pupils. (Pennsylvania State University, 1972.) Dissertation Abstracts International 33A: 3378; January 1973.

Fennema, Elizabeth. Mathematics Learning and the Sexes: A Review. Journal for Research in Mathematics Education 5: 126-139; May 1974.

Gawronski, Jane Donnelly. Inductive and Deductive Learning Styles in Junior High School Mathematics: An Exploratory Study. Journal for Research in Mathematics Education 3: 239-247; November 1972.

Gore, Dolores A. Sex-Related Differences in Relation to Teacher Behavior as Wait-Time During Fourth-Grade Mathematics Instruction. (University of Arkansas, 1981.) Dissertation Abstracts International 42A: 2546; December 1981.

Gray, Roland F. An Approach to Evaluating Arithmetic Understandings. Arithmetic Teacher 13: 197-191; March 1966.

Hartley, Susan Strait. Meta-Analysis of the Effects of Individually Paced Instruction in Mathematics. (University of Colorado at Boulder, 1977.) Dissertation Abstracts International 38A: 4003-4004; January 1978.

Herrington, James Patrick. Experimental Analysis of Cognitive Style and Information Processing Variables in Mathematical Problem Solving. (Southern Illinois University at Edwardsville, 1980.) Dissertation Abstracts International 41A: 973-974; September 1980.

Hollis, Joseph Henry. A Pilot Study of a Personalized Mathematics Program for Fourth, Fifth, and Sixth Grade Students to Explore the Relationship Between Cognitive Style-Performance Objectives and Achievement. (Wayne State University, 1974.) Dissertation Abstracts International 35A: 4130; January 1975.

Holmes, Darrell and Harvey, Lois. An Evaluation of Two Methods of Grouping. Educational Research Bulletin 35: 213-222; November 1956.

Horak, Virginia M. A Meta-Analysis of Research Findings on Individualized Instruction in Mathematics. Journal of Educational Research 74: 249-253; March/April 1981.

Ivey, John O. Computation Skills: Results of Acceleration. Arithmetic Teacher. 12: 39-42; January 1965.

Jacobs, James N.; Berry, Althea; and Leinwohl, Judith. Evaluation of an Accelerated Arithmetic Program. Arithmetic Teacher 12: 113-119; February 1965.

Johnson, David W.; Johnson, Rogert T.; and Scott, Linda. The Effects of Cooperative and Individualized Instruction on Student Attitudes and Achievement. Journal of Social Psychology 104: 207-216; April 1978.

Keane, Edward Joseph. The Interactive Effects of Global Cognitive Style and General Mental Ability with Three Instructional Strategies on a Mathematical Concept Task. (University of Southern California, 1979.) Dissertation Abstracts International 40A: 5375; April 1980.

King, F. J.; Roberts, Dennis; and Kropp, Russell P. Relationship Between Ability Measures and Achievement Under Four Methods of Teaching Elementary Set Concepts. Journal of Educational Psychology 60: 244-247; June 1969.

Klausmeier, Herbert J. Effects of Accelerating Bright Older Elementary Pupils: A Follow Up. Journal of Educational Psychology 54: 165-171; 1963.

Koback, Ronald Graham. An Aptitude-Treatment Interaction Curriculum Study of the Mutually Mediating Effects of Cognitive Style and Lesson Structure and Pace Among Fifth Graders in Learning Mathematics. (University of Miami, 1975.) Dissertation Abstracts International 36A: 2597; November 1975.

Lawton, Peter Andrew. Peer Relationships and Mathematical Achievement: A Study of the Relationship of Peer Acceptance and Peer Acceptability to Achievement in Mathematics. (New York University, 1970.) Dissertation Abstracts International 31A: 6281; June 1971.

Madden, Nancy Abigail. Effects of Cooperative Learning on the Social Acceptance of Mainstreamed Academically Handicapped Students. (The American University, 1980.) Dissertation Abstracts International 41B: 2332-2333; December 1980.

Passy, Robert A. Socio-Economic Status and Mathematics Achievement. Arithmetic Teacher 11: 469-470; November 1964.

Poggio, John Paul. Prediction of Mathematics Achievement at the Sixth Grade Level Employing Selected Personality Characteristics of Students and Their Teachers. (Boston College, 1972.) Dissertation Abstracts International 33A: 6240-6241; May 1973.

Powell, Marilyn Cox. The Relationship of Cognitive Style, Achievement, and Self-Concept to an Indicated Preference for Self-Directed Study. (Memphis State University, 1976.) Dissertation Abstracts International 37A: 3383; December 1976.

Prielipp, Ronald Walter. Partner Learning in Secondary School Mathematics. (University of Oregon, 1975.) Dissertation Abstracts International 36A: 5898-5899; March 1976.

Radatz, Hendrik. Some Aspects of Individual Differences in Mathematics Instruction. Journal for Research in Mathematics Education 10: 359-363; November 1979.

Roberts, Gerhard H. The Failure Strategies of Third Grade Arithmetic Pupils. Arithmetic Teacher 15: 442-446; May 1968.

Schoen, Harold L. Self Paced Mathematics Instruction: How Effective Has It Been? Arithmetic Teacher 23: 90-96; February 1976. See also ERIC: ED 137 103.

Scott, Allen Wayne. An Evaluation of Prescriptive Teaching of Seventh-Grade Arithmetic. (North Texas State University, 1969.) Dissertation Abstracts International 30A: 4696; May 1970.

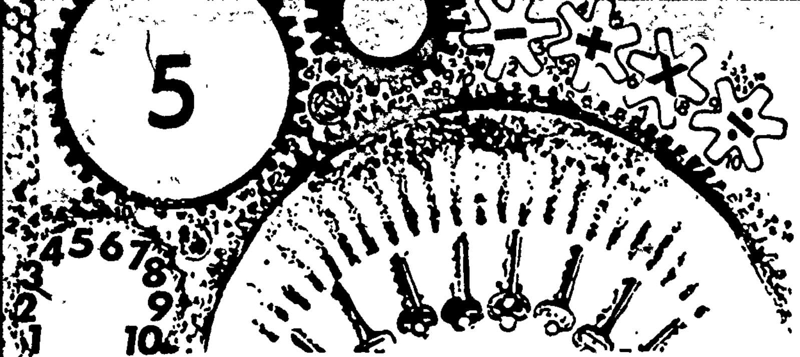
Snyder, Henry Duane, Jr. A Comparative Study of Two Self-Selection-Pacing Approaches to Individualizing Instruction in Junior High School Mathematics. (University of Michigan, 1966.) Dissertation Abstracts 28A: 159-160; July 1967.

Threadgill, Judith Ann. The Relationship of Field-Independent/Dependent Cognitive Style and Two Methods of Instruction in Mathematics Learning. Journal for Research in Mathematics Education 10: 219-222; May 1979.

Unkel, Esther. A Study of the Interaction of Socioeconomic Groups and Sex Factors with the Discrepancy Between Anticipated Achievement and Actual Achievement in Elementary School Mathematics. Arithmetic Teacher 13: 662-670; December 1966.

Vaidya, Sheila and Chansky, Norman. Cognitive Development and Cognitive Style as Factors in Mathematics Achievement. Journal of Educational Psychology 72: 326-330; June 1980.

Westbrook, Helen Rose. Intellectual Processes Related to Mathematics Achievement at Grade Levels Four, Five, and Six. (University of Georgia, 1965.) Dissertation Abstracts 26: 6520; May 1966.



INSTRUCTIONAL MATERIALS AND MEDIA

What has been learned from analyses of mathematics textbooks?

Textbooks are the primary determinant of mathematics curriculum. From an extensive review of the literature, Suydam and Osborne (1977) concluded that, in most classrooms, a single textbook is used with all students, rather than referring to multiple textbooks or varying text use by group or individual needs. That the textbook influences what is learned was supported by Begle (1973), who reported that different patterns of achievement were associated with the use of different textbooks.

Elementary mathematics textbooks have periodically been analyzed to ascertain the content included at each grade level, physical features, points of emphasis, and teaching methodology, as well as comparison between books and trends across books. Thus, Marksberry et al. (1969) checked cognitive objectives in textbooks with committee-suggested objectives and with questions and activities in teacher's manuals, and Callahan and Passi (1972) analyzed text materials to ascertain the cognitive level of activities. (Not unexpectedly, they found that low-level activities were more frequent than high-level activities; manipulating of symbols dominated the activities; while the frequency of translating, analyzing, synthesizing, and evaluating levels was low.) Still others compared textbooks with tests to ascertain the objectives covered. Thus, Dahle (1970) found that one text series corresponded more closely to an expected distribution of objectives than did two standardized tests. When Rogers (1981) compared minimum competency test objectives with two textbook series, she found that computational objectives were adequately covered, but problem-solving objectives were not.

In a comparison of two seventh-grade textbooks, McLaughlin (1970) found that students scored significantly higher using the book which included more explanation and discussion of subject matter, made greater use of symbolic notation, and provided more examples with the explanations.

Evidence from a study by Freeman et al. (1980) indicates that among three fourth-grade textbooks and five standardized tests, important differences in what might be taught are found. All materials dealt with addition, subtraction, multiplication, division, and geometry, but there the similarities ended. The texts and tests varied greatly on such topics as fractions, number sentences, estimation, and metric measurement. Out of a total of 385 topics identified in the materials, only six were included in all four textbooks and on all five tests.

What do studies on the vocabulary of textbooks show?

The vocabulary level of textbooks has also occasioned the interest of researchers. Wide ranges in the number of words used and the frequency of their use were found across textbooks (e.g., Browning, 1971; Stevenson, 1971). Moreover, the readability levels may range above the grade level of many of the students in the grades in which they are used (e.g., Smith, 1971). Earp and Tanner (1980) found that sixth graders could only comprehend half of the words in their textbooks.

In a different type of vocabulary study, Olander and Ehmer (1971) administered a test from 1930 to pupils in 1968. On the text, 1968 pupils achieved higher scores on 74 of 100 items in grade 4, 59 items in grade 5, and only 48 items in grade 6 than did pupils who had taken the test in 1930. On a test of contemporary terms, mean scores were 49 for grade 4, 58 for grade 5, and 64 for grade 6 on the 100 items. Again, this is evidence of the need for teachers of mathematics to be teachers of reading — at least in-so-far as their students need to learn to read mathematics vocabulary.

How useful are mathematics tests?

In addition to the test-textbook comparisons cited above, some studies considered only tests. After analyzing a standardized mathematics achievement test for grades 2-5, Gridley (1971) reported that the test measured several clusters of achievement items. The clusters varied from grade to grade, and subtest headings did not represent distinct clusters. The meaningfulness of the total score, as well as the subtest scores, was questioned, since several skills or abilities were being measured. Mercer (1978) also analyzed two standardized tests to ascertain the proportion of coverage of various content, and Knifong (1980) found considerable variation in computational procedures and difficulty level among the word problems in eight standardized tests.

Does programmed instruction facilitate achievement?

Programmed instruction materials purportedly allow each pupil to progress at his or her own rate. It appears from the research that programmed materials are effective to supplement the classroom teacher. As Goebel (1966) indicated, teachers devoted much more of their time (68%) to work with individuals when programmed materials were used, compared to only 3% of their time devoted to individuals when a conventional approach was used.

Research on programmed instruction (e.g., Jamison et al., 1974) indicated that achievement with programmed materials was essentially comparable to traditional instruction.

What is the role of concrete or manipulative materials?

Throughout these bulletins, much evidence is cited which indicates that the use of concrete materials appears to be essential in providing a firm foundation for developing mathematical ideas, concepts, and skills. Generally, we are bound philosophically to their use; but research increasingly indicates that we need to analyze when they are used, with whom they are used, what types should be used, and how they are used.

In an analysis of a large number of studies, Suydam and Higgins (1977) found that lessons using manipulative materials have a higher probability of producing greater mathematical achievement than do lessons in which manipulative materials are not used. This was true across a variety of mathematics topics, at every grade level (K-8), at every achievement level, at every ability level.

There is also evidence which indicates that having children manipulate materials themselves may not be necessary for all topics -- or for all children (e.g., Gilbert, 1975; Steger, 1977; Trueblood, 1968; Zirkel, 1981). Watching the teacher use the materials in a demonstration mode was often at least as effective as manipulating the materials themselves. The reason for this may lie with the fact that it is easier to direct children's attention to important points when the teacher is in control of the materials.

Many of the studies provide at least partial support that the use of materials should proceed in stages from concrete to semi-concrete (that is, pictorial) to abstract or symbolic (e.g., Olley, 1974). Investigations by Johnson (1971), Portis (1973), Carmody (1971), and Punn (1974) indicate that use of either or both physical and pictorial aids result in significantly higher achievement than when only symbolic materials are used. Fennema (1972) concluded that the research she reviewed appeared to indicate that the ratio of concrete to symbolic models used to convey mathematical ideas should reflect the developmental level of the learner. It might be that alternative models should be available so the learner can select the one most meaningful for him or her.

Many researchers have focused on the use of Cuisenaire rods (e.g., Crowder, 1966; Hollis, 1965; Lucow, 1964; Robinson, 1978). Generally, pupils using these materials scored as high or higher than those not using them. Prior background, length of time, and the specific topic may account for differences in the effect of using Cuisenaire materials.

Another point of concern is whether the number of different materials (or embodiments) affects achievement. The evidence on this is equivocal. Wheeler (1972) concluded that children proficient in using three or more concrete embodiments had a significantly higher level of understanding than children without this proficiency with concrete aids. Edge (1980), Gau (1973), and Beardslee (1973) found that pupils working with one, two, or three embodiments could operate wtih symbols and generalize essentially the same.

When materials should be used is also of concern. For instance, Weber (1970) reported no significant differences between groups of first graders who used manipulative materials for follow-up activities and those who used paper-and-pencil activities at that point. Perhaps materials provide a foundation, but at some point they are no longer needed by children.

The cost of materials was studied in relation to their effect by Harshman et al. (1962). First graders were taught for one year using either a collection of inexpensive commercial materials, a set of expensive commercial materials, or materials collected by the teacher. When significant differences in achievement were observed, they were always in favor of the third program. It was concluded that spending a lot of money for manipulative materials does not seem justified.

What is the effect of using calculators?

In the 1977-78 calculator surveys conducted by NAEP (Carpenter et al., 1981), 75% of the students aged 9 and 80% of the students aged 13 indicated that they either had their own calculators or had access to one. Other studies indicate this percentage may be even higher. Thus, calculators are available which could be used for mathematics instruction. However, while a large number of teachers now believe that calculators should be used in schools, far fewer actually use them (Reys et al., 1980; Weiss, 1978).

Over 100 studies on the effects of calculator use have been conducted since 1975 (Suydam, 1981). This is more investigations than on almost any other topic or tool or technique for mathematics instruction during this century. Many of these studies had one goal: to ascertain whether or not the use of calculators would harm students' mathematical achievement. The calculator did not appear to affect achievement adversely. In all but a few instances, achievement scores were as high or higher when calculators were used for mathematics instruction as when they were not used for instruction. The decrease in time spent on paper-and-pencil practice did not appear to harm the achievement of students who used calculators.

Some studies explored the role of the calculator in relation to problem solving. Wheatley (1980), for instance, considered the effect of calculator use on problem-solving strategies employed by children in grade 5. It appears that different strategies and solution methods are used with calculators than are used without calculators. In particular, the calculator makes the exploration of hypotheses feasible.

When beliefs and attitudes are surveyed, however, it becomes obvious that many persons ignore the evidence that calculators are useful instructional tools. The Priorities in School Mathematics Project (PRISM, 1981), devoted about 20% of its items to ascertain ways in which educators at all levels from primary through college, parents, and school board members felt about the use of calculators. Educators were much more supportive of increased use of calculators than were lay persons: 54% of the professional samples but only 36% of the lay samples would increase emphasis on calculators during the 1980s. Support was strong for using calculators for checking answers, doing a chain of calculations, computing area, making graphs, solving word problems, learning why algorithms work, and doing homework.

Data from the NAEP calculator assessment (Carpenter et al., 1981) indicate that students performed routine computation better with the aid of a calculator. The calculator aided every age group for subtraction, multiplication, and division with calculators. In several cases, performance was nearly 50 percentage points higher when a calculator was available. Performance on all nonroutine computation exercises was poor, with no improvement shown when a calculator was available. It was evident from the data that problem solving also requires more than computational skills: in general, the problem-solving performance of both 9- and 13-year-olds with a calculator was poorer than that of students without a calculator.

What is the effect of using computers?

Computers have been used in schools since the 1960s. In the late 1970s, however, microcomputers appeared in ever-increasing numbers in elementary school classrooms.

We know from the research of the past 20 years that computers can be used effectively for problem solving, drill and practice, tutorial instruction (CAI), management, games, programming, and simulations.

Data from PRISM (1981) indicated that nearly 75% of the professional samples and 80% of the lay samples believed that the use of computers and other technology should be increased during the 1980s; 78% indicated that the emphasis on computer literacy should be increased. Having computers or computer access for students was given strong support (95%) at the secondary school level and moderately strong support (77%) at the elementary school level. Strong support (84%) was also shown for having several microcomputers for each class, teaching about the roles of computers in society, and knowing about the types of problems computers can solve. The idea that knowledge of computers is only needed by specialists was strongly opposed (by 89%).

Data from NAEP (Carpenter et al., 1981) indicate that only 14% of the 13-year-olds reported they had used computers when studying mathematics.

It would appear that the evidence from the computer uses of the past 20 years can be applied to the use of microcomputers. But the microcomputer is far more accessible to students, in the home as well as in the school. Therefore, its effects may be very different, especially as children have the time to explore its capabilities, with all this means in terms of problem solving.

List of Selected References

Beardslee, Edward Clarke. Toward a Theory of Sequencing: Study 1-7: An Exploration of the Effect of Instructional Sequences Involving Enactive and Iconic Embodiments on the Ability to Generalize. (Pennsylvania State University, 1972.) Dissertation Abstracts International 33A: 6721; June 1973.

Begle, E. G. Some Lessons Learned by SMSG. Mathematics Teacher 66: 207-214; March 1973.

Browning, Carole Livingston. An Investigation of Selected Mathematical Terminology Found in Five Series of Modern Mathematics Texts Used in Grades Four, Five, and Six. (The Florida State University, 1970.) Dissertation Abstracts International 32A: 2899-2900; December 1971.

Callahan, LeRoy and Passi, Sneh Lata. Textbooks, Transitions, and Transplants. Arithmetic Teacher 19: 381-385; May 1972.

Carmody, Lenora Marie. A Theoretical and Experimental Investigation into the Role of Concrete and Semi-Concrete Materials in the Teaching of Elementary School Mathematics. (The Ohio State University, 1970.) Dissertation Abstracts International 31A: 3407; January 1971.

Carpenter, Thomas P.; Corbitt, Mary Kay; Kepner, Henry S., Jr.; Lindquist, Mary Montgomery; and Reys, Robert E. Results from the Second Mathematics Assessment of the National Assessment of Educational Progress. Reston, Virginia: National Council of Teachers of Mathematics, 1981.

Crowder, Alex Belcher, Jr. A Comparative Study of Two Methods of Teaching Arithmetic in the First Grade. (North Texas State University, 1965.) Dissertation Abstracts 26; 3778; January 1966.

Dahle, Mary McMahon. A Procedure for the Measurement of the Content Validity of Standardized Tests in Elementary Mathematics. (University of Southern California, 1970.) Dissertation Abstracts International 30A: 5336; June 1970.

Earp, N. Wesley and Tanner, Fred W. Mathematics and Language. Arithmetic Teacher 28: 32-34; December 1980.

Edge, Douglas Richard Montgomery. The Effects of Using Multiple Embodiments of Selected Multi-Digit Numeration Concepts on the Understanding and Transferability of Those Concepts. (University of Maryland, 1980.) Dissertation Abstracts International 41A: 1990; November 1980.

Fennema, Elizabeth H. The Relative Effectiveness of a Symbolic and a Concrete Model in Learning a Selected Mathematical Principle. Journal for Research in Mathematics Education 3: 233-238; November 1972.

Freeman, Donald J. and Others. The Fourth-Grade Mathematics Curriculum as Inferred from Textbooks and Tests. Research Series No. 82. East Lansing: Institute for Research on Teaching, Michigan State University, 1980. ERIC: ED 199 047.

Gau, Gerald Elmer. Toward a Theory of Sequencing: Study 1-6: An Exploration of the Effect of Instructional Sequences Involving Enactive and Iconic Embodiments on the Attainment of Concepts Embodied Symbolically. (Pennsylvania State University, 1972.) Dissertation Abstracts International 33A: 6728; June 1973.

Gilbert, Robert Kennedy. A Comparison of Three Instructional Approaches Using Manipulative Devices in Third Grade Mathematics. (University of Minnesota, 1974.) Dissertation Abstracts International 35A: 5189-5190; February 1975.

Goebel, Laurence Gayheart. An Analysis of Teacher-Pupil Interaction When Programmed Instructional Materials are Used. (University of Maryland, 1966.) Dissertation Abstracts 27A: 982; October 1966.

Gridley, John David, Jr. An Expirical Investigation of the Construct of Mathematics Achievement in the Elementary Grades Based on the Method of Homogeneous Keying. (Fordham University, 1971.) Dissertation Abstracts International 32A: 1914; October 1971.

Harshman, Hardwick, W.; Wells, David W.; and Payne, Joseph N. Manipulative Materials and Arithmetic Achievement in Grade 1. Arithmetic Teacher 9: 188-191; April 1962.

Hollis, Loye Y. A Study to Compare the Effect of Teaching First and Second Grade Mathematics by the Cuisenaire-Gattegno Method with a Traditional Method. School Science and Mathematics 65: 683-687; November 1965.

Jamison, Dean; Suppes, Patrick; and Wells, Stuart. The Effectiveness of Alternative Instructional Media: A Survey. Review of Educational Research 44: 1-67; Winter 1974.

Johnson, Robert Leo. Effects of Varying Concrete Activities on Achievement of Objectives in Perimeter, Area, and Volume by Students of Grades Four, Five, and Six. (University of Colorado, 1970.) Dissertation Abstracts International 31A: 4624; March 1971.

Knifong, J. Dan. Computational Requirements of Standardized Word Problem Tests. Journal for Research in Mathematics Education 11: 3-9; January 1980.

Lucow, William H. An Experiment with the Cuisenaire Method in Grade Three. American Educational Research Journal 1: 159-167; May 1964.

Marksberry, Mary Lee; McCarter, Mayme; and Noyce, Ruth. Relation Between Cognitive Objectives from Selected Texts and from Recommendations of National Committees. Journal of Educational Research 62: 422-430; May/June 1969.

McLaughlin, John Paul. Mathematics Achievement in Relation to Textbook Communication Style and Ability Group Membership: A Comparison of Two Seventh Grade Mathematics Textbooks. (The University of Texas at Austin, 1969.) Dissertation Abstracts International 30A: 2911; January 1970.

Mercer, Maryann. The Content of Two Mathematics Achievement Subtests. School Science and Mathematics 78: 669-674; December 1978.

Olander, Herbert T. and Ehmer, Charles L. What Pupils Know About Vocabulary in Mathematics -- 1930 and 1968. Elementary School Journal 71: 361-367; April 1971.

Olley, Peter George. The Relative Efficacy of Four Experimental Protocols in the Use of Model Devices to Teach Selected Mathematical Constructs. (Washington State University, 1973.) Dissertation Abstracts International 34A: 4993; February 1974.

Portis, Theodore Roosevelt. An Analysis of the Performances of Fourth, Fifth and Sixth Grade Students on Problems Involving Proportions, Three Levels of Aids and Three I.Q. Levels. (Indiana University, 1972.) Dissertation Abstracts International 33A: 5981-5982; May 1973.

PRISM. Priorities In School Mathematics: Executive Summary of the PRISM Project. Reston, Virginia: National Council of Teachers of Mathematics, 1981.

Punn, Avtar Kaur. The Effects of Using Three Modes of Representation in Teaching Multiplication Facts on the Achievement and Attitudes of Third Grade Pupils. (University of Denver, 1973.) Dissertation Abstracts International 34A: 6954-6955; May 1974.

Reys, Robert E.; Bestgen, Barbara J.; Rybolt, James F.; and Wyatt, J. Wendell. Hand Calculators -- What's Happening in Schools Today? Arithmetic Teacher 27: 38-43; February 1980.

Robinson, Evelyn Barron. The Effects of a Concrete Manipulative on Attitude Toward Mathematics and Levels of Achievement and Retention of a Mathematical Concept Among Elementary Students. (East Texas State University, 1978.) Dissertation Abstracts International 39B: 1335; September 1978.

Rogers, Barbara Dillard. North Carolina's Minimum Competency Testing Program and Mathematics Textbooks. (Duke University, 1981.) Dissertation Abstracts International 42A: 2061; November 1981.

Smith, Frank. The Readability of Sixth Grade Word Problems. School Science and Mathematics 71: 559-562; June 1971.

Steger, Clarence Donald. The Effects of Two Classes of Sensory Stimuli and Race on the Acquisition and Transfer of the Mathematics Principle Place Value. (University of South Florida, 1976.) Dissertation Abstracts International 37A: 4120; January 1977.

Stevenson, Erwin Francis. An Analysis of the Technical and Semi-Technical Vocabulary Contained in Third Grade Mathematics Textbooks and First and Second Grade Readers. (Indiana University, 1971.) Dissertation Abstracts International 32A: 3012; December 1971.

Suydam, Marilyn N. The Use of Calculators in Pre-College Education: Fourth Annual State-of-the-Art-Review. Columbus, Ohio: Calculator Information Center, 1981. ERIC: ED 206 454.

Suydam, Marilyn N. and Higgins, Jon L. Activity-Based Learning in Elementary School Mathematics: Recommendations from Research. Columbus: ERIC/SMEAC, September 1977. ERIC: ED 144 840.

Suydam, Marilyn N. and Osborne, Alan. The Status of Pre-College Science, Mathematics, and Social Science Education: 1955-1975. Volume II, Mathematics Education. Columbus: ERIC/SMEAC, 1977. ERIC: ED 153 878.

Trueblood, Cecil Ross. A Comparison of Two Techniques for Using Visual-Tactual Devices to Teach Exponents and Non-Decimal Bases in Elementary School Mathematics. (The Pennsylvania State University, 1967.) Dissertation Abstracts 29A: 190-191; July 1968.

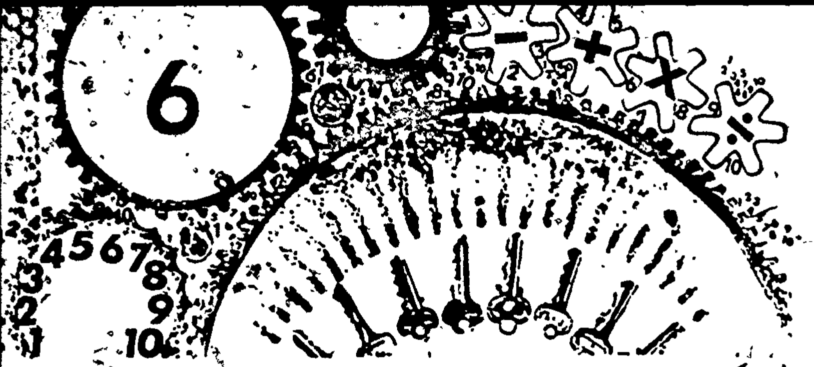
Weber, Audra Wheatly. Introducing Mathematics to First Grade Children: Manipulative Vs. Paper and Pencil. (University of California, Berkeley, 1969.) Dissertation Abstracts International 30A: 3372-3373; February 1970.

Weiss, Iris R. Report of the 1977 National Survey of Science, Mathematics, and Social Studies Education. Final Report. Research Triangle Park, North Carolina: Research Triangle Institute, March 1978. ERIC: ED 152 565.

Wheatley, Charlotte L Calculator Use and Problem-Solving Performance. Journal for Research in Mathematics Education 11: 323-324; November 1980.

Wheeler, Larry Eugene. The Relationship of Multiple Embodiments of the Regrouping Concept to Children's Performance in Solving Multi-Digit Addition and Subtraction Examples. (Indiana University, 1971.) Dissertation Abstracts International 32A: 4260; February 1972.

Zirkel, Ronald Elmer. The Role of Manipulatives in Schematic Approaches to Area Measurement for Middle School Students. (Virginia Polytechnic Institute and State University, 1980.) Dissertation Abstracts International 41A: 3933; March 1981.



ADDITION AND SUBTRACTION WITH WHOLE NUMBERS

What foundation for addition and subtraction do children have upon entering school?

As teachers are well aware, some background for the development of skills in addition and subtraction is formed before children begin systematic study of these operations. The ability to count is of particular importance: children often use counting as a primary means of ascertaining and verifying addition and subtraction facts (e.g., Gelman and Tucker, 1975; Houlihan and Ginsberg, 1981). The ability to recognize the number of a set without counting is also helpful.

Through the years, many investigations have been conducted to ascertain the counting skills and other mathematical abilities possessed by the pre-first-grade child (e.g., Brace and Nelson, 1965; Buckingham and MacLatchy, 1930; Hendrickson, 1979; Mott, 1945; Priore, 1957; Rea and Reys, 1971). In some studies, it was found that many children could solve simple addition and subtraction examples in an oral or problem context. Across the studies, wide differences were found in children's ability to count. While some children could count to 100 or beyond, a few had difficulty counting to 10. Thus, the classroom teacher cannot assume that all children have the counting and other skills which appear necessary to work with addition and subtraction. Teachers must assess the attainment of the individual children in their classes.

Whether rote counting or rational counting should be taught first is a recurrent question, but has not been explicitly answered by research. Generally, the pre-school child learns first to say the number names with sets of objects. Brush (1973) found that informal notions about addition and subtraction were picked up through everyday experiences.

Is conservation of number a necessary condition for understanding addition and subtraction?

Piaget's work with conservation appears to have some applicability to the teaching of addition and subtraction. Steffe (1968) found that first graders who did better on tests of addition facts and problems had the ability to conserve (that is, they knew when two sets had the same number of objects even though they were arranged differently), and Thaeler (1981) found a significant relationship between developmental level and the strategy used in getting answers to addition facts. LeBlanc (1968) reported a similar conclusion with respect to subtraction problems. However, Mpiangu and Gentile (1975) are among those who have concluded that number conservation was not a prerequisite to the development of mathematical understanding, suggesting that the two develop simultaneously. Similarly, Hiebert et al. (1980) reported that some children who had not yet developed a particular cognitive ability solved at least some problems of each type, calling into question the use of these cognitive tasks as readiness variables for instruction.

What is the relative difficulty of addition and subtraction facts?

At one time, especially when stimulus-response theories of learning were prevalent, there was great interest in ascertaining whether some basic number facts or combinations -- e.g., 5 + 2 = 7, 9 + 6 = 15, 8 - 3 = 5, 17 - 9 = 8 -- were more difficult than others. Textbook writers as well as classroom teachers used the results of such research to determine the order in which facts would be presented. The assumption was that if the combinations were sequenced appropriately, the time needed to memorize them could be reduced. Varying procedures to ascertain the difficulty level resulted in a lack of agreement among the studies. Some common findings were evident, however (Suydam and Dessart, 1976):

3 + 4
4 + 3

(1) An addition combination and its "reverse" form tend to be of equal difficulty.

6 + 8
17 - 9

(2) The size of an addend rather than the size of the sum is the principal indicator of difficulty.

6 + 6
9 - 1

(3) The doubles and those combinations in which 1 is an addend appear to be easiest in addition; those with differences of 1 or 2 are easiest in subtraction.

(4) Subtraction combinations are more difficult than addition combinations.

Some researchers have used the data-gathering ability of the computer to explore the relative difficulty of the basic facts (e.g., Suppes and Groen, 1967). The findings have been used to sequence programs for drill and practice available in paper-and-pencil form and on computers.

Swenson (1944) questioned whether results on relative difficulty obtained under repetitive drill-oriented methods of learning are valid when applied in learning situations not so definitely drill-oriented. She found that the order of difficulty seemed to be a function of the teaching method, at least in part. This finding was confirmed by Marotta (1974). He noted that the teacher should provide specific learning experiences for individual students, realizing that the basic facts "may be organized for instruction in many different ways, using different approaches and materials, and different means to achieve the goal of computational skill with understanding."

How do children "solve" basic facts?

A number of years ago, Brownell (1928) found that pupils use various ways of obtaining answers to combinations -- guessing, counting, and solving from known combinations, as well as immediate recall. Brownell stated, "Children appear to attain 'mastery' only after a period during which they deal with procedures less advanced (but to them more meaningful) than automatic responses." Grouws (1974) verified that these processes are still frequently used, but his list of ways third graders solved open addition and subtraction sentences included others: addend-sum relationships, equivalent sentences, random or systematic substitution, counting on or back, tallying, inverse relationship, and simplifying (by considering a sentence of the same type but usually with smaller numbers).

Research by Thornton (1978) supported the use of thinking strategies in teaching basic facts. Rathmell (1978) proposed activities for teaching such thinking strategies for addition as counting on, one more or less, and compensation.

First-grade pupils taught counting on did significantly better on timed tests of addition facts than did pupils using a set approach or a textbook (Leutzinger, 1980), while emphasis on the counting relationship among addition facts resulted in higher achievement than no such emphasis (Carnine and Stein, 1981). Sauls and Beeson (1976) found that about 62% of the fourth graders they studied still used counting when adding and subtracting.

Beattie (1979) investigated how students in grades 4, 5, and 6 derived answers to unknown subtraction facts: counting forward, counting backward, derivation from a known fact, and bridging using 10 as an intermediate step. He noted that errors occurred when children's procedures were faulty, inefficient, or too complex.

Evidence from the 1977-78 National Assessment (Carpenter et al., 1981) indicated that most pupils do learn the addition and subtraction facts: approximately 90% of the

9-year-olds knew addition facts and 79% knew subtraction facts. At age 13, the percentages were over 90% for each set of facts.

Are open addition and subtraction sentences all of the same difficulty?

Various investigations pertaining to the difficulty level of open sentences have been reported (e.g., Engle and Lerch, 1971; Grouws, 1974; Weaver, 1971.) A synthesis of findings suggest that:

(1) open subtraction sentences are more difficult to solve than open addition sentences;

(2) sentences of the form $\square - b = c$ or $c = \square - b$ are clearly the most difficult of all types;

(3) sentences with the operation sign on the right-hand side of the equals sign are more difficult than those with the operation sign on the left-hand side;

(4) sentences with numbers between 20 and 100 are more difficult than those that are within the context of basic facts;

(5) children's methods of solving open sentences vary from type to type; and

(6) their solution methods also vary within each particular type of sentence.

Teachers should be careful of the order in which open-sentence types are introduced and studied. It is likely that within each column below, the types of open sentences are listed in order of increasing conceptual difficulty:

$a + b = \square$	$\square = a + b$	$a - b = \square$	$\square = a - b$
$a + \square = c$	$c = a + \square$	$a - \square = c$	$c = a - \square$
$\square + b = c$	$c = \square + b$	$\square - b = c$	$c = \square - b$

Teachers also may need to be careful of the pace at which open-sentence types are mixed. Lindvall and Ibarra (1980) found that most pupils' errors appeared to be associated with an incorrect and inconsistent reading of number sentences. In a study with first graders, Nibbelink (1981) found that they scored 66% correct on horizontal open sentences and slightly better (75%) on the vertical form.

Should addition and subtraction be taught at the same time?

It is somewhat surprising, considering how frequently this question is asked, to find that little research has been done on the topic. In early studies, it was found that higher achievement resulted when addition and subtraction facts were taught together. In a study in grades 1 and 2, Spencer (1968) reported that there may be some intertask interference, but emphasis on the relationship between the operations facilitated understanding.

Wiles et al. (1973) investigated the effect of a sequential and an integrated approach to the introduction of two algorithms for addition and subtraction examples involving renaming. For second graders, no evidence was found to support any advantage of an integrated approach in which the two algorithms were introduced more or less simultaneously. The subtraction-then-addition instructional sequence generally produced the poorest performance; the tendency of all groups was to learn addition first.

What type of problem situation should be used for introductory work with subtraction?

Gibb (1956) explored ways in which pupils think as they attempt to solve subtraction problems. In interviews with 36 second graders, she found that pupils did best on "take-away" problems and poorest on "comparative" problems. For instance, when the question was "How many are left?", the problem was easier then when it was "How many more does Tom have than Jeff?". "Additive" problems, in which the question might be "How many more does he need?", were of medium difficulty and took more time. She reported that the children solved the problems in terms of the situation, not realizing that one basic idea occurred in all applications.

While Schell and Burns (1962) found no difference in performance on the three types of problems, "take-away" situations were considered by pupils to be easiest. Thus, they are generally considered first in introductory work with subtraction.

Coxford (1966) and Osborne (1967) found that an approach using set-partitioning, with explicit use of the relationship between addition and subtraction, resulted in greater understanding than the "take-away" approach.

How should subtraction with renaming be taught?

Over the years, researchers have been very concerned about procedures for teaching subtraction involving renaming (once commonly called "borrowing"). The question of most concern has been whether to teach subtraction by equal additions or by decomposition.

How do you do this example?

$$\begin{array}{r} 91 \\ -24 \\ \hline 67 \end{array}$$

You're using decomposition if you do it this way:

11 - 4 = 7 (ones); 8 - 2 = 6 (tens)

If you do it this way, you're using equal additions:

11 - 4 = 7 (ones); 9 - 3 = 6 (tens)

In a classic study involving 1 400 third-grade pupils, (Brownell, 1947; Brownell and Moser, 1949), the comparative merits of two algorithms (decomposition and equal additions), in combination with two methods of instruction (meaningful and mechanical), were investigated:

	meaningful	mechanical
decomposition	a	b
equal additions	c	d

It was found at the time of initial instruction that:

(1) Meaningful decomposition [a] was better than mechanical decomposition [b] on measures of understanding and accuracy.

(2) Meaningful equal additions [c] was significantly better than mechanical equal additions [d] on measures of understanding.

(3) Mechanical decomposition [b] was not as effective as either equal additions procedure [c or d].

(4) Meaningful decomposition [a] was superior to each equal additions procedure [c,d] on measures of understanding and accuracy.

It was concluded that whether to teach the equal additions or the decomposition algorithm depends on the desired outcome.

In a study focused on how to teach the decomposition algorithm more effectively, Trafton (1971) reported that for third graders more extensive development of the decomposition algorithm was better than a procedure that included work with concepts and the use of the number line before the algorithm was taught.

Cosgrove (1957), in a study with sixth graders who had learned the decomposition algorithm, found that they could change to the equal-additions algorithm without significant interference effects. Hypothesized speed and accuracy advantages for equal addition, found in other research, were not observed, however. Sherrill (1979) did find that the decomposition algorithm was superior in accuracy to the equal-additions algorithm with third graders.

Several researchers have studied algorithms in which a minimum of memory is needed. For instance:

8 (8 + 5 = 13; write 3 on the ones side, 1 on the tens side)

5

1 3 (3 + 9 = 12; write 2 on the ones side, 1 on the tens side)

9

1 2 (2 + 7 = 9; write 9 on the sum of the ones, then add the two 1s from the tens side)

7

29

Hutchings (1975) and Lester (1979) reported some affirmative results for such algorithms, and Alessi (1974) found significant differences favoring the cited addition algorithm with fourth graders when the number of columns attempted and the number correct were the criteria for success. Such algorithms might be considered for students who are having difficulty with other algorithmic forms.

What is the role of materials in developing understanding and skill in addition and subtraction?

It is as important to use concrete materials in introducing algorithms as it is in introducing basic facts. Ekman (1967), for instance, reported that when third graders manipulated materials before the presentation of an addition algorithm, both understanding and ability to transfer increased. Using materials before introducing or developing the algorithm was better than using only pictures or using neither aid. Punn (1974) similarly found that third-grade groups using manipulative materials and symbols, or materials, symbols, and pictures, achieved significantly higher than those using only pictures and symbols.

Wheeler (1972) indicated that second-grade pupils proficient in regrouping two-digit addition and subtraction examples with three or more concrete materials scored significantly higher on multidigit tests than those not proficient in using materials. In a study with second graders, Knaupp (1971) found that both teacher-demonstration and student-activity modes with either blocks or sticks resulted in significant gains in achievement on addition and subtraction algorithms and ideas of base and place value. After testing primary-grade children on two-digit addition examples, Brownell (1928) concluded that thorough understanding based on the manipulation of concrete materials resulted in an easier transition to abstract work.

However, Gibb (1956) reported that although children performed poorest on problems presented in an abstract context, they performed better on subtraction examples presented in a semiconcrete context than on examples presented with concrete materials. Nevertheless, she noted that children had less difficulty solving problems when they could manipulate objects with which the problems were directly associated than when they had to solve the problems entirely on a verbal, abstract basis. However, Moser (1980) found that most successful problem solvers used physical materials, even when they probably could have used more sophisticated strategies.

Delaying the use of symbols while providing much work with materials seemed particularly important for lower achievers (Hamrick, 1979).

What is the role of drill in teaching addition and subtraction?

Prior to the 1930s, much research was done on the effectiveness of various types of drill, often isolated from other instruction or even the primary mode of instruction. For instance, Knight (1927) and Wilson (1930) reported on programs of drill in which the distribution of practice on basic facts was carefully planned -- no facts were neglected, but more difficult combinations were emphasized. Accuracy has been and is accepted as a goal in mathematics, and it is in an attempt to meet this goal that drill is stressed. The back-to-the-basics movement promoted accuracy, and thus drill-oriented instruction, to a preeminent goal.

However, many studies have shown that drill per se is not effective in developing mathematical concepts. For instance, programs stressing relationships and generalizations among the addition and subtraction combinations were found to be preferable for developing understanding and the ability to transfer.

Brownell and Chazal (1935) summarized their research work with third graders by stating that drill must be preceded by meaningful instruction. The type of thinking which is developed and the child's facility with the process of thinking is of greater importance than mere recall. Drill in itself makes little contribution to growth in quantitative thinking, since it fails to supply more mature ways of dealing with numbers.

What types of errors do pupils commonly make?

Many studies have identified errors pupils make. Lankford (1972), using written tests, oral interviews, and analysis of written work, indicated that the most frequently found errors were with basic facts and with zero, subtracting the minuend from the subtrahend, and adding a carried number later.

Other research has indicated that regrouping errors in addition and reversing digits in subtraction are the most common errors, along with basic fact errors. Failure to regroup correctly and use of incorrect operations or algorithms are also found frequently (e.g., Engelhardt, 1977; Brown and Burton, 1978).

Cox (1975) reported that 5 percent of the errors in addition and 13 percent of the errors in subtraction can be clssified as systematic: that is, errors that are consistently made. Such systematic errors are potentially remediable: first of all, teachers must look for patterns in the responses of children having difficulty with computational skills. Then the task must be analyzed into its component subtasks, so that the subtasks causing the error can be retaught.

Should non-paper-and-pencil practice be provided?

Many mathematical problems which arise in everyday life must be solved without pencil and paper. Providing a planned program of non-paper-and-pencil practice on both examples and problems has been found to be effective in increasing achievement in addition and subtraction, as for other topics in the curriculum (e.g., Flournoy, 1954). Other researchers have suggested that certain "thinking strategies" especially suited to such practice should be taught. For instance, a left-to-right approach to finding the sum or difference is useful, rather than the right-to-left approach used in the written algorithm. "Rounding," using the principle of compensation, and renaming are also helpful. Increased understanding of the process may result.

Should children "check" their answers?

The answers which research has provided to this question are not in total agreement. We encourage children to check their work, since we believe that checking contributes to greater accuracy. There is some research evidence to support this belief.

However, Grossnickle (1938) reported data which should be considered as we teach. He analyzed the work of 174 third graders who used addition to check subtraction answers. He found that pupils frequently "forced the check," that is, made the sums agree without actually adding; in many cases, checking was perfunctory. Generally, there was only a chance difference between the mean accuracy of the group of pupils when they checked and their mean accuracy when they did not check.

What does this indicate to teachers? Obviously, children must understand the <u>purpose</u> of checking -- and what they must do if the solution in the check does not agree with the original solution. With the increasing use of hand-held calculators, it is imperative that children attain this understanding.

List of Selected References

Alessi, Galen James. Effects of Hutchings's 'Low Fatigue' Algorithm on Children's Addition Scores Compared under Varying Conditions of Token Economy Reinforcement and Problem Difficulty. (University of Maryland, 1974.) Dissertation Abstracts International 35A: 3502; December 1974.

Beattie, Ian D. Children's Strategies for Solving Subtraction-Fact Combinations. Arithmetic Teacher 27: 14-15; September 1979.

Brace, Alec and Nelson, L. Doyal. The Preschool Child's Concept of Number. Arithmetic Teacher 12: 126-133; February 1965.

Brown, John Seely and Burton, Richard R. Diagnostic Models for Procedural Bugs in Basic Mathematical Skills. Cognitive Science 2: 155-192; 1978.

Brownell, William A. The Development of Children's Number Ideas in the Primary Grades. Supplementary Educational Monographs 35: 1-241; August 1928.

Brownell, William A. An Experiment on "Borrowing" in Third-Grade Arithmetic. Journal of Educational Research 41: 161-171; November 1947.

Brownell, William A. and Chazal, Charlotte B. The Effects of Premature Drill in Third-Grade Arithmetic. Journal of Educational Research 29: 17-28; September 1935.

Brownell, William A. and Moser, Harold E. Meaningful vs. Mechanical Learning: A Study in Grade III Subtraction. Durham, North Carolina: Duke University Press, No. 8, 1949.

Brush, Lorelei Ruth. Children's Conceptions of Addition and Subtraction: The Relation of Formal and Informal Notions. (Cornell University, 1972.) Dissertation Abstracts International 33B: 4989; April 1973.

Buckingham, B. R. and MacLatchy, Josephine. The Number Abilities of Children When They Enter Grade One. In Report of the Society's Committee on Arithmetic; Part II, Research in Arithmetic. Twenty-ninth Yearbook. National Society for the Study of Education. Bloomington, Illinois: Public School Publishing Company, 1930.

Carnine, Douglas W. and Stein, Marcy. Organizational Strategies and Practice Procedures for Teaching Basic Facts. Journal for Research in Mathematics Education 12: 65-69; January 1981.

Carpenter, Thomas P.; Corbitt, Mary Kay; Hepner, Henry S., Jr.; Lindquist, Mary Montgomery; and Reys, Robert E. Results from the Second Mathematics Assessment of the National Assessment of Educational Progress. Reston, Virginia: National Council of Teachers of Mathematics, 1981.

Cosgrove, Gail Edmund. The Effect on Sixth-Grade Pupils' Skill in Compound Subtraction When They Experience a New Procedure for Performing This Skill. (Boston University School of Education, 1957.) Dissertation Abstracts 17: 2933-2934; December 1957.

Cox, L. S. Systematic Errors in the Four Vertical Algorithms in Normal and Handicapped Populations. Journal for Research in Mathematics Education 6: 202-220; November 1975.

Coxford, Arthur Frank, Jr. The Effects of Two Instructional Approaches on the Learning of Addition and Subtraction Concepts in Grade One. (University of Michigan, 1965.) Dissertation Abstracts 26: 6543-6544; May 1966.

Ekman, Lincoln George. A Comparison of the Effectiveness of Different Approaches to the Teaching of Addition and Subtraction Algorithms in the Third Grade (Volumes I and II). (University of Minnesota, 1966.) Dissertation Abstracts 27A: 2275-2276; February 1967.

Engelhardt, J. M. Analysis of Children's Computational Errors: A Qualitative Approach. British Journal of Educational Psychology 47: 149-154; June 1977.

Engle, Carol D. and Lerch, Harold H. A Comparison of First-Grade Children's Abilities on Two Types of Arithmetical Practice Exercises. School Science and Mathematics 71: 327-334; April 1971.

Flournoy, Mary Frances. The Effectiveness of Instruction in Mental Arithmetic. Elementary School Journal 55: 148-153; November 1954.

Gelman, Rochel and Tucker, Marsha F. Further Investigations of the Young Child's Conception of Number. Child Development 46: 167-175; March 1975.

Gibb, E. Glenadine. Children's Thinking in the Process of Subtraction. Journal of Experimental Education 25: 71-80; September 1956.

Grossnickle, Foster E. The Effectiveness of Checking Subtraction by Addition. Elementary School Journal 38: 436-441; February 1938.

Grouws, Douglas A. Solution Methods Used in Solving Addition and Subtraction Open Sentences. Arithmetic Teacher 21: 255-261; March 1974.

Hamrick, Kathy B. Oral Language and Readiness for the Written Symbolization of Addition and Subtraction. Journal for Research in Mathematics Education 10: 188-194; May 1979.

Hendrickson, A. Dean. An Inventory of Mathematical Thinking Done by Incoming First-Grade Children. Journal for Research in Mathematics Education 10: 7-23; January 1979.

Hiebert, James; Carpenter, Thomas P.; and Moser, James M. Cognitive Development and Performance on Verbal Addition and Subtraction Problems. Technical Report No. 560. Madison: Wisconsin Research and Development Center for Individualized Schooling, August 1980.

Houlihan, Dorothy M. and Ginsburg, Herbert P. The Addition Methods of First- and Second-Grade Children. Journal for Research in Mathematics Education 12: 95-106; March 1981.

Hutchings, Barton. Low-Stress Subtraction. Arithmetic Teacher 22: 226-232; March 1975.

Knaupp, Jonathan Elmer. A Study of Achievement and Attitude of Second Grade Students Using Two Modes of Instruction and Two Manipulative Models for the Numeration System. (University of Illinois at Urbana-Chapaign, 1970.) Dissertation Abstracts International 31A: 6471; June 1971.

Knight, F. B. The Superiority of Distributed Practice in Drill in Arithmetic. Journal of Educational Research 15: 157-163; March 1927.

Lankford, Francis G., Jr. Some Computational Strategies of Seventh Grade Pupils. Charlottesville, Virginia: University of Virginia, October 1972. ERIC: ED 069 496.

LeBlanc, John Francis. The Performance of First Grade Children in Four Levels of Conservation of Numerousness and Three I.Q. Groups When Solving Arithmetic Subtraction Problems. (University of Wisconsin, 1968.) Dissertation Abstracts 29A: 67; July 1968.

Lester, Louise Rainey. An Investigation into the Acceleration of Basic Quantitative Skills of Educationally Disadvantaged Rural Primary School Children Using Hutchings Low Stress Procedures. (University of South Carolina, 1978.) Dissertation Abstracts International 39A: 5908-5909; April 1979.

Leutzinger, Larry Paul. The Effects of Counting on the Acquisition of Addition Facts in Grade One. (The University of Iowa, 1979.) Dissertation Abstracts International 40A: 3765; January 1980.

Lindvall, C. Mauritz and Ibarra, Cheryl Gibbons. Incorrect Procedures Used by Primary Grade Pupils in Solving Open Addition and Subtraction Sentences. Journal for Research in Mathematics Education 11: 50-62; January 1980.

Marotta, George A. A Survey of the Teaching of the Basic Addition Facts in Elementary School Mathematics in the United States. (Rutgers University, The State University of New Jersey, 1973.) Dissertation Abstracts International 34A: 3700-3701; January 1974.

Moser, James M. A Longitudinal Study of the Effect of Number Size and Presence of Manipulative Materials on Children's Processes in Solving Addition and Subtraction Verbal Problems. April 1980. ERIC: ED 184 880.

Mott, Sina M. Number Concepts of Small Children. Mathematics Teacher 38: 291-301; November 1945.

Mpiangu, Benayame and Gentile, J. Ronald. Is Conservation of Number a Necessary Condition for Mathematical Understanding? Journal for Research in Mathematics Education 6: 179-192; May 1975.

Nibbelink, William H. A Comparison of Vertical and Horizontal Forms for Open Sentences Relative to Performance by First Graders: Some Suggestions. School Science and Mathematics 81: 613-619; November 1981.

Osborne, Alan Reid. The Effects of Two Instructional Approaches on the Understanding of Subtraction by Grade Two Pupils. (University of Michigan, 1966.) Dissertation Abstracts 28A: 158; July 1967.

Priore, Angela. Achievement by Pupils Entering First Grade. Arithmetic Teacher 4: 55-60; March 1957.

Punn, Avtar Kaur. The Effects of Using Three Modes of Representation in Teaching Multiplication Facts on the Achievement and Attitudes of Third Grade Pupils. (University of Denver, 1973.) Dissertation Abstracts International 34A: 6954-6955; May 1974.

Rathmell, Edward C. Using Thinking Strategies to Teach the Basic Facts. In Developing Computational Skills (Marilyn N. Suydam, Ed.). 1978 Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Rea, Robert E. and Reys, Robert E. Competencies of Entering Kindergarteners in Geometry, Number, Money, and Measurement. School Science and Mathematics 71: 389-402; May 1971.

Sauls, Charles and Beeson, B. F. The Relationship of Finger Counting to Certain Pupil Factors. Journal of Educational Research 70: 81-83; November/December 1976.

Schell, Leo M. and Burns, Paul C. Pupil Performance with Three Types of Subtraction Situations. School Science and Mathematics 62: 208-214; March 1962.

Sherrill, James M. Subtraction: Decomposition Versus Equal Addends. Arithmetic Teacher 27: 16-17; September 1979.

Spencer, James Edward. Intertask Interferences in Primary Arithmetic. (University of California, Berkeley, 1967.) Dissertation Abstracts 28A: 2570-2571; January 1968.

Steffe, Leslie P. The Relationship of Conservation of Numerousness to Problem-Solving Abilities of First-Grade Children. Arithmetic Teacher 15: 47-52; January 1968.

Suppes, Patrick and Groen, Guy. Some Counting Models for First-Grade Performance Data on Simple Addition Facts. Research in Mathematics Education. Washington, D.C.: National Council of Teachers of Mathematics, 1967.

Suydam, Marilyn N. and Dessart, Donald J. Classroom Ideas from Research on Computational Skills. Reston, Virginia: National Council of Teachers of Mathematics, 1976.

Swenson, Esther J. Difficulty Ratings of Addition Facts as Related to Learning Method. Journal of Educational Research 38: 81-85; October 1944.

Thaeler, John Schropp. The Relationship of Piagetian Number Conservation and Concepts to Levels of Processing of the Basic Addition Facts. (The Florida State University, 1981.) Dissertation Abstracts International 42A: 2548-2549; December 1981.

Thornton, Carol A. Emphasizing Thinking Strategies in Basic Fact Instruction. Journal for Research in Mathematics Education 9: 214-227; May 1978.

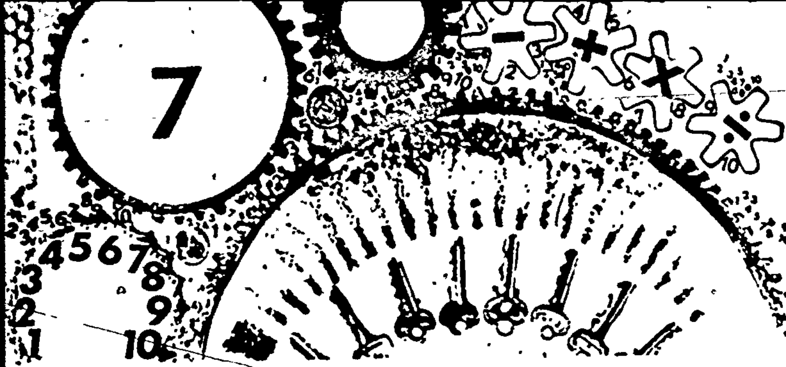
Trafton, Paul Ross. The Effects of Two Initial Instructional Sequences on the Learning of the Subtraction Algorithm in Grade Three. (The University of Michigan, 1970.) Dissertation Abstracts International 31A: 4049-4050; February 1971.

Weaver, J. Fred. Some Factors Associated with Pupil's Performance Levels on Simple Open Addition and Subtraction Sentences. Arithmetic Teacher 18: 513-519; November 1971.

Wheeler, Larry Eugene. The Relationship of Multiple Embodiments of the Regrouping Concept to Children's Performance in Solving Multi-Digit Addition and Subtraction Examples. (Indiana University, 1971.) Dissertation Abstracts International 32A: 4260; February 1972.

Wiles, Clyde A.; Romberg, Thomas A.; and Moser, James M. The Relative Effectiveness of Two Different Instructional Sequences Designed to Teach the Addition and Subtraction Algorithms. Journal for Research in Mathematics Education 4: 251-262; November 1973.

Wilson, Gary M. New Standards in Arithmetic: A Controlled Experiment in Supervision. Journal of Educational Research 22: 356-360; December 1930.



MULTIPLICATION AND DIVISION WITH WHOLE NUMBERS

How should children be encouraged to learn the basic multiplication facts?

At an appropriate time in the learning sequence, it is desirable that children strive to achieve immediate recall of basic multiplication facts. About 60% of the 9-year-olds and over 90% of the 13-year-olds knew basic multiplication facts on the 1977-78 National Assessment (Carpenter et al., 1981).

Findings from a comprehensive investigation with children in grades 3 to 5 by Brownell and Carper (1943) suggested that activities and experiences which contribute to pupils' understanding of the mathematical nature of multiplication should <u>precede</u> work which focuses on memorization of facts.

Teachers know that the number of specific basic facts to be memorized is reduced substantially if pupils are able to apply relevant properties of multiplication. Thus, learning that 4 x 3 = 12 should not be distinct from learning that 3 x 4 = 12; knowing that $\square$ x 0 = 0 and $\square$ x 1 = $\square$ makes it unnecessary to learn specific instances of those properties.

Ascertaining the relative difficulty of the multiplication facts was once a matter of great interest. Little commonality of levels of difficulty was evident among the studies except on two points: (1) combinations involving 0 presented difficulty; (2) the size of the product was positively related to difficulty.

In his analysis of data on difficulty levels, Jerman (1970) reported that students appeared to use different strategies for different multiplication combinations and that the strategy used may be a function of the combination itself. Strategies used in grade 3 appeared to be the ones used for the same combinations in grade 6 in 72% of the cases. Thornton (1978) affirmed the usefulness of learning thinking strategies for the basic facts, and Rathmell (1978) proposed activities for teaching such thinking strategies for multiplication as skip counting, repeated addition, splitting the product into known parts, facts of 5, and patterns.

Investigations pertaining to the relative difficulty of open multiplication and division sentences have been conducted by Grouws and Good (1976) and by McMaster (1976). Findings parallel those from analogous investigations summarized earlier for open addition and subtraction sentences: the form of the sentence, the size of the numbers, and the position of the placeholder. Not surprisingly, division sentences were more difficult than multiplication sentences.

How should multiplication be conceptualized for children?

Multiplication of whole numbers is often conceptualized for children in terms of combining equal-sized groups and the addition of equal addends. For instance, "4 x 7" has been interpreted as "4 groups of 7" and "7 + 7 + 7 + 7". Some research has investigated the feasibility of using other conceptualizations of multiplication. At the third-grade level, Schell (1964) compared the achievement of pupils who used rectangular array representations exclusively for their introductory work with multiplication with that of pupils who used a variety of representations. He found no conclusive evidence of a difference in achievement levels.

A comparison of repeated addition and an approach using ordered pairs of numbers was conducted by Tietz (1969). They, too, appeared to be equally effective. Hervey (1966) compared equal addends and Cartesian-product approaches. Second-grade pupils found equal-addends problems easier to conceptualize and solve than Cartesian-product problems.

Is attention to distributivity helpful in early work with multiplication?

We know $3 \times (4 + 7) = (3 \times 4) + (3 \times 7)$. This is an instance of the distributive property of multiplication over addition which (in one form or another) is used to some extent in many programs of mathematics instruction. Specific instances of this property often are illustrated with arrays.

From a study with third-grade pupils and their beginning work with multiplication, Gray (1965) found that an emphasis upon distributivity led to better transfer and retention than an approach that did not include work with this property. The superiority was statistically significant on three of four measures: a posttest of transfer ability, a retention test of multiplication achievement, and a retention test of transfer. On the remaining measure -- a posttest of multiplication achievement -- children who had worked with distributivity scored higher than those who had not, but the difference was not statistically significant.

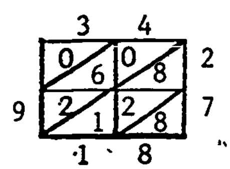
Gray's findings added support to the evidence on advantages to be expected from instruction which emphasizes mathematical meaning and understanding. The "pay-off" may not always be particularly evident in terms of the achievement immediately following instruction. Rather, the pay-off is much more clearly evident in relation to factors such as comprehension, transfer, and retention. Thus, Johnston (1978) found a significant correlation between distributivity scores and success on later work with two- and three-digit multiplication.

However, in a recent survey in grades 4 through 7, Weaver (1973) found that pupils could not use distributivity in solving examples varied in context, form, format, and number. Hobbs (1976) found a similar lack in his investigation, which was based on structured interviews with individual pupils in grade 5. This suggests that more emphasis must be placed on this property if we are to expect a "pay-off."

What multiplication algorithms have been found to be effective?

Hazekamp (1977) found that when multiplication was taught with an emphasis on grouping and base ideas, greater achievement resulted for fourth graders than when the emphasis was on place-value representations.

On the basis of multiple criteria, Schrankler (1967) also evaluated the relative effectiveness of two algorithms for teaching multiplication to fourth-grade pupils. He concluded that algorithms using general ideas based on the structure of the number system were more successful than other algorthims investigated in achieving the objectives of increased computational skills, understanding of processes, and problem-solving abilities associated with the multiplication of whole numbers between 9 and 100.



$$\begin{array}{r} 34 \\ \times\ 27 \\ \hline \end{array}$$

Hughes and Burns (1975) investigated the teaching of multidigit multiplication to fourth graders using the lattice method and the distributive method. The groups using the lattice method were able to compute multidigit multiplication exercises in significantly less time and more accurately than groups using the distributive method. (Whether or not the time to draw the lattices was included in the test time is unspecified.) No significant differences on tests of understanding of multiplication were found.

Other forms of a multiplication algorithm, involving less memorization and more paper-and-pencil recording, are reported effective with low achievers.

Which is it better to teach: the subtractive or the distributive form of the division algorithm?

Two algorithms for division are used in many elementary school mathematics programs. One is often called the distributive algorithm:

```
        2
  23) 552          First think
      46           '2's in 5?'
      92
      etc.
```

The other is a multiplicative and subtractive approach to the division algorithm:

```
        2
  23) 552
      230    10 x 23
      322
      230    10 x 23
       92
      etc.
```

In one investigation comparing use of the distributive and subtractive algorithms, Van Engen and Gibb (1956) reported that there were some advantages for each. They evaluated pupil achievement in terms of understanding the process of division, transfer of learning, retention, and problem-solving achievement. Among their conclusions were:

(1) Low-ability children taught the subtractive algorithm had a better understanding of the process or idea of division than low-ability children taught the distributive algorithm.

(2) Children taught the distributive algorithm achieved higher problem-solving scores.

(3) Use of the subtractive algorithm was more effective in enabling children to transfer to unfamiliar but similar situations.

(4) The two algorithms appeared to be equally effective on measures of retention of skill and understanding. This seems to be more related to teaching procedures, regardless of the algorithm used.

Kratzer and Willoughby (1973) prepared two instructional units, both involving meaningful instruction. One used the distributive algorithm and the other used the subtractive algorithm, each as a method of keeping records while manipulating bundles of sticks. No significant differences between the algorithms were found on fourth graders' achievement of familiar problems on immediate or retention tests. There was, however, a significant difference between the algorithms on achievement of unfamiliar problems on both types of test: those using the distributive approach displayed a better understanding of the process. Jones (1976) did not support this latter conclusion, but he did find that those using the distributive algorithm had higher computation scores.

Dilley (1970) also compared the teaching of division of grade 4 using the distributive algorithm and the subtractive algorithm. Significant differences were found on an applications test favoring use of the subtractive algorithm, and on a retention test favoring the distributive algorithm.

In another study with fourth graders, Rousseau (1972) studied the effect of four algorithms based on varied foundations: (1) mathematical, based on the distributive property of division over addition; (2) real-world, based on the physical act of "quotitioning"; (3) real-world, based on the physical act of partitioning; and (4) rote, based on the memorization of routines. No significant differences in retention of algorithms were found. For extensions to cases of slightly greater difficulty, the rote algorithm was superior. For problems of greater difficulty, however, the quotitive and distributive algorithms were better than the rote and partitive algorithms.

Thus, there appear to be some advantages for each algorithm. The needs of individual pupils must be considered in deciding which algorithm to use.

What is the most effective method of teaching pupils to estimate quotient digits?

Inefficient algorithms need to be shortened to gain proficiency in division. Then pupils must be able to estimate quotient digits systematically. Several methods have been advocated: (1) the "apparent" or "round-down" method, in which the divisor is rounded to the next lower multiple of 10; and (2) the "increase-by-one" methods, in which the divisor is rounded to the next higher multiple of 10, (a) either "round-both-ways," depending on whether the digit in units' place is less or greater than 5, or (b) "round-up," no matter what. Which method do you use?

	apparent or round-down	increase-by-one: round-up	increase-by-one: round-both ways
$42\overline{)216}$	$4\overline{)21}$	$5\overline{)21}$	$4\overline{)21}$
$47\overline{)216}$	$\begin{array}{r}5\\4\overline{)21}\end{array}$	$\begin{array}{r}4\\5\overline{)21}\end{array}$	$\begin{array}{r}4\\5\overline{)21}\end{array}$

To resolve the issue of which method is best, researchers have focused on analysis and comparison of the success of each method on a specified population of division examples. Hartung (1957) critically reviewed such analytic studies. He concluded that "round-up" was the most useful method, because of the advantages of obtaining an estimate that is less than the true quotient (which decreased the need for erasing), and because of the relative simplicity of a "one-rule" method.

In one of the few experimental investigations on this topic, Grossnickle (1937) studied the achievement of groups taught by "round-down" and "round-both-ways." He concluded that there were no significant differences between the scores of the two groups.

How children apply the method was studied by Flournoy (1959), who found that "round-both-ways" was used as effectively as the "round-down" method. She stressed that perhaps not all children should be taught the "round-both-ways" method. Carter (1960) reported that pupils taught this method were not as accurate as those taught a one-rule method — nor did pupils always use the method taught.

What is the difficulty level of division?

Little research has been done on the difficulty level of the basic division facts; however, we do know that by age 13, 81% responded correctly on the National Assessment (Carpenter et al., 1981). Great attention was given in the 1940s to the difficulties inherent in the algorithm and much of it still seems relevant. Osburn (1946) noted 41 levels of difficulty for division examples with two-digit divisors and one-digit quotients. Pupils' ability to divide with two-figure divisors has been found to involve a considerable variety of skills varying widely in difficulty (Brownell, 1953; Brueckner and Melbye, 1940). Examples in which the apparent quotient is the true quotient (as in $43\overline{)92}$) are (of course) much easier than those requiring correcting (such as $43\overline{)81}$), with difficulty increasing as the number of digits in the quotient increases.

What is the role of measurement and partition situations in teaching division?

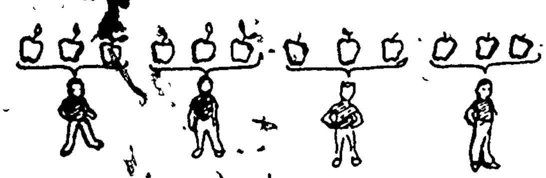
Measurement problems involve situations such as:

> If each boy is to receive 3 apples, how many boys can share 12 apples? (Find the number of equivalent subsets.)

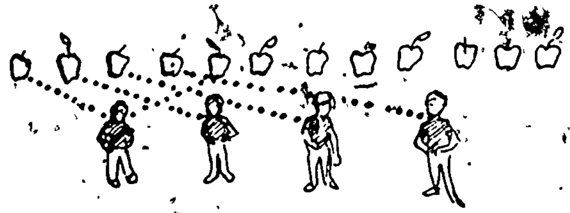
Partition problems involve situations like this:

> If there are 4 boys to share 12 apples equally, how many will each boy receive? (Find the number of elements in each equivalent subset.)

In a study with second graders (chosen since children at this level have usually had little experience with division which would interact with the teaching in the research study), Gunderson (1953) reported that problems involving partition situations were more difficult than problems involving measurement situations. The ease of visualizing the measurement situation probably contributes to this. For instance, for the illustration above, a picture like this could be formed:



For the partition situation, the drawing might be:



and so on!

Zweng (1964) also found that partition problems were significantly more difficult for second graders than measurement problems. She further reported that problems in which two sets of tangible objects were specified were easier than those in which only one set of tangible objects was specified.

In the study in which they compared two division algorithms, Van Engen and Gibb (1956) found that children who used the distributive algorithm had greater success with partition situations, while those who used the subtractive algorithm had greater success with measurement situations.

Taking one step more, Scott (1963) used the subtractive algorithm for measurement situations and the distributive algorithm for partition situations. He suggested that: (1) use of the two algorithms was not too difficult for third-grade children; (2) two algorithms demanded no more teaching time than only one algorithm; and (3) children taught both algorithms had a greater understanding of division.

Bechtel and Weaver (1975) used structured interviews with second-grade children to ascertain ways in which they manipulated objects to solve measurement and partitive problem situations prior to formal instruction on division. The findings confirmed that these situations are conceptually different for young children and suggested that systematic instruction should be designed accordingly.

They also found that problem situations with non-zero remainders were no more difficult for children to cope with than were problem situations with zero remainders, suggesting that no sharp dichotomy should be made between such instances when providing pre-division experiences in an instructional program.

List of Selected References

Bechtel, Robert D. and Weaver, J. Fred. A Preliminary Investigation of the Ability of Second-Grade Children to Solve Division Problems Presented in a Physical-Object Mode and in Different Partitive-Measurement Sequences. Madison: Wisconsin Research and Development Center for Cognitive Learning, 1976. ERIC: ED 179 337.

Brownell, William A. The Effects of Practicing a Complex Arithmetical Skill upon Proficiency in Its Constituent Skills. Journal of Educational Psychology 44: 64-81; February 1953.

Brownell, William and Carper, Doris V. Learning the Multiplication Combinations. Durham, North Carolina: Duke University Studies in Education, Duke University Press, No. 7, 1943.

Brueckner, Leo J. and Melbye, Harvey O. Relative Difficulty of Types of Examples in Division with Two-Figure Divisors. Journal of Educational Research 33: 401-414; February 1940.

Carpenter, Thomas P.; Corbitt, Mary Kay; Kepner, Henry S., Jr.; Lindquist, Mary Montgomery; and Reys, Robert E. Results from the Second Mathematics Assessment of the National Assessment of Educational Progress. Reston, Virginia: National Council of Teachers of Mathematics, 1981.

Carter, Mary Katherine. A Comparative Study of Two Methods of Estimating Quotients When Learning to Divide by Two-Figure Divisors. (Boston University, 1959.) Dissertation Abstracts 20: 3317; February 1960.

Dilley, Clyde Allan. A Comparison of Two Methods of Teaching Long Division. (University of Illinois at Urbana-Champaign, 1970.) Dissertation Abstracts International 31A: 2248; November 1970.

Flournoy, Frances. Children's Success with Two Methods of Estimating the Quotient Figure. Arithmetic Teacher 6: 100-104; March 1959.

Gray, Roland F. An Experiment in the Teaching of Introductory Multiplication. Arithmetic Teacher 12: 199-203; March 1965.

Grossnickle, Foster E. An Experiment with Two Methods of Estimation of the Quotient. Elementary School Journal 37: 668-677; May 1937.

Grouws, Douglas A. and Good, Thomas L. Factors Associated with Third- and Fourth-Grade Children's Performance in Solving Multiplication and Division Sentences. Journal for Research in Mathematics Education 7: 155-171; May 1976.

Gunderson, Agnes G. Thought-Patterns of Young Children in Learning Multiplication and Division. Elementary School Journal 55: 453-461; April 1953.

Hartung, Maurice L. Estimating the Quotient in Division (A Critical Analysis of Research). Arithmetic Teacher 4: 100-111; April 1957.

Hazekamp, Donald Wayne. The Effects of Two Initial Instructional Sequences on the Learning of the Conventional Two-Digit Multiplication Algorithm in Fourth Grade. (Indiana University, 1976.) Dissertation Abstracts International 37A: 4933; February 1977.

Hervey, Margaret A. Children's Responses to Two Types of Multiplication Problems. Arithmetic Teacher 13: 288-292; April 1966.

Hobbs, Charles Eugene. An Analysis of Fifth-Grade Students' Performance When Solving Selected Open Distributive Sentences. (The University of Wisconsin-Madison, 1975.) Dissertation Abstracts International 36A: 5770; March 1976.

Hughes, Frank G. and Burns, Paul C. Two Methods of Teaching Multidigit Multiplication. Elementary School Journal 75: 452-457; April 1975.

Jerman, Max. Some Strategies for Solving Simple Multiplication Combinations. Journal for Research in Mathematics Education 1: 95-128; March 1970.

Johnston, William Francis. Interrelationships of Fourth-Grade Teachers' and Pupils' Comprehension of the Distributive Property of Multiplication Over Addition in the Set of Whole Numbers. (The University of Wisconsin-Madison, 1977.) Dissertation Abstracts International 38A: 5973; April 1978.

Jones, Wilmer Lee. An Experimental Comparison of the Effects of the Subtractive Algorithm Versus the Distributive Algorithm on Computation and Understanding of Division. (University of Maryland, 1976.) Dissertation Abstracts International 37A: 3395-3396; December 1976.

Kratzer, Richard O. and Willoughby, Stephen S. A Comparison of Initially Teaching Division Employing the Distributive and Greenwood Algorithms with the Aid of a Manipulative Material. Journal for Research in Mathematics Education 4: 197-204; November 1973.

McMaster, Mary Jane. Differential Performance of Fourth- Through Sixth-Grade Students in Solving Open Multiplication and Division Sentences. (The University of Wisconsin-Madison, 1975.) Dissertation Abstracts International 36A: 7853; June 1976. See also ERIC: ED 116 967, ED 116 968.

Osburn, Worth J. Levels of Difficulty in Long Division. Elementary School Journal 46: 441-447; April 1946.

Rathmell, Edward C. Using Thinking Strategies to Teach the Basic Facts. In Developing Computational Skills (Marilyn N. Suydam, Ed.). 1978 Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Rousseau, Leon Antonio. The Relationship Between Selected Mathematical Concepts and Retention and Transfer Skills with Respect to Long Division Algorithms. (Washington State University, 1972.) Dissertation Abstracts International 32A: 6750; June 1972.

Schell, Leo Mac. Two Aspects of Introductory Multiplication: The Array and the Distributive Property. (State University of Iowa, 1964.) Dissertation Abstracts 25: 1561-1562; September 1964.

Schrankler, William Jean. A Study of the Effectiveness of Four Methods for Teaching Multiplication of Whole Numbers in Grade Four. (University of Minnesota, 1966.) Dissertation Abstracts 27A: 4055; June 1967.

Scott, Lloyd. A Study of Teaching Division Through the Use of Two Algorithms. School Science and Mathematics 63: 739-752; December 1963.

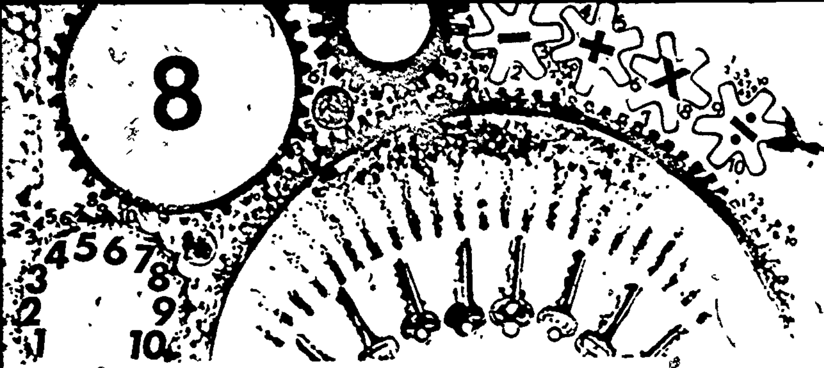
Thornton, Carol A. Emphasizing Thinking Strategies in Basic Fact Instruction. Journal for Research in Mathematics Education 9: 214-227; May 1978.

Tietz, Naunda Meier. A Comparison of Two Methods of Teaching Multiplication: Repeated-Addition and Ratio-to-One. (Oklahoma State University, 1968.) Dissertation Abstracts International 30A: 1060; September 1969.

Van Engen, Henry and Gibb, E. Glenadine. General Mental Functions Associated with Division. Educational Service Studies, No. 2. Cedar Falls, Iowa: Iowa State Teachers College, 1956.

Weaver, J. Fred. Pupil Performance on Examples Involving Selected Variations of the Distributive Idea. Arithmetic Teacher 20: 697-704; December 1973.

Zweng, Marilyn J. Division Problems and the Concept of Rate. Arithmetic Teacher 11: 547-556; December 1964.



RATIONAL NUMBERS: FRACTIONS AND DECIMALS

Since several interpretations of the above words are possible, let's clarify how we're using them. We shall use the word fraction to refer to a number: a number that may be expressed in the form $\frac{a}{b}$, where $\underline{a}$ and $\underline{b}$ are whole numbers and $b \neq 0$. The word decimal will be used to refer to a particular kind of fraction: one that is expressed in our familiar positional place-value notation, with the implicit denominator being some power of 10.

Assessment data indicate that children do not achieve as well with fractions and decimals as they do with whole numbers. Thus, on the 1977-78 National Assessment (Carpenter et al., 1981):

- About one-third of the 13-year-olds could add 3/4 and 1/2.
- About one-half could add 3.57 and 1.2.
- Many students apparently did not understand the <u>concepts</u> involved in work with fractions and decimals.

Can young children learn fractional concepts?

We have found from surveys of what children know about mathematics upon entering school that at least 50% can recognize halves, fourths, and thirds, and have acquired some facility in using these fractions. Campbell (1975) surveyed five-, six-, and seven-year-olds on their understanding of the fractions one-half, one-third, and one-fourth, prior to formal instruction. The children consistently showed a higher level of understanding of "fraction of a whole" than of "fraction of a set" or "division" interpretations. More evidence of understanding was shown when concrete materials were used rather than semi-concrete representations. Gunderson and Gunderson (1957) interviewed 22 second graders following their initial experience with a lesson on fractional parts of circles. The investigators concluded that fractions could be introduced at this grade level, with the use of manipulative materials and through oral work with no symbols used.

Sension (1971) reported that area, set-subset, and combination representations for introducing rational number concepts appeared to be equally effective on tests containing items consistent with the experimental instruction. However, the combination treatment produced a higher level of generalization to a number-line model.

Kieren (1976) has intensively studied the various interpretations of fractions: both the number of interpretations and our lack of understanding of them have resulted in difficulties for students. As the interpretations are clarified, instructional approaches will presumably become more evident.

A planned, systematic program for developing fractional ideas seems essential as readiness for work with symbols. Furthermore, the use of manipulative materials appears vital in this preparation -- and is still imperative in work on operations with fractions with older children.

What sequence should be used in teaching fractions?

Investigators have focused on this question in varying ways. Novillis (1976), for instance, developed a hierarchy with 23 steps. Eighteen of these were found to be appropriate; that is, they depended on previously learned ideas. For example, associating fractions with part-whole and part-group models were prerequisite to associating a fraction with a point on a number line. Another hierarchy was formulated by Uprichard and Phillips (1977).

Bohan (1971) tried three approaches to teaching skills and concepts related to equivalent fractions in grade five. He found that approaches in which equivalent fractions were introduced with the aid of diagrams and sets of objects, followed by addition and then multiplication, resulted in higher achievement than an approach in which multiplication with fractions was taught first, then applied to equivalent fractions, followed by addition.

Four approaches to teaching comparison of fractions were investigated by Choate (1975): (1) pupils were taught a rule, without conceptual development; (2) the rule was developed meaningfully; (3) conceptual work for comparing fractions using diagrams preceded presentation of the rule; and (4) only the conceptual work was included, with no algorithmic work. He concluded, "The crucial consideration is the time of presentation of the algorithm in relation to the conceptual development." He suggested that the third approach, with conceptual work preceding presentation of the rule, would provide the strongest base.

In another study, Ellerbruch (1976) found that the rule-first approach appeared to be better on skill items, but the model-first conceptual approach was better for understanding. A suggested sequence for teaching fraction ideas was also developed (Ellerbruch and Payne, 1978).

What procedures are effective in work with addition and subtraction with fractions?

There is little evidence on the effectiveness of procedures for finding the common denominator in addition with fractions, and even less for subtraction with fractions. Anderson (1966) analyzed errors made by 26 fifth-grade classes using two procedures for finding the least common denominator when adding two "unlike" fractions: by setting up rows of equivalent fractions, and by factoring the denominators. There were no significant differences between the two procedures on tests of four kinds of addition with fractions examples. Furthermore, Anderson reported that errors connected with (1) "reducing," (2) determining the numerator, and (3) addition occurred most frequently, with the greatest frequency of error in examples in which the least common denominator was not apparent.

Fifth graders who were taught either the factoring method or the "inspection" method used in a textbook series were compared by Bat-haee (1969). Those taught by the factoring method scored significantly higher on the experimental posttests.

Howard (1950) reported on a study with pupils in grades 5 and 6 who were taught addition of fractions by three methods differing in the amount of emphasis on meaning, use of materials, and practice. Pupils retained better when they learned fractional work through extensive use of materials and with considerable emphasis on meaning, plus provision for practice. Other researchers also strongly support the importance of using meaningful methods and materials for work with fractions. It may not be necessary for children to handle materials, noted Bisio (1971). He conducted a study on addition and subtraction of like fractions with fifth graders, and found that having pupils watch the teacher use manipulative materials was as effective as using materials themselves and better than non-use of materials.

Carney (1973) taught four classes of fourth graders to add and subtract fractions, using 30 lessons based on field postulates and other properties, and taught four other classes 30 lessons based on objects and the number line. The approach using the field postulates and other properties was more effective than the object-and-number-line approach.

In another study with fourth graders, Coburn (1974) reported that, while achievement on some concepts related to equivalent fractions was comparable for the two groups, students using the region approach achieved significantly better on adding and subtracting unlike fractions and on some retention and attitude measures.

How can we most effectively develop the algorithm for multiplication with fractions?

Surprisingly little research has been done on this question. It may be that it seems to present less difficulty to teachers: frequently, such topics receive less attention from researchers.

The use of two different approaches for teaching multiplication with fractions in grade 5 was investigated by Green (1970). An approach based on the area of a rectangular region was more effective than one based on finding a fractional part of a region or set. Each approach was studied in relation to two different modes of representation (diagrams and cardboard strips); the "area" approach taught with diagrams was most successful, the "fractional part" approach taught with cardboard strips was second, and the "fractional part" approach taught with diagrams was poorest.

Much of the other research on multiplication with fractions has involved the use of programmed instruction: the purpose of the investigation was to compare various programming strategies, while fractions served merely as the content vehicle (e.g., Kyte and Fornwalt, 1967; Miller, 1964).

DiVincenzo (1979) reported that teaching all four operations with fractions simultaneously (in sixth grade) was more effective than teaching them separately. No one else has explored this point, however.

What algorithm shall we use for division with fractions?

Much more attention has been given to this question -- one which teachers frequently ask.

Bergen (1966) prepared booklets designed to teach pupils by complex-fraction, common denominator, or inversion algorithms, but each was significantly superior to the common denominator algorithm on most types of examples, a finding supported by Smith (1979). Bergen concluded that division of fractions should be introduced by the complex-fraction method, with the inversion method saved as a useful shortcut to facilitate rapid calculations.

common denominator

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4} = \frac{3 \div 2}{4 \div 4} = \frac{3 \div 2}{1} = 3 \div 2 = \frac{3}{2}$$

inversion

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{6}{4} = \frac{3}{2}$$

Teaching the common denominator and inversion algorithms with and without explanation of the reciprocal principle as the rationale behind inversion was compared by Sluser (1963). The group given the explanation scored lower on tests of division with fractions than a group merely taught to invert and multiply. He suggested that only above-average pupils could understand the principle. However, a large percentage of errors occurred because pupils performed the wrong operation. Krich (1964) reported no significant differences on immediate posttests for pupils taught why the inversion procedure works, as compared with those merely taught the rule. On retention tests requiring recall, however, the group taught with meaning scored significantly higher.

complex fraction

$$\frac{3}{4} \div \frac{1}{2} = \frac{\frac{3}{4}}{\frac{1}{2}} = \frac{\frac{3}{4} \times \frac{2}{1}}{\frac{1}{2} \times \frac{2}{1}} = \frac{\frac{6}{4}}{1} =$$

$$\frac{6}{4} = \frac{3}{2}$$

In a study by Capps (1963), the effectiveness of the common denominator and inversion algorithms was also compared. There were no significant differences in achievement on tests of addition, subtraction, and division with fractions. However, pupils taught the inversion algorithm scored significantly higher on immediate posttests and on retention tests of multiplication with fractions than did those taught the common denominator algorithm. This retroactive effect on multiplication was also reported by Bidwell (1968). He found that the inverse operation procedure was most effective, followed by complex fraction and common denominator procedures. The complex fraction procedure was better for retention, while the common denominator procedure was poorest.

In a further analysis of the three algorithms; Bidwell (1971) noted that the common-denominator approach did not have inherent advance organizers to help learners move from their current level of understanding. Thus, it was not as readily learned as the other two methods which had such advance organizers.

Is it helpful to analyze errors pupils make with fractions?

Many early studies were concerned primarily with the specific errors children make. In general, it was found that, for all operations with fractions, the major errors were caused by (1) difficulty with "reducing," (2) lack of comprehension of the operation involved, and (3) computational errors. Such findings frequently influenced the material included in textbooks.

Another survey on errors is characterized by the details it provides on how students responded. Lankford (1972) reported the incorrect solutions given by seventh graders who were interviewed as they attempted various examples with fractions. This information can be very helpful to teachers in deciding what to stress as operations with fractions are taught.

From other interviews with pupils in grade 6, Peck and Jencks (1981) expressed concern. They found that the children lacked conceptual understanding of fractions. They appeared to "sift through rules" that seemed almost meaningless to them, to find one that might work. Perusal of the student responses by Lankford strengthens this conclusion.

Ways of helping children recognize (and then correct errors) was the focus of some research. Thus, Aftreth (1958) had sixth-grade pupils identify and correct errors imbedded in 19 completed sets of examples in addition and subtraction with fractions, while a control group worked the examples. No significant differences on either immediate or delayed recall tests were found for addition with fractions, while some significant differences favoring the group working the examples were found for subtraction with fractions. The author suggested that having pupils correct their own errors might be more effective than having them correct imbedded errors.

Romberg (1968) reported that among sixth graders who used a correct algorithm to multiply fractions, many pupils either did not express products in simplest form (as directed) or made errors in doing so. He attributed this difference to pupils' failure to "cancel," and suggested that the cancellation process is important -- even essential -- if efficiency in multiplication with fractions is one of the desired outcomes of instruction.

Is it helpful to relate decimals with fractions or place value?

Because of increasing use of calculators, attention has been directed to the sequencing of fractions and decimals. Can decimals be taught before fractions, or is the fraction-then-decimal sequence necessary? Research has provided some indication of an answer to this question. However, it remains to be studied more carefully with the integral use of calculators.

One indication is found in the study by Faires (1963). He introduced some pupils to decimals through a sequence based on an orderly extension of place value, with no reference to common fraction equivalents, while others were taught fractions before decimals, as is usually done. Gains in computational achievement and at least as good an understanding of fraction concepts resulted. Faires indicated that "computation with decimals is [apparently] more nearly like computation with whole numbers than fractions"; thus reinforcement of whole number computational skills is provided.

O'Brien (1968) reported that pupils taught decimals with an emphasis on the principles of numeration, with no mention of fractions, scored lower on tests of computation with decimals than those taught either (a) the relation between decimals and fractions, with secondary emphasis on principles of numeration, or (b) rules, with no mention of fractions or principles of numeration. On later retention measures, the numeration approach was significantly lower than use of the rules approach, but not significantly different from the fraction-numeration approach.

With fifth graders, Willson (1972) compared the fraction-then-decimal sequence with the decimal-then-fraction sequence. No significant differences in achievement were found, although greater raw-score gains were made by those having the decimal-then-fraction sequence. Thus, it would appear at least plausible to consider teaching decimals before fractions.

How should we teach children to place the decimal point in division with decimals?

Only early studies have considered this question. Brueckner (1928) and Grossnickle (1941) analyzed the difficulties with decimals which children have, citing misplacing of the decimal point in division as one of the major sources of error. Flournoy (1959) compared sixth-grade classes taught to locate the decimal point in the quotient by (1) making the divisor a whole number by multiplying the dividend by the same number, or (2) subtracting the number of decimal places in the divisor from the number of places in the dividend. Multiplying by a power of 10 resulted in greater accuracy, as Grossnickle had concluded earlier.

List of Selected References

Aftreth, Orville B. The Effect of the Systematic Analysis of Errors in the Study of Fractions at the Sixth Grade Level. Journal of Educational Research 52: 31-34; September 1958.

Anderson, Rosemary C. A Comparison of Two Procedures for Finding the Least Common Denominator in the Addition of Unlike, Unrelated Fractions. (University of Iowa, 1965.) Dissertation Abstracts 26: 5901; April 1966.

Bat-haee, Mohammad Ali. A Comparison of Two Methods of Finding the Least Common Denominator of Unlike Fractions at Fifth-Grade Level in Relation to Sex, Arithmetic Achievement, and Intelligence. (Southern Illinois University, 1968.) Dissertation Abstracts 29A: 4365; June 1969.

Bergen, Patricia M. Action Research on Division of Fractions. Arithmetic Teacher 13: 293-295; April 1966.

Bidwell, James King. A Comparative Study of the Learning Structures of Three Algorithms for the Division of Fractional Numbers. (University of Michigan, 1968.) Dissertation Abstracts 29A: 830; September 1968.

Bidwell, James K. Some Consequences of Learning Theory Applied to Division of Fractions. School Science and Mathematics 71: 426-434; May 1971.

Bisio, Robert Mario. Effect of Manipulative Materials on Understanding Operations with Fractions in Grade V. (University of California, Berkeley, 1970.) Dissertation Abstracts International 32A: 833; August 1971.

Bohan, Harry Joseph. A Study of the Effectiveness of Three Learning Sequences for Equivalent Fractions. (The University of Michigan, 1970.) Dissertation Abstracts International 31A: 6270; June 1971.

Brueckner, Leo J. Analysis of Difficulties in Decimals. Elementary School Journal 29: 32-41; September 1928.

Campbell, Bobby Gene. The Child's Understanding of Three Interpretations of Certain Unit Fractions Prior to Formal Instruction. (The University of Oklahoma, 1974.) Dissertation Abstracts International 35A: 4855; February 1975.

Capps, Lelon R. A Comparison of the Common Denominator and Inversion Method of Teaching Division of Fractions. Journal of Educational Research 56: 516-522; July/August 1963.

Carney, Harold Francis. The Relative Effectiveness of Two Methods of Teaching the Addition and Subtraction of Rational Numbers. (New York University, 1973.) Dissertation Abstracts International 34A: 659-660; August 1973.

Carpenter, Thomas P.; Corbitt, Mary Kay; Kepner, Henry S., Jr.; Lindquist, Mary Montgomery; and Reys, Robert E. Results from the Second Mathematics Assessment of the National Assessment of Educational Progress. Reston, Virginia: National Council of Teachers of Mathematics, 1981.

Choate, Stuart Alan. The Effect of Algorithmic and Conceptual Development for the Comparison of Fractions. (The University of Michigan, 1975.) Dissertation Abstracts International 36A: 1410; September 1975.

Coburn, Terrence Gordon. The Effect of a Ratio Approach and a Region Approach on Equivalent Fractions and Addition/Subtraction for Pupils in Grade Four. (The University of Michigan, 1973.) Dissertation Abstracts International 34A: 4688; February 1974.

DiVincenzo, Robert M. Clustering-Operations: A More Efficient Alternative for Teaching Common Fraction Operations. School Science and Mathematics 79: 328-332; April 1979.

Ellerbruch, Lawrence Walter. The Effects of the Placement of Rule and Concrete Models in Learning Addition and Subtraction of Fractions in Grade Four. (The University of Michigan, 1975.) Dissertation Abstracts International 36A: 6441-6442; April 1976.

Ellerbruch, Lawrence W. and Payne, Joseph N. A Teaching Sequence from Initial Fraction Concepts through the Addition of Unlike Fractions. In Developing Computational Skills (Marilyn N. Suydam, Ed.). 1978 Yearbook. Reston, Virginia: National Council of Teachers of Mathematics, 1978.

Faires, Dano Miller. Computation with Decimal Fractions in the Sequence of Number Development. (Wayne State University, 1962.) Dissertation Abstracts. 23: 4183; May 1963.

Flournoy, Francis. A Consideration of Pupils' Success with Two Methods for Placing the Decimal Point in the Quotient. School Science and Mathematics 59: 445-455; June 1959.

Green, Geraldine Ann. A Comparison of Two Approaches, Area and Finding a Part of, and Two Instructional Materials, Diagrams and Manipulative Aids, on Multiplication of Fractional Numbers in Grade Five. (University of Michigan, 1969.) Dissertation Abstracts International 31A: 676-677; August 1970.

Grossnickle, Foster E. Types of Errors in Division of Decimals. Elementary School Journal 42: 184-194; November 1941.

Gunderson, Ethel and Gunderson, Agnes C. Fraction Concepts Held by Young Children. Arithmetic Teacher 4: 168-173; October 1957.

Howard, Charles F. Three Methods of Teaching Arithmetic. California Journal of Educational Research 1: 25-29; January 1950.

Kieren, Thomas E. On the Mathematical, Cognitive, and Instructional Foundations of Rational Numbers. In Number and Measurement (Richard A. Lesh and David A. Bradbard, Eds.). Columbus, Ohio: ERIC/SMEAC, 1976. ERIC: ED 120 027.

Krich, Percy. Meaningful vs. Mechanical Method, Teaching Division of Fractions by Fractions. School Science and Mathematics 64: 697-708; November 1964.

Kyte, George C. and Fornwalt, James E. A Comparison of Superior Children with Normal Children in the Rate Mastery of the Multiplication of Fractions. Journal of Educational Research 60: 346-350; April 1967.

Lankford, Francis G., Jr. Some Computational Strategies of Seventh Grade Pupils. Charlottesville, Virginia: University of Virginia, October 1972. ERIC: ED 069 496.

Miller, Jack W. An Experimental Comparison of Two Approaches to Teaching Multiplication of Fractions. Journal of Educational Research 57: 468-471; May/June 1964.

Novillis, Carol F. An Analysis of the Fraction Concept into a Hierarchy of Selected Subconcepts and the Testing of the Hierarchial Dependencies. Journal for Research in Mathematics Education 7: 131-144; May 1976.

O'Brien, Thomas C. An Experimental Investigation of a New Approach to the Teaching of Decimals. (New York University, 1967.) Dissertation Abstracts 28A: 4541-4542; May 1968.

Peck, Donald M. and Jencks, Stanley M. Conceptual Issues in the Teaching and Learning of Fractions. Journal for Research in Mathematics Education 12: 339-348; November 1981.

Romberg, Thomas A. A Note on Multiplying Fractions. Arithmetic Teacher 15: 263-265; March 1968.

Sension, Donald Bruce. A Comparison of Two Conceptual Frameworks for Teaching the Basic Concepts of Rational Numbers. (University of Minnesota, 1971.) Dissertation Abstracts International 32A: 2408; November 1971.

Sluser, Theodore F. A Comparative Study of Division of Fractions in Which an Explanation of the Reciprocal Principle is the Experimental Factor. (University of Pittsburgh, 1962.) Dissertation Abstracts 23: 4624-4625; June 1963.

Smith, Carol Annette. Effects of a Meaningful Treatment for Division of Fractions: A Comparative Study. (The University of Texas at Austin, 1979.) Dissertation Abstracts International 40A: 1329; September 1979.

Uprichard, A. Edward and Phillips, E. Ray. An Intraconcept Analysis of Rational Number Addition: A Validation Study. Journal for Research in Mathematics Education 8: 7-16; January 1977.

Willson, George H. Decimal-Common Fraction Sequence Versus Conventional Sequence. School Science and Mathematics 72: 589-592; October 1972.

MEASUREMENT, GEOMETRY, AND OTHER TOPICS

Is there agreement on what measurement and geometry content will be presented?

Measurement has long been accepted as a topic that should be and is taught in the elementary school mathematics program, largely because of its usefulness in practical situations (PRISM, 1981). Geometry has come to be considered by curriculum developers as an important component in the elementary school mathematics curriculum more recently. However, the importance of geometry is not always apparent in classroom practice. One reason for including geometry is that geometric ideas are used to facilitate some number ideas -- for instance, in developing area representations and for work with a number line. The teachers and other educators who responded to the PRISM questionnarie on preferences and priorities for school mathematics in the 1980s gave strong support to three goals which can apply to elementary school geometry: to develop logical thinking abilities, to develop spatial intuitions, and to acquire knowledge for future study.

Much of the evidence on the content about measurement and geometry included in the curriculum comes from textbook analyses. Thus, Paige and Jennings (1967) surveyed 39 textbook series, summarizing the measurement content. They noted that there were few experiences in which students created their own units of measure, too little emphasis on practical application, and too few problems requiring actual measuring. To determine the status of geometric content in their curricula, Neatrour (1969) analyzed 16 textbook series and surveyed 156 middle schools. He found that while the amount of geometric content varied greatly, three times as much was included as in 1900, with an emphasis on informal geometry. Compartmentalization of geometric content into two- and three-dimensional ideas was common.

The responses from PRISM (1981) indicate strong support for four topics: the metric system, the use of measurement devices, estimation, and the use of both standard and nonstandard units of measure. For geometry, properties of triangles and rectangles, parallel and perpendicular lines, symmetry, and similar figures were given strong support.