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ABSTRACT

The papers contained in this document were originally presented at the May 1978 conference on Modeling Mathematical Cognitive Development sponsored by the Models of Learning Mathematics Working Group of the Georgia Center for the Study of Learning and Teaching Mathematics. Most have been revised to reflect comments and suggestions made at the meeting. The view of models presented includes the thinking of representatives of psychology, science, educational psychology, and philosophy, as well as mathematics educators. The efforts of those outside of mathematics education towards modeling as represented in this work are seen to be of great assistance in moving towards better models. Individual papers are titled: (1) What is a Model? Modeling and the Professions; (2) The Conception and Perc ption of Number; (3) Cognitive Microanalysis: An Approach to Analyzing Intuitive Mathematical Reasoning Processes; (4) An Information Processing Approach to Research on Mathematics Learning and Problem Solving; and (5) Reflections of Interdisciplinary Research Teams. Reactions to the first four titles are included. (MP)

Modeling Mathematical Cognitive Development

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Mathematics Education Reports

Mathematics Education Reports are developed to disseminate information concerning mathematics education. These reports fall into three broad categories. Research reviews summarize and analyze recent research in specific areas of mathematics education. Resource guides identify and analyze materials and references for use by mathematics teachers at all levels. Compilations of references announce the availability of documents and review the literature in selected areas of mathematics education. Reports in each of these categories may be targeted for specific subpopulations of the mathematics education community. Priorities for the development of future Mathematics Education Reports are established by the Advisory Board of the Clearinghouse, which includes representatives of the National Council of Teachers of Mathematics, the Special Interest Group for Research in Mathematics Education, supervisors, and teachers. Individual comments on past Reports and suggestions for future Reports are always welcomed.

We are pleased to make this collection of papers on modeling available.

Marilyn N. Suydam
Associate Director
Mathematics Education

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Constructing Models: A Prefatory Note

The papers contained in this volume were originally presented at the May 1978 conference on Modeling Mathematical Cognitive Development sponsored by the Models of Learning Mathematics Working Group of the Georgia Center for the Study of Learning and Teaching Mathematics. Most of the papers have been revised to reflect comments and suggestions made at the meeting.

The Models Group has been in existence since 1975, and this latest conference, held at the University of Georgia, represented an attempt to bring talent and resources from outside the group to bear on the problem of formulating models of mathematics learning. Membership in the group has varied, but the core of the group has remained fairly constant and is composed primarily of mathematics educators. The Models Group works as a "whole" only through meetings such as this one. Subsets of group members have collaborated closely in some areas, as evidenced by our two previous monographs (Fuson & Geeslin, 1979; Osborne, 1976). Consequently, our work has not been in a single direction, but has usually represented the thinking of mathematics educators as opposed to other relevant disciplines. The meeting reported here attempted to display a broader view of models by including colleagues from psychology, educational psychology, science, and philosophy. It was hoped that these people would broaden our focus, call attention to relevant work we had overlooked, correct any blatant misconceptions on our part, and rekindle our enthusiasm for pursuing models.

Before proceeding with the papers, it is important that the reader understand how our group operates. We do not focus on specific results of single studies. We are in the formulation stage. Models are not monolithic and, in our view, almost anyone can work on models. We do not feel we can afford to wait for an "Einstein" to propose a grand model. Model building is a slow, painstaking process. As we learn from our research, the models develop. The probability of obtaining a useful model from a one-shot study is essentially zero.

Models are intended to assist us in predicting behavior, simulating behavior, locating causal factors of specific behaviors, and perhaps even in observing behavior systematically. "The cogent assumption is that model construction and validation will lead to better understanding of children's mathematical learning which, in turn, could lead to more effective classroom instruction" (Geeslin, 1979, p.1). It is assumed that a rational explanation of classroom behavior can be constructed, at least in a probabilistic sense.

As Edgerton notes, we cannot merely copy methods from the physical sciences. However, we are attempting to adopt the "scientific method" for use in our research. Richards (1979) states:

> Two characteristics are common to all models:
>
> (1) There is a similarity in structure between the model and what is being modeled...and
>
> (2) There is a clear and obvious difference between the model and [what is being modeled]. (p.5)

Currently we do not insist on having "models" in the strict sense outlined by Richards. The papers in this volume, however, are intended to guide us in that direction.

Some evidence of success in modeling exists. Beginnings of developmental models can be found in the work of Steffe, Richards, and von Glasersfeld (1979), Fuson (1979), Carpenter, Hiebert, and Moser (Note 1), and Mierkiewicz (1979). Likewise, a foundation for information processing models can be found in the work of Geeslin and Shar (1979), Geeslin and Shavelson (1975a, 1975b), Shavelson (1973), Greeno (1979), and Branca (1980). A lot of this work is on "less complex" mental behavior, but it represents a beginning. It also serves as a reminder that "less complex" behavior, such as mathematical computation and spatial perception, are neither simple nor well understood.

To date there do not appear to be many unifying threads to modeling. Two distinct methodologies, i.e., clinical and empirical, exist. Also there are the strict statistical analysis of a single data set and meta-analysis of combinations of studies. These are not opposing philosophies, but all necessary parts of the same process. All these methods are "research" in the sense that they are not development, not focused on method, and clearly not mere "armchair theorizing." In each case, data on students and classrooms are obtained and these data are used to mold theory. Empirical work is characterized by experimental control, probabilistic reproducibility, and generalizability. These characteristics are necessary for proper classroom application of the findings (or models). Research on models is a form of systematic testing of hypotheses. These hypotheses may arise from "gut reaction" (Geeslin & Shar, 1979), mathematics (Mierkiewicz, 1979), clinical observation (Steffe, Richards, & von Glasersfeld, 1979; Mick & Brazier, 1979), empirical trial and error (Carpenter, Hiebert, & Moser, Note 1), or a combination of the preceding. In each case a cycle of observation, proposing a model, and intervention exists. As this cycle is repeated, the model should become more generalizable and more applicable to the classroom.

Obviously, many difficulties lie ahead. Modeling is long-term work with the usual problems of maintaining interest, financial resources, cooperative efforts, etc. Insight is needed for qualitative advances in theory. Unfortunately, insight neither occurs at regular intervals nor is necessarily available as needed. Cooperative research efforts are essential for educationally significant advances. Nonetheless, systematic theory development is the way to achieve continuous progress in improvement of instruction. Theory development, in turn, requires models. The efforts of our colleagues outside mathematics education represented by the papers in this monograph are of great assistance in moving toward better models.

William E. Geeslin
Models Working Group Leader

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WHAT IS A MODEL?
MODELING AND THE PROFESSIONS

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The term model is a very prestigious label. It is derived mainly from descriptions offered of some scientific activities. Those who pursue these activities claim for their investigations the authority of science, that is, the authority of reason and experience. Those who borrow the term and attach it to their studies would like to imply, or perhaps even claim, similar authority for their intellectual endeavors. Since the prestige of science is currently very high, the frequency of borrowing its vocabulary is high also. As a result it is possible to find almost everything and anything termed a model. For example, it is not unusual to encounter such disparate things as conceptual networks, operational definitions, flow charts, and diagrams all labeled as models.

But where does this phenomenon come from? And how may models function in intellectual activity?

What Is a Model?

As early as the 19th century, but primarily in the 20th century, a group of philosophers and some scientists with a philosophical bent decided that the achievements of the physical sciences had been so extraordinary that it would be helpful to identify the structural components of known theoretical systems. Being intellectual reformers, they thought if they could analyze science that was already developed and establish criteria for its assessment, they would thereby assist what they saw as underdeveloped areas of science, especially the social sciences, in their quest for

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The viewpoint expressed in this paper is indebted to the studies of Sir Karl R. Popper in the philosophy of science. Professor Henry J. Perkinson served as a helpful critic. Interpretations of the professions and the conclusions reached are my own.

prediction and control. They raised the question, "What are the logical and empirical components of science?" They answered with a flurry of articles and books delineating the structure of science. To name just a few of the participants and the structures they identified in this intellectual pursuit, Duhem (1962) wrote on the "crucial experiment," Bridgman (1961) on the "operational definition," Campbell (1952) on "What is science?", Hempel (1952, 1966) and Hempel and Oppenheim (1953) on "explanations" and "concept formation," and Hesse (1966) on "models and analogies."

The structure of interest here is, of course, model. The debate over this particular structure took place between the followers of Duhem and the followers of Campbell. Hesse has codified the issue, interpreting the original argument and extending it.

Her answer to the question, "What is a model?", may be put grossly but simply as follows: Models are ways of theorizing which follow the pattern of analogies. In making analogies, people go from familiar intellectual objects to those that are unfamiliar. This activity has three dimensions. The first is the negative analogy, which describes properties of the familiar object that do not seem to belong to the unfamiliar object of interest. The second dimension is the positive analogy, which ascribes properties to the unfamiliar object that it shares with the familiar one. Finally, there are neutral properties which cannot be identified as positive or negative. It is claimed that these neutral properties offer avenues of discovery about the phenomenon of interest. Modeling, according to this view, is conceptualizing through the vehicle of a familiar object or situation. Some philosophers have labeled these familiar objects or situations models.

What is important is not so much the answers to the question of structural components these 20th century, "logical positivist" scholars evolved, but rather their suggestion that the identification of the structures of science would assist researchers in creating accurate theoretical systems with high predictability. Going even further, their intellectual activity has implied and sometimes persuaded those in the social sciences and the professions that all science should look like the science they envisaged and thought they found, largely in the physics extant by the mid-20th century (cf. Brodbeck, 1963).

Often the questions people raise and the answers they dictate lead to intellectual orientations which are not only faulty but often not helpful. "What is a model?" is, I think, such a question. This question asks for a definition of the "nature" of an activity, as if having that definition would reveal something special, something about past and possibly future scientific practice (cf. Popper, 1962, 1974).

I am not saying that terms should not be defined or that viable forms of discovery should not be considered in the quest for knowledge. Scholars should be encouraged to develop as many ways of interpreting entities, states of affairs, and events as are within their creative powers. They should at the same time define their terms as a means to clearer communication and increased understanding. It is wise, however, to realize that delineating traits, characteristics, or properties of an idea or thing will not uncover its hidden function. Thinking which contributes to the notion that revealing the "essence" or "form" of a phenomenon--disclosing its proper nature and its attributes--results in discovering its assumed powers is, I believe, misleading and ineffectual.

To put the matter another way, some people believe that a true analysis of model would assist them in finding this intellectual entity which could then offer them some basic and mystical insight into their scholarly investigations. Quite frankly, as I have explained, I fear this orientation is unfruitful.

But, then, how else may models be viewed? What questions should be asked? The question I suggest is: "How can we use abstract conceptualizations to help us interpret phenomena we are studying?"

Bacon and the context of discovery. To understand an interesting but, under some circumstances, misleading notion of model, it is first necessary to understand the thought of Francis Bacon concerning scientific investigations. Bacon (1605/1944) raised the question, "What is the scientific method?", and answered it with a formula for scientific activity. True scientific laws were said to emerge inductively from pure observation. Bacon believed that the process by which one discovers theoretical statements established, guaranteed, and justified its product. Hence, the steps of the truly scientific rain dance assured a heavy scientific downpour. It is this viewpoint of scientific discovery which led eventually to the development of sundry methodologies. Modeling can be seen, I think, as such a methodology.

In this context, models are intellectual structures that help in interpreting various situations of interest. They are attempts to offer a guide to the discovery of new ideas. Modeling, then, is much akin to the utilization of conceptual sets that may help in gaining insight into a situation that is not as yet fully understood. Such an activity seeks to interpret the unfamiliar with the somewhat familiar or at least with a known dimension taken from an understood context.

It is clear then that, seen as a legacy of Baconian thought, modeling becomes an attempt to discover some new information by superimposing a framework on a not-as-yet-understood state of affairs. This manner of viewing models is close to the notion that modeling is a strategy for the discovery of knowledge. It may be compared to the drawing of analogies. The important and helpful element of this view is the idea of borrowing conceptualizations in order to discover new interpretations. Its drawback is to go beyond these expectations and see the "correct" form of borrowing as a criterion for evaluating newly gained insights.

Models as theory construction. There is a quite different notion of modeling which, I think, is closer to invention (Edgerton, 1973). This orientation suggests that a theory is conjectured about the not-as-yet-understood situation. What happens, then, is that a conceptual structure is offered by which someone may try to explain an interesting situation. This approach is, to put it simply, an attempt at theory construction. Models in this context are seen as weak theories. The assumptions of these theories or the set of deductive statements which follow from them are not known. The "theory" is basically only a few concepts linked together to provide a viewpoint for looking at a situation from a newly acquired perspective. As though seen through a new pair of glasses, things are perceived differently than they were previously.

But of what worth is a discussion of the activity of modeling to the professions? In particular, of what worth is it to educators? Answers to these questions can be better understood by considering the relationship between the professions and science.

Science and the Professions

Professions have always argued that they should be given the power to determine policy in their designated domains. They have protected their social territories by claiming the authority of expertise gained in schooling. In the science-based professions it has been argued that this expertise comes from the possession of science. Predictability and, therefore, control in professional settings have been an assumed outcome of their scientific acumen.

The fact that these professions have not had this intellectual power has seldom stopped them from making their claims to a specified professional turf, but the absence of this power has caused continual discomfort and has allowed various invaders to shift their chosen realms of responsibility. Social workers, for example, have not successfully defended their territory. Nurses, therapists, former addicts and alcoholics, and many others have eroded the social workers' claims to expertise in the handling of the socially deprived and needy.

Just as claims for legitimacy in the professions have been made and defended largely by reference to the possession of scientific knowledge, claims to the knowledge of causal regularities in social matters have created an orientation in the professions towards scientific research. This orientation has manifested itself in the increased push toward the attainment of higher degrees with research as a primary requirement. As curricula for advanced degrees have evolved, "theory construction" has become a major target in research. The need to uncover the social regularities thought necessary to predict and control social settings, and thereby increase the possibility of successful practice, has become a higher priority as public and professional criticism has prompted the demand for accountability. At the same time, these degree requirements have been used to bolster the claim of social authority for the profession and its members. Model building in this context has been recognized basically as a means to more, and more powerful, theory. Importantly, model building has also been used to convey the impression--often a false one--of having mature scientific accomplishment.

This pattern of the development of arguments for professional authority has been as true in the education profession as it has been in nursing, engineering, and medicine. Educators want scientific theory to answer their professional questions of administration, curriculum, and instruction as well as bolster their claims for legitimacy. They have attempted to attain these aims by developing and utilizing theoretical foundations in several forms.

Modeling in the Education Profession

Professionals in education have been interested primarily in answering two questions: "How do students learn?" and "How should we teach them?" These questions have been answered in recent years by attempts to devise techniques for learning which have been labeled models of instruction. These models have had a number of dimensions. They have been rules for instruction which consider not only the logic of teaching and subject matter but also the psychological and sociological foundations of learning. These efforts suggest that there is a "science of education"; that is, theoretical regularities exist that educators can discover or borrow which, when applied to an identified social situation, will generate prediction and control. These regularities have been sought by investigating the social setting itself or by borrowing theory, thought to be true, from the disciplines. Sometimes, of course, both directions have been taken.

This movement has been a major development in education during the middle and last decades of this century. It is attested to not only by research activity but also by the development of curricula for advanced degrees which emphasize psychological, sociological, historical, and philosophical foundations. It should be pointed out too that, in general, other professions, like nursing and medicine, have followed the same lines of intellectual development.

In view of these pursuits, the question of whether there is a science of education becomes an important point of focus for educators and, in a way, for other aspiring professions.[1] And, if there is such a science, it then becomes important to learn how to find, or discern, it and how to utilize it.

I can comment definitely on two approaches educators should not take:

(1) They should not look for logical and empirical structures, which may not exist, and which may not characterize past and present scientific achievement in the physical or natural sciences,[2] and

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[1]A number of professions have grown increasingly interested in their philosophies of education but have not resolved the problems surrounding the "training" or "educating" of practitioners. Because professionals practice in a social setting, it is necessary to understand not only the constraints under which they operate but also how competence and responsibility are garnered in their education. The medical profession, for instance, is becoming increasingly interested in educating young physicians in ethics.

[2]Feyerabend (1968) and Kuhn (1962) have discussed scientific practice and its contribution to answering the question, "How does our knowledge grow?" Arguing from historical examples, each of them has interpreted past scientific activity in ways that the actual contributors have seldom claimed. Although these studies analyze descriptions of scientific practice, the fundamental issue is, "How ought scientists practice?" Kuhn seems to argue that, since science has been so enormously successful, how they have succeeded is important as a prescription for future activity. Feyerabend contends that some practices of some scientists have been quite different from those that they or the logical positivists have described. He thinks certain practices should be encouraged and others should not. In particular, he attacks rigid deduction and meaning invariance and suggests that the philosophers who argue for these principles are encouraging a dogmatic metaphysics in science.

(2) They should not, in a Baconian manner, think there are methodologies to be discovered which, if found, would, like formulas, deliver true scientific laws.

The first approach emphasizes the discovery of "meta" objects that go beyond linguistic expressions and suggests that scientific activities in the professions are the same as in the pure sciences. The second approach confuses the question, "How may we discover theory?", with the question, "How do we evaluate theory?" It suggests that the manner in which discoveries are made is the criterion for evaluating what is discovered.

But, then, what may educators do that will help them in their quest?

Alternatives for the Education Profession

The first, and least promising, alternative is to seek a conceptual framework for a science of education. Such a framework would consist of a most basic and abstract conceptual network together with the accompanying assumptions. This framework would attempt to describe theoretically the richest conceptualizations of solutions to the major questions of educators. Much ingenuity would be required for such an intellectual endeavor by educational theorists. The resulting framework might center, for instance, around the key notions of rationality, health, and learning. One philosopher of education (McMurray, 1955) began such an attempt by calling for "an autonomous discipline of education." It should be noted that the development of a conceptual framework is an orientation currently being fostered in the nursing profession (Rogers, 1970; Smith, 1979).

Another, and a more promising, alternative would be to see more problems of education in an applied science setting; that is, educators would view themselves a little more as engineers do, at least some of the time. This orientation would mean taking a problem-solving approach in practitioner settings. Practitioners would be educated to see themselves as theorists in problem-solving settings in which they would be called upon to conceptualize their problems, consider social constraints, and devise solutions. But let me explain in some detail what I mean.

A striking feature of professional literature is the crude way in which problem solving in the applied science setting has been cast. The impression that the problem solver just grabs some pure theory, usually developed elsewhere, and applies it is, of course, a very naive view of rationality at the applied science level (cf. Smith, 1951).

Problem solving in a practitioner setting consists of a complex series of judgments about a considerable number of factors. Some of these factors are public, professional, and private values. What has been called "applied science" in actuality is an activity which at least involves aims, theoretical information, theoretical interpretations, and values. Conceptualizing applied science in this way raises the question, "How should scientific inquiry be carried on in the professions?" An adequate response, I think, can be given.

In the first place the problem solver works within a framework of aims. These aims usually involve the elements of risk and safety. The problem solver is asked to find a solution that minimizes social and private risk and maximizes safety. The problem exists in a setting about which theoretical interpretations must be made, and the problem must be conceptualized and resolved within a context of technical and value parameters. Therefore, the problem solver must identify theoretical information to be used, recast it, and even generate more theory, as well as identify and resolve the value issues involved.

A helpful example to illustrate this process is that of an engineer building a bridge. One aim of the engineer is to build a bridge that is safe. The bridge must be constructed in such a way as to handle both Volkswagons and tractor-trailers. Therefore, the engineer must have technical information that guarantees neither overbuilding nor underbuilding the bridge. Financial, political, and even aesthetic factors must also be considered: how much money may be spent, where the bridge will be constructed, and whether or not it will be pleasing to the eye. To accomplish all of these aims, the engineer must be a theorizer, not just an applier, of knowledge.

Professionals need to help practitioners resolve problems seen in this way and with these kinds of parameters. It is, therefore, essential that practitioner settings be investigated and explored. Learning how to theorize in real situations is another primary task. Doing this, I think, means learning how to evolve theoretical information in a social context instead of in some pure science form which is seen as applicable in every practitioner setting.

These intellectual activities are a matter of developing a theory of practitioner rationality. Interestingly, such a project involves a philosophy of applied science. Guidelines for minimizing risk and maximizing safety must be developed as well as appropriate epistemologies.

Educators, of course, must delineate their aims and develop a philosophy of applied science that will suit their needs. Then, they must examine practitioner settings and create theory within them. The resolution of value choices and value conflicts must be a major study.

If this kind of perspective is adopted by the education profession, the consequences of inheriting faulty perspectives which surround our descriptions of the activities of pure science may be avoided. In my opinion, the sooner educators see that they are involved in a different kind of activity, the sooner they will resolve their professional problems.

If "the philosophy of knowledge and the professions" outlined in this paper were to be called a model, I would laugh but I would not object.

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Reaction

to

WHAT IS A MODEL? MODELING AND THE PROFESSIONS

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Research in mathematics education would be more productive, Professor Edgerton argues, if it were patterned after applied, as opposed to pure, science. Specifically, educators require methods for theorizing in practitioner settings; that is, methods for evolving "theoretical information in a social context instead of in some pure science form which is seen as applicable in every practitioner setting."

Professor Edgerton then poses two alternatives for educational research:

(a) to seek a conceptual framework for a "science of education," and

(b) to see more of the problems of educators in an applied science setting.

The former, she argues, is "the least promising" but has gained wide acceptance, primarily because science is considered a normative term. Consequently, there has been a rush to adopt many of the appurtenances of science in order to provide the appearance of respectability. Modeling, which is part of what science does, is thus, prima facie, respectable behavior. We are warned, however, that even if modeling is appropriate for the hard sciences, and that is not at all clear, it is not necessarily appropriate for the softer sciences--those dealing with people and messy data.

While I sympathize with the thrust of this argument, following the line of thought a bit farther provides a context for appreciating the value, role, and function of modeling.

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It is useful, and perhaps important, to stress the "applied science" quality of research in mathematics education, but it is a mistake to consider the two paths suggested above as real, clearly distinguishable, alternative directions for this research.

Science, Pure and Applied

There is no clear dichotomy between conceptual frameworks and applied science. Rather, applied science must be grounded in a conceptual framework, which, in turn, is at least partially motivated by the requirements of the practical setting. The conceptual framework provides a common context for the variety of applied problems and a common ground, or conceptual link, between the various theories or disciplines which are being applied.

The establishment of this common ground is particularly appropriate to issues in mathematics education, which is intrinsically interdisciplinary in nature. It is essential to glean the pertinent ideas from relevant research in any field, but there are real problems with adopting simultaneously, say, a behavioral theory of learning, a developmental theory of education, a cognitive theory of language, and a dualistic philosophy of mind. Instead, a conceptual framework provides a unified basis for research where the psychology, mathematics, education, and philosophy can complement each other. The conceptual framework and the resulting operative theory determine the problems to be addressed, the methods to apply, and the nature and scope of the data. Applied science, especially, must develop within the context of, in Imre Lakatos' (1970) terms, a methodological research programme.

When applied science is conducted outside such a framework, the product has the appearance of a patchwork quilt. Solve this problem this way--that problem that way--use this teaching method here--try the following analogy there. This is the direct result of adopting Professor Edgerton's "problem-solving approach in practitioner settings," without also seeking a conceptual framework which can bind the solutions into a viable program. Unfortunately, too much of mathematics education research is already conducted in the absence of such a framework, and reading through the literature is very much like scanning a patchwork quilt. While the character of applied science has remained largely uninvestigated, it is certainly more than a patchwork quilt. It is a complex endeavor which, as Professor Edgerton explains, "involves, at least, aims, theoretical information, theoretical interpretations, and values."

I do not mean to suggest that we need to develop a "science of education" as in the first alternative suggested above. It is not even clear what a science of education would be. Rather, it is necessary to develop a coherent methodology; that is, it is necessary to develop ways to study the educational process which provide a basis for conducting research profitably. It is through the development of conceptual frameworks, and the development of appropriate methodologies for these frameworks, that the patchwork quilt gains meaning, coherence, and value.

Models and Data

The applied science image Professor Edgerton suggests has much to offer. In particular, I wish to focus on the applied science notion of a functional model. These models are constructed when the researcher does not have access to what is being modeled. For example, in designing a bridge, the engineer uses models to evaluate the design before the bridge itself is constructed.

This functional aspect of a model depends on a partial similarity between the model and what is being modeled (the prototype). The model bridge has a similarity of structure and design with the intended bridge. The model serves the engineer's purpose of examining stress or, perhaps, planning the construction. The model simplifies the situation and provides an image with which to work.

Adopting the applied science notion of functional models is probably best motivated by the nature of the data in mathematics education and the social sciences. The implication throughout Professor Edgerton's paper is that the richness of the data overwhelms the researcher. That is, the contextual setting of the problems is so rich that it prevents generalization--of methods, results, or theory.

On the contrary, in a very important sense it is not the richness, but the paucity, of our data which overwhelms us. We are overwhelmed because the central focus of our study, say the child's construction of whole number concepts, is beyond our grasp. We cannot observe this construction, or any other mental process, directly. Moreover, we must observe the dynamic character of the learning process from a perspective which is distant from this process. To account for this dynamic process and to study the cognitive constructions of a child, we build models. These models postulate connections which might account for the appearance of new behaviors and alterations of old behaviors.

It is important to note that this use of models is much closer to applied science and the black box model of cybernetics than to the model/theory relationship of pure science, at least as this relationship is portrayed in the reconstruction of theories in physics or the philosophy of science. From this applied perspective a model is functional--it serves a purpose and has a particular role to play in research. As educational researchers we do not, and cannot, have access to anything corresponding to cognitive structures and mechanisms. More importantly, cognitive structures and mechanisms are theoretical entities which we postulate, and we construct models which describe the operation of these cognitive mechanisms and account for changes in cognitive structures. These models then foster discussion of underlying operations and processes which, from the perspective of our theory, make aspects of their mathematical experiences meaningful for children. The problem, then, is to build models which make behavior and transformations in behavior interpretable. This enterprise requires a coherent cognitive framework.

If we are building models to create an idealized science of education to be a reprint of some picture which we have of "hard" science, then of course Professor Edgerton is correct, and our program is bankrupt. But, there are other reasons for seeking a conceptual framework or constructing models, a I have set forth in these brief remarks. Moreover, while I have sympathy with Professor Edgerton's proposal for adopting a problem-solving approach in practitioner settings, it seems clear to me, first, that a coherent conceptual framework is essential for this effort to be productive and, second, that using models is an effective method for working in situations with so many variables and even more unknowns.

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THE CONCEPTION AND PERCEPTION OF NUMBER

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> Thus much is true, that of natural forms, such as we understand them, quantity is the most abstracted and separable from matter.
>
> Francis Bacon (1623/1891)

For some 50 years Piaget has been saying that the process of perception does not seem feasible unless we assume that the perceiver has some prior structure which can assimilate sensory experience. Though there are empirical findings that corroborate that hypothesis, its strength springs from the epistemological foundation on which Piaget has built his entire theory of cognition.

Unlike scientists a hundred years ago, some of us today have come to believe that what we call "data" invariably and necessarily presupposes some theoretical structure to direct and inform our observation. Hanson (1958) said it very simply: "Observation of x is shaped by prior knowledge of x" (p. 19; also cf. Bridgman, 1961; Feyerabend, 1975; Kuhn, 1970). Psychological research seems to be among the last of the investigative activities to be affected by this shift in attitude toward the philosophy and practice of science. Somehow psychology still perpetuates doctrines inspired by the intellectual climate of the 19th century, in that they view organisms as essentially passive objects whose changes of state and activities can be explained as effects of causes in an objective environment. Consider, for example, the opening

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The many discussions with other members of the team--Les Steffe, John Richards, Izzy Weinzweig, and Pat Thompson--have helped me a great deal to sharpen the still developing ideas I have expressed in this paper. I am grateful also for the critical comments Roger Thomas and Michael Tomasello made on an early draft. Finally, I want to stress that the pioneering work on the role of attention in the construction of concepts was done 30 years ago by Silvio Ceccato. The present research was supported by the National Science Foundation (SED 78-17365) and the generous allotment of research time in my Department.

sentence in a recent article by Skinner (1977): "The variables of which human behavior is a function lie in the environment" (p. 1). To a lesser extent, and with some exceptions, this behavioristic orientation prevails even in cognitive psychology. There is still a widespread explicit, or implicit, belief that the result of cognitive processes, knowledge, is in some sense a picture or replica of the cognizing organism's environment and that this picture contains "information" which the organism has obtained through its senses. I have shown elsewhere that this scenario of cognition, by assuming the transfer of information from a ready-made environment into an organism, inevitably leads to a self-contradictory model of cognition and epistemology. I have argued that a radically constructivist view of knowledge provides a more promising approach (Richards & von Glasersfeld, 1979; von Glasersfeld, 1974, 1976, 1979a, 1979b, in press).

In this paper I shall apply the constructivist approach to the analysis of the concept of number and outline a hypothetical model that attempts to explicate the structure of numerical concepts in terms of activities carried out by the cognizing subject. In this view, things like unity, plurality, number, and set are not independent entities that "exist" ready-made in an objective "reality" but, instead, are conceptual constructs, the results of a subject's specific ways of operating. This perspective does not imply that the construction of numerical concepts may not involve perceptual processes, but it does imply that such perceptual processes must be considered constructive, rather than a passive reception of "facts" that are numerical in themselves and, as such, belong to an ontological reality.

Hence, two main efforts will be made in this paper: One, to analyze what we may have in mind when we say "number"; the other, to consider the extent to which numbers can be perceived. Though these two problems are related in many ways, I shall concentrate on the conceptual problem in the next four sections of the paper and devote the remainder to the perceptual problem.

Concerning Hypothetical Models

There are three points that have to be cleared up at the outset. First, as constructivists, we must remain constantly aware of our basic assumption that concepts and conceptual structures are necessarily hypothetical items and are doubly hypothetical whenever they are attributed to other people. At best, we can know of them only to the extent that the owner or user tells us about them or, alternatively, acts in a way that leads us to infer them. Both of these ways of access, however, are subject to a general restriction which, although it is traditionally disregarded by realists of every denomination, must be taken very seriously by constructivists. In its simplest form, the restriction amounts to this: Whenever we interpret what someone says or does, we interpret what we hear or see in terms of elements that are part of our own experience. I deliberately speak of "elements of our own experience" because I do not wish to imply that we are limited to imputing to others the same, and only the same, operational procedures that we impute to ourselves. That way of viewing others would be an extreme form of what Piaget calls "egocentrism." Rather, I mean that whatever procedures we hypothetically impute to other people will be composed of experiential elements that are conceivable to us on the basis of our own experience. To use a drastic example, a congenitally blind person's interpretation of sighted friends will necessarily be composed of elements within the blind person's experience. It may contain correlations, regularities, and probabilities that are radically different from those the blind person normally uses, but it cannot possibly contain elements derived from visual experience. This limitation is important to remember when interpreting observations about children, because children may construe perceptual and conceptual items in ways to which the adult, in spite of all efforts to decentrate, can no longer return.

Second, there is the problem of "models." If we are not satisfied with mere descriptions of observable behavior but want to formulate theories as to <u>how</u> the observed behaviors come about, the obvious procedure would be to open up the behaving organism so we could observe what goes on inside. Living organisms, however, have the awkward peculiarity that their more interesting functions either cease when we cut them open or are still so mysterious that the investigator has no idea what to look for (e.g., the function of memory). From a cyberneticist's point of view, therefore, living organisms (and children in particular) are "black boxes" whose internal functions are simply not accessible to observation. But both cyberneticists and cognitivists still want to go farther: They want to see if they can set up hypothetical operations and ways of combining them into larger operational structures that yield the same results as the behavior manifested by the organism. Such "models" are, and remain, hypothetical and should never be said to depict or replicate what actually goes on inside the organism. But, as long as the organism's actual functioning remains inaccessible to observation, it is extremely useful to know at least one way in which what it does <u>could</u> be done. This way of proceeding, it should be noted, is essentially the same as that of the modern physicist who postulates hypothetical entities with hypothetical properties, such as spin, charm, color, etc.--all of which are outside the range of direct observation but nevertheless enable the physicist to construct theories and make predictions about observable events. For example, Lieberman (1979) discusses the distinction between real and hypothetical objects and comes to the conclusion that "the dividing line between real and hypothetical, then is partly a matter of convention, for scientists to draw wherever they find it most convenient" (p. 329).

Third, when some phenomenon pertaining to another organism is to be explained developmentally, differences must be found between what the organism is doing now and what was observed before or what will be observed later. Moreover, development is always studied with some idea of what is being developed. That is to say, there is some idea of a resulting end-state or product--otherwise it would simply be a study of change. Therefore, when we speak of development in children, we need to specify a plausible succession of changes that may characterize in a generalizable fashion children's progression from an original, primitive way of acting in, or responding to, certain experiential situations to the accepted adult way of acting or responding.

In order to formulate hypotheses as to how or why a child's action or response is different from an adult's, we must have some sort of model of what goes on in the adult. Thus, in order to investigate children's development of numerical concepts, it will be indispensable to have an explicit model of the adult concepts of number and the related constructs.

I have made these three points as explicit as I could, partly as an admonition to myself, partly as an attempt to forestall any "realist" interpretation of what I intend to present in the main body of the paper. I shall develop a hypothetical model that does not purport to be the description of any reality. At best, it may turn out to be compatible with such observations as have been, or will be, made. If that should be the case, the model may be used to make predictions and guide the development of didactic methods.

What Is a Number?

Whenever a question of the type, "What is an x?", is formulated, there is more than one path towards an answer. In the case of "What is a number?", one answer might be: "Well, ...one, two, fifteen, and thirty-eight are numbers." That response would be equivalent to answering "Pippins, Winesap, and Golden Delicious" to the question, "What is an apple?" It would not be much help to a child who has never experienced an apple. Instead of the verbal reply, one could go to a well-stocked pantry, come back with specimens of Pippin, Winesap, and Golden Delicious apples, and say, "All of these are apples!" Those examples would contribute to an extensional definition. If the child then asked "Why?", one could point out that all of these objects are relatively round and smooth; are of certain size and weight; have skin, flesh, and core; and have a smell and a taste that one can learn to recognize. In other words, giving part of an intensional definition would, by and large, be successful in specifying some of the characteristics that, once abstracted, make up the concept apple.

When the question concerns number, there are immediate difficulties. If the child asks, "Why are 'one,' 'two,' and 'fifteen,' numbers?", we may at first put one glass, two spoons, and fifteen toothpicks on the table. At some point we will realize that this procedure is unlikely to work. We may then have an inspiration: We push everything aside and arrange toothpicks (or whatever) in lots of one, two, and fifteen. Now, we feel, it should be obvious. Fortunately, children rarely pursue the question further. If they did, we certainly could not tell them what characteristics they have to abstract in order to form the concept number.

Peano (1891a), at the beginning of his brief essay on the principles of mathematical logic, observes that names such as 1, 2, 3/4, and $\sqrt{2}$ are proper nouns, whereas words like number, polygon, and equilateral refer to classes and are, therefore, common nouns. That statement is illuminating because it clearly brings out the difficulty: Individuals have to be characterized by individual characteristics, classes by common ones. What are these characteristics in the case of one, two, and three, and what are they in the case of number? Peano was well aware of that problem:

> I primi numeri che si presentano, e con cui si formano tutti gli altri, sono gli interi e positivi. E la prima questione si è: possiamo noi definire l'unità, il numero, la somma di due numeri? La definizione commune di numero, che è l'Euclidea, "numero è l'aggregato di più unità," può servire come schiarimento, ma non è soddisfacente come definizione. Invero un bambino, a pochi anni usa le parole uno, due, tre, ecc.; in seguito adopera la parola numero; solo molto piu tardi nel suo dizionario comparisce la parola aggregato....Quindi, dal lato pratico la questione parmi risoluta; ossia, non conviene in un insegnamento dare alcuna definizione del

numero, essendo questa idea chiarissima agli allievi, e ogni definizione non avendo che l'effetto di confonderla.[1] (Peano, 1891b, pp. 90-91)

He then discusses the theoretical aspect and concludes, "Il numero non si può definire" (p. 91); that is, number cannot be defined.

Peano and those who, after him, joined the effort to formalize a logical foundation of the number system and mathematics, were intent upon defining properties and relationships within the formal system rather than upon an analysis of how we come to have concepts of unity, plurality, and number, whose experiential generation they simply--and, given their interest, justifiedly--took for granted. Psychologists and investigators of the development of numerical thinking have, as a rule, allowed themselves to be trapped in these formalistic definitions rather than undertaking a conceptual analysis. A recent representative example is Saxe (1979): "As used here, number is a category of knowledge that is defined by relations of one-to-one correspondence and order" (p. 74). That statement is much like saying, "A telegraph pole is a kind of object that is defined by wire connections to no less than two other things." Statements like these may be interesting for all sorts of purposes but can hardly be considered definitions of the objects in question.

What, then, is a number? Maybe Euclid's "clarification" is helpful, after all. It became clear, for instance, in the example of the toothpicks that it is not any characteristic of the objects that matters, but their arrangement in lots, their aggregation. But against that, it could be said that, if four toothpicks are placed one in every corner of the room, they could still be considered as four. Where, then, is the aggregation? The answer to that question is both old and

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[1]My translation: The first numbers that present themselves, and with which all others are formed, are integers and positive. The first question is: Can we define unity, number, the sum of two numbers? The usual definition of number, which is Euclid's "number is the aggregate of several unities," may serve as clarification but is not satisfactory as definition. In fact, a child of few years uses the words one, two, three, etc.; later it uses the word number; only much later the word aggregate appears in its vocabulary.... Hence, from a practical point of view, the question seems to me resolved; that is, in the course of instruction it would not be advisable to give any definition of number, since that idea is perfectly clear to the students, and any definition would only have the effect of confusing the idea.

frequently disregarded. The oldest statement of it that I have found is also the most elegant and the most convincing. It comes from Caramuel (1670/1977), Bishop of Vigevano, who has recently been credited with the first formulation, in the Western world, of the binary number system:

> Un tizio parlava nel sonno, e quando l'orologio suono le quattro disse: "Uno, uno, uno, uno. Quest'orologio e matto: ha suonatc quattro volte l'una." Il tizio, dunque, aveva contato quattro volte un colpo, e non quattro colpi. Ovvero aveva in mente non il quattro, ma l'uno per quattro volte. Cio dimostra che contare e considerare piu cose contemporaneamente sono attivita diverse. Se io infatti avessi nella mia bibliotheca quattro pendole, e se tutte suonassero contemporaneamente l'una, no diro che hanno suonato le quattro, ma che hanno suonato quattrc volte l'una. Questa differenza non e insita nelle cose, non e indipendente dalle operazioni della mente: dipende anzi dalla mente di colui che conta. L'intelletto dunque "fa" i numeri non li "trova"; considera diverse cose come distinte ciascuna in se, e come intenzionalmente unite dal pensiero.[2] (pp. 43-44)

Berkeley (1706/1708-1930), some 30 years later, made a note to himself: "Number not without the mind in anything, because 'tis the mind by considering things as one that makes complex ideas of them" (p. 24); and Dewey, before the beginning of this century, concluded: "Number is a rational process, not a sense fact" (McLellan & Dewey, 1895/1908,

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[2]My translation: There was a man who talked in his sleep. When the clock struck the fourth hour, he said: "One, one, one, o. . That clock must be mad--it has struck one four times." The man clearly had counted four times one stroke, not four strokes. He had in mind, not a four, but a one taken four times which goes to show that to count and consider several things contemporaneously are different activities. If I had four clocks in my library, and all four were to strike one at the same time, I should not say that they struck four, but that they struck one four times. This difference is not inherent in the things, independent of the operations of the mind. On the contrary, it depends on the mind of him who counts. The intellect, therefore, does not find numbers but makes them; it considers different things, each distinct in itself, and intentionally unites them in thought.

p. 23). Thus, there is an active mind that takes distinct things and unites them by an operation. But that still leaves the question where the "things" come from or, rather, how the mind distinguishes things in such a way that it can unite them. Dewey has every intention of going farther and specifying operations:

> In the simple recognition, for example, of three things as three the following intellectual operations are involved: The recognition of the three objects as forming one connected whole or group--that is, there must be a recognition of the three things as individuals, and of the one, the unity, the whole, made up of the three things. (p. 24)

That is the closest he gets. By using the word "recognition," he unwittingly blocks any further operational analysis. To speak of recognizing the threeness or oneness of things inevitably implies that threeness and oneness belong to the things as some kind of perceivable property. That is unfortunate, because at a later point Dewey again says that number arises "from certain rational processes in construing, in defining and relating the material of sense perception" (p. 35). It is these operations of defining and relating that need to be specified if there is to be a model of the number concept.

We have nevertheless come a long way. It seems clear now that separating and uniting are the crucial activities. There must be an operation that creates unitary items that can be seen as discrete unities, and there must be an operation that takes several such unities and unites them so that they can be seen as another unity. Hence, the question now is: How do we come to have a unity or unitary item? The physicist Bridgman (1961) formulated the same question when he asked: What is the thing that we count? His answer was that of a constructivist:

> It is obviously not like the objects of common sense experience--the thing that we count was not there before we counted it, but we create it as we go along. It is the acts of creation that we count. (p. 103)

From a constructivist point of view, then, unity is the result of an act of creation, an operation carried out by the subject, not a perceptual property of an object. That may sound absurd, because it would seem that the subject establishes the unity of an item on the basis of a particular sensory characteristic that makes the item distinguishable from the experiential background. In fact, it is very likely that infants first come to attribute "thinghood"[3] to items that are visually easy to discriminate from the field. But, although sensory signals may well be helpful in the development of the concept of unitary item or thing, the operation that actually constitutes such a unity cannot be dependent on sensory signals. Even in purely visual experience, there are examples that illustrate this distinction. For instance, in looking at the wave line shown in Figure 1, the subject can alternatively see it as one continuous curve, as three crests, as two troughs, or as a multitude of dots. The sensory signals remain the same throughout, yet they can be organized into different unities.

Figure 1

The wave line, one might object, nevertheless provides some sensory basis for each of the organizations and, indeed, determines what organizations are possible. But that, again, is an illusion. The straight line shown in Figure 2 can be seen as one piece; but in spite of its perfect sensory homogeneity, it can also be seen as two halves, three thirds, or four quarters. And if we work a little harder, we can also see that there are roughly five inches in it or, with more practice in the metric system, about 13 centimeters. Excepting the segment of line, none of these units are determined by sensory signals, and that fact leads to the inevitable conclusion that they are the result of some quite independent operation.

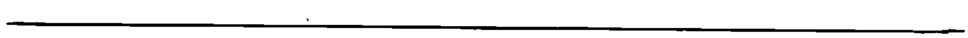

Figure 2

[3]The expression "thinghood" is intended to designate merely the unitary separation of an item from the experiential field, much as in the realms of vision and art a "figure" is separated from the "ground." This idea must not be confused with the concept of object permanence, a far more complex structure that involves both externalization and representation, neither of which is required in "thinghood."

The Attentional Component

> A *unit* is that by virtue of which each of the things that exist is called one. (Euclid, Book VII)

If we assume that the operation that creates unitary items is, indeed, independent of sensory signals, it is tempting to suppose that it involves motion in some way. Piaget has long maintained that the perception of patterns is the result of active composition of sensory data by means of motion. Dividing a line into unitary sections might plausibly be achieved by movement alternating with pauses, and the same could be said in the case of the visual perception of items such as toothpicks lying on a table. By means of more or less special additional hypotheses, that idea can even be extended to situations in which there is no direct perceptual scanning. There is, however, a considerable body of evidence showing that figural composition can take place without any actual eye or body movement. Köhler (cited in McCulloch, 1951), Lashley (1951), Pritchard, Heron and Hebb (1960), and Zinchenko and Vergiles (1972) have independently noted that scanning of the visual field can take place by the movement of *attention* when the field is stabilized on the retina, that is, when there is no eye movement. From the theoretical point of view, these findings are revolutionary. They indicate that a perceiver's attention can focus on specific parts of the visual field and shift focus from one part to another, without any corresponding change in the position of the sensory organ or of the signals in the visual field. Accepting this mobility of the focus of attention provides, on the one hand, an alternative to physical motion in the composition or integration of perceived patterns and, on the other hand, an active agent in the experiencer's organization of his or her experience.

Instead of tying the generation of unitary items to movements and pauses in the actual perceptual process, we can now attempt to account for it by shifting and alternating the focus of attention. This approach has the immediate advantage of enabling us to posit one and the same operational procedure regardless of the item being unitized. In other words, we know experientially that we can conceptually divide into several unities, or consider as a single unity, not only any array of perceptual signals, but also, for instance, last night's sleep or the rest of our lives. And the same is true of irnumerable other items that, by their nature, cannot cont in perceptual signals to guide the unitizing operation.

The idea that the structure of certain abstract concepts could be interpreted as *patterns of attention* was first proposed by Ceccato (1962, 1964-1966, 1974). In the pages that follow, I shall outline a possible application of that idea to numerical concepts.[4]

Attention, in this model, is conceived as a pulse-like activity that picks out, for further processing, some of the signals from the more or less continuous multitude of signals which the organism's nervous system supplies. That is to say, a single pulse or moment of attention can be, but need not be, focused on a particular signal. When it is unfocused, it does not pick out particular signals, but that does not mean that there are no signals that could have been picked out. On the other hand, attention can focus on items that are not present as active sensory-motor signals, but as records of such signals (or composites of them) that have been "picked out" on some prior occasion.

It should be clear that I am using the word attention in a way that is somewhat different from ordinary usage. Ordinary expressions like "focusing attention on a diagram" or "on the sunset" are used in situations in which the speaker has posited such things as a diagram or sunset and an organism that interacts perceptually with those items. Similarly, saying that "an organism focuses attention on signals in its nervous system" has meaning only if it is assumed that organisms can operate on several levels. There will be a level of sensation that comprises the generation and transmission, within the neural network, of sensory-motor signals. Then there will be the level of attentional activity where focused pulses pick out particular sensory-motor signals, while unfocused pulses create discontinuities or intervals. Finally, there will be a level of records where the results of the attentional activity can be maintained in such a way that they, in turn, can become the object of attentional activity.[5] In short, attention in my model

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[4]Ceccato's idea of the constitutive role of attention in the construction of concepts has also recently been elaborated by Vaccarino (1977), another member of Ceccato's early study group.

[5]Such a system of three levels is obviously far too crude and simple to account for most of the results which a human organism can produce. There seems to be evidence, for example, that there must be a level on which sensory-motor signals are recorded regardless of whether they have, or have not, been attended to; and the recent work of Hilgard (1974) indicates that there are several relatively independent levels of attentional activity.

refers to a selective activity just as it does in ordinary usage. But in ordinary usage the items which attention focuses on and selects are things that exist in a reality outside the attending organism; in my model they are items or events within the organism.

Given this model that operates on several levels, one can attempt to map--as a very crude approximation, to be sure--how a person could come to have something like the "concept" of, say, an apple. The partial definition of apple I proposed earlier was composed of a number of characteristics. Some of them, like taste and smell, would be represented by sensory signals; shape, size, and texture would be combinations of visual, tactual, and proprioceptive (motor) signals; weight would be tactual and proprioceptive; and the characteristic arrangement of skin, flesh, and core would probably involve color and other visual, tactual, and proprioceptive signals. If the subject were to discover in its experiential records that an aggregate of moments of attention focused on these specified sensory-motor signals was a recurrent event, the process of concept formation could be implemented through the simple extraction of those signals that are common to all, or at least most, of the occurrences. In some cases there might be an obligatory order for some of the signals; in others it could be just a list. In all cases, however, there would be one further condition: Whatever the pattern of sensory-motor signals involved, it must be such that it constitutes a consecutive sequence of focused moments of attention; for if it were not consecutive, if it contained an interval of unfocused moments, it could never be categorized as a "thing," a "whole," or a "unitary item." It is the two moments of unfocused attention at the beginning and the end that provide the closure and the cohesion of a unitary item.

Crudely and provisionally, I shall map the conceptual structure of a perceptual item, like an apple, by the sequence:

$$O \begin{pmatrix} I & I & I & I \ldots & I \\ a & b & c & d \ldots & n \end{pmatrix} O$$

where "O" designates unfocused moments of attention, "I" focused moments, and "a, b, c, ... n" different sensory-motor signals individually picked out by consecutive focused moments of attention. This map is obviously a crude approximation because, as mentioned above, even a relatively simple concept like apple involves substructures in which sensory and motor elements are combined in specific, characteristic ways. These substructures could be represented by parentheses or some other notational device. However, because the main concern here is the construction of number concepts, I shall disregard the intricacies of the sensory-motor discrimination of different kinds of objects and concentrate, instead, on the experiential features they must all have in common if they are to be individuated as unitary items.

What makes the conceptual mapping of a perceptual item like apple into a unitary item is the attentional pattern that consists of an unfocused moment, an unspecified sequence of focused moments, and a terminal unfocused moment. In my notation, that pattern can be represented as:

O I I ... I O

or, minimally, when a single sensory-motor signal distinguishes an item from other perceptual items:

O I O

Though this role of attention is, at present, no more than an ad hoc assumption, I shall provide arguments in the next section to show why this assumption seems reasonable and even plausible.

Abstraction of Numerical Concepts

> A number is a multitude composed of units. (Euclid, Book VII)

It is generally assumed that concepts like redness, softness, sweetness, etc. are derived by abstraction from experiential situations in which particular sensory signals recur that are associated with the respective words. This process of abstraction is essentially the same as the one I have postulated for the generation of concepts like apple. In all cases the process consists of extracting common elements from a collection of experiences. Concepts of perceptual things will be combinations of specific sensory-motor signals that are recurrently experienced conjunctively and are, in some sense, detachable from the rest of the experiential field. Concepts of perceptual properties will be derived from single or multiple sensory-motor signals that are recurrently experienced as components of perceptual things. The model I am proposing makes it possible to construct the concept of unitary item using the same process of abstraction, but applied to elements that are not of sensory-motor origin. I am suggesting that this concept of unitary item is derived by abstracting the characteristic attentional pattern that is recurrently experienced as an essential part of "things."

In the case of the four clocks in Caramuel's library, for instance, each clock could strike a different note--so there would be four different sensory signals--but they would still be considered four single, equivalent unities, because each one would be experienced with the same attentional pattern, namely O I O. The whole experience could be mapped as:

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O I O    O I O    O I O    O I O
  a        b        c        d
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where "a, b, c, d" are the different sensory signals picked out by focused moments of attention. When only the attentional pattern is considered and the sensory signals are disregarded, then each of the strokes is experienced as an instantiation of <u>one</u>; a succession of "ones" constitutes a <u>plurality</u>. If there is no initial unfocused moment and no terminal one that can serve as boundaries and provide closure, the unit patterns remain individuals connected by nothing but their contiguity in experience.

This approach at once provides the key to an ambiguity of which we are always more or less dimly aware: <u>One</u> seems to refer to two concepts. Their difference becomes apparent when <u>one</u> is opposed to <u>many</u> and then to <u>two</u>, <u>three</u>, etc. The first opposition is the same as that between singular and plural or between unity and plurality. The second is not an opposition at all, but merely the difference between one number and other numbers. Caramuel's insight that "to count and to consider several things contemporaneously are different activities" is uncannily correct. The mere repetition of the attentional pattern that creates unitary items is not counting but just establishing a plurality. In order to count, Caramuel says, "the intellect...considers different things, each distinct in itself, and intentionally unites them in thought." A plurality is, indeed, made up of different items, each a discrete unity separated from the others by two moments of unfocused attention--one, the terminal moment of the first item; the other, the initial moment of the next. In order to unite such discrete items, one has to carry out another operation that I shall call <u>attentional iteration</u>.

If the concept of unity, in the model, is constituted by the attentional pattern O I O, the operation of unitizing a plurality must produce something that corresponds to that pattern. In other words, it must be analogous to the operation that encloses a sequence of sensory-motor signals to form a unitary perceptual thing. In that operation, as suggested above, a string of different signals, each the focus of a moment of attention, is bounded by two unfocused moments. Consider now the following situation: You are in the pantry facing a shelf on which apples are stored. Your companion has asked you to get half a dozen apples but added, "Only red ones!" This common use of the word <u>one</u> is highly significant. Having established that you are, in fact, facing a plurality of complex perceptual unit items that conform to your concept of apple, you are now reducing the string of sensory-motor characteristics that enabled you to identify those unit items as apples, and you are reconstituting unit items of the minimal type, that is, items with only one moment of attention focused on the signal associated with the word red. You are actually constructing red ones within a plurality of individual items that you have already categorized as apples. This reduction can be mapped like this:

$$\begin{array}{ccccccccl} O & (I & I & I & \ldots & I) & O & & \\ & (a & b & c & \ldots & n) & & & (= \text{apple}) \\ & & & & & & & & \\ O & (O & & \overset{I}{r} & & O) & O & & (= \text{one red one}) \end{array}$$

where "a, b, c, ... n" are the sensory-motor signals that constitute appleness, whereas "r" is the signal that constitutes redness. (Note that the signal focused on to form the minimal unit items need not be a criterial signal for the classification of the perceptual objects.)

In my model, the operation that transforms a plurality into the kind of composite unity that can be considered a number is once more an analogy of an operation carried out with perceptual material. Now, however, the reduction concerns the attentional pattern rather than sensory-motor signals, but it again requires a reconsideration of what has already been constructed: The unities that composed the plurality are reprocessed, not as individual unities, but simply as an iteration of focused and unfocused moments of attention, bounded by unfocused moments that provide closure and produce a composite unity. For example, the transformation of four ones into a unity of four can be mapped as follows:

$$\begin{array}{ll} \underline{O\ I\ O} \quad \underline{O\ I\ O} \quad \underline{O\ I\ O} \quad \underline{O\ I\ O} & (= \text{plurality of four ones}) \\ O \quad \overline{(O\ I\ O\ I\ O\ I\ O\ I\ O)} \quad O & (= \text{unity of four}) \end{array}$$

I believe this process of reconsideration is analogous to the conceptual operations which Piaget, without individually specifying them, subsumes under the term reflective abstraction (e.g., Piaget & Inhelder, 1969). Moreover, this transition from a conceptually unbounded plurality to a unity composed of unities is precisely what McLellan and Dewey described by saying: "The child's own activity is conceiving a whole of parts and relating parts in a definite whole" (pp. 30-31).

This activity of "relating" has two aspects. First, it creates a composite, a "whole of parts," by instituting a relation of similarity, or even equivalence, between separate items. The items are taken to have something in common. Though the common element may ultimately be any characteristic whatever, it is in the child's development initially limited to the perceptual sphere. It may be the redness of apples, the figural aspect of toothpicks, or an acoustic feature like the striking clock. Later, on a higher level of abstraction, the common element does not have to be perceptual or sensory-motor but can be supplied by the attentional pattern that, within a given context, has made the items into discrete units, as when we say there are <u>n</u> things on the table. This aspect of <u>qualitative</u> commonality in the components that are being related to form a composite whole is, of course, the link that ties the concept of number to the concept of class.

Second, the activity of "relating" creates a composite unity by reprocessing separate unitary items, not as separate items, but only as pulses of focused and unfocused attention in a simple, alternating iteration that provides a homogeneous continuity between an initial and a terminal unfocused pulse. Both this continuity and the qualitative commonality of components are indicated in my notation by parentheses.

Once such a compound unity has been constructed, it has a definite <u>numerosity</u> because it is bounded. The fact that conceptually it has numerosity, however, does not mean that the specific numerosity has been established or that it has been counted--but now it <u>can</u> be. And because this possibility is an inherent feature, it is this conceptual structure of the compound unity that is the proper referent of the set, whereas an unbounded plurality is not. Moreover, the specification of its numerosity constitutes it as the concept of an individual <u>cardinal number</u>. We can now map how the attentional patterns that constitute the individual numbers are derived from pluralities by attentional iteration and bounding:

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O I O --------------> O (O I O) O            named one

O I O  O I O --------> O (O I O I O) O        named two

O I O  O I O  O I O -> O (O I O I O I O) O    named three

etc.
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The analysis I have presented concerns the structure of the concepts, not the way in which an individual may come to have them (cf. von Glasersfeld, 1981). Thus, for instance, awareness of the mutual interconnection of the number concepts (e.g., that one is "contained" in two, two in three, and so on), though clearly manifest in the notation, is not a necessary requirement for possession of the concepts <u>one</u>, <u>two</u>, <u>three</u>, etc. Similarly, the semantic connection between, say, the conceptual structure O(OIOIO)O and the word <u>two</u> or the numeral <u>2</u> is not a prerequisite for possessing <u>that</u> conceptual structure; and, conversely, the mere fact that a child can recite "one, two, three" or even use these words appropriately in response to certain particular perceptual situations is no guarantee that the child possesses the numerical concepts that are necessarily associated with those words in the adult.

The developmental aspects and the theoretical linkage between this view of number concepts and the various types of counting that Steffe, Richards and von Glasersfeld (1979; Note 2) have isolated in children's behavior are the subject of ongoing investigations. In the remaining sections of this paper I shall discuss what has been called the perception of number, the connections between number words and sensory-motor material.

The Question "How Many?"

> The numbers from 1 through 6 are perceptibles; others, only countables. (McCulloch, 1965)

Any attempt to find out, in a given situation, how many of something there are implies certain presuppositions:

(a) "How many?" makes sense only when dealing with unitary items;

(b) The subject must have the belief that, in the given situation, the unitary items have numerosity; that is to say, they must be conceived as a bounded plurality;

(c) The subject must have a conceptual system of numbers;

(d) The subject must also have a conventional system of number words to record or communicate the results arrived at;

(e) Even taking for granted that the conventions under (d) guarantee a fixed semantic linkage between individual number words and individual number concepts, the additional belief is needed that there is a reliable method for establishing univocal links between the number concepts and the specific numerosities of perceptual things that are being experienced.

The first three points have been discussed at some length in the preceding sections. The fourth was barely suggested in the table of the numerical progression on page 63. Conventional systems of number words and numerals are, however, the least controversial and best known part of arithmetic and mathematics, and they have been explicated innumerable times.

In the context of this paper, I want to stress only two features of our conventional system of number words. One is that a more or less extended sequence of number words can be memorized in exactly the same way as a poem or any other sequence of words. The other feature sets the sequence of number words apart from most other conventional word sequences. It resides in the fact that the number word sequence can be indefinitely extended, once one has grasped the system according to which number words are constructed.

The important point here is that neither the rote memorizing of number words nor knowledge of the system by means of which they can be constructed is in any way dependent on the construction of number concepts. That is to say, the observation that a child produces number words, either singly or in the conventional order, is no indication whatsoever that the child has acquired numerical concepts. This point is one to which I shall return.

The fifth presupposition listed above involves the problem of establishing links between number words and experiential things. That problem is far more complex than it might appear at first sight. Much has been said and written about the need to create one-to-one correspondences in every act of counting, and there is little doubt that some such correspondence is always involved. But it has become equally clear that the term <u>counting</u> has, until recently, been used indiscriminately for the processing of item sequences of very disparate kinds. There are sequences of unitary objects on a table; sequences of extended or flexed fingers; sequences of taps, pointing motions, or nods of the head; sequences of spoken or written numerals; and, last but not least, there are sequences of imaginary elements that are part of a conceptual number representation, regardless of whether that representation is thought to consist of attentional pulses or of something else. A one-to-one correspondence can be set up between any two of these sequences, and though the term <u>counting</u> has traditionally been used for all of them, it is obvious that they are very different activities and must, therefore, be distinguished as different types of counting (Steffe et al., 1979; Note 2).

Establishing one-to-one correspondences, though it may involve perceptual processes, is not a direct perception of number, but an activity aimed at the determination of a specific numerosity. Hence, I shall leave aside the various activities or operations that can be considered counting and, instead, turn to those circumstances in which numbers or numerosities appear to be perceptibles.

There are two particular ways of perceiving numerosity: First, the perceptual assessment of uncounted but physically circumscribed pluralities (called gross quantity by Piaget & Szeminska, 1941/1967); second, the perceptual recognition of visual patterns that are thought to embody a specific small numerosity (called subitizing by Kaufman et al., 1949).

Intuition of Quantity

Throughout their classic work on the genesis of number in the child, Piaget and Szeminska (1941/1967) discriminate two different ways of arriving at, or construing, quantitative judgments and, therefore, two different concepts of quantity. On the one hand, there is the concept of intensive quantity, comprising all judgments of gross quantity (quantité brute) that are derived directly from the intuitive evaluation of some sensory-motor activity. On the other hand, there is the concept of extensive quantity, comprising all judgments based on a compositional activity in which the resulting quantity is conceived as a summation of more or less constant parts. The Genevans hypothesized these two processes and conjectured their developmental interaction because it enabled them to construct a coherent model for the interpretation of their observations. It was years later that research in perception hit upon the idea that in the vertebrate visual system there are, indeed, two physiologically different and functionally relatively independent processes of perception. They were then described in these terms:

> On the one hand, we all acknowledged the important capacity to recognize and to distinguish among objects on the basis of their shapes and motions. But, on the other hand, we each had our own reasons for believing that the ability to orient to these objects, or otherwise relate movement of the body to their loci in space, was an independent capacity. (Held et al., 1967, p. 42)

Trevarthen (1967) calls the two functions focal and ambient and says of the first that it consists "of resolving detail of form subtending fractions of a minute of arc and is sensitive to the very slightest difference in position, orientation, luminance or hue" (p. 328). This mode of functioning clearly fits the perceptual component in the phenomenon of subitizing. Of the other function, Trevarthen says that it continuously maps the behavioral space around the body, is "driven by locomotion or turning the head," and that "angular velocities ranging from about 1° per second to about 100 times this are measured and compared in this process" (p. 328). Thus, ambient vision apprehends distance, size, and relative proportions of objects and derives quantitative judgments about them from the movements the organism carries out in the process of perceiving, not by unitizing the perceived object. This mode of visual functioning admirably fits Piaget's conception of gross quantity, and since it involves proprioceptive rather than perceptual signals, it is not surprising that its results have often been called intuitive.[6]

One further step has to be taken. There are occasions when one makes a judgment about gross quantity that does not seem derivable from motor signals. Let us say you walk down a long corridor, look out through a window at the beginning, and see some trees (plurality). Then, way down the corridor through another window, you again see trees. You have not counted the trees either time nor in any way estimated their numerosity. Yet, you would be willing and able to judge, within certain limits of accuracy, whether you saw more, fewer, or roughly the same number of trees through one window as through the other. The actual signals (from locomotion and scanning) are exactly the same both times. Your judgment must therefore be based on something else. I propose that it is based on awareness of the appropriate frequency of attentional unit-patterns executed in the visual context of each window. The comparison that has to be made to decide the question of more or fewer trees does not concern numbers but rather the intuitive assessment of the two frequencies. The awareness of the frequencies of unitary items constructed is analogous to an awareness of muscular effort in perceptual activity, except that it is now attentional effort.

This addition provides the possibility of conceptually distinguishing gross numerosity from gross continuous quantity, since the first can now be derived from the attentional construction of plurality; the second, from the motor activity involved in a perceptual process. Hence, the notion of gross numerosity should be interpreted as a supplement to, not a contradiction of, Piaget's gross quantity.[7]

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[6]Particularly convincing confirmation of the assumption of different visual processes comes from the work on visual adaptation by Mikaelian and Held (1964) and Malatesta and Mikaelian (Note 1).

[7]The concept of gross numerosity is also useful in the interpretation of certain intermediary phenomena in the child's transition from judgments of gross quantity to the conservation of number in linear arrays of varied extension.

Perception of Small Lots

In the paper in which Kaufman et al. (1949) coined the term subitizing for the spontaneous attribution of number words to arrangements of up to six stimulus dots, they contrasted the new term with estimating, which they reserve for the quantitative assessment, but not counting, of more numerous arrangements. They introduced the distinction because results of their own experiments, as well as of earlier ones, showed a sharp deterioration of subjects' speed, accuracy, and confidence of response when faced with more than six items. This phenomenon led the authors to postulate two different mechanisms for the visual discrimination of numerousness. They stress that this postulate was based exclusively on their functional findings and they add, somewhat regretfully, that while

> the duplexity theory of vision presents an example of a neat relation between functional and anatomical findings, there is no such relation in numerousness; we do not know of separate organs or pathways for subitizing on the one hand and estimating on the other. (p. 523)

The reason that subitizing and estimating were not associated with different visual processes by these authors--or by others who later investigated the same phenomena--is, I would argue, that number was always considered a property of the stimulus and its perception an event in which the perceiver played a passive/receptive rather than an active/constructive role.

If, instead, one assumes a constructivist point of view, two different visual processes can immediately be linked to the two different kinds of numerical response. I have elaborated the connection between ambient vision and the intuitive estimation of gross numerosity in the last section. It remains to be shown that there is also a connection between focal vision and subitizing. In order to do that, we shall have to examine more closely what actually goes on in the situations in which subitizing has been observed.

Beckwith and Restle (1966) surveyed the older literature on what they call "the immediate apprehension of number" (p. 438). The studies they reviewed range from 1897 to 1961. Nearly all used arrangements of dots as stimuli and found that subjects' responses reliably indicated that arrangements of more than five or six dots were processed differently than smaller numbers of dots. Subsequent to that review, Schaeffer, Eggleston and Scott (1974) speak of "pattern recognition of small numbers" (p. 358); and Gelman and Gallistel (1978), who agree that "young children accurately abstract the numerosity of small sets but rapidly lose accuracy as set size becomes greater than four or five" (p. 67), believe that there is "evidence that number representations are first obtained by counting rather than by subitizing" (p. 69). Yet, two pages later, these last authors concede that "perhaps at some stage children do subitize numerosity without being able to count" (pp. 71-72).

There are, then, a good many investigators who have observed something that seems to fit their various definitions of subitizing. All of these definitions can, in fact, be reduced to the one given at the beginning of this section, that is, the spontaneous attribution of a number word to a small lot of perceptual items. This formulation has the advantage that it involves neither the concept of number nor that of numerosity. It refers only to the association of number words and specific perceptual items, and that, I believe, is all that could actually be inferred from the experiments in question. I am, of course, not suggesting that the adult subjects of Kaufman et al. or Beckwith and Restle did not have concepts of numbers and numerosity; but I am suggesting that these concepts were not required by the tasks set in the experiments and could, therefore, not be inferred from the subjects' performance. Where adults are concerned, however, that may be considered irrelevant. They admittedly possess numerical concepts, and the question of whether or not they use them in subitizing is at best of minor theoretical interest. But since subitizing has been ascribed to young children and is persistently linked with numerical concepts and not merely with number words, a further analysis may, indeed, prove worthwhile.

Number words, like other words, can be learned qua vocal products long before the concepts have been formed that will later be associated with them as their meaning. Brownell (1928), for instance, referring to children of preschool age, places knowledge of some numerals at "the earliest beginnings of number knowledge" (p. 1), and Piaget and Szeminska (1941/1967) stress that "the verbal numeration which the social milieu at times imposes on the child at the earliest stage (i.e., any time before the age of 4-5 years) remains entirely verbal and without operational significance" (p. 48). This observation has been corroborated by many more recent studies (e.g., Ginsburg, 1977; Pollio & Whitacre, 1970; and Potter & Levy, 1968).

By the time children learn to say the first few number words, they have long since acquired relatively fixed habits concerning the segmentation of their perceptual fields. There is massive evidence from many different investigations that children learn to discriminate and retain certain visual shapes early during their first year (e.g., Bower, 1966; Fantz, 1961/1972; Piaget, 1937; Yendovitskaya, Zinchenko & Ruzskaya, 1974). Children can construct and retain unitary items; they recognize recurrent visual patterns and have at least summary representations of the figural composites, as well as of a great many "permanent objects" (Piaget, 1937); and they can complete or reconstruct whole figures from partial perceptual material. They are also rapidly expanding their linguistic ability and are constantly forming new semantic associations between thing-representations and word-representations.

A three-year-old whose parents have provided a set of wooden or plastic block capitals and numerals and have occasionally used the names of these objects--e.g., "Give me the A," "There is the three," "That's the five, not the S"--will have formed quite stable semantic links between some of the number words and the figural representations of the corresponding numerals. These figural representations of the numerals will thus be linked to the acoustic representations of the words one, two, three, etc., in exactly the same way as figural representations of the words spoon, ball, and pineapple. That is to say, there is as yet no abstract number concept nor anything properly numerical associated with the number words.

Similarly, a child who has been given dominoes or dice to play with will soon come to associate particular figural dot patterns with the words three, four, five, etc., without any conception of number or numerosity. One could say that in these cases the numeral is "apprehended," "recognized," and "perceived" as a sensory-motor object and not at all as a symbol that means a numerical concept. That point can be substantiated by the fact that the recognition of numeral patterns as objects rather than as symbols is by no means restricted to children. Bridge and poker players immediately recognize playing cards as twos, threes, sevens, etc. (see Figure 3); but in spite of the number words associated with them, the patterns on the cards do not represent numerosities in the context of bridge or poker. They function as ordinals, and therefore a two and a three cannot be added to form a five, nor would it make sense to say that a seven is more than two threes.

This recognition of playing cards is also a strong argument against the assumption that subitizing means "to abstract the numerosity of small sets" (Gelman & Gallistel, 1978, p. 67) or involves an "operator used to quantify small collections" (Klahr & Wallace, 1973, p. 304). Klahr and Wallace (1976) have presented a revised version of their "information processing" analysis of subitizing, but their basic position does not seem changed: Unities, pluralities, and numerosities are still considered "information" that comes ready-made from the stimulus into the perceiving organism, whose "processing" consists of coding and recoding it. Numerosities or quantification have nothing to do with the

Figure 3

An experienced card player immediately recognizes the partially exposed cards and can say their names, which, of course, are number words. The card player sees parts of five diamond designs and says, "seven"; sees two club designs and says, "three"; and for the partially visible five spade designs says, "ten." The player certainly does not perceive the numerosities associated with the number words uttered.

association of a specific number word with a specific figural pattern, and, what is more, as Figure 3 demonstrates, the perceptual stimulus does not even need to be an instantiation of the numerosity associated with the number word attributed to it. In other words, we can recognize a visual pattern as a "ten" in spite of the fact that, if we count what we see, we cannot get ten as the result. Thus Schaeffer, Eggleston and Scott (1974) come much closer to the mark when they suggest that children between 2 and 3-1/2 years

> can probably learn to recognize by sight the patterns formed by the small numbers. An array of one is a dot, an array of two forms a straight line, an array of three usually forms a triangle, and an array of four usually forms a quadrilateral. (p. 371)

Even children who do not play with cards, dice, or dominoes have ample opportunity to associate the first few number words with perceptual patterns and hence with figural representations. Some of these associations will turn out to be more useful than others and will, therefore, be strengthened and become relatively permanent as the basis of adult subitizing. To my knowledge, Brownell (1928) did the only study that was extensive enough to throw some light on the question of which patterns are most frequently associated with a particular number word. But Brownell believed that numerosity is inherent in stimulus patterns and he merely wanted to find out whether or not the "apprehension of visual concrete number" (p. 3) becomes more difficult as the numerosity increases. Hence he deliberately mixed patterns and even criticized Howell (1914) for having used only one type of "number picture." Both Howell and Brownell, however, provide some confirmation that the patterns mentioned in the passage quoted above from Schaeffer et al. are used more often than others.

An investigation to establish which patterns are most frequently associated with a particular number word would, however, have only an indirect connection with the study of numerical concepts and their development. This indirect connection arises in that a subject who already possesses a concept of the numerosity designated by <u>four</u> can, and does, attribute that numerosity to any figural pattern that becomes associated with the number word <u>four</u>. But, to reiterate, such attribution of numerosity takes place after the visual signals have been unitized to form the pattern associated with the number word, and it is not the perceptual process that constitutes the numerosity.

The observations and experiments reported in the literature on subitizing do not justify the assumption implied in the stated definitions of the phenomenon. Subitizing is most appropriately defined as the attribution of numerals to small lots, because the phenomenon neither requires nor provides a conception of numerosity in the subitizer. The subitizer may, of course, interpret the subitized result numerically on the strength of specific numerosities otherwise associated with the number words, but that interpretation is not part of the subitizing activity. This view is substantiated by the fact that behavioral phenomena very similar to subitizing have been observed in nonhuman primates and can be explained on the basis of perceptual and associational mechanisms (e.g., Ferster, 1964; Rohles & Devine, 1966, 1967).

How concepts of numerosity are acquired by the child is a question beyond the scope of this paper. The various activities that go under the name of counting, the differences among them, and the elements they have in common will undoubtedly be among the main building blocks in a theory of that development. There is, however, one other area that could yield valuable insights, though developmentally it precedes the formation of numerical concepts by as much as several years, namely, the sensory-motor coordinations that the infant constructs during the first efforts to organize spontaneous motor acts. An infant playing with blocks, for instance, may come to discriminate a one-block percept from a two-block percept by the sensory-motor difference that the one requires one hand, the other both hands, whenever the goal is to displace the blocks or to pick them up. Analogous differences in the required motor activity would crop up and could be registered in the case of three-block and four-block percepts, once repetition of movements, especially rhythmical movements, have become part of the motoric repertoire. One hand, one block; two hands, two blocks; two hands twice, four blocks; and so on, constitutes a sensory-motor precedent of one-to-one coordination that could well serve as a starting point for the development, via pointing and head nodding, of the fully abstracted coordinations necessary for later numerical operations (cf. von Glasersfeld, 1981).

Summary

After a cursory justification of the constructivist approach, three points concerning the epistemological status of theoretical models of cognitive development were outlined. First, while conceptual structures are always hypothetical, a double hypothesis is involved whenever an observer attributes conceptual structures to others; for whatever others do or say is inevitably interpreted in terms of constructs derived from, and applied to, the observer's own experience. Second, explanatory models concerning the unobservable internal

functioning of living organisms should never be understood as descriptions of what "really" goes on in the organisms, but merely as hypothetical mechanisms that would yield similar results under similar circumstances. As such, the cognitivist's models have the same theoretical status as the conceptual models of physics. Third, when constructing models of cognitive development, it must be remembered that the child may do all sorts of things in his or her mind that are no longer conceivable to the adult. This, of course, will not, and should not, stop anyone from modeling the child's conceptual activities--but before embarking on that task, we must at least have some fairly well-defined model of our own concepts.

An examination of the concepts of number, unity, and numerosity shows that these concepts cannot be satisfactorily derived from perceptual material. Instead, I propose that they are the result of an active attentional pulse analogous to, but not identical with, the well-established theory that the perception of shapes and patterns is the result of the perceiver's actions in the perceptual process rather than of properties of the sensory signals.

With regard to the perception of number, the fact is emphasized that number words and their systematic production can be learned without any involvement of numerical concepts. Two phenomena are then discussed that have frequently obscured the relative independence of number words, numerals, numerical concepts, and perception. One is the intuitive apprehension of quantity, the other the activity called subitizing.

I argue that, while intuitive judgments of quantity of the continuous type are based on an evaluation of perceptual work, as Piaget has maintained, intuitive judgments of gross numerosity (i.e., a quantity consisting of unities) are based on an evaluation of the attentional effort.

Subitizing is more appropriately characterized as an associative attribution of number words to figural patterns than as an activity that involves numerical concepts. The child who responds "three" to a stimulus of dots, apples, or cookies provides evidence of an association between that number word and a figural pattern, but not a concept of numerosity. In an older subject, the numerosity is associated with the number word or numeral and is attributed to a percept, once that percept has been recognized as one of the patterns associated with the word <u>three</u>.

In conclusion, I would add that in this paper I have persistently (and, I hope, consistently) argued for what I believe to be a fundamental paradigm in this branch of research: Any assumption that unitary items or numbers "exist" prior to, or independent of, the experiencer's activity is simply a way of burying the question of how the experiencer might come to have such concepts.

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Reaction

to

THE CONCEPTION AND PERCEPTION OF NUMBER

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Professor von Glasersfeld's application of Ceccato's attentional model to numerical concepts surely qualifies as a landmark development in furthering our understanding of children's acquisition of early number concepts. His contribution will not be diminished in the least by noting certain points that need to be clarified and certain aspects that are not addressed by his model.

Because the radical constructivist philosophy is so fundamental to Professor von Glasersfeld's perspective on learning and, consequently, intrinsic to his model, it is only fair that I try to identify my own epistemological philosophy in order to put my remarks into their proper context. "Radical eclectic" is about as close as I can get. That is, in the contrast between constructivism and realism, I am torn by two competing phenomena: On the one hand, the difficulty that human beings have in communicating on any but the most superficial level provides clear evidence that individuals do indeed construct their own concepts their own subjective reality; on the other hand, it has to be more than coincidence that almost everybody behaves much of the time as though they were operating with remarkably similar concepts and principles, ideas that seem to reflect some objective reality.

This realist-constructivist kind of binocular perspective can induce parallax when viewing things at close range. The comments that follow should be interpreted with that caveat in mind.

The Conception of Number

Children construct a concept of (cardinal) number in several distinct steps, as Professor von Glasersfeld points out. First, they seem to notice the distinction between one and more than one, probably before they have the linguistic capacity to label either case. Next, they begin to associate number names with small sets, just as they associate color names with objects. This ability to subitize numbers is initially limited to numbers less than six and seems to appear in the order two, one, three, four, five. As a third step in constructing number, children learn to count and thereby acquire a procedure for determining a number name to associate with a set whose cardinality is beyond the subitizing range. Finally, they develop the notion of gross quantity, an estimation that may be based on a variety of factors, such as density, area, time, or loudness. In this paper Professor von Glasersfeld applies the attentional model to only the first two of these steps in detail. He discusses the construction of the general concepts of unity and plurality, and he considers the construction of the specific number concepts one, two, three, ...

He begins by comparing the concept of number to the concept of apple in order to highlight the difficulty (i.e., impossibility) of illustrating concepts that are properties of sets of objects (number) in the way that concepts of objects themselves (apple) can be illustrated. His example of apple is helpful later in describing the concepts of unity and plurality as seen from the perspective of the attentional model. However, stretching the analogy to the point of equivalence classes is not particularly helpful, because children develop a concept of number by first understanding equivalence classes (one, two, three, ...), whereas they refine their concept of apple by later identifying equivalence classes (Pippins, Winesap, Golden Delicious, ...).

It might have been instructive to consider the concept of color at some point. As a property of objects, color is somewhere intermediate between apple and number. Moreover, children begin to develop the concepts of color and number at about the same time and via the same equivalence class route (red, blue, green, ...; one, two, three, ...). That is, they develop a concept of color by learning to associate specific color names with appropriate objects; they develop a concept of number, at least in part, by learning to associate specific number names with appropriate (small) collections of objects, apparently long before they learn to count.

The Attentional Model

The example of Caramuel's striking clocks is absolutely enchanting--what a perfect illustration of the quintessence of number! How, indeed, do children learn to identify separate unitary items and then reunite them into sets according to the conventions of their culture? How is it that everyone who reads the Caramuel story sees the same anomaly? Admittedly, these questions reflect a realist's view of the world, as do my questions about the attentional model.

Separating. Professor von Glasersfeld identifies separating and uniting as the crucial activities in creating countable sets of items. He partially motivates his attentional model by claiming that the operation of separating, which creates unitary items, is completely independent of sensory or proprioceptive signals. He supports his contention by citing the example of a homogeneous line segment which, he claims, can be regarded as two halves, three thirds, four quarters, etc. I would say, yes, a person with very sophisticated notions of number can "see" the line segment as two halves, in the sense that one whole is equivalent to two halves, but, in this regard, the line segment is incidental to the abstract number property 1 = 2/2. That is, a person who already knows this number property can think of anything--any whole--as two halves in this same sense.

Furthermore, the act of separating a whole into two countable halves is not an example of the kind of separating into unit items that is pertinent to an early concept of number. Mentally dividing a whole into an arbitrary number of pieces and then reuniting them into a countable plurality reduces the whole to a "set" with which any cardinal number at all could be associated. The important question is how children construct a concept of number that enables them to associate the same (unique) cardinal number with a given set, as everyone else does.

Since early number concepts are developed in situations that involve sensory signals, proprioceptive signals, passage of time, and the like, it is extremely difficult to argue that the act of separating into unit items is independent of these influences. Moreover, it does not seem particularly important, except perhaps to the constructivist perspective, to try to make this argument. On the contrary, the possibility of dialectical interaction between the child's mental constructs and the sensory signals derived from a "real world" may help explain the two remaining points I would like to raise relative to the model.

First, what is a "moment of attention"? Professor von Glasersfeld leaves the word <u>moment</u> undefined. This omission would not be so disquieting were it not for the fact that the term <u>moment</u> carries a connotation of elapsed time. This connotation interferes with the model in two respects: (a) in the case of subitizing, it is not clear that there are unfocused "moments" of attention to separate the items; (b) in most other circumstances, the element of time is critical and must be considered explicitly. Using the word <u>moment</u> to categorize intervals of time that vary widely across, and within, situations is either too simplistic or else so distracting in its connotation that it subverts the intent of the model. Deleting the word <u>moment</u> and referring simply to <u>focus of attention</u> might permit essentially the same description without confounding the time factor.

<u>Uniting</u>. Except for the usage of the word <u>moment</u>, I think the attentional model describes rather nicely the operation of separating sensory unit items. However, the

operation of uniting may actually be the more critical activity in developing the concept of number. Certainly in the case of the clock "striking one" four times, it was the uniting operation that went awry. How do children learn to put boundaries on sets in conventional ways?

If the operation of uniting is a process of attentional iteration, as Professor von Glasersfeld suggests, then uniting unitary items into a set is essentially the process of separating sets, and it follows that there is, in a sense, only one operation, not two.

Regardless of whether uniting is considered to be uniting objects or separating sets, the model seems less adequate for describing this process than it does for describing the separation of sensory unit items. Its weakness in describing uniting stems largely from its failure to account for two critical factors: time and the novelty of situations. Ignoring the time factor in separating objects can be excused on the basis that the passage of time contributes to the separating process; ignoring the time factor in uniting objects cannot be readily excused because the passage of time interferes with the uniting process, as the example of the clock so "strikingly" illustrates. Accounting for the novelty of situations is also important: Children's ability to identify conventionally defined sets that are continually novel is as remarkable as their ability to produce grammatical sentences that are continually novel.

The Perception of Number

One aspect of Professor von Glasersfeld's paper that needs to be clarified is the discussion of subitizing. For example, does subitizing depend upon familiar figural patterns? The examples of playing cards, dice, and dominoes seem to suggest that it does, but observations of very young children's ability to associate number names with arbitrary small sets of objects, apparently without counting, seem to suggest that subitizing does not depend on familiar patterns. Also, is the identification of a familiar pattern, upon seeing only a portion of it, the same process as subitizing? Again, the playing card example suggests that it is, whereas an analysis of the factors involved suggests that pattern identification is not the same process as subitizing. Finally, is it subitizing that provides the basis for the beginning abstraction of numerical concepts? The description of number concepts in terms of the attentional model seems to refer to subitizing, as suggested by the logical development from unity (apple) to plurality (red ones) to cardinal number (four). In the end, however, the relationship between the conception of number, as described by the model, and the perception of number, in the case of subitizing, is left for the reader to infer.

Now, I realize that many of my comments and questions relate to "how" children develop the concept of number, and Professor von Glasersfeld disclaims any intention of addressing that issue. Nevertheless, it is the "how" questions that are ultimately of most interest to mathematics educators. Perhaps the new perspective provided by the attentional model will eventually facilitate the formulation of answers to these questions.

COGNITIVE MICROANALYSIS: AN APPROACH TO ANALYZING INTUITIVE MATHEMATICAL REASONING PROCESSES

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This paper is divided into five main sections: an introduction to the methodology of cognitive microanalysis, observations from the protocol of a third-grader's word problem solution, a detailed model of the subject's cognitive processes, a revised model, and conclusions and implications. A point of departure for the study is Piaget's theory of cognitive functioning based on action-oriented schemes, assimilation, accommodation, disequilibrium, and symbolization processes. Because of the magnitude of the task of constructing a general theory of intellectual development, Piaget tended to focus his attention on long-term developmental processes rather than on the details of mental functioning during problem solving in everyday contexts. The present paper represents an attempt to narrow this gap by illustrating an approach to the study of cognition which I will call cognitive microanalysis. A major purpose of the paper is to assemble an adequate set of concepts as well as a diagramming system for describing certain intuitive reasoning processes. To do this it will be necessary to refine the meaning of the Piagetian concepts mentioned above and to draw on other theoretical concepts as well, such as hill-climbing, recursion, and internalized actions on images. The analysis will be restricted to a single protocol in order to give as detailed an example as possible. Additional protocol data which are analyzed using the same theoretical approach are described in Clement (1977).

The protocol analyzed here is that of an eight-year-old student working on a division word problem about sharing some objects. The analysis models the child's reasoning in terms of action-oriented cognitive structures that remain active in parallel over a period of time. Reasoning processes in this model do not take the form of manipulation of internal

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statements according to the rules of a formal logic. Reasoning takes place when schemes coordinate to form action sequences that were not specified by a predetermined procedure. A method of diagramming is used that allows the tracking of such processes as they occur over time. The analysis is therefore an extension of Piaget's attempt to provide a theory of thinking based primarily on the coordination of actions and only secondarily on the manipulation of verbal symbols. However, the concepts used here are related to concepts found in both Piagetian and information processing theories of cognition. Thus the paper also suggests a framework within which these theories might fruitfully interact.

Still another purpose of this paper is to develop models of intuitive reasoning processes that are grounded in clinical observations of behavior, rather than in a prior analysis of the subject area. It is assumed here that the logical exposition of a certain area of mathematics is not necessarily identical in form to the knowledge structures children can develop most easily. If the teacher's role is to facilitate a process of knowledge construction that takes into account the ideas children bring to school, then it becomes important for educators to know something about the intuitive conceptions children construct. Children's intuitive understanding may be concrete, practical, or inconsistent, where the discipline is abstract, logical, and consistent. Constructivists assume that certain ideas children construct may never have been identified before. This assumption stems from a recognition of each individual child's creative potential and from the Piagetian position that children construct ideas partially on their own. The exploratory clinical interview is then a search for the authentic ideas of the child, whether or not those ideas fit into the mold of standard mathematics. The present analysis will identify practical, action-oriented conceptions used by the subject and will raise the question of whether such intuitive conceptions might be tapped as starting points for building mathematical ideas in the classroom.

Methodology of Cognitive Microanalysis

Methodology and diagramming techniques related to cognitive microanalysis have been discussed by Clement (1977, 1979), Driver (1973), Easley (1974), Knifong (1971), and Witz and Easley (1978). A related, but somewhat different, approach to protocol analysis is described by Newell and Simon (1972).

Characteristics of cognitive microanalysis. Cognitive microanalysis is marked by several characteristics:

(a) The basic experimental tool is the taped clinical interview in which subjects are encouraged to think aloud while solving a problem, giving an explanation, or playing spontaneously;

(b) The investigator avoids specifying predefined response categories and avoids experimental situations which greatly restrict the range of possible responses; the subject is encouraged to give creative and natural responses in relatively unstructured problem-solving interviews; observations of unorthodox responses are valued as clues to the structure of intuitive conceptions;

(c) The investigator avoids making prior assumptions about the form or functioning of a child's cognitive structures; instead, the investigator attempts to construct a model of structures during intensive observation of the child's spontaneous behavior;

(d) The investigator strives in this way to map out conceptions as they exist in the child rather than to test the degree to which the child's conceptions conform to those of an adult.

Many scientific theories attain a significant part of their explanatory power from the use of visualizable models such as molecules, waves, fluids in circuits, etc. (Hesse, 1966). In cognitive microanalysis, diagrams are an important tool for representing visualizable models of cognitive processes. Diagrams will be used in this paper to:

(a) Model the cognitive structures used by the child;

(b) Model the child's reasoning by mapping the interaction of cognitive structures during the interview;

(c) Exhibit explicit ties between theorized cognitive structures and the protocol observations they account for.

The protocol analysis comprising the main body of this paper is divided into two sections: a section describing observations derived from the protocol and a section describing a model of cognitive processes which can account for these observations. These two separate sections reflect another important characteristic of the method, the attempt to separate observations from theory as clearly as possible, i.e., the attempt to separate descriptions of external behavior from models of internal cognitive events.

Observations from a Problem-Solving Protocol

This protocol is from an eight-year-old student (referred to here as David) who solved a word problem about sharing some objects. At the time of the interview David was in the third month of third grade and lived in a working class district of a middle-sized town in midwestern United States. His teacher characterized his general level of mathematical performance as well above average. He had not yet studied multiplication or division in school.

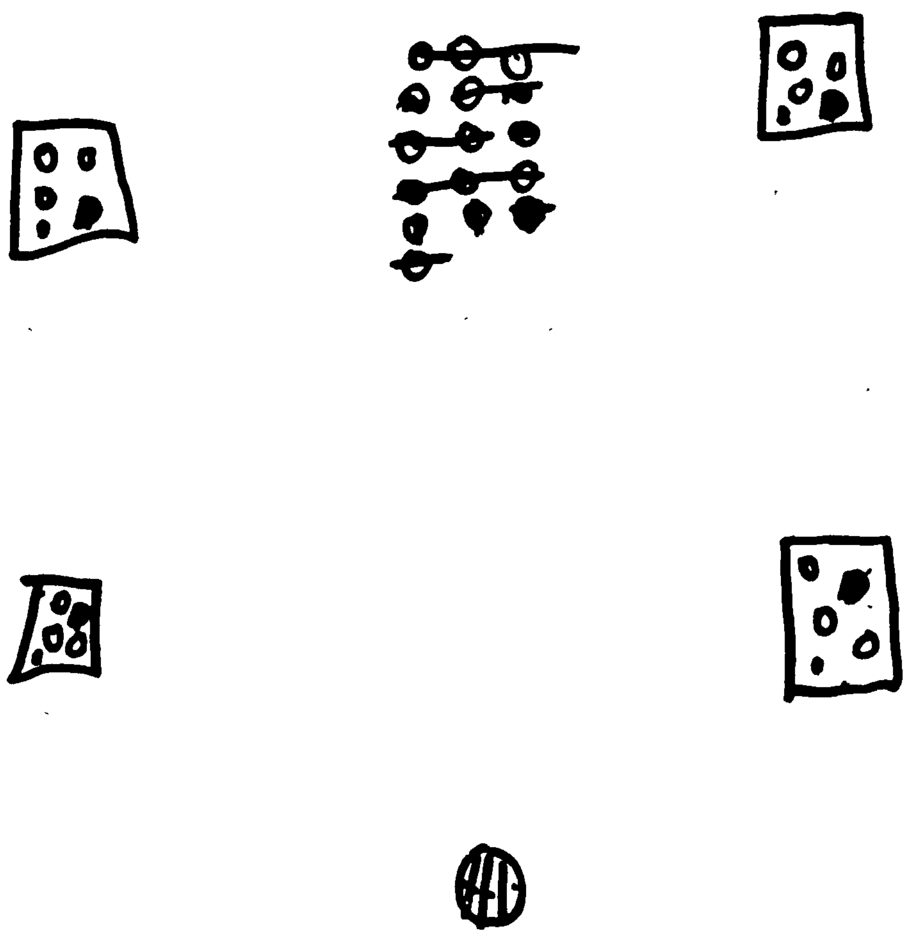


Figure 1. David's completed drawing.

David's Protocol. The subject's completed drawing is shown in Figure 1. Arrows indicate points in the transcript corresponding to stages in the drawing.

Section A

1. David: (Reads the problem) "Jim and his 4 friends found a green paper bag about 2 feet from a rabbit hole.

2. Inside they found 15 green stones.

3. They want to share them equally.

4. How many green stones should each one get?"

5. Oh no --

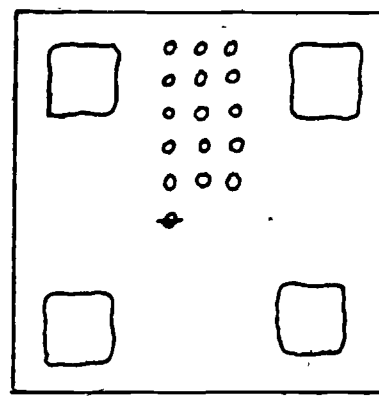
6. Investigator: Tough?

7. D: Uh-huh

8. I: How can we start on it?

9. D: 15 green stones--(draws 15 circles in rows of 3, and a 16th, recounts them and crosses out the 16th).

10. OK, now we want to divide it by 4.



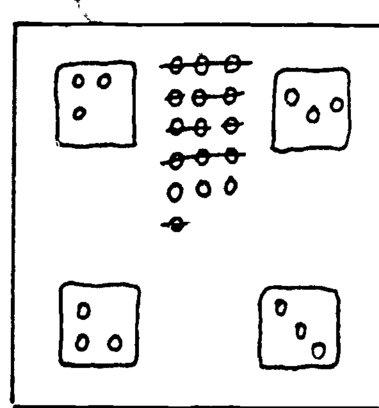
11. I: What does that mean?

12. D: Here's one sack--little can (draws a square), another, another, another (draws 3 more squares).

13. OK, one for each--1,2,3,4, (draws small circle in each square).

14. OK, 4 are gone (crosses off 4 circles in center group).

15. Now--we divide 4 more--1, oops, 2,3,4 (draws small circle in each square).



16. (Crosses off 4 more circles in center).

17. Now we divide this by 4 more (adds circle to each square).

18. Everybody's got 3.

19. (Crosses off 4 circles in center).

Section B

20. D: And there's 3 more!
[concerned tone]

21. I: What's wrong?

22. D: Cut one in half, put it in here and here (draws a circle in 2 of the 4 squares).

23. I: And this is another half?

24. D: Cut this in half [referring to second circle in fifth row of central group] and here, and here (draws a circle in each of the remaining 2 squares).

25. I: Now, what are those you just put here [the last piece of stone put in each box], are these whole stones?

26. D: Half--half stones.

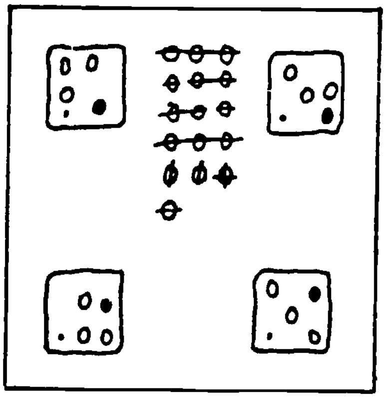
27. I: Let's blacken those in so we know they're halves.

28. Are there any more?

29. D: (Blackens half circles in the 4 squares).

30. (Draws vertical lines through 2 of the 3 circles remaining uncrossed in center).

31. There's just one more.



32. So we'll put little chunks of that one in each box (puts a dot in each square, puts two crossed lines on last circle in center).

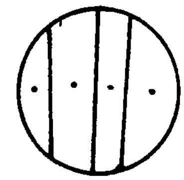
Section C

33. I: OK, how big are the little chunks?

34. D: Little--like chunk, chunk, chunk.

35. I: Could you draw that last stone down at the bottom--make a big --great big thing for the last stone--show me how you--

36. D: 1,--2,3,--1,2,3,--4 chunks--
divided (draws a large circle,
divides it into 4 parts with
vertical lines, puts a dot in
each part).



37. I: What can we call those chunks?

38. D: I don't know.

39. I: A half of a stone?

40. D: Uh-huh--half of half of half of a stone.

41. I: A half of a half of a stone? What does that mean?

42. D: I don't know. Half of a half of a half of a stone.

43. I: Half of a half of a half of a stone--is that what they get?

44. D: I don't know.

45. I: Is there any way to write what you did with numbers?

46. D: I don't know.

47. I: That was a rough one, huh?

48. D: Yeah, I think I needed bigger cans.

David's solution was precocious in the sense that he solved a story problem ordinarily thought of as a division problem even though he had not had multiplication or division in school. David's intuitive solution illustrates an important finding: In solving story problems, students do not always formulate an arithmetic problem to be solved. David, for example, seemed to "act out" the solution instead.

The problem David was given contains some extra information, stating that the stones to be shared were found "about 2 feet away from a rabbit hole." David successfully ignored this information, but, as might be expected, he interpreted the text as designating a total of 4 people sharing the stones instead of the 5 people described in the problem. It is not clear why he used 4 people, but one explanation could be that David may have been in the habit of always using the printed numerals as they appear in story problems in school. This demonstrates that a story problem cannot be assumed to be a neutral, standard stimulus for all subjects. The subject's perception of the problem will depend on the form of the structures in the subject that assimilate the problem. Analysis can proceed, however, on the assumption that David was solving a problem involving four people.

Overview of the protocol. In Section A of the transcript, David read the problem and immediately drew a group of 15 circles, then saw his error and crossed out the last one. This group of circles will be referred to as the source group. He then drew 4 squares which he called "sacks" or "cans" and transferred (by drawing) 12 of the circles from the source group to the squares. He did this in lots of 4 circles, drawing one in each square and then crossing off 4 circles in the source group before distributing the next 4 circles.

In Section B David distributed the 3 circles that remained. He cut 2 of the circles in half and distributed a half stone to each square. For the single remaining stone he said, "We'll put little chunks of that one in each box." Some children would have been content to leave 3 objects as an unused portion or let one person go short by one, but David found a more interesting solution. Thus he shifted spontaneously to a new method when the initial method of repeatedly giving one stone to each became inapplicable.

In Section C the interviewer probed for a more detailed description of the "little chunks" from the last stone. David was uncertain about their size but said they could be called a "half of a half of a half of a stone."

The protocol raises a number of interesting theoretical questions, such as: If David was not using an arithmetic operation, what method was he using? How should his concept of sharing be modeled? His concept of cutting in half? His concept of cutting in chunks? What kind of mental reasoning process tied these concepts together? Did he use heuristics? It does not occur to many children to cut the stones; what triggered this idea in David? How can his reasoning, "half of a half of a half of a stone," be modeled? The analysis which deals with these questions begins with some general protocol observations.

Observation 1. David acted out the problem situation relatively explicitly. Had he made a more realistic drawing, or found some real stones to use, we would say that he was even more explicit. Conversely, if he had mentioned only numbers and number operations we would say that he had not explicitly acted out the situation.

Observation 2. He did not refer to any arithmetic problems. A possible exception appears in line 10, "OK, now we want to divide it by 4." However, it appears from the transcript that the antecedent of "it" was not a number but the group of stones. Thus, that statement probably was not an expression of an arithmetic problem. David may have been trying to make his comments "sound mathematical" by using the word divide but there is no evidence that he was thinking about dividing one number by another number.

Observation 3. David constructed a drawing as part of his solution. In this case, it was a "skeleton" drawing, with only selected aspects of the story represented.

Observation 4. David changed the drawing as he solved the problem. He verbally related aspects of the story to different parts of his drawing and to changes he made in the drawing.

Observation 5. Several sections of transcript can be identified which show that David repeatedly referred to or acted on several distinct groups of objects in his drawing. These include Groups 1-6 shown in Figure 2. Groups 1,2,3,4 and 5 are drawn with their members spatially contiguous. Group 1's members are drawn sequentially in rows. Groups 2,3,4, and 5 were referred to when David said, "Everybody's got 3." Groups 1 and 6 were also referred to verbally. (This observation provides evidence that David attended mentally to various specific groups of objects at specific times.)

Observation 6. David referred to the squares differently at different times during the interview, calling them sacks, cans, and boxes and apparently associating them with people in the statement, "Everybody's got 3." This behavior indicates that the abstract figures in his drawing are flexible to a certain extent as symbols for imaginal variations on the story.

Observation 7. In several places, David described actions he was about to perform before he manipulated (made a change in) the drawing. These include, for example, line 13, "OK, one for each (draws small circle in each square)"; line 32, "So we'll put little chunks of that one in each box (puts a dot in each square)"; as well as lines 10, 15, 17, 22 and 24. (This phenomenon will be interpreted as anticipations which occurred internally before he represented them on the drawing.)

Observation 8. David exhibited several repeated behavior patterns. (a) In Section A, David drew a circle in each of the 4 boxes and then crossed off 4 circles in the center group. This behavior pattern was repeated 3 times. (b) When 3 circles were left in the central group, David referred to cutting one in half, and put a circle in 2 squares. This pattern was repeated once more and each time he crossed off only 2 circles in the central group. (c) There was a more general behavior pattern of repeatedly transferring identical objects to each of the 4 squares. The 4 squares were each assigned one circle, then 2, then 3, then 3-1/2, and finally 3-1/2 and a "little chunk." These actions together form another behavior pattern that was executed 5 times--each time David drew an identical object in each of the 4 squares and then crossed off one or more circles in the central group. This last behavior pattern is shown more explicitly in Table I

It should be noted that these behavior categories were not defined before the interview. They were formulated from the child's behavior by the analyst as he viewed the tape.

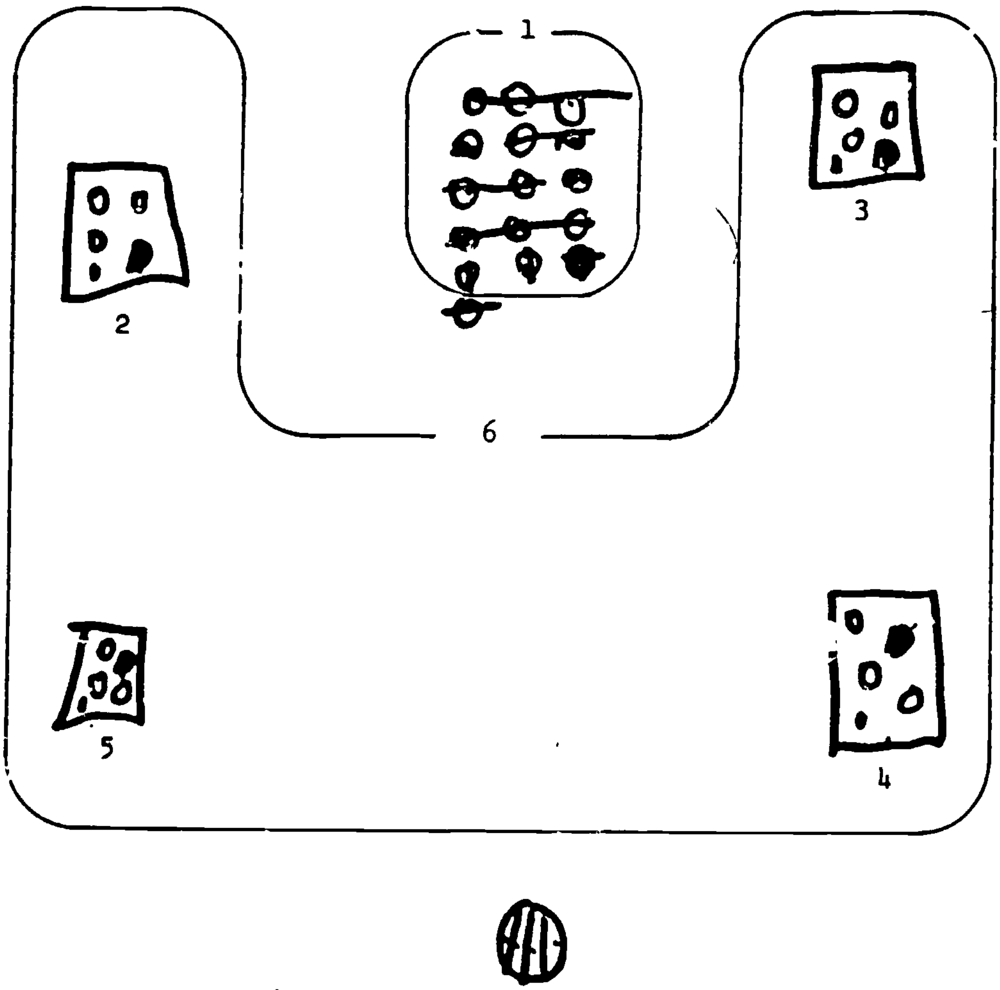


Figure 2. Groups David refers to.

TABLE I

Behavior Pattern Chart

General Behavior Pattern[a]	Line	Excerpt from Protocol
X	13	Draws small circle in each square.
Y	14	Crosses off 4 circles in center group.
	15	"Now we divide 4 more."
X	15	Draws small circle in each square.
Y	16	Crosses off 4 more circles in center.
	17	"Now we divide this by 4 more."
X	17	Adds circle to each square.
	18	"Everybody's got 3."
Y	19	Crosses off 4 circles in center.
	22	"Cut one in half."
	22	Draws a circle in 2 of the 4 squares.
X	24	"Cut this in half."
	24	Draws a circle in each of the remaining 2 squares.
Y	30	Draws vertical lines through 2 circles in center.
	32	"We'll put little chunks of that one in each box."
X	32	Puts a dot in each square.
Y	32	Puts 2 crossed lines on last circle in center.

[a]Key: X - put an indentical object in each square.
Y - crossed off circles in center group.

Cognitive Process Model

In this section the above observations will be interpreted by constructing a model of the cognitive processes going on in David during the interview. Observations 1-4 concern acting out the problem without arithmetic operations and suggest that the model should involve knowledge structures for basic practical actions such as sharing and cutting in half, rather than knowledge structures for arithmetic operations. The observed behavior pattern of distributing four objects to the squares five times can be used to describe David's basic solution method as solving the problem in parts by distributing manageable portions of the source group to the four squares. This method contrasts with a single-step solution of dividing the number 4 into 15 to obtain 3-3/4.

However, as an example of solving a problem in parts, David's approach was of a particular kind. He did not give evidence of going through a preliminary process of defining all of the subproblems before beginning to solve each part. Rather, he seemed to "slice off" a new piece of the problem as he disposed of the previous piece. An additional characteristic of his approach was that each act of sharing small groups of objects contributed to the overall goal of using up the source group of stones. A name used for this type of approach is hill-climbing. This metaphor refers to the simplest strategy for finding one's way through a forest to the top of a mountain by simply taking each step in a direction that goes uphill, each step being thought of as a piece of the solution. Hill-climbing is a well-known problem-solving heuristic (Wickelgren, 1974). It can be described more precisely as a cycle with the following form:

(a) The current situation is viewed and an action-oriented structure is activated that contributes directly toward the goal;

(b) The action is performed within the story situation constraints;

(c) Steps (a) and (b) are repeated until the solution is completed.

David's spontaneous solution had this cyclic characteristic even though he was probably not conscious of it as a general strategy; his solution process thus included an intuitive heuristic. The cycle coincides with behavior pattern X-Y in Table I. Each time he distributed a single object to each person, he moved directly toward the goal of using up the source group. In the model to be developed it will not be assumed that this cycle was produced directly by a general cognitive structure for hill-climbing. Rather, the cycle will be described as a property of David's processing that emerged from the recurring assimilatory activity of an action-oriented structure for sharing objects.

Enabling actions. David cut two objects in half in order to enable him to share them. This cutting action did not contribute directly toward the goal of depleting the source group. Instead, the cutting-in-half structure acted "in the service of" the sharing structure to generate the four objects of equal size that enabled the sharing structure to act again. For this reason, cutting in half will be called an enabling action. Cutting in chunks was also an enabling action in this same sense. These enabling actions allowed David to fill in a missing precondition for the operation of the sharing structures used in the main hill-climbing sequence of actions. This type of enabling action appears to be a fundamental reasoning process. Hill-climbing actions and enabling actions, then, were the two basic components of David's solution process.[1]

David's structure for sharing. Having identified these important components of David's solution process, the second step in the modeling task is to specify a cognitive mechanism that can account for the basic hill-climbing cycle. The observed pattern of behavior in which David repeatedly transferred identical objects to each of the four squares suggests that the initial reading of the problem activated a cognitive structure in David that embodied the idea of sharing some things fairly by giving each person an identical piece.

Following Witz and Easley (1978), a cognitive structure will be defined as a unit of knowledge which can assimilate certain aspects of the environment and provide a interpretation or a response to them. Structures which are activated can play a role in the current thinking process and remain activated on their own for a short period of time. A structure constitutes a stable unit of knowing that can be remembered; it is presumably realized as a neural pattern of activity which, if activated, can be repeated even after months of disuse. In this model, learning, then, would involve permanent changes in cognitive structures.

There are, however, a variety of ways to think about sharing (cutting, dealing, transferring, etc.) and the particular way in which David thought about it should be specified. The data here indicate that the basic concept David used is extremely simple, namely, the idea that equal portions should be transferred to each person from a source. David performed this basic act repeatedly during his solution as he gave one object to each person five times.

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[1]These processes are related to the theory of means-ends analysis proposed by Newell, Shaw, and Simon (1959), but they are developed here as emergent properties of groups of autonomous, action-oriented schemes.

There is a competing hypothesis that passing out the first 12 stones in groups of 4 was governed by an established procedure for "dealing" which assimilated the entire source group. Such a procedure would contain some automatic looping mechanism that caused it to repeat the action of distributing the stones. Although this procedural model does offer an alternative interpretation, a model involving a less sophisticated structure which simply gives one portion to each person is preferable for several reasons. It is important here to rely on the protocol for direction rather than on intuition about how a general method for sharing might be defined. David's behavior contrasts to that of children who deal out 12 or more of the stones in a circle continuously without a break. He passed out groups of 4 stones at a time in action episodes that were clearly separated by crossing off 4 stones each time. He also began each of these episodes with a punctuating expression such as, "Now, we're going to divide 4 more," apparently indicating the start of a new task. His later acts of passing out pieces of stones also involved 4 objects (one-half or one-quarter to each). Thus, the action of passing out one object to each person appears to be a coherent and independent unit of action in the protocol. This observation will be modeled by showing a simple give-one-portion-to-each-person sharing structure acting five separate times in the diagrammed model.

Initial diagram model. Figure 3 shows a simplified model of David's cognitive structure activity during the interview. In this diagram time runs from left to right. Roughly, what is going on inside the subject's head appears above the wavy line and what is going on externally appears below the wavy line. The diagram as a whole reads somewhat like a musical score, with different instruments (cognitive structures) coming in at different points and playing roles for varying amounts of time.

More precisely, the investigator's observations of events during the interview are shown below the wavy line. These observations include statements by the interviewer and by the subject, actions performed by the subject (written in parentheses), and aspects of the environment (shown in enclosed regions). The investigator's model of the subject's mental activity is shown above the wavy line. The structures shown as being active at various points in this diagram would correspond roughly to structures currently in short-term memory in an information-processing model.

The activity of the sharing structure can account for the repeated actions of passing out stones. Figure 3 shows the structure operating five times. Each time the action-oriented substructure labeled give one portion to each person assimilated a source group of 4 stones and distributed them to 4 people.

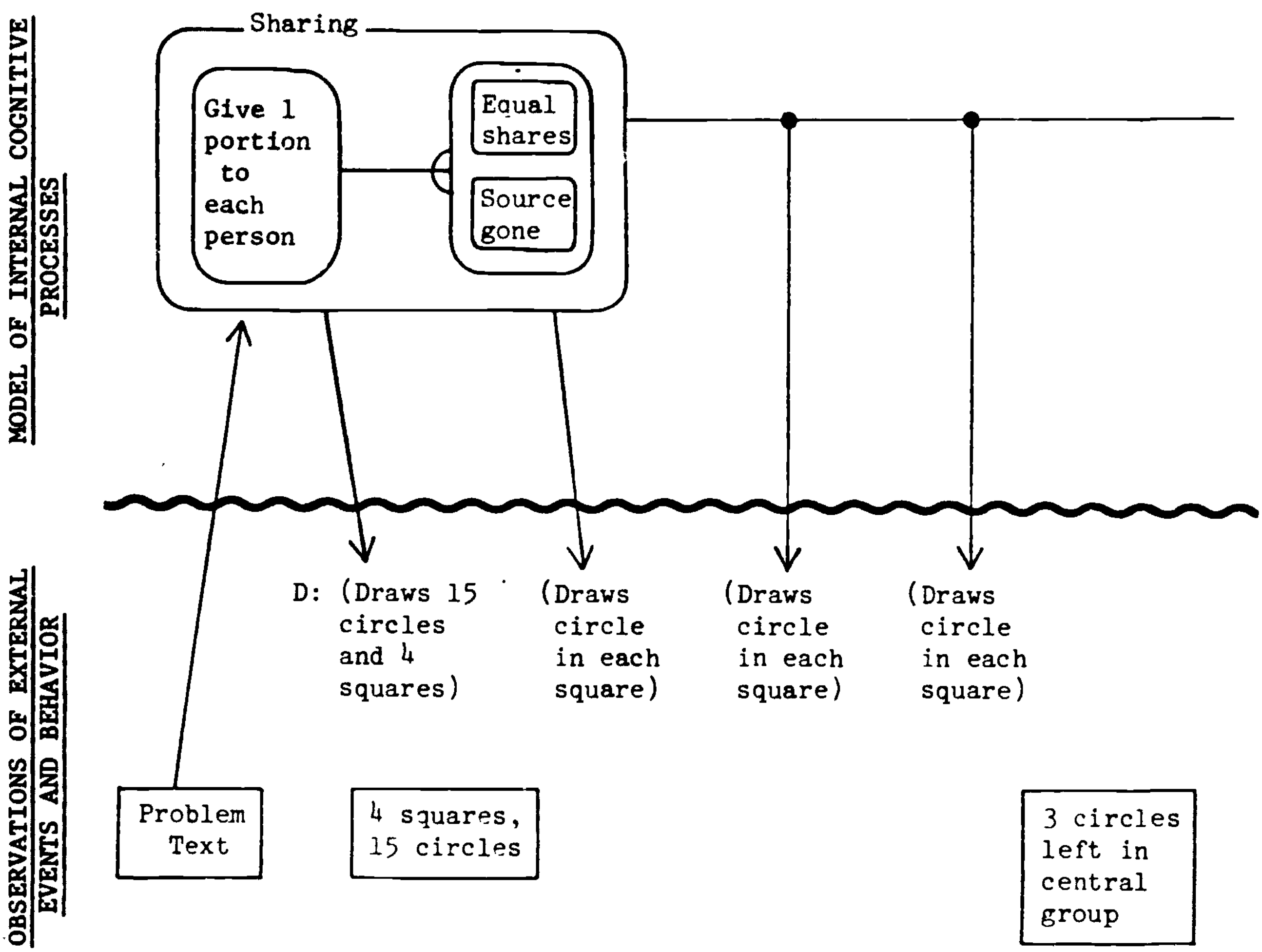


Figure 3. Initial diagram - David's solution process

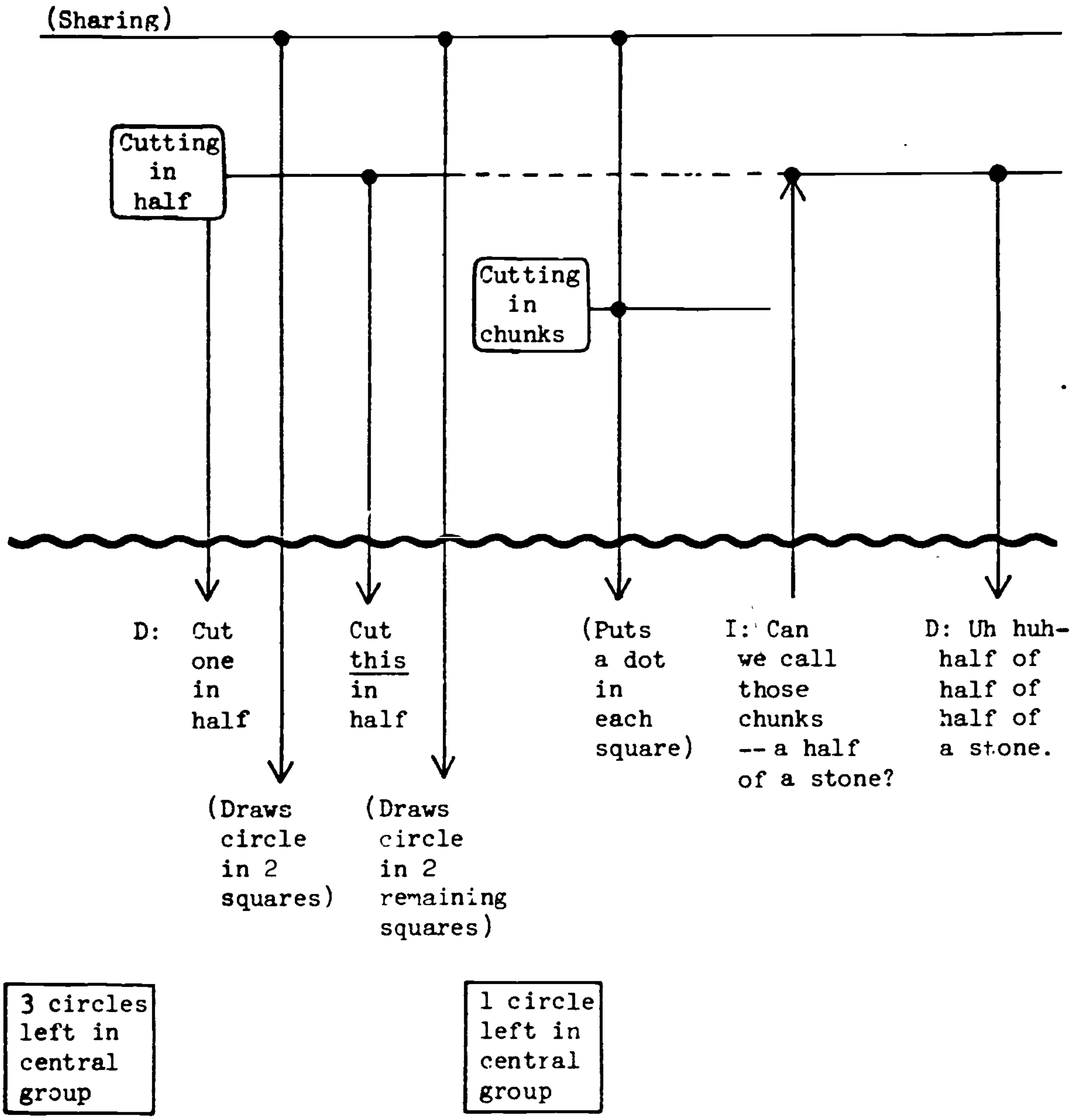


Figure 3 (cont.)

When the sharing structure could not assimilate the last 3 stones, the structure labeled cutting in half became active in parallel with the sharing structure as two of the circles were cut in half and four half-circles were passed out, two at a time. (How the cutting-in-half structure was activated will be the focus of a later discussion.) Another structure labeled cutting in chunks accounts for the way David handled the last remaining stone as he said, "So we'll put little chunks of that one in each box." The cutting-in-chunks structure was related to, but less differentiated than, the cutting-in-half structure, and it anticipated the size of each of the resulting chunks with less precision. When asked about the size of the "chunks," David's cutting-in-half structure appeared to operate recursively, causing him to describe the chunks as a "half of a half of a half of a stone." Lower level perceptual and motor output structures can be assumed to have been operating as well but are not shown explicitly in the diagram. Only the higher level "mediating process" is represented.

The sharing structure is represented by a closed region above the wavy line, labeled sharing. The horizontal activity trace line extending to the right from this structure indicates its extended, continuous activity, stretching almost to the end of the protocol. Vertical lines connect this structure to the aspects of observed behavior that it accounts for below the wavy line. Thus, beginning from the left side of Figure 3, the sharing structure was activated as David read the problem, and it played a part in producing his behavior throughout most of the solution process. The model of action-oriented structures used here implies that if David were to pass out 4 real objects to 4 real people one would see the behavioral output of this same structure. It can be assumed that, when he put a circle in each square in the drawing, this structure was operating in the same way, except that instead of feeding low level motor commands for moving objects the structure fed perceptual motor routines for producing drawings.

The sharing structure can be viewed as a scheme in Piagetian terms. (I use the Piagetian spelling for action-oriented structures here, although others may use schema.) The sharing structure is considered a scheme when thought of as a unit of knowledge, a unit that controlled David's sharing behavior and monitored the status of several groups--a group of people, the source group, and the groups of material each person received. The sharing structure is treated as a process when it remains active over a period of time and controls behavior or participates in reasoning activity.

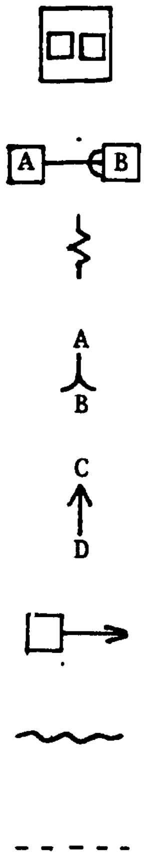
The diagram indicates an important link between the give-one-portion-to-each-person substructure of the sharing structure and the other two substructures which create the expectations that each person should have an equal amount and that the source group should be used up. The action-expectation form of this model for scheme structures is indicated by the notation $a \in (B)$, meaning: Do a, then expect B. Structures of this form have been shown by Knifong (1971) and Witz (1976) to account for the spontaneous behavior of 3- to 5-year-olds manipulating simple pieces of apparatus such as a hook balance. The term expectation is used here to mean that certain perceptual substructures are activated ("warmed up"), ready to assimilate an external event. There is a kind of tension condition set up within the B substructure, and this tension condition is relaxed after the act, if the expectation is fulfilled, that is, if the expected event is assimilated to the waiting substructure.

It has not been assumed that any kind of external or internal verbal activity was necessary on David's part in order for a structure to be active or in order for David to act or think about acting. A special effort has been made to avoid thinking of the sharing structure or any other structure as a piece of static information or some kind of verbal statement. Instead, a structure should be thought of as a stable, action-oriented unit of functioning in the child. As a unit of knowledge, it is closer to "knowing how" than to "knowing that."

Detailed diagram model. Figure 4 includes a number of features missing from the model in Figure 3. Two separate levels of cognitive activity are included: action-oriented structures and perceptual structures. Also, ties between cognitive structures and behavior are shown in greater detail by vertical lines; these multiple ties provide empirical support for the model constructed above the wavy line. Arrows pointing downward indicate those places where David's observable actions or statements are initiated and controlled by one or more cognitive structures.

Arrows pointing upward indicate external assimilation; they show aspects of the environment below the wavy line that are assimilated by cognitive structures above the wavy line. In assimilating an external object or event, a structure orients to the object and provides an interpretation for it. The assimilation of a group of four stones, for example, is then a temporary relationship wherein the structure interprets, attends to, and keeps track of the group over a period of time (in this case about ten seconds). It is assumed in this model that a similar relationship can also occur between internal structures, such as the sharing structure and a perceptual structure, and the symbol $\perp$ indicates an internal assimilation. It should be noted that assimilation is not a process whereby there is a one-way causal link from an object to an internal structure. The form of the assimilating structure will determine which objects are

Key to Figure 4



Cognitive structure and substructures

Doing A leads to expectation B

Disequilibrium

A assimilates B internally

C assimilates D externally

Activity trace

Structure in tension

Low level activity

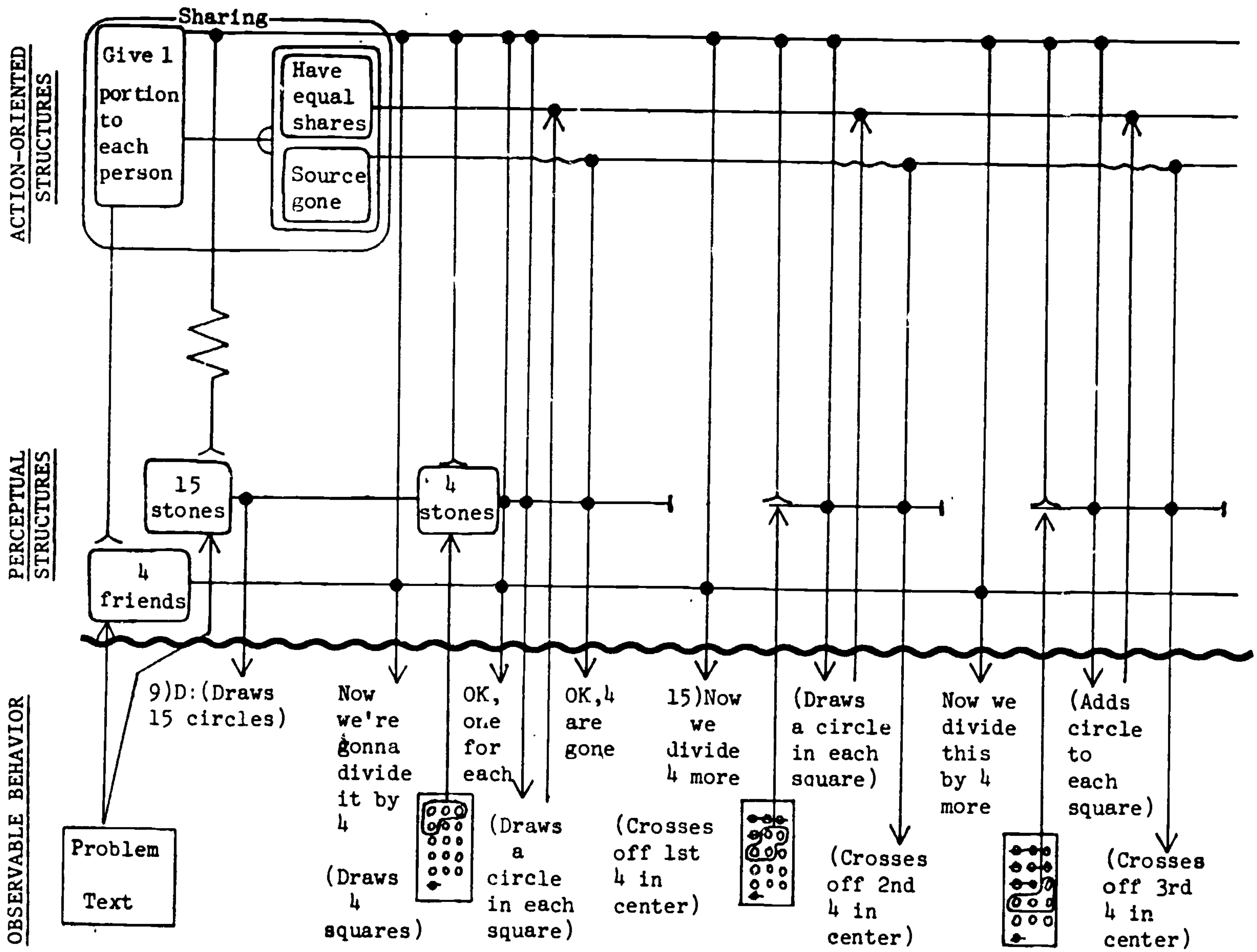


Figure 4. Detailed diagram - David's solution process

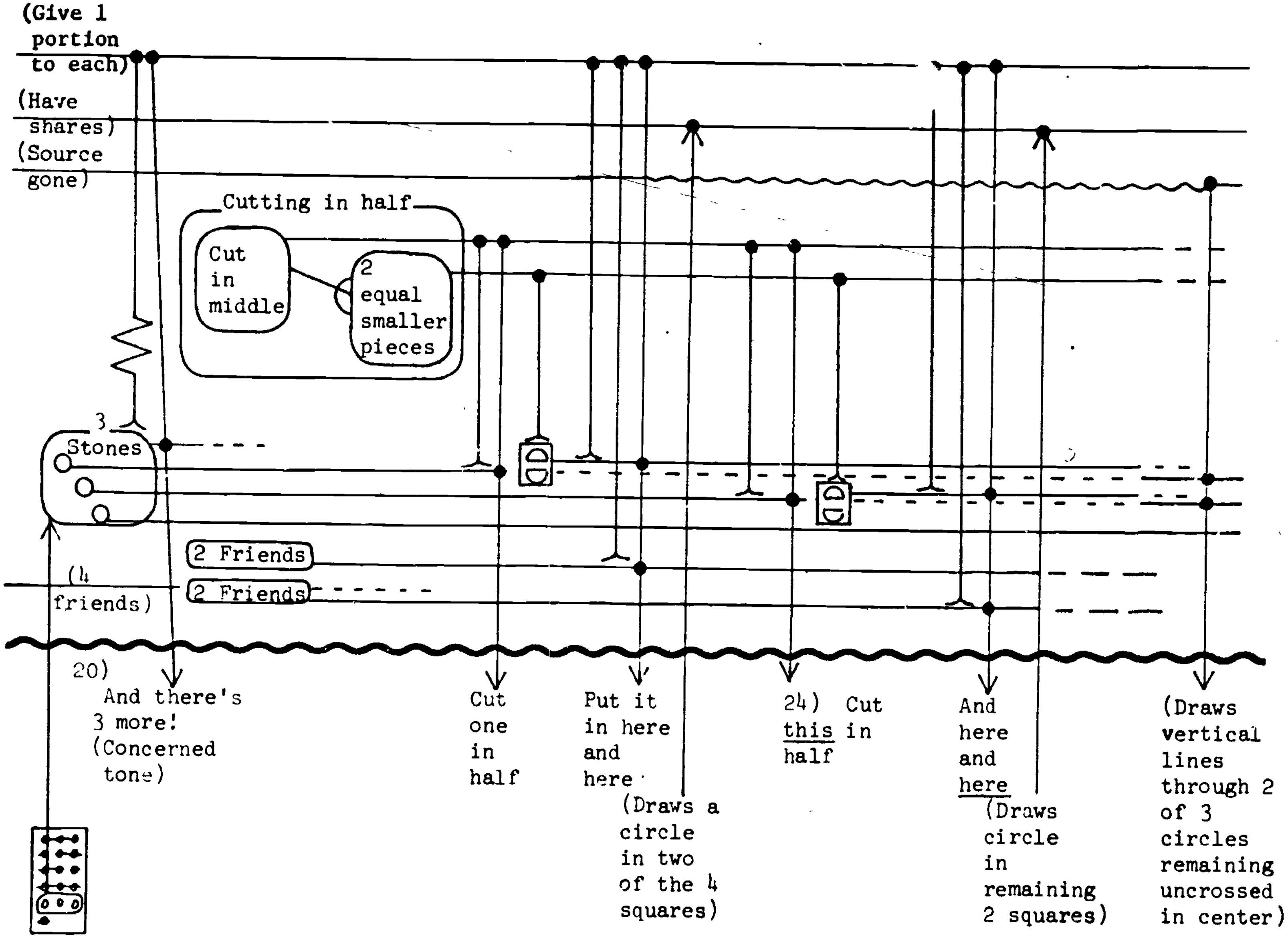


Figure 4 (cont.)

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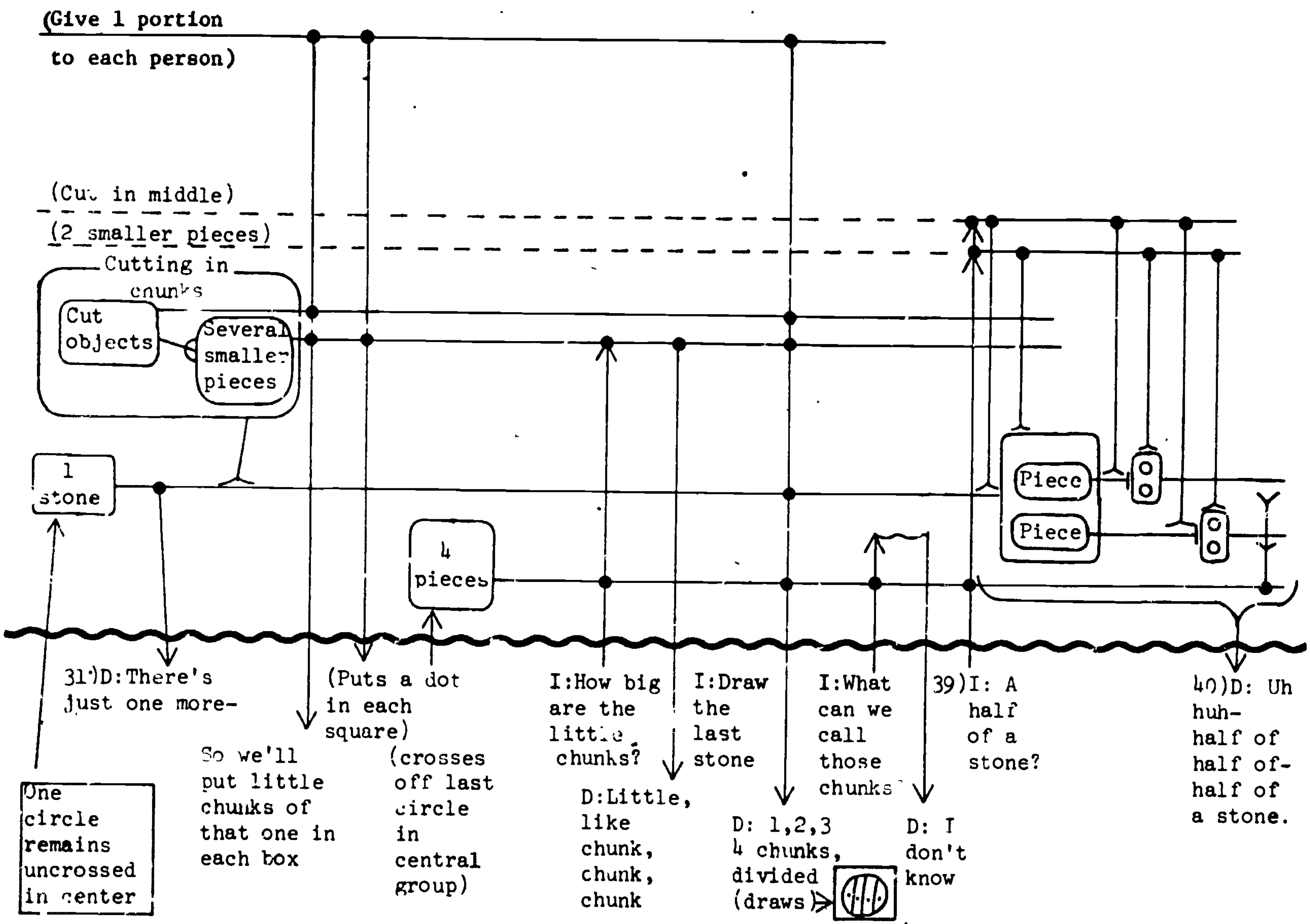

Figure 4 (cont.)

assimilated and how they are interpreted. Thus, external assimilation, for example, is indicated by a unidirectional, upward-pointing arrow only for reasons of notational simplification.

To account for the way in which David began the basic hill-climbing approach, the detailed diagram in Figure 4 first shows the 15-stones structure being assimilated to the give-one-portion-to-each-person substructure. But, there was no evidence of any action being taken on the source group as a whole. Instead, as the diagram shows, the give-one-portion-to-each-person substructure assimilated a group of 4 stones, signifying that David st ped attending to the 15 stones and focused on only the first 4 stones as a source group of manageable size.

The give-one-portion-to-each-person substructure reassimilated new groups of 4 stones and distributed them until only 3 were left, at which point it could no longer assimilate a group compatible with the 4-friends structure. This kind of situation, in which a structure attempts to make an assimilation but cannot do so, creates a type of disequilibrium condition that I will call vertical disequilibrium, as distinct from horizontal disequilibrium, in which two higher-order structures compete for the assimilation of a lower-order structure. The vertical disequilibrium condition shown at the left of Figure 4b presumably encouraged another structure to become active and resolve the difficulty, namely, the cutting-in-half structure became active and resolved the disequilibrium condition by providing enough half stones for the sharing structure to distribute. How the cutting-in-half structure was activated is still to be discussed.

Action-expectation activity over time. Figure 5 is an elaboration of a portion of Figure 4b and shows a method of diagramming the activity of cutting in half, a typical action-expectation structure, over a time period of 20 seconds or so. The substructure labeled object embodies the knowledge of a precondition, that one must first focus on an object (perhaps with specific properties) to be cut. (Such preconditions were omitted from Figures 3 and 4 to conserve space.) Two equal, smaller pieces embodies the knowledge of what one expects to see after the cutting takes place. The tension condition that exists until this expectation is fulfilled is indicated by the oscillating horizontal line emanating from the structure. In Figure 5 the structure is shown performing a real action. However, it is hypothesized that the same internal processes could take place in solving a word problem when no real actions take place, as will be discussed further. It is not assumed here that preconditions are best modeled as precise sets of discrete features. Some preconditions may be more Gestalt-like and flexible. In the case of physical actions some preconditions are embedded in the proprioceptive

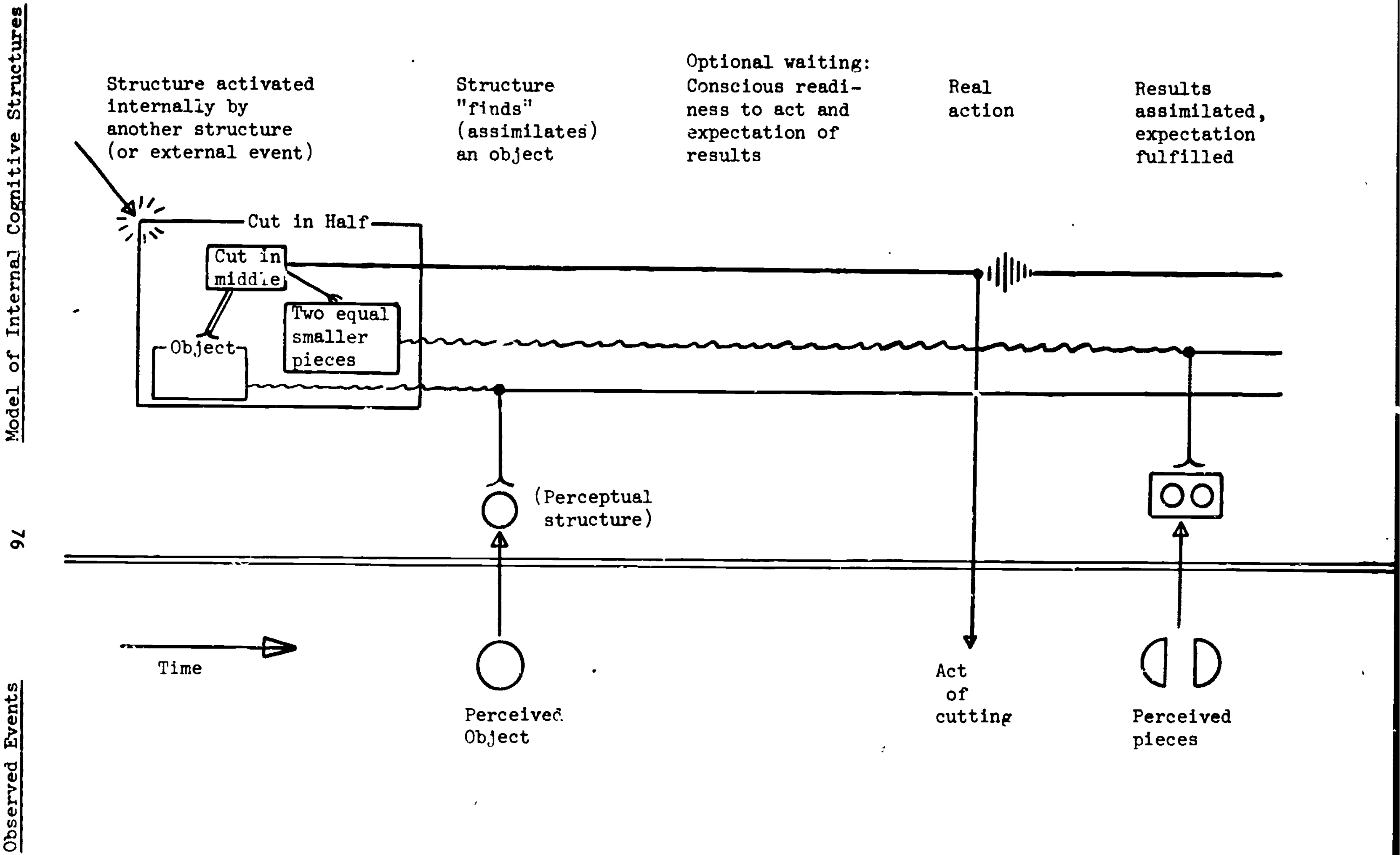

Figure 5. Activity of an Action-Expectation Structure over Time

orienting activities that constitute the first stages of the action itself and are not easily modeled as discrete features symbolized internally in some static form.

Anticipation and internalized action. The fact that David could actually anticipate what would happen when an object was cut in half (leading to his use of the idea) can be explained by assuming that David went through the internalized action of cutting the stone in half. In Figure 4 the diagramming technique developed so far is powerful enough to show some aspects of this internalized action in detail. It is assumed that, in thinking about small numbers of objects or groups (approximately 1-5), David was capable of holding active a separate perceptual structure for each object or group. For example, the structures represented by two semi-circles inside a box in Figure 4b were responsible for the perceptual expectation of having two smaller pieces as a result of cutting one stone in half. The same perceptual structure that would have been active if David had actually been viewing a small stone is assumed to have been active here, even though there were no real stones present. That is, David imagined the presence of a stone. Similarly, he imagined cutting the stone in half when the cutting-in-half structure was active without actually producing cutting movements in output mode. It is assumed that these internally activated perceptual structures were what enabled him to make a drawing (in conjunction with appropriate hand movement structures not modeled here). More importantly, they helped him anticipate the beneficial results of cutting in half as an enabling action that would allow the sharing structure to continue operating.

The interpretation represented in the diagram is that the ability to perform a mental action on an image is basically a nonverbal, perceptual-motor anticipation. David knew how to cut something in half, and his cutting-in-half scheme activated his perceptual structures in a top-down manner to assimilate two new, smaller objects of equal size even before the actual cutting occurred. That is, he was already imagining the two halves when he drew them. This interpretation contrasts with the idea that David might have been using a memorized arithmetic fact in a verbal form like "1 divided by 1/2 is 2." I am instead inferring that David's mental activity in this case was very similar to what it would have been had he actually been sharing and cutting real objects.

This model is consistent with the Piagetian view that there is a basic "logic of actions" level of reasoning that is more fruitfully modeled, not as manipulations of verbal symbols or abstract propositions, but rather as the coordination of internalized actions. While aspects of

David's behavior might be accounted for by a model which uses only verbal representations of abstract features, a reason for taking the internalized-actions point of view here is the smoothness and ease with which David constructed and interacted with his visual drawing. The fact that David focused on his drawing during the entire interview and worked so closely with it is consistent with the hypothesis that the drawing, as a spontaneous mode of symbolization, was tied very closely to the internal imagery processes he was using and was, for him, a fairly direct symbolization of those processes. Of course, it is clear that there were eventually more items represented in the drawing than he could hold in mind at once, and that is why the drawing was useful.

On the other hand, it might be argued that the drawing itself was the representation David was acting on and that there is no need for positing internal image structures. However, anticipations via internalized actions on images are included in the model in order to account for David's insight that cutting two objects in half was the right thing to do and to account for his description of the last distribution involving "half of a half of a half of a stone." Although confirmation of this interpretation will require much more research on the theory of knowledge representations, it appears to be the most plausible interpretation in the case of the present protocol.

An important task for future research is to determine the limitations of this internalized-actions-on-images system. The conjecture that the system can only operate effectively on less than six objects, or groups of objects, at a time is consistent with research on children's subitizing ability--the ability to enumerate groups of less than six objects very quickly without counting (Klahr, 1973). For example, David could not have been expected to keep track of changes in the source group of 15 stones without using a drawing, but he might have been expected to handle 6 stones mentally by focusing first on 4 stones and then on 2. It is further conjectured that objects are not imagined in detail--that only gross characteristics are imagined. These conjectures point to some important questions for future research.

Goals. The model developed here to explain how goals are set up and maintained is a vertical disequilibrium, or tension, model. In Figure 4 the oscillating portions of the horizontal line to the right of the source-gone structure indicate a state of tension. The activation of the sharing structure by the task creates the expectation that the source group must be used up in order to arrive at a solution to the problem. In general, we assume that a perceptual structure S_1 can act as a goal when it is "held active" by some basic (possibly chemical) drive or by some other continually activated structure S_2 for which there is an expectation that the satisfaction of S_1 will lead to the satisfaction of S_2. Since the source-gone substructure cannot assimilate an empty source group at the beginning of the solution, the structure is initially in tension. The tension caused by this "assimilation gap" is reduced each time an action brings the number in the source group closer to zero.

A tension condition is hypothesized to have two effects: It makes the structure a strong competitor for attention in the organism; and it motivates the subject to act so as to satisfy the structure by trying out various possible actions mentally. When he can imagine one of these actions contributing to the goal, this action becomes the dominant focus of attention. Thus, the model explains goal-oriented behavior in terms of high activations maintained in specific knowledge structures, rather than as the transfer of a static symbol to a "place" labeled "the current goal." A consequence of this model is that it is natural to imagine the possibility of several competing goals, at different tension levels, with the strongest tension dominating at any moment.

Recursion. As described earlier, the last single "stone" remaining in David's source group is assumed to have been assimilated by a cutting-in-chunks structure. The phrase half of a half of a half of a stone, used by David in response to a question about the size of the resulting "little chunks," has several possible interpretations. It appears that David was applying the cutting-in-half structure recursively. Roughly, this means that the structure was applied to its own output. More precisely, recursion refers to the activity of a structure $\underline{S}$ assimilating a perceptual situation $\underline{P_1}$ and activating the expectation of another perceptual situation $\underline{P_2}$ (see Figure 6). When $\underline{S}$ is reapplied by assimilating $\underline{P_2}$ to its own action component, then $\underline{S}$ is said to be applied recursively.

David's situation was slightly more complicated than that shown in Figure 6 because the first expected effect of cutting in half the single stone was to produce two objects. Each of these could have been assimilated by the same cutting-in-half structure to yield four equal pieces of smaller size. However, it is not clear that David was able to imagine this with precision. He indicated his uncertainty (line 42) by saying, "I don't know," before saying "half of a half of a half of a stone." Yet, there is a certain definiteness to his response, countering as it does the interviewer's probing (line 41), "a half of a half of a stone? What does that mean?" (indicating only two "halvings"). Two possible interpretations of David's statement are as follows:

(a) He had an appreciation for the possibility of generating four equal pieces from one stone by

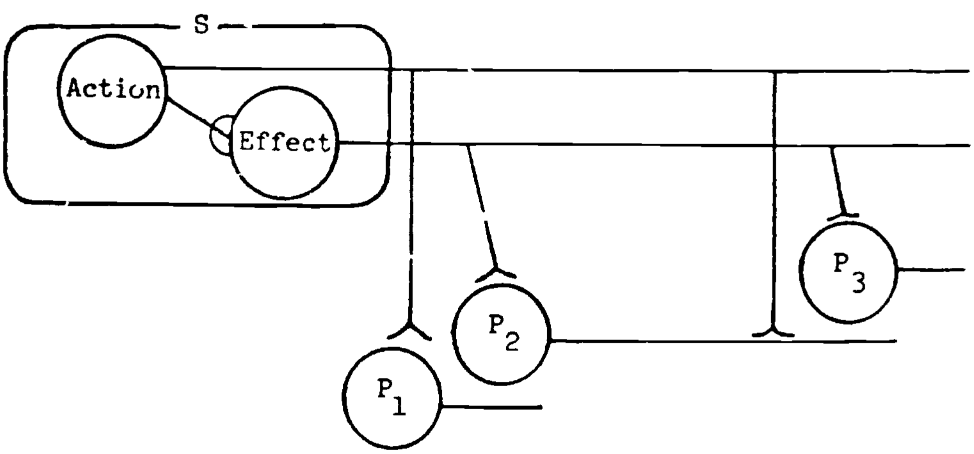

Figure 6. Recursion

applying the cutting-in-half structure recursively, but the exact sequence and number of halvings required were unclear to him;

(b) He comprehended the chain of actions required, but described them linguistically in a nonstandard format; there were three acts of halving required to generate the four pieces he drew, so he said "half of" three times.

It is the latter interpretation that is represented in Figure 4c. This interpretation is consistent with an assumed tendency on David's part to focus on the act of cutting a piece in half as opposed to focusing on the resulting half-pieces. In either case, the fact that David appreciated the possibility of recursively applying the cutting-in-half structure to produce four identical pieces is an impressive example of intuitive reasoning.

Improving the Model

An important characteristic of structure-interaction diagrams is that they are criticizable as theories. Figure 4, for example, is the end result of many cycles of constructing a model, checking it against the exact sequence of events in the tape, and modifying it. However, the present map can still be criticized for leaving certain protocol data unexplained. Several improvements can be made to produce a slightly more complex model that would describe David's mental activity in the following sequence (asterisks indicate new steps):

*(a) A general, frame-like structure activated the goal of sharing 15 objects among 4 people;

(b) David realized that he did not know how to do this in a single step;

(c) A specific sharing structure repeatedly oriented to sharing 4 objects among 4 people to account for the basic hill-climbing cycle;

(d) The first 12 stones were distributed;

*(e) The general frame-like sharing structure also weakly activated the cutting-in-half and cutting-in-chunks structures as actions which are related to a sharing context,

*(f) David planned the action of cutting 2 objects in half in order to produce 4 objects to satisfy a precondition for the sharing structure that required 4 objects for distribution;

(g) He executed this plan in two steps, giving a piece to each person after cutting each stone in half;

(h) Cutting in chunks operated on the last stone in order to produce 4 more pieces for distribution by the sharing structure;

(i) Cutting in half operated recursively on the last stone to provide a more precise solution.

These changes in the model involve three issues: first, accounting for the goals in the hill-climbing cycle via two separate sharing structures at different levels of generality; second, the extent to which planning occurs via internalized actions; and third, choosing between an autonomous-schemes model and a more structured, frame-oriented model to account for the initial activation of structures. These issues will be discussed in turn below.

Levels of structure. In addition to leaving certain data unexplained, the model shown in Figure 4 does not account for the goal that motivated the hill-climbing cycle which David followed in repeatedly distributing a single object to each of four people. One might theorize that this goal was embodied in the continuing tension in the source-gone expectation in the sharing structure, but, as David's focus switched from the original source group to the first subgroup of four objects, the source-gone expectation would have been fulfilled after distributing four objects; there would have been no source of tension to drive further distributions. This difficulty can be resolved by theorizing that there were two levels of sharing structures (see Figure 7). The first structure to be activated, shown in the upper left corner of the diagram, was a generalized conception of the gross qualitative features of a sharing episode, where fairness is desired, while the second structure embodied a specific method for sharing, where fairness is guaranteed. (The latter is equivalent to the sharing structure modeled in Figure 4.) Related models of action-oriented structures at differing levels of generality have been discussed by Newell, Shaw, and Simon (1959) and by Witz (1973).

The hill-climbing cycle that David followed can be explained by positing that the general sharing structure assimilated the entire source group of 15 stones. A tension or goal-seeking condition persisted in the source-gone expectation of this structure during the entire problem solution. Any action which transferred stones from the source group to the people reduced this tension. Such an action was provided by the second, more specific sharing structure shown in Figure 7. This structure repeatedly assimilated only four stones at a time and gave one to each person.

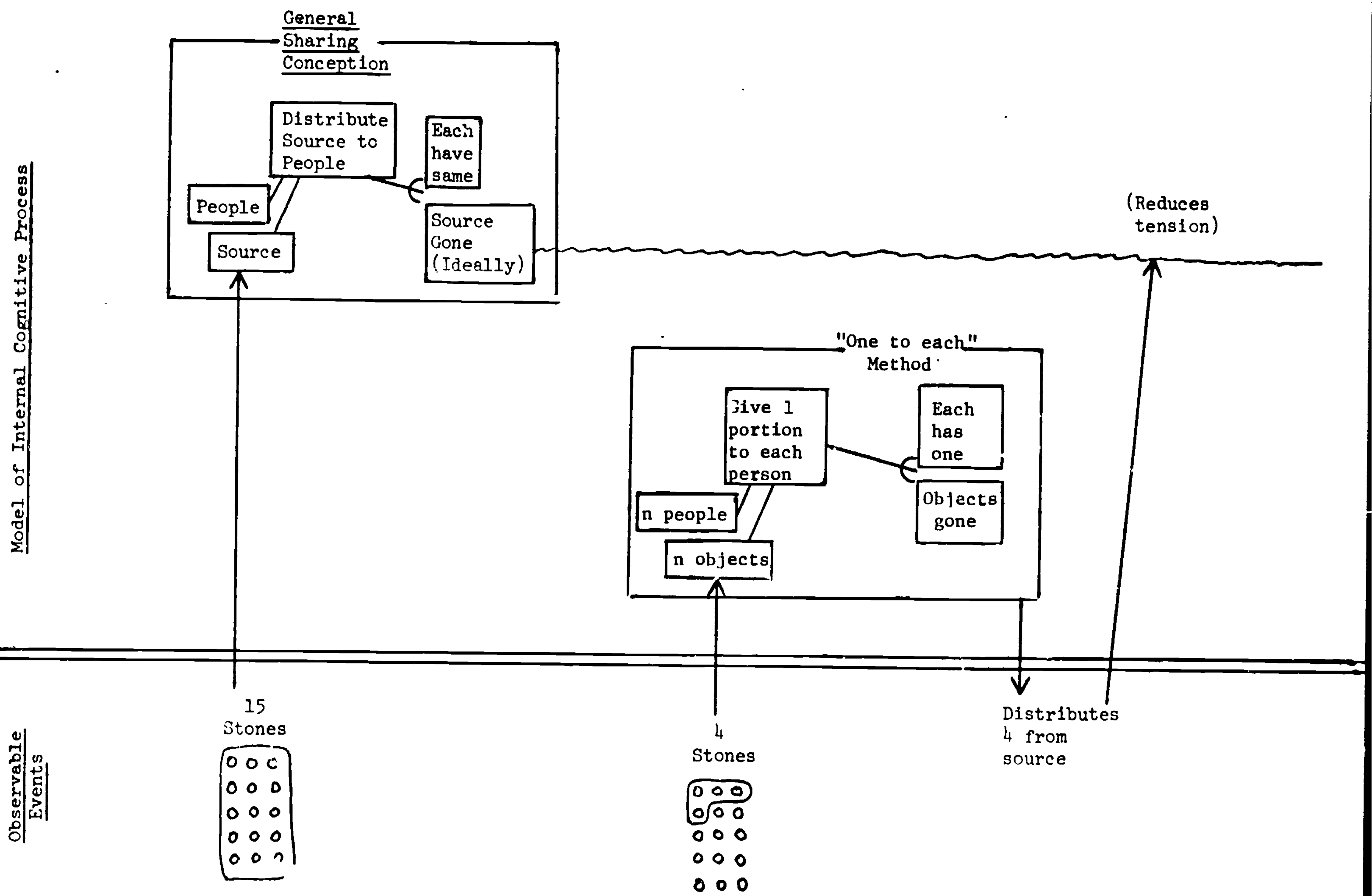

Figure 7. Generalized and Specific Sharing Structures

An interesting feature of this tension-reduction model is that it eliminates the need for an "executive procedure" which organizes the problem-solving process in a centrally controlled way. Such a procedure might have been used to model a general hill-climbing or difference-reduction strategy explicitly and sequentially carried out by the problem solver. In the present model the tension-reduction mechanisms are thought of as being built into the "hardware" of the system in a more distributed way, that is, built into the "drive-to-assimilate" property of the relatively autonomous cognitive structures themselves. The hill-climbing cycle then emerges as a systemic property of a distributed system rather than being determined locally by an established procedure with controlled looping.

Chaining of structures. The use of cutting in half as an enabling action can also be thought of as "working backwards" or chaining backwards from a subgoal. If finding four objects was the goal when David had three stones left, then cutting the stones in half constituted working backwards from the goal of sharing four objects to the idea of finding four objects to the idea of cutting in half. Internalized actions offer a mechanism for explaining how such chains of action could be planned mentally ahead of time by the subject. If a goal tension condition exists in a perceptual structure P, an action-expectation structure A with an expectation substructure identical or similar to P will tend to be activated and tried out internally. If the precondition substructure in A is uninstantiated, it will in turn become a new primary goal and this explains how chaining can occur spontaneously. An intriguing question to ask at each point in the protocol is: How much did David plan ahead of time--how far ahead could he look?

It is not possible to answer this question definitively in the absence of announced predictions or plans on the part of the subject. The position represented in the diagram in Figure 4 is that most results of action chains were not anticipated beyond the current drawing cycle, a drawing cycle being a relatively well-defined burst of drawing activity in the protocol, separated by pauses and spontaneous verbalization on the part of the subject.

One probable exception to this limitation was the sequence in which David cut two stones in half to distribute to four people. It has already been inferred that he anticipated that cutting one stone in half would provide equal pieces for two people. How much more did he anticipate here--that cutting two stones in half would give just enough for the four people? The diagram as drawn in Figure 4b does not reflect this anticipation, but it could easily be changed to model it by compressing and shifting to the left the cognitive activity associated with each cutting action, thereby implying more internalized actions on images before the drawing of "half stones."

The major Piagetian theme is that, given an appropriate object, active schemes will operate autonomously on the object to produce an expected result, even when there is no special reason to do so. This tendency of structures to apply themselves autonomously is most apparent in the relatively goal-free play behavior of children, who are always "trying out" things with (applying action schemes to) different objects. The present model assumes that when a scheme like cutting in half was activated, it automatically tried itself out internally on the images of the circles already drawn. Thus the beneficial results (four equal objects to share) were anticipated. This sequence outlines a mechanism by which planning can occur in a spontaneous manner, as the result of active assimilations on the part of autonomous structures.

Activation of structures. There remains the question of how the cutting-in-half and cutting-in-chunks structures were activated--the memory access, or scheme activation, problem. Successful problem solving involves at least two considerations, namely, determining what actions are relevant to the problem and determining what sequence of these actions will work. The discussion so far has focused largely on the latter consideration, but the question of determining actions relevant to a problem should not be ignored, as it too often is in psychological research. Several possible answers to this question will be described below.

At one extreme, it might be that structures call other structures directly, as is done in many everyday computer programs. An established, higher-order procedure would determine the order in which these structures come into play using direct subroutine calls. A weakness in such a highly structured format is the difficulty of obtaining robust procedures that adapt flexibly, as David seemed to do, to new problem-solving situations.

At the opposite extreme, there might be many autonomous structures (variously called schemes, demons, or productions) each of which determines its own relevance to the current situation and competes for attention if the relevance is high. The search for a match between structure and situation would presumably have to occur in parallel, given the vast number of structures in the organism. Models along these lines have been proposed by Newell (1973), Selfridge (1959), and Witz and Easley (1978).

Between these extremes is a scenario in which structures related to the same context are activated together, but their sequence of deployment is not determined ahead of time. Models of this last type have been proposed by Witz (1973), who models "framework" structures in young children from a Piagetian perspective, and Minsky (1975), who argues for the use of "frame" structures in an artificial intelligence system.

Action-oriented structures might be activated in any of these ways. Their effects, or short chains of several actions and effects, could be anticipated internally and evaluated on the basis of current goals. In David's case the initial sharing structure was probably activated directly via the linguistic process involved in reading the problem text. It is unlikely that the cutting-in-half and cutting-in-chunks structures were called at a certain point by an established procedure for solving sharing problems, because of the flexibility with which David coordinated sharing and cutting in half in parallel and the novelty of his solution. If this had been his approach, then in one sense he could not have been considered to be engaged in problem solving, since he would have already had an established procedure for doing the task. The model favored here is the framework model in which the cutting-in-half and cutting-in-chunks structures were initially activated by a general sharing structure which included associations to relevant enabling actions. However, the short length of the protocol precludes a definitive judgment on this issue. Interviewing strategies that will make such judgments possible need to be developed. One strategy might be to examine a series of solutions to related problems by the same subject.

Conclusions and Summary

The proposed model of David's reasoning posits relatively little high-level structuring, that is, no general macro-procedure which would specify how sharing objects, cutting in half, and cutting in chunks are to be sequenced in a given situation. This lack of structuring is made up for by a mechanism of spontaneous reasoning interactions. Reasoning takes place when schemes coordinate to form action sequences that are not specified by an established, predetermined procedure. A combination of limited knowledge structures and spontaneous reasoning potential might be a more powerful (adaptive, flexible, general) configuration for problem solving than a set of highly structured procedures oriented to specific tasks.

David's behavior was accounted for via three practical, action-expectation structures: sharing, cutting in half, and cutting in chunks. Each of these structures has basically the same form, consisting of an action substructure connected to a perceptual substructure comprising an expected effect.

Internalized actions involve the activity of these structures in the absence of external output. David's overall approach was described in terms of a hill-climbing cycle, each cycle corresponding to an act whereby the sharing structure reassimilated a new group of identical objects and distributed them. It was hypothesized that internalized actions can be chained to provide for planning and that a structure can function recursively by reassimilating its own effects.

This model of David's thought processes is not based on stored arithmetic facts or other passive structures but on action-oriented cognitive structures that remain active in parallel for varying lengths of time during a problem-solving episode. Reasoning processes in this model do not take the form of manipulation of internal statements according to the rules of a formal logic but, rather, consist of cooperation between structures during internal assimilation and conflict between structures during disequilibrium. The modeling of both knowledge structures and their interactions as dynamic processes preserves and reflects one of Piaget' most important insights into cognition. Many concepts discussed in this paper are related to concepts found in both Piagetian and information-processing theories of cognition and provide poi t which these theories might fruitfully interact.

ntensively analyzing a single short protocol, some aspe of a model of the child's cognitive processes can be supported, while other aspects remain conjectural. The most firmly supported cognitive processes in the present model, for example, are those that have been tied to several different observations from the transcript, as shown explicitly in Figure 4 by the vertical lines connecting structures to observable behavior. The primary objective here, however, has been to raise key questions and to illustrate a variety of techniques and concepts that might be applied to a richer data base.[2] One promising direction for future research is to take a single subject and analyze several related problem solutions. Such case studies should provide more data for deciding between alternative models.

Educational implications. The existence of action-oriented approaches like the one observed here provides evidence for the assertion that children have intuitive conceptions about mathematical situations that are more basic than the four arithmetic operations. It is suggested here that, if traditional arithmetic operations are to have meaning for children, it must be in terms of similar types of

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[2]Additional data which motivated the development of the concepts used in this paper are provided in Clement (1977).

underlying intuitive conceptions. It is also suggested that these intuitions are the natural source of the semantic interpretations for number operations needed to accompany the syntactic rules for symbol manipulation that are so heavily emphasized in school.

More exploratory clinical interviews are needed to map intuitive mathematical structures and to study the relationship between formal and intuitive mathematics in the classroom. For example, the author has found that many third graders are able to solve practical story problems involving multiplication and division with small numbers before studying these operations in school (Clement, 1977). These students use a variety of methods such as skip counting, drawings, concrete materials, etc. Because some children have more difficulty than others in using intuitive methods, it would seem to make sense to have children strengthen their intuitive conceptions with problems like the one discussed here before they learn the operations as facts and algorithms. This intuitive approach could be a step toward making arithmetic more meaningful for children and a step toward remedying the difficulties many students have in applying their knowledge of arithmetic to practical problems.

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Reaction

to

COGNITIVE MICROANALYSIS: AN APPROACH TO ANALYZING INTUITIVE MATHEMATICAL REASONING PROCESSES

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John is to be commended for his efforts in using cognitive microanalysis as an approach for analyzing children's intuitive mathematical reasoning procedures in problem-solving situations. Furthermore, the method of diagramming that he used to describe David's cognitive processes seems to be a viable means of representing cognitive interactions as they occur. I also commend John for identifying David's intuitive reasoning processes and then relating these processes to concepts in the appropriate Piagetian and information-processing models of cognition.

Difficult as it may have been, keeping observed events and theoretical constructs separate enables one to focus on a realistic set of concepts for explaining children's cognitive processes without the interference of restrictions in expected behavior based on the theoretical models. Indeed, the method of cognitive microanalysis provides, in my judgment, a useful approach for research. This procedure encourages children to think aloud while solving problems, avoids predefined categories and predetermined interpretations of children's natural responses, and does not make prior assumptions about the form or functioning of children's cognitive structures. If we are to continue building our knowledge base of children's intuitive cognitive processes, then we must pursue the mapping of conceptions as they exist in children rather than testing the degree to which their conceptions conform to those of an adult or some other model derived from theoretical analysis.

Let us now consider David's protocol specifically, models of cognitive processes generally, and the support of this kind of research for developing mathematical concepts in the classroom.

David's Protocol

I was most interested in David's behavior in the situation that he experienced, not only as described in the paper but also as viewed from the videotape. For the most part, his behavior was that which one would expect of an eight-year-old child. The experience of sharing should be a stable and familiar experience in an eight-year-old's world of reality. To represent the problem by making a picture of 15 circles for the 15 stones and 4 squares (which he referred to as sacks, cans, or people) for the 4 friends also seems a natural response. Eight-year-olds and even seven-year-olds can be expected to feel more comfortable with pencil and paper drawings than with the use of manipulative objects, such as the blocks that were available in this case (viewed in the tape).

The fact that David overlooked Jim as a member of the group and saw the problem as one of sharing 15 stones among 4, rather than 5, people was unexpected. Did David identify with Jim and thus exclude himself as a member of the group--a behavior observed on the part of younger children? I am inclined to agree with John's conjecture that "school experience conditioning" probably prompted David to focus on the numbers in the problem. Nevertheless, his failure to see 5 people provided an interesting and informative opportunity for analyzing his cognitive processes in this problem situation, particularly as he continued the hill-climbing cycle to share those last three stones. Furthermore, it provided insight into David's lack of stable fraction concepts. Although he used two stones to give a half stone to each of the four friends, his "half of a half of a half of a stone" seemed to be associated with the three cuts made on the last stone to get the four "chunks." These ideas seemed to be unstable, as confirmed in David's last statements in the interview. One would expect him to have had more stable concepts of both halves and fourths, based on school experience, if not the world of reality.

Models of Cognitive Processes

Whether or not the sharing structure was activated as David read the problem seems questionable. Rather, it seemed to be activated during lines 5, 6, and 7 of the protocol and probably between his response of "Oh no" and the interviewer's asking "How can we start on it?" In retrospect, it may have been unfortunate to have asked that question at that particular time. Part of the thinking may have been lost with the suggestion that action should be getting underway.

Clearly, David's internal cognitive processes included the structure of sharing a given collection among four people so that each person gets an equal amount and the collection is exhausted. His output seemed to exhibit sharing throughout the problem solution, including the cutting of the stones. However, I feel somewhat uncomfortable with the introduction of the structure labeled cutting in chunks. Without further information, cutting in chunks seems to be an enabling action in order to continue with the sharing structure. For model development and refinement, more analyses of David's processes in similar and different problem-solving situations, as well as analyses of other children's protocols in these same situations, are needed especially if these models are to have educational implications.

John has indicated that additional research is also needed to determine the limitations of the system of internalized actions on images. It is not completely clear how this system is related to the subitizing process; in particular, it is not clear why it should operate effectively with only 6 or fewer objects. Personally, I am not convinced that David's protocol offers much support for the internalized-actions interpretation. The information available seems to suggest that the goal of tension-reduction in the source-gone expectation may be an adequate explanation of the motivation for David's behavior.

Developing procedures whereby children really think aloud as they engage in problem-solving activities is not easy. Even when we think that children are telling us what they are thinking, we can be quite "taken-aback," as I was in one interview with a child. After this child had made what seemed like "thinking-aloud" comments reflecting his internal cognitive processes while engaged in a proble -solving situation, he later commented to me, "That was what I could tell you. It really wasn't what I was thinking." That, from a seven-year-old! We might do well to keep our models more general.

Educational Implications

John has made the assumption that a constructivist model of learning is more valid than a "blank slate" model of the learning process. I, too, support that assumption. On the other hand, he seems to regard the arithmetic operations as symbolic records written on paper rather than internalized generalizations of abstract ideas which children learn to communicate orally and in written symbols. Examining the pages of a school mathematics textbook may lead one to believe that children's intuitive ideas are overlooked in developing concepts and that ideas the children have already attained are ignored when the same concepts are later refined or expanded. Realistically, it cannot be expected that intuitive concepts will be found in children's textbooks, unless research over a broad population supports the inclusion of those concepts. Moreover, it cannot be denied that there do exist classrooms in which the mathematics instruction is, essentially, the textbook. However, in general, there are classroom activities that help prepare children for working in their textbooks.

Most teachers, drawing upon their professional expertise and suggestions in the teacher's guide, can, and do, use children's intuitive thinking as a basis for concept development. These teachers provide experiences for children that are exploratory in nature--what some may regard as "messing around"--to help children arrive at generalizations based on experience. They show children how to express oral ideas in written mathematical language before turning to the textbook. In their textbooks, children see the language in print and compare the textbook developmental lessons to their exploratory experiences before engaging in individual work to develop further insight and skill in using ideas.

These professional teachers are very much aware of children's intuitive concepts about mathematical situations. To say that these concepts are more basic than the four arithmetic operations reflects a poor communication between the realities of a large share of classrooms and the research community. These intuitive ideas are the beginnings upon which the mathematical operations are made meaningful to children before developing techniques and skill in using facts and algorithms. Building on the foundation that children have to assimilate new knowledge and to modify and alter previous knowledge not only could be, but is, an approach used in children's mathematics classes. Perhaps the difficulty that children have in applying their intuition to practical problems is a consequence of the fact that attention to problem solving is not always appropriately balanced with understandings and skills in school mathematics programs.

Although we have already seen the influence of knowledge about children's intuitive cognitive processes on school mathematics curricula, certainly more work is needed. I would expect further knowledge, or refinement of previous knowledge, to continue to influence mathematics programs as ' .il as facilitate the building of general models of these processes.

AN INFORMATION PROCESSING APPROACH TO RESEARCH ON MATHEMATICS LEARNING AND PROBLEM SOLVING

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This paper is a report to mathematics educators concerning the methods used by psychologists to study learning and problem solving in mathematics and the progress made to date. We find the need for such a paper somewhat embarrassing and quite telling. One explanation for this need is that most educational psychologists have sought (and many still seek) to find general laws of behavior that pertain to schooling, so they have used particular subject matters in their research largely for convenience rather than as fields worthy of study in themselves. Unfortunately, general laws of behavior have been hard to formulate because behavior seems to depend upon so many things--the particular task at hand, the nature of the people involved, etc. (cf. Cronbach, 1975).

As a consequence of the complexity of human behavior and the success of research on human problem solving in specific tasks (e.g., Newell & Simon, 1972), some educational psychologists are now focusing their attention on particular subject matters such as mathematics. They are attempting to describe what students learn as a consequence of mathematics instruction, how they learn it, and how they use what they have learned to solve mathematical problems. Regardless of whether their approach to the study of mathematics is experimental (Mayer & Greeno, 1972), ethnographic (Lave, 1977), or descriptive, using mathematical models (Groen & Resnick, 1977) or computer simulations (Greeno, 1976), some educational psychologists now recognize the necessity of working closely with mathematicians and have begun to do so (e.g., Geeslin & Shavelson, 1975a, 1975b; Lave, 1977; Shavelson & Geeslin, 1975). However, this trend is only in its infancy and a communication gap still exists. In providing mathematics educators with a sketch of how some educational psychologists study learning and problem solving in mathematics, our goal is to provide a foundation for communication. Perhaps then, together, we can get on with the important task of improving curriculum and instruction.

The approach described here has its roots in cognitive psychology and artificial intelligence (computer simulation of human intelligence). Briefly stated, this approach attempts to describe the structure of a mathematical task presented to a student (e.g., adding single-digit numbers without carrying), the way in which the student mentally represents the structure of the task, and the processes the student brings to bear on transforming the initial representation (e.g., "the sum of two and three") in order to reach some goal (in this case the sum, five). Since this approach draws heavily on a psychological model of how students process information in instruction or in solving a problem, the next section of this paper provides a brief sketch of one such model. Of necessity, this model is a compromise between several possible models but, hopefully, does justice to each. The second section presents some methods for representing a mathematical task, since the representation of the external task is just as important in this psychological model as the student's internal representation of it (cf. Newell & Simon, 1972; Shavelson, 1972, 1974a, 1974b). The third section applies the information processing model to students' learning and problem solving in mathematics and reviews illustrative studies. And the final section briefly summarizes the major points of the paper.

Information Processing View of Humans[1]

Recently, psychologists have characterized humans as processors of information. Norman (1976) summarizes this view well:

> In particular, we are concerned primarily with verbal, meaningful information in acoustical and visual form. The aim is to follow what happens to information as it enters the human and is processed by the nervous system. The sense organs provide us with a picture of the physical world. Our problem is to interpret the sensory information and extract its psychological content. To do this we need to process the incoming signals and interpret them on the basis of our past experiences. Memory plays an active role in this process. It provides the information about the past necessary for proper understanding of the present. There must be temporary storage facilities to maintain the incoming information while it is being interpreted and it must be possible to add information about presently occurring events into permanent memory. We then make decisions and take actions on the information we have received. (p. 3)

[1]For a readable introduction to information processing psychology, see Mayer (1977) or Norman (1976). See also Neisser (1976).

One possible, simplified model of human information processing is presented in Figure 1. (For an alternative model, see Craik & Lockhart, 1972; but also see Baddeley, 1978). The information processing model can be divided into two general components: perception and memory. "By perception, we include those processes in the initial transduction of the physical signal into some sensory image, the extraction of relevant features from the sensory image, and the identification of that list of features with a previous learned structure" (Norman & Rummelhart, 1970, p. 21). In Figure 1, perception is represented by an arrow to indicate its presence but secondary importance for the purposes of this paper.

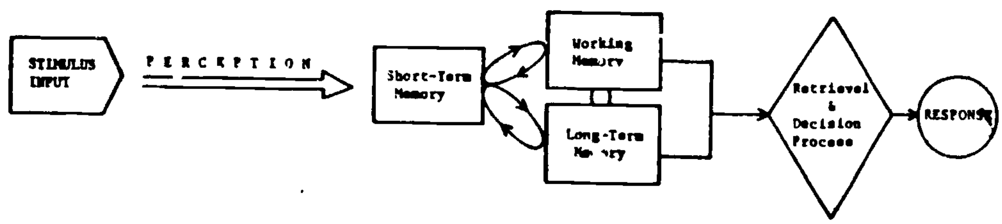

Figure 1. Simplified model of human information processing. (From Shavelson, 1974a, p. 233.)

The second component of the model is memory. By <u>memory</u> we mean those processes which serve to retain information from the perceptual component. Memory can be characterized by four subcomponents: short-term memory (STM), working memory (WM), long-term memory (LTM), and a retrieval and decision process. (Actually, many models of memory combine the last three processes and label them LTM.) In general, STM is a small capacity memory which serves as a buffer between perception and both WM--a fairly large capacity, malleable component--and LTM--an unlimited capacity, highly organized, permanent information store. The retrieval and decision processes search for and retrieve information appropriate in a given context.

Short-Term Memory

In most cases, perceptual information is initially stored in short-term memory. STM is limited in its capacity to hold information; that is, it can store about five to seven words from a long string of unrelated words (cf. Miller, 1956). The amount of time any word or symbol remains in STM without further processing is on the order of seconds. Therefore, a symbol in STM must be transferred to WM or LTM, rehearsed, or lost. Rehearsal takes one of two forms: It may be rote repetition of a string of symbols (e.g., repetition of a telephone number) or an elaborative (cf. Norman, 1976) combining of symbols in some meaningful way (Craik & Lockhart, 1972; Craik & Tulving, 1975; Craik & Watkins, 1973). In the latter case, STM and LTM are linked in order to form meaningful clusters or "chunks" of symbols. The repetitive form of rehearsal maximizes the amount of information held in STM, while the elaborative form maximizes the retention of information over time. Finally, the choice of the form of rehearsal can be controlled by the individual (cf. Bjork, 1972, 1975). STM, then, can be characterized as a communication channel between the perceptual apparatus and the central memory processes whose function is to provide time for further processing to take place. "The net result of such an immediate memory mechanism...is that the total processing system has a very narrow 'focus of attention,' that is, the central processes can attend only to a minuscule portion of the external stimulus environment at any time" (Feigenbaum, 1970, p. 455).

Working Memory

In contrast to short-term memory, working memory is capable of storing moderate amounts of information for hours, days, or perhaps weeks. It holds an internal representation of the stimuli being learned (Feigenbaum, 1970), as well as information from LTM copied intact (cf. Erickson & Jones, 1978). The information in WM is thought to be represented in tree structures with the nodes of the tree representing concepts, ideas, or images, and the lines representing (hierarchical) relationships between them. These representations are pragmatic in the sense that the structure of WM corresponds closely to the sequence of the task. No attempt is made to restructure WM for efficiency or logical order. (This conceptualization of WM may be consistent with students' verbal reports that they learned the material for the examination and then, immediately afterward, "dumped" it!)

Long-Term Memory

In contrast to short-term and working memory, long-term memory is permanent, well-organized, and unlimited in storage capacity. For convenience in describing LTM, a semantic and an algorithmic component can be distinguished.

Semantic component. The semantic component of LTM contains facts about various things, events, and states of the world. While the exact nature of the representation of these facts is currently being debated (e.g., Anderson, 1978), there seems to be agreement that, in using the contents of semantic memory, the facts may be represented either as verbal propositions or as images, analogical mirrors of the world (cf. Baddeley, Grant, Wright, & Thompson, 1975; Brooks, 1968; Kosslyn, 1973; Moyer, 1973; Norman, 1976).

The semantic component is often expressed as a complex directed graph or network (e.g., Anderson, 1976; Anderson & Bower, 1973; Feigenbaum, 1970; Frijda, 1972; Greeno, 1978; Kintsch & Van Dijk, 1978; Norman, Rummelhart, and the LNR Research Group, 1975; Schank, 1975; Shavelson, 1972, 1974a). One possible example of such a network is shown in Figure 2. The nodes in the network represent ideas or concepts or images. The lines represent various relationships between the nodes. In general, three important characteristics of the network are: (a) a relational structure linking nodes by

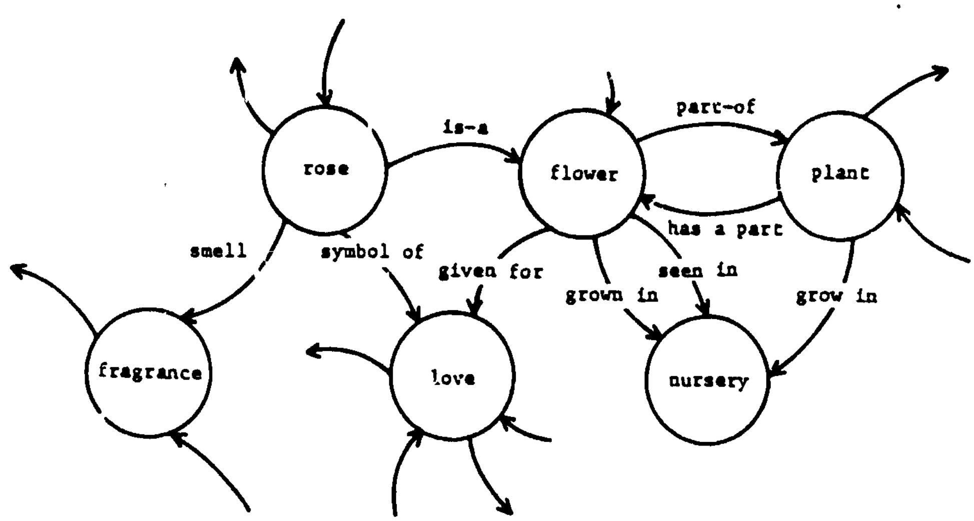

Figure 2. Fragment of information network. (From Frijda, 1972, p. 4.)

specific types of relations (cf. Norman, 1976), (b) a hierarchical feature in which one node may represent a set of nodes (cf. Wickelgren, 1974), and (c) a structure of implicit information. "The organization that does exist is implicit in the pattern of linkages between nodes, which may be direct or indirect over other nodes, and which may give rise to impor- tant local differences in network density" (Frijda, 1972, pp. 5-6).

Gagné (1978) provides a clear example of how one model of semantic memory (Anderson, 1976) would represent a statement like "Bach wrote baroque music":

> Ideas expressed by the sentence...can be represented as the set of nodes and links shown in [our Figure 3]. In this structure, the ideas "Bach," "wrote," "baroque" and "music" (Nodes 4-7) constitute nodes that are associated through links. In addition, a complex node (Node 1) represents the idea formed by the entire proposition, another complex node (Node 2) represents the idea of "wrote baroque music" and a final complex node (Node 3) represents the idea of "baroque music." (pp. 631-632)

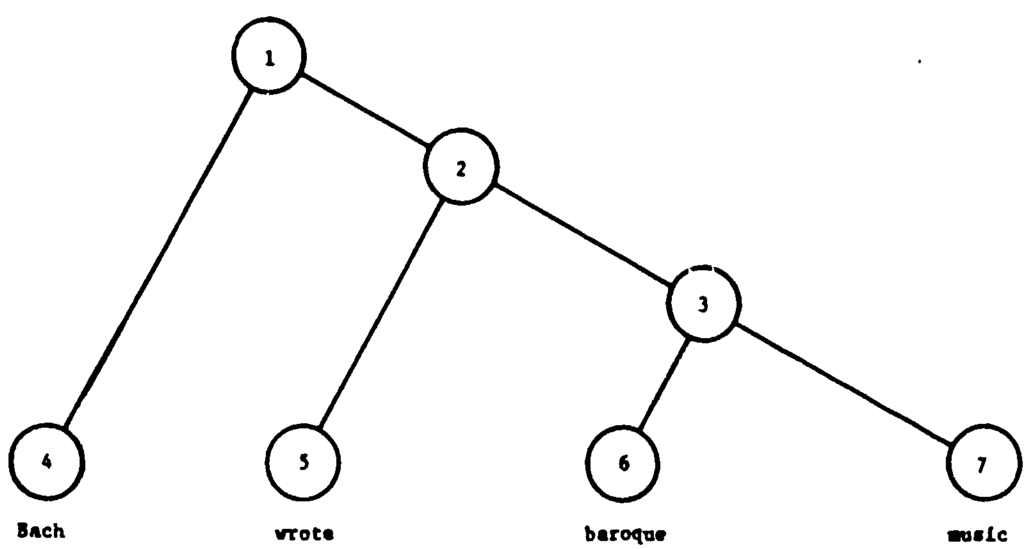

Figure 3. Proposition network for "Bach wrote baroque music." (From Gagné, 1978, p. 632.)

Algorithmic component. The algorithmic component of LTM contains procedural knowledge, i.e., step-by-step procedures needed to reach some goal. Greeno (1978) formalized algorithms as production systems in which "an intellectual skill is represented as a set of productions" and "each component of the skill is a production rule consisting of a condition and an action" (p. 268). Newell and Simon (1972) described a person walking down a street and crossing the street as a simple production system:

Condition		Action
Red light	----------->	stop;
Green light	--------->	move;
Move (left foot on pavement)	----->	step with right foot;
Move (right foot on pavement)	---->	step with left foot.

(pp. 32-33)

Relationship between components. The distinction between the semantic and algorithmic components of memory is one of convenience. They systematically interact. At any point in time, a finite part of the semantic component is assumed to be working. Some of the nodes in the active part of semantic memory are linked to conditions of algorithms. If a condition of an algorithm is activated, the corresponding action is taken until the algorithm has been executed. "The result of the interaction of the [semantic] network and the production system [algorithm] depends upon the part of the propositional network that is active and upon the particular actions generated by the productions whose conditions are met in this active network" (Gagné, 1978, p. 633). Together, these two systems account for what is commonly called *cognition*.

Abstract similarity structure. When curriculum theorists speak of learning a structure, they mean more than just an algorithm or some aspect of semantic structure. They are interested in the abstract relationships among important facts (etc.) in a subject matter. Put another way, they are interested in the student's view of the forest in contrast to the trees. We believe curriculum theorists are interested in similarity structures in students' memories--almost an analog representation of the structure of a subject matter (cf. Shepard, 1978; Shepard & Chipman, 1970). For example, Shavelson (1972) examined students' "views" of the structure of concepts in Newtonian mechanics and represented the similarity structure geometrically as shown in Figure 4. From the curricular point of view, the abstract similarity structure is probably an important component of long-term memory even if it can be derived from more primitive components.

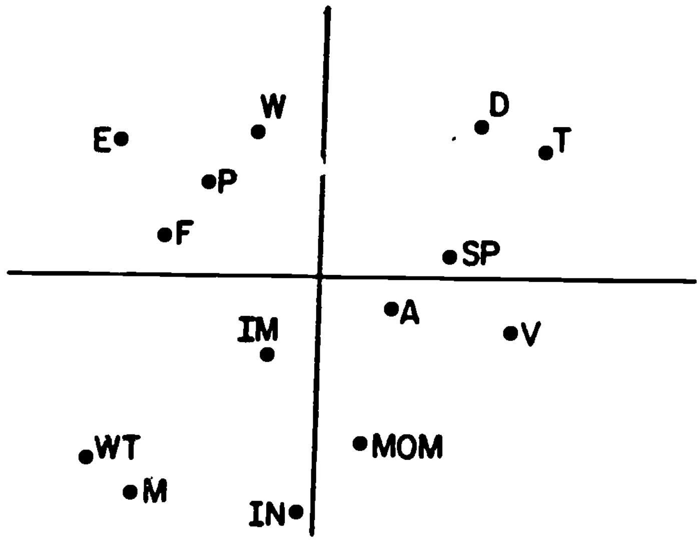

Figure 4. Multidimensional scaling of data representing students' abstract similarity structure in Newtonian mechanics.

Abbreviations:

A = acceleration	MOM = momentum
D = distance	P = power
E = energy	SP = speed
F = force	T = time
IM = impulse	V = velocity
IN = inertia	W = work
M = mass	WT = weight

(From Shavelson, 1970, p. 80.)

Retrieval of Information

With LTM conceived as an unlimited, permanent store of information, "forgetting" occurs because information cannot be retrieved. While there is agreement that limited STM, retrieval problems, and other limitations in human memory lead to errors in problem solving and decision making (e.g., Landa, 1976; Slovic, Fischoff, & Lichtenstein, 1977), there is by no means agreement on how information is retrieved (e.g., Anderson, 1976; Brown & McNeill, 1966; Frijda, 1972; Kintsch, 1970; Mandler, 1967, 1972; Neisser, 1976; Ratcliff, 1978; Tversky & Kahneman, 1974; Wickelgren, 1976). A brief synthesis of several views, especially that of Collins and Quillan (1972), is given below, as well as an indication of the direction in which work in this area is going.

In recall, information must first be retrieved from LTM. Then a decision must be made as to whether the retrieved information is what was required. Suppose a student is asked, "In Newtonian mechanics, force is like what other concepts?" To retrieve information to answer this question, the node representing the concept of force is accessed in LTM. From this node, a search radiates along the lines relating the force node to other nodes. Although the search proceeds along all of the lines leading from the force node, the ease with which the links are traversed is influenced by the context provided by the question. In this example the first concepts retrieved will be "like" force and constrained by the context of Newtonian mechanics.

As each concept is retrieved, it is checked against some subjective criterion the student has as to the type of response required by the question. Thus, the student decides to respond "push" or "mass times acceleration" and not "teacher" or "police." In this manner, the student continues until: (a) all concepts meeting the response criterion have been exhausted, (b) the task is completed, or (c) a time limit for responding is reached.

There are several critical features of the retrieval and decision process. One feature is the importance of the context established by the task and the instructions to the subject. They influence the student's search of LTM and establish a criterion against which alternative responses can be tested. "It should be clear that the appropriate search and decision strategy (or decision rule) varies in different cases, depending on syntax and task instructions, and even the range of stimuli used" (Collins & Quillan, 1972, p. 329). A second feature is that the order of concepts retrieved and the clusters of retrieved concepts are influenced by the structure of LTM. And a third feature is that, in retrieving information from memory, people use strategies or heuristics to simplify the task. For example, in classifying a person or object, an individual will judge the similarity of the person's features to the features of a category held in memory. While this heuristic may work in many circumstances, it can also lead to errors. Thus, in judging the occupation of an individual described as "very shy and withdrawn, invariably helpful, but with little interest in people or in the world of reality" (Tversky & Kahneman, 1974, p. 1124), people will search memory for features similar to those described and conclude, for example, that the individual is a librarian, even though this occupation is held by relatively few people in the labor force.

Representations of Subject-Matter Structure

In describing what students learn as a consequence of mathematics instruction and how they use what they learn to solve mathematical problems, the first step is to describe, as thoroughly as possible, the nature of the subject matter to be learned. For purposes of comparing the nature of the subject matter with students' views of it and how they use it, the subject matter can be represented by the same structure as long-term memory. Hence, a subject-matter structure can be analyzed into a semantic component, an algorithmic component, and an abstract similarity component.

At first blush, forcing a subject matter into these components would seem to distort it, but does not, for at least two reasons: one, logical relationships between the concepts and procedures of a subject matter are retained; two, a subject matter is ultimately psychological in nature since it was, and is, conceived by the human mind. Thus, a subject-matter structure may be thought of as a representation of the agreed-upon structure of the knowledge of the experts in the field at some particular point in time (cf. Shavelson & Stanton, 1975). Over time, this structure is expected to change as new knowledge is gained. And so, the representation of the subject matter structure should also change.

Abstract Similarity Structure

One possible view of the structure of a subject matter is what Shavelson (1974a) termed content structure: "the web of facts (words, concepts) and their interrelations in a body of instructional material" (p. 231; see also Shavelson, 1972, 1974b). This representation is obtained by identifying key terms (representing concepts) in the instructional material, syntactically parsing the instructional material to obtain relationships between the key terms, and mapping these relationships onto a directed graph, or digraph (Harary, Norman, & Cartwright, 1965) via a set of rules (Shavelson & Geeslin, 1975). When the digraph is represented as a key-term by key-term distance matrix, the underlying similarity relationships between key terms may be examined with nonmetric scaling procedures or clustering algorithms to produce a spatial representation of the content structure.

This procedure for examining content structure has been successfully applied to subject matter in physics (Shavelson, 1972; Shavelson & Geeslin, 1975), probability (Geeslin & Shavelson, 1975a, 1975b), and operational systems (Branca, 1980; Shavelson, 1974b; Shavelson & Stanton, 1975). For example, an operational system (OS) is defined as "a set together with a binary operation on the set. An OS (e.g., whole numbers under addition) may possess any or none of the following properties: associativity, commutativity, identity element, and roundness (inverses)" (Shavelson, 1974a, p. 244).

From OS curriculum materials, key terms were identified and the digraph analysis performed with the resulting structural representation shown in Figure 5. Construct validation studies (Shavelson & Stanton, 1975) suggest that the interpretation of Figure 5 as a representation of content structure is warranted.

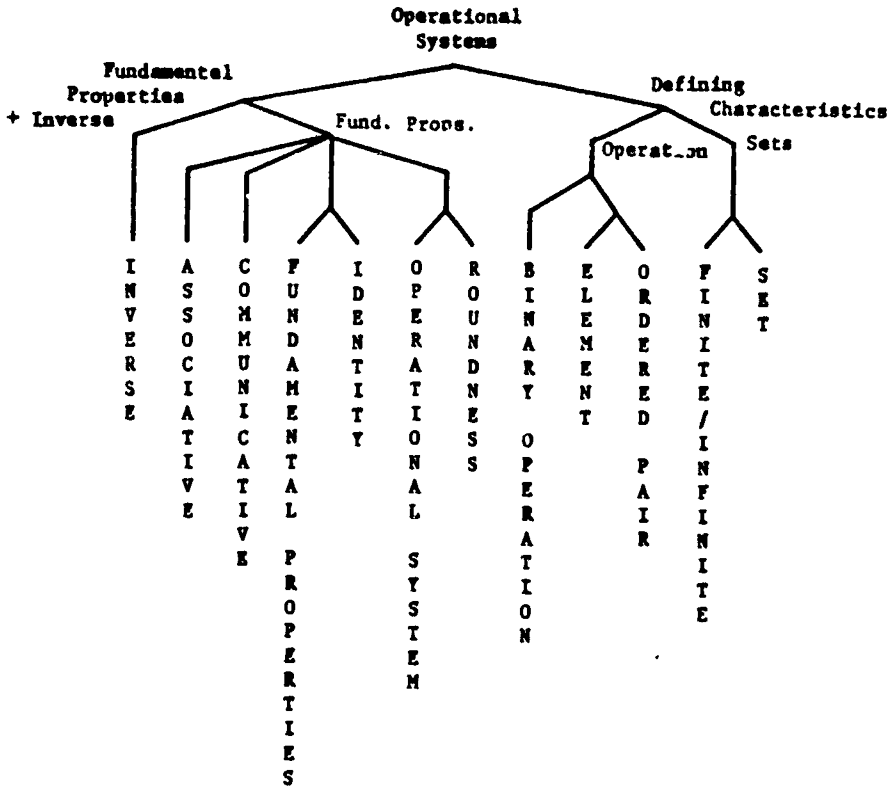

Figure 5. Digraph analysis of content structure. (From Shavelson, 1974a, p. 246.)

Algorithmic Structure

The term algorithmic structure denotes a step-by-step procedure used to solve a problem in mathematics. In school, algorithms often are not taught in their entirety to students, although bits and pieces of them are embedded in the mathematics curriculum. It is often assumed that students will somehow acquire algorithms for computing solutions to problems and that they will select the appropriate ones for a particular problem on a given occasion and execute them accurately and efficiently.

An example of an algorithmic structural analysis for the multiplication of mixed numbers is shown in Figure 6.[1] The top part of the algorithm transforms all quantities into the desired a/b form, where a is the numerator and b is the denominator of the fraction. For example, the mixed number 1-1/2 would be rewritten as 3/2. (This portion of the algorithm is never presented explicitly in any single chapter of Peters et al., 1974. It is one of those procedures that is assumed to be acquired by the students.) The next step is to perform the multiplication:

$$\frac{a_1}{b_1} \times \frac{a_2}{b_2} = \frac{a_1 \times a_2}{b_1 \times b_2}$$

For example, $\frac{3}{2} \times \frac{2}{9}$ is rewritten as $\frac{3 \times 2}{2 \times 9}$ and then computed.

The algorithm assumes that the prerequisite skill of multiplication of whole numbers has already been mastered, so that $\frac{3 \times 2}{2 \times 9}$ will yield $\frac{6}{18}$. The final task in the multiplication is reducing the answer to lowest terms. Figure 6a indicates one procedure for doing so by recognizing that 6 is the highest common factor of 6 and 18 so that $\frac{6}{18}$ reduces to $\frac{1}{3}$.

Given two or more nonnegative rational numbers to be multiplied, the algorithm presented will always provide the correct answer.

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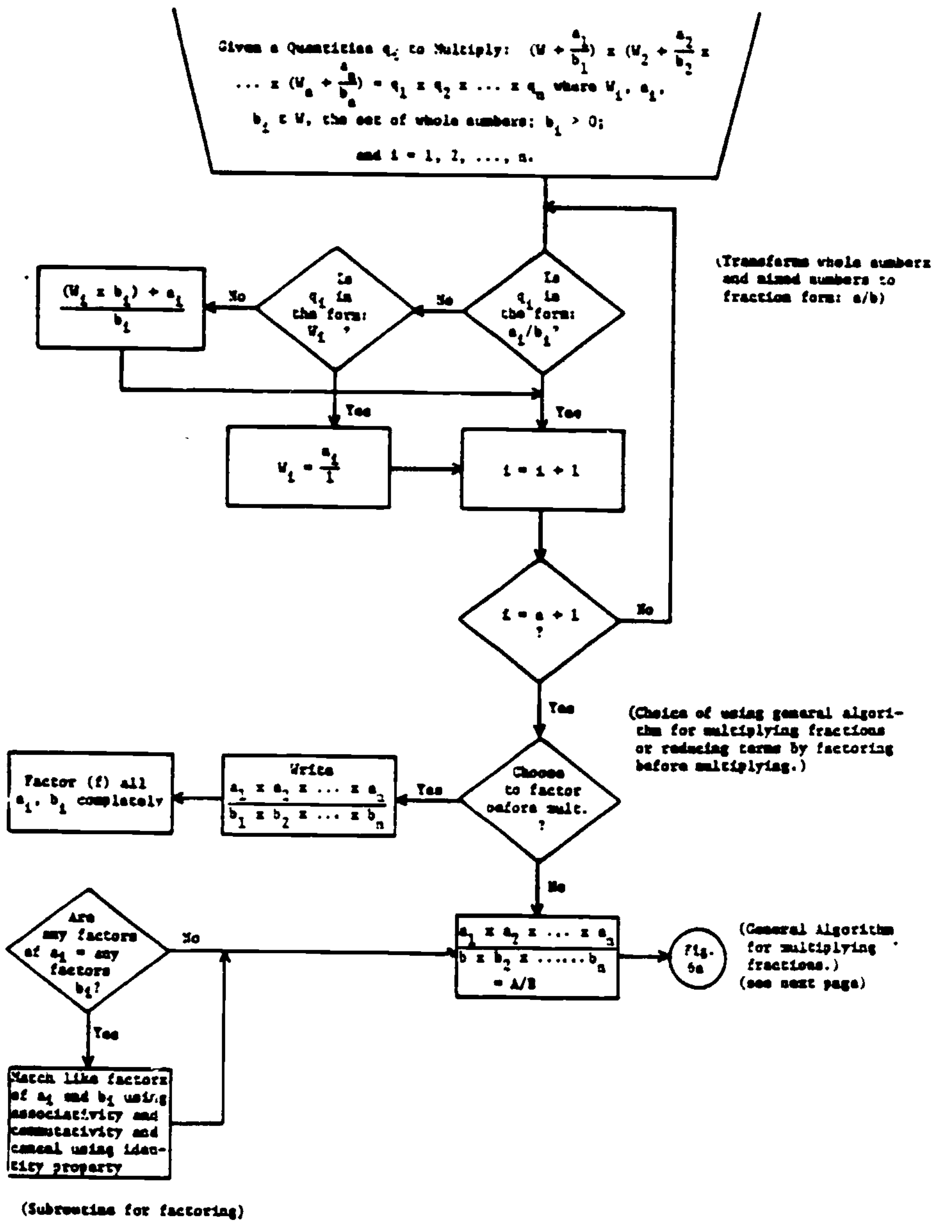
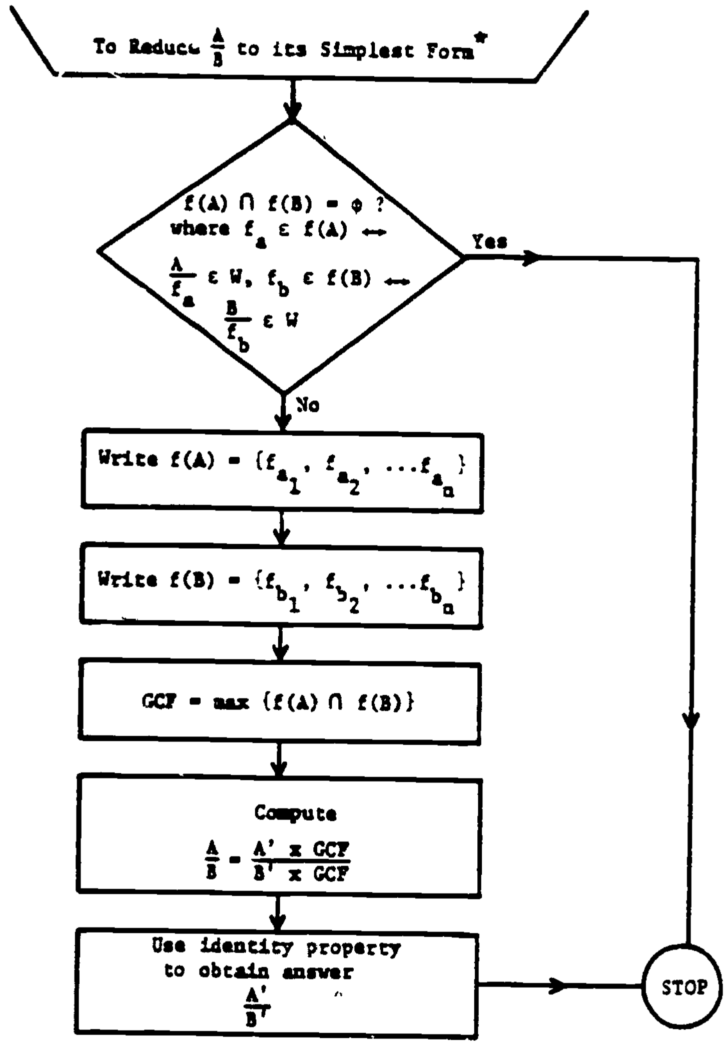
[1]Note that the algorithm incorporates what is presented in several chapters of a textbook. It is not necessarily an "ideal" algorithm in its comprehensiveness or efficiency; however, it does represent the algorithm the students are expected to learn.



*Inefficient if a_i, b_i are relatively prime.

Figure 6. General algorithm for multiplying mixed numbers as presented in a junior high school mathematics textbook (Peters, Schor, Mang, & Wayne, 1974).



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*A and B are whole numbers and B > 0;
f_a is a factor of A and f_2 is a factor of B;
f(A) is the set of all factors of A and f(B) is the set of all factors of B;
W is the set of whole numbers; and GCF is the Greatest Common Factor.

Figure 6a. Subroutine for reducing the fraction A/B to its simplest terms.

It may appear as though multiplication of nonnegative rational numbers should be divided into subdomains. The students must learn how to multiply proper fractions (less than 1), improper fractions (greater than 1), mixed numbers, and combinations of fractions (including whole numbers). Many texts, including Peters et al. (1974), treat some of these cases separately. However, a nonnegative rational number is defined as a number which can be written as the quotient of two whole numbers, excluding division by zero. Since proper fractions, improper fractions, mixed numbers, and whole numbers can all be written in the form a/b, where a and b are whole numbers ($b \neq 0$), one general algorithm can be applied to all possible cases, as shown in Figure 6. (However, note that we are speaking of a subject-matter structure. In an analysis of a student's algorithmic memory structure, separation into different categories might be quite prevalent.)

The bottom section of Figure 6 indicates that the student has the option of immediately multiplying the given quantities once they are written in the correct form or factoring and simplifying before multiplying, thereby obtaining a solution in lowest terms. Furthermore, the student need not choose one of the algorithmic procedures presented in the text. The learner may devise other strategies or heuristics to simplify the problem-solving process. These options will be considered more fully later in this paper.

Semantic Structure

Understanding mathematical concepts is basically understanding language. Semantic structure is the representation of the meaning of text material and word problems. According to Greeno (1978), understanding a text requires the construction of a representation in memory of the information in it. Although his work focuses on cognitive structure, it can be interpreted in terms of subject-matter structure. For example, Figure 7 presents Greeno's propositional structure for the "idea that division is the inverse operation of multiplication.... [In this figure] multiplication and division are represented as actions that cause changes in the value of a quantity. I suggest that if a goal of instruction is to have students understand this idea, then one achievement that is desired is that students have a schema like [the one shown in Figure 7] in their cognitive [memory] structures" (p. 266).

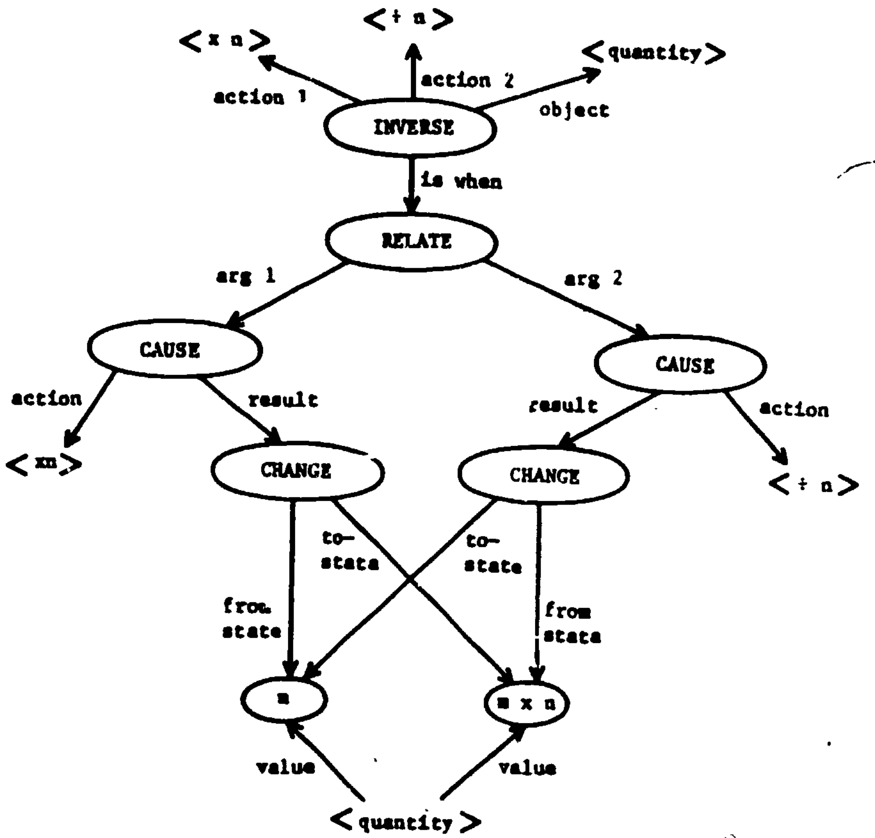
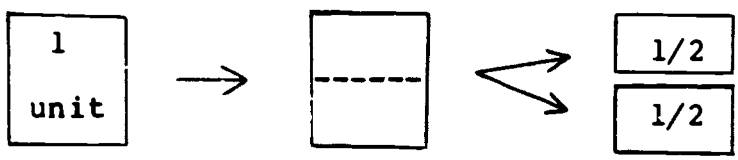


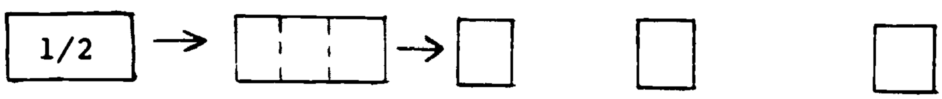
Figure 7. Propositional structure for the idea that division is the inverse of multiplication. (From Greeno, 1978, p. 266.)

Semantic structures in mathematics texts also include spatial representations of arithmetic concepts. So an analysis of semantic structure should be able to provide an analogical representation as well as a propositional structure. For example, textbooks convey the meaning of rational numbers and operations on them in a variety of ways.

Most books rely heavily on imaginal representations. Children may first be taught to translate 1/3 x 1/2 into "1/3 of 1/2." This proposition presumably gets stored in memory along with the multiplication algorithm and countless other propositions in mathematics. The meaning of multiplying two rational numbers in fractional form may then be taught by dividing boxes. For example, start with one box and divide it into halves:

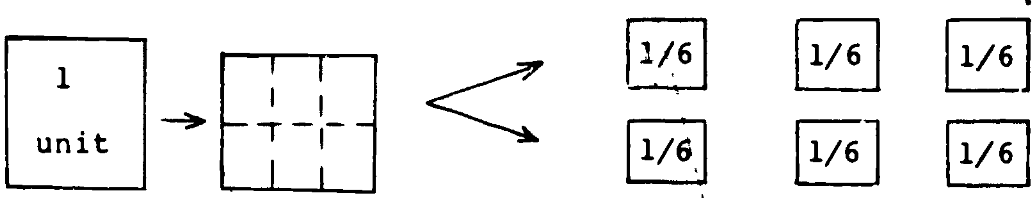


Divide one of these halves into thirds:



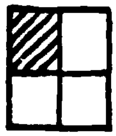
1/3 of 1/2 1/3 of 1/2 1/3 of 1/2

These two operations give the same result as dividing the entire box into six equal parts:

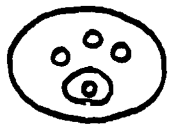


Therefore, 1/3 of 1/2 must be the same as 1/6. Thus, words are used in conjunction with pictures to represent the meaning of 1/3 x 1/2. This representation is intended to give meaning to the multiplication of rational numbers and to provide the foundation for learning the algorithm for multiplying fractions. Once the topic has been developed conceptually, a new algorithm is presented: $a/b \times c/d = \frac{a \times c}{b \times d}$. It is evident that the algorithmic structure is embedded in the semantic structure of a subject matter.

Other textbooks represent fractions in other ways. For example, the fraction 1/4 can be interpreted as a region:

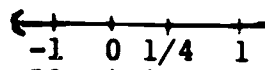


; part of a collection:

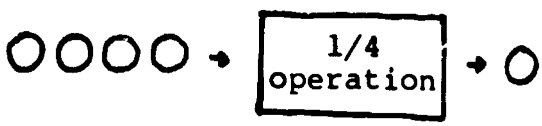


; a segment:

a point on a number line:



; an ordered pair: (1,4); or a quotient: "1 divided by 4." A particularly interesting representation is the fraction as an operator (Kieren, in press) in which quantities are inputs and outputs of a machine which performs the operation of "fractioning" on them:



Syntactic Structure

<u>Syntactic structure</u> can be defined as the representation of a sequence of mathematical symbols. Syntactic knowledge is required for understanding the relationship among quantities in an arithmetic expression. Although the students may already possess the semantic knowledge necessary for understanding fractional relationships, they need to learn an abstract notational system for describing these relationships. They must be able to recognize the equivalence of 1/2, $\frac{1}{2}$, and possibly (1,2). They need to distinguish between $1/2 \times 3/4 = ?$ and $1/2 \times ? = 3/4$.

Syntactic and semantic structure are strongly interrelated. In fact, it is difficult to say where syntax ends and semantics begins. For example, recognition that the particular product $1/4 \times 1/6$ is the same as $1/6 \times 1/4$ follows a syntactic rule, but knowledge that $a/b \times c/d = c/d \times a/b$ for all whole numbers a, b, c, d, (b, $d \neq 0$) implies understanding of the commutative property.

Representation of Students' Knowledge of Mathematics

In this section studies of mathematics learning and problem solving are reviewed. They are related to one another in that they are motivated by information processing models. Since the model outlined earlier has implications in many areas of mathematics but has not yet been applied systematically to any one domain, we will draw examples from several different domains. However, the topic of rational numbers will serve as our primary example wherever possible. Thus, we will begin to develop a cognitive theory of rational numbers by modeling the underlying psychological processes. Studies from other areas of mathematics will be drawn upon as supplementary examples of the application of the information-processing model.

We begin by assuming that the student is an information-processing system, confronted by a well-structured mathematics problem that is difficult but not unsolvable. Methods for solving the problem are accessed, relatively slowly, from long-term memory (LTM). The structural components of LTM, built upon prior knowledge and skills acquired during learning, enter into the problem-solving process. We posit three structural components (cf. Greeno, 1978)--an algorithmic structure, a semantic structure, and a syntactic structure--and tie them together with an overarching, abstract similarity structure (cf. Shavelson, 1972, 1974a, 1974b). These components guide our analysis of the cognitive structure of individual students.

We use the term <u>cognitive structure</u> to refer broadly to the memory structures and processes involved in learning and solving problems. Cognitive structure, broadly defined, is "a hypothetical construct referring to the organization (relationships) of concepts in memory" (Shavelson, 1974a, p. 232). It is built in the process of learning a subject matter and influences problem solving (e.g., Mayer & Greeno, 1972; Shavelson, 1972).

We speak of more than one structural component of cognitive structure because the particular component will depend on the task at hand. For example, if the task is to teach a student to find the answer to the problem, 1/2 x 1/3, it is possible to teach a relatively simple algorithm which involves multiplying the numerators and denominators to obtain 1/6. If the complexity of the task is increased somewhat, say 1/2 x 2/3, the algorithm will serve our purposes with the inclusion of a component for reducing 2/6 to 1/3 or a component for simplifying, making use of the commutative and identity properties (e.g., $\frac{1}{2} \times \frac{2}{3} = \frac{1 \times 2}{2 \times 3} = \frac{2 \times 1}{2 \times 3} = \frac{2}{2} \times \frac{1}{3} = 1 \times \frac{1}{3} = \frac{1}{3}$).

However, the algorithmic representation of the problem does not capture all that is involved in, say, a word problem such as: "If Johnny receives one-third of half of a pie, how much of the pie did he receive?" In this case, the task is to decode the sentence for its meaning so that an appropriate computational algorithm can be identified. Thus, another representation of the structure, a semantic representation, is needed. And, as Kieren (in press) suggests, that semantic structure may be represented by at least five different models of rational numbers. A student may or may not decode the word problem correctly depending on the model learned and the model underlying the word problem. Finally, when mathematicians speak of the structure of mathematics, they mean more than just an algorithm or some aspect of semantic structure (cf. Begle, 1971; Geeslin & Shavelson, 1975a, 1975b; Shavelson, 1974a). They are interested in, for example, the relations between operations. For instance, addition and multiplication of whole numbers and rational numbers are logically the same. However, psychologically, multiplication of whole numbers can be conceived of as successive additions, while multiplication of rational numbers cannot. With rational numbers, addition and subtraction are similar in the need to find the (least) common denominator, and multiplication and division are similar in that once the fraction is inverted in a division problem, multiplication is used to find the answer. Moreover, mathematicians consider knowledge of the fundamental properties of operational systems (e.g., associativity, commutativity) essential in working with fractions. These properties play an important role in justifying a method for, say, multiplying fractions or factoring terms.

Different but interrelated aspects of cognitive structure can be identified. They correspond to the distinctions made with respect to subject-matter structure: (1) abstract similarity structure, (2) algorithmic structure, (3) semantic structure, and (4) syntactic structure. Each is discussed below.

Abstract Similarity Structure

The organization of key concepts in memory, according to their similarity, is one aspect of a student's cognitive structure. With respect to fractions, abstract similarity structure refers to the student's view of the key concepts (addition of fractions, least common denominator, associativity, etc.) as a whole.

There are a number of methods available for examining a student's similarity structure: (a) word association, (b) similarity judgments, (c) card sorting, and (d) graph building (cf. Fillenbaum & Rapoport, 1971; Shavelson, 1974a; Shavelson & Stanton, 1975). These measurement techniques have been successfully applied to diverse areas such as physics

(Cox, Johnson, & Curran, 1970; Johnson, 1964, 1965, 1967, 1969; Preece, 1976; Shavelson, 1972), educational psychology (Konold & Bates, Note 2), statistics and psychometrics (Traub & Hambleton, 1974), botany (Rudnitsky, 1976), and color perception (Fillenbaum & Rapoport, 1971). Figures 8 and 9 depict results of a hierarchical cluster analysis of data obtained from the word association and graph construction measures of cognitive structure, respectively, using the subject area of operational systems. (See Figure 5 for a representation of the similarity structure of this subject matter. For information on validating construct interpretations of these measures, see Shavelson and Stanton, 1975.)

With respect to problem solving, some previous research suggests a low, positive correlation between "goodness of similarity structure" (i.e., goodness of the fit between representations of content and cognitive structure) and achievement test scores (e.g., Geeslin & Shavelson, 1975a, 1975b; Shavelson, 1972). Other studies have shown a stronger relationship (e.g., Konold & Bates, Note 2). Figure 10 provides data on the correspondence between content and cognitive structure for students learning probability and a control group learning an unrelated mathematical topic. These data indicate that, after instruction, experimental subjects' cognitive structures corresponded much more closely to the content structure than they had prior to instruction, and that this correspondence was closer than that of the control group.

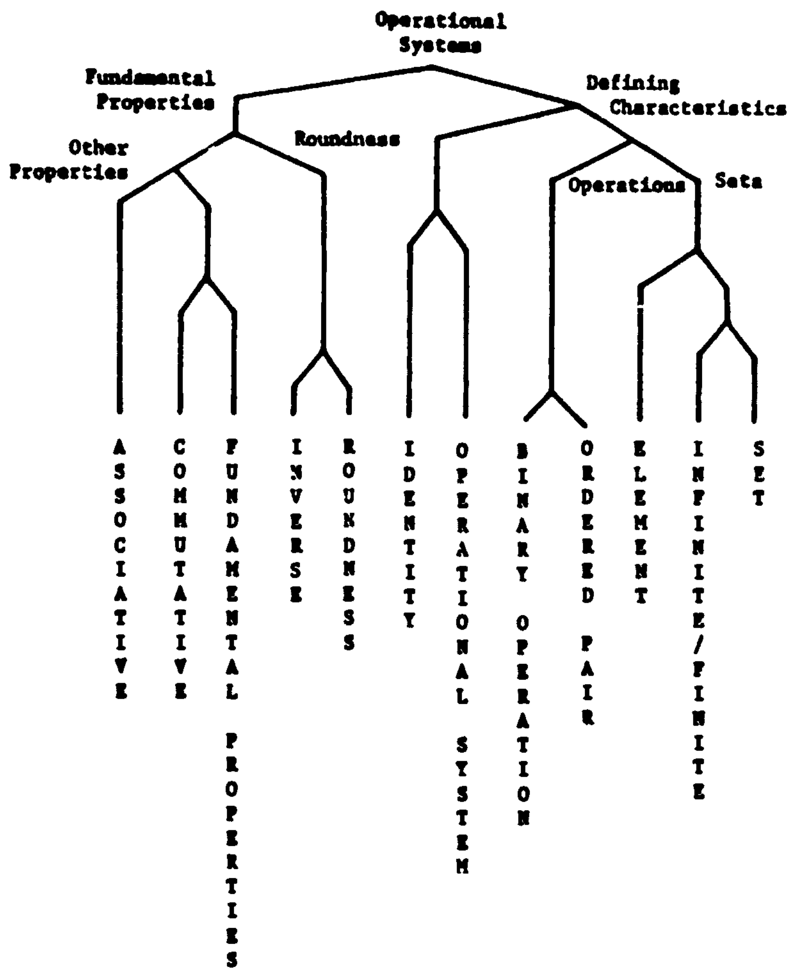


Figure 8. Word associatiๅn m?asure of cognitive structure: Results of the hie:archical cluster analysis of proximities (relatedness coefficients) between key concepts in operational systems. (From Shavelson, 1974a, p. 247.)

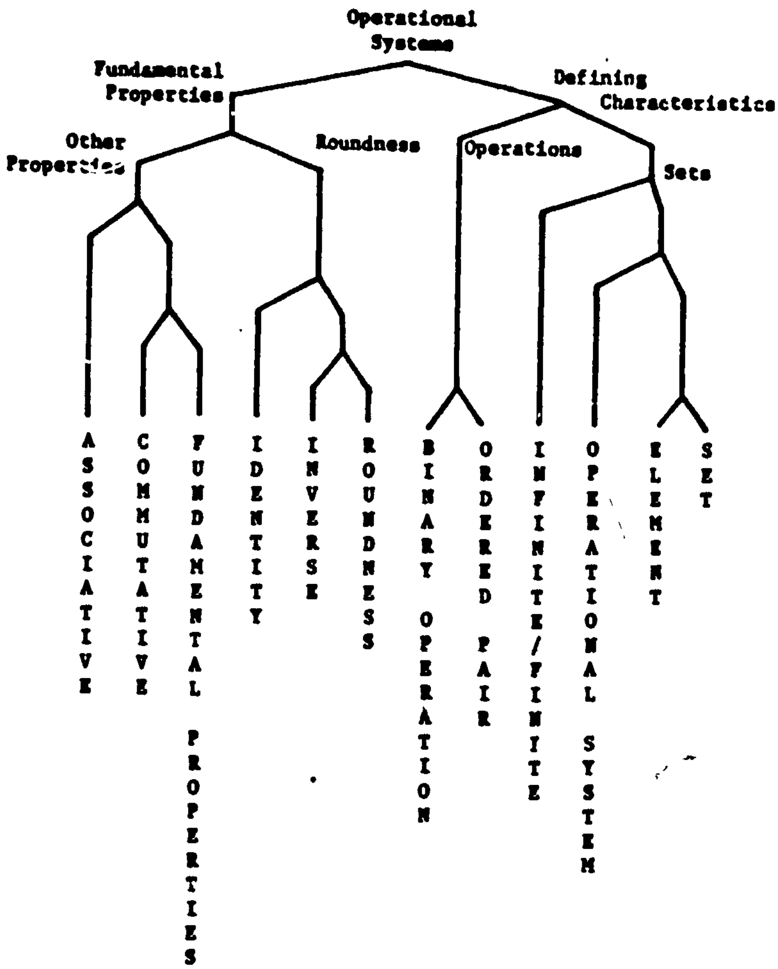


Figure 9. Graph construction measure of cognitive structure: Results of the hierarchical cluster analysis of graph distances between key concepts in operational systems. (From Shavelson, 1974a, p. 247.)

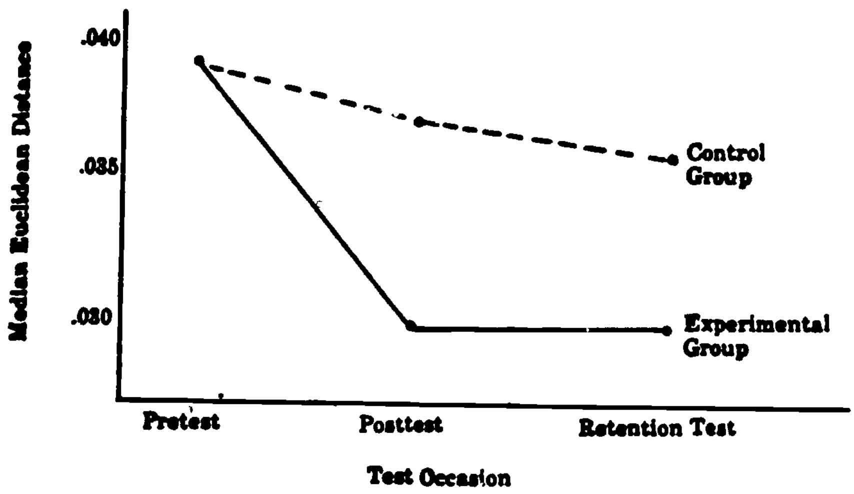


Figure 10. Median Euclidean distances between cognitive structure and content structure. (From Geeslin & Shavelson, 1975b, p. 35.)

Algorithmic Structure

Another method of representing the knowledge acquired by children learning fractions is the analysis of their algorithmic procedures. As discussed earlier, the term algorithm denotes a step-by-step procedure used to solve a problem in mathematics. Prior research (Brown, Burton, & Hausman, 1977; Ginsburg, 1977; Groen & Resnick, 1977; Lankford, Note 3) has shown that algorithms children use in their computations are often unlike those they were taught. Resnick (1980) and Groen and Resnick (1977) provide evidence of invention on the part of young children learning simple addition where exposure to instruction was controlled. They were taught to add two numbers, $m + n$, by setting a "counter" in the head to zero, incrementing it m times, then, without resetting, incrementing it n more times. However, some of the children apparently set the counter to whichever of the two numbers was greater and then incremented the counter by the other (smaller) addend. The invented procedure (heuristic) is more complex but much more efficient than the one they had been taught.

Woods, Resnick, and Groen (1976) postulated five possible algorithms used to solve single-digit subtraction problems of the form $m - n$, where $m \geq n$. They are:

1. The counter is set to 0, incremented m times, then decremented n times. The solution is the final value in the counter.

2. The counter is set to $\underline{m}$ and then decremented $\underline{n}$ times. The solution is the final value in the counter.

3. The counter is set to $\underline{n}$ and then incremented ($\underline{m}$ - $\underline{n}$) times, until $\underline{m}$ is reached. The solution is the number of times the counter has been incremented.

4. The counter is set to 0, incremented $\underline{n}$ times and then incremented until $\underline{m}$ is reached. The solution is the number of times the counter has been incremented after $\underline{n}$ is reached.

5. Either procedure 2 or 3 is used, depending on which requires fewer operations. (p. 18)

They found that most children used the heuristic procedure (#5), but that some of the younger children used the taught algorithm (#2). This finding suggests a developmental trend in which children progress from using fixed algorithms to solve subtraction problems to using more efficient heuristics involving judgment and estimation.

Sometimes the algorithms invented by children are fraught with errors or "bugs" (Brown, Burton, & Hausman, 1977). These bugs may initially appear to be random errors but are actually systematic and predictable errors of procedure. Davis (Note 1) cites examples in which the procedures themselves have flaws (e.g., $\underline{c}[\underline{x}^2 + \underline{y}^2] = \underline{cx}^2 + \underline{y}^2$), as well as instances of activating the wrong procedure for a specific problem even though it is performed correctly (e.g., in .3 + .4 = .07, counting decimal places and adding them). These mistaken strategies can arise from misunderstandings of the syntactic, semantic, and/or algorithmic structure of the instructional material.

A process-tracing approach[3] can be used to characterize algorithms employed by children in solving problems involving rational numbers. In the process-tracing method (Shulman & Elstein, 1975), children are asked to "think aloud" as they work through a problem. Protocols are written from their oral descriptions which are then translated into step-by-step procedures.

The algorithm for multiplying mixed numbers, shown in Figure 11, was produced when the process-tracing method was used with one of the authors of this paper. In this

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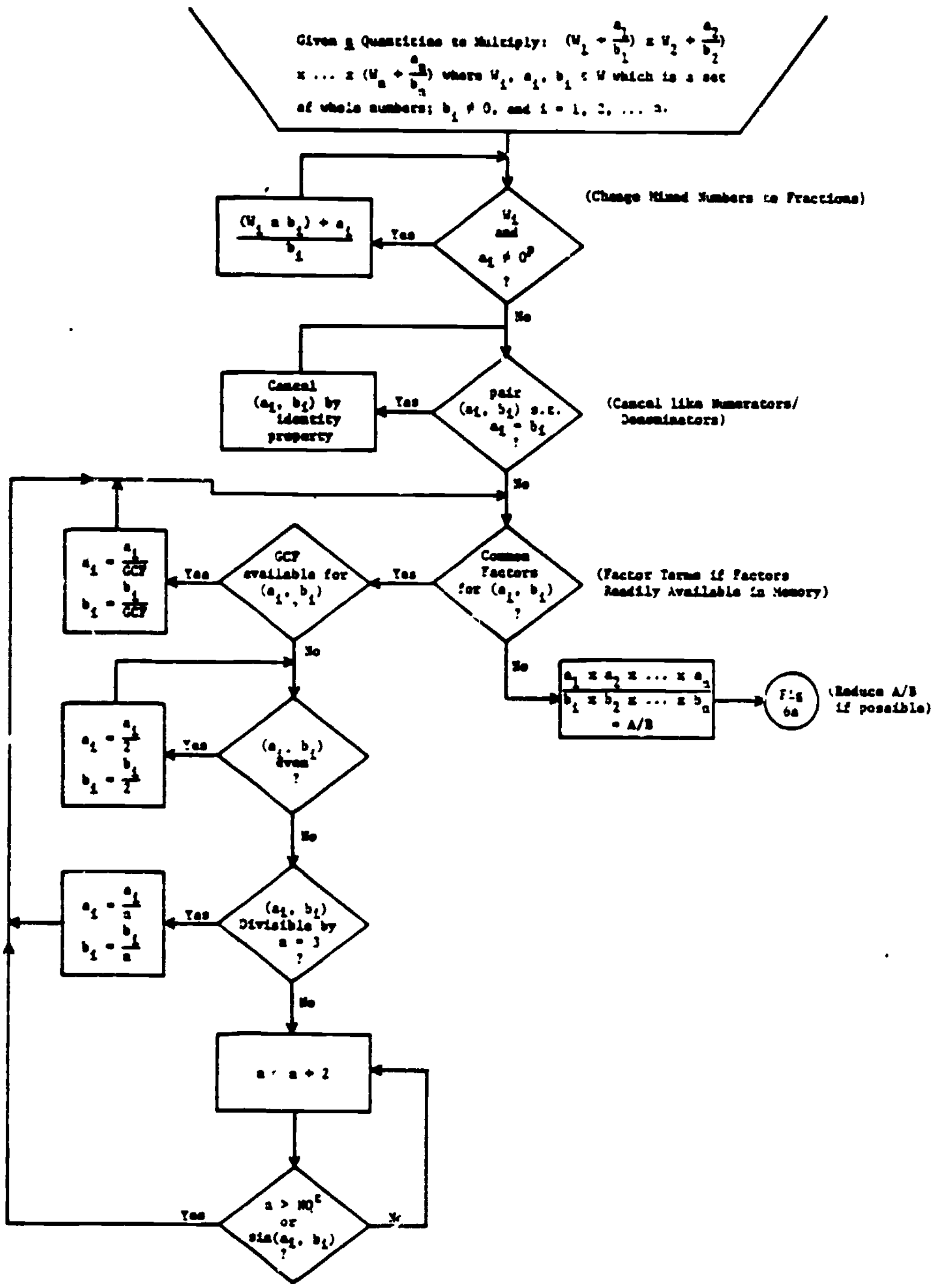
[3]This technique has been criticized (e.g., Erickson & Jones, 1978) for imposing a serialization of processing which may not necessarily exist.



P If $a_i = 0$, W_i replaces a_i and $b_i = 1$.

t NQ = nauseousness quote—decide cannot find common factor.

Figure 11. Representation of one author's algorithm for multiplying fractions.

algorithm, all cancellations which are obvious are performed before the multiplication is carried out. These include cancelling equal numerators and denominators and using the greatest common factor to simplify the expression. If the greatest common factor is not readily available, a search for factors is conducted until no additional factors are found or the nauseousness quota (NQ), determined by the individual, is reached. When the search is completed, the multiplication is carried out using the general algorithm:

$$\frac{a_1 \times a_2 \times \ldots \times a_n}{b_1 \times b_2 \times \ldots \times b_n}\ .$$

The result is then reduced to its lowest terms if necessary.

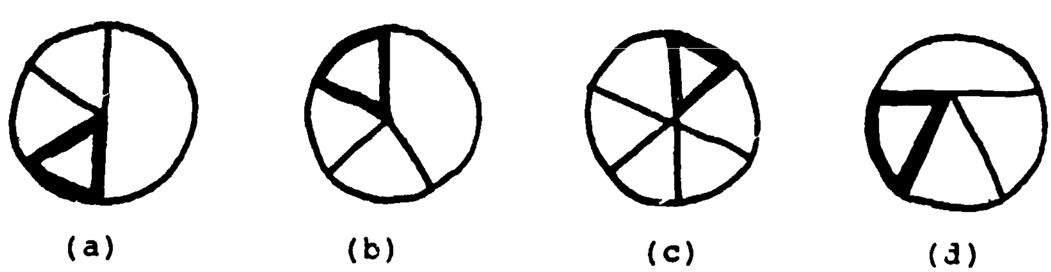
The analysis of "bugs" in students' algorithms provides data upon which instructional treatments can be developed to test our model of this aspect of cognitive structure. These treatments would address common errors in algorithms as well as attempt to provide "inventions" (cf. Resnick, 1980) as heuristics for solving problems. Of particular concern is the link of this aspect of structure with the other representations.

Semantic Structure

A third method of representing a student's cognitive structure is semantic structure. This component of memory contains information about the "meaning" of, say, a fraction such as 1/2 or of fractions such as 1/2 of 1/4. It also provides the means for decoding prose material and word problems. And, finally, it points to algorithms and similarity structures involved in solving problems with fractions.

Empirical research is needed to test the flexibility of semantic representation of fractions in students' memories. Is it the case that students will differ in their answers to the following question based on how fractions were originally presented to them?

Which pie correctly represents the solution to 1/3 x 1/2? You may circle more than one answer.



(a) (b) (c) (d)

Circling (a) alone would indicate rigidity in the interpretation of expressions involving fractions.

Thus, the semantic structure or schema corresponding to a student's understanding of simple expressions with fractions may differ depending on the method used to teach the representation of these expressions. This idiosyncratic interpretation can also be used as the basis for understanding a word problem such as "John has six marbles. One-half of them are red. How many marbles are red?" One student may need to draw a picture to represent the situation and count the red marbles, whereas another student may be able to translate the sentences into symbols directly and multiply.

In order for a student to understand multiplication of rational numbers, Greeno's (1978) criterion of connectedness must be satisfied. Connectedness refers to relating new information to prior knowledge including familiar experiences, specific mathematical procedures, and general mathematical concepts. Thus, learning of fractions should be facilitated by the student relating them to personal experience with such things as pies, boxes, and money. Furthermore, the student should realize that multiplication is an operation on a pair of numbers, as are addition and subtraction, and that any two numbers can be multiplied whether they are whole numbers, mixed numbers, or fractions. The general algorithm for multiplication then, will not be understood unless the student integrates it with procedures already learned, such as multiplication of whole numbers. However, it is also possible that the student who learns that rational numbers are operators themselves may relate them in an abstract similarity structure to other operators like "+" or "-."

Comparisons between subject-matter structure and students' semantic structures can be made in order to ascertain the range of knowledge the students have and their ability to decode a word problem and select an appropriate algorithm for solving it. This comparison will form the basis for developing instructional treatments and connecting verbal statements to algorithms and so provide a test of the validity of this representation of structure.

Syntactic Structure

Syntactic structure in students' memories was previously defined as the cognitive representation of the order of mathematical symbols. Errors in students' computational strategies may be revealed in their acquired syntactic structures. The student who responds "2" to $1/2 \times ? = 4$ may have executed the wrong algorithm because of a failure to understand the significance of symbol position. However, the error might arise from a misunderstanding of the concept of inverse operation or from not knowing that division is the inverse operation of multiplication. Or, the error might result from a bug in the division algorithm. Uncovering the source of error is not an easy task. If the child rewrites 4-1/2 as 4 x 1/2, is this an error in syntax or semantics? Syntactical errors are related to bugs in algorithmic structure, semantic structure, or both. Instructional treatments for correcting not just the syntactical bugs but also the algorithmic and semantic ones need to be developed.

Summary

This paper sketched some of the methods used by educational psychologists to study mathematical learning and problem solving from an information-processing perspective. The student was characterized as an information-processing system that (a) received initial information, such as a mathematical problem to be solved, through perceptual processes, (b) represented the concept or problem internally in memory structures, (c) transformed the internal representation as learning and problem solving progressed, and (d) responded (e.g., gave a problem solution) overtly on the basis of this internal representation.

A general model of memory processes was then set forth, emphasizing long-term memory (LTM; a permanent, unlimited, highly organized information store) and two major structural components of LTM: semantic structure (knowledge of things, events, and states of the world) and algorithmic structure (knowledge of step-by-step procedures needed to reach some goal). This model was used to map the structure of the task confronted by the student (e.g., the problem to be solved) and the way in which the student represented the task internally and transformed it in order to reach some goal (e.g., finding a solution to the problem). Finally, representative research both in the structural analysis of the task and in the structural analysis of memory processes was described.

The emphasis of the information-processing approach on the analysis of the structure of the task as well as on the analysis of the student's internal representation of the task makes the collaboration between mathematics educators and psychologists essential. Mathematics educators are best able to identify the many possible ways to conceive of a mathematical concept or problem, while psychologists are best able to model the processes the student uses in order to learn mathematics and solve mathematical problems. If this paper fosters such collaborative research in some small way, it will have fulfilled its purpose.

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Reaction

to

AN INFORMATION PROCESSING APPROACH TO RESEARCH ON MATHEMATICS LEARNING AND PROBLEM SOLVING

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The reader should note that Geeslin and Shavelson have worked together on occasion and that Geeslin is in general agreement with many tenets of information processing theory. Thus, the reactions presented here do not question the validity of information processing models as plausible descriptors of cognitive behaviors or processes. Rather, this paper will focus on problems of communication between mathematics educators and psychologists or educational psychologists and the incompleteness of the model(s) outlined by Shavelson and Porton.

The Shavelson and Porton paper apparently fulfills quite well its stated purpose of being "a report to mathematics educators concerning the methods used by some educational psychologists to study learning and problem solving in mathematics." A reader who is totally unfamiliar with the methods and research discussed in the paper might still have questions after reading the paper, but the extensive list of references should provide the answers to most questions. The language and examples used in the paper are such that most readers should not only understand the "spirit" of the type of research being presented but also comprehend the theory and current results. Information processing appears to be a promising methodology for furthering our understanding of human behavior, particularly when this behavior involves mathematics. Of course, many unanswered questions about learning remain, but Shavelson and Porton outline promising avenues by which some of these questions may be answered. I believe this is the most one can ask of any emerging scientific theory.

Even though this reactor greatly appreciates the synthesis of research achieved by Shavelson and Porton, this reaction will be useful only if it focuses on questions raised by the paper and on aspects of the theory not clearly explained or known. To this end, I will consider three issues raised in reading the paper: (a) a problem of language and/or terminology, (b) the failure to account for higher level processing, and (c) the lack of information concerning retrieval from long-term memory.

Terminology

Shavelson and Porton discuss the fact that many educational psychologists use mathematical content in their research without having an inherent interest in mathematics per se. I think one reason mathematics is often used as a vehicle in research is that many people view it as a highly structured content area compared to other common school subject-matter areas; they believe that unique, correct solutions exist to mathematical problems or tasks. However, this view is at least partially incorrect. Even though current school mathematics programs may often emphasize the "right answer" and the "correct algorithm," mathematics can be operated upon (or constructed) at a higher cognitive level in any school grade. Searching for patterns, developing new algorithms, proving conjectures, and revising assumptions are all examples of "mathematical" activity. At least some desired mathematical processing implies less, rather than more, structured mental operations as the result of such processing. The psychologist, who is familiar primarily with common school mathematics, may well have a different view (concept) of addition than the mathematician. As a consequence, the psychological model may not do well at predicting mathematical problem-solving behavior even if it does simulate cognitive processing related to computation. Thus, the mathematics educator may complain about the inappropriateness of the model for classroom application, or worse yet, ignore the model because mathematical terms clearly are misused (the example of Piaget's work comes to mind). The crucial question is: What meaning do the psychologists attach to various mathematical terms? We should not be overly concerned with whether these meanings are "correct"; we should be more concerned with whether the ideas are helpful. In turn, mathematics educators may believe they understand this presentation of information processing, yet have distorted concepts of the psychological terms used. In fact, we will not know if Shavelson and Porton succeeded in explaining information processing until mathematics educators begin to use and refine the psychological theory in their own research. It appears that joint research ventures are a necessity for overcoming the "communications gap" alluded to by Shavelson and Porton.

Higher Level Processing

One question concerning the Shavelson and Porton paper arises from the mathematically simple examples that are discussed. Is the restriction to simple examples merely a convenience in elucidating the psychological theory, or does it imply that the information processing model is applicable only to lower level cognitive processes such as computation and simple school text applications? The most troublesome aspect of much information processing research is the often low correlation between cognitive structure variables and achievement (particularly problem solving) variables. The research demonstrates rather clearly that we can change students' cognitive structures in terms of relationships between concepts; however, the connection between cognitive structure measures and noncomputational performance measures seems tenuous at best. The failure of the information processing model to deal satisfactorily with higher level processing is not in itself a fatal flaw, but it does suggest that a significant amount of study is needed to perfect the model. Many mathematics educators are more concerned about problem solving and applications of mathematics than they are about computation.

Information Retrieval

The weakest aspect of information processing theory appears to lie in the area of information retrieval. The various clustering and graph notions provide rather clear and useful models of concept formation and/or concept organization We believe we can observe indirectly changes in cognitive organization, and we have some theory to explain these changes and the procedures which cause these changes to take place. Yet, this reader does not understand what happens when students are presented with a mathematical problem and begin to use their cognitive structures, i.e., their current organization of concepts (information). Some students are consistently more successful than others in efficiently recalling appropriate information and using this information to formulate solutions to a problem. Admittedly it is extremely difficult to form hypotheses concerning behavior that is largely unobservable. However, it is precisely this area of processing that is of most concern to mathematics educators. It is also this area in which mathematics educators might make the most contributions to psychological theory, since their mathematical understanding should make them aware of the variety of processes possible and thus lead to ways of eliminating certain hypotheses. Models are needed to show the process of change in cognitive structures and the process of using cognitive structures.

In summary, the Shavelson and Porton paper provides a framework for considerable research. The concise and illuminating presentation of information processing theory is a welcome one. We seem to have made some progress toward understanding children's thinking. We are still short of being able to use this theory (model) for classroom applications. That is, little evidence has been presented which implies that classroom use of the model leads to a high probability of successful student learning. Nonetheless, Shavelson and Porton have pointed us in a potentially useful direction. Mathematics educators should not ignore this theory simply because we are in a theory development stage rather than a classroom application stage.

REFLECTIONS ON INTERDISCIPLINARY RESEARCH TEAMS

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> ....we believed that the divisions between the sciences were convenient administrative lines for the apportionment of money and effort, which each working scientist should be willing to cross whenever his studies should appear to demand it. Science, we both felt, should be a collaborative effort.
>
> (Norbert Wiener concerning his working relationship with Arturo Rosenblueth, the Mexican neurologist as described in I Am a Mathematician.)

After several years of intensive and remarkably rewarding collaboration, each of us feels deeply committed to interdisciplinary research. At the same time we have become aware of some of the problems inherent in any interdisciplinary effort, problems that cannot be solved by enthusiasm alone. The conditions under which an interdisciplinary research team may become productive, let alone flourish, are in an important way different from those that foster other research, and they are not at all obvious.

In principle, there is much to be gained by initiating and maintaining a dialogue between disciplines, but such a dialogue is not as easy to establish as it might seem. We contend that only when the disciplinary boundaries are breached in each member of the research team will interdisciplinary research produce a measure of success. Each discipline has its own problems, its own methods, its own picture of the world, and its own language. It is difficult to identify all the hazards involved in crossing the uncharted waters between disciplines. Like Columbus setting out for China, one may land in America. The approach to any new shore, be it the expected or an unforeseen one, is marked by a shift in climate. Even if one lands unscathed from the journey, the terrain is unfamiliar, the culture alien, and the language far less translatable than it might seem at first. These obstacles hinder mutual understanding--but even if understanding is achieved, it is not sufficient as a basis for joint research. The individuals involved must not only become bi- or multilingual, they must also be resilient, forgiving, and willing to relinquish--at least for the time being--a good many of their habitual patterns of thought. The work of an interdisciplinary research team, as opposed to other research teams, begins with a search for mutually acceptable patterns of thought, and the first task in this search is to establish a language in common. It is not surprising that there are few interdisciplinary research teams. In contrast, there are many individuals who operate quite successfully in more than one discipline and who, as individuals, have made the transition from a single discipline to productive interdisciplinary work. Clashes within one person are, as a rule, less bloody than clashes between two or more people.

We know of no efforts in the past to document the successes and failures of the interdisciplinary aspect of research. We have found no analyses in the literature of either the problems or their solutions. Our recent experience of continuing intensive interaction has forcibly clarified several points for us, and the ideas that have sprung from the process may be of interest to others.

What is Interdisciplinary Research?

Interdisciplinary research is research conducted *among* two or more disciplines. It requires a shared research program--that is, a shared language, methodology, problem priorities, and epistemology (Lakatos, 1970). Interdisciplinary research is to be distinguished from *cross-disciplinary* research, which is simultaneous or parallel research conducted *within each* of several disciplines separately. Cross-disciplinary research occurs when there is a temporary set of shared problems and a pooling of results, but a lack of interaction. For example, as Harry Beilin (1976) points out, there are logical, mathematical, and linguistic models which all attempt to account for the development of concepts in the child, and all serve separately as bases for alternative programs for teaching and learning mathematics. These parallel models have been developed in different disciplines, each having its own problems and using its own methods. There is understandably only a slight chance that one model will ever modify another, let alone that all of them may be synthesized into a single program. Interdisciplinary research is also to be distinguished from *undisciplined* research which occurs in that no-man's-land *between* the disciplines and thus creates an entirely new field of study.

Interdisciplinary research is unique in its manner of relating the disciplines involved. They are not viewed as competing, and they are not taken as alternatives. No parent, or sponsoring, discipline can have a greater investment than the others. Rather, each discipline must be equally committed to the importance of the joint research. If the research problems are peripheral to any of the disciplines, then that discipline will necessarily assume a secondary, supporting role. Its team member will have less stake in the results and will be more reluctant to relinquish the assumptions customary in his or her discipline.

Much of the current clamor for interdisciplinary research is bolstered by drawing an analogy with *consultation*, that is, the borrowing of specific results and methods from a separate discipline. Obvious benefits may be gained when an investigator in one discipline seeks advice on a specific topic from an authority in another discipline. For example, a psychologist may consult with a statistician on the design of an experiment; in this situation the statistician is used as a resource, much like a volume in a library. The analogy between interdisciplinary research and consultation breaks down, however, because their structures are different. In consultation the information flows essentially one way, in that the discipline consulted provides information to the researcher. In interdisciplinary research the information flows both ways, so that each discipline affects the other.

Preconditions for Interdisciplinary Research

Researchers turn to interdisciplinary research out of dissatisfaction and frustration not only with their own research and lack of progress but, perhaps mainly, with the state of research in their disciplines as a whole. They turn to other disciplines for methods that might be more promising than their own. Thus, interdisciplinary research arises from a rejection of currently accepted lines of research and methods of inquiry. As with any significantly new turn, it may, if successful, produce a revolution (cf. Kuhn, 1970). However, because there is a distinctly self-conscious awareness that revolution is a highly probable result, interdisciplinary research is perhaps best characterized as a *planned revolution*.

Although any revolution ends, by definition, in a nonstandard program, interdisciplinary research begins that way. The difficulty of planning such a program is sufficient to account for the failure of many interdisciplinary research projects. The initial program has two essential aspects, one negative and one positive. A planned revolution must take into account both of these aspects if it is to have any chance of success.

Negative program. There must be an adequate critique of existing programs of research. This critique must extend to the protected dogmatic core of the accepted programs. Because it is an attack on this core, the critique might best be gauged by the vigor with which it is rejected and by the animosity generated in the established research community. The better the critique, the more it will be perceived as an attack on the research program itself and not merely as a competing theory that might become part of the existing program.

Positive program. There must be an adequate basis for establishing new methods, new theories, new metaphysics--in short, a new methodological research program. This basis takes the form of a radically different perspective that provides new kinds of data and creates new distinctions.

While these two preconditions exist in almost any period of ferment within a discipline, there are several additional characteristics which mark the turning from a tradition and which are needed for interdisciplinary research to develop. These conditions for success are more clearly fortuitous than the two preconditions, yet they provide the basis for the interdisciplinary resolution, as opposed to the development of a new program within a single discipline.

Conditions for Successful Interdisciplinary Research

Simultaneity. For interdisciplinary research to develop, the preconditions described above must be met by research in several disciplines at the same time. Frustrations must arise simultaneously if the revolution is not to be a takeover of one discipline by another. It is precisely this shared feeling of inadequacy which brings researchers whose previous work has been unrelated together in search of a common ground.

In contrast, the takeover of one discipline by another is often manifested in a "bandwagon" effect, when an established research program is adopted hook, line, and sinker by researchers in another discipline. A takeover must not occur if a synthesis is to be achieved. Schwab (Note 1) argues that in contexts where several disciplines are involved it is essential to avoid the "arrogances of specialism." These arrogances are a necessary and important part of the conduct of science. They are a result of the confidence created by the security of the research program. This confidence is essential if an investigator is to spend a lifetime on the program's problems. The program would break down under constant questioning and prolonged challenging of the presuppositions. Challenges undermine the foundations and destroy the security needed for lifelong commitments. Schwab argues:

Collegiality will arise only to the extent that a minimal capacity for shame and a degree of humility characterize each member of the group. (p. 29)

But it is precisely these qualities, humility and shame, which are suppressed by an established methodological research program.

The takeover of one discipline by another either literally, through some form of reductionism, or figuratively, by a wholesale adoption of methods, problems, and techniques, precludes the establishment of interdisciplinary research in the proximity of a firmly established and flourishing discipline. The tradition of the successful discipline, by the very force of its success, prevents the genuine consideration of alternatives. An established program has set answers to attacks on its protected core, and it has an established hierarchy of problems that are not easily ignored or superseded.

Relatedness. The positive programs (see above) of the disaffected researchers must share a common perspective on philosophy and method. Although interdisciplinary team members need not have identical philosophies or subscribe to a common method entirely, they must have a shared perspective. They must see similar things and must approach issues from the same angle. The shared perspective does not and cannot prevent the formulation of competing, perhaps incompatible, subtheories, but it does provide a context for resolving conflicts. It is only through the existence of a shared core of beliefs that domains can be restricted, because the belief system provides the categories that determine domains for research. Once this common ground is established, conflicts are no longer likely to threaten the cohesion of the group.

Equality. Each team member must perceive the others' research problems as significant and must perceive that each discipline has potential for contributing to the research enterprise. This requirement is perhaps the most important and the one most dependent on the idiosyncratic development of the investigators. In interdisciplinary research, as in society generally, "separate but equal" cannot work. This doctrine fails in research for the same reasons as in society. If disciplines are maintained separately, there is little basis for familiarity and less for cooperation or compromise. Inevitably some become more equal than others, and the collegiality so desperately needed is destroyed.

Equality in this sense can only be achieved when the boundaries of the disciplines are breached in (each) of the participating individuals. Thus, every team member must develop competence in each of the separate disciplines. A precondition for our own project was that each of us deliberately operate outside our original field of competence. Another place in which research has successfully transcended disciplinary boundaries is the Genevan school. In a survey of interdisciplinary research Piaget (1973) observes:

> Geneva has always encouraged psychologists to collaborate with logicians, mathematicians, cyberneticians, physicists and so on....the link between a 'higher' (in the sense of more complex) and a 'lower' field results neither in a reduction of the first to the second nor in greater heterogeneity of the first, but in mutual assimilation such that the second explains the first, but does so by enriching itself with properties not previously perceived, which afford the necessary link. (p. 67)

Specificity. Interdisciplinary research must be directed at specific problems that are seen as significant from the perspective of each of the separate disciplines. Moreover, each discipline must view the problems in terms of specific issues which require the participation of other disciplines. However, there need not be a common solution, nor agreement on method, nor agreement on exactly which aspects of the problems are to be attacked. Each investigator approaches the issues with the training and tools of his or her respective discipline. Strange configurations of issues may be produced initially, but it is the interdisciplinary history of each team member which provides the basis for an eventual integration of ideas.

Whether the two preconditions and four conditions *really* coexist at any time for any particular project is difficult to establish and is, in fact, beside the point. What is essential is that the researchers themselves *believe* that the conditions are right. Often, the first years of a project are best directed towards achieving the conditions.

Our Own Project

The groundwork for our project was laid during five years of interaction and collaboration on several joint efforts. In retrospect, it is clear that we were becoming acquainted with and carefully evaluating each other's disciplines but had not yet formed a research team. The formulation of our proposal for an interdisciplinary study of an experimental model of learning and teaching whole numbers (Steffe, Richards & von Glasersfeld, 1979; Note 2) marks the point at which our ideas first came together to form a "methodological research programme" (cf. Lakatos, 1970). Our mutual acceptance of a constructivist epistemology provided a shared perspective, even though constructivism meant different things to each of us. Most important of all, we had developed complementary positive and negative programs.

For our project we identified a common problem--the construction of whole number concepts in children--which interested each of us for vastly different reasons. Other topics might have provided the content for our investigation, but attacking this issue in mathematics education was attractive for several reasons. To begin with, mathematics education is just the sort of pragmatic area that invites the contributions of other disciplines. Moreover, it is beset right now by various crises both in curriculum and practice, at the same time that it suffers a complete absence of agreement on research methods or philosophy. Thus there seemed to be an open area which not only fit our respective backgrounds but also was susceptible to planned revolution.

Although our project has produced a great deal in a short time, and we all feel tremendous excitement at what we have accomplished so far, there is a negative undercurrent resulting from the extended nature of the project. The development of new methods, new language, and a new research program takes a good deal of time. In two years we have begun to begin; the major work is still to be done. There is a strong temptation to "take the money and run." That is, there is a strong temptation to treat our work as cross-disciplinary to take what we have individually learned and return to our separate disciplines. While we each might contribute much to our separate disciplines, such a move would abort the planned revolution. On the other hand, there is a good deal of societal pressure to stop. Whether our project continues depends on several factors, most of which are only peripherally related to the intellectual endeavor.

Can an Interdisciplinary Research Program Succeed?

Planned revolutions rarely succeed. Revolutions largely depend on the confluence of events that appear to be out of anyone's control. Revolutions are mostly fortuitous happenings.

Work in any research program requires a deep commitment. Successful programs in accepted disciplines have available support systems, both material and emotional, that allow research to continue in the absence of positive results. New, nonstandard research programs within a single discipline face overwhelming odds, but there are structures that support nonstandard programs--tenured positions, research monies, some amenable journals, and so on. There is no comparable support system once individuals leave the confines of a discipline.

Publications, research monies, hiring and promotional decisions all require evaluations, and these evaluations are made by members of the affected disciplines. It is not difficult to anticipate the results. Interestingly enough, when members of a discipline do acknowledge the benefits of the new research, they attempt to separate the benefits from the interdisciplinary program. The contributions of the other disciplines are minimized and regarded as being of secondary importance or as having a parasitic existence, making little, or no, real contribution.

The current trend in universities and research centers is to cut back services and programs in response to reduced budgets. Financial exigencies exert pressure on nonstandard research programs. Moreover, the political division of universities into departments isolates the interdisciplinary investigator. Even when decisions are made above the department level, it is with the aid of departmental representatives. An interdisciplinary project rarely has a "protector" and becomes part of the fat which can be trimmed. Funding agencies, too, are departmentalized into disciplines, in spite of recent attempts at broadening their scope. When control resides with those who perceive their role as protectors of their discipline, it goes without saying that they see little value in interdisciplinary work.

Is it possible to produce short-term gains sufficient to buy time? Attacking specific problems provides some hope for success but is not a guarantee. Publishing nonstandard articles is difficult for two reasons. First, a new paradigm must not only be conceived but also communicated, often using a language that is still far from adequate. Second, the written product, to be accepted for publication, must survive the scrutiny of readers who frequently see little need for revolution in their discipline.

The most essential factor in the long-term survival of a project is attracting other researchers to the project. A research program takes time to succeed (cf. Feyerabend, 1975) or even to fail. If the work produced provokes others, then interest and support will develop within the several affected disciplinary communities. To remain interdisciplinary a project must not be limited in its attraction to a single community. The absence of wide-ranging support will reduce the project to just another movement in a particular discipline. The broader project--that of a planned revolution--will be forced underground.

Reference Notes

1. Schwab, J. J. The practical 3: Translation into curriculum. Paper presented at the Melton Center, Jewish Theological Seminary of America, New York, about 1973.

2. Steffe, L. P., Richards, J., & von Glasersfeld, E. Learning and teaching whole numbers: An interdisciplinary study of an experimental model. Proposal for a research project funded by the National Science Foundation (SED78-17365), 1978.

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