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ABSTRACT

This document deals with the observation of students in a direct translation scheme in the solution of word problems in a university freshman-level Intermediate Algebra class. It is felt that since successful problem solvers of algebraic equations often have as much difficulty in solving word problems as do other students in the classes, the problem for most students must exist in the translation of words into algebraic expressions and equations. The focus of this study is directed towards the translation aspect of the solution of word problems. The observations detailed are based on the collected papers of 84 students. Ten problem samples and associated pupil errors are detailed. Among the recommendations made: (1) Students should be encouraged to set up an equation in phrases containing all the variable names for unknown quantities; (2) Teachers should stress the phrase structure of sentences and then proceed with a direct translation scheme; and (3) The use of auxiliary cues and representations should be encouraged, including pictures, diagrams, and flowcharts. It is noted that human problem solving is a complex task and insights into teaching strategies can be developed by understanding the thinking process. (MP)

Error Analysis in Solving Algebra Word Problems

by

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Error Analysis in Solving Algebra Word Problems

One of the most difficult topics to teach at all levels of mathematics is the solution of word problems. Because there are varied and complex tasks involved at different stages of the process, the procedure must be separated into several steps. The textbook will usually outline a step-by-step guide for students to follow, such as:

1. Read the problem carefully to determine what facts are given and precisely what is to be found.
2. Draw diagrams if possible to help interpret the given information and the nature of the solution.
3. Express the given information and the solution in mathematical symbols and/or numbers.
4. Express the problem in the form of an equation.
5. Solve the equation.
6. Check the solution to see if it satisfies the equation and to see if it makes sense with regard to the story problem.

The procedure seems simple enough, but all teachers of mathematics are aware of the difficulties students face in attempting the solution. Most students prefer such a guide rather than a "trial-and-error" or inductively reached answer. This step-by-step procedure essentially involves, first, translating the sentences into algebraic equations, and second, solving the equation. Since experience has shown that successful problem solvers of algebraic equations quite often have as much difficulty in solving word problems as do other students in the classes, the problem for most students must exist in the translation of words into algebraic expressions and equations. Hence attention in this paper is directed toward the translation aspect of the solution of word problems.

One of the most promising theories that influence this area of study is known as information processing. An information processing theory presents a detailed set of procedures that account for the observed behavior. Students are observed solving problems and the theory will then provide an explanation of the student behavior. This information processing approach focuses attention on cognitive processes by building a theory of behavior that describes a problem-solving model. This details the problem structure, the relationship among the problem elements and the eventual arrangement of the elements into a solution. This theory has been found to be useful in the development of computer programs that simulate human thinking patterns in the solution of word problems. One scheme that has been proposed is a direct translation procedure. A direct translation process transforms the statement into a mathematically equivalent equation. This would be, roughly, the same as a translation process of any natural language into another. Jeffery Paige and Herbert Simon (1966) have identified several successive transformations which will occur in the process. The transformation rules alter the current status of the problem and identify the operations relevant to the solution. Some of these are mandatory substitutions (such as 2 times for twice), function tags (indicating the grammatical function of the word), identification of conventional and relational naming of variables, and the use of auxiliary representations and cues. Conventional naming would be exemplified by the use of a new variable to identify an additional unknown, whereas relational naming would connect or relate the information using a single variable. Auxiliary cues use a conventional knowledge structure called frames that help account for the ability to recognize and interpret the relevant information in a setting. Cues can provide the trigger for processing the heuristics needed to classify problems and retrieve from memory useful information. A direct translation scheme in its purest form has to be augmented by specific semantic knowledge to insure full understanding of the problem. These processes have been adapted for the purposes of this paper and will provide the theoretical basis of discussion. For the purposes of this paper, function tags indicate not only the grammatical use of the word but also the operational function of the word.

The following observations are based on the collected papers of 84 Intermediate Algebra students enrolled in a four-year university. A large majority (75%) have had at least one previous algebra course (mainly high school Algebra I) within the last five years. Although the university has a three-year high school mathematics entrance requirement, few students have taken a second year high school algebra course.

Students were asked to read each problem and write an algebraic equation reflecting the given information for the problem. They were instructed not to solve the equation. The intent was only to observe the translation process. (Problems are from Munem/Tschirhart, pp.150-152.)

Problem 1: Three more than twice a certain number is 57. Find the number.

Eighty-three of the 84 students attempted this problem and 64, or seventy-seven percent, presented a correct equation. Error types for the remaining 19 students were:

Error pattern #1: $x + (3 + 2x) = 57$

Error pattern #2: $x^3(x2) = 57$

Error pattern #3: $3x + 2 = 57$

Error pattern #4: $3x + 2x = 57$

Error pattern #5: $x + 3 = 57$

Error pattern #6: $3 + 2x^2 = 57$

Error pattern #7: $x^2 + 3 = 57$

Error pattern #8: $2(x + 3) = 57$

Only one student did not correctly identify the function tag of "three more than." (Error pattern #2) This fact is interesting when compared to a following problem which involves the phrase "greater than." The operation is incorrectly translated in that problem. Others had difficulty with the tag "twice", translating it as "square." (Error patterns #6 and #7). The most common error in this problem, Error pattern #1, stresses the relational naming procedure. This exemplifies the fact that students feel they must use all the information they have accumulated, rather it fits the problem or not. Thirty-three percent of the papers with this error had a correct legend:

x: the number
$3 + 2x$

but then continued and wrote the equation $x + (3 + 2x) = 57$.

Problem #2: Find three consecutive even integers such that the first plus twice the second plus four times the third equals 174.

Eighty-two students attempted the problem and 49 of these students (or 60%) found a correct equation for the problem. Common error patterns were:

Error pattern #1: $x + 2x + 4x = 174$

Error pattern #2: $x + 2(x + 1) + 4(x + 2) = 174$

Error pattern #3: $x + (2x + 2) + (4x + 4) = 174$

Error patterns #1 and #2 violate several of the transformations needed in the correct formation of the equation. The relational naming and function tag of "three consecutive integers" proved to be the main cause of errors. One student had in his legend:

x: first even integer
$x + 2x$: second even integer

and then wrote the equation $x + 2x + 4(3) = 174$.

One of the most interesting errors was

$$x + x^2 + x^4 = 174.$$

This is an error in the mandatory substitution phase of the function tagging.

All students correctly identified the operation of addition. Fifteen percent of students writing an incorrect equation knew of the functional connection between the numbers but were unable to make the relational connection. Instead they used the conventional naming scheme and wrote an equation of the form:

$$x + 2y + 4z = 174.$$

Problem #3: One-fourth of a number is 3 greater than one-sixth of it. Find the number.

Thirty-five of the 76 attempting this problem were able to give a correct equation. Two major error patterns emerged from the findings.

Error pattern #1: $\frac{1}{4}x = 3(\frac{1}{6}x)$

Error pattern #2: $\frac{1}{4}x + 3 = \frac{1}{6}x$

Thirteen errors were attributed to the inability to connect "3 greater than" with the correct process. This is unusual because most students were able to make the correct connection in Problem #1. It may be conjectured that other variable names interfered with this connection.

The second common error pattern is the inability to place the relational naming process with the correct variable name. The function tagging is correct but the relational aspect is not. This problem also embodies the principle of conservation, as outlined by Piaget in his work in the development of cognitive processes of children. Auxiliary cues could have been used to identify the disparity between the sentence and the equation. Unusual errors are:

$$x(\frac{1}{4}) + 3(\frac{1}{6}) = x$$

$$(\frac{1}{4}n + 3) + \frac{5}{6}n = 42$$

$$\frac{1}{4}x + 3 - \frac{1}{6}x = x$$

Problem #4: A payphone slot receives quarters, dimes and nickels. When the phone box was emptied, there was $6.50 in coins. If there are 4 more dimes than quarters and three times as many nickels as dimes, find the number of coins of each kind.

Seventy-six students attempted this problem, yet only 10 worked it correctly. There were over 50 different error types. Twenty-one students had an error involving "4 more than", which they translated to be 4x. Another prominent error type excluded the money value of variables. A typical legend would read:

x = quarters
$x + 4$ = dimes
$3(x + 4)$ = nickels

But the equation would be $x + x + 4 + 3(x + 4) = \$6.50$. Operationally to the student there was no difference between the <u>number</u> of coins and the <u>value</u> of these coins. The auxiliary information involving the value of a quarter, nickel or dime was not called forth into the thinking process of the student. This provides some contrast with the next problem, which is also a money problem. Students were better able to work with the value of the coins rather than the number.

Problem #5: A parking meter slot receives dimes and nickels. Emptying the box produced 70 coins worth \$4.85. How many nickels and how many dimes were there?

Five students, out of 52 students attempting the problem, were able to write a correct equation. No student attempted to use the conventional naming scheme for the solution so that the variables were relationally named. Yet only 5 students were correctly able to use this process.

Thirty-five different errors were identified. All of these errors involved improper use of the total 70. Students simply did not know what to do with the total. Error types included:

$$70 \div 10x + 5x = 485$$

$$70 - x = \$4.85$$

$$5x + 10y = 4.85$$

$$25x + 70 = 4.85$$

$$.5x + .25x = \frac{4.85}{70}$$

$$70x = \$4.85$$

$$.10x + .5x = 4.85$$

In this problem the auxiliary cue of the value of the coins was present for the majority of students solving the problem but the process of relational naming was not. Only 7 errors occurred without some use of the money value associated with the variable. This is in contrast with the previous problem. Also, several students had the following rationale:

x = nickel
$2x$ = dimes

The students have operationally implied the money association in this legend, yet they were unable to correlate correctly back to the problem. One equation using the above legend was: $70(x + 2x) = \$4.85$.

Problem #6: One pipe can fill a tank in 18 minutes and another pipe can fill it in 24 minutes. The drain pipe can empty the tank in 15 minutes. With all pipes open, how long will it take to fill the tank?

Twenty-six students attempted the problem; no student was able to provide a correct equation. Only one of the students tried a fractional approach:

$$\frac{1}{18} + \frac{1}{24} - \frac{1}{15}$$

Yet the student could not generate an equation.

Sample error types were:

$$18x + 24x = 15x$$

$$x = 18 + 24 - 15$$

$$18x + 24x - 15 =$$

$$x + x + 6 - x + 3 =$$

$$\frac{(x + 3)(x + 9)}{x} = \text{FULL}$$

$$x + 18 + x + 24 = x - 15$$

Students had difficulty in naming the variables conventionally or relationally. Several students tried a conventional approach in their legend:

x = pipe fills tank in 18 minutes
y = pipe fills tank in 24 minutes

Yet they were unable to form an equation. Others tried the relational approach:

$18 + x$ = first pipe
$24 + x$ = another pipe

The equation given from this legend was:

$32 + x - 15$ = time to fill tank

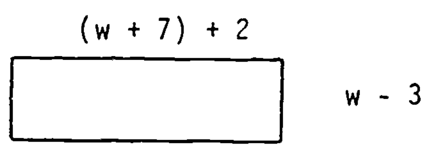
(It is assumed that the 32 is an incorrect result of the sum of 18 and 24.)

Students were unable to identify the unknown or assign to x a particular notation. In handling this word problem students did not make use of any auxiliary cue to direct them to a possible solution.

Problem #7: The length of a rectangle is 7 inches greater than its width. If its length is increased by 2 inches and its width is decreased by 3 inches, its area is decreased by 37 square inches. Find its dimensions.

Four students of the 31 who attempted this problem were able to give a correct equation representing the problem. This problem, more than any other, had more examples of conventional naming of variables rather than relational naming. The function tagging was incorrectly applied also since students did not pick up on the auxiliary cue of area representing a multiplication process.

Only two students used the spatial and physical cue of providing a diagram or drawing of the rectangle. An example of one drawing is



$$1 \times w = A$$

$$((w + 7) + 2)(w - 3) = 37$$

Two students correctly used the operation of decreasing the area by 37.

Problem #8: A woman has $8,000 invested at 5 percent and $2,000 invested at 7 percent. How much must she invest at 8 percent to make an average of 6 percent on her investment?

One student of the fourteen who attempted the problem gave a correct equation. The most common error pattern was:

$$8000\ (.05) + 2000\ (.07) = x(.08)$$

All students who attempted this problem were able to use the mandatory substitution of percent into its correct decimal representation. Even though this auxiliary information was processed correctly, the common error type was an inappropriate use of the conservation process.

The word "average" caused the grammatical cue of the definition of this word to interfere with the correct processing in one case

The student wrote the equation

$$\frac{8000(.05) + 2000(.07) + x(.08)}{2} = .06$$

to incorporate "average."

Problem #9: A grocer has 100 pounds of candy selling at \$1.80 per pound. How many pounds of a different candy worth \$3.00 per pound should he mix with the 100 pounds in order to have a mixture worth \$2.40 per pound?

Eight students attempted this problem and 4 of these provided a correct equation. Errors were:

$100\ (1.80) + x(3.00) =$

$100\ (1.80)\ x + 3.00 = 2.40$

$100\ (1.80) + x(3.00) = 100\ (2.40)$

None of the students used a legend to indicate the translated value of x. These errors involved postulates about the physical relations in this problem. Students were unable to comprehend the total and were not applying the principle of conservation in their equation formation. No diagrams or other forms of auxiliary representations were observed.

Problem #10: A man started for a town 75 miles away at the rate of 30 miles per hour. After traveling part of the way, road construction forced him to reduce his speed to 22 miles per hour for the remainder of the trip. If the entire trip took 2 hours and 38 minutes, how far did he travel at the reduced speed?

One student attempted this problem but did not succeed in writing a correct equation. Students were unable to use the grammatical cue of rate or distance and the auxiliary information of the relation between distance, rate and time to set up an equation. (Only one student recognized this as a distance problem and wrote

$$d = r \times t$$

on his paper. He was unable to do anything else with the problem.) No one drew a diagram or presented any form of physical representation for the problem. Students did not use the cues in the problem to connect the distance formula in memory to use in the direct translation process of solving the equation.

OBSERVATIONS AND IMPLICATIONS

This paper has dealt with the observation of students in a direct translation scheme in the solution of word problems in a university freshman-level Intermediate Algebra class. Suggestions for teacher use in classroom situations follow.

Students should be encouraged to set up an equation in phrases, containing all the variable names for unknown quantities. Teachers must stress the <u>phrase structure</u> of the sentences and then proceed with a direct translation scheme. Appropriate substitutions must be made for functional and operational notation. Words and phrases such as "greater than" or "consecutive even integers" should be part of a direct process, yet in translation attention must be given to the properties of conservation. For example, in Problem #7 phrase translation could result in several conventionally named variables (l=length; w=width; A=area) being restructured into a relationally named equation.

"The length of a rectangle is 7 inches greater than its width" can be written (by direct translation) $l = 7 + w$. "If its length is increased by 2 inches and its width is decreased by 3 inches, its area is decreased by 37 square inches" can then be translated as:

$$(l + 2) \times (w - 3) = A - 37$$

The substitution of $l = 7 + w$ and $A = l \times w$ (which requires the auxiliary cue of area representing length times width) leads to the last equation:

$$((7 + w) + 2) \times (w - 3) = (7 + w)(w) - 37$$

This type of phrase translation can aid in combining quantities in a correct procedure and insuring the equality of the resulting statement.

Use of auxiliary cues and representations should be encouraged, including pictures, diagrams and flowcharts. Distance problems should be recognized during the first reading of the problem and students should automatically be cued to the distance formula. Work problems all follow a similar format and this procedure should be emphasized to students, so that grammatical cues could help in a word or phrase recognition. Students would then immediately be aware of the correct process. Money problems also have similar procedures that are used in all problems of this type. Students need to be aware of the number and value aspect of the coins involved in these problems and that both conditions must be considered in the solution to the problem. Conservation techniques should be emphasized. Students should be taught explicit procedures to ensure the equality in equation formation. Students need information about problem categories, including problem structure and classification. Equation formation is facilitated by relevant information retrieved from memory, such as knowledge about equations, diagrams and appropriate procedures. Students will need to fit a problem into a schema or frame so that underlying principles of comprehension can be used.

Human problem solving is a complex task and involves careful and detailed analysis of the procedures. By understanding the thinking process and information processing theories related to problem solving, insights into appropriate teaching strategies can be developed.

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