DOCUMENT RESUME

ED 207 787 SE 033 648

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TITLE Algebraic and Trigonometric Equations With Applications. A Study Guide of the Science and Engineering Technician Curriculum.
INSTITUTION Saint Louis Community Coll. at Florissant Valley, Mo.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 76
GRANT NSF-GZ-3378; NSF-HES74-22284-A01; NSF-SED77-17935
NOTE 114p.; For related documents, see SE 033 647-657. Not available in paper copy due to copyright restrictions. Contains occasional marginal legibility.
AVAILABLE FROM National Science Teachers Association, 1742 Connecticut Ave., N.W., Washington, DC 20009 (write for correct price).

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *Algebra; *College Mathematics; Engineering Education; Higher Education; Instructional Materials; Interdisciplinary Approach; Mathematical Applications; Mathematics Curriculum; Mathematics Education; *Science Course Improvement Projects; Science Education; *Study Guides; *Technical Education; *Trigonometry; Two Year Colleges
IDENTIFIERS *Science and Engineering Technician Curriculum

ABSTRACT

This study guide is part of an interdisciplinary curriculum entitled the Science and Engineering Technician (SET) Curriculum designed with the objective of training technicians in the use of electronic instruments and their applications. The curriculum integrates elements from the disciplines of chemistry, physics, mathematics, mechanical technology, and electronic technology. This guide provides that part of the mathematics content related to algebraic and trigonometric equations and their applications. The following topics are included: (1) linear equations in two unknowns; (2) trigonometric equations and vectors; (3) systems of linear equations; (4) quadratic equations; (5) complex numbers-imaginary roots of quadratic equations; (6) equations containing fractions; and (7) exponential and logarithmic equations. (Author/SK)

SEDR DATA FORMAT	1. Award Number SED77-17915	2. Award Date September 28, 1977	3. Termination Date July 31, 1979

6. Title
Algebraic and Trigonometric Equations with Applications
A Study Guide of the Science and Engineering Technician Curriculum

4. Amount of Award
$70,400 (Cum. Amt.)

5. Type
Final Technical Report

7. Performing Organization
SET Project
St. Louis Community College at Florrissant Valley
St. Louis, MO 63135

8. Pagination
112 pages

9. Accession Number
00317

11. Principal Investigator; Field or Specialty
Donald R. Mowery
Bill G. Aldridge
Lawrence J. Wolf

10. Performing Organization Report Number

12. NSF Program Manager
Gene D'Amour

13. SEDR Program
DISE

14. SEDR Subprogram

15. Abstract

This study guide is part of an interdisciplinary course study entitled the Science and Engineering Technician Curriculum (SET). The curriculum integrates elements from the disciplines of chemistry, physics, mathematics, mechanical technology, and electronic technology, with the objective of training technicians in the use of electronic instruments and their applications.

This guide provides that part of the mathematics content related to algebraic and trigonometric equations and their applications. The following topics are included: (1) linear equations in two unknowns; (2) trigonometric equations and vectors; (3) systems of linear equations; (4) quadratic equations; (5) complex numbers-imaginary roots of quadratic equations; (6) equations containing fractions; and (7) exponential and logarithmic equations.

16. Descriptors

Study Guides
Instructional Materials
Junior Colleges
Associate Degrees
Mathematics
College Mathematics
Mathematics Curriculum
Algebra
Trigonometry

17. Identifiers
Science and Engineering Technician Curriculum (SET)

18. Field
Algebra or Number Theory / Mathematical Sciences

19. Target Audience
Two-Year College Students

20. Availability
Limited number of copies available from St. Louis Community College at Florissant Valley, St. Louis, MO 63135.

21. Supplemental Notes
One copy

ALGEBRAIC AND TRIGONOMETRIC EQUATIONS WITH APPLICATIONS

A STUDY GUIDE OF THE SCIENCE AND ENGINEERING TECHNICIAN CURRICULUM

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The materials contained herein were developed under Grant Numbers HES74-22204 A01, GZ-3378, and SED77-17935.

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CHAPTER ONE

LINEAR EQUATIONS IN TWO UNKNOWNS

1. Definition of Function - Function Notation.

One variable, say y, is a function of a second variable, say x, if and only if for each value of x there corresponds one and only one value of y. Denote y as a function of x by $y = f(x)$, read "y equals f of x". A corresponding pair of values of x and y is written as an ordered pair (x,y).

Examples.

1.1 In the formula $C = 2\pi r$, the circumference of a circle C is a function of the radius r since a given radius r corresponds to one and only one circumference C. This function can be denoted as $C = f(r)$ or $f(r) = 2\pi r$. If a value of the radius is given, say 5 meters, then the notation $C = f(r)$ is changed to $C = f(5)$ where $f(5) = 2 \cdot \pi \cdot 5 = 10\pi$ meters. Thus, the circumference 10π meters corresponds to a radius of 5 meters forming the ordered pair $(5,10\pi)$.

1.2 The correspondence between two variables s and t given by $s^2 = 4t$ does not represent s as a function of t because a value of t, say 25, corresponds to two values of s, namely -10 and 10.

1.3 In example 1.2, t is a function of s since for each value of s there corresponds exactly one value of t. Some corresponding pairs are (4,4), (100,25), and $(-1,\frac{1}{4})$.

2. Linear Function.

The variable y is a linear function of the variable x if the difference between any two values of x divided into the difference of the corresponding values of y is constant. This functional relationship is expressed by an equation of the form $y = ax + b$ or $f(x) = ax + b$ where a and b are constants.

Examples.

2.1 The x and y values in the table below define y as a linear function of x. The ratio of the differences in corresponding values of x and y is the constant 2.

e.g. $\frac{11-9}{4-3} = \frac{2}{1} = 2$, $\frac{3-11}{0-4} = \frac{-8}{-4} = 2$, etc.

The equation $y = 2x + 3$ for $x = 0, 1, 2, 3,$ and 4 symbolizes this function.

x	0	1	2	3	4
y	3	5	7	9	11

2.2 The table of values shown at the right does not define s as a linear function of t since not every ratio of differences in corresponding values of s and t is the same; that is, $\frac{16-8}{4-2} = \frac{8}{2} = 4$, while $\frac{36-24}{8-6} = \frac{12}{2} = 6$.

s	t
2	8
4	16
6	24
8	36

2.3 The equations $f(x) = 3x + 14$, $y = -14x$, $s = -3t - 5$, and $p(t) = 3$ $(p(t) = 0 \cdot t + 3)$ express a first variable as a linear function of a second variable.

2.4 The equations $y = 4x^2 - 2x$, $F(t) = 3t^4$, and $s = \sqrt{t}$ represent nonlinear functions.

Exercise Set 1

1. Determine if y is a linear function of x from the given table of values. Also, express each linear relationship in equation form.

a.

x	0	1	2	3	4
y	0	3	6	9	12

b.

x	10	20	30	40	50
y	22	42	62	82	102

c.

x	-5	-4	-3	-2	-1
y	-5	-3	-1	1	3

d.

x	3	6	9	12	15
y	5	11	17	20	29

2. Which of the following equations define a linear function?

   a. $y = -x^2$ b. $r = 10 - t$ c. $f(z) = z + 9^2$

   d. $f(z) = z^2 - 9$ e. $p = q$ f. $q = 3$

3. An automobile traveling a distance of 600 kilometers has an average speed $\bar{v}$ and time t given by the equation $\bar{v} = 600/t$. Is the relationship between $\bar{v}$ and t linear?

4. The Celsius (C) and Fahrenheit (F) temperature conversion formula is $C = 5/9(F - 32)$. Is the relationship between Celsius and Fahrenheit temperature linear?

5. The formula $V_1 = V_o(1 + .00366T)$ is applied in the study of expansion of gases. If V_o is a constant, is V_1 a linear function of T?

3. <u>Linear Equations in Two Unknowns - A Different Form of Linear Function.</u>

The linear function $y = 2x/3 - 1/3$ can be manipulated algebraically to give $2x - 3y = 1$. The variable y is still considered a linear function of x in this new equation form, called a <u>linear equation in two unknowns</u>.

In general, a linear equation in two unknowns has the standard form <u>$ax + by = c$</u> where a, b, and c are constants. Examples of equations of this type are $3x + y = 7$, $2m - n = 0$, $3y = 6$, and $2t - 11 = 0$.

A <u>solution</u> of a linear equation in two unknowns x and y is an ordered pair (x,y) which makes the equation true. For example, the ordered pair (5,3) is a solution of the equation $2x - 3y = 1$. To find a solution, give x a value, substitute this value into the equation and solve for the corresponding value of y. To find a solution of $5x - 2y = 7$, let x be some value, say 3, then $5(3) - 2y = 7$ from which $y = 4$. The solution is (3,4).

Exercise Set 2

1. For the given equation and a value of one variable, find the corresponding value of the second variable and write an ordered pair solution of the equation.

   a. $3x + 5y = 0,\ x = 5$ b. $4s - t = 10,\ s = \frac{3}{4}$

   c. $-8x - 3y = 15,\ x = 0.6$ d. $9x,\ y = 16$

2. Given $3t - p = 1$, complete the table and write the five associated solutions of the equation.

t	0	5	-3	$\frac{1}{3}$	4.6
p					

3. Given the equation $2m - 5n = 18$, find five ordered pairs of the form (m,n) which satisfy the equation.

4. If (3,4) is a solution of $ax - y = 26$, what is the value of the constant a?

4. The Rectangular Coordinate System.

Two number lines, one representing the values of x, a second the values of y, placed perpendicular in a plane so their origins cooincide, form a rectangular coordinate system (See Figure 1.1). The number lines are called coordinate axes with positive directions to the right for x and upward for y. The axes divide the coordinate plane into four quadrants which are enumerated with Roman Numerals as shown in the figure. The point of intersection of the axes is called the origin.

Each point in a coordinate plane is the graph of an ordered pair (x,y). For example, a point in the plane is the graph of (-5,3) where -5 is the x - coordinate and 3 is the y - coordinate of the point.

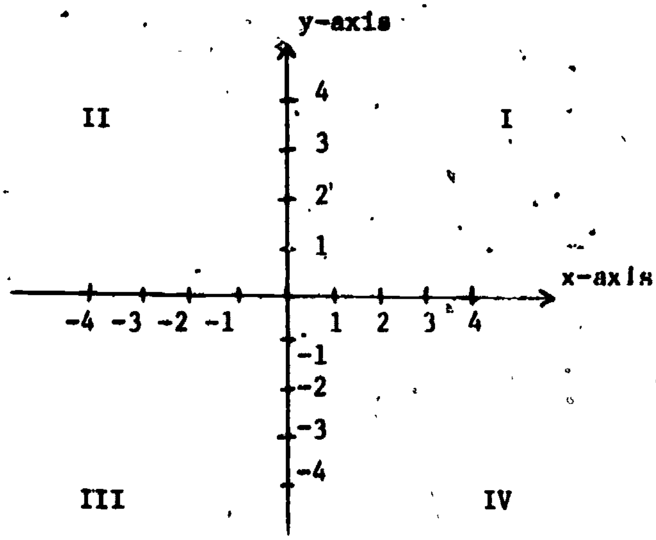


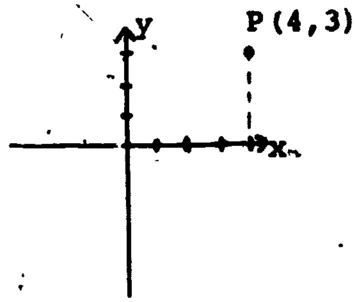
Figure 1.1

A point P in the plane whose coordinates are (x,y), written P(x,y), is located by starting from the origin and moving to the x - coordinate on the x - axis. From this point, move vertically in the direction and distance indicated by the y - coordinate. This terminal point is the graph of P(x,y).

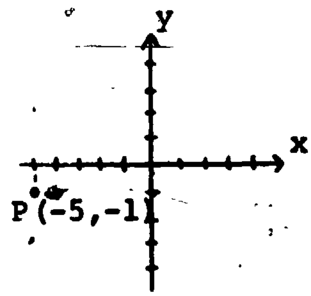
Examples.

4.1 Locate the point P(x,y) whose coordinates are

a. (4,3) . . . From the origin, move right 4 units on the x - axis, then up 3 units to P.



b. (-5,-1) . . From the origin, move left 5 units, then down 1 unit to P.



c. (0,-3) . . From the origin, do not move left or right but move down 3 to P.

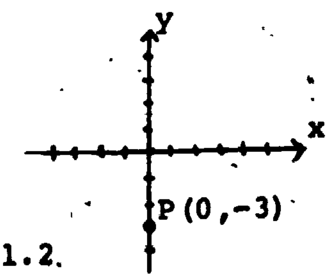


Figure 1.2.

4.2 Graph five solutions of the equation $3x - y = 1$.

Step 1. Complete a table of values for 5 values of x.

x	0	-1	1	-2	2
y					

x	0	-1	1	-2	2
y	-1	-4	2	-7	5

Step 2. Form five solutions from the table: (0,-1), (-1,-4), (1,2), (-2,-7), (2,5).

Step 3. Graph the solutions on a coordinate system.

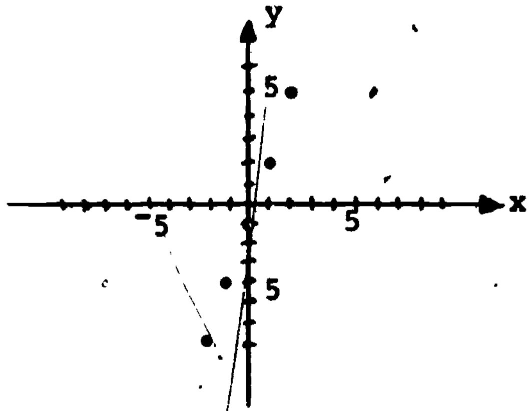


Figure 1.3

Exercise Set 3

1. Plot the graph of each ordered pair and state the quadrant it is in.

   a. (3,1) b. (5,-7) c. (-8,-2) d. (-6,5)

2. Graph the five solutions of the equation $x - 2y = 5$ corresponding to $x = 0, -2, 3, -5,$ and 4.

3. Pass a straight line through the points $P_1(0,0)$ and $P_2(2,1)$ in a coordinate system. Determine which of the following points lie on this line.

   a. P(3,5) b. Q(4,2) c. R(-2,1)

   d. S(-2,-1) e. T(4,-2) f. M(7,9)

5. Graph of a Linear Function in Two Unknowns.

Let y be a function of x given by $y = ax + b$. The graph of all solutions of this equation is called the graph of the function.

Similiarly, if the linear function is expressed in the form $ax + by = c$, the graph of all solutions of this equation is called the graph of the equation.

The names 'linear function' and 'linear equation' are derived from the fact that their graph is a straight line.

Examples.

5.1 Graph the function $y = 2x - 3$.

Step 1. Give x values and find the corresponding values of y. Even though 2 points are sufficient to determine the line, a third point will be found to check for possible error.

x	0	4	-1
y			

x	0	4	-1
y	-3	5	-5

Step 2. From the table, list solutions of the given equation: (0,-3), (4,5), (-1,-5)

Step 3. Graph the solutions from step 2 and using a straight edge, pass a straight line through these points. This line is the graph of $y = 2x - 3$.

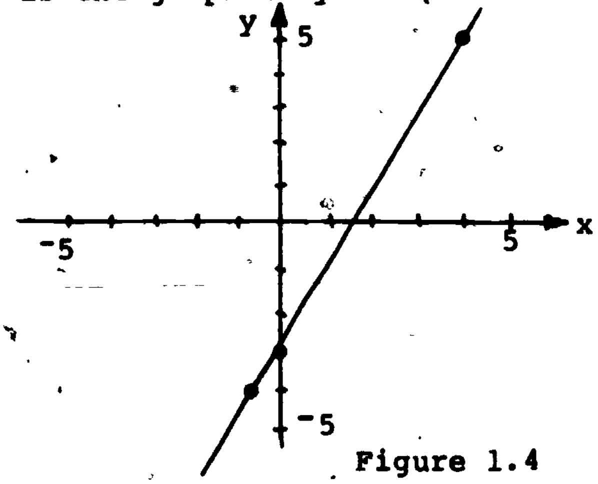


Figure 1.4

5.2 Graph $2x + 3y = 6$.

Step 1. Give x values and evaluate y.

x	0	6	-3
y			

x	0	6	-3
y	2	-2	4

Step 2. List solutions of $2x + 3y = 6$ from the table: (0,2), (6,-2), (-3,4)

Step 3. Pass a straight line through the graphs of the solutions. This line is the graph of $2x + 3y = 6$.

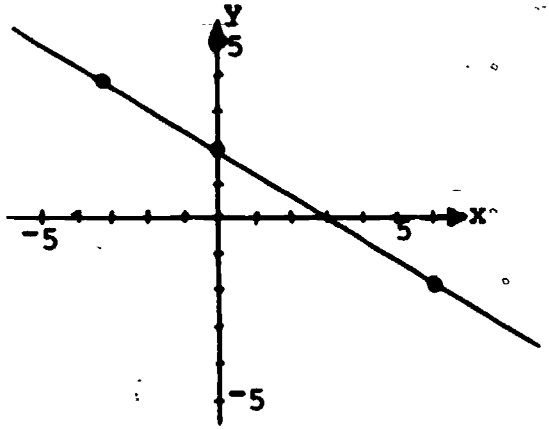


Figure 1.5

5.3 Graph $f(x) = 2$.

Step 1. Assign x values and solve for $f(x)$. Consider $f(x) = 0 \cdot x + 2$.

x	-3	4	0
f(x)			

x	-3	4	0
f(x)	2	2	2

Step 2. Ordered pair solutions are (-3,2), (4,2), and (0,2).

Step 3. Pass a line through the graphs of (-3,2) and (4,2) which is the graph of the function $f(x) = 2$.

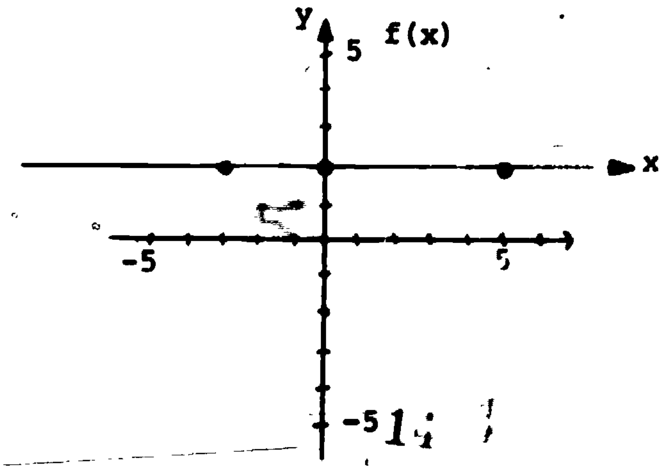


Figure 1.6

5.4 The Celsius and Fahrenheit temperature conversion formula, $C = \frac{5}{9}(F - 32)$, expresses C as a linear function of F. Graph this linear function for values of F between 0° and 212°.

Step 1. Give F values within the indicated range 0° to 212° and solve for corresponding values of C. It is convenient to let $F = 0^{\circ}$ and 212° to establish the range of values of C for the graph.

F	0	212	32
C			

F	0	212	32
C	$-\frac{160}{9}$	100	0

Step 2. Connect the graphs of $(0, \frac{-160}{9})$, (212,100), and (32,0) with a straight edge to produce the graph of the given linear function for the specified values of F.

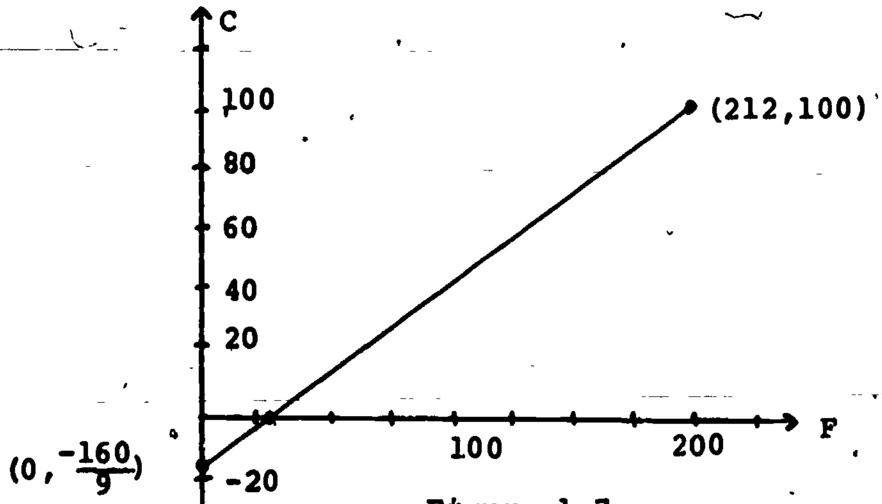


Figure 1.7

Exercise Set 4

1. Graph the following linear functions.

a. $y = 2x - 4$

b. $s = -4t + 5$

c. $y = -4$

d. $V = T$

2. Graph each linear equation in two unknowns.

a. $3x - 4y = 12$ b. $35m - 70n = 300$

c. $4x - y = 0$ d. $x = 0$

3. A car, traveling at a constant speed of 30 kilometers per hour, goes a distance of d kilometers in t hours given by $d = 30t$. Graph d as a linear function of t for t between 1/2 and 4 hours.

6. <u>Distance Between Two Points in The Plane</u>.

When two points lie on a horizontal or vertical line, a <u>directed distance</u> from one point to another or an <u>undirected distance</u> between the points can be calculated. Figure 1.8 and the table illustrates the types of distances and how they can be determined.

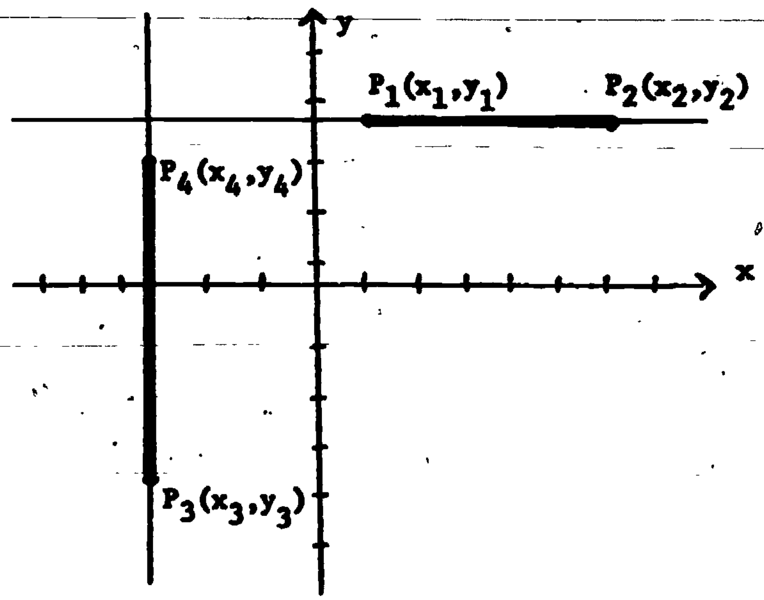


Figure 1.8

Description of Distance	Movement	Symbol for Distance	Distance Calculation	Sign of Distance
Directed, P_1 to P_2	Horizontally, left to right	$\overline{P_1P_2}$	x_2-x_1	positive
Directed, P_2 to P_1	Horizontally, right to left	$\overline{P_2P_1}$	x_1-x_2	negative
Directed, P_3 to P_4	Vertically, up	$\overline{P_3P_4}$	y_4-y_3	positive
Directed, P_4 to P_3	Vertically, down	$\overline{P_4P_3}$	y_3-y_4	negative
Undirected, between P_1 and P_2	None	$\lvert\overline{P_1P_2}\rvert$	$\lvert x_1-x_2\rvert = \lvert x_2-x_1\rvert$	positive
Undirected, between P_3 and P_4	None	$\lvert\overline{P_3P_4}\rvert$	$\lvert y_4-y_3\rvert = \lvert y_3-y_4\rvert$	positive

Examples.

6.1 Find the directed distance from $P_1(9,3)$ to $P_2(27,3)$.

Step 1. Since the second coordinates are the same, the line containing the points is horizontal. Hence, the equation of the line is $y = 3$.

Step 2. This positive distance is found by the left to right movement on the line (from lesser to greater values of x).

Step 3. Subtract the lesser x value from the greater x value. $\overline{P_1P_2} = x_2 - x_1 = 27 - 9 = 18$ units.

6.2 Find the undirected distance between $P_1(8,-4)$ and $P_2(8,-16)$.

Step 1. Since the first coordinates are the same, the points lie on a vertical line whose equation is $x = 8$.

Step 2. $|\overline{P_1P_2}| = |-4 - (-16)| = |12| = 12$ units which also equals $|-16 - (-4)| = |-12| = 12$ units.

The distance between any two points in the plane can be found by applying the Pythagorean Theorem which states 'In a right triangle, the square of the hypotenuse (side opposite the right angle) equals the sum of the squares of the other two sides'.

In Figure 1.9, the distance between $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is the length of the hypotenuse of the right triangle whose other two sides are $y_2 - y_1$ and $x_2 - x_1$. By the Theorem,

$$|\overline{P_1P_2}|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

from which $|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

This relationship is called the distance formula.

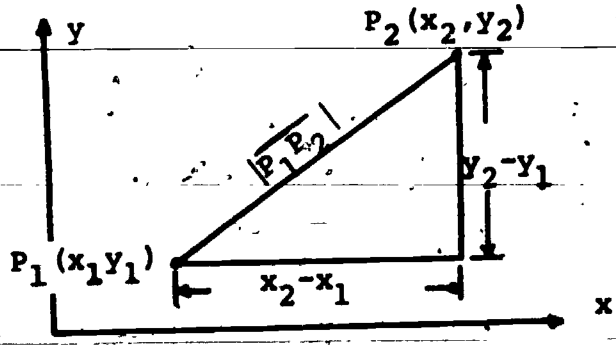


Figure 1.9

Example.

6.3 The undirected distance from $P_1(3,5)$ to $P_2(-6,9)$ is $|\overline{P_1P_2}| = \sqrt{(-6 - 3)^2 + (9 - 5)^2} = \sqrt{81 + 16}$
$= \sqrt{97}$ units

Exercise Set 5

1. Refer to figure 1.10 to find the indicated directed or undirected distance.

a. $\overline{OP_1}$ b. $\overline{P_1O}$ c. $\overline{OP_3}$

d. $\overline{P_3O}$ e. $\overline{OP_4}$ f. $\overline{P_2P_1}$

g. $|\overline{OP_2}|$ h. $|\overline{P_3P_2}|$ i. $|\overline{P_3P_1}|$

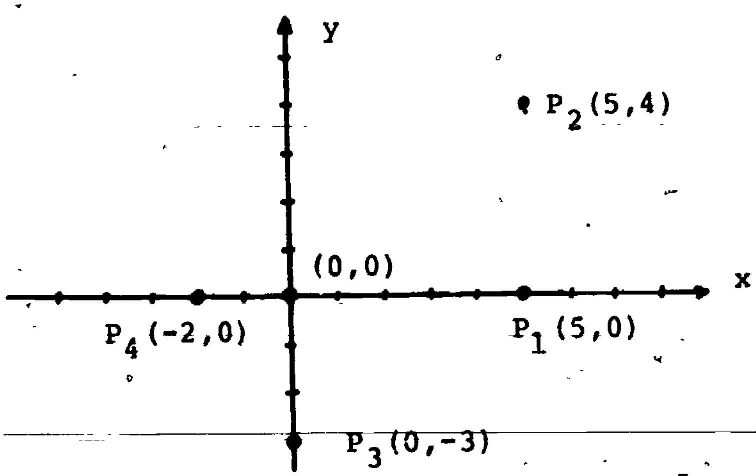


Figure 1.10

.7. Slope of a Line.*

On a nonhorizontal straight line, as the values of x increase, the corresponding values of y will either increase or decrease. In Figure 1.11, as x increases from 1 to 2, the corresponsing values of y increase for lines A and B and decrease for line C. This increase or decrease in the values of y provide a measure for the 'steepness' of a line. This 'steepness' is indicated by the term slope.

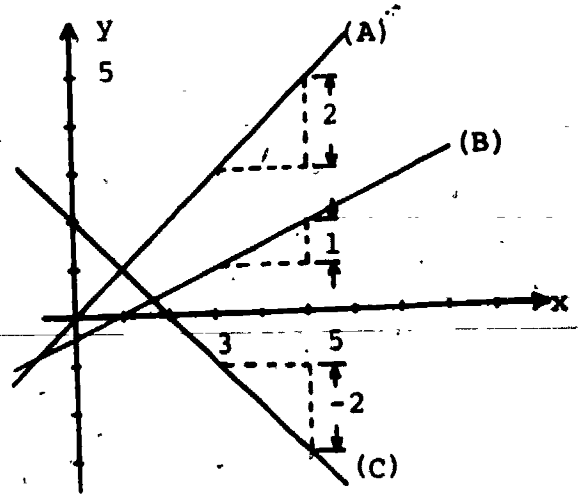


Figure 1.11

*Only straight lines are considered in this chapter.

The slope, m, of a straight line is the quotient of any increase in the values of x divided into the corresponding increase (or decrease) in values of y.

Symbolically, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ for any two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ on the given straight line. See Figure 1.12.

The difference in the x - coordinates is called a change in x or run expressed as Δx (read delta x) and the corresponding difference in the y - coordinates is called a change in y or rise, written Δy.

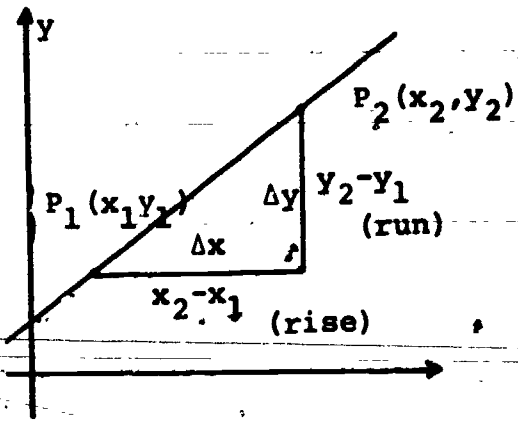


Figure 1.12

Examples.

7.1 The points (3,-9) and (-7,8) lie on a line. The slope of the line is

$$m = \frac{-9 - 8}{3 - (-7)} = \frac{-17}{10}.$$

7.2 Find the slope of the line whose equation is $6x - 2y = 9$.

Step 1. Find two solutions of the equation. They are coordinates of two points on the line.

x	0	3
y	$-\frac{9}{2}$	$\frac{9}{2}$

$P_1(0,-\frac{9}{2})$ $P_2(3,\frac{9}{2})$

Step 2. By definition,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{9}{2} - \left(-\frac{9}{2}\right)}{3 - 0} = 3.$$

7.3 Verify that the slope of the horizontal line having the equation $y = 13$ or $0x + y = 13$ is zero.

Step 1. Find the coordinates of two points on the line.

x	-4	2
y	13	13

$P_1(-4,13)$ $P_2(2,13)$

Step 2. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{13 - 13}{2 - (-4)} = \frac{0}{6} = 0.$

7.4 Show that a horizontal line whose equation is $y = C(0x + y = C)$, C a constant, has a slope equal to zero.

Step 1. Find the coordinates of two points on the line.

x	0	5
y	C	C

$P_1(0,C)$ $P_2(5,C)$

Step 2. The slope is $m = \frac{C - C}{5 - 0} = \frac{0}{5} = 0.$

In Figure 1.13, lines A and B are parallel. The slope of (A) is 1/2 and the slope of (B) is 1/2. In general, two lines having 2 slopes m_1 and m_2 are <u>parallel</u> if and only if $m_1 = m_2$.

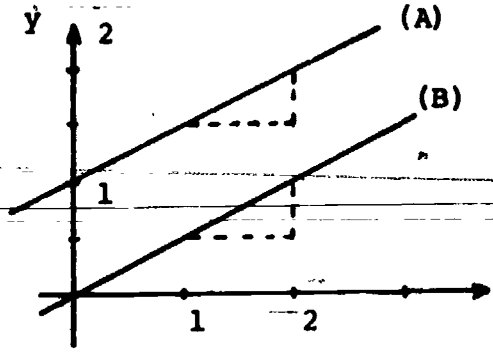


Figure 1.13

In Figure 1.14, lines (A) and (B) are perpendicular. The slope of (A) is 3/1 while (B) has slope $\frac{-1}{3}$ or $-\frac{1}{3}$.

The slope of (A) is the negative reciprocal of the slope of (B). In general, two lines having slopes m_1 and m_2 are <u>perpendicular</u> if and only if

$m_1 = -\frac{1}{m_2}$ or equivalently,

$m_1 \cdot m_2 = -1.$

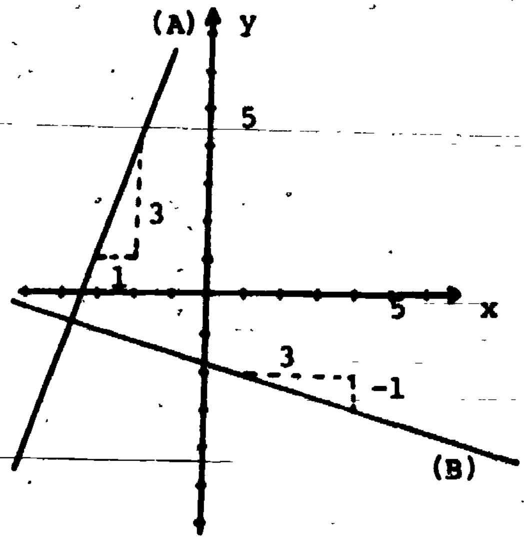


Figure 1.14

Examples.

7.5 Two lines have equations $4x + y = 3$ and $x - 4y = 8$. Show that the lines are perpendicular.

Step 1. Find the coordinates of two points on each line.

x	0	1
y	3	-1

$P_1(0,3)$ $P_2(1,-1)$

x	0	4
y	-2	-1

$P_3(0,-2)$ $P_4(4,-1)$

Step 2. Find the slope of each line.

$m_1 = \frac{-1 - 3}{1 - 0} = -4$ $\qquad$ $m_2 = \frac{-1 - (-2)}{4 - 0} = \frac{1}{4}$

Step 3. Since $-4 \cdot \frac{1}{4} = -1$ ($m_1 \cdot m_2 = -1$), the lines are perpendicular.

Exercise Set 6

1. Find the slope of the line passing through the given points.

   a. $P_1(8,5)$, $P_2(-7,0)$ $\qquad$ b. $P_1(0,0)$, $P_2(3,4)$

   c. $P_1(-4,-6)$, $P_2(-5,-3)$ $\qquad$ d. $P_1(4.8,3.9)$, $P_2(6.5,-8.1)$

2. A line passes through $P_1(-4,2)$ and $P_2(5,11)$. A second line passes through $P_3(-6,1)$ and $P_4(3,10)$. Are the lines parallel?

3. Show that a vertical line having the equation $x = c$ has no slope.

4. A line passes through $P_1(0,7)$ and $P_2(3,-1)$. A second line contains the points $P_3(4,-1)$ and $P_4(-12,2)$. Are the lines perpendicular?

5. Two lines have slopes of -2.8 and $5/14$. Are the lines perpendicular?

6. The graph of the Celsius - Fahrenheit temperature relationship $C = 5/9(F - 32)$ is a line. What is the slope of the line?

8. Equations of a Line.

An equation of a straight line can be determined if two points on the line are known.

Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two given points on the line. If $P(x,y)$ is a point on the line, then the slope of the line is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Multiplying both sides of the equation by $x - x_1$ gives

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

called the two - point form of an equation of a line.

Example.

8.1 Determine an equation of the line passing through the points $P_1(4,-3)$ and $P_2(-3,1)$.

Step 1. Substitute the coordinates into the two - point form.

$$(y - (-3)) = \frac{-3 - 1}{4 - (-3)} \cdot (x - 4)$$

Step 2. Simplify the equation to $4x + 7y = -5$.

An equation of a line can be found if the slope m and a point $P_1(x_1,y_1)$ on the line are known. Changing the slope expression to m in the two - point form, gives $(y - y_1) = m \cdot (x - x_1)$, called the slope - point form of an equation of a line.

Example.

8.2 Write an equation of the line with slope $-\frac{1}{4}$ passing through the point $P(7,-1)$.

Step 1. Substituting the given into the slope - point form yields $y - (-1) = (-\frac{1}{4}) \cdot (x - 7)$.

Step 2. Simplifying the equation of Step 1 gives $x + 4y = 3$.

A line intersecting the y - axis at the point (0,b) has a y - intercept of b. For a line having slope m and y - intercept b (it passes through (0,b)), the equation $(y - b) = m\cdot(x - 0)$ can be manipulated algebraically into $y = mx + b$, called the slope - intercept form of an equation of a line.

Example.

8.3 Write an equation in standard form of the line with slope -2/5, passing through the point (0,8).

Step 1. From the point (0,8), the y - intercept is 8.

Step 2. Substituting the values $m = -2/5$ and $b = 8$ into $y = mx + b$ gives $y = (-2/5)\cdot x + 8$ from which $2x + 5y = 40$.

A vertical line passing through the point (c,0) has an equation $x = c$.

A horizontal line with a y - intercept b has an equation $y = b$.

A line passing through the origin (0,0) with slope m has an equation $y = mx$.

Example.

8.4 A line passing through the point (-1,13) which

a. is vertical also passes through (-1,0). Hence, the equation is $x = -1$.

b. is horizontal also passes through (0,13) and has a y - intercept of 13. Thus, the equation is $y = 13$.

c. also passes through the origin has slope $m = \frac{13 - 0}{-1 - 0} = -13$. The equation of the line is $y = -13x$.

Exercise Set 7

1. Write an equation of the line passing through the given points.

   a. $P_1(8,5)$, $P_2(-7,3)$ b. $P_1(0,0)$, $P_2(3,4)$

   c. $P_1(-4,6)$, $P_2(-4,10)$ d. $P_1(7,5)$, $P_2(6,5)$

2. Write an equation of the line with the given slope and passing through the given point.

   a. $m = 3$, $P_1(4,7)$ b. $m = 0$, $P_1(8,-5)$

   c. $m = 4/7$, $P_1(5,1)$ d. $m = -5.7$, $P_1(0.5,6.3)$

3. Write an equation for the line having the given slope and y - intercept.

   a. $m = -6$, $b = -5$ b. $m = -1/3$, $b = 7$

   c. $m = 1$, $b = 0$ d. $m = 0$, $b = 0$

4. Find the slope and y - intercept of the line whose equation is given.

   a. $y = 5x - 10$ b. $y = -x$

   c. $y = -8.6$ d. $f(t) = 9 \cdot t - 1$

   e. $C = 5/9(F - 32)$ f. $x + y = 0$

5. What is an equation of the line passing through the origin which is parallel to the line having the equation $3x - 7 = y$?

6. A line intersects the y - axis at (0,5) and it is perpendicular to the line $5x - 2y = 3$. What is an equation of this line?

7. A line is perpendicular to the line $ax - cb = gy$ and passes through the origin. What is an equation of the line?

9. Direct Variation (Data Analysis).

Experimental data can sometimes be analyzed by determining a linear relationship between the experimental variables. The data is first graphed to determine whether the relationship is linear. Any form of the line equation (two - point, point - slope, or slope - intercept) may then be used to establish the relationship.

Example.

9.1 In the laboratory the pressure (P) of a fixed volume of gas is measured for various temperatures (T). The following data is obtained:

Temperature (^{o}C)	0	23	58	84	100
Pressure (N/m^2)	22.7	24.6	27.5	29.7	31.0

Write an equation to express the relationship between the pressure and the temperature of the gas.

Step 1. Graphing the data suggests that the relationship is linear.

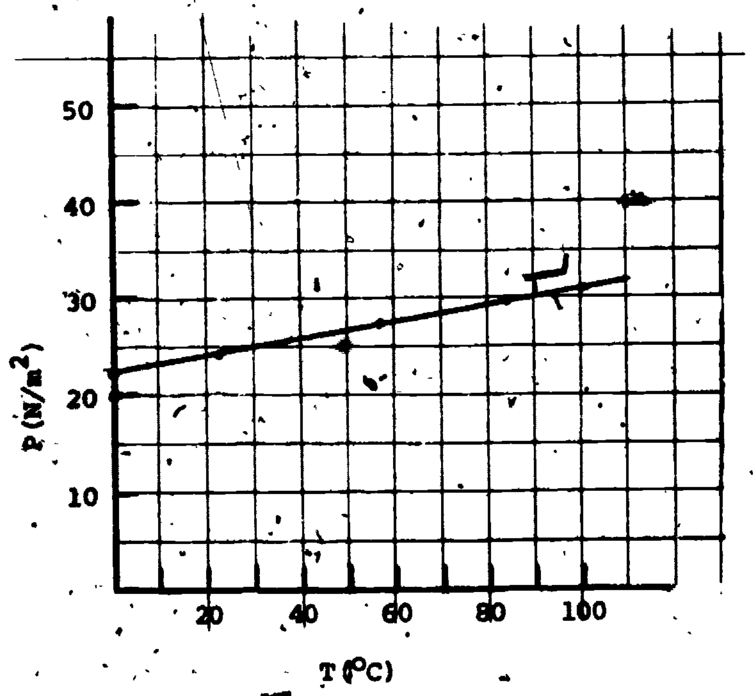


Figure 1.15

Step 2. The slope (m) is determined from the points (0,22.7) and (100,31.0):

$$m = \frac{31.0 - 22.7}{100 - 0} = \frac{8.3}{100} = 0.083$$

Step 3. From the point (0,22.7) the intercept (b) is 22.7.

Step 4. Substituting m = 0.083 and b = 22.7 into the slope - intercept form gives: P = 0.083 T + 22.7

Exercise Set 8

1. In the laboratory, the length of a spring (L) is measured when various weights (W) are suspended on it. The following data is obtained:

Weight (oz)	0	1	2	4	5	7
Length (in)	6.0	6.5	7.0	8.0	8.5	9.5

Write an equation to express the relationship between the length of the spring and the weight on it.

CHAPTER TWO

TRIGONOMETRIC EQUATIONS AND VECTORS

1. Angles and Their Measure.

An angle is generated by rotating a half-line from an initial position, about its endpoint, to a terminal position. The initial position of the half-line is called the initial side of the angle; the terminal position is called the terminal side. The fixed endpoint is called the vertex of the angle. If the rotation is counterclockwise, the angle is said to be positive. A clockwise rotation generates a negative angle.

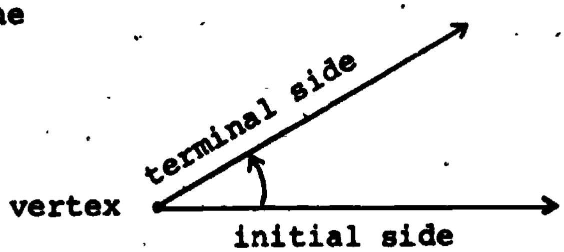


Figure 2.1

An angle in standard position has its vertex at the origin and its initial side is the positive portion of the x - axis. An angle θ (read theta) appears in standard position in Figure 2.2.

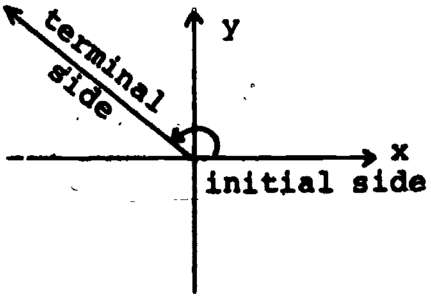


Figure 2.2

Two units of measure of an angle are the revolution and the degree. If the initial side of an angle rotates counterclockwise so that the terminal position coincides with the initial position, the measure of the angle generated is 1 revolution or 360 degrees, written 360°. Thus, 1° equals 1/360 of a revolution.

The degree is divided into 60 equal parts called minutes and each minute is divided in 60 equal parts called seconds. An angle of measure 35 degrees, 13 minutes, and 43 seconds is written $35^{\circ}13'43''$.

A third unit of measure of an angle is the radian. If the radius of a circle is rotated so that the intercepted arc of the circle is equal in length to the radius, the angle generated has the measure 1 radian (See Figure 2.3). Since there are 2π radii in the circumference of a circle, there are 2π or approximately 6.28 radians in 1 revolution. One radian is about 57.3° and 2π radians equals 360°.

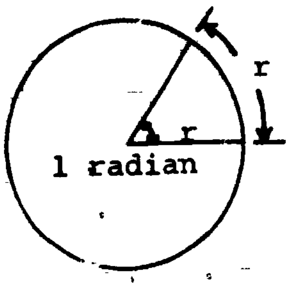


Figure 2.3

Examples.

1.1 Four angles in standard position are shown below.

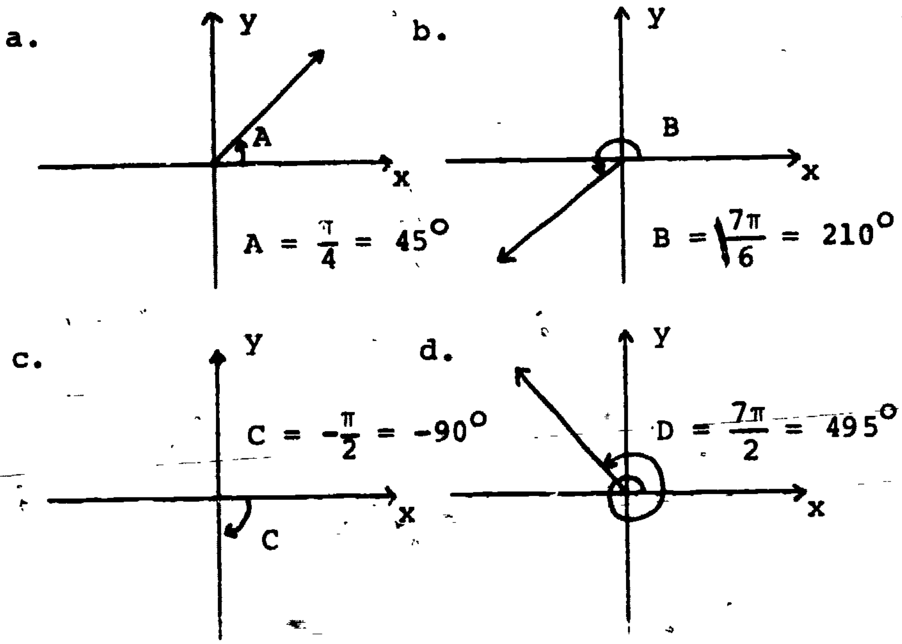


figure 2.4

1.2 Convert 273.2558° to degree - minute - second notation.

Step 1. $273.2558^{\circ} = 273^{\circ} + 0.2558^{\circ}$

$$0.2558 \cancel{\text{degrees}} \times \frac{60 \text{ minutes}}{1 \cancel{\text{degree}}} = 15.348 \text{ minutes}$$

Step 2. $15.349' = 15' + 0.349'$

$0.349 \cancel{\text{minutes}} \times \frac{60 \text{ seconds}}{1 \cancel{\text{minute}}} = 21$ seconds

Step 3. Conclusion: $273.2558^\circ = 273^\circ 15' 21''$

1.3 Express $18^\circ 52' 14''$ in degrees (to the nearest thousandth).

Step 1. $18^\circ 52' 14'' = 18^\circ + \frac{52}{60}^\circ + \frac{14}{3600}^\circ$

$= 18^\circ + 0.867^\circ + 0.004^\circ$

$= 18.871^\circ$

1.4 Add $35^\circ 48' 56''$ and $43^\circ 40' 23''$.

Step 1.
$$\begin{array}{r} 35^\circ 48' 56'' \\ \underline{43^\circ 40' 23''} \\ 78^\circ 88' 79'' \end{array}$$

Step 2. $78^\circ 88' 79'' = 78^\circ 89' 19''$

Step 3. $78^\circ 89' 19'' = 79^\circ 29' 19''$

1.5 Subtract $13^\circ 46' 27''$ from $61^\circ 10' 15''$.

Step 1. $61^\circ 10' 15'' = 60^\circ 70' 15''$

$= 60^\circ 69' 75''$.

Step 2.
$$\begin{array}{r} 60^\circ 69' 75'' \\ \underline{-13^\circ 46' 27''} \\ 47^\circ 23' 48'' \end{array}$$

1.6 Find $\frac{1}{2}$ of $135^\circ 24'$.

Step 1. $\frac{1}{2}(135^\circ 24') = \frac{1}{2}(135^\circ + 24')$

$= 67.5^\circ + 12'$

$= 67^\circ 30' + 12' = 67^\circ 42'$

An alternate method is

Step 1. $\frac{1}{2}(135^\circ 24') = \frac{1}{2}(134^\circ 84')$

$= 67^\circ 42'$

Exercise Set 1

1. Sketch the angle having the given measure.

   a. $\frac{\pi}{3}$ radians b. -45° c. π radians d. $\frac{5}{6}$ revolution

   e. $\frac{3\pi}{4}$ radians f. 585° g. -330° h. 3π

2. Express

   a. $17^{\circ}21'50''$ in decimal form.

   b. 5.50° in degrees, minutes, seconds.

   c. 47.36° in degrees, minutes, seconds.

3. Perform the indicated operations.

   a. Add $39^{\circ}42'18''$ and $51^{\circ}51'51''$.

   b. Subtract $46^{\circ}31'12''$ from $63^{\circ}7''$.

   c. $\frac{1}{2}(146^{\circ}40')$

   d. $\frac{1}{2}(7^{\circ}14'50'')$

   e. $\frac{2}{3}(62^{\circ}35'30'')$

2. Conversion From One Angular Measure to Another.

When converting from one type of angular measure to another, a conversion factor derived from the relations $180^{\circ} = \pi$ radians, $360^{\circ} = 1$ revolution, and 2π radians $= 1$ revolution can be used.

If an angular measure type A is to converted to a measure type B, then A · B/A gives B where B/A is a known conversion factor.

Examples.

2.1 To convert 18° to radians,

$$18 \text{ \cancel{degrees}} \times \frac{\pi \text{ radians}}{180 \text{ \cancel{degrees}}} = \frac{18\pi}{180} \text{ radians}$$

$$= \frac{\pi}{10} \text{ radians}$$

2.2 Convert $484^{\circ}30'$ to revolutions (rev).

Step 1. $482^{\circ}24' = 484.4^{\circ}$

Step 2. $482.4\ \cancel{\text{degrees}} \times \dfrac{1\ \text{rev}}{360\ \cancel{\text{degrees}}}$

$$= \frac{482.4}{360}\ \text{rev} = 1.34\ \text{rev}.$$

2.3 Change $\frac{15\pi}{2}$ radians to degrees.

$$\frac{15\pi}{2}\ \cancel{\text{radians}} \times \frac{180\ \text{degrees}}{\pi\ \cancel{\text{radians}}} = \frac{15 \cdot 180 \cdot \pi}{2 \cdot \pi}\ \text{degrees}$$

$= 1350^{\circ}$

Exercise Set 2

1. Convert each of the following.

a. 30° to radians

b. 18.7 rev to degrees

c. $\frac{\pi}{18}$ radians to rev

d. 30° to rev

*e. 45° to radians

*f. 15 rev to radians

g. $\frac{4}{5}$ rev to degrees

*h. 3.14 radians to degrees

*Express as decimals to the nearest thousandth.

3. The Trigonometric Functions.

Each angle in Figure 2.5 shows a line segment $\overline{PQ}$ drawn perpendicular to the x - axis from a point P(x,y) on its terminal side. The right triangle formed has six ratios of its sides which are functions of the terminal side and thus, the angle itself. The distance from the vertex to the point P(x,y), called the radius vector r, is the

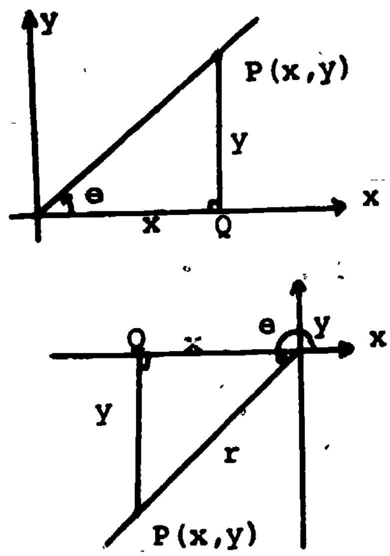


Figure 2.5 (cont.)

hypotenuse of the right triangle.

The six functions, called <u>trigonometric functions</u>, are defined below.

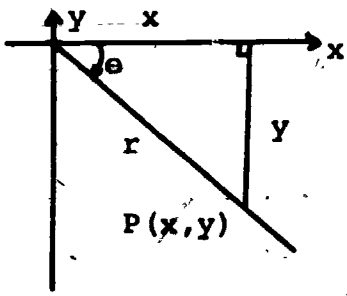


Figure 2.5

Function	Abbreviation	Ratio	Sides Related to Angle θ
sine of θ	sin θ	$\frac{y}{r}$	$\frac{\text{opposite}}{\text{hypotenuse}}$
cosine of θ	cos θ	$\frac{x}{r}$	$\frac{\text{adjacent}}{\text{hypotenuse}}$
tangent of θ	tan θ	$\frac{y}{x}$	$\frac{\text{opposite}}{\text{adjacent}}$
secant of θ	sec θ	$\frac{r}{x}$	$\frac{\text{hypotenuse}}{\text{adjacent}}$
cosecant of θ	csc θ	$\frac{r}{y}$	$\frac{\text{hypotenuse}}{\text{opposite}}$
cotangent of θ	cot θ	$\frac{x}{y}$	$\frac{\text{adjacent}}{\text{opposite}}$

The values of the trigonometric functions can be found using the calculator. Some calculators, however, do not compute values of the cosecant, secant, or cotangent functions directly. To do this, the following <u>reciprocal relations</u> are applied.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Examples.

3.1 The value of

a. $\csc 40^\circ = \frac{1}{\sin 40^\circ} = \frac{1}{0.643} = 1.556$

b. $\sec \frac{\pi}{20} = \frac{1}{\sin \frac{*\pi}{20}} = \frac{1}{0.156} = 6.392$

*Convert $\pi/20$ to 9° if the calculator does not accept radian measure.

c. $\cot(-215^{\circ}) = \frac{1}{\tan(-215^{\circ})} = \frac{1}{-0.700} = -1.428$

3.2 The point $P(-2, -\sqrt{5})$ lies on the terminal side of an angle θ. The values of the 6 trigonometric functions of θ are

a. $\sin \theta = \frac{-\sqrt{5}}{3}$

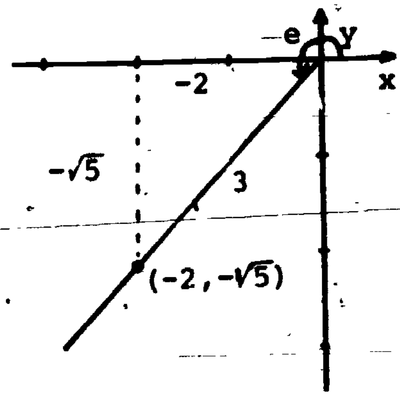
b. $\cos \theta = \frac{-2}{3}$

c. $\tan \theta = \frac{-\sqrt{5}}{-2} = \frac{\sqrt{5}}{2}$

d. $\csc \theta = \frac{3}{-\sqrt{5}} = -\frac{3\sqrt{5}}{5}$

e. $\sec \theta = \frac{3}{-2} = -\frac{3}{2}$

f. $\cot \theta = \frac{-2}{-\sqrt{5}} = \frac{2\sqrt{5}}{5}$



"figure 2.6"

3.3 The function value (to the nearest thousandth) for the given angle using the calculator are

a. $\sin 30^{\circ} = 0.500$

b. $\cos 170^{\circ} = -0.985$

c. $\tan 36^{\circ} = 0.727$

d. $\sec 150^{\circ} = -1.155$

e. $\csc(-\frac{3\pi}{2}) = -1.000$

f. $\cot(2.6 \text{ rev}) = 1.376$

g. $\sin 36.617^{\circ} = 0.596$

h. $\cos(-6^{\circ}15") = 0.994$

3.4 Find a positive angle A less than 360° so that $\tan A = 0.538$.

The calculator procedure is $\tan^{-1}(0.538)$ or Arc $\tan(0.538)$ which gives 28.280° or 0.494 radians.

Exercise Set 3

1. Evaluate each trigonometric function to the nearest thousandth.

   a. $\sin 45^\circ$ b. $\cos \frac{\pi}{4}$ c. $\tan 195^\circ$

   d. $\sec(-60^\circ)$ e. $\cot \frac{5\pi}{3}$ f. $\csc 220^\circ 30'$

   g. $\sin(-3\pi)$ h. $\tan 88^\circ 15' 45''$ i. $\sec 135^\circ$

2. Find a positive angle A less than 360° given that

   a. $\sin A = 0.500$ b. $\cos A = 0.346$ c. $\tan A = -15.318$

3. Find the trigonometric functions of an angle whose terminal side passes through the point $(-3, \sqrt{7})$.

4. Solving Right Triangles.

To solve a right triangle means to find the lengths of its sides and measures of its angles.

Example.

4.1 Solve the right triangle shown in Figure 2.7.

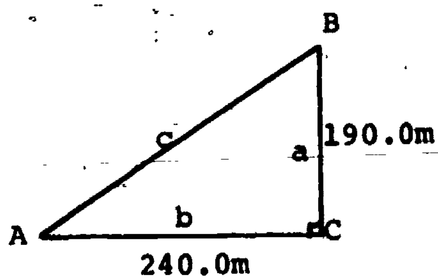


Figura 2.7

Step 1. To find angle A, $\tan A = 190.0/240.0 = .7917$. Thus, $A = 38.37^\circ$ or $37^\circ 22' 12''$

Step 2. The sum of A and B is 90.00°, so $B = 90.00 - 28.37^\circ = 51.63^\circ$ or $51^\circ 37' 48''$.

Step 3. Using the Pythagorean Theorem,
$C = \sqrt{(240.0)^2 + (190.0)^2}$ from which
$C = \sqrt{94,700} = 306.1$ meters.

4.2 A cross-sectional view of a metal shaft with a tapered end is shown in Figure 2.8. Find the diameter of the shaft $\overline{BC}$ and the slant length $\overline{AB}$.

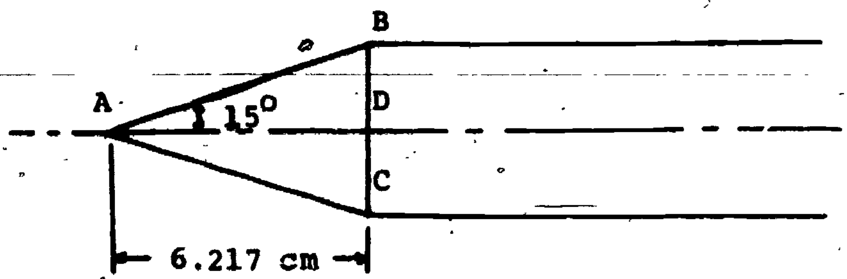


Figure 2.8

Step 1. $\tan 15^\circ = \dfrac{\overline{BD}}{6.217}$ and

$$\begin{aligned}\overline{BD} &= (\tan 15^\circ)\cdot(6.217)\\ &= (.2679)\cdot(6.217)\\ &= 1.666 \text{ cm.}\end{aligned}$$

Step 2. $\overline{BC} = 2 \cdot \overline{BD} = 2 \cdot (1.666) = 3.332$ cm

Step 3. $\sec 15^\circ = \dfrac{\overline{AB}}{6.217}$ and $\overline{AB} = (\sec 15^\circ)\cdot(6.2\ 7)$

$$\begin{aligned}&= (1.035)\ (6.217)\\ &= 6.436 \text{ cm.}\end{aligned}$$

Exercise Set 4

Assume that in each exercise 1-5, the right triangle has its parts labeled as shown in Figure 2.9. Solve for the remaining parts.

1. $c = 5$ m, $b = 4$ m
2. $c = 25$ cm, $a = 20$ cm
3. $A = 30^\circ$, $a = 1$ km
4. $A = \dfrac{\pi}{4}$, $b = 1.000$ cm
5. $B = \dfrac{3}{40}$ rev, $a = 390$ mm

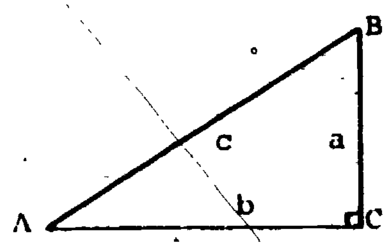


Figure 2.9

6. An "impedance" right triangle is used in analyzing alternating current circuits. In a particular circuit, $X = 1655$ and $Z = 5135$. Find the phase angle θ.

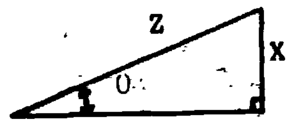


Figure 2.10

5. <u>Applications of Radian Measure.</u>

The length of an arc S of a circle is directly proportional to the measure of the central angle θ expressed in radians. This relationship is expressed by the equation $S = \theta r$ where r is the radius of the circle.

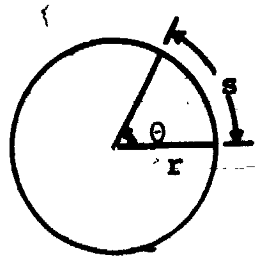


Figure 2.11

Examples.

5.1 An arc length of 43.70 cm corresponds to the central angle of 250°. What is the radius?

Step 1. Converting 250° to radians gives 4.363 radians.

Step 2. Substituting into $S = \theta r$, $43.70 = 4.363r$. The radius $r = \dfrac{43.70}{4.363} = 10.02$ cm.

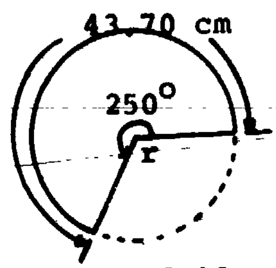


Figure 2.12

5.2 A road curves along the arc of a circle with a radius of 400.0 meters and a central angle of 62°. A steel cable costing \$4.16 per meter is to be placed on the curve of the road as a protective barrier. How much will the cable cost?

Step 1. $62^\circ = 1.082$ radians

Step 2. In the formula $S = \theta r$, $S = (1.082)(400.0) = 432.8$ meters.

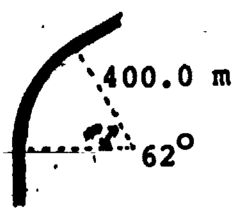


Figure 2.13

Step 3. The cost is 432.8 $\cancel{\text{meters}} \cdot \frac{\$4.16}{1\ \cancel{\text{meter}}}$
$= \$1800.45$

The area A of a sector of a circle in terms of the radius r and the central angle θ in radians is given by $A = \frac{1}{2}\theta r^2$.

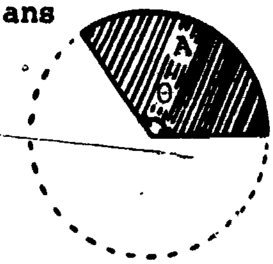


Figure 2.14

Example.

5.3 A cone is formed by joining the two radii of a sector of a circle together. How much does it cost to make a cone from sheet metal costing \$12.50 per square meter if a sector having a radius of 25.00 centimeters and a central angle of 240° is used?

Step 1. Convert 240° to 3.927 radians.

Step 2. The cone is made from a sector of area given by $A = \frac{1}{2}\theta r^2$.

$A = \frac{1}{2} \cdot (3.927)(25.00)^2 = 1227\ \text{cm}^2$.

Step 3. Convert 1227 cm^2 to square meters.

$1227\ \cancel{\text{cm}^2} \times \frac{1\ \text{m}^2}{10{,}000\ \cancel{\text{cm}^2}} = 0.1227\ \text{m}^2$.

Step 4. The cost is $0.1227\ \cancel{\text{m}^2} \cdot \frac{\$12.50}{1\ \cancel{\text{m}^2}} = \$1.53$.

An object moving on a circular path has an average angular velocity $\bar{\omega}$ (omega) defined to be the angular displacement θ through which a body rotates divided by the time t elapsed. That is, $\bar{\omega} = \theta/t$.

The equation $V = \bar{\omega} \cdot r$ relates average linear velocity $\bar{V}$ of an object moving along the arc of a circle of radius r with the average angular velocity $\bar{\omega}$ of the object.

Example.

5.4 A stereo turntable with a diameter of 1/3 meter rotates at 78 rpm. Find the angular velocity $\bar{\omega}$ in radians per second and the corresponding linear velocity of a point on the circumference of the turntable.

Step 1. $\frac{78\ \cancel{rev}}{1\ \cancel{min}} \cdot \frac{2\pi\ rad}{1\ \cancel{rev}} \cdot \frac{1\ \cancel{min}}{60\ sec} = 8.17\ rad/s$

Step 2. To find linear velocity, substitute in $\bar{V} = \bar{\omega} \cdot r$.

$$\bar{V} = \frac{8.17}{sec} \cdot \frac{1}{6}\ meter = 1.36\ meters/s$$

Exercise Set 5

Find the unknown quantities listed in the table.

	Angular Velocity $\bar{\omega}$	Time t	Central Angle θ
1.	? rad/sec	20 sec	10π rad
2.	5 rad/sec	? sec	85 rad
3.	π/2 rad/sec	7 min 30 sec	? rad

	Linear Velocity $\bar{V}$	Angular Velocity $\bar{\omega}$	Radius
4.	? m/sec	40 rad/sec	6 m
5.	200 m/min	? rad/sec	0.5 m
6.	4.5 m/min	150π rad/min	? mm

7. Points P_1, P_2, and P_3 are located at various points along a radius of the flywheel as shown in the figure. Point P_1 on the perimeter is 25 cm from the center while P_2 and P_3 are one-half and one-fourth of the radius distance from the center. If the flywheel rotates at 300 revolutions per minute, what is the linear velocity of each point in centimeters per second?

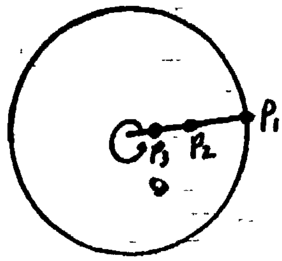


Figure 2.15

8. The arrowhead design represented in the figure was formed using arcs of concentric circles of radii 30 cm and 16 cm and five consecutive radii 15^{o} apart. The design is to be made from sheet metal costing $13.50 per square meter.

   a. What is the cost of material to produce one arrowhead if there is no scrap metal loss due to recycling?

   b. A metal border costing $40 per meter is to surround the arrowhead for decoration. What is the cost of putting a border on one arrowhead?

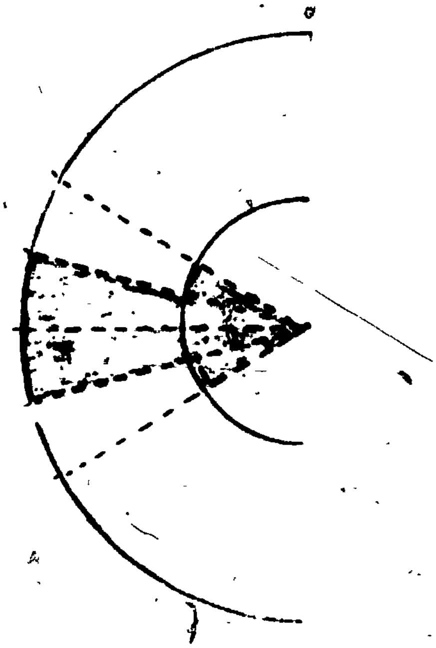


Figure 2.16

9. When two pulleys are belted together, the ratio of the diameter of the largest pulley to the diameter of the smallest pulley equals the ratio of the angular velocity of the smallest pulley to the angular velocity of the largest. i.e.--the larger the diameter of a pulley the slower it must rotate. $D_A/D_B = V_B/V_A$

If pulley B has a diameter of 35 cm and is rotating clockwise at 18 revolutions per minute, what is the angular velocity in radians per second and revolutions per minute of pulley A which has a diameter of 98 cm?

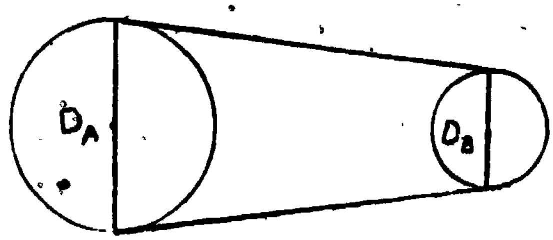


Figure 2.17

6. Vectors - Geometric Interpretation.

Physics and engineering are concerned with quantities that can be characterized by both magnitude and direction. Examples of these quantities, called vector quantities, are velocity and force.

The geometric representation of a vector quantity is a directed line segment from a point $P(x_1,y_1)$ to a point $Q(x_2,y_2)$, written $\overrightarrow{PQ}$, called a vector. The point P is called the initial point and the point Q is called the terminal point. Two vectors from P to Q and from R to S are equal if they have the same direction and length, written $\overrightarrow{PQ} = \overrightarrow{RS}$ (See Figure 2.18). A single letter with an arrow, denoted $\vec{V}$, is also used to symbolize a vector.

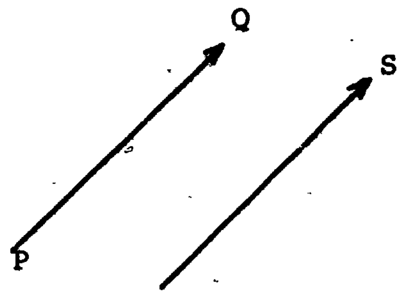


Figure 2.18

If the directed line segment $\overrightarrow{PQ}$ is vector $\vec{A}$ and $\overrightarrow{PQ} = \overrightarrow{RS}$, then $\overrightarrow{RS}$ also represents vector $\vec{A}$. That is, a vector quantity can be represented by any directed line segment, regardless of its location in the plane, as long as it has a specific length and direction.

The length of a vector is proportional to the magnitude of the vector quantity it represents. The direction of a vector is the same as the direction of the vector quantity.

Example.

6.1 A force F of 50 newtons is being exerted at an angle of 35^o with the horizontal. This vector quantity can be represented by the vector $\vec{F}$ as shown in Figure 2.19

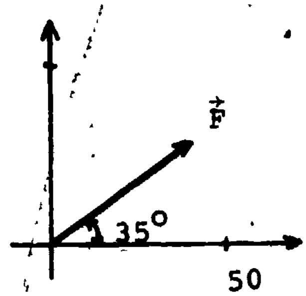


Figure 2.19

Addition of two vectors can be accomplished in the plane by a parallelogram method. The two vectors to be added are placed so that their initial points coincide. Form a parallelogram having the two vectors as adjacent sides. The diagonal of the parallelogram is the sum vector, sometimes called the resultant, written $\vec{R}$. See Figure 2.20.

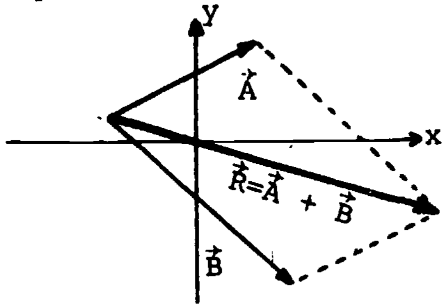


Figure 2.20

A triangle method can be used to add two vectors in the plane. The initial point of one vector is placed at the terminal point of the other. The resultant becomes the third side of the triangle formed with the two given vectors as adjacent sides. See Figure 2.21.

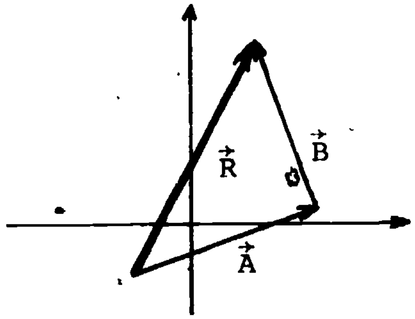


Figure 2.21

Example.

6.2 A directed line segment from the initial point $I_1(-4,6)$ to the terminal point $T_1(3,9)$, call it vector $\vec{V}_1$, is added to a second vector, $\vec{V}_2$, having $I_2(9,-6)$ and $T_2(4,0)$ as initial and terminal points, respectively. The sum of $\vec{V}_1$ and $\vec{V}_2$ is the resultant vector $\vec{R}$ shown below and found by the

a. parallelogram method:

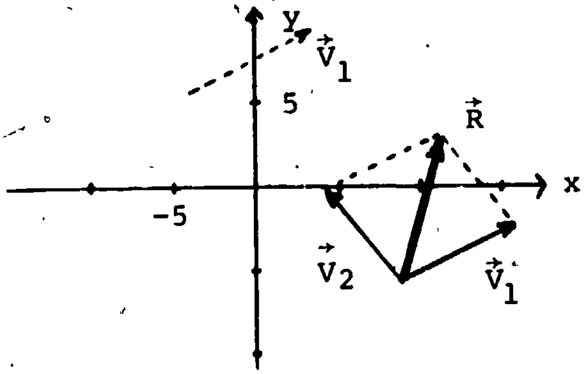


Figure 2.22

b. triangle method:

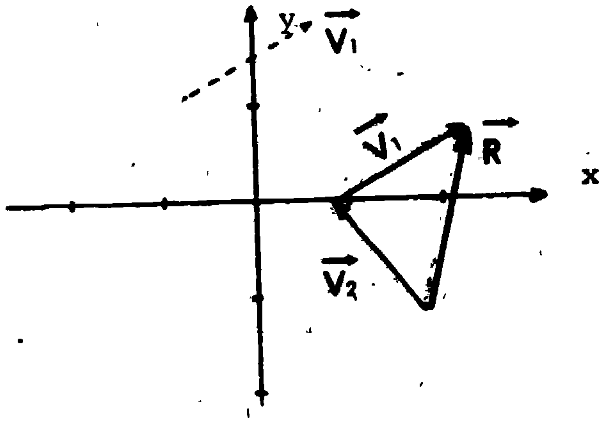


Figure 2.23

Exercise Set 6

1. The initial point I and terminal point T of directed line segments used to define vectors are given. Find the resultant vector of the given vectors by the parallelogram or triangle method. Match the resultant with one of the vectors in Figure 2.24.

   a. $I_1(-8,-6)$ $T_1(-6,4)$; $I_2(-6,-8)$ $T_2(0,-2)$

   b. $I_1(-2,0)$ $T_1(-2,-3)$; $I_2(-9,-4)$ $T_2(-2,-4)$

   c. $I_1(8,5)$ $T_1(0,-2)$; $I_2(-9,-3)$ $T_2(-1,-3)$

   d. $I_1(-$[illegible]$,4)$ $T_1(-8,7)$; $I_2(-9,-3)$ $T_2(-1,-3)$

   e. $I_1(-10,1)$ $T_1(-4,5)$; $I_2(-4,5)$ $T_2(-2,1)$;

      $I_3(6,0)$ $T_3(3,8)$

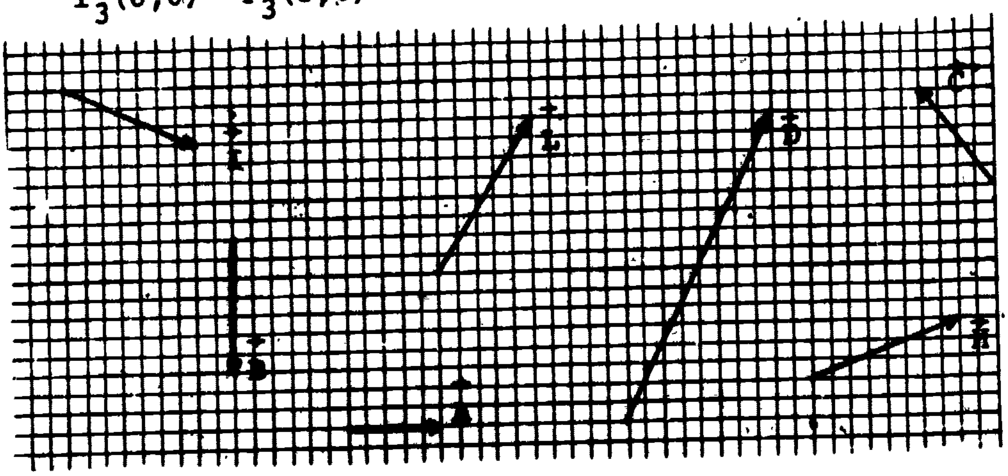


Figure 2.24

7. Vectors as Ordered Pairs.

A vector in the plane is an ordered pair of real numbers, written <x,y>. The first member is called the x-component and the second is called the y-component.

A vector with an initial point $P_1(x_1,y_1)$ and a terminal point at $P_2(x_2,y_2)$ has the form $<x_2-x_1, y_2-y_1>$. If the initial point is the origin (0,0), then the vector i written as $<x_2,y_2>$.

Examples.

7.1 The vectors represented geometrically in figure 2.25 can be expressed as

$\vec{A} = <3,4>$

$\vec{B} = <-5,2>$

$\vec{C} = <2-(-2), -4-(-1)>$

$= <4,-3>$.

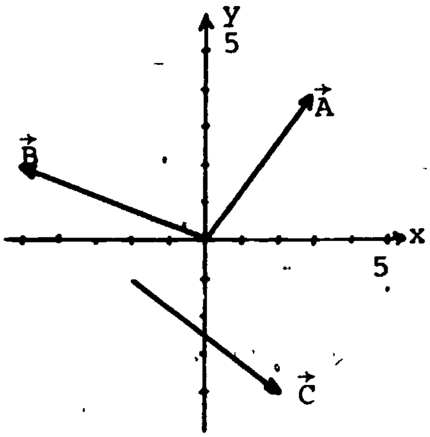


Figure 2.25

The direction of a vector $\vec{V} = <x,y>$ is the angle θ found by solving the equation tan θ = y/x. The magnitude of $\vec{V}$ written $|\vec{V}|$, equals $\sqrt{x^2 + y^2}$.

Examples.

7.2 A vector $\vec{V}$ has an initial point $P_1(2,-1)$ and a terminal point $P_2(5,-7)$. $\vec{V} = <5 - 2, -7-(-1)>$ of $<3,-6>$. The direction of $\vec{V}$ is found by tan θ = -6/3 which gives θ = -63.44°. The magnitude

$|\vec{V}| = \sqrt{3^2 + (-6)^2} = \sqrt{45}$.

7.3 The vector $\vec{A} = <0,-5>$ has a terminal point at (0,-5). Its direction is 270°.

$|\vec{A}| = \sqrt{0^2 + (-5)^2} = 5$.

A vector can be described by its magnitude and direction. A vector with magnitude 15 and direction 57° is written $(15,57^\circ)$. In general, for a vector $\vec{V}$ with direction θ, $\vec{V} = (|\vec{V}|, \theta)$.

To express $\vec{V} = (|\vec{V}|,\theta)$ in the component form $<x,y>$, Figure 2.26 shows that

$\cos\theta = \dfrac{x}{|\vec{V}|}$ from which $x = |\vec{V}|\cdot\cos\theta$;

$\sin\theta = \dfrac{y}{|\vec{V}|}$ from which $y = |\vec{V}|\cdot\sin\theta$;

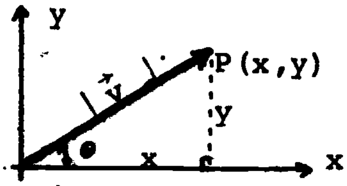


Figure 2.26

Thus, $(|\vec{V}|,\theta) = <|\vec{V}|\cdot\cos\theta, |\vec{V}|\cdot\sin\theta> = <x,y>$.

Example.

7.4 A vector given by $(30,\pi/15)$ has an x-component of $30 \cdot \cos\pi/15$ and a y-component of $30 \cdot \sin\pi/15$. Thus, $x = 30\cdot(0.978) = 29.3$ and $y = 30\cdot(0.208) = 6.24$. Hence, $(30,\pi/15) = <29.3,6.24>$.

Two vectors $\vec{A} = <a_1,a_2>$ and $\vec{B} = <b_1,b_2>$ are equal if and only if $a_1 = b_1$ and $a_2 = b_2$.

If c is any real number, called a scalar, and $\vec{V} = <x,y>$, then the product of c and $\vec{V}$ is a vector denoted by $c\vec{V} = c<x,y> = <cx,cy>$.

For two vectors $\vec{A} = <a_1,a_2>$ and $\vec{B} = <b_1,b_2>$, the sum of $\vec{A}$ and $\vec{B}$, written $\vec{A} + \vec{B}$, is the vector $<a_1+b_1,a_2+b_2>$. Geometrically, $\vec{A} + \vec{B}$ is the resultant vector found by using the parallelogram or triangle method to add $\vec{A}$ and $\vec{B}$.

Examples.

7.5 For two vectors $\vec{A} = <-2,5>$ and $\vec{B} = <3,8>$,

a. $\vec{A} + \vec{B} = <-2,5> + <3,8>$

$= <-2+3,5+8>$

$= <1,13>$.

b. $3\vec{A} + (-2)\vec{B} = 3<-2,5> + (-2)<3,8>$

$= <-6,15> + <-6,-16>$

$= <-6+(-6),15 + (-16)>$

$= <-12,-1>$.

Exercise Set 7

1. Express each given vector in the component form. Write each component to the nearest hundredth.

   a. $(7,85^{\circ})$ b. $(13.6,160^{\circ})$ c. $(0.58,\pi/18)$

2. Given that $\vec{A} = \langle 3,4 \rangle$, B has the initial point $I(-6,3)$ and the terminal point $T(7,-1)$, $\vec{C} = \langle 10,0 \rangle$, and $\vec{D} = \langle -5,-4 \rangle$, find the following.

   a. $\vec{A} + \vec{C}$ b. $3\vec{A} + \vec{B}$ c. $\vec{C} + (-3)\vec{D}$

3. Express $\vec{D}$ from exercise 2. in the form $(|\vec{D}|,\theta)$.

4. Two pool sharks, Harold III and Felix X, hit a ball at exactly the same time. Harold III hits the ball NW with a force sufficient to give the ball a speed of 70.3 centimeters per second. Felix X hits the ball SW with a force sufficient to produce a speed of 70.3 centimeters per second. What direction does the ball travel? How fast? Use the component method of vector addition.

5. A picture hangs crookedly as shown in the figure. The upward tensions in wires A and B are vector quantities having magnitudes of 60 newtons and 45 newtons, respectively. If Wire A makes an angle of 140° with the horizontal, and Wire B an angle of 50°, what are the components of A and B? Add the Vertical components to find the vertical force on the nail.

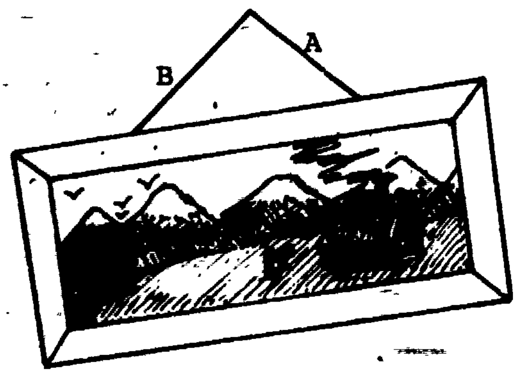


Figure 2.27

8. Computer and Calculator Applications.

Some calculators (Hewlett-Packard 45 and 55, Texas Instrument SR-51, Monroe Beta 326, Sharp PC-1002) have the capability of converting vector representation from the magnitude - direction to the component form. If you have such a calculator, do the following exercises.

Exercise Set 8

1. Convert each given vector to the component form.

   a. $(6.00, 120^{\circ})$ b. $(10.00, 45^{\circ})$ c. $(700.00, 4\pi/3)$

   d. $(85.00, 310^{\circ})$ e. $(67.00, 0^{\circ})$

2. If possible, use a calculator to add the following vectors.

   a. $(10.00, 25^{\circ})$ $(15.00, 150^{\circ})$

   b. $(20.00, 205^{\circ})$ $(8.00, 30^{\circ})$

   c. $(100.00, 75^{\circ})$ $(120.00, 310^{\circ})$ $(40.00, 90^{\circ})$ $(85.00, 110^{\circ})$

CHAPTER THREE

SYSTEMS OF LINEAR EQUATIONS

1. Two Linear Equations in Two Unknowns

Two or more linear equations of the form $ax + by = c$ considered together are called a system of equations. Symbolically,

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

is a system of two linear equations in the unknowns x and y where a_1, b_1, c_1, a_2, b_2, and c_2 are constants.

If an ordered pair (x,y) satisfies both equations of the system, then it is called a solution to the system. To solve a system means to find its solution(s).

Example.

1.1 In the system $\begin{cases} 4x - 2y = 14 \\ 3x + 5y = 4 \end{cases}$,

the ordered pair $(3,-1)$ is a solution since it satisfies both of the equations. On the other hand, $(2,-3)$ satisfies the equation $4x - 2y = 14$ but does not satisfy $3x + 5y = 4$. Therefore, $(2,-3)$ is not a solution of the system.

2. Solution of a System by Graphing

By graphing the equations of a system, it is possible to locate the point of intersection of the two line-graphs. A point of intersection is the graph of a solution of the system. Keep in mind that inaccuracies can occur in the drawing and reading of graphs to make this method of solving a system less appealing than other methods to be presented later.

Examples.

2.1 Steps leading to the solution of the system

$$\begin{cases} 4x - 3y = 3 \\ 2x + 6y = 19 \end{cases}$$ are:

Step 1. Graph the equations.

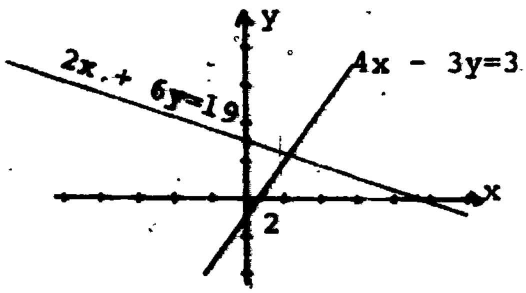


Figure 3.1

Step 2. Project the point of intersection of the lines onto the x-axis and y-axis to find the approximate values of x and y, respectively.

Step 3. The graphical solution of the system is estimated to be (2.5,2.3). [Actually, the solution is $(\frac{5}{2},\frac{7}{3})$]

2.2 $$\begin{cases} \frac{3}{2}x - \frac{1}{2}y = 2 \quad . . . (1) \\ -6x + 2y = -8 \quad . . . (2) \end{cases}$$

Step 1. Graph the equations.

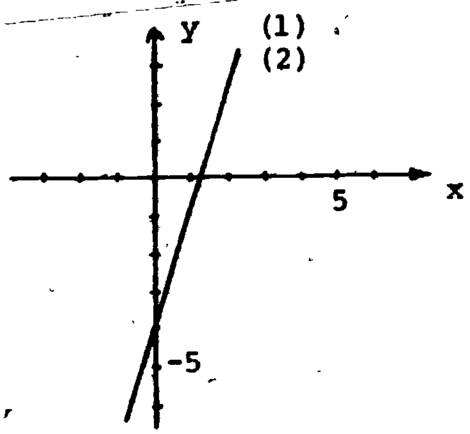


Figure 3.2

Step 2. Both equations of the system have the same line-graph. The points of intersection are the line itself.

Step 3. Conclusion. The solution of the system is the set of coordinates of all points on the line. The solution set is infinite and the system is called <u>dependent</u>.

2.3
$$\begin{cases} 5s + 2t = 12 \quad . . . (1) \\ 10s + 4t = 8 \quad . . . (2) \end{cases}$$

Step 1. Graph the equations.

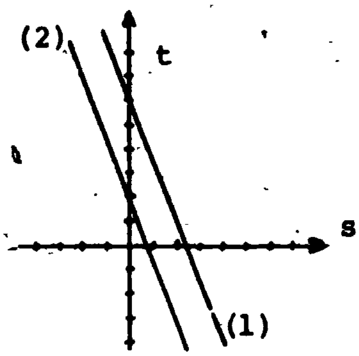


Figure 3.3

Step 2. The lines are parallel; thus, they have no points of intersection.

Step 3. Conclusion. Without points of intersection, there cannot be common solutions of both equations of the system. Therefore, the system has no solutions and it is called <u>inconsistent</u>.

<u>Exercise Set 1</u>

Use the graphing method to solve the given system.

1. $\begin{cases} 2x + 3y = 5 \\ x + y = 1 \end{cases}$

2. $\begin{cases} 4x - 2y = 5 \\ 2x - y = -3 \end{cases}$

3. $\begin{cases} 2m + n = 6 \\ 10m - 2n = -5 \end{cases}$

4. $\begin{cases} x + 4y = 8 \\ \frac{1}{2}x - 4 = 2y \end{cases}$

3. Solving a System By Elimination--Addition or Subtraction Method

Exact solutions of a system can be found by eliminating one of the variables in order to get an equation in the other unknown. The examples to follow illustrate this elimination procedure.

Examples.

3.1 $\begin{cases} 3E - 2V = 6 \ . \ . \ . \ (1) \\ 4E - 5V = 1 \ . \ . \ . \ (2) \end{cases}$

Step 1. To eliminate the variable E, multiply the sides of (1) by 4, the sides of (2) by 3, and subtract.

4 times (1) is $12E - 8V = 24$

3 times (2) is $\underline{12E - 15V = 3}$

$7V = 21$ or $V = 3$.

Step 2. Substitute $V = 3$ into either (1) or (2) and solve for E.

$4E - 5(3) = 1$ or $E = 4$

Step 3. Conclusion. The solution of the system is $(E,V) = (4,3)$.

3.2 $\begin{cases} 4R_1 - R_2 = 0 \ . \ . \ . \ (1) \\ 2R_1 = .5R_2 \ . \ . \ . \ (2) \end{cases}$

Step 1. Multiply the sides of (2) by 10 to remove the decimal coefficients and write (2) in standard form.

$$4R_1 - R_2 = 0 \ . \ . \ . \ (3)$$

$$20R_1 - 5R_2 = 0 \ . \ . \ . \ (4)$$

Step 2. Multiply (3) by 5 and subtract to eliminate either R_1 or R_2.

$$20R_1 - 5R_2 = 0$$

$$\underline{20R_1 - 5R_2 = 0}$$

$$0 = 0$$

Step 3. Conclusion. Since both variables R_1 and R_2 were eliminated resulting in a true equation, the system is <u>dependent</u>. Therefore, the solution of the system is the infinite set of ordered pairs which satisfy either equation of the system.

3.3 $\begin{cases} 3s - 5t = 4 \ldots (1) \\ 9s - 15t = 7 \ldots (2) \end{cases}$

Step 1. Eliminate either s or t by multiplying the sides of (1) by 3 and subtracting.

3 times (1) gives $9s - 15t = 12$

$$\begin{array}{r} 9s - 15t = 12 \\ 9s - 15t = 7 \\ \hline 0 = 5 \end{array}$$

Step 2. Conclusion. Both variables s and t were eliminated resulting in the false equation $0 = 5$. The system is <u>inconsistent</u> with no solutions.

<u>Exercise Set 2</u>

Solve each given system by the addition or subtraction method.

1. $\begin{cases} 5x - y = 3 \\ 4x + y = 6 \end{cases}$

2. $\begin{cases} 3s - 6t = 5 \\ s - 3t = 5 \end{cases}$

3. $\begin{cases} 3x + 2y = 7 \\ 2x - 4y = -22 \end{cases}$

4. $\begin{cases} .2x + .3y = 1 \\ .04x + .06y = .2 \end{cases}$

5. Using Kirchoff's laws, the following equations were set up to solve an electrical circuit. (I = current in amperes E = potential difference in volts). Solve for I_G and I_B.

$$\begin{cases} 16 = E_S = 2.2\ I_G + 2\ I_B \\ 12 = E_B = 2\ I_G + 2.8\ I_B \end{cases}$$

4. Solving a System By Elimination--Substitution Method

This method is convenient to use if one variable in an equation of the system can be easily expressed in terms of the other variable.

Examples.

4.1 $\begin{cases} 10x + 5y = 2 \ . \ . \ . \ (1) \\ 8x + y = 1 \ . \ . \ . \ (2) \end{cases}$

Step 1. Solve (2) for y in terms of x to get $y = 1 - 8x$.

Step 2. Substitute $1 - 8x$ for y in equation (1) and solve for x.

$$10x + 5(1 - 8x) = 2$$

$$-30x = -3$$

$$x = \frac{1}{10}$$

Step 3. Substitute $x = \frac{1}{10}$ into either (1) or (2) and solve for y.

$$10\left(\frac{1}{10}\right) + 5y = 2$$

$$5y = 1 \text{ and } y = \frac{1}{5}$$

Step 4. Conclusion. The solution of the system is $\left(\frac{1}{10}, \frac{1}{5}\right)$.

Exercise Set 3

Solve by the substitution method.

1. $\begin{cases} 8x - 5y = -10 \\ x - 3y = 6 \end{cases}$

2. $\begin{cases} x - y = 10 \\ 3x - 13y = 20 \end{cases}$

3. $\begin{cases} 2a - 3b = 1 \\ 2a - b = -9 \end{cases}$

4. A chemist wants to calculate the atomic weight of nitrogen and oxygen. He knows the atomic weight of dinitrogen pentoxide N_2O_5 is 108 amu and the atomic weight of nitrogen dioxide NO_2 is 46 amu. So he sets up the system of equations:

$$\begin{cases} 2\cdot N + 5\cdot O = 108 \\ N + 2\cdot O = 46 \end{cases}$$

Solve the system and find the atomic weights of nitrogen and oxygen in atomic mass units (amu).

5. Solving a System Using Determinants--Cramer's Rule

A determinant is a square array of numbers, called elements, which symbolize the sum of certain products of these elements. This sum is called the value of the determinant. A determinant of order 2 is symbolized and evaluated as shown in the equation below. An element is a member of the row indicated by the left digit of the subscript and it is a member of the column indicated by the right digit. Thus, a_{21} is in the second row, first column.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}$$

Examples.

5.1 $\begin{vmatrix} 4 & -3 \\ 6 & -1 \end{vmatrix} = 4(-1) - (-3)(6) = 14$

5.2 $\begin{vmatrix} 0.8 & -1 \\ -6 & 6.5 \end{vmatrix} = (0.8)(6.5) - (-1)(-6) = -0.8.$

The values of x and y which satisfy the system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

are found using determinants by a method called Cramer's rule given by:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2}$$

Example.

5.3 Solving the system $\begin{cases} 4x - 5y = 1 \\ 6x + 10y = 5 \end{cases}$ using Cramer's rule gives

$$x = \frac{\begin{vmatrix} 1 & -5 \\ 5 & 10 \end{vmatrix}}{\begin{vmatrix} 4 & -5 \\ 6 & 10 \end{vmatrix}} = \frac{1\cdot 10 - (-5)(5)}{4\cdot 10 - (-5)(6)} = \frac{35}{70} = \frac{1}{2}$$

$$y = \frac{\begin{vmatrix} 4 & 1 \\ 6 & 5 \end{vmatrix}}{70} = \frac{4\cdot 5 - 1\cdot 6}{70} = \frac{14}{70} = \frac{1}{5}$$

The solution of the system is $\left(\frac{1}{2}, \frac{1}{5}\right)$.

If either fractional value of a variable using Cramer's rule is of the form $\frac{0}{0}$, the system is dependent. If a value has the form $\frac{a}{0}$, where $a \neq 0$, the system is inconsistent.

Exercise Set 4

1. Evaluate each order 2 determinant.

a. $\begin{vmatrix} 63 & 30 \\ 2 & 1 \end{vmatrix}$ b. $\begin{vmatrix} 0 & 0 \\ 6 & 4 \end{vmatrix}$ c. $\begin{vmatrix} -4 & 10 \\ \frac{1}{5} & \frac{1}{2} \end{vmatrix}$ d. $\begin{vmatrix} 0.31 & 10 \\ 0.6 & 200 \end{vmatrix}$

2. Solve each system using Cramer's rule.

a. $\begin{cases} 2x - 3y = 5 \\ x + y = 40 \end{cases}$ b. $\begin{cases} 4x - 9y = 3 \\ 2x + 3y = 5 \end{cases}$

c. $\begin{cases} 2m - 5 = 0 \\ 4m - n = 7 \end{cases}$ d. $\begin{cases} d - 3e = 10 \\ 2d - 6e = 5 \end{cases}$

3. A boat can go 10 km down stream in 45 minutes and return in 75 minutes. How fast is the current travelling?

4. A barge traveled 20 km downstream in 50 minutes. A speed-boat takes the same time to go upstream but takes only 15 minutes to go downstream. How fast are the current and the two boats moving?

6. <u>Three Linear Equations in Three Unknowns</u>

A <u>system of three linear equations in three unknowns</u> x, y, and z has the standard form

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

A <u>solution</u> is an ordered triple (x,y,z) which satisfies each equation. To solve a system means to find all of its solutions.

Example.

6.1 The ordered triple (-2,1,5) is a solution of the system

$$\begin{cases} 2x + 3y + z = 4 \\ x - 2y + 2z = 6 \\ 5x \quad\quad + 3z = 5 \end{cases}$$

because x = -2, y = -1, and z = 5 satisfy all three equations.

7. Solving a System--Elimination Method

An elimination procedure similar to the addition-subtraction method of Section 3 can be used to solve a system of three linear equations.

Example.

7.1 The steps in solving the system

$$\begin{cases} 4x - y - 2z = 1 & \ldots (1) \\ 3x + 2y + z = 5 & \ldots (2) \\ 2x + 3y + 3z = 10 & \ldots (3) \end{cases}$$ are:

Step 1. Choose a pair of equations and eliminate one of the variables. Selecting (1) and (2), eliminate z by multiplying (2) by 2 and adding.

$$\begin{array}{rl} 4x - y - 2z &= 1 \\ \underline{6x + 4y + 2z} &\underline{= 10} \\ 10x + 3y \qquad &= 11 \quad \ldots (4) \end{array}$$

Step 2. Select another pair of equations, say (2) and (3), and eliminate z by multiplying (2) by 3 and subtracting.

$$\begin{array}{rl} 9x + 6y + 3z &= 15 \\ \underline{2x + 3y + 3z} &\underline{= 10} \\ 7x + 3y \qquad &= 5 \quad \ldots (5) \end{array}$$

Step 3. Solve the system of two equations (4) and (5) by any convenient method of the previous section. Subtracting (5) from (4) eliminates y.

$$\begin{array}{rl} 10x + 3y &= 11 \\ \underline{7x + 3y} &\underline{= 5} \\ 3x \qquad &= 6 \text{ or } x = 2. \quad y = -3. \end{array}$$

Step 4. Substitute x = 2 and y = -3 into an original equation of the system and solve for z. Using equation (3),

$$2(2) + 3(-3) + 3z = 10 \text{ or } z = 5.$$

Step 5. Conclusion. The solution of the system is $(2,-3,5)$.

Exercise Set 5

Solve the given systems.

1. $\begin{cases} 4x + 3y - 2z = 11 \\ 2x - 3y - 3z = 5 \\ 5x + y - z = 8 \end{cases}$

2. $\begin{cases} 4x + 2y = -2 \\ x + 5y - z = 12 \\ 3y - 2z = -5 \end{cases}$

8. Solving a System Using Determinants--Cramer's Rule

A determinant of order 3 has the standard form of

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

where the elements are real numbers.

The minor of an element a_{ij} $(i = 1,2,3,\ j = 1,2,3)$, written M_{ij}, is the smaller determinant whose elements are formed by eliminating the row and column in which a_{ij} is a member. The cofactor of an element a_{ij} is $(-1)^{i+j}\cdot M_{ij}$.

Example.

8.1 The minor of the element 5 in the determinant

$$\begin{vmatrix} 4 & 2 & -1 \\ 3 & -4 & 5 \\ 6 & 1 & 7 \end{vmatrix} \text{ is } M_{23} = \begin{vmatrix} 4 & 2 \\ 6 & 1 \end{vmatrix} = -8 \text{ formed by}$$

eliminating the second row $(i = 2)$ and third column $(j = 3)$ of the given determinant.

8.2 The cofactor of the element 5 in the determinant of 8.1 above is $(-1)^{2+3}\cdot M_{23} = -1\cdot\begin{vmatrix} 4 & 2 \\ 6 & 1 \end{vmatrix}$

$$= -1\cdot(-8)$$

$$= 8.$$

The value of a determinant of order 3 (or higher) is found by taking the sum of the products of elements in any row (or column) multiplied by their corresponding cofactors. This procedure is sometimes called LaPlace's expansion after the originator.

Example.

8.3 Evaluate $\begin{vmatrix} 4 & 3 & -2 \\ 2 & -3 & -3 \\ 5 & 1 & -1 \end{vmatrix}$

Step 1. Select a row or column, say row 2, and multiply the elements of this row by their corresponding cofactors.

$$2\cdot(-1)^{2+1}\cdot\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix} = 2\cdot(-1)\cdot(-1) = 2$$

$$-3\cdot(-1)^{2+2}\cdot\begin{vmatrix} 4 & -2 \\ 5 & -1 \end{vmatrix} = -3\cdot1\cdot6 = -18$$

$$-3\cdot(-1)^{2+3}\cdot\begin{vmatrix} 4 & 3 \\ 5 & 1 \end{vmatrix} = -3\cdot(-1)\cdot(-11) = -33$$

Step 2. The value of the determinant is the sum of the products $2 + (-18) + (-33) = -49$.

<u>Cramer's rule</u> states that the value of each variable in a system of three equations is the ratio of two order 3 determinants. The denominator is the determinant of coefficients of the variables in the system. The numerator is the coefficient determinant in which the coefficients of the variable being solved for are replaced by the constants of the equations.

Example.

8.4 Solve $\begin{cases} 3x + y - 2z = 1 \\ 2x + 4y + 6z = 10 \\ x - 2y - 7z = -5 \end{cases}$ using Cramer's rule.

Step 1.

$$x = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 10 & 4 & 6 \\ -5 & -2 & -7 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -2 \\ 2 & 4 & 6 \\ 1 & -2 & -7 \end{vmatrix}} = \frac{24}{-12} = -2$$

$$y = \frac{\begin{vmatrix} 3 & 1 & -2 \\ 2 & 10 & 6 \\ 1 & -5 & -7 \end{vmatrix}}{-12} = \frac{-60}{-12} = 5$$

$$z = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 2 & 4 & 10 \\ 1 & -2 & -5 \end{vmatrix}}{-12} = \frac{12}{-12} = -1$$

Step 2. The solution is (-2,5,-1).

Exercise Set 6

Solve each system using Cramer's rule.

1. $\begin{cases} 4x - 3y + z = 1 \\ 2x + 3y - 4z = -4 \\ x - y + 2z = 5 \end{cases}$

2. $\begin{cases} 3R_1 - 2R_2 + 4R_3 = -3 \\ 6R_1 + 3R_2 - 2R_3 = 8 \\ 2R_2 - 5R_3 = 4 \end{cases}$

9. Computer and Calculator Applications

There are many calculator and computer programs which aid in the solution of systems of equations. There are two types of programs. One will evaluate determinants for solution of the equations by Cramer's rule. The other type solves the system by giving the values of the variables when you enter the coefficients and right-side constants.

Exercise Set 7

1. Use the program "DET" to evaluate the determinants necessary to solve the following system of equations by Cramer's rule.

$$\begin{cases} x + y + z = 0 \\ 2x - 5y - 3z = 10 \\ 4x + 8y + 2z = 4 \end{cases}$$

2. Use the program "SIMEQU" to find the solutions to the following system of equations.

$$\begin{cases} 3x + 2y - 2z = 1 \\ -x + y + 4z = 13 \\ 2x - 3y + 4z = 8 \end{cases}$$

CHAPTER FOUR

QUADRATIC EQUATIONS

1. Quadratic Functions in One Unknown.

A function f(x) is called a quadratic function of x if it assumes the form $f(x) = ax^2 + bx + c$ where a, b, and c are constants, and $a \neq 0$.

Examples.

1.1 The functions $f(x) = 3x^2 + 6x + 1$, $g(t) = -4t^2$, and $h(s) = s^2 - 3s$ are quadratic.

1.2 The functions $f(x) = 3x - 4$ and $g(t) = 4t^3 - 0.5t$ are not quadratic functions.

The graph of a quadratic function $f(x) = ax^2 + bx + c$ is the set of all points whose coordinates (x,y) satisfy the equation $y = ax^2 + bx + c$. It is customary to graph a few selected solutions of this equation and join these points with a smooth curve to arrive at the graph of the function.

Example.

1.3 Graph the function $y = 2x^2 + x - 15$.

Step 1. Using a table and given values of x, find the corresponding values of y

x	-4	-3	-2	-1	0	1	2	3	4
y	13	0	-9	-14	-15	-12	-5	6	21

Step 2. From the table, list solutions of the given equation: (-4,13) (-3,0) (-2,-9) (-1,-14) (0,-15) (1,-12) (2,-5) (3,6) (4,21).

Step 3. Graph the solutions in Step 2.

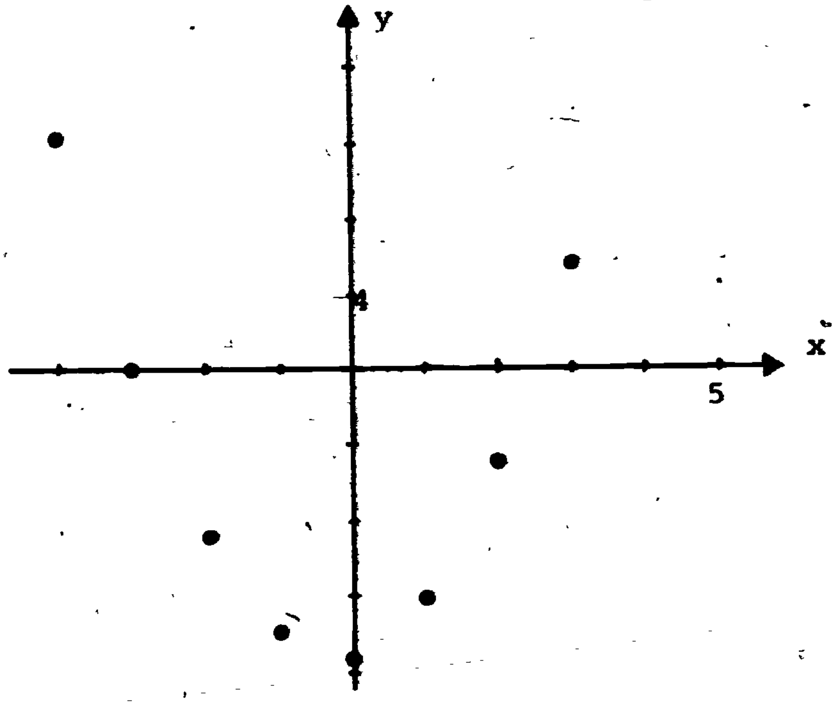


Figure 4.1

Step 4. Join the points with a smooth curve.

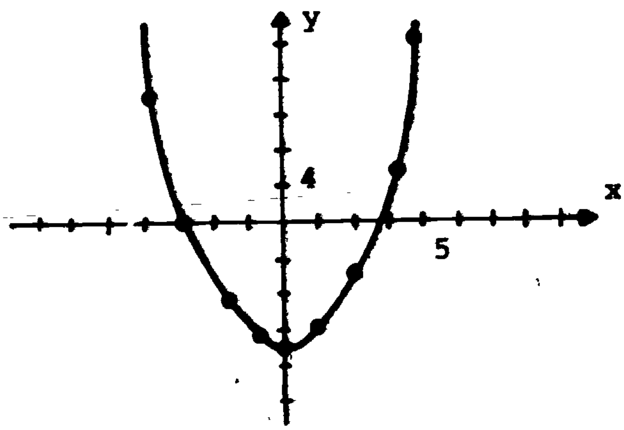


Figure 4.2

Exercise Set 1

1. Using graph paper, sketch the graph of each given function.

a. $y = x^2 + 3x - 4$ b. $f(x) = -2x^2 - x - 15$

c. $y = x^2 - 6x + 8$

2. Roots and Zeros.

If a quadratic function $f(x) = ax^2 + bx + c$ is set equal to some value, say d, then the solution(s) of the resulting equation $d = ax^2 + bx + c$ are called roots of the equation.

Examples.

2.1 If the function $f(x) = x^2 - x - 6$ is set equal to 6, then the solutions of $6 = x^2 - x - 6$, namely $x = -3$ and $x = 4$, are roots of $6 = x^2 - x - 6$.

Graphically, the roots can be found by taking the x-coordinates of the intersection points of the graphs of $y = x^2 - x - 6$ and $y = 6$. (See Figure 4.3).

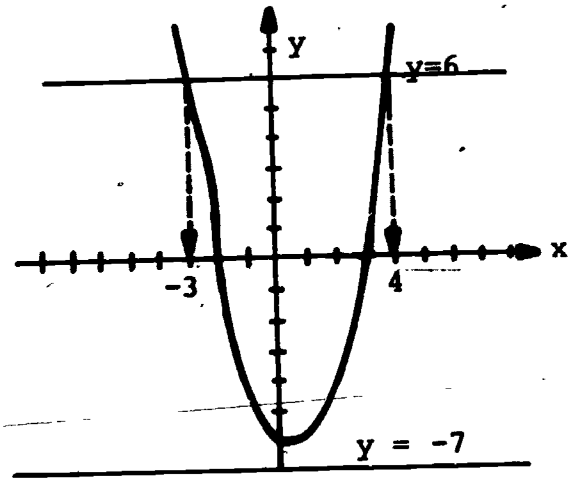


Figure 4.3

2.2 If the function $f(x) = x^2 - x - 6$ is set equal to -7, the equation $-7 = x^2 - x - 6$ has no real roots since the graphs of $y = x^2 - x - 6$ and $y = -7$ do not intersect. (See Figure 4.3).

If a quadratic function $f(x) = ax^2 + bx + c$ is set equal to zero, then the solution(s) of the equation $0= ax^2 + bx + c$ are called zeros of the function.

Examples.

2.3 If the function $f(x) = 2x^2 + x - 10$ is set equal to zero, then the solutions of $0 = 2x^2 + x - 10$, namely -2.5 and 2, are zeros of the function.

Graphically, the zeros are the x-coordinates of the points of intersection of the x-axis and the graph of $y = 2x^2 + x - 10$. (See figure 4.4)

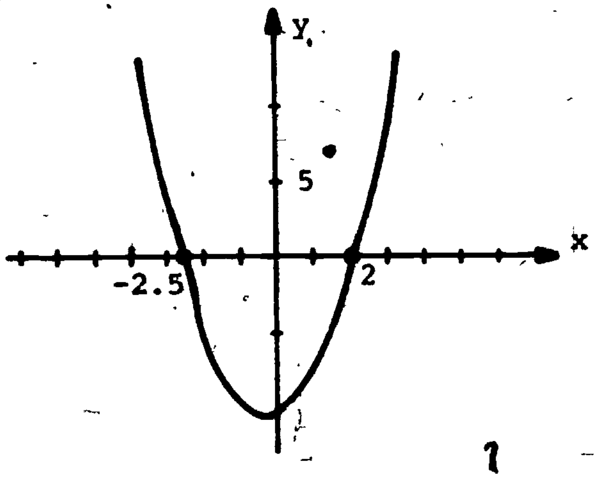


Figure 4.4

2.4 The zero of $f(x) = x^2 - 4x + 4$ is $x = 2$; the solution or root of $0 = x^2 - 4x + 4$ is $x = 2$.

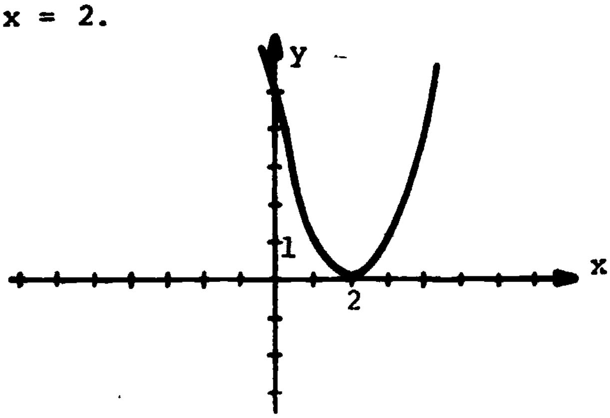


Figure 4.5

Exercise Set 2

1. Determine which of the given values of the variable are roots of the given equation.

   a. $3 = x^2 - 3x - 7;\ x = 0, -2, 3, 5$

   b. $12 = 3t^2;\ t = -6, -2, 2, 7$

   c. $20 = 4x^2 - \sqrt{5}\,x + 5;\ x = -3, 0, 1, \sqrt{5}$

2. Determine which of the given values of the variable are zeros of the given function.

   a. $f(x) = 4x^2 - 20x;\ x = 3, 5, -5, 0$

   b. $g(t) = 5t^3 - 20t;\ t = -3, 1, 2, 0$

   c. $h(s) = 3s^2 + 11s - 4;\ s = -4, 0, \frac{1}{3}, 2$

3. Find the zeros of $f(x) = 2x^2 + 7x - 4$ graphically.

3. Finding Zeros of a Quadratic Function.

The present section deals with finding the roots of the equation $ax^2 + bx + c = 0$ (or zeros of $f(x) = ax^2 + bx + c$; they are the same). ~~Consider~~ this equation to be the standard form of a quadratic equation in x. To solve a quadratic equation is to find its roots.

A quadratic equation can be solved by graphing. Locate the points where the graph of the equation intersects the x-axis. These point~~(s) have~~ coordinates of the form (x,0) where x is a root.

Example.

3.1 Solve $2x^2 + x - 10 = 0$ by graphing.

Step 1. Graph the function $y = 2x^2 + x - 10$.

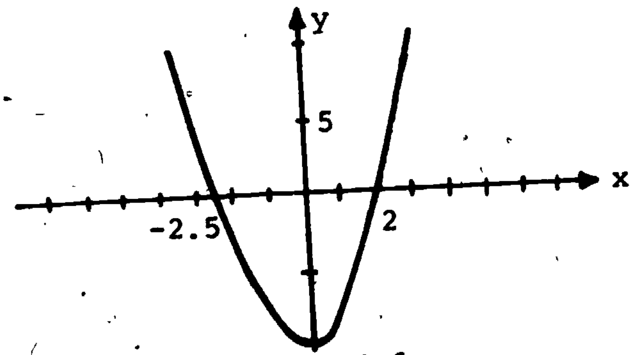


Figure 4.6

Step 2. The roots of $2x^2 - x - 10 = 0$ are the x-intercepts of the graph, namely $x = -2$ and $x = 2.5$.

A factoring method can be used to solve a quadratic equation whenever the expression $ax^2 + bx + c$ is factorable. Write the equation in standard form. Factor the quadratic side and set each factor equal to zero. Solving these new equations individually results in the zeros of the quadratic function and the roots of the original equation.

Examples.

3.2 Solve $2t^2 = t + 10$.

Step 1. Write the equation in standard form, $2t^2 - t - 10 = 0$.

Step 2. Factor the left side and set each factor equal to zero. Solve the resulting equations.

$$(2t - 5)\cdot(t + 2) = 0$$

$$2t - 5 = 0 \text{ or } t + 2 = 0$$

$$t = \frac{5}{2} \text{ or } t = -2$$

Step 3. The roots of $2t^2 = t + 10$ are $\frac{5}{2}$ and -2.

3.3 Solve $4t^2 = 9t$.

Step 1. Standard form of the equation is $4t^2 - 9t = 0$.

Step 2. $t\cdot(4t - 9) = 0$

$$t = 0 \text{ or } 4t - 9 = 0$$

$$t = 0 \text{ or } t = \frac{9}{4}$$

Step 3. The roots of $4t^2 = 9t$ are $t = 0$ and $t = 9/4$.

If a quadratic equation cannot be solved by factoring, a second method known as completing the square method can be used. The procedure is illustrated by example.

Example.

3.4 Solve $2x^2 + 5x = 12$ by completing the square.

Step 1. Write the equation in standard form and divide both sides of the equation by the coefficient of x^2 if it is not one:

$$2x^2 + 5x - 12 = 0$$

$$x^2 + \frac{5}{2}x - 6 = 0$$

Step 2. Move the constant to the right side to get $x^2 + \frac{5}{2}x = 6$.

Step 3. Square $\frac{1}{2}$ of the coefficient of x and add the result to both sides. That is, add $(\frac{1}{2} \cdot \frac{5}{2})^2 = \frac{25}{16}$ to both sides.

$$x^2 + \frac{5}{2}x + \frac{25}{16} = 6 + \frac{25}{16}$$

Step 4. The left side trinomial is the square of the binomial whose first term is x and whose second term is 1/2 of the coefficient of the x-term in the trinomial. The equation can now be written as $(x + \frac{5}{4})^2 = \frac{121}{16}$.

Step 5. Take the square roots of the sides and solve the resulting equations for x.

$$x + \frac{5}{4} = \pm \frac{11}{4}.$$

Thus, $x + \frac{5}{4} = \frac{11}{4}$ or $x + \frac{5}{4} = \frac{-11}{4}$ from which $x = \frac{3}{2}$ or $x = -4$.

Step 6. The roots of $2x^2 + 5x = 12$ are $x = \frac{3}{2}$ and $x = -4$.

If the quadratic equation $ax^2 + bx + c = 0$ is solved by completing the square, the roots are found to be

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

usually written as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This equation is called the quadratic formula.

The roots of a quadratic equation can be found by identifying the values of the constants a, b, and c from the standard form, substituting these values into the quadratic formula, and simplifying.

Examples.

3.5 Solve $2x^2 = 1 - x$ using the quadratic formula.

Step 1. Write the equation in standard form and determine the values of a, b, and c.

$2x^2 + x - 1 = 0$; $a = 2$ $b = 1$ $c = -1$

Step 2. Substitute $a = 2$, $b = 1$, and $c = -1$ into the quadratic formula and simplify.

$$x = \frac{-1 \pm \sqrt{1^2 - 4\cdot(2)\cdot(-1)}}{2\ 2}$$

$$x = \frac{-1 \pm \sqrt{9}}{4} = \frac{-1 \pm 3}{4} = \frac{-1 - 3}{4}$$

or $\frac{-1 + 3}{4}$

$x = -1$ or $\frac{1}{2}$.

Step 3. The roots of $2x^2 = 1 - x$ are $x = -1$ and $x = \frac{1}{2}$.

3.6 Four 2 decimeter squares are cut from the corners of a rectangular piece of sheet metal that is 7 decimeters longer than it is wide. A tray having a volume of 156 cubic decimeters is formed by bending up the sides and soldering the seams. What are the dimensions of the piece of sheet metal?

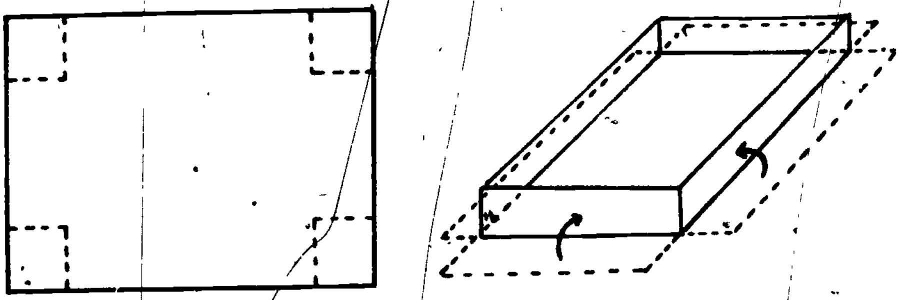


Figure 4.7

Step 1. A labeled diagram of the piece of metal may provide insight into the translation of the problem into an equation form.

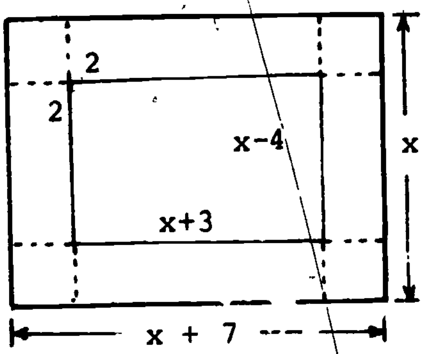


Figure 4.8

Step 2. The volume of the tray is the product of its length, width, and depth. Thus, $156 = (x + 3)\cdot(x - 4)\cdot 2$

Step 3. Simplifying the equation from step 2 gives

$$x^2 - x - 90 = 0.$$

Step 4. Substituting a = 1, b = -1, and c = -90 into the quadratic formula,

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-90)}}{2 \cdot 1}$$

$$= \frac{1 \pm \sqrt{361}}{2} = \frac{1 \pm 19}{2}.$$

The roots are 10 and -9.

Step 5. The width of the piece of metal is 10 decimeters and the length is 17 decimeters.

Exercise Set 3

Solve the quadratic equations in exercises 1 - 4 by the method suggested.

1. Graphing Method:

   a. $x^2 + 4x = 0$ b. $6x^2 + 11x = 7$

2. Factoring Method:

   a. $x^2 - 3x = 0$ b. $t^2 = 6t - 8$

   c. $2x^2 = x + 10$ d. $6x^2 + 7x = 20$

3. Completing the Square Method:

   a. $2y^2 = 4 - 7y$ b. $x^2 = 2\sqrt{10}x - 10$

4. Quadratic Formula:

   a. $3s^2 - s = 4$ b. $5m^2 - 3m - 5 = 0$

5. The formula $d = V_ot + 1/2gt^2$ gives the distance (d) from a starting point after a time (t) of an object which falls with an initial velocity (V_o), under the accleration of gravity (g = 9.8 meters/second/second). If a woman is pushed with a velocity of 2 meters/sec

from the window of her flaming hotel bedroom (25 meters from the ground), how long will it take her to hit ground?

6. Dizzy Dean pitches with a velocity of 20 meters per second. His hand is 2.2 meters above ground when the ball leaves it at an angle of 30° to the horizontal.

   a. Find the times when the ball is 4 meters above the ground.

   b. Find the time when the ball is 1 meter above the ground.

   c. Why is there only one answer to b?

   Hint: Use the quadratic equation $y = (V_o \cdot \tan \theta)t - 1/2gt^2 + y_o$, where t is time in seconds, y is the height above ground in meters, y_o is the distance from the point of projection to a zero point, θ is the angle from the horizontal, and g is the acceleration of gravity, 9.8 m/sec^2.

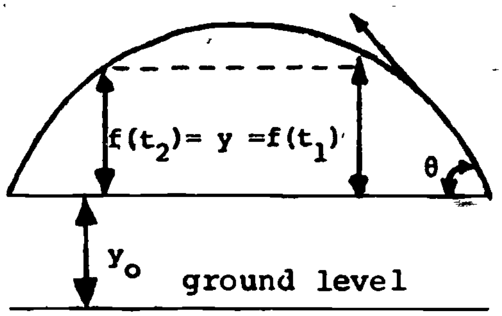


figure 4.9

7. a. Find the radius r of the circle shown in figure 4.10.

   b. Find the length of the arc of the circle from point A to point B on the circumference.

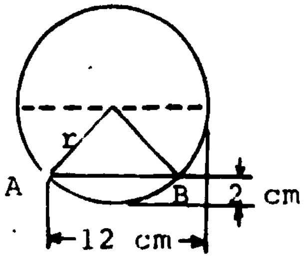


Figure 4.10

CHAPTER FIVE

COMPLEX NUMBERS - IMAGINARY ROOTS OF QUADRATIC EQUATIONS

1. Complex Numbers

No negative number has a real number as its square root. Equivalently, no real number squared is a negative real number. Therefore, a new system of numbers, called complex numbers, is introduced to remove this deficiency.

The symbol j is defined to be the imaginary unit having the property that $j^2 = -1$. It follows intuitively, that $j = \sqrt{-1}$.

Any number which can be expressed in the form a + bj where a and b are real numbers is called a complex number. The 'a' part is called the real part and 'bj' is called the imaginary part.

The real number 'a' is a complex number of the form a + 0j. Thus, any real number is a complex number whose imaginary part is 0j.

A pure imaginary number is a complex number of the form 0 + bj or, more simply, bj.

An imaginary number is a complex number where $b \neq 0$.

Examples.

1.1 The complex number -4 - 8j is an imaginary number whose real part is -4 and whose imaginary part is -8j.

1.2 The real numbers 0 and 13 are complex numbers of the form 0 + 0j and 13 + 0j, respectively.

1.3 The imaginary number 7i is a pure imaginary number (a = 0).

1.4 The diagram below illustrates the relationship among complex numbers.

Complex Numbers $(a + bj)$

Real Numbers $(b = 0)$

$(3, -9, \sqrt{13}, \frac{1}{3}, 8.5,$ etc.)

Imaginary Numbers $(b \neq 0)$

$(3 - 2j, -\frac{1}{4}j, 5 + \sqrt{3}j,$ etc.)

Pure Imaginary $(a = 0)$

$(6j, -\frac{1}{4}j, -\sqrt{5}j,$ etc.)

Figure 5.1

The square roots of negative numbers can be represented in terms of j. If p is a positive number, then the square roots of the negative number -p are given by

$$\sqrt{-p} = \sqrt{p} \cdot j \text{ and } \sqrt{-p} = -\sqrt{p} \cdot j.$$

The expression $\sqrt{p} \cdot j$ is called the <u>principal square root</u> of -p. When finding the square root of a negative number, express the principal square root unless specified otherwise.

Examples.

1.5 The two square roots of -25 are 5j and -5j. The principal square root of -25 is 5j.

1.6 The square roots of -1 are j and -j, with j the principal square root.

1.7 $\sqrt{-17} = \sqrt{17}j$.

<u>Exercise Set 1</u>

1. State whether the given number is a real number or an imaginary number. If it is imaginary, indicate if it is a pure imaginary number.

a. $\sqrt{-9}$	b. $3 - 8j$	c. $\sqrt{49}$	d. $3 - \sqrt{5}$
e. $5 + \sqrt{-4}$	f. 0	g. $-5j$	h. $\sqrt{(-7)^2}$

2. Write the given numbers in the form $a + bj$.

a. $\sqrt{-100}$ b. $3 - \sqrt{-36}$ c. $5 + \sqrt{-5}$

2. <u>Operations Involving Complex Numbers.</u>

Equality of two complex numbers $a + bj$ and $c + dj$ is defined as $a + bj = c + dj$ if and only if $a = c$ and $b = d$. For example, $x + yj = 3 - \sqrt{5}j$ implies that $x = 3$ and $y = -\sqrt{5}$.

<u>Operations on Complex Numbers.</u> If $a + bj$ and $c + dj$ are complex numbers, then

i.) $(a + bj) + (c + dj) =$
$(a + c) + (b + d)j$ .... <u>addition</u>

ii.) $(a + bj) - (c + dj) =$
$(a - c) + (b - d)j$ .... <u>subtraction</u>

*iii.) $(a + bj) \cdot (c + dj) =$
$(ac - bd) + (ad + bc)j$............ <u>multiplication</u>

iv.) $\frac{a + bj}{c + dj} = \frac{a + bj}{c + dj} \cdot \frac{c - dj}{c - dj} =$

$\frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}j$............... <u>division</u>

Examples.

2.1 $(3 + 4j) + (\frac{1}{2} + 12j) =$
$(3 + \frac{1}{2}) + (4 + 12)j = 3\frac{1}{2} + 16j$

2.2 $(-3 + 4j) - (6 - 2j) =$
$(-3 - 6) + (4 - (-2))j = -9 + 6j$

2.3 $(-2 + 3j) \cdot (-5 - j) =$
$10 + 2j - 15j - 3j^2 =$
$10 - 3(-1) + 2j - 15j =$
$13 - 13j$

* Multiply as two binomials with $j^2 = -1$.

2.4 $3j \cdot (2 + 13j) = 6j + 39j^2 =$
$6j + 39(-1) = -39 + 6j$

**2.5 $\sqrt{-8} \cdot \sqrt{-5} = \sqrt{8}\, j \cdot \sqrt{5}\, j =$
$\sqrt{40} j^2 = \sqrt{40}(-1) = -\sqrt{40} = -2\sqrt{10}$

2.6 $\dfrac{4 - 6j}{5j} = \dfrac{4 - 6j}{5j} \cdot \dfrac{-5j}{-5j} =$

$\dfrac{-20j + 30j^2}{-25j^2} = \dfrac{-30 - 20j}{25} =$

$\dfrac{-6}{5} - \dfrac{4}{5}j$

2.7 $\dfrac{3 - 4j}{6 - j} = \dfrac{3 - 4j}{6 - j} \cdot \dfrac{6 + j}{6 + j} =$

$\dfrac{18 + 3j - 24j - 4j^2}{36 - j^2} =$

$\dfrac{(18 + 4) - 21j}{37} = \dfrac{22}{37} - \dfrac{21}{37}j$

2.8 $-\sqrt{-4} - (3 - 4j) + (4 + j)(-6 - 2j) =$

$-2j - 3 + 4j + (-24) - 8j - 6j - 2j^2 =$

$-3 - 24 - 2(-1) - 2j + 4j - 8j - 6j =$

$-25 - 12j$

Exercise Set 2

Perform the indicated operations and simplify.

1. $(3 - 5j) - (-.7 + 3j)$
2. $\sqrt{-4} + \sqrt{-36} - (3 + j)$
3. $-\sqrt{-5} \cdot \sqrt{-20}$
4. $(6 - j) \cdot (4 + 3j)$
5. $\sqrt{-25} \div 5j$
6. $\dfrac{3 - 8j}{4j}$
7. $\dfrac{6 - 4j}{3 + 2j}$
8. $\sqrt{-25} - (2 - j)(-4 + j)$

** It is important that imaginary numbers be expressed in the standard complex form $a + bj$ (whenever $b \neq 0$) before operations are performed.

3. Imaginary Solutions of Quadratic Equations.

The expression $b^2 - 4ac$ in the quadratic formula is called the discriminant. If

$b^2 - 4ac > 0$, then the roots are real, unequal;

$b^2 - 4ac = 0$, the roots are real, equal;

$b^2 - 4ac < 0$, the roots are imaginary.

As stated above, if $b^2 - 4ac < 0$, then the solutions of a quadratic equation are imaginary. When this occurs, the factoring and graphing methods of finding solutions cannot be used; however, completing the square and the quadratic formula methods can be applied.

Example.

3.1 Solve $x^2 = 8x - 20$.

Step 1. Standard form is $x^2 - 8x + 20 = 0$ with $a = 1$, $b = -8$, and $c = 20$.

Step 2. Substituting into the quadratic formula,

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(20)}}{2 \cdot 1}$$

$$= \frac{8 \pm \sqrt{-16}}{2}$$

$$= \frac{8 \pm 4j}{2}$$

$$= 4 \pm 2j.$$

Step 3. The roots of $x^2 = 8x - 20$ are $4 + 2j$ and $4 - 2j$.

Exercise Set 3

Solve the following equations which have imaginary solutions.

1. $x^2 + 1 = 0$
2. $x^2 = 6x - 10$
3. $x^2 - 8x = -25$
4. $36x^2 - 36x + 13 =$

CHAPTER SIX

EQUATIONS CONTAINING FRACTIONS

1. Rational Expressions.

A polynomial is the sum of products of real numbers and nonnegative integral powers of the variable(s). In general, a polynomial in one variable has the form

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_o \text{ where}$$

$a_n, a_{n-1}, \ldots, a_o$ are real numbers and n is an integer, $n \geq 0$.

Examples.

1.1 The algebraic expressions $x^2 - 3x + 4$, $7x - 2x^4$, 34, and x are polynomials in one variable.

1.2 The expressions $\sqrt{3x}$, $\frac{1}{x}$, and $\frac{x + 2}{3x - 5}$ are not polynomials.

A rational expression is the quotient of two polynomials where the denominator is non-zero.

Examples.

1.3 Rational expressions and excluded value(s) of the variable for which the denominator is zero are

a. $\frac{3x + 5}{x - 6}$, $x \neq 6$ b. $\frac{4t}{t^2 - t - 12}$ $(t \neq -3, 4)$

c. $\frac{x^2 - 4x + 7}{x}$, $x \neq 0$ d. $\frac{3x^2 - 2}{5}$

A fraction will be considered in lowest terms when the numerator and denominator have no common factor other than 1 or -1. To remove common factors, factor both numerator and denominator and apply the Fundamental

<u>Principal of Fractions</u>, which states

If a, b, and c are real numbers with $b \neq 0$ and $c \neq 0$, then

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$

The common factor c can be removed (cancelled) to give the reduced form a/b.

Examples.

1.4 $\frac{15x^3y^2}{5xy^2} = \frac{3 \cdot \cancel{5} \cdot \cancel{x} \cdot x^2 \cdot \cancel{y^2}}{\cancel{5} \cdot \cancel{x} \cdot \cancel{y^2}} = 3x^2;\ x \neq 0, y \neq 0.$

1.5 $\frac{m^2 - 5m - 6}{m^2 - 1} = \frac{\cancel{(m + 1)}(m - 6)}{\cancel{(m + 1)}(m - 1)} = \frac{m - 6}{m - 1};\ m \neq -1, 1$

1.6 $\frac{t^2 + 10t + 25}{2t^3 + 7t^2 - 15t} = \frac{\cancel{(t + 5)}(t + 5)}{t\cancel{(t + 5)}(2t - 3)} = \frac{t + 5}{t(2t - 3)};$

$t \neq 0, -5, \frac{3}{2}$

<u>Exercise Set 1</u>

Reduce each fraction to lowest terms and state the excluded value(s) of the variable for which the denominator is zero.

1. $\frac{-12x^4}{3x^2}$

2. $\frac{x^3 - x^2}{x^3 - x}$

3. $\frac{R^2 + 8R + 12}{R^2 - 36}$

4. $\frac{6t^2 - 13t + 6}{3t^2 + 10t - 8}$

2. <u>Operations Involving Rational Expressions.</u>

If M, N, P, and Q represent polynomials, then rational expressions of the form M/N and P/Q can be <u>multiplied</u> according to

$$\frac{M}{N} \cdot \frac{P}{Q} = \frac{M \cdot P}{N \cdot Q} \text{ where N and Q are not zero.}$$

The product should be written in lowest terms.

Division is performed using the rule

$$\frac{M}{N} \div \frac{Q}{P} = \frac{M}{N} \cdot \frac{P}{Q} = \frac{M \cdot P}{N \cdot Q}$$ where N, Q, and P are not zero.

The quotient should be expressed in lowest terms.

Examples.*

2.1 $$\frac{4x^3}{3} \cdot \frac{1}{2x^2 - 2x} = \frac{4x^3}{3 \cdot (2x^2 - 2x)}$$

$$= \frac{2 \cdot \cancel{2} \cdot \cancel{x} \cdot x^2}{3 \cdot \cancel{2} \cdot \cancel{x}(x - 1)} = \frac{2x^2}{3(x - 1)}$$

2.2 $$\frac{x^2 - 25}{3x - 2} \cdot \frac{12x - 8}{x^2 - 2x - 35} = \frac{(x^2 - 25) \cdot (12x - 8)}{(3x - 2) \cdot (x^2 - 2x - 35)}$$

$$= \frac{(x - 5) \cdot \cancel{(x + 5)} \cdot 4 \cdot \cancel{(3x - 2)}}{\cancel{(3x - 2)}\ (x - 7)\ \cancel{(x + 5)}} = \frac{4(x - 5)}{x - 7}$$

2.3 $$\frac{9p^2}{2p - 4} \div \frac{3p^3 + 9p^2 + 6p}{p^2 - 4}$$

$$= \frac{9p^2}{2p - 4} \cdot \frac{p^2 - 4}{3p^3 + 9p^2 + 6p}$$

$$= \frac{9p^2 \cdot (p^2 - 4)}{(2p - 4)\ (3p^3 + 9p^2 + 6p)}$$

$$= \frac{3 \cdot \cancel{3} \cdot p \cdot \cancel{p} \cdot \cancel{(p - 2)} \cdot \cancel{(p + 2)}}{2\cancel{(p - 2)} \cdot \cancel{3} \cdot \cancel{p}\cancel{(p + 2)}\ (p + 1)}$$

$$= \frac{3p}{2(p + 1)}$$

* Throughout the remainder of the chapter, assume that values of variables for which denominators are zero have been excluded.

Exercise Set 2

Perform the indicated operations and express the result in lowest terms.

1. $\frac{3x^2 - 27}{x^2} \div \frac{x^2 + 6x + 9}{x^3}$

2. $\frac{T^2 - T - 2}{4T^2} \cdot \frac{-12T^3}{3T^2 - 2T - 8}$

3. $\frac{(2x + 16)}{3x + 9} \cdot \frac{x + 3}{x + 8}$

4. $\frac{5x^2}{x^2 - 4} \div \frac{15x}{x^2 + 3x - 10} \cdot \frac{x^2 + 4x + 4}{2x^2 + 9x - 5}$

Addition and subtraction of two rational expressions which have a common denominator is given by the rules

$$\frac{M}{N} + \frac{P}{N} = \frac{M + P}{N} \quad \text{and} \quad \frac{M}{N} - \frac{P}{N} = \frac{M - P}{N}, \text{ where } N \neq 0.$$

Seldom are the denominators the same however. When two different denominators are involved, a common denominator must be found, or better yet, the least common denominator (abbreviated L.C.D.) should be found.

To find the L.C.D., factor the denominators of the fractions to be added or subtracted. Choose each factor the greatest number of times it occurs in any one denominator to be part of the L.C.D.

Once the L.C.D. has been determined, the fractions to be added or subtracted will be changed in form, if necessary, using the Fundamental Principle of Fractions so that they will have the same denominator, namely, the L.C.D. The addition or subtraction operation can now be performed following the rules stated above.

Examples.

2.4 If two denominators are $4x^2 - 64$ and $10x^2 - 80x + 160$, the L.C.D. is found by

Step 1. Factoring both denominators completely.

$$4x^2 - 64 = 2^2 \cdot (x - 4)(x + 4)$$

$$10x^2 - 80x + 160 = 2 \cdot 5(x - 4)^2$$

Step 2. Each type of factor present, from step 1, must be a factor in the L.C.D. Thus, the L.C.D. has the factors 2, 5, x - 4, and x + 4.

Step 3. Each factor in L.C.D. must occur the greater number of times it occurred in either denominator. Therefore the L.C.D. $= 2^2 \cdot 5 \cdot (x - 4)^2 \cdot (x + 4)$.

2.5 Subtract: $\dfrac{5x - 2}{8x^2} - \dfrac{3}{36x^3}$

Step 1. The denominators are not the same. They can be factored as $2^3 \cdot x^2$ and $2^2 \cdot 3^2 \cdot x^3$ to give a L.C.D. of $2^3 \cdot 3^2 \cdot x^3$ or $72x^3$.

Step 2. By the Fundamental Principle, the fractions are expressed in an equivalent form having the L.C.D.:

$$\frac{(5x - 2) \cdot 3^2 \cdot x}{8x^2 \cdot 3^2 \cdot x} - \frac{3 \cdot 2}{36x^3 \cdot 2}$$

Step 3. The fractions can now be subtracted to give:

$$\frac{(5x - 2) \cdot 9x - 6}{\text{L.C.D.}}$$

Step 4. Simplifying, the difference is

$$\frac{45x^2 - 18x - 6}{72x^3} = \frac{\cancel{3}(15x^2 - 6x - 2)}{\cancel{3} \cdot 24x^3}$$

$$= \frac{15x^2 - 6x - 2}{24x^3}$$

2.6 $$\frac{3n}{n^2 - 2n - 3} + \frac{2n - 1}{n^2 + 2n + 1} - \frac{1}{n^2 + n}$$

$$= \frac{3n}{(n - 3)(n + 1)} + \frac{2n - 1}{(n + 1)^2} - \frac{1}{n(n + 1)}$$

$$(\text{L.C.D.} = n(n + 1)^2(n - 3))$$

$$= \frac{3n \cdot n(n + 1)}{(n - 3)(n + 1) \cdot n(n + 1)} + \frac{2n - 1 \cdot n(n - 3)}{(n + 1)^2 \cdot n \cdot (n - 3)}$$

$$- \frac{1 \quad \cdot (n - 3)(n + 1)}{n(n + 1) \cdot (n - 3)(n + 1)}$$

$$= \frac{3n \cdot n \cdot (n + 1) + (2n - 1) \cdot n \cdot (n - 3) - (n - 3)(n + 1)}{\text{L.C.D.}}$$

$$= \frac{5n^3 - 5n^2 + 5n + 3}{n(n + 1)^2(n - 3)}$$

Exercise Set 3

Perform the indicated addition or subtraction and express the result in lowest terms.

1. $\frac{x^2}{x^2 - 4} - \frac{4}{x^2 - 4}$

2. $\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

3. $\frac{t}{t + 4} - \frac{32}{t^2 - 16}$

4. $1 - \frac{2z_1}{z_1 + z_2}$

5. $\frac{7}{16x^2} - \frac{1}{24x} + \frac{5}{36x^3}$

6. $\frac{3x - 4}{x^2 + 6x + 5} - \frac{x}{2x^2 - 2}$

3. Solving Equations Containing Fractions.

In solving an equation which contains fractions, multiply both sides by the L.C.D. of all the denominators in the equation: This will result in an equivalent equation containing no fractions.

Examples.

3.1 Solve $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R.

Step 1. The L.C.D. of the fractions in the equation is $R \cdot R_1 \cdot R_2$.

Step 2. Multiply both sides of the equation by the L.C.D. and cancel where possible.

$$\cancel{R} \cdot R_1 \cdot R_2 \cdot \frac{1}{\cancel{R}} = R \cdot \cancel{R_1} \cdot R_2 \cdot \frac{1}{\cancel{R_1}} + R \cdot R_1 \cdot \cancel{R_2} \cdot \frac{1}{\cancel{R_2}}$$

Step 3. Simplify and proceed to solve for the indicated unknown.

$$R_1 R_2 = R R_2 + R R_1$$

$$R_1 R_2 = R(R_2 + R_1)$$

$$\frac{R_1 \cdot R_2}{R_2 + R_1} = R$$

(It is understood that R, R_1, R_2 are non-zero; $R \neq R_1$ and $R \neq R_2$.)

3.2 Solve $\frac{p}{d} = \frac{m}{d} - \frac{m^2}{2\pi r^2}$ for d.

Step 1. The L.C.D. of the fractions involved is $2\pi r^2 d$.

Step 2. Multiply both sides by $2\pi r^2 d$ and cancel each denominator.

$$2\pi r^2 \cancel{d} \cdot \frac{p}{\cancel{d}} = 2\pi r^2 \cancel{d} \cdot \frac{m}{\cancel{d}} - \cancel{2\pi r^2} d \cdot \frac{m^2}{\cancel{2\pi r^2}}$$

Step 3. Simplify and solve for d.

$$2\pi r^2 p = 2\pi r^2 m - m^2 d$$

$$m^2 d = 2\pi r^2 m - 2\pi r^2 p$$

$$d = \frac{2\pi r^2 (m - p)}{m^2}$$

3.3 Solve $\dfrac{3}{y^2 - 16} - \dfrac{2}{y - 4} = \dfrac{4}{y + 4}$

Step 1. The L.C.D. is $(y - 4)(y + 4)$.

Step 2. Multiply the sides by $(y - 4)(y + 4)$ and reduce each fraction.

$$\cancel{(y - 4)(y + 4)} \cdot \frac{3}{\cancel{y^2 - 16}}$$

$$- \cancel{(y - 4)}(y + 4) \cdot \frac{2}{\cancel{y - 4}}$$

$$= (y - 4)\cancel{(y + 4)} \cdot \frac{4}{\cancel{y + 4}}$$

Step 3. Simplify and solve for y.

$$3 - (y + 4) \cdot 2 = (y - 4) \cdot 4$$

$$3 - 2y - 8 = 4y - 16$$

$$-6y = -11$$

$$y = \frac{11}{6}$$

3.4 The perimeter of the rectangle shown in figure 6.1 is 18 centimeters. What is the length and width of the rectangle?

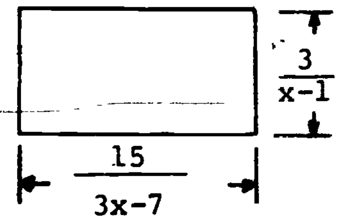


Figure 6.1

Step 1. By definition of perimeter,

$$18 = 2 \cdot \frac{15}{3x - 7} + 2 \cdot \frac{3}{x - 1}.$$

Dividing both sides by 6 simplifies the equation to

$$3 = \frac{5}{3x - 7} + \frac{1}{x - 1}.$$

Step 2. Multiplying both sides by the L.C.D. $(3x - 7) \cdot (x - 1)$ gives

$$3 \cdot (3x-7)(x-1) = \cancel{(3x-7)}(x-1) \cdot \frac{5}{\cancel{3x-7}} + (3x-7)\cancel{(x-1)} \frac{1}{\cancel{x-1}}$$

Step 3. Simplifying yields $9x^2 - 38x + 33 = 0$.

Step 4. Letting $a = 9$, $b = -38$, $c = 33$, and substituting into the quadratic formula,

$$x = \frac{38 \pm \sqrt{1444 - 1188}}{18} = \frac{38 \pm \sqrt{256}}{18}$$

$$= \frac{38 \pm 16}{18}.$$

$$x = 3 \text{ and } x = \frac{11}{9}$$

Step 5. Replacing x by 11/9 in the length $15/(3x - 7)$ results in a meaningless value of -4.5. Letting $x = 3$, the length is 7/2 centimeters and the width is 3/2 centimeters.

Exercise Set 4

1. Solve the following equations containing fractions.

   a. $\frac{w}{4} + \frac{1}{3} = \frac{w}{2}$ b. $\frac{t + 4}{5} = \frac{t - 2}{2}$

   c. $\frac{1}{R} = \frac{3}{R} - \frac{5}{R - 1}$ d. $\frac{3}{4F^2 - 9} - \frac{4}{2F^2 - F - 3} = 0$

2. The total resistance R in a set of parallel resistors in a circuit with resistance r_i $(i = 1, 2, \ldots, n)$ is given by the equation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \ldots \frac{1}{r_n}$$

   a. If there are three resistors in the circuit (i.e. $i = 1, 2, 3$), solve for the total resistance R in terms of r_1, r_2, and r_3.

   b. What does R equal in terms of r_1 and r_2 if there are three resistors r_1, r_2, and r_3 with $r_2 = r_3$?

   c. Two resistors $r_1 = 900$ ohms $r_2 = 1000$ ohms are arranged in parallel. What is the value of the total resistance R?

d. What is the total resistance R where $r_1 = 900$, $r_2 = 1000$, and $r_3 = 1000$ ohms?

3. Find the length and width of the rectangle in Figure 6.2 whose perimeter is 2 meters.

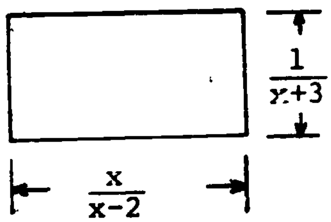


Figure 6.2

4. According to the Doppler effect the observed frequency produced by a moving source is different from its actual frequency. i.e. the frequency that would be observed if the object was stationary with respect to its observer. (This is why the pitch of a car horn seems to change as a car passes by.) For sound waves

$$f_o = \frac{v + w + v_o}{v + w + v_s} \cdot f_s$$ where f_s is the actual

frequency of the source, f_o is the observed frequency, v is the velocity of sound in the medium, v_o is the velocity of the observer, v_s is the velocity of the source, and w is the velocity of the wind. Sound travels in air with a velocity of 331 m/sec. A car is traveling at 20 m/sec in the same direction as a 20 m/sec wind. As the car approaches, a stationary opera singer (with perfect pitch) thinks that the car's horn is sounding a middle C (256 cycle/sec). What is the actual frequency of the car's horn?

CHAPTER SEVEN

EXPONENTIAL AND LOGARITHMIC EQUATIONS

1. Exponential Form - Laws of Exponents.

Real numbers can be expressed as the product of a single factor taken several times. For example, the number 16 equals the product of 4 twos or $16 = 2 \cdot 2 \cdot 2 \cdot 2$. To abbreviate this product and still be able to indicate the factor and the number of times it occurs, the numeral 2^4, is used. The factor 2 is called the base, the 4 is called the exponent or power, and the complete form 2^4 is called an exponential form of 16. Exponential forms of 64 are 8^2, 4^3, and 2^6 while $8/27 = (2/3)^3$.

In general, for any real number x, the product $x \cdot x \cdot x \cdot \ldots \cdot x \cdot x = x^n$ for n factors of x. The number x is called the base, n is called the exponent, and x^n is the nth power of x.

Examples.

1.1 $5^2 = 5 \cdot 5 = 25$

1.2 $(0.2)^4 = (0.2)(0.2)(0.2)(0.2) = 0.0016$

1.3 $81 = 9 \cdot 9 = 9^2$

1.4 $\frac{64}{125} = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} = (\frac{4}{5})^3$

Exercise Set 1

1. Write the number whose exponential form is given.

a. 2^2	d. $(0.3)^3$	g. $(\frac{1}{3})^4$
b. 5^3	e. 2^{10}	h. $(4\frac{1}{2})^2$
c. 10^4	f. 1^9	i. $(50)^3$

2. Write an exponential form for each of the following.

a. 100	c. 36	e. 0.008
b. 16	d. $\frac{4}{49}$	f. 10,000

In order that an exponential form of a number can be used in certain calculations, the following <u>laws of exponents</u> are introduced. Assume that a and b are real numbers; m and n are positive integers.

i)	$a^m \cdot a^n = a^{m+n}$		(product of powers)
ii)	$\dfrac{a^m}{a^n} = a^{m-r}$, $a \neq 0$, $m > n$		(quotient of powers)
iii)	$\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$; $a \neq 0$, $n > m$		(quotient of powers)
iv)	$(a^m)^n = a^{m \cdot n}$		(power of a power)
v)	$(a \cdot b)^n = a^n \cdot b^n$		(power of a product)
vi)	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$; $b \neq 0$		(power of a quotient)

Examples. The laws of exponents being illustrated are given in parentheses in each example.

1.5	$x^5 \cdot x^3 = x^8$	(i)
1.6	$(4y)^3 = 4^3 \cdot y^3 = 64y^3$	(v)
1.7	$(t^4)^2 = t^{4 \cdot 2} = t^8$	(iv)
1.8	$y^7 \div y^5 = y^{7-5} = y^2$	(ii)
1.9	$\left(\dfrac{3}{a}\right)^4 = \dfrac{3^4}{a^4} = \dfrac{81}{a^4}$	(vi)
1.10	$t^4 \div t^{10} = \dfrac{1}{t^{10-4}} = \dfrac{1}{t^6}$	(iii)
1.11	$(3x^4)^3 = 3^3 \cdot (x^4)^3$	(v)
	$= 27x^{12}$	(iv)

Exercise Set 2

Use the laws of exponents to perform the indicated operations and simplify.

1. $(2x)^5$
2. $x^4 \cdot x^5 \cdot x$
3. $t^4 \div t^{11}$
4. $(\frac{5}{x^2})^2$
5. $(w^5)^3 \cdot w^2$
6. $(-2)^3 \cdot (-2)^2$
7. $(3x^4)^2 \cdot (5x^3)^2$
8. $y^9 \div y$
9. $(\frac{8}{m})^2$
10. $(\frac{2x^2}{t^3})^4$

2. Zero, Negative, and Fractional Exponents.

The exponential form x^0 equals one for all nonzero real numbers x. That is $x^0 = 1$, $x \neq 0$.

Suppose n is a positive integer. Then -n is a negative integer and the exponential form x^{-n} is defined by

$$x^{-n} = \frac{1}{x^n},\ x \neq 0.$$

The laws of exponents also apply to zero and negative exponents.

Examples.

2.1 $x^{-5} \cdot x^3 = x^{-5+3} = x^{-2} = \frac{1}{x^2}$

2.2 $(3^{-1}t^4)^{-2} = (3^{-1})^{-2} \cdot (t^4)^{-2}$
$3^{(-1)(-2)} \cdot t^{4\cdot(-2)} = 3^2 \cdot t^{-8} = \frac{9}{t^8}$

2.3 $\left(\frac{2^{-3}}{t^{-2}}\right)^2 = \frac{2^{-3\cdot 2}}{t^{-2\cdot 2}} = \frac{t^4}{2^6} = \frac{t^4}{64}$

Exercise Set 3

Write the following in simple form without zero or negative exponents.

1. 5^{-1}
5. 8^0
9. $(\frac{1}{x^{-2}}) \cdot 4^{-2}$

2. 10^{-2}
3. $\frac{1}{2^{-3}}$
4. 4^{-3}
6. x^{-4}
7. $\frac{1}{15^{-1}}$
8. $(5^{-2})^{-2}$
10. $(16)^{0} \cdot 1^{-5}$
11. $x^{-6} \cdot x^{3} \cdot x^{-5}$
12. $(2x^{-2})^{-3}$

The laws of exponents, valid for rational powers, show that $25^{\frac{1}{2}} = \sqrt{25}$. That is, $25^{\frac{1}{2}} \cdot 25^{\frac{1}{2}} = 25^{\frac{1}{2}+\frac{1}{2}} = 25$. Since $\sqrt{25} \cdot \sqrt{25} = 25$, $25^{\frac{1}{2}} = \sqrt{25}$. In general,

$$x^{\frac{1}{n}} = \sqrt[n]{x} \text{ where } x \geq 0 \text{ and n is even.}$$

If m and n are integers and $\sqrt[n]{x}$ is a real number, then a rational power m/n has meaning given by

$$x^{\frac{m}{n}} = x^{\frac{1}{n} \cdot m} = (\sqrt[n]{x})^{m} \text{ and } x^{\frac{m}{n}} = (x^{m})^{\frac{1}{n}} = \sqrt[n]{x^{m}} .$$

Note: Irrational exponents will not be discussed because there does not seem to be a need.

Examples.

2.4 $16^{3/4} = (\sqrt[4]{16})^{3} = 2^{3} = 8$ or $16^{3/4}$

$= \sqrt[4]{16^{3}} = \sqrt[4]{4096} = 8$

2.5 $25^{3/2} = (\sqrt{25})^{3} = 5^{3} = 125$ or $\sqrt{25^{3}}$

$= \sqrt{15,625} = 125$

2.6 $(k^{2} + 5)^{3/4} = \sqrt[4]{(k^{2} + 5)^{3}}$

2.7 $8^{-5/3} = \frac{1}{8^{5/3}} = \frac{1}{(\sqrt[3]{8})^{5}} = \frac{1}{2^{5}} = \frac{1}{32}$

2.8 $(27a^{-3}b^{6})^{-2/3} = 27^{-2/3} \cdot a^{(-3)\cdot(-2/3)} \cdot b^{(6)(-2/3)}$

$= \frac{1}{27^{2/3}} \cdot a^{2} \cdot b^{-4} = \frac{a^{2}}{9b^{4}}$

2.9 $4s^{-3/2} + 6s^{-1/2} = \frac{4}{s^{3/2}} + \frac{6}{s^{1/2}} = \frac{4}{s^{3/2}} + \frac{6 \cdot s}{s^{1/2} \cdot s}$

$= \frac{4}{s^{3/2}} + \frac{6s}{s^{3/2}}$

$= \frac{4+6s}{s^{3/2}}$

Exercise Set 4

Find the value of each expression.

1. $400^{1/2}$ 2. $4^{3/2}$ 3. $32^{1/5}$ 4. $16^{-1/2}$

5. $100^{-5/2}$ 6. $M^{-1/2} \cdot N^{3/4}$ where M = 64, N = 16

7. $\dfrac{y^{-2/3}}{x^{-1/2}}$ where x = 81, y = 27

8. $(x^2 - 17)^{-1/2}$ where x = 9

Use the laws of exponents to simplify and express the results without negative exponents.

9. $(t^{-3})^{5/3}$ 10. $(8x^6y^3)^{-2/3}$

11. $\left(\dfrac{25t^{-2}}{36}\right)^{3/2}$ 12. $\left(\dfrac{x^{-2} \cdot y^2}{y^{-3}}\right)^{-2}$

3. The Exponential Function $y = b^x$.

Quantities which change (usually with time) are sometimes determined by an exponential function represented by $f(x) = b^x$ or $y = b^x$ where b is a positive real number different from one and x is a rational number. Some such quantities are radium which decays in time, bacteria which grow in time, and bodies whose temperature varies according to the temperature of the surrounding medium.

A solution of the equation $y = b^x$ is an ordered pair (x,y) which satisfies the equation. For example, the equation $y = 10^x$ has solutions (0,1), $(1/2, \sqrt{10})$, and (3,1000). The set of all solutions of $y = 10^x$ is the solution set of the equation.

The graph of an exponential function has two general forms depending upon the value of the base b. These forms are illustrated in the graphs of Figure 7.1. The point - graphs of solutions for rational x are joined by a smooth curve.

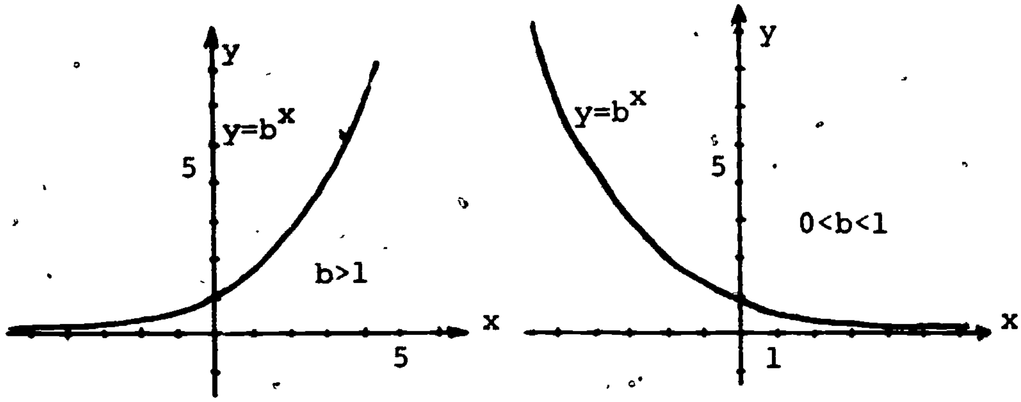


Figure 7.1

Example.

3.1 Plot the graph of $y = 3^x$.

Step 1. Construct a table of values for given values of x.

x	-3	-2	-1	0	1	2
y						

x	-3	-2	-1	0	1	2
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Step 2. Plot the solutions of $y = 4^x$ taken from the table of step 1. Join the points with a smooth curve.

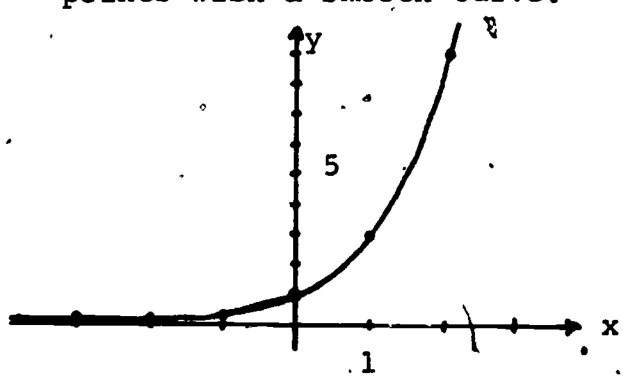


Figure 7.2

Exercise Set 5

Sketch the graph of each function for values of x between -3 and 3.

1. $y = 2^x$ 2. $y = (\frac{1}{2})^x$ 3. $y = 2.5^x$

4. Logarithms - Properties of Logarithms.

In the exponential equation $y = b^x$, the exponent x is called the logarithm of y to the base b, written $x = \log_b y$ $(y > 0)$.

Examples.

4.1 Since $10^3 = 1000$, 3 is the logarithm of 1000 to the base 10.

4.2 In the equation $16^{1/4} = 2$, 1/4 is the logarithm of 2 to the base 16.

4.3 The value of $\log_5 125$ is 3 because $5^3 = 125$.

4.4 The logarithmic equation which means the same as $4^3 = 64$ is $\log_4 64 = 3$.

4.5 $\log_5 (1/5) = -1$ since $5^{-1} = 1/5$.

4.6 $\log_8 8^{9.3} = 9.3$.

If the base b in the equation $y = b^x$ is 10, then $y = 10^x$ and the logarithm of y to the base 10 is called a common logarithm, written log y. The numeral 10 designating the base is omitted, i.e. $\log_{10} y = \log y$.

Another base which has widespread applications is the base e, an irrational number with an approximate value of 2.71828. The logarithm of y to the base e, called a natural logarithm, is written ln y, i.e. $\log_e y = \ln y$.

Examples.

4.7 $\log 100 = 2$ since $10^2 = 100$.

4.8 $\log 0.0001 = -4$ because $10^{-4} = 0.0001$.

4.9 $\ln e = 1$ $(e^1 = e)$ and $\ln 1 = 0$ $(e^0 = 1)$.

4.10 The values of the following logarithms are found using a calculator.

a. $\log 8.312 = 0.9197$

b. $\ln 8.312 = 2.1177$

c. $\log 0.523 = -0.2815$

d. $\ln 8351 = 9.030$

Exercise Set 6

1. Find the value of the logarithm of the given number.

a. $\log 1000$ b. $\log_9 9$ c. $\log_2 32$

d. $\log_7 \frac{1}{49}$ e. $\log_{25} 5$ f. $\ln e^3$

g. $\log_8 1$ h. $\log 0.1$ i. $\log\frac{1}{100}$

j. $\ln 83$ k. $\log 0.051$ l. $\ln 4.35$

2. Express each given equation in the equivalent logarithmic equation form.

a. $16^{1/2} = 4$ b. $10^{-1} = \frac{1}{10}$ c. $6^0 = 1$

3. Express each given logarithmic equation in exponential equation form.

a. $\log_3 9 = 2$ b. $\ln 8349 = 9.030$

c. $\log 10{,}000 = 4$ d. $\log 8.312 = 0.9197$

Since logarithms are exponents, the properties of logarithms listed below follow from the laws of exponents.

i. $\log_b (M \cdot N) = \log_b M + \log_b N$ (logarithm of a product)

ii. $\log_b \frac{M}{N} = \log_b M - \log_b N$ (logarithm of a quotient)

iii. $\log_b M^N = N \cdot \log_b M$ (logarithm of a power)

Examples.

4.11 $\log_8 xy = \log_8 x + \log_8 y$ (i)

4.12 $\log_2 \frac{1}{6} = \log_2 1 - \log_2 6$

$= 0 - \log_2 6 = -\log_2 6$ (ii)

4.13 $\log_5 \frac{1}{10} = \log_5 10^{-1} = -1 \cdot \log_5 10$ (iii)

4.14 $\log_2 \sqrt[3]{48} = \log_2 48^{1/3}$

$= 1/3 \log_2 48$ (iii)

$= 1/3 \log_2 (16 \cdot 3)$

$= 1/3(\log_2 16 + \log_2 3)$ (i)

$= 1/3 \cdot 4 + 1/3 \cdot \log_2 3$

$= 4/3 + 1/3 \cdot \log_2 3.$

Exercise Set 7

Express each given logarithm as a sum product, or multiple of logarithms and simplify when possible.

1. $\log_3 27x^3$
2. $\log_4 \frac{4}{x^5}$
3. $\log_2 \sqrt{64y^3}$
4. $\log_{10} 5000$
5. $\log_{10} 0.0003$
6. $\log_5 \frac{3}{125}$

5. Application of Logarithms - Solving Exponential Equations.

To solve an equation ~~containing an unknown~~ exponent, common logarithms or natural logarithms are employed. In this way, the calculator can be used to evaluate the logarithms of numbers.

Examples.

5.1 Solve the equation $6^{x-4} = 13^{2x}$.

Step 1. Take the common logarithm of both sides.

$$\log 6^{x-4} = \log 13^{2x}$$

Step 2. By the property of logarithms (iii),

$$(x-4)\log 6 = 2x\cdot\log 13$$

Step 3. Solving the equation from step 2 for x,

$$x\cdot\log 6 - 2x\cdot\log 13 = 4\cdot\log 6$$

and

$$x = \frac{4\cdot\log 6}{\log 6 - 2\log 13}$$

Step 4. Evaluating the logarithmic expressions,

$$x = \frac{4\cdot(.7782)}{.7782 - 2(1.114)} \text{ or } x = -2.147.$$

5.2 Solve the equation $55 = e^{0.08t}$.

Step 1. Take the natural logarithm of both sides.

$$\ln 55 = \ln e^{0.08t}$$

Step 2. Since $\ln(e^{0.08t}) = 0.08t$,

$$\ln 55 = 0.08t \text{ and } t = \frac{\ln 55}{0.08}$$

Step 3. Evaluating ln 55 to be 4.00 (nearest hundredth),

$$t = \frac{4.01}{0.07} = 57.29.$$

Exercise Set 8

Solve the given exponential equation.

1. $3^x = 7$

2. $4^{2x+5} = 75$

3. $10^{4x-2} = 6^{3x}$

4. $75 = 31 \cdot e^{-0.50t}$

6. The Logarithmic Function.

If the relationship between x and y in the exponential function $y = b^x$ is reversed, that is, the value of the exponent logarithm function is formed. In general, whenever $y = b^x$, the logarithm function is $x = \log_b y$.

Since a functional relationship between two quantities is independent of the variables used to represent them, the x and y are interchanged so that the general form of the logarithmic function is $y = \log_b x$.

A solution of the equation $y = \log_b x$ is an ordered pair (x,y) which satisfies the equation. The equation $y = \log_2 x$ has solutions (1/4, -2), (1,0), (2,1), and (64,6).

The graph of a logarithmic function, like that of the exponential function, is of two forms depending upon the base b. These forms appear in Figure 7 3.

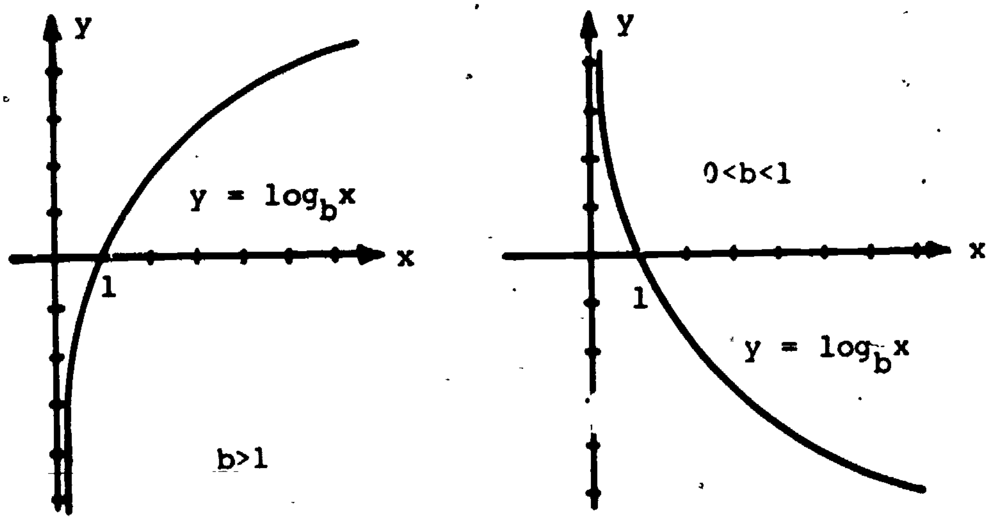


Figure 7.3.

Example.

6.1 Plot the graph $y = \log_3 x$.

Step 1. Construct a table of values for given values of x.

x	0	1/9	1/3	1	3	9
y						

x	0	1/9	1/3	1	3	9
y	*?	-2	-1	0	1	2

* Not defined.

Step 2. Graph the solutions derived from the table in Step 1. Join the points with a smooth curve.

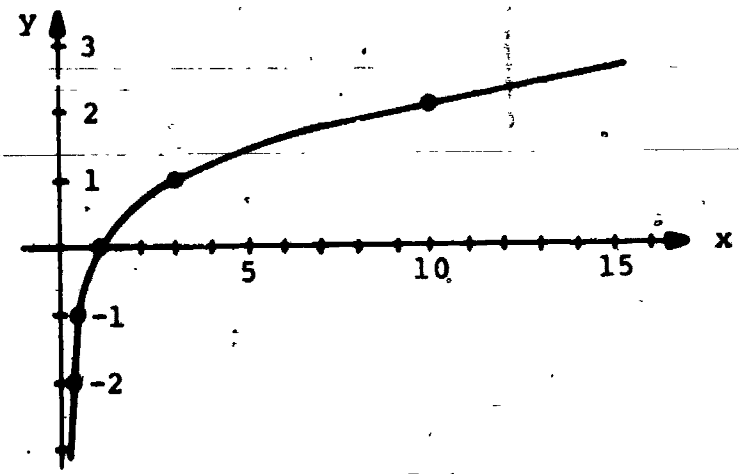


Figure 7.4

Exercise Set 9

Sketch the graph of each function.

1. $y = \log_{0.5} x$ 2. $y = \log_2 x$ 3. $y = \log_{10} x$

ANSWERS TO EXERCISES

CHAPTER ONE

Set 1 (Page 2)

1. a. yes, $y = 3x$
   b. yes, $y = 2x + 2$
   c. yes, $y = 2x + 5$
   d. no

2. b. c. e. f.

3. no

4. yes

5. yes

Set 2 (Page 4)

1. a. $(5,-3)$
   b. $(\frac{3}{4},-7)$
   c. $(0.6,-6.6)$
   d. $(\frac{2}{3},16)$

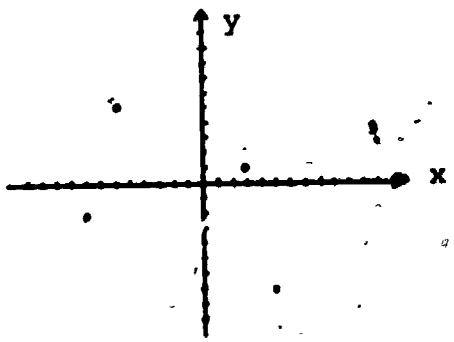
2. $(0,-1)$, $(5,14)$, $(-3,-10)$, $(\frac{1}{3},0)$, $(4.6,12.8)$

3. answers vary

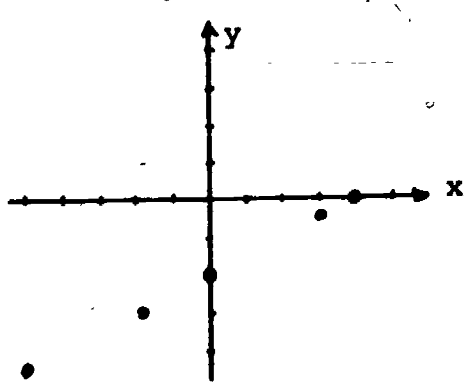
4. 10

Set 3 (Page 6)

1.



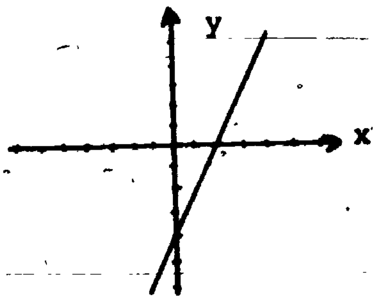
2.



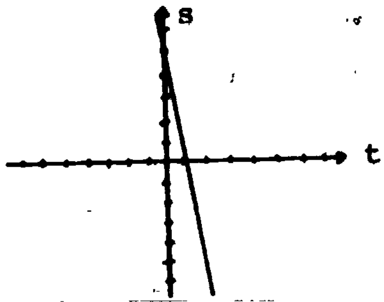
3. Q;S

Set 4 (Page 9)

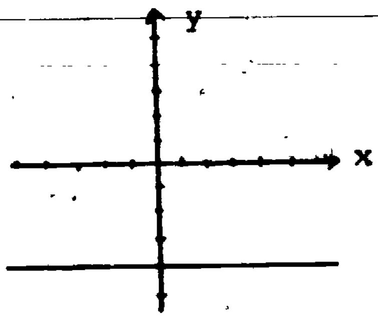
1. a.



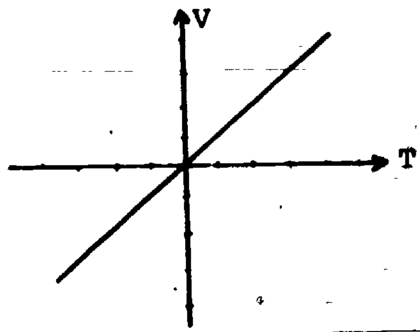
b.



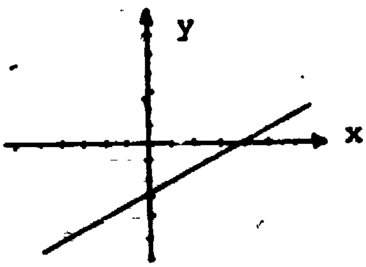
c.



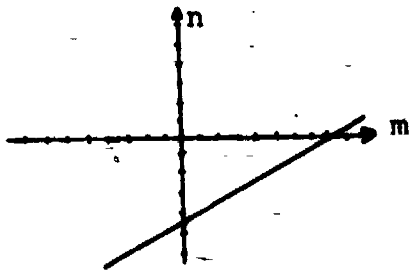
d.



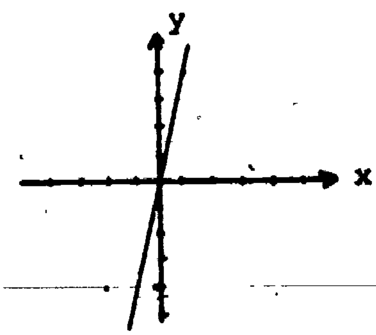
2. a.



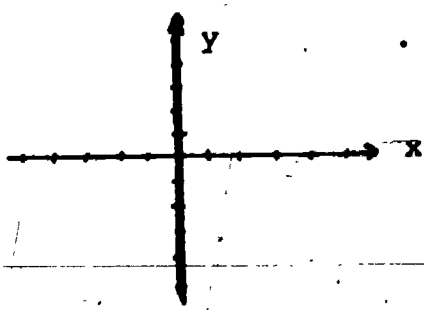
b.



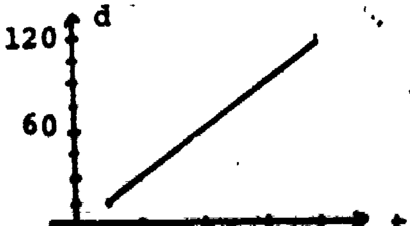
c.



d.



3.



Set 5 (Page 13)

1. a. 5 b. -5 c. -3
   d. 3 e. -2 f. -4
   g. $\sqrt{41}$ h. $\sqrt{74}$ i. $\sqrt{34}$

Set 6 (Page 16)

1. a. $\frac{1}{3}$ b. $\frac{4}{3}$ c. -3
   d. $\frac{-120}{17}$

2. yes 4. no 5. yes

6. $\frac{5}{9}$

Set 7 (Page 19)

1. a. $2x - 15y = -59$ b. $y = \frac{4}{3}x$
   c. $x = -4$ d. $y = 5$

2. a. $3x - y = 5$ b. $y = -5$
   c. $4x - 7y = 13$ d. $57x + 10y = 91.5$

3. a. $6x + y = -5$ b. $x + 3y = 21$
   c. $x - y = 0$ d. $y = 0$

4. a. $m = 5$, $b = -10$ b. $m = -1$, $b = 0$
   c. $m = 0$, $b = -8.6$ d. $m = r$, $b = -k$
   e. $m = \frac{-5}{9}$, $b = \frac{-160}{9}$ f. $m = -1$, $b = 0$

5. $3x - y = 0$ 6. $2x + 5y = 25$

7. $gx + ay = 0$

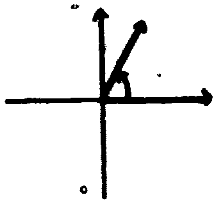
Set 8 (Page 21)

1. $L = \frac{1\text{ in}}{2\text{ oz}} \cdot W + 6\text{ in}$

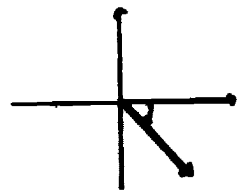
CHAPTER TWO

Set 1 (Page 25)

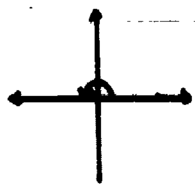
1. a.



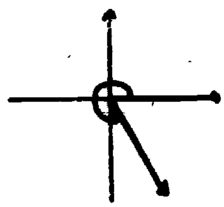
b.



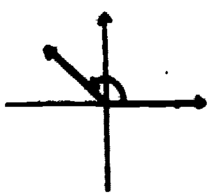
c.



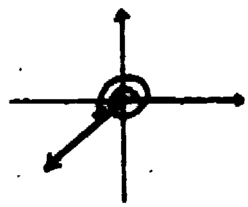
d.



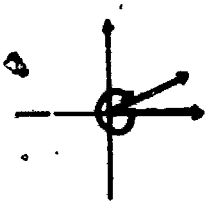
e.



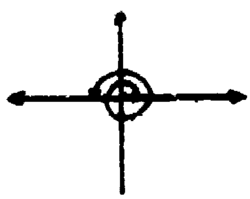
f.



g.



h.



2. a. 17.364° b. 5°30'0" c. 47°21'36"

3. a. 91°34'9" b. 16°28'55" c. 73°20'

d. 3°37'25" e. 41°43'40"

Set 2 (Page 26)

1. a. $\frac{\pi}{6}$ b. 6732° c. $\frac{1}{36}$ rev

   d. $\frac{1}{12}$ rev e. 0.785 f. 94.248

   g. 288° h. 179.909°

Set 3 (Page 29)

1. a. 0.707 b. 0.707 c. 0.268

   d. 2.000 e. -0.577 f. -1.540

   g. 0 h. 32.966 i. -1.414

2. a. 30°, 150° b. 69.757°, 290.243° c. 273.735°, 93.735°

3. $\sin\theta = \frac{\sqrt{7}}{4}$ $\cos\theta = \frac{-3}{4}$ $\tan\theta = -\frac{\sqrt{7}}{3}$ $\csc\theta = \frac{4\sqrt{7}}{7}$

   $\sec\theta = -\frac{4}{3}$ $\cot\theta = \frac{-3\sqrt{7}}{7}$

Set 4 (Page 30)

1. a = 3 m
   B = 53.130°
   A = 36.870°

2. b = 15 cm
   A = 53.130°
   B = 36.870°

3. B = 60°
   b ≅ 1.732 km
   c = 2 km

4. B = $\frac{\pi}{4}$
   a = 1.000 cm
   c ≅ 1.414 cm

5. A = 63°
   b = 199 mm
   c = 438 mm

6. 18.802°

Set 5 (Page 33)

1. $\frac{\pi}{2}$ 2. 17 3. 225π

4. 240 5. 6 2/3 6. 9.549

7. 250π, 125π, 62.5π 8. a. 41¢

9. 6.429, 0.673 b. $33.63

Set 6 (Page 37)

1. a. $\vec{D}$ b. $\vec{I}$ c. $\vec{B}$ d. $\vec{H}$ e. $\vec{I}$

Set 7 (Page 40)

1. a. <0.61, 6.97> b. <12.78, 4.65> c. <0.57, 0.10>

2. a. <13, 4> b. <22, 8> c. <25, 12>

3. ($\sqrt{41}$, 218.66°) 4. 180°, 99.419 cm/sec

5. $\vec{A}$ = <-45.963, 38.567>, $\vec{B}$ = <28.925, 34.472>, 73.039

Set 8 (Page 41)

1. a. <3.00, 5.20> b. <7.07, 7.07> c. <-350.00, -606.22>

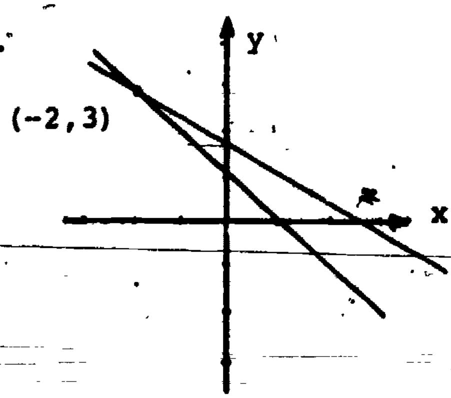
d. <54.64, -65.11> e. <67.00, 0.00>

2. a. (12.37, 108.52°) b. (12.05, 201.67°) c. (144.84, 59.30°)

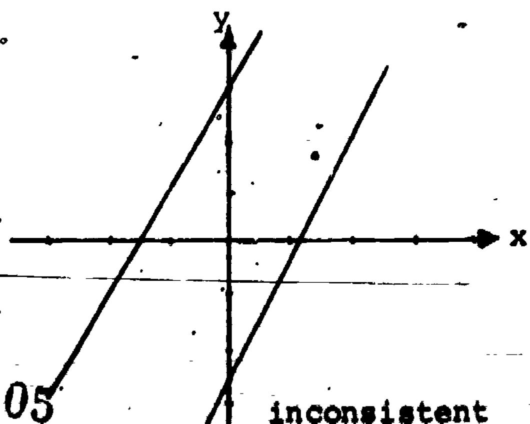
CHAPTER THREE

Set 1 (Page 44)

1.



2.



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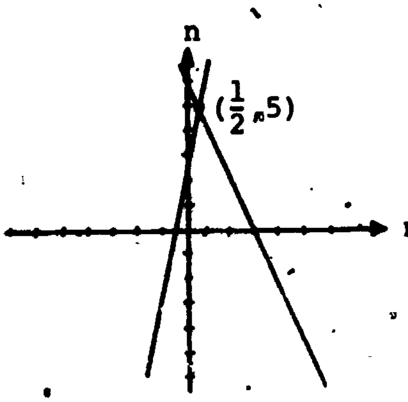

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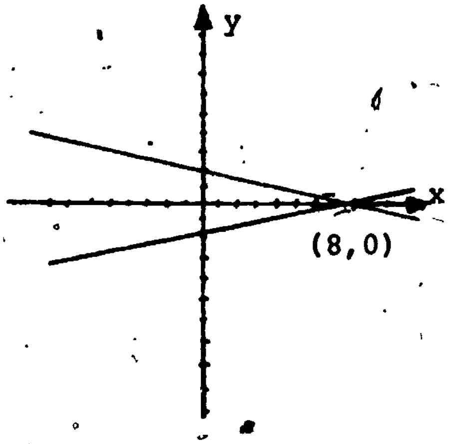


Set 2 (Page 46)

1. (1,2)

2. $(-5,\frac{-10}{3})$

3. (-1,5)

4. Dependent

5. $(\frac{260}{27}, \frac{-70}{27})$

Set 3 (Page 47)

1. (0,-2)

2. (11,1)

3. (-7,-5)

4. (14,16)

Set 4 (Page 49)

1. a. 3   b. 0   c. -4   d. 56

2. a. (25,15)   b. $(\frac{9}{5},\frac{7}{15})$ c. $(\frac{5}{2},3)$ d. Inconsistent

3. $\frac{2}{45}$ km/min

4. $\frac{7}{15}$ km/min (river); $\frac{13}{15}$ km/min (speedboat); $-\frac{1}{15}$ km/min (barge)

Set 5 (Page 52)

1. (1,1,-2)

2. (-3,5,10)

Set 6 (Page 54)

1. (1,2,3)

2. $(\frac{1}{3},2,0)$

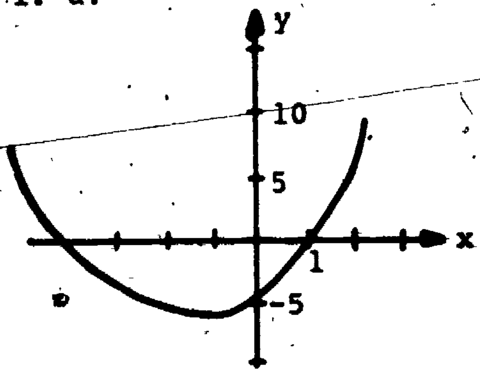
Set 7 (Page 55)

1. (2,0,-2) 2. (1,2,3)

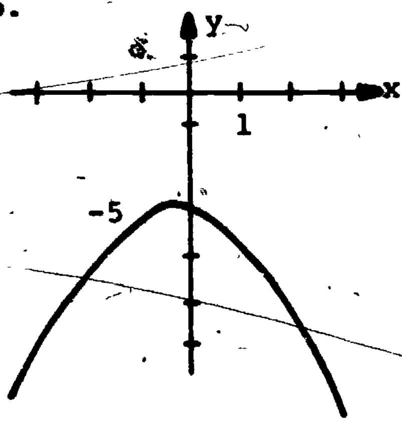
CHAPTER FOUR

Set 1 (Page 57)

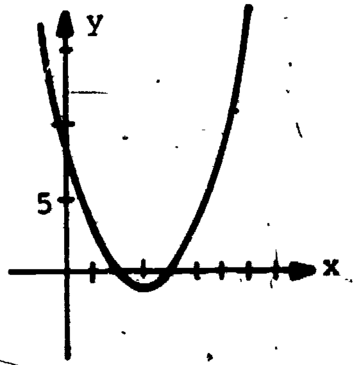
1. a.



b.



c.



Set 2 (Page 60)

1. a. -2, 5 b. -2,2 c. $\sqrt{5}$

2. a. 5,0 b. 2,0 c. -4 , $\frac{1}{3}$

Set 3 (Page 65)

1. a. 0,-4 b. $-\frac{7}{3}, \frac{1}{2}$

   b. 2, 4

2. a. 0,3 c. -2, $\frac{5}{2}$ d. $\frac{4}{3}, -\frac{5}{2}$

3. a. -4, $\frac{1}{2}$ b. $\sqrt{10}$

4. a. -1, $\frac{4}{3}$

   b. $\frac{(3 \pm \sqrt{109})}{10}$

5. 2.064 sec

6. a. 0.168 sec, 2.189 sec b. 2.456 sec

7. a. 7.068 cm   b. 10.91 cm

CHAPTER FIVE

Set 1 (Page 68)

1. a. imaginary (pure)   b. imaginary
   c. real   d. real
   e. imaginary   f. real
   g. imaginary (pure)   h. real

2. a. $0 + 10j$ b. $3 - 6j$ c. $5 + \sqrt{5} \cdot j$

Set 2 (Page 70)

1. $3.7 - 8j$
2. $-3 + 7j$
3. 10
4. $27 + 14j$
5. 1
6. $-2 - \frac{3}{4}j$
7. $\frac{10}{13} - \frac{24}{13}j$
8. $7 - 1j$

Set 3 (Page 71)

1. $\pm j$ 2. $3 \pm j$ 3. $4 \pm 3j$ 4. $\frac{1}{2} \pm \frac{1}{3}j$

CHAPTER SIX

Set 1 (Page 73)

1. $-4x^2;\ x \neq 0$ 2. $\frac{x}{x + 1};\ x \neq 0, -1, 1$

3. $\frac{R + 2}{R - 6}$; $R \neq 6, -6$ 4. $\frac{2t - 3}{t + 4}$; $t \neq \frac{2}{3}, -4$

Set 2 (Page 75)

1. $\frac{3x(x - 3)}{x + 3}$ 2. $\frac{-3T(T + 1)}{3T + 4}$

3. $\frac{2}{3}$ 4. $\frac{x(x + 2)}{3(2x - 1)}$

Set 3 (Page 77)

1. 1   2. $\frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}$ 3. $\frac{t - 8}{t - 4}$

4. $\frac{Z_2 - Z_1}{Z_1 + Z_2}$ 5. $\frac{63x - 6x^2 + 20}{144x^3}$ 6. $\frac{5x^2 - 19x + 8}{2(x + 1)(x - 1)(x + 5)}$

Set 4 (Page 80)

1. a. $\frac{4}{3}$ b. 6   c. $-\frac{2}{3}$ d. $-\frac{9}{5}$

2. a. $\frac{r_1r_2r_3}{(r_2r_3 + r_1r_3 + r_1r_2)}$ b. $\frac{r_1r_2}{(r_2 - 2r_1)}$

   c. $\frac{9000}{19}$ ohms   d. $\frac{4500}{14}$ ohms

3. $\frac{3}{5}, \frac{2}{5}$ 4. 271 cycle/sec

CHAPTER SEVEN

Set 1 (Page 82)

1. a. 4   b. 125   c. 10,000   d. 0.027

   e. 1024   f. 1   g. $\frac{1}{81}$ h. $\frac{81}{4}$

   i. 125,000

2. a. 10^2 b. 4^2, 2^4 c. 6^2 d. $(\frac{2}{7})^2$

e. $(0.2)^3$ f. 10^4, $(100)^2$

Set 2 (Page 84)

1. $32x^5$
2. x^{10}
3. $\frac{1}{t^7}$
4. $\frac{25}{x^4}$
5. w^{17}
6. -32
7. $225x^{14}$
8. y^8
9. $\frac{64}{m^2}$
10. $\frac{16x^8}{t^{12}}$

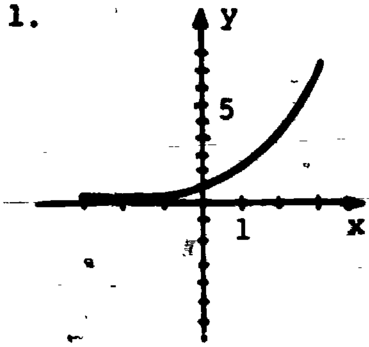
Set 3 (Page 84)

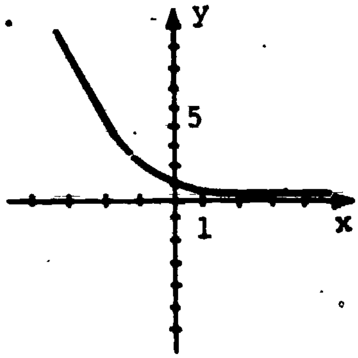
1. $\frac{1}{5}$
2. $\frac{1}{100}$
3. 8
4. $\frac{1}{64}$
5. 1
6. $\frac{1}{x^4}$
7. 15
8. 625
9. $\frac{x^2}{16}$
10. 1
11. $\frac{1}{x^8}$
12. $\frac{x^6}{8}$

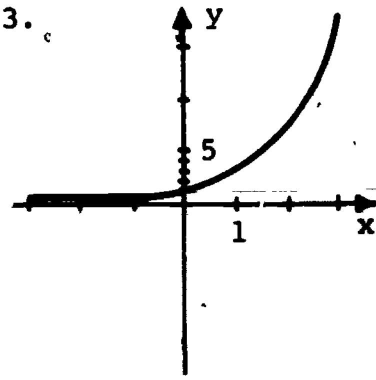
Set 4 (Page 86)

1. 20
2. 8
3. 2
4. $\frac{1}{4}$
5. $\frac{1}{100,000}$
6. 1
7. 1
8. $\frac{1}{8}$
9. $\frac{1}{t^5}$
10. $\frac{x^4}{4y^2}$
11. $\frac{125}{216t^3}$
12. $\frac{x^4}{y^{10}}$

Set 5 (Page 88)

1. 

2. 

3. 

Set 6 (Page 89)

1. a. 3 b. 1 c. 5 d. −2

e. $\frac{1}{2}$ f. 3 g. 0 h. −1

i. −2 j. 4.419 k. −1.292 l. 1.470

2. a. $\log_{16} 4 = \frac{1}{2}$ b. $\log_{10} \frac{1}{10} = -1$ c. $\log_6 1 = 0$

3. a. $3^2 = 9$ b. $e^{9.030} = 8349$

c. $10^4 = 10{,}000$ d. $10^{0.9197} = 8.312$

Set 7 (Page 90)

1. $3 + 3 \log_3 x$ 2. $1 - 5 \log_4 x$ 3. $3 + \frac{3}{2} \log_2 y$

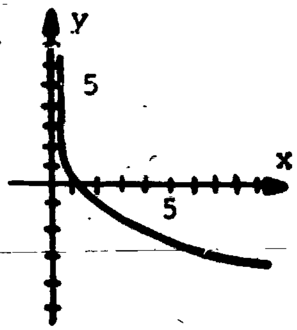
4. $3 + \log 5$ 5. $-4 + \log_{10} 3$ 6. $\log_5 3 - 3$

Set 8 (Page 92)

1. 1.771 2. −0.943 3. 1.201 4. −1.767

Set 9 (Page 100)

1.



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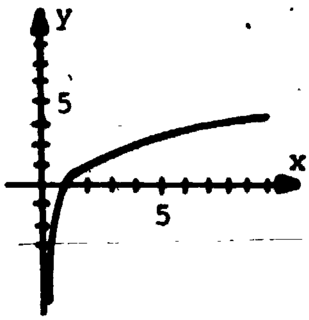

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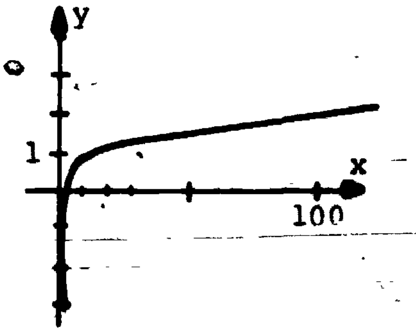

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