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ABSTRACT

Provided is an examination of the methodology used to study the problems of learning addition and subtraction skills used by developmental researchers. The report has sections on categories of theory and their methodologies, which review: (1) Behaviorist, Neo-Behaviorist and Piagetian Theories: (2) the Behaviorist and Piagetian Paradigms: (3) Soviet Studies, Constructivism, and Teaching Experiment: and (4) Artificial Intelligence, Information Processing, and Computer Simulation. A summary section draws together the discussions of the report by suggesting the need to make a global distinction between two categories of theory. A new model of intelligence is offered as an alternative to the psychometric group of models that dominate the field. The document concludes with some notes on the terminology used and references. (MP)

Project Paper 81-3

THEORIES AND METHODOLOGIES

by

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Report from the Mathematics Work Group

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- diversity as a fundamental question in American social thought, through studies of social policy related to education

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Preface

This paper is a revision of an essay prepared for a Seminar on Initial Learning of Addition and Subtraction held at the Wingspread Conference Center in Racine, Wisconsin, November 1979. A brief version containing some of the elements of this paper is Chapter 13, "Type 1 Theories and Type 2 Theories in Relationship to Mathematical Learning" in the book, Addition and Subtraction: A Developmental Perspective, T. P. Carpenter, J. M. Moser, and T. A. Romberg (Eds.), Lawrence Erlbaum Associates, Hillsdale, New Jersey, 1981.

Thomas A. Romberg
Editor

My brief for the present paper was:

> An examination of the methodology used to study the problems of learning addition and subtraction skills used by developmental researchers.

In this introduction, I shall outline how I have set about this task.

To supplement the literature which I already had available on the subject, a computer search using ERIC was initiated. The key words used were: Addition, Subtraction, Research. From the computer printout, 28 papers were selected as relevant to the present study. (Others, for example, dealt with the addition and subtraction of integers or fractions.) Reprints were obtained of 18 of these papers: in the case of the other 10, the abstracts in the printout were clear and full enough to show what methodology had been used. Three issues of The Journal of Children's Mathematical Behaviour also provided such valuable material that I give them special mention. These are Vol. 1, No. 2 (Autumn 1973), Vol. 1, No. 3 (Summer 1975), and Vol. No. 4 (Autumn 1974).

From the great amount of data thus obtained, I have tried to extract certain general ideas which may be used as a basis for further thinking.

Methodology

Methodology refers to the set of techniques by which a researcher constructs (builds and tests) a theory. This includes both constructing a new theory ab initio, and improving an existing theory by extending its domain or increasing its accuracy and completeness. Methodology and theory are thus closely related, and the construction of a successful theory will depend largely on the use of appropriate methodology. It might therefore be expected that researchers would refer explicitly to this relationship. This is not usually the case, though there are notable exceptions. For example, Steffe (1977) writes:

> Constructivism, an epistemological theory, has not yet produced a theory of mathematics learning. However, several principles central to constructivism have been used to provide powerful analogies for building models in the teaching and learning of the whole number system. The central purpose of this paper is to outline a continuation of the construction of such models using a methodology called "the teaching experiment".

Ginsburg (1977) also is explicit in his statement of his theoretical position and methodology.

> In the spirit of Piaget, I try to show how the child's mind operates and develops as he or she encounters mathematical problems in and out of school. . . . The primary method is the in-depth interview with children as they are in the process of grappling with various sorts of problems. (pp. iii-iv)

Where a researcher has not explicitly indicated the grounds for his choice of methodology, there are several possible reasons.

1. It may be that all those whom he expects to read his report use the same theory with its associated methodology, which he takes for granted and does not seek to challenge. This is usually the case with researchers

in the natural sciences, such as electricity and magnetism, chemistry, atomic physics. Research of this kind falls into the category which Kuhn (1970) calls "normal science" It is certainly not the case with mathematics educational research, nor with the psychological research often used by educational researchers as their starting point. In both of these fields it is easy to identify a number of alternative theories, none of which is so universally accepted that it may be taken for granted that both writer and reader are using it.

2. It may be that a theory, or at least a general theoretical position, is clearly implied by the content of the report. For example, the title of a paper by Allardice (1977), "The development of written representations for some mathematical concepts," makes it clear that the author takes the position of the cognitive psychologists, in which concepts and symbols are importantly different, rather than that of the behaviorists, to whom a concept is a common response to a class of stimuli and may be equated with its symbol.

3. The researcher may be at the stage of making systematic observations, not yet organized into a theory. Even so, a theoretical position, that is to say a category or kind of theory, is implicit in the kind of observations which were made and the conditions under which they were made. For example, written tests administered to groups of children imply one kind of theoretical stance, while naturalistic observation and individual in-depth interviews imply a different kind.

4. Often, however, it is difficult to avoid the conclusion that the researcher has used a particular method without having considered it in relation either to a methodology or to an associated theory. By a method

I mean what a researcher does, his plan of action; by a methodology I mean the more general body of knowledge or beliefs from which he derives a particular method and by which he can justify it. A person who uses a method unrelated to a methodology is thus in somewhat the same position as a pupil who uses algorithms in mathematics without having the underlying mathematical conceptual structures from which the algorithm is derived, and by which it can be understood as a correct procedure.

Both the second and third are acceptable positions; the fourth in my view is not. In the natural sciences, position 1 also is acceptable, but not in the field with which we are at present concerned (though it might be so within certain groups, such as the members of a particular research group).

Categories of Theory and Their Methodologies

1. Behaviorist and Neo-Behaviorist Theories: Methodology Based on the Natural Sciences

The powerful, even dominant, influence which this school exercised over many years has not diminished. Nevertheless, as was shown by the computer search, the bulk of what is currently in print still falls into this category, and though relatively little of what is innovative in current research is behaviorist, this still provides an important example of the relationship between theory and methodology. Moreover, there are important lessons to be learned by analyzing the errors which, with hindsight, we can see to be inherent in the behaviorist approach. If we do not learn from these, we are in danger of falling into the same errors in new disguises.

The growth of this school is closely associated with the efforts of academic psychologists to establish psychology as an accepted science. It is understandable that these efforts took as their model the natural sciences, which even in the early days of psychology were proving their power in enabling us to shape our physical environment, and since then have shown an exponential rate of growth.

Characteristic methods in all of the physical sciences are:

1. the replicable experiment, by which others can verify the results of an individual researcher as a precaution against experimental error and as a prerequisite for the general acceptance of these results;

2. measurement in standard units, without which experimental conditions and results cannot be described accurately enough for the above;

3. the isolation and manipulation of independent variables, so that their separate effects on the dependent variables can be measured;

4. quantitative as well as qualitative statement of results.

To use these methods in experimental psychology (and subsequently in the application of this kind of psychology to educational research), adaptations were necessary. To take a simple example, an experiment in the electrolysis of a saline solution is replicable because two samples of NaCl, and two samples of pure water, are identical, and the electrical force and current can be measured by test instruments internationally standardizable with a high degree of accuracy. But no two persons are identical, so it becomes necessary to work with groups of subjects on the assumption that individual differences which affect the result of the experiment are random, and that their overall effect on the dependent variable, when averaged, is close to zero. Thus, while it is not expected that experiments will be replicable with single subjects, it is so expected with comparable groups of subjects. This introduces the need for simple statistical treatment of the results. The separate manipulation of independent variables is also sometimes hard to achieve with groups of human subjects; so instead, their effect is teased out from the set of measures which represent the outcome of the experiment by more sophisticated statistical techniques such as analysis of variance or factor analysis. Another procedure designed to ensure replicability is operational definition of the variables in terms of publicly observable behavior of the experimenters and of their subjects.

Because of the need for brevity, and on the assumption that most readers will already be familiar with them, examples of experiments conforming to behaviorist paradigms are not given here.

To reject behaviorist models because they are mechanistic is understandable but, in my view, not a good reason. Carpenter (1979) points out that "the relevant question is pragmatic. Which model is more fruitful for adequately explaining and predicting behavior?" (p. 6). And though behaviorist models have been remarkably successful in bringing about the learning of bar-pressing by rats, and kicking a ping-pong ball by pigeons, it is a hard fact that they have been remarkable unsuccessful in explaining, predicting, or controlling the higher forms of learning, in which man most differs from the laboratory rat and pigeon, and of which mathematics is a particularly clear example.

In addition to the pragmatic objection to behaviorist models, which is that they haven't worked, there are other criticisms to be made, the grounds for which may be called category errors.

In constructing psychological and educational models similar to those which have proved so successful in the natural sciences, an implicit assumption has been made which on examination appears questionable. This is, that the kinds of objects whose qualities we seek to discover, abstract, and embody in our models are the same in both cases: or in other words, that different though the objects themselves may be, these differences are not such that a different kind of model is required. To give an analogy, although English, Russian, and Greek are written in different scripts, these scripts all consist of a basic set of symbols from which are constructed words, the words then being put together to make sentences. So a person whose first language was English would not have to make any major change in his thinking in order to learn to write either of the others. Japanese writing, however, is not put together in the same way. Whereas in English, Russian, and Greek the separate letters represent sounds (albeit rather loosely), in Japanese the characters represent meanings. This would be explained at the outset to a new student of Japanese. If nobody explained this difference, and the student never managed to figure it out, continuing to think of Japanese writing as being in the same category as the other three would make learning nearly hopeless.

The first of the category errors which I beleive to be inherent in any behaviorist model is that whereas our physical environment is indifferent to our activities in shaping it, our fellow humans are not. Any attempt by A to shape the behavior of B implies some degree of loss of freedom for B, whether this be realized or not. This raises the possibility (to put it at its least) that consciously or unconsciously, B will seek to remain as autonomous as possible by resisting the efforts of A. Whether or not B resists, and how much, will be likely to vary between individuals, and will depend partly on how each construes the situation, again not necessarily consciously. Where this factor exists, or where there is a strong prima facie possibility of its existence, I suggest that to ignore this possibility is a category error.

A second category error is made when symbols are equated with concepts, when a sound or a mark on paper is equated with its meaning. Since a symbol is publicly observable, while a concept is not, the former certainly comes closer to what is acceptable as scientific evidence. Nevertheless, any mathematician would assert that the differences between

$$x^2 - y^2 = (x - y)\ (x + y) \text{ and}$$
$$a^2 - b^2 = (a - b)\ (a + b)$$

are unimportant compared with the fact that (to a mathematician) their meanings are identical. The mathematician could, moreover, generate an almost indefinite variety of symbols representing this same meaning. So for researchers into mathematical education, the distinction between symbols and concepts is one which is essential to preserve.

The third and most important category error which I believe to be characteristic of behaviorist models is that they fail to distinguish between what I shall call type 1 theories and type 2 theories. This distinction is the subject of the whole of the last section of this paper, and it will therefore not be elaborated here.

2. Piagetian Theory: Methodology of the Diagnostic Interview *

Strongly contrasted to behaviorism both in methodology and theory is the work of Piaget, his associates, and his adherents. A clear and concise

* For an explanation of my choice of this terminology, please see the Notes on Terminology at the end of this paper.

account of Piaget's methodology, and its origins, is to be found in Opper (1977), from which the following extracts are taken. (A clear and concise account of Piaget's theory is another matter.)

> In the mid-1920's, at the start of his career, Piaget worked in Simon's psychological laboratory in Paris where one of his duties was to standardize a French version of a series of Burt's reasoning tests (Piaget, 1966). While engaged in this work, Piaget became particularly interested in the incorrect responses given by the younger children and decided to carry out cognitive studies in order to discover the underlying reasons for incorrect answers in younger children and correct ones in older children. (p. 90)
>
> Since no adequate research method existed for the type of studies he wished to conduct, Piaget created his own. Familiarity with the clinical interviews used in the medical field led him to design a similar method for the study of reasoning in children.... (p. 91)
>
> The essential character of the method is that is constitutes a hypothesis-testing situation, permitting the interviewer to infer rapidly a child's competence in a particular aspect of reasoning by means of observation of his performance at certain tasks.... For the most part the experiment involves both a concrete situation with objects placed in front of the child and a verbally presented problem related to this situation... At the start of each session, the interviewer has a guiding hypothesis about the types of thinking that the child will engage in.... For each item the interviewer then asks a series of related

questions which are aimed at leading the child to predict, observe, and explain the results of the manipulations performed on the concrete objects. It is these predictions, observations, and explanations that provide useful information on the child's view of reality and his thought processes.... (pp. 92-93)

The interviewer then tests his original hypothesis on the basis of the child's verbal responses and actions. If further clarifications are required, he asks additional questions or introduces extra items. Each successive response of the child thus guides the interviewer in his formation of new hypotheses and consequently in his choice of the subsequent direction of the experiment. (p. 93)

The foregoing methodology may be contrasted, point by point, with the characteristic methods of behaviorist methodology l[illegible]ed in the preceding section. Instead of a replicable experiment, we now have individual interviews, no two of which are exactly alike. Instead of experimental designs carefully planned in advance, and executed so far as possible according to those plans in every detail, we have experiments in which only the initial situation and hypothesis are prepared in advance, new hypotheses and procedures being successively introduced according to the results of the experiment thus far. Instead of the outcome being measured in standardized units, it is presented descriptively. Often extracts from the child's verbal responses are given verbatim, together with the experimenter's inferences from them. Moreover, in the behaviorist methodology the experimental results are usually given as an array of figures, such as a correlation matrix, table of means and standard deviations, or analysis of variance, together with significance levels, from which conclusions are derived that the experimental hypothesis stated at the outset is confirmed or refuted. In contrast, the outcome of a Piagetian experiment is presented in the form of some general statement giving a synthesis or overview of the final state of the experimenter's thinking, resulting from the successive modifications of the original hypothesis during the course of the experiment. And finally, the Piagetian approach is much more time-consuming, relative to the number of subjects from whom data is collected, than the behaviorist. The amount of experimenter's time required is a major practical difficulty in Piagetian-style research.

What are the implicit assumptions underlying these sharply contrasted paradigms? Concentrating on those most directly relevant to the present volume, and over-simplifying for the sake of emphasis, I suggest that these assumptions may be summarized as follows.

Behaviorist paradigm. The behaviorist is interested in subjects' publicly observable behavior, and this is mainly dependent on conditions external to the subjects. These conditions can be controlled with a fair degree of precision by an experimenter or teacher. Factors internal to the subjects, and especially those particular to individuals, are random in their occurrence and can therefore be eliminated by appropriate statistical techniques.

Piagetian paradigm. What the Piagetian is interested in is the mental processes which give rise to the subject's observable behavior, and these are mainly the result of processes internal to the subject. These vary between different individuals, and between the same individual at different ages, and the differences are as important as the likenesses. To investigate these we need to work with individuals in a one-to-one relationship with the experimenter, making hypotheses about underlying mental processes which are tested against a variety of observable behaviors.

> Here is an example of the first paradigm (Uprichard & Collura, 1977). The independent variable was method of instruction. The dependent variable was the child's ability to perform on a test constructed by the investigators to measure computation skills (addition and subtraction), place value and number concepts.... Past research ...seems to support the premise that "meaningful and developmental" instruction yields high achievement in mathematics. In this investigation "meaningful or developmental" instruction was operationally defined in terms of mathematical structure.... Within a class four subjects were randomly assigned to an experimental group and four to a control group.... Analysis of covariance was used to analyse the data with the pretest acting as a covariate for the post test. (pp. 2-5)

Implicit in this paradigm is the assumption that short-term learning, and long-term development regarded as the sum of all the short-term learning which has taken place, can be shaped as chosen by an experimenter or teacher provided only that we can find out how. "Meaningful" is defined operationally in terms of what the instructor does, not cognitively in terms of how this is understood by the pupils. So the possibility that what is meaningful to some may not be meaningful to others, or that different meanings may be attached to the same instruction, is not envisaged or investigated.

Ginsburg, in contrast, is particularly interested in the informal knowledge of mathematics which young children have before they come to school, and its effects on formal and systematic instruction. To investigate this in the context of addition, three types of problem were devised (Brush &

Ginsburg, 1971; quoted in Ginsburg, 1975). These were intended to discover the degree of maturity of children's conception of addition, and of their available strategies, including the amount of information which they could take into account and relate. A mature strategy would be one which took into account and made appropriate use of all the relevant information, while an immature strategy would be one which centered on one (or possibly more) aspects of the problem, these being insufficient for a correct solution.

In the case of addition, an immature strategy which Ginsburg calls "absolute addition" might be described somewhat like this (my paraphrase of Ginsburg's ideas): "Adding makes things more. So a set which has been added to is always more than a set which hasn't been added to." A mature strategy, in contrast, would take into account the relative numbers initially and also the number added to one of them. A majority of the children were able to solve both the second and third of the problems, indicating that they already had available a relatively mature strategy. This above experiment is characteristically Piagetian in that its aim was to identify the mental processes underlying children's observable responses by means of suitably devised problems given individually. The full Piagetian methodology was however not employed (or if it was, not reported) in that there was no "question leading to answer leading to further question" sequence, by which the experimenter's hypotheses about the children's mental strategies could have been further supported. This sequence is well shown in the context of addition and subtraction by an experiment by Kennedy (1977). She was investigating how young children used written symbolism to solve simple verbal problems in addition and subtraction. This child, Liam, was at that time aged 6 years and 2 months. He was seated at a table, with paper, pencil, small marshmallows, and M & M candies available for his use.

I: Let's pretend it is your birthday. We have invited twelve children, but all we have are seven cups to put the candies in. How many more cups do we have to buy? You can figure it out any way you want. Use the paper and pencil, marshmallows, or M & M's. (p. 129)

There then followed an interaction between experimenter and child throughout which hypotheses were being made and tested about the mental processes which gave rise to the child's observable (vocal and written) activities. Summarizing these, Kennedy writes:

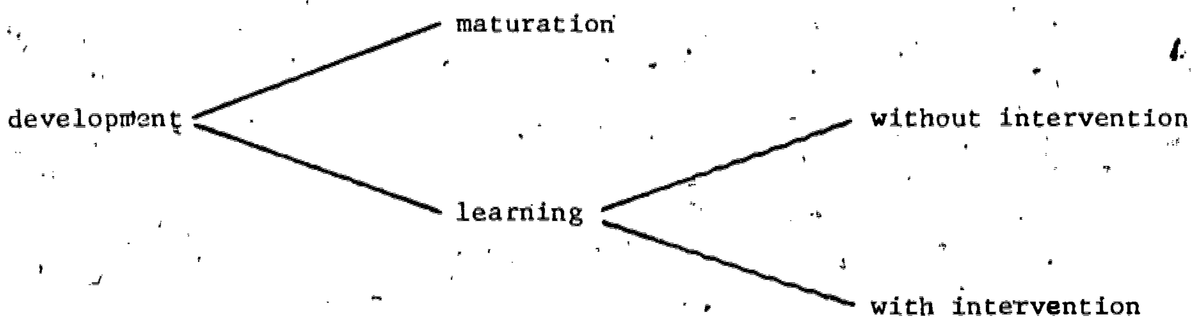
Liam did not use counting on. Instead, he assimilated the new problem into the already existing scheme of one-to-one correspondence...Further, Liam's diagram indicated that he has acquired certain key skills necessary for symbolic representation. The circle represents either a cup or a child; the values of the symbol can be interchanged easily. Thus, the five circles that initially represent children eventually stand for cups. In effect, Liam has invented a symbolic subtraction machine. (p. 130)

A similar hypothesis, that young children enter school with knowledge and skills which they have already learned informally, underlies (and is substantiated by) the research reported by Carpenter and Moser (1979). "Our basic interest is in the strategies children use, both before they receive formal (i.e., school) instruction and during and after they receive initial instruction, in the operations of addition and subtraction" (p. 19). The method was also that of individual interviews; but "The interview procedures were not clinical in the sense described by Opper (1977). Rather, they could be considered as an attempt at naturalistic observation. If a student's strategies could be directly observed, no follow-up questions were posed. If not, the interviewer followed a standardized routine for questioning children and coding responses" (p. 19).

The two authors just cited would not, I think, regard themselves as Piagetian. This suggests that we need a different name to include the wider category of researchers who, while using individual interviews to generate and test hypotheses about children's thinking, are not necessarily committed to Piaget's theories about cognitive development. It is for this reason that I have suggested the term diagnostic interview to describe this wider category of methodology. The term includes not only verbal interaction, but observation of children's activities such as finger counting, point counting, manipulation of physical objects, drawing.

It is also desirable that the term development be examined more closely, since it is used to describe both a process and its result. Piaget's well known developmental stages refer to the latter meaning; and his methodology, based on the diagnostic interview, has undoubtedly helped us toward a better understanding of the nature of children's thinking at each of these stages. His concept of equilibration, however, by which he explains the way children's thinking develops from one of these stages to another, is one which I have always found unsatisfactory; nor have I been able to identify a methodology by which this part of his model has been tested.

If by development we mean in the present context the process or processes by which a child's thinking reaches more advanced levels of knowledge and skills within a particular field, these may be analyzed as follows:



We need to know whether the achievement of a particular developmental stage--say, the conservation of number--is dependent primarily on maturation or on learning before we can decide whether or not it should be taught. This information, moreover, though necessary is not sufficient, for there is also a very real possibility that some kinds of learning take place at least as well without intervention as with it. Ginsburg (1977) writes: "Children can learn in apparently adverse circumstances. Children learn a great deal about numbers outside of school, without instruction or special help" (p. 10). My copy has a marginal annotation: "Perhaps lack of teaching isn't always adverse!" This annotation received further support from Carpenter and Moser's (1979) paper already cited. "As a final comment to this section, it is interesting to contrast the performance of the children we have studied and the problem solving abilities of older students. We have found that young children very carefully analyze problems and base their solutions on the structure and content of the problem. This analytic ability is precisely what older children lack. Although they are generally successful in solving simple addition, subtraction, multiplication, and division word problems, they have a great deal of difficulty with even simple non-routine problems that involve anything more than a straightforward application of a single arithmetic operation" (p. 40).

Powerful as the diagnostic interview has been, we need both theory and methodology to help us answer, in particular applications, questions of the form: How do children make progress in their knowledge and skills? And how can we try to ensure that the teaching they receive is truly helpful to this progress?

Soviet studies, constructivism, and teaching experiment. It has already been suggested that Piagetian theory takes little account of the function of instruction. Indeed, one of the features of the diagnostic interview is the care taken by the experimenter not to teach. As Opper (1977) writes:

> A particularly delicate aspect of the method, and one against which every interviewer must be on the alert, is the tendency to suggest answers to the child. Inexperienced interviewers, and sometimes even experienced ones, often forget how easy it is to convey to the child cues as to how they expect him to react.... It is essential, therefore, for the interviewer to remain neutral during the interview session in order to pick up the spontaneous thinking of the child, and avoid channeling the child's responses in the direction he believes these responses should be expressed. (pp. 97-98)

This emphasis results from the experimenter's intention that the observation shall make as little change as possible on what is observed. But questioning can be a powerful method of teaching, as was demonstrated long ago by Socrates; even when teaching is not intended, questioning can have the effect of initiating lines of thinking which might not have happened if the questions had not been put. This is one reason why seminars with one's graduate students are so beneficial to oneself.

Nor is intentional teaching incompatible with the aim and methodology of the diagnostic interview. Trying to teach a child something new can bring to light lack of understanding or misconceptions in the child's existing knowledge which might not show up in the performance of tasks based on this earlier level. For example, a child might do correctly additions not involving regrouping, such as 23 + 45, with imperfect understanding that the 2 and 4 had different significance from the 3 and the 5. This lack of understanding would show up if an experimenter tried to teach additions like 27 + 46. From performance at a particular level it may be difficult to infer whether a child has relational* or instrumental* understanding, but the inherent lack of adaptability in instrumental understanding makes it an inadequate basis for further learning, since it is not based on mathematical structures capable of assimilating further mathematical ideas.

In Soviet research, full emphasis is given to the function of teaching. Like Piaget's research, much of the Soviet research has relied on qualitative methods and other processes that children use to solve problems. However, whereas Piaget and most Western psychologists have focused on concepts that presumably develop independently of the school curriculum, the Soviets maintain that cognitive development and school learning are inexorably linked.

> In the final analysis, a pupil's mental development is determined by the content of what he is learning. Existing intellectual capabilities must therefore be studied primarily by making certain changes in what children learn at school. (El'Konin & Davydov, 1975, p. 2)

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*See Skemp (1976).

> Thus, stages of development are not viewed as absolute, and it is believed that changes in the curriculum can result in significant changes in the nature of the developmental stages through which a child passes. The types of misconceptions that Piaget identifies in early stages of development are attributed to shortcomings in the curriculum, and much of the Soviet research is directed at identifying such misconceptions and reconstructing the curriculum so that they do not develop. (Carpenter, 1979, pp. 54-55)

An example of a study based on the foregoing theory and methodology may be found in the present volume (Davydov, 1981).

A neo-Piagetian approach which also uses the teaching experiment as its methodology is constructivism. A summary of six principles of constructivism is given in Steffe, Richards, and von Glasersfeld (1979), and it is hard to do justice to these in any attempt to summarize them still further. Nevertheless the attempt must be made. Among the key features, as they appear to the present writer, are the following:

> Knowledge is viewed as pertaining to invariances in the living organism's experience rather than to entities, structures, and events in an independently existing world. (p. 29)
>
> Mental operations are part of a total structure, and structure is seen in the organization of operations. Different surface behaviors of a child may be interpreted as springing from the same cognitive structure. (p. 30)
>
> The structure of the learning environment must be considered within two frames of reference. On the one hand

there are the operational systems controlling the child's experiences and, on the other, there is the content to be learned. (p. 30)

Concepts, structures, skills, or anything that is considered "knowledge" cannot be conveyed ready-made from teacher to student or from sender to receiver. They have to be built up, piece by piece, out of elements which must be available to the subject. (p. 31)

The methodology of the teaching experiment may be regarded as an extension of that of the diagnostic interview, in which the purpose is to make and test hypotheses not only about the nature of a child's thinking at a particular time, but about how this thinking is developed from one stage to another. It is summarized by Steffe (1977) as follows:

1. daily teaching of small groups of children by the experimenters,

2. intensive observation of individual children as they engage in mathematical behavior,

3. prolonged involvement with the same children over periods ranging from about six weeks to the academic year,

4. clinical interviews with children, and

5. detailed records of observations through video taping and the written work of the children.

A salient characteristic of this methodology is that it takes up a great deal of the experimenter's time, and the data thus derived come from a relatively small number of children. When a theoretical model has been built up in this way, everything will then hinge on the generalizability of these findings. To assess generalizability, it may be necessary to revert to the technique of the experimental psychologist, involving substantial numbers of children in group-administered experiments. It will be interesting to see whether the theoretical position under present consideration can be maintained undistorted when linked with a methodology originally developed to serve behaviorist types of theory. Prima facie there are two reasons for expecting that the findings will be generalizable. First, the mathematical structures to be learned are the same, or nearly so, for all the children whose learning we hope eventually to help. Second, though children themselves vary both in their learning abilities and in the schemata which they have available for each new learning task, there are regularities in the learning process itself, both between different children and between different contents of learning. "A constructivist approach emphasizes similarities among seemingly disparate events or fields of knowledge acquisition" (Steffe, et al., 1979, p. 30). These regularities will need to be embodied in whatever theory is eventually constructed, using teaching experiments as the initial methodology; so it is to be hoped that the regularities constructed initially by research with small numbers will prove to be those which also apply to learning of mathematics by children in general. In any case, I think it is a risk (in investment of time and expertise) which is rightly taken, since I believe that by these means, better understanding of the learning and teaching of mathematics is likely to be reached than by any other way devised so far. My reasons for thinking this will be given in the last section.

For an example of a research study based on the foregoing theory and methodology, we need look no further than the present volume, and to the paper by Steffe et al., already cited. Another teaching experiment is that of Resnick (1979). This was successful on two fronts, theoretical and remedial. It confirmed an experimental hypothesis of much practical importance to the teaching profession, namely that the way to correct children's errors in arithmetic is first, to identify and if necessary correct these at the semantic level of mathematical concepts and conceptual structures; and then to ensure that children know the "correct" (i.e., conventionally agreed) ways of expressing their mathematical ideas in symbols. Also, it included a successful piece of remedial teaching. I would here like personally to endorse the professional ethic expressed by the experimenter, that when children who are helping us by taking part in our experiments themselves need help which it is appropriate and practically possible for us to give, then we own it to them to take the time to give it.

Resnick's study makes no mention of constructivism; nor do various other studies which make use of the methodology of the teaching experiment, such as those of Herscovics (1979) and of Kieran (1979). This suggests that the methodology of the teaching experiment relates not to single theory, but to a category of theories, a point which will be considered furth[illegible] the last section.

<u>[illegible]ficial intelligence, information processing, and computer simulatio[illegible]</u> Neither the results of the computer search, nor any of the other literature consulted before the preparation of this paper, included reports research into the learning of addition and subtraction skills. A major [illegible]udy of this kind was, however, presented at the Wingspread conference by Brown and Burton (1981). With the rapid advance of microprocessor technology, researchers into the learning and teaching of mathematics will wish to examine any kind of learning model which offers the possibility of useful applications in this field. Moreover, an examination of this

theoretical stance and methodology raises a number of points which need to be considered even by those whose own approach falls within one of the other categories. The present analysis takes as its starting point the summary and discussion in Carpenter (1979, pp. 58-64). He quotes the following passage from Klahr and Wallace (1976).

> Tasks must be analyzed in much more detail than is provided by a description of their conventional logical structure. The general problem is to determine exactly how the input is encoded by the subject and what transformations occur between encoding and decoding. The objective task structure alone does not yield a valid description of the solution performance, and it is necessary to diagnose the actual psychological processes in great detail to obtain minute descriptions or well supported inferences about the actual sequences and content of the thinking processes. (pp. 3-4)

But the sequences may, and indeed do, differ between subjects. Jones (1975) found that different children used a wide variety of methods to perform the same simple arithmetical tasks. For example, he identified 17 different methods by which the subtraction 83 - 26 was correctly evaluated. Twenty-five children correctly used one of three standard methods which they had been taught, but 50 children got the right answer by using one of the other 14 methods. In such a case, which of these should the computer program model?

The answer implicit in Brown and Burton (1979) is "the standard algorithm". But as Resnick has shown, the level at which procedural mistakes can best be corrected is not necessarily the syntactic level, and as cogently argued by by Ginsburg (1975), one of our most important tasks as teachers is to help children to relate their own conceptual structures to culturally accepted procedures and formal expressions. This will not be achieved by confining our attention to the latter.

That the same correct answer may be reached by a variety of different mental paths which are mathematically equivalent depends on the properties of the natural number system. These are at two levels of generality: number facts such as 13 = 10 + 3, 13 = 12 + 1, 5 x 12 = 60; and the five properties which apply to all numbers, that addition and multiplication are associative and commutative, and multiplication is distributive over addition. Possession of this mathematical knowledge, at a formal and reflective level (as in our case) or at a more informal and intuitive level (as in the case of children), is knowledge of a kind which I call knowledge-that. Any particular method, such as the 17 identified by Jones, I call knowledge-how. Knowledge-that is descriptive (e.g., multiplication is distributive over addition); knowledge-how is prescriptive (if you want to calculate 5 x 13, you do this and this and this). From one knowledge-that schema, not only one but several appropriate methods can be derived for each of a wide variety of tasks. In concentrating on performance alone, a computer model would omit this important dimension of mathematical ability.

Another human dimension distinguishes knowing-how from being-able. For a computer in good order, this distinction does not exist. Every program (corresponding to a particular piece of know-how) is correctly executed. Anything it knows-how to do it is able to do. This is far from being the case with children, or indeed adults. These may have a correct plan for performing a given mathematical task, but it is yet another kind of ability by which this plan is translated into action. Skill is two-dimensional: Having the right plan is one dimension, and being able to execute it accurately and also speedily is quite another. In computer model, speed and accuracy are automatically present, and so the processes by which they are acquired, important in school learning, are not embodied in the model.

Shortly afterward Ginsburg (p. 60) also states:

> The most extensive attempt to generate computer simulations of developmental phenomena is provided by the work of Klahr and Wallace (1976). Their general modus operandi can be described as follows:
>
> Faced with a segment of behavior of a child performing a task, we pose the question: "What would an information-processing system require in order to exhibit the same behavior as the child?" The answer takes the form of a set of rules for processing information: a computer program. This program constitutes a model of the child performing the task. It contains explicit statements about the capacity of the system, the complexity of the processes, and the representation of information--the data structure--with which the child must deal. (p. 5)

This invites an analysis of the term <u>developmental</u>. It may accurately be used with a number of distinct though related meanings. In our present context, the learning of mathematics includes

1. a change of knowledge-how, i.e., the acquisition of an improved repertoire of good plans;

2. a change of being-able, i.e., an improvement in the translation of the plans into action;

3. a change in knowing-that, i.e., an improvement of the mathematical schema from which plans are devised; and finally,

4. whatever brings about or makes possible these changes.

Major goals of mathematical instruction include helping to bring about changes of all the kinds listed above under items 1-3. The computer program described, however, appears not to embody any of them, so it is hard to understand how it can be regarded as a simulation of developmental phenomena.

> At the metaphorical level, one of '-s most viable information-processing models has been proposed by Pascual-Leone (1970, 1976). The principal forms of this theory regard the capacity of the central processor. Pascual-Leone (1970) hypothesizes that the basic intellectual limitation of children is the number of schemes, rules or ideas they can handle simultaneously--a capacity that increases regularly with age. The maximum number of discrete chunks of information that a child can integrate is assumed to grow linearly in an all-or-none manner as a function of age. From the early preoperational stage (3 to 4 years), a child's information-processing capacity, or M-power, grows at a rate of one chunk every two years until the late formal operational stage (about 15 to 16 years). (Carpenter, 1979, p. 61)

This passage focuses on an important feature of intelligence which shows particularly clearly in mathematics: the ability to form and use models containing more and more information. Mathematics itself, however, indicates that an important way this is done is not only by increasing the capacity for handling the same kind of information, but also by qualitative changes in the kind of information which is processed. This change consists of the formation of higher-order concepts. For example, the statement

$$(a + b)x = ax + bx$$

contains the same information as an infinity of statements like

$$(7 + 3)(5) = (7)3 + (5)3$$

$$(6 + 28)17 = (6)17 + (28)17$$

$$(2591 + 864)3065 = (2591)3065 + (864)3065.$$

A student's ability to process the information in the statement

$$(a + b)x = ax + bx$$

is a result, not of the ability to infinitely expand the capacity to process information, but of the ability to change the kind of information processed into a more condensed, thus more instructive, expression.

It is understandable that the enormous power of computers to store and process information should have led to exploration of ways in which they might be used as models for human intellectual processes. And it has already been fully demonstrated that computers can replicate human mathematical performance of many kinds and levels, ranging from simple addition to the calculation of regression equations. But for research into mathematical education, we need models which replicate human learning of mathematics: which replicate not only performance, but processes by which performance is improved. At present, improvements in computer programs are made by a human programmer, outside the computer. Will the artificial intelligence theorists be able successfully to simulate this?

Type 1 Theories and Type 2 Theories

In this last section, I shall draw together the discussions of the preceding four sections by suggesting that we need to make a global distinction between two categories of theory which I shall call type 1 and type 2, and that if this distinction is not made, there is likelihood of methodological errors, not only of detail but of principle.

My own realization of this distinction has followed the construction of a new model of intelligence, offered as an alternative to the psychometric group of models which have dominated the field for 70 years. In the course of discussions with graduate students about their proposed research based on this model, and particularly when discussing how it could be tested, it became apparent that an inference from this model was that the methods by which it should be tested were not necessarily those traditionally used by experimental psychologists. Further thinking along these lines led to the view that this was also the case for other theories, and in particular for learning theories and those of developmental psychologists. There is thus reasonable hope that the line of thinking which follows will be of interest to others who are interested in the same problems, without their necessarily accepting in detail my own model.

As a starting point we need an outline of this new model, and I shall try to give the briefest account which will be adequate for the present purpose. (A full exposition is to be found in Skemp, 1979a.) The model assumes, as a matter of observation, that much, possibly most, of human behavior is goal-directed, which implies that if we want adequately to understand what people are doing, we need to go beyond the outward and easily observable aspect of their actions, and ask ourselves what is their goal.

To help in thinking about how people direct their actions toward the achievement of their goals, a model of a director system was developed which synthesizes ideas from cybernetics and cognitive psychology. Its essential features include (a) some kind of sensor which takes in the present state of the operand (the operand being whatever is to be taken from the present state to the goal state); (b) some kind of internal representation of the goal state; (c) a comparator, which compares these two; and (d) a plan by which energy is applied to the operand to diminish the difference between its present state and its goal state until these coincide.

The changes from present state to goal state take the operand through a succession of intermediate states, each of which becomes a present state. All of these, in turn, have to be represented within the system so that they can be compared with the goal state. It is a short step from this to the need for a mental representation of the path from the present state to the goal state. This is a minimal requirement. More effective, particularly in a varying environment, is to have not just (an image of) a particular path, but a cognitive map from which a variety of paths can be constructed, as required, to meet the requirements of different starting points and environmental conditions. A schema, or conceptual structure, is simply a further development of the idea of a cognitive map, including concepts at different levels of abstraction and a symbol system for retrieving and manipulating these.

In the lower animals, many of these director systems are innate, the result of natural selection. But there is an upper limit to what can be transmitted genetically, and there are other disadvantages, such as slowness to adapt to environmental change. So it is not surprising that some species have evolved the ability to set up new director systems, and to improve the ones they have. This is how <u>learning</u> is conceptualized within the present model. Other animals can learn too, but we have also evolved a more advanced kind of learning which is qualitatively different from those studied in animal laboratories and embodied in theories such as operant conditioning. It is the ability to learn in this more advanced kind of way which I now call <u>intelligence</u>. A major feature of intelligence is the construction (building and testing) of the schemas (conceptual structures) which was shown earlier to be an important part of the more advanced kinds of director systems.

The new model uses the concept of a director system at two levels

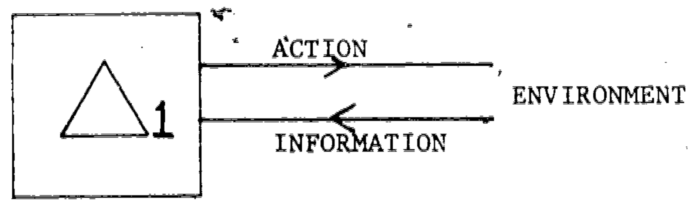

Figure 1

Leaving out all the interior detail, delta-one is a director system whose operands are physical objects in the outside environment.

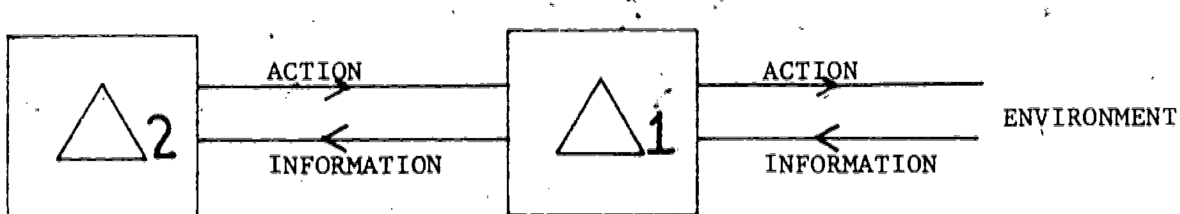

Figure 2

Delta-two is a second-order director system, which has delta-one as its operand. Its function is to take delta-one to states in which delta-one itself can function better. This includes not only improving director systems, but bringing new ones into existence. In brief, delta-two optimizes delta-one. Learning is one of the long-term ways in which this is done; making particular plans for particular situations is another, short term; building up a stock of ready-made plans for regularly encountered situations is another. Algorithms are an example of the last.

With the help of the foregoing, we can now distinguish two major categories of theories.

A type 1 theory is a somewhat abstract, general mental model of regularities in the physical world. It is thus a particular kind of schema (for we can imaginatively construct other kinds of schemas which do not and are not intended to represent anything having physical existence). A type 1 theory is used by delta-one as a basis for goal-directed action on operands in the physical environment. In terms which have been used already, a theory is a cohesive and abstract body of formal knowledge (knowledge-that) from which we can, as required, derive particular procedures (knowledge-how) to achieve particular goals in particular situations. Knowledge-how is a particularly important case of prediction. A prediction states that initial state A, without intervention, is followed by state B. Example: astronomical theory. Knowledge-how takes this a step further, and states that initial state A, with intervention based on plan P, will result in state B. Example: theory of electronics. Knowledge-how is a necessary but not a sufficient condition for being able. The intervention prescribed by the knowledge-how may be beyond our ability to translate into action.

All the natural sciences such as chemistry, astronomy, metallurgy, aerodynamics, electromagnetic theory, or genetics, are type 1 theories; in their respective fields of application, they are very successful in helping us to direct our actions successfully at the delta-one level, which is to say in achieving goal states of operands in the physical environment.

A type 2 theory is a model of regularities in the ways by which type 1 theories are constructed; and by which plans of action (for execution by delta-one) are derived from these theories. It is a mental model of the mental-model-building process. From an appropriate type 2 theory, we may hope to derive knowledge-how; possibly we shall also be able to intervene helpfully in children's learning of mathematics. Example of type 2 theories are: constructivism; my own theory of intelligence; any theory about the learning and teaching of mathematics which recognizes that teaching is an intervention in someone else's learning, i.e., that regards learning as a goal-directed activity with an important degree of autonomy in the subject, rather than regarding behavior as being shaped by the environment.

A type 1 methodology is concerned with constructing (building and testing) the models which delta-one requires for its successful functioning. When constructed, these models are type 1 theories. Each of the natural sciences has its own methodology, though these have much in common.

A type 2 methodology is concerned with constructing (building and testing) models of how type 1 theories are constructed, and how particular plans of action are derived from these. When constructed, these models are type 2 theories.

The importance of the foregoing for our present analysis is that if type 1 and type 2 theories belong to different categories, then we must be very alert to the possibility that they require different methodologies. Failure to make this distinction may result in the application of inappropriate methodologies, leading to unsound theories. In addition, it may result in the wrong overall conception of what one is trying to construct, so that while working on a type 2 thoery a person is all the time trying to make it look like a type 1 theory. These, I believe, are two of the ways in which behaviorism went wrong.

To show how a type 2 methodology needs to differ from a type 1 if it is to succeed, the following summary of type 1 methodology is offered as a starting point (modified from Skemp, 1979a, p. 174).

	Construction	
BUILDING	Mode	TESTING
One's own experience of the physical world.	1	One's own experiments on physical objects, involving the testing of predictions.
Communications from others: personal, lectures, journals; searching the literature.	2	Comparing one's own ideas with those of others, often involving discussion; seminars, conferences.
From within, by working on and with existing ideas: synthesis, extrapolation, imagination, intuition. Creativity.	3	Comparison with one's own existing knowledge and beliefs: internal consistency.

Although a correspondence can be seen between the three kinds of building and of testing, any one or more of the former can be used in conjunction with any one or more of the latter in the construction of a theory. The natural sciences use all three modes of building and all three modes of testing. However, the ultimate appeal is always to testing by mode 1, experiment. This fits in with the present model. If the purpose of constructing (which includ s improving) type 1 theories is to increase the powers of delta-one relative to the physical world, the physical world is where they must prove their success. Other criteria, such as economy, coherence, intelligibility, are also important. They help to make a theory more usable by facilitating the conversion of knowledge-that into knowledge-how.

Popper (1976) proposes that the term "scientific" should be reserved for theories tested by mode 1. This would be to equate all sciences with the natural sciences, and any scientific theory would thus be a type 1 theory. Scientific method would in this case be a body of particular methods derived from the methodology summarized above, with mode 1 testing as an essential component. I do not yet know whether I myself accept this position.

What kind of theory is mathematics? We need at least a partial answer to this; how can we usefully think about teaching it if we do not know what kind of a theory we are trying to teach? Mathematics seems to me to be a type 1 theory of an unusual, perhaps unique, kind. Though it can make good use of mode 1 at the outset, e.g., in the building of the concept of order and in the initial construction of the natural numbers, it rapidly abandons mode 1 and relies entirely on modes 2 and 3. Thus, correct or incorrect predictions of physical events play no part in confirming or refuting a mathematical theory as they do for other type 1 theories. Rather, the discovery of an internal inconsistency would refute a mathematical theory. The discovery that new ideas were consistent with the accepted body of mathematical knowledge would help to confirm them, and a demonstration that they were a necessary consequence of certain parts of this knowledge would constitute a proof, in the mathematical sense.

Although mathematics is not itself one of the natural sciences, it can be regarded as a conceptual 'kit' of great generality and versatility, so valuable to anyone who wants to construct a scientific theory as to be almost indispensable. The conversion from a mathematical statement to a theoretical model is often a very short one, requiring only the attachment of units. For example, E = IR is a mathematical statement if E, I, R represent pure numbers. But if they represent numbers of units of e.m.f., current, and resistance respectively, it becomes Ohm's law. These very close links, and the ease of transition both ways, suggest that mathematics may be regarded as a type 1x theory, having all the characteristics of a type 1 theory except mode 1 testing. Note that mode 1 building may be present, as in the construction of the natural numbers. Calculus offers another example.

Some of the reasons this has been a difficult paper to write are now becoming apparent, and I mention them here because these or similar reasons may apply to some readers also. First, it was necessary to put a certain distance between myself and the ways of thinking acquired as a mathematician, with physics in my case as a supporting subject. The years spent in these disciplines were followed by a period of 18 years as an psychologist. My initial orientation was that of an experimental psychologist, but during this period, using hindsight I realize that I was engaged in making the transition from a type 1 theorist to a type 2 theorist. This is a transition which others have been making. But we who are making this transition are in a different position from persons working on type 1 theories, for though they are at the frontiers of knowledge, they have well established methods of exploration. We are at two frontiers at the same time, the second one being a frontier of methodology. We need a methodology for investigation, not of children's observable performance, but of whatever brings about changes in their ability to perform. These changes may result (a) from increase of their knowledge-that, the construction and improvement of their mathematical schemas; (b) from their having now succeeded in deriving a new plan from their existing knowledge; (c) from increasing their repertoire of plans, eliminating for a greater number of tasks the necessity for (b).

These changes (in terms of the present model) take place within the child's delta-one, which by its nature cannot be observed by the experimenter. And whatever brings about these changes (in the present model, it is the higher-order system delta-two) is even more inaccessible to observation. By the activity of reflective intelligence, delta-two can sometimes observe, and even report on, activities within delta-one. But the activities of delta-two itself can only be inferred from changes in delta-one and the circumstances leading to such changes.

However, if we could find some way of observing the concepts and schemas within a child's delta-one, even indirectly and by inference, we would have made a substantial beginning. These observations, both for building and testing our theory-in-the-making, would then replace the model methods described previously.

Our starting point toward a method is a consideration of the function of symbols. These act as an interface of two kinds. The first is between the child's mathematical schemata* (located in the child's delta-one), and the experimenter's. The second is between the conscious and unconscious levels of the child's own thinking. As I have suggested elsewhere (Skemp, 1979a, pp. 157-158) it is questionable whether secondary concepts and schemata can be observed directly, even by their possessor: our sense organs are directed outwards, toward the physical world. The process of making a concept conscious seems to be closely connected with associating the concept with a symbol. So it is by symbols that the child knows what is in his own mind, as well as enables the experimenter to know what is in the child's mind. This knowledge is only partial, but it is the best we can get.

From the foregoing analysis, diagnostic interviews and teaching experiments both emerge as methods appropriate for the construction of a type 2 theory.

In the diagnostic interview, the experimenter set up in his own delta-one tentative images of what might be in the child's delta-one, and tests these by the symbolic interactions between himself and the child. In other words, the experimenter tries to get inside the child's mind by

*For an explanation of the term "mathematical schema", see the Notes on Terminology at the end of this paper.

forming a mental image of the child's mind, and interpreting what the child says in relation to this image. This interpretation is tested by another question, and if necessary the image is corrected. In this way, the experimenter may hope to construct images of the thinking of a number of children, and in particular of the ways in which they construct plans from their available schemata. From these images the instructor will then try to abstract regularities, and put these together into a theory.

The method of the teaching experiment takes this process a step further. The experimenter forms a mental model of the present state of the mathematical schemata of the learner, and also decides on a goal state for the learner to reach. The experimenter can conceptualize the goal state; the learner cannot, or can do so only vaguely. Next, the experimenter makes a conceptual analysis of the concepts belonging to the goal schema, and reanalyses these in turn, setting up a dependency network showing which concepts are prerequisite for others, or at least showing a working hypothesis about these relationships. On the basis of this the experimenter sets up in delta-one a path connecting the starting schema with the goal schema. This path will be a psychological path, not a logical one; it will be a sequence of schemata, each of which can be reached from the one before by the expansion of existing concepts, the formation of new concepts, or extrapolation. It will not be a sequence of logical inferences, for this involves examining implications between concepts which a person already has: It is not a process by which new concepts can be formed. As indicated earlier, the experimenter should examine this path carefully to find out which transitions can be made by enlargement of the existing schema, involving only the processes described, and which if any will require the much more difficult process of reconstruction.

The next preliminary to the teaching experiment is to devise material, which, if assimilated, will lead the learner through the necessary expansion of existing concepts and formation of new ones. If restructuring is required before assimilation can take place, particular thought must be given to the teaching methods by which it is hoped to bring this about.

The teaching experiment itself will involve trying to take the learner along this path, by two means: presentation of the material which has been devised, and additionally where necessary by explanations and direct information which help the learner to assimilate the new material to currently available schema. The method of the diagnostic interview will be used at every stage to compare the desired state of the learner's schema, as imaged on the path within the teacher's delta-one, with the state the learner has in fact arrived at. In this way, the experimenter will try to continually correct the initial teaching plan until one is developed which does, so far as indicated by the diagnostic interviews, achieve the desired learning goal. It will then be necessary to discover whether these plans are effective for other teachers and learners. This is the field in which a type 2 theory for the teaching of mathematics will have to prove itself, corresponding to the proving ground in the physical world of a type 1 theory.

The foregoing combination of methods appears to me an appropriate replacement for mode 1 building and testing as a first step in the conversion of the type 1 methodology to a type 2 methodology. Modes 2 and 3 do not need replacing, but their relationship may need to be revised in other ways. These methods are already in use, having been devised quite independently of the new model of intelligence which has been the starting point for the foregoing analysis. This convergence of thinking I find encouraging.

Notes on Terminology

Construction (of a theory). I use this term to mean both building and testing.

Diagnostic interview. The same as Piaget's clinical method, and Ginsburg's in-depth interview. I wanted to get away from the medical connotations of the former. "Diagnostic", from the Greek dia meaning through, and gignōskō recognize, seems to me more general.

Mathematical schema. A personal mathematical conceptual structure, as distinct from the general body of accepted mathematical knowledge.

Methodology. A prescriptive theory for theory construction, from which particular methods are derived and by which they are justified.

Model. In the present context, this always refers to a mental model.

Schema. A conceptual structure. Can be derived from the idea of a cognitive map, if we regard a shcema as analagous to a cognitive atlas in which (e.g.) a dot representing London or New York on a map of U.K. or U.S.A. can itself be expanded into a map. Not quite the same as Piaget's "scheme".

Teaching. A (conscious and intentional) intervention in the learning process of another.

Theoretical model. The same as Theory.

Theory. A mental model which is more abstract and general than those used in everyday thinking.

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