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ABSTRACT

This report opens by asking how operations in general, and addition and subtraction in particular, are characterized for elementary school students, and examines the "standard" instruction of these topics through secondary schooling. Some common errors and/or "sloppiness" in the typical textbook presentations are noted, and suggestions are made that these problems could lead to pupil difficulties in understanding mathematics. The ambiguity of interpretation of number sentences of the forms "a+b=c" and "a-b=c" leads into a comparison of the familiar use of binary operations with a unary-operator interpretation of such sentences. The second half of this document focuses on points of view that promote varying approaches to the development of mathematical skills and abilities within young children. The importance of meaning and understanding, particularly among young pupils, is promoted. An interpretation of symbolic notation within the domain of natural or whole numbers is emphasized which stresses "change-of-state" and is thought to be a neglected and potentially useful approach to number operations for young children. Investigations are called for to test instructional methods that focus on states and operators rather than operations per se, to see if the proposed change is indeed fruitful. (MP)

Project Paper 79-7

"ADDITION," "SUBTRACTION" AND MATHEMATICAL OPERATIONS

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Report from the
Mathematics Work Group

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Mathematics Work Group

"Addition," "Subtraction" and Mathematical Operations

J. F. Weaver

The University of Wisconsin-Madison

A Paper Prepared for the Seminar on

THE INITIAL LEARNING OF ADDITION AND SUBTRACTION SKILLS

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PREFACE

This paper has been organized in two principal parts:

Part I. Some Mathematical Considerations

Part II. Some Research Considerations

Although the content of Part I is relatively unsophisticated, it does go well beyond "addition and subtraction skills" at the level of "initial learning." Most importantly, this background emphasizes an all-too-frequently overlooked or neglected interpretation of number operations that, I believe, has nontrivial import for past, ongoing, and future research and instructional considerations pertaining to "The Initial [and Subsequent] Learning of Addition and Subtraction Skills."

In preparing this paper I have profited greatly from numerous discussions with one of my Ph.D. advisees, Glendon W. Blume. However, I am very quick to absolve Glen of any responsibility for the paper's content and for points of view it may advance. I alone accept the partisan's role.

Mathematical Operations

How are *operations* in general, and *addition* and *subtraction* in particular, characterized for elementary-school students?

In many textbooks (and similar materials) no attempt is made to do so in any explicit verbal or symbolic form. This may be wise, since certain efforts to characterize operations or addition or subtraction leave much to be desired. For instance:

1. Milton & Leo (1975) assert the following:

"Number operations. An operation is a *rule* for *combining numbers*. Addition and subtraction are operations." (Italics mine).

2. Eicholz, O'Daffer & Fleenor (1978) refer to "combining" only in connection with addition, and make no mention of any "rule":

"Addition: An operation that combines a first number and a second number to give exactly one number called a sum." (p. 342)

"Subtraction. An operation related to addition as illustrated:

$$7 + 8 = 15 \qquad \begin{matrix} 15 - 8 = 7 \\ 15 - 7 = 8 \end{matrix}$$ " (p. 344)

3. SMSG (School Mathematics Study Group, 1965) eschewed both "rule" and "combining":

"Addition and subtraction are two operations of mathematics. . . .

"An operation on two numbers is a *way of thinking* about two numbers and getting one and only one number. When we think about 9, 5 and get 14, we are adding. We write 9 + 5 = 14. When we think about 9, 5 and get 4, we are subtracting. We write 9 - 5 = 4."[1] (p. 72, italics mine).

The most charitable thing that can be said about most of the preceding characterizations is that they are *vacuous*. Some are in fact misleading or even erroneous when viewed in the light of more advanced or sophisticated interpretations.

How are *operations* defined "ultimately;" i.e., for secondary or post-secondary students?

Feferman (1964) has indicated that "In algebra it is customary to use the word *operation* instead of *function*, but these have exactly the same meaning." (p. 50)

The essential features of a *function* have been characterized clearly by Allendoerfer & Oakley (1963), for instance, who also identify it as "a special case of *relation*" which in turn "is a set of ordered pairs." (p. 195, italics mine). Specifically:

"A *function* f is a relationship between two sets: (1) a set X called the *domain of definition* and (2) a set Y called the *range*, or *set of values*, which is defined by (3) a rule that assigns to each element of X a unique element of Y.

"This definition may be more compactly stated as follows:

"A *function* f is a set of ordered pairs (x,y) where (1) x is an element of a set X, (2) y is an element of a set Y, and (3) no two pairs in f have the same first element." (p. 189)

[Note that there is nothing in this definition that precludes the possibility that $Y = X$, for instance, or that X is itself a product set.]

Because of things to follow in this paper, it will be helpful to distinguish as Hess (1974) has done between various kinds or types of operations:

"An n-ary operation on a set A is a function from $A \times A \times \cdots \times A$ (n factors) into a set A." (p. 281)

And as Lay (1966) has indicated,

"According to whether $n = 1, 2, 3, \cdots, n$ the operation is said to be unary, binary, ternary, $\cdots$, n-ary." (p. 198)

This paper will be concerned principally with *unary* and *binary* operations which have been characterized in the following ways by Fitzgerald, Dalton, Brunner & Zetterberg (1968), for instance, for second-year algebra students:

"A unary operation, *defined on a set X, is the set of ordered pairs which is determined by a mapping of each element of X to one and only one element of X.*" (p. 70)

"By definition, a binary operation defined on a set X is a mapping of each ordered pair (x_1,x_2), which may be formed with the elements of set X, to *one and only one element* x_3 in the same set. A more concise way of stating this is to say: For all x_1 and x_2 in X, each ordered pair (x_1,x_2) is mapped to a *unique* x_3 in X." (p. 76)

A binary operation also may be viewed as a mapping from $X \times X$ into X, where (x_1,x_2) is in $X \times X$ and x_3 is in X. A binary operation, then, is a set of ordered pairs, each of the form $((x_1,x_2),x_3)$, where the first component of each ordered pair is itself an ordered pair, (x_1,x_2).

Scandura (1971) has given an equivalent characterization in these words:

"*A binary operation is a set of ordered triples of elements such that there are no two triples such that the first two elements are the same and the third one different. In effect, the first two elements of any triple specify a unique third element.*

"Sets of ordered triples, of course, are nothing but ternary relations. Hence, *binary operations* may be defined as (certain) *ternary relations*." (p. 95)

Except for Scandura's, each definition cited thus far for "binary operation" makes it explicit that a binary operation <u>is</u> *a set of ordered pairs*. Thurston's (1956) definition of (binary) operation, however, does not equate an operation with a set per se:

"An operation can be formally defined as follows: it is a *rule* whereby to each ordered pair of elements of the set there corresponds a third element of the set." (p. 13, italics mine).

Birkhoff & MacLane (1965) also equate "operation" with "rule":

"A *binary operation* "$\circ$" on a set S of elements a, b, c, $\cdots$ is a rule which assigns to each ordered pair of elements a and b from S a uniquely defined third element $c = a \circ b$ in the same set S." (p. 28)

[Buck (1970) has cautioned that "it is not a formal definition to equate 'function' [or "operation"] with 'rule' if the latter is left undefined." (p. 253). This is equally true when "operation" is equated with a "set of ordered pairs." There is no need in this paper, however, to carry the preceding characterizations to the point of formal definitions,[2]--although such would be necessary under certain other circumstances.]

Finally, in connection with this consideration of "ultimate" characterization of operations, Armstrong (1970) has indicated that

"By a binary operation on a set S of objects, we mean a *process* that enables us to produce a single object of the set S from any pair of objects of the set S that we might be given." (p. 35, italics mine).

The principal distinguishing feature among the preceding characterizations is that for some *an operation is a rule* (that generates a set of ordered pairs) whereas for others *an operation is a set of ordered pairs* (involving assignments that might be made arbitrarily but more often are generated in accord with a rule). In this paper I shall adhere to the latter characterization rather than the former as I turn now to the questions,

What is *addition*? What is *subtraction*?

"Addition" and "subtraction" commonly are associated with *numbers* of one kind or another and often are identified as *binary operations* applied to such numbers. More explicitly:

Given a set S of numbers, *addition* as a binary operation on S is a mapping: it is the set of all correspondences $((a,b),c)$ for which each (a,b) in $S \times S$ has a unique image c in S such that $c = a + b$.

And:

Given a set S of numbers, *subtraction* as a binary operation on S is a mapping: it is the set of all correspondences $((a,b),c)$ for which each (a,b) in $S \times S$ has a unique image c in S such that $c = a - b$.

We're still somewhat unenlightened about addition and subtraction as binary operations, however. For instance: Addition may qualify as an operation on some set of numbers but not on another. The same may be true for subtraction. And what is the(?) assignment rule for addition? for subtraction? The nature of such would seem to have a bearing upon whether addition, or subtraction, qualifies as an operation.

In this paper interest centers upon both the set of *natural* or *counting* numbers, $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \cdots\}$, and the set of *whole* numbers, $W = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\}$.[3] For these sets the following may be used:

An assignment rule for *addition* of *natural* or *whole* numbers:

Select sets A and B such that $A \cap B = \phi$, $n(A) = a$ and $n(B) = b$. Then $c = n(A \cup B) = a + b$, where $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

An assignment rule for *subtraction* of *natural* or *whole* numbers:

Select sets A and B such that $B \subseteq A$,[4] $n(A) = a$ and $n(B) = b$. Then $c = n(A \setminus B) = a - b$, where $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\} = A'$.

Strictly speaking, then: *addition* is a binary operation on N and also on W; *subtraction* is not a binary operation on either N or W. Tables 1, 2, 3 and 4 may help in further consideration of this fact.

Insert Tables 1, 2, 3, 4 about here

TABLE 1

Domain of Definition for "Natural-number Addition"

×		*N*							
		1	2	3	4	5	6	7	
N	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	
	7	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	

Note.--The pattern of the table continues without end.

TABLE 2

Domain of Definition for "Whole-number Addition"

×		W						
		0	1	2	3	4	5	6
W	0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)
	1	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,0)	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Note.--The pattern of the table continues without end.

TABLE 3

Domain of Definition for "Natural-number Subtraction"

×		*N*						
		1	2	3	4	5	6	7
N	1							
	2	(2,1)						
	3	(3,1)	(3,2)					
	4	(4,1)	(4,2)	(4,3)				
	5	(5,1)	(5,2)	(5,3)	(5,4)			
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)		
	7	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	

Note.--The pattern of the table continues without end.

TABLE 4

Domain of Definition for "Whole-number Subtraction"

×		W						
		0	1	2	3	4	5	6
W	0	(0,0)						
	1	(1,0)	(1,1)					
	2	(2,0)	(2,1)	(2,2)				
	3	(3,0)	(3,1)	(3,2)	(3,3)			
	4	(4,0)	(4,1)	(4,2)	(4,3)	(4,4)		
	5	(5,0)	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	
	6	(6,0)	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Note.--The pattern of the table continues without end.

Every member of $N \times N$ (Table 1) has an image in N under *addition*, and every member of $W \times W$ (Table 2) has an image in W under *addition*.

Some but not all members of $N \times N$ (Table 3) have images in N under *subtraction*, and some but not all members of $W \times W$ (Table 4) have images in W under *subtraction*.

In this paper I shall take a slight(?) liberty with mathematical correctness or preciseness and refer to both *addition* and *subtraction* as binary *operations*, recognizing that when applied to natural or whole numbers the domain of definition is different for the two operations.

It also is important to recall that in this paper the distinguishing characteristic of an operation is to be found in a mapping,--a set of correspondences,--rather than in a rule. It is possible that a particular set of correspondences may be generated by markedly different rules;[5] and in such instances we are not dealing with different operations,--but with one and the same operation.

For instance:

Previously in this paper natural-number addition was associated with a "union-of-disjoint-sets" assignment rule. The same operation,--the same set of assignments,--can be derived from a "concatenated segments" assignment rule, for instance:

Select distinct collinear points X, Y, Z such that Y is between X and Z, $m(\overline{XY}) = a$, and $m(\overline{YZ}) = b$. Then $m(\overline{XZ}) = c = a + b$.

Regardless of the assignment rule associated with natural-number addition, natural-number subtraction can be characterized directly in terms of the addition operation rather than in terms of a "set difference" assignment rule (as was done previously in this paper) or whatever:

$a - b = c$ means that there exists a natural number c such that

$c + b = a$ or $b + c = a$. (Thanks to the commutativity of addition.)

Whole-number subtraction and addition are related in a similar way, of course.

A *trivial* distinction?

If you were to examine mathematical texts at a "teachers level," for instance, you would observe the following:

Some texts establish, in effect, that

(1) $a - b = n \leftrightarrow n + b = a$ or $a = n + b$

as the basic or primary way of defining subtraction in terms of addition, and may or may not also make explicit that

(2) $a - b = n \leftrightarrow b + n = a$ or $a = b + n$.

Other texts, in effect, state the defining condition in terms of (2), and may or may not make an explicit statement of (1).

In fact, as I have identified in Appendix A, eight of 25 texts take the former position; 17, the latter.

This may be a trivial distinction at *our* level of mathematical comprehension; but, as I shall explain later in this paper, it may be *non*trivial for *young children* in their development of ideas about addition and subtraction.

At present, however, I turn next to a different conceptual matter.

The Ambiguity of "$a \nabla b = c$"

Let a, b, and c be members of a set S of numbers such that

$$a \nabla b = c,$$

where "∇" ("wedge") signifies a *binary* operation (e.g., + or -) that assigns to the pair a in S and b in S,--i.e., to the ordered pair (a,b) in $S \times S$,-- a unique image c in S.

It is unfortunate (in my judgment) that only rarely (e.g., Lay, 1966) do texts on relatively elementary mathematical content present and discuss at length any alternative(s) to the preceding *binary-operation* interpretation of sentences of the form "$a \nabla b = c$." But there is at least one, and (depending on the nature of ∇) possibly two, other interpretation(s) of the same "$a \nabla b = c$":

(1) The *post*- or *right*-operator "∇b" (b in S) signifies a *unary* operation that assigns to operand a in S a unique image c in S. (There are times in this paper when I make that interpretation explicit by writing a sentence in the form "$a \underline{\nabla b} = c$.")

And possibly

(2) The *pre*- or *left*-operator "$a \nabla$" (a in S) signifies a *unary* operation that assigns to operand b in S a unique image c in S. (There are times in this paper when I make that interpretation explicit by writing a sentence in the form "$\underline{a \nabla} b = c$.")

These three different interpretations of "$a \nabla b = c$" involve three different operations and may be portrayed (to advantage, I believe) by picturing function or operation "machines" as in Figure 1.

- - - - - - - - - - - - - - - -

Insert Figure 1 about here

- - - - - - - - - - - - - - - -

Notice that 1.1 and 1.2 are but slightly(?) different ways of picturing the ordered-pair input to which the binary operation ∇ is applied. But 1.3 and 1.4 depict operations that are different from each other as well as from ∇, although the same image is generated in each instance.

Figure 2 pictures the ambiguity of interpretation of "$a + b = c$;" and interpretations of "$7 + 2 = 9$" in particular, for instance, are pictured in Figure 3.

- - - - - - - - - - - - - - - - - -

Insert Figures 2 and 3 about here

- - - - - - - - - - - - - - - - - -

In Figure 2 "+" and "$+ b$" and "$a +$" signify three different operations, just as do "+" and "$+ 2$" and "$7 +$" in Figure 3.

Binary

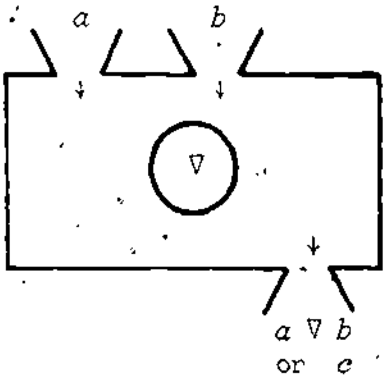

Figure 1.1

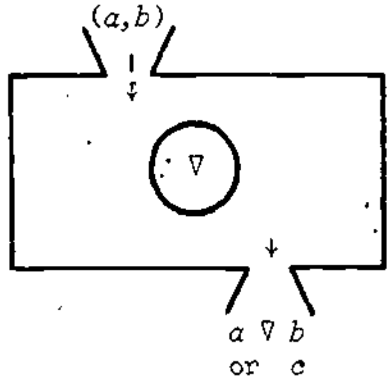

Figure 1.2

Unary

Post operator

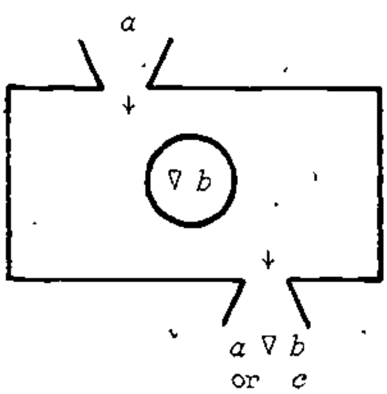

Figure 1.3

Pre operator

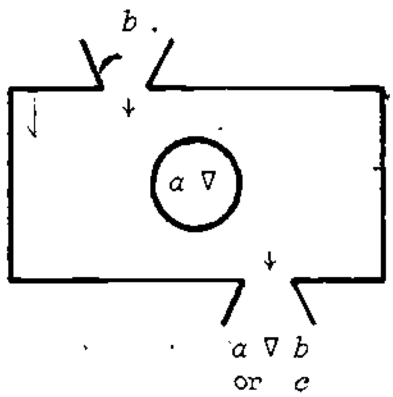

Figure 1.4

Figure 1. Function- or operation-machine interpretations of $a \nabla b = c$

Binary

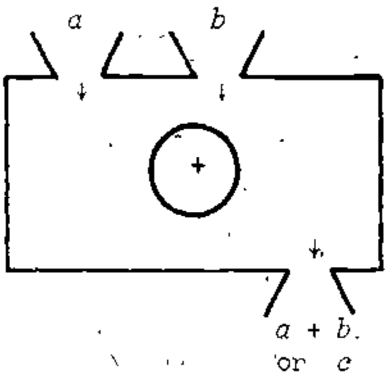

Figure 2.1

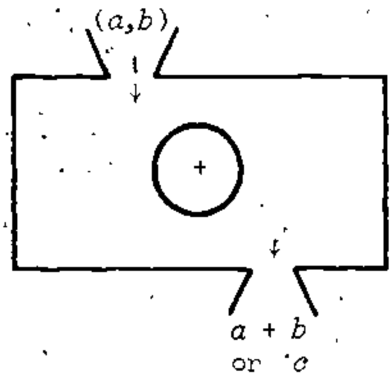

Figure 2.2

Unary

Post operator

Pre operator

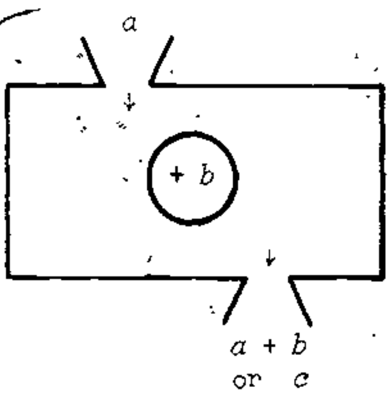

Figure 2.3

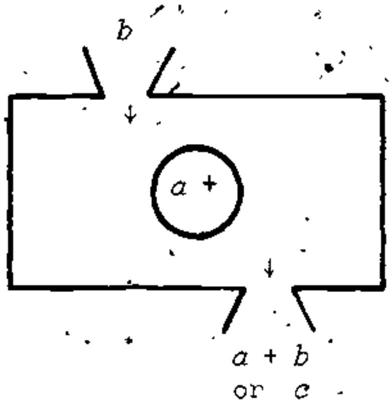

Figure 2.4

Figure 2. Function- or operation-machine interpretations of $a + b = c$

Binary

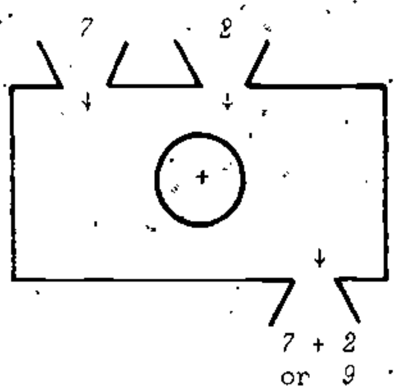

Figure 3.1

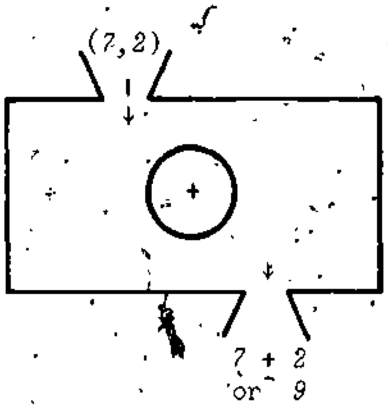

Figure 3.2

Unary

Post operator

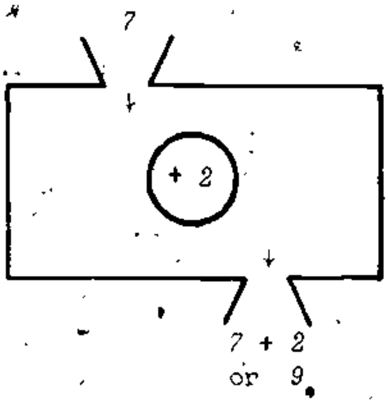

Figure 3.3

Pre operator

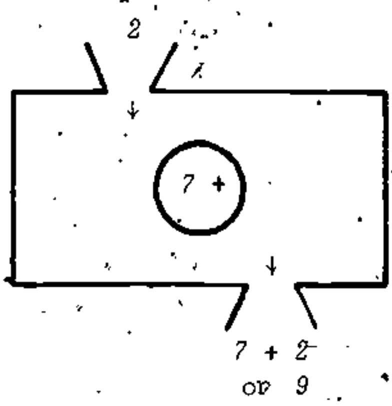

Figure 3.4

Figure 3. Function- or operation-machine interpretations of $7 + 2 = 9$

Figures 4 and 5 emphasize that more restricted interpretations must be placed upon "$a - b = c$" and, for instance, upon "$7 - 2 = 5$" than was true for "$a + b = c$" (Figure 2) and, for instance, for "$7 + 2 = 9$" (Figure 3).

- -

Insert Figures 4 and 5 about here

- -

In Figure 4, the unary-operator sentence "$a \underline{- b} = c$" associated with 4.3 is compatible with the binary interpretation of "$a - b = c$" associated with 4.1 and 4.2. However, the unary-operator phrase or expression "$\underline{a - b}$" associated with 4.4 conveys a different meaning that is *in*compatible with the preceding interpretations.

More particularly in Figure 5, the unary-operator sentence "$7 \underline{- 2} = 5$" associated with 5.3 is compatible with the binary interpretation of "$7 - 2 = 5$" associated with 5.1 and 5.2. However, the unary-operator expression or phrase "$\underline{7 - 2}$" conveys a different meaning that is *in*compatible with the preceding interpretations of Figure 5.

Table 5 summarizes the principal differences or ambiguities involved in the interpretations conveyed by Figures 2, 3, 4 and 5; and also uses an "arrow notation" as an alternative *un*ambiguous form of symbolization.

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Insert Table 5 about here

- - - - - - - - - - - - - - - - -

It is imperative in connection with Table 5 (and with Figures 2, 3, 4 and 5) that symbols such as "$+ b$" and "$- b$" and "$+ 2$" and "$- 2$" be interpreted as *unary operators*, and NOT as "signed" or "directed" numbers, which are very markedly different things conceptually,--as Lay (1966) has emphasized,--or as directed segments or vectors.

Binary

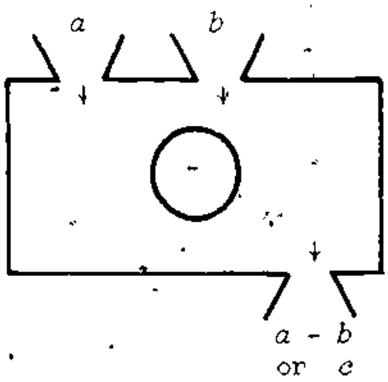

Figure 4.1

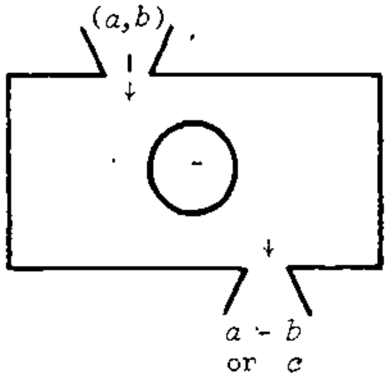

Figure 4.2

Unary

Post operator

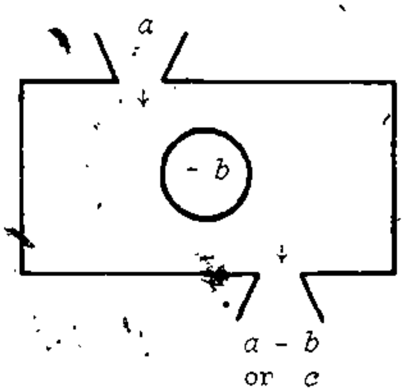

figure 4.3

Pre operator

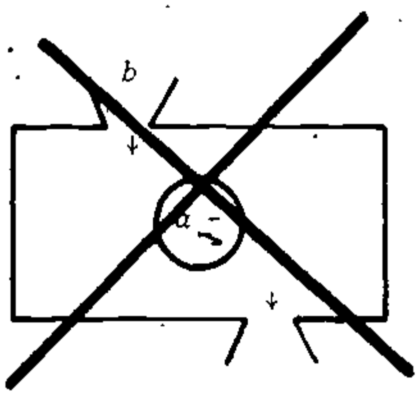

Figure 4.4

Figure 4. Function- or operation-machine interpretations of $a - b = c$

Binary

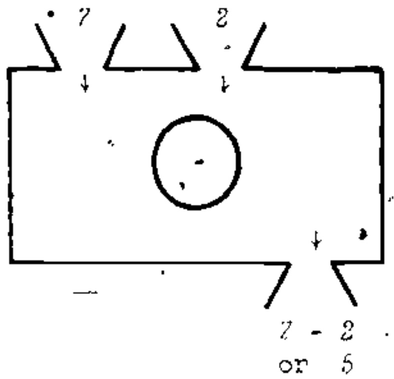

Figure 5.1

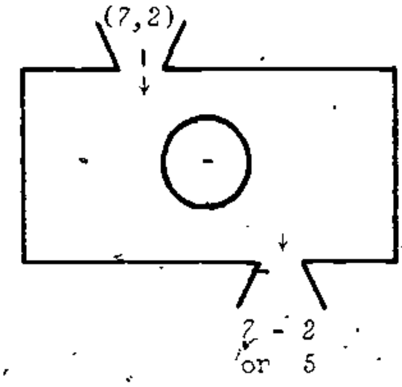

Figure 5.2

Unary

Post operator

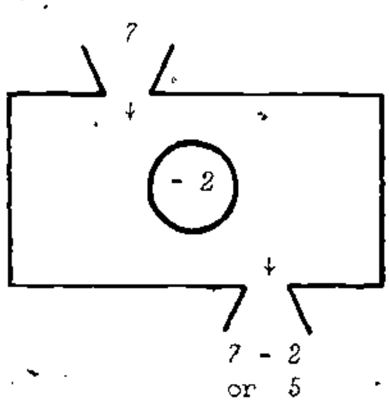

Figure 5.3

Pre operator

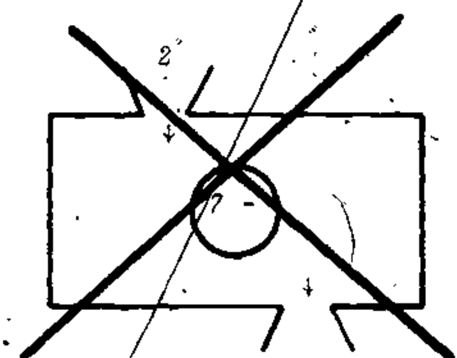

Figure 5.4

Figure 5. Function- or operation-machine interpretations of 7 - 2 = 5

TABLE 5

Binary and Unary Interpretations of $a + b = c$ and $a - b = c$

(For Instance, of $7 + 2 = 9$ and $7 - 2 = 5$)

Binary interpretation			Unary interpretation					
			Right- or post-operator interpretation			Left- or pre-operator interpretation		
Input	Operator	Output	Input	Operator	Output	Input	Operator	Output
a and b or (a,b)	+	$a + b$ or c	a	+ b	$a + b$ or c	b	a +	$a + b$ or c
	-	$a - b$ or c		- b	$a - b$ or c			
For instance:								
7 and 2 or (7, 2)	+	7 + 2 or 9	7	+ 2	7 + 2 or 9	2	7 +	7 + 2 or 9
	-	7 - 2 or 5		- 2	7 - 2 or 5			

Alternative notation

Binary interpretation	Right- or post-operator interpretation	Left- or pre-operator interpretation
$(a,b) \xrightarrow{+} c$	$a \xrightarrow{+\ b} c$	$b \xrightarrow{a\ +} c$
$(a,b) \xrightarrow{-} c$	$a \xrightarrow{-\ b} c$	
For instance:		
$(7,2) \xrightarrow{+} 9$	$7 \xrightarrow{+\ 2} 9$	$2 \xrightarrow{7\ +} 9$
$(7,2) \xrightarrow{-} 5$	$7 \xrightarrow{-\ 2} 5$	

Note. Unary *operators* are *symbols* that *name* particular unary operations or classes of unary operations. Binary *operators* are *symbols* that name particular binary operations.

Henceforth in this paper I shall dispense with *left-* or *pre-*operators to signify unary operations and shall adhere to the more commonly used *right-* or *post-*operator interpretation.

<u>What other "operational" distinctions are to be made?</u>

There are several.

1. An operation has been characterized as a mapping,--as a set of assignments,--defined for a specified domain. Therefore, as evidenced from Tables 6 and 7, the binary operation of *natural*-number addition is *not* the same operation as the binary operation of *whole*-number addition. Also, as evidenced from Tables 8 and 9, the binary operation of *natural*-number subtraction is *not* the same operation as the binary operation of *whole*-number subtraction.

- -

Insert Tables 6, 7, 8 and 9 about here

- -

Furthermore, the properties associated with *natural-* and *whole*-number addition are not identical, nor are the properties associated with *natural-* and *whole*-number subtraction.

There simply is no such thing as *THE* addition operation, or *THE* subtraction operation.

2. Tables such as 10 and 11 are *conceptually* rather than "cosmetically" different from Tables 7 and 9, respectively,--each of which is a set of assignments defining *one* binary operation. But each of Tables 10 and 11 consists of a *multiplicity* of sets of assignments defining a *multiplicity* of unary operations.

- -

Insert Tables 10 and 11 about here

- -

TABLE 6

$a + b = c$ for Natural Numbers

+		*b*											
		1	2	3	4	5	6	7	8	9	10	11	12
	1	2	3	4	5	6	7	8	9	10	11	12	13
	2	3	4	5	6	7	8	9	10	11	12	13	14
	3	4	5	6	7	8	9	10	11	12	13	14	15
	4	5	6	7	8	9	10	11	12	13	14	15	1
	5	6	7	8	9	10	11	12	13	14	15	16	17
	6	7	8	9	10	11	12	13	14	15	16	17	18
a	7	8	9	10	11	12	13	14	15	16	17	18	19
	8	9	10	11	12	13	14	15	16	17	18	19	2
	9	10	11	12	13	14	15	16	17	18	19	20	21
	10	11	12	13	14	15	16	17	18	19	20	21	22
	11	12	13	14	15	16	17	18	19	20	21	22	2
	12	13	14	15	16	17	18	19	20	21	22	23	24
	13	14	15	16	17	18	19	20	21	22	23	24	25
	14	15	16	17	18	19	[illegible]	21	22	23	24	25	[illegible]

TABLE 7

$a + b = c$ for Whole Numbers

+		b											
		0	1	2	3	4	5	6	7	8	9	10	11
a	0	0	1	2	3	4	5	6	7	8	9	10	11
	1	1	2	3	4	5	6	7	8	9	10	11	1
	2	2	3	4	5	6	7	8	9	10	11	12	13
	3	3	4	5	6	7	8	9	10	11	12	13	14
	4	4	5	6	7	8	9	10	11	12	13	14	15
	5	5	6	7	8	9	10	11	12	13	14	15	1
	6	6	7	8	9	10	11	12	13	14	15	16	17
	7	7	8	9	10	11	12	13	14	15	16	17	1
	8	8	9	10	11	12	13	14	15	16	17	18	19
	9	9	10	11	12	13	14	15	16	17	18	19	20
	10	10	11	12	13	14	15	16	17	18	19	20	21
	11	11	12	13	14	15	16	17	18	19	20	21	22
	12	12	13	14	15	16	17	18	19	20	21	22	2
	13	13	14	15	16	17	18	19	20	21	22	23	24

TABLE 8

$a - b = c$ for Natural Numbers

a \ b	1	2	3	4	5	6	7	8	9	10	11	12
1												
2	1											
3	2	1										
4	3	2	1									
5	4	3	2	1								
6	5	4	3	2	1							
7	6	5	4	3	2	1						
8	7	6	5	4	3	2	1					
9	8	7	6	5	4	3	2	1				
10	9	8	7	6	5	4	3	2	1			
11	10	9	8	7	6	5	4	3	2	1		
12	11	10	9	8	7	6	5	4	3	2	1	
13	12	11	10	9	8	7	6	5	4	3	2	1
14	13	12	11	10	9	8	7	6	5	4	3	2

TABLE 9

$a - b = c$ for Whole Numbers

	b											
a	0	1	2	3	4	5	6	7	8	9	10	11
0	0											
1	1	0										
2	2	1	0									
3	3	2	1	0								
4	4	3	2	1	0							
5	5	4	3	2	1	0						
6	6	5	4	3	2	1	0					
7	7	6	5	4	3	2	1	0				
8	8	7	6	5	4	3	2	1	0			
9	9	8	7	6	5	4	3	2	1	0		
10	10	9	8	7	6	5	4	3	2	1	0	
11	11	10	9	8	7	6	5	4	3	2	1	0
12	12	11	10	9	8	7	6	5	4	3	2	1
13	13	12	11	10	9	8	7	6	5	4	3	2

TABLE 10

Some Unary Operations Associated with a + b = c

		+ b											
		+ 0	+ 1	+ 2	+ 3	+ 4	+ 5	+ 6	+ 7	+ 8	+ 9	+ 10	+ 11
a	0	0	1	2	3	4	5	6	7	8	9	10	11
	1	1	2	3	4	5	6	7	8	9	10	11	12
	2	2	3	4	5	6	7	8	9	10	11	12	13
	3	3	4	5	6	7	8	9	10	11	12	13	14
	4	4	5	6	7	8	9	10	11	12	13	14	15
	5	5	6	7	8	9	10	11	12	13	14	15	16
	6	6	7	8	9	10	11	12	13	14	15	16	17
	7	7	8	9	10	11	12	13	14	15	16	17	18
	8	8	9	10	11	12	13	14	15	16	17	18	19
	9	9	10	11	12	13	14	15	16	17	18	19	20
	10	10	11	12	13	14	15	16	17	18	19	20	21
	11	11	12	13	14	15	16	17	18	19	20	21	22
	12	12	13	14	15	16	17	18	19	20	21	22	23
	13	13	14	15	16	17	18	19	20	21	22	23	24

TABLE 11

Some Unary Operations Associated with $a \underline{-\ b} = c$

		- b											
		- 0	- 1	- 2	- 3	- 4	- 5	- 6	- 7	- 8	- 9	- 10	- 1
a	0	0											
	1	1	0										
	2	2	1	0									
	3	3	2	1	0								
	4	4	3	2	1	0							
	5	5	4	3	2	1	0						
	6	6	5	4	3	2	1	0					
	7	7	6	5	4	3	2	1	0				
	8	8	7	6	5	4	3	2	1	0			
	9	9	8	7	6	5	4	3	2	1	0		
	10	10	9	8	7	6	5	4	3	2	1	0	
	11	11	10	9	8	7	6	5	4	3	2	1	0
	12	12	11	10	9	8	7	6	5	4	3	2	1
	13	13	12	11	10	9	8	7	6	5	4	3	2

In Table 10, for instance, the assignments whose images are in the column headed "+ 0" define *one* operation; the assignments whose images are in the column headed "+ 1" define a *different* operation; the assignments whose images are in the column headed "+ 2" define *another different* operation; etc., ad infinitum.

Similarly in Table 11, the assignments whose images are in the columns headed "- 0," "- 1," "- 2," etc., ad infinitum, define *different* operations,--no two of which are the same.

3. A further nontrivial difference between certain binary and unary operations may be seen in connection with commutativity.

It is well known that natural- or whole-number *addition* is *commutative*: i.e., for every natural or whole number a and for every natural or whole number b it is true that

$$a + b = b + a$$

which permits us to write equivalent sentences such as those in Figure 6.

- - - - - - - - - - - - - - - -

Insert Figure 6 about here

- - - - - - - - - - - - - - - -

Within the natural- or whole-number domain it also is valid to assert that

$$a \underline{+ b} = b \underline{+ a}$$

which on the surface *looks like* "commutativity" but isn't. Except for the special case in which $a = b$, the operators "+ a" and "+ b" signify *different* operations; thus, this "pseudocommutativity" is a valid property but not about *an* operation. Figure 7, therefore, is markedly different conceptually from Figure 6.

- - - - - - - - - - - - - - - -

Insert Figure 7 about here

- - - - - - - - - - - - - - - -

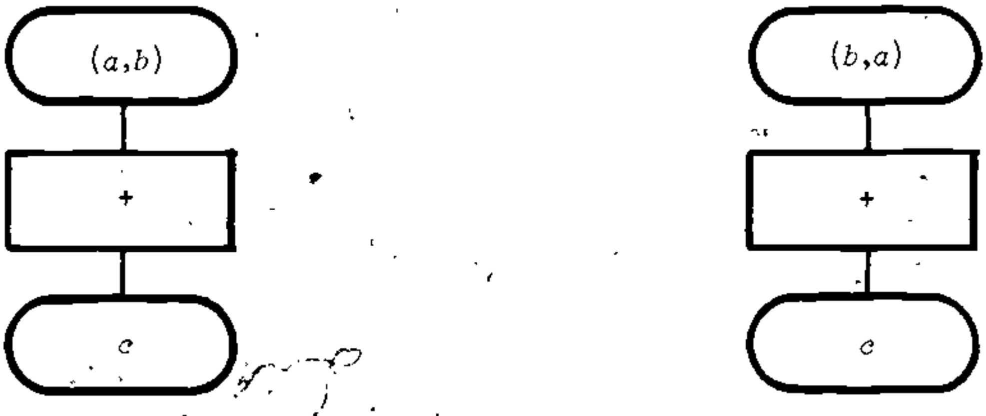

Figure 6.1. $a + b = c \leftrightarrow b + a = c$

Figure 6.2. $7 + 2 = 9 \leftrightarrow 2 + 7 = 9$

Figure 6. Some equivalent sentences that are based upon the commutativity of natural- or whole-number addition.

Figure 7.1. $a \underline{+ b} = c \leftrightarrow b \underline{+ a} = c$

Figure 7.2. $7 \underline{+ 2} = 9 \leftrightarrow 2 \underline{+ 7} = 9$

Figure 7. Some equivalent sentences that are based upon a "pseudocommutative" property of certain natural- or whole-number unary operations. (Note that the operators "+ a" and "+ b" signify classes of unary operations, whereas the operators "+ 2" and "+ 7" sig- signify particular unary operations.)

Just as "binary subtraction" is *non*commutative, so "unary subtraction" is *non*pseudocommutative.

Within the *natural*-number domain, where a and b are natural numbers,

$$a - b \neq b - a.$$

This is equally true within the *whole*-number domain except for those whole numbers a and b such that $a = b$.

Again, where a and b are *natural* numbers,

$$a \underline{- b} \neq b \underline{- a}.$$

This is equally true for all *distinct* whole numbers a and b.

In connection with unary operations, however, there is a significant property in which unary addition and subtraction *operators* are *commutative*. This is illustrated in Table 12.

- - - - - - - - - - - - - - - - - - -

Insert Table 12 about here

- - - - - - - - - - - - - - - - - - -

4. It is commonplace to assert, *erroneously*, that binary addition and subtraction are "inverse operations." It will be clear from Figures 8 and 9 why such an assertion is untrue.

- -

Insert Figures 8 and 9 about here

- -

In no way does Figure 8.1 imply 8.2, or Figure 8.3 imply 8.4, or Figure 9.1 imply 9.2, or Figure 9.3 imply 9.4. Figures 8.2, 8.4, 9.2 and 9.4 are, in fact, nonsensical. A binary operation (- or + in these instances) is *not* a mapping from a single number to an ordered pair of numbers, as each of the questionable figures suggests, which is "backwards" from the

TABLE 12

Commutativity of Certain Unary Operators

1	$x \underline{+a} \underline{+b} = x \underline{+b} \underline{+a}$; i.e., $(x \underline{+a}) \underline{+b} = (x \underline{+b}) \underline{+a}$
2*	$x \underline{+a} \underline{-b} = x \underline{-b} \underline{+a}$; i.e., $(x \underline{+a}) \underline{-b} = (x \underline{-b}) \underline{+a}$
3*	$x \underline{-a} \underline{+b} = x \underline{+b} \underline{-a}$; i.e., $(x \underline{-a}) \underline{+b} = (x \underline{+b}) \underline{-a}$
4*	$x \underline{-a} \underline{-b} = x \underline{-b} \underline{-a}$; i.e., $(x \underline{-a}) \underline{-b} = (x \underline{-b}) \underline{-a}$

* The stated property is valid for each *proper* (Lay, 1966) operand; i.e., each operand for which a particular unary operation (or class of unary operations) is defined.

For instance

1	$9 \underline{+2} \underline{+6} = 9 \underline{+6} \underline{+2}$; i.e., $(9 \underline{+2}) \underline{+6} = (9 \underline{+6}) \underline{+2}$
2	$9 \underline{+2} \underline{-6} = 9 \underline{-6} \underline{+2}$; i.e., $(9 \underline{+2}) \underline{-6} = (9 \underline{-6}) \underline{+2}$
3	$9 \underline{-2} \underline{+6} = 9 \underline{+6} \underline{-2}$; i.e., $(9 \underline{-2}) \underline{+6} = (9 \underline{+6}) \underline{-2}$
4	$9 \underline{-2} \underline{-6} = 9 \underline{-6} \underline{-2}$; i.e., $(9 \underline{-2}) \underline{-6} = (9 \underline{-6}) \underline{-2}$

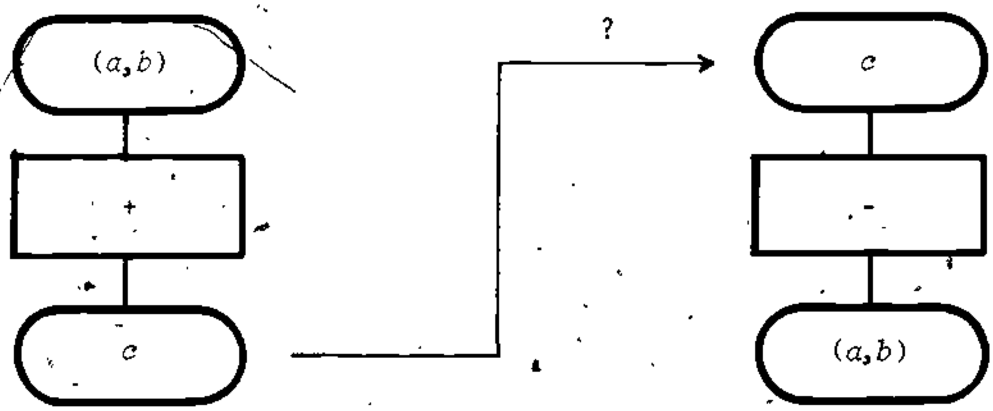

Figure 8.1

Figure 8.2

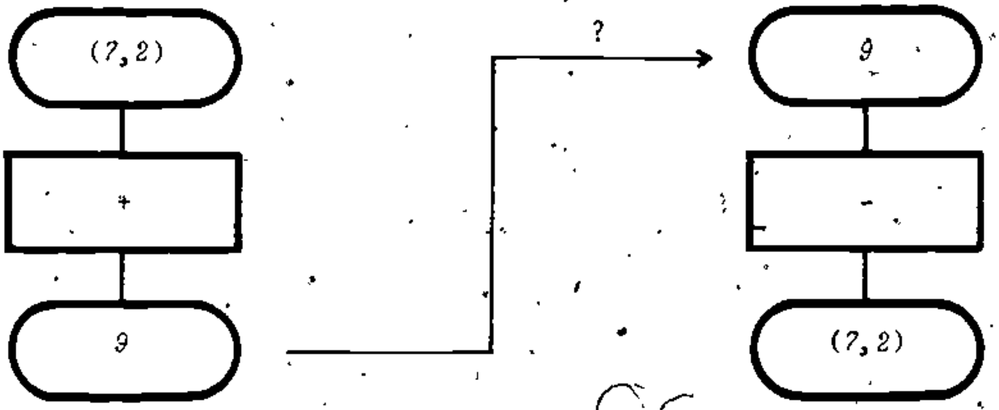

Figure 8.3

Figure 8.4

Figure 8. Binary subtraction is *not* the *inverse* of binary addition.

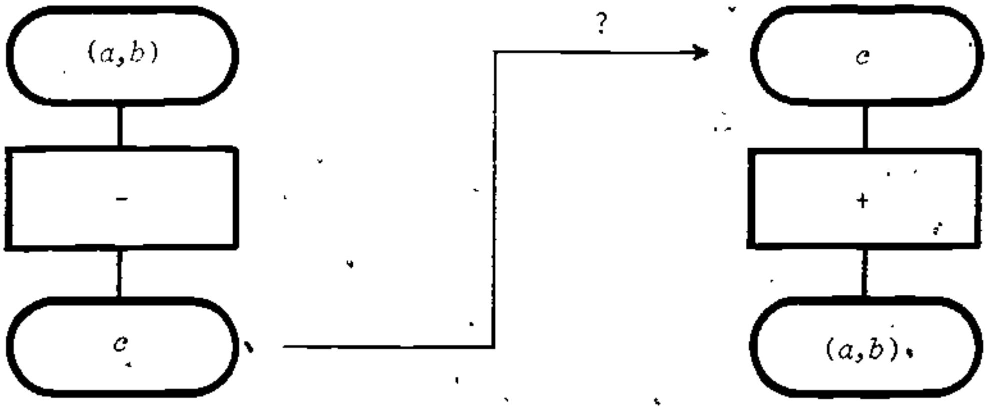

Figure 9.1

Figure 9.2

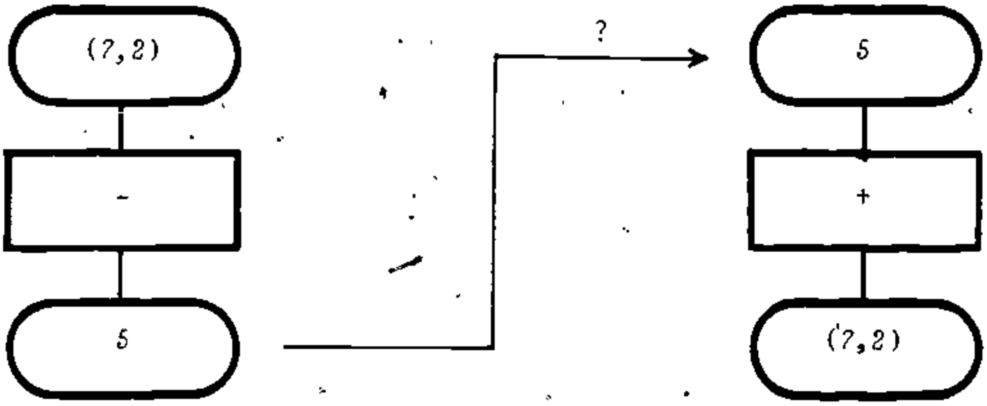

Figure 9.3

Figure 9.4

Figure 9. Binary addition is *not* the *inverse* of binary subtraction.

correct interpretation of binary operations suggested by Figures 8.1, 8.3, 9.1 and 9.3.

However, there are infinitely many pairs of *unary* operations that *are* inverses of each other, exhibiting for proper operands the relationships inherent in Figure 10,--which become more particularized in Figures 11 and 12.

Insert Figures 10, 11 and 12 about here

The relationships governing Figures 10, 11 and 12 are those that permit us to make assertions such as the following within the natural- and whole-number domains (taking cognizance of proper operands when necessary):

1. $a \; \underline{\nabla\, b} \; \underline{\Delta\, b} = a$
2. $a \; \underline{\Delta\, b} \; \underline{\nabla\, b} = a$
3. $a \; \underline{+\, b} \; \underline{-\, b} = a$
4. $a \; \underline{-\, b} \; \underline{+\, b} = a$
5. $7 \; \underline{+\, 2} \; \underline{-\, 2} = 7$
6. $7 \; \underline{-\, 2} \; \underline{+\, 2} = 7.$

(It is so tempting to use the preceding statements as an excuse to get into the *composition* of unary operators, starting with something like Figures 13 and 14, but I shall resist the urge to go any further with that.)

Insert Figures 13 and 14 about here

---

5. Within the domain of natural or whole numbers, consider assignments or correspondences of the forms

$$(a,b) \xrightarrow{+} c \quad \text{and} \quad (a,b) \xrightarrow{-} c\,.$$

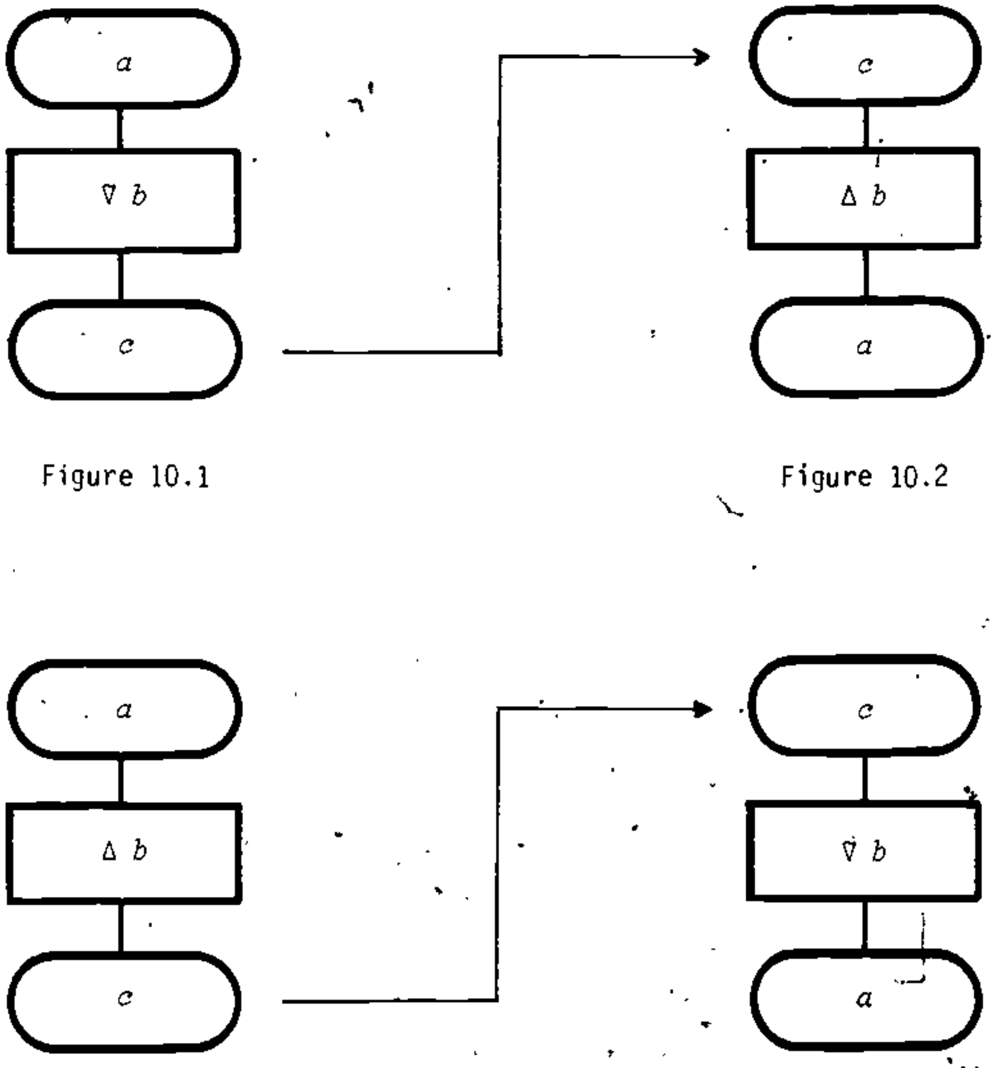

Figure 10.1 Figure 10.2

Figure 10.3 Figure 10.4

Figure 10. For proper operands, pairs of unary operations that are related to each other as illustrated by 10.1 and 10.2 and by 10.3 and 10.4 are *inverses* of each other.

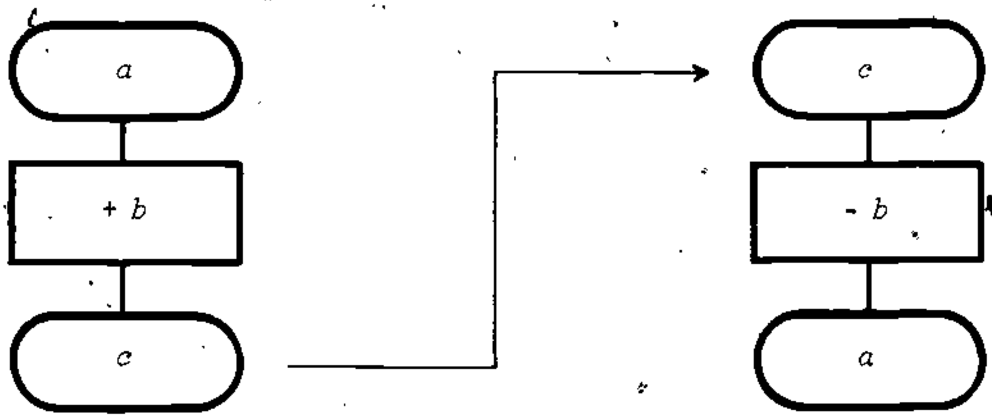

Figure 11.1

Figure 11.2

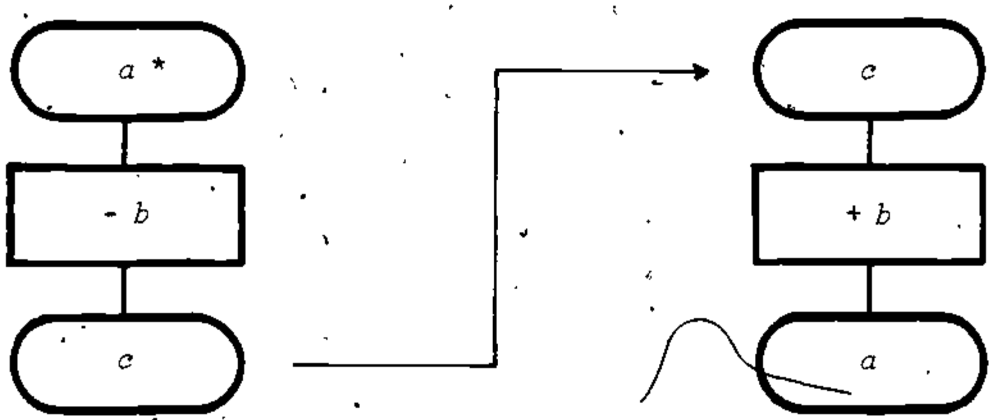

Figure 11.3

Figure 11.4

Figure 11. Unary operations associated with the operators "+ *b*" and "- *b*" are inverses of each other.

* It is assumed that *a* is a proper operand.

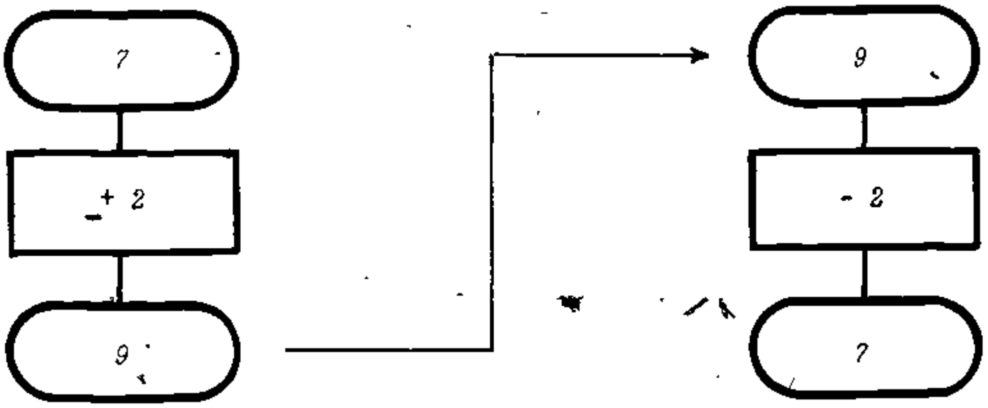

Figure 12.1 Figure 12.2

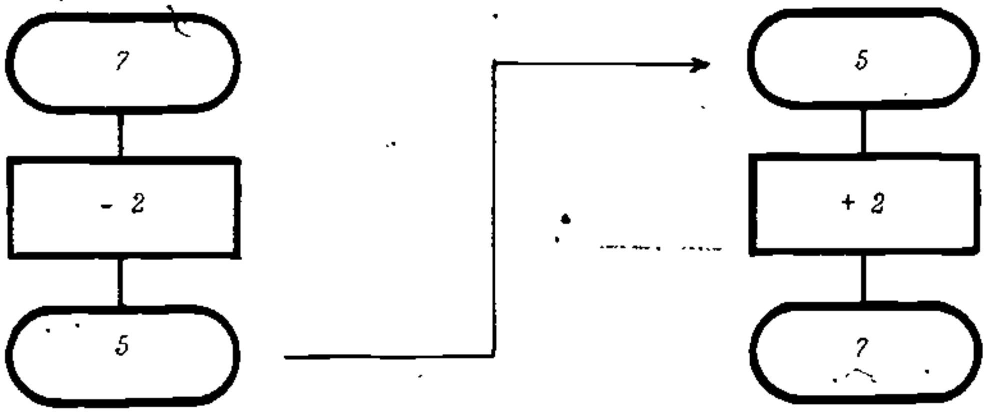

Figure 12.3 Figure 12.4

Figure 12. An illustration of the unary operations associated with the operators "+ 2" and "- 2" as inverses of each other.

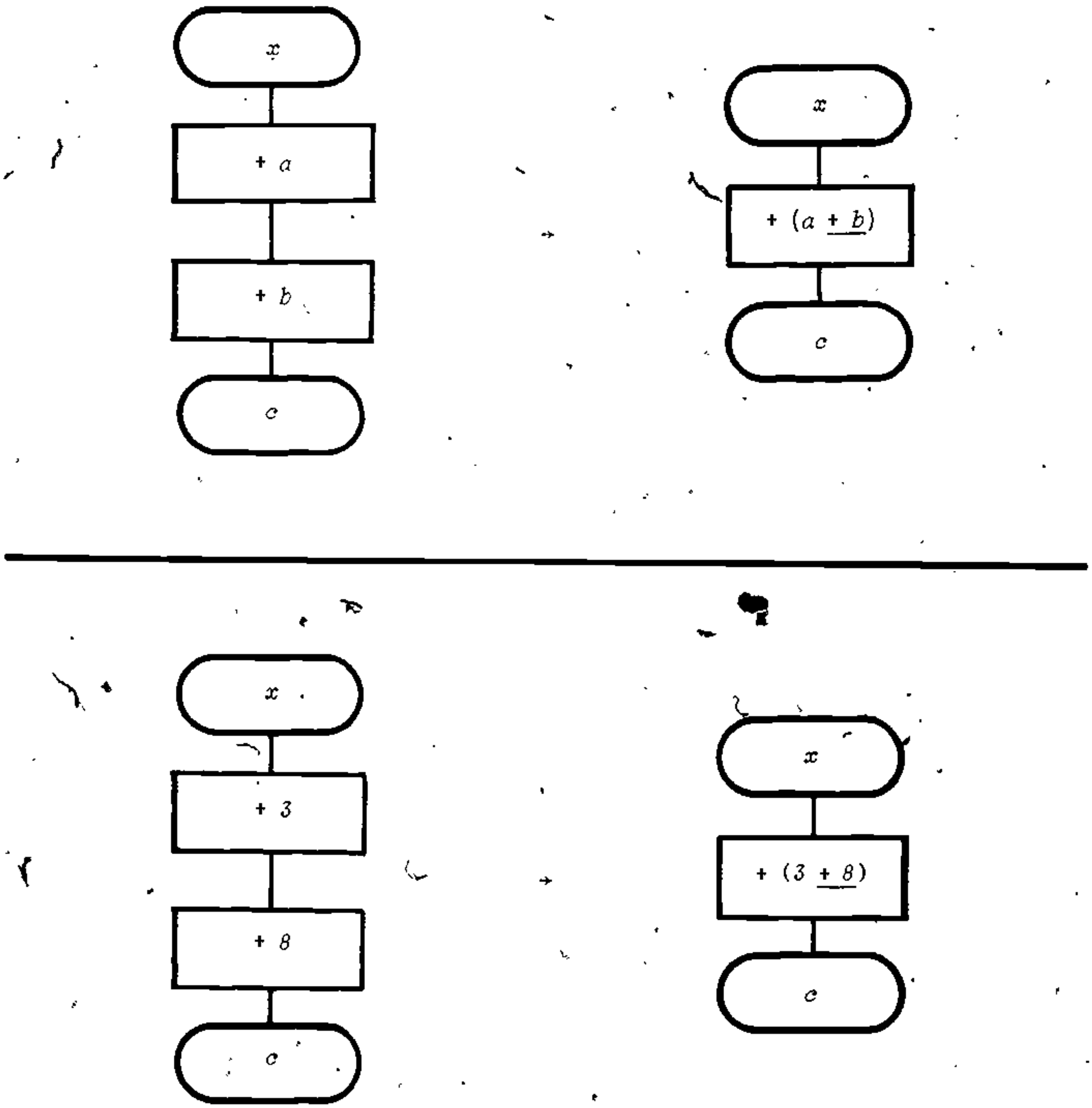

Figure 13. Compositions of unary addition operators

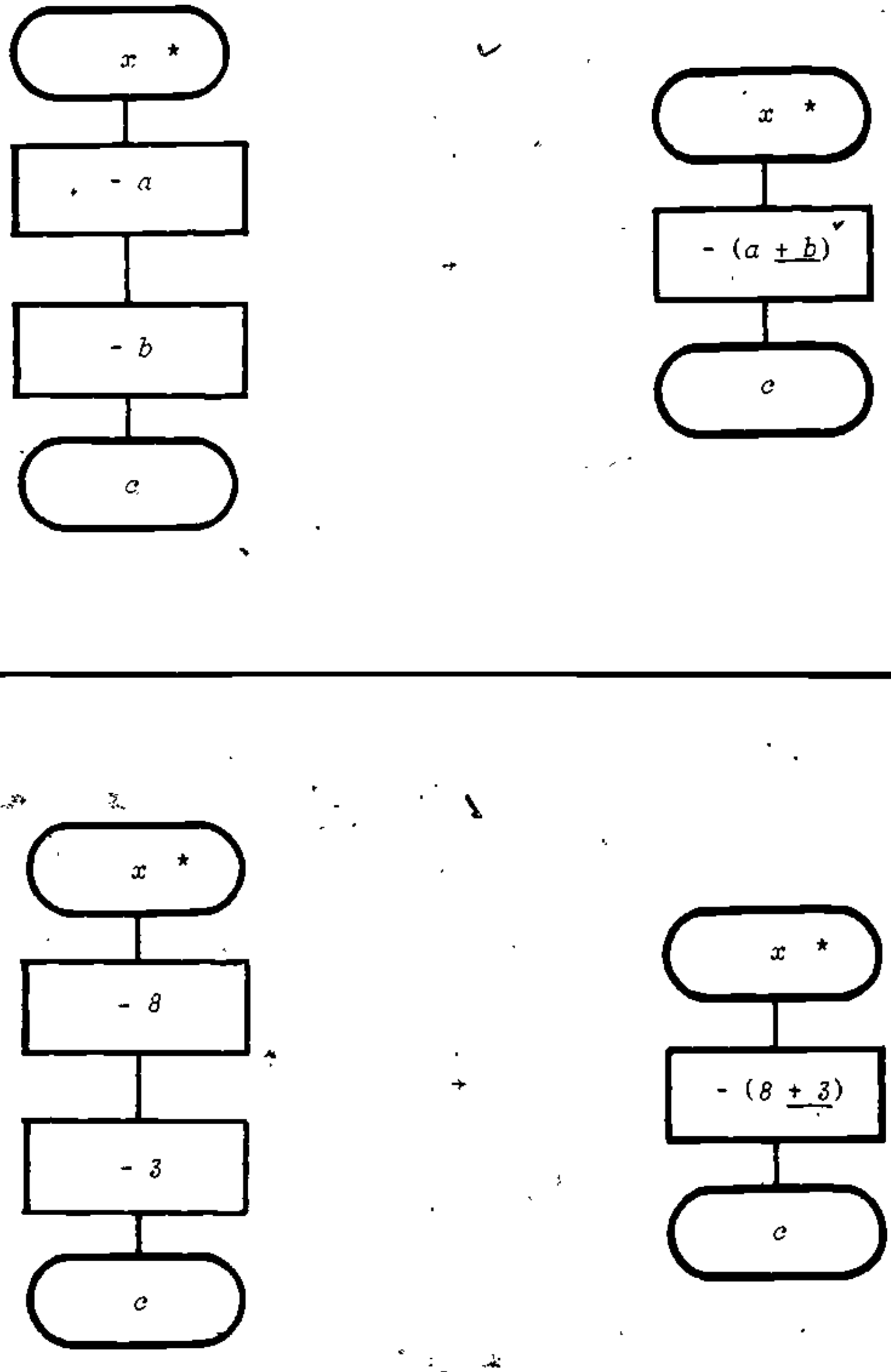

Figure 14. Compositions of unary subtraction operators

* x is assumed to be a proper operand.

In the case of any binary addition assignment, it is pointless to even raise the question of whether "adding" makes more; and in the case of any binary subtraction assignment, it is pointless to raise the question of whether "subtracting" makes less.

In neither case is there a basis for comparing the magnitude of c, a single number, with that of the ordered pair (a,b). In no case can it be asserted that $c > (a,b)$ or that $c = (a,b)$ or that $c < (a,b)$. In each instance the relational expression is senseless.

The situation is somewhat different, however, for unary operations.

First consider mappings of the form

$$a \xrightarrow{+\ b} c.$$

Within the natural-number domain, for every b,--i.e., for every unary operation,--it is true for every a that $c > a$. Within that domain, then, the <u>process</u> of "adding b" *always* "makes more." The same is true for the whole-number domain except when $b = 0$.

Now consider mappings of the form

$$a \xrightarrow{-\ b} c.$$

Within the natural-number domain, for every b,--i.e., for every unary operation,--it is true for every <u>proper</u> <u>operand</u> a that $c < a$. Within that domain, then, the <u>process</u> of "subtracting b" *always* "makes less." The same is true for the whole-number domain except when $b = 0$.

These "change of state" interpretations associated with unary operators of the forms "+ b" and "- b" will receive more extended consideration in Part II of this paper.

In Conclusion

In Part I of this paper I have emphasized an ambiguity of interpretation of number sentences of the forms $a + b = c$ and $a - b = c$ within the domains of natural and whole numbers. Particular attention has been given to the relatively neglected unary-operator interpretation of such sentences as contrasted with more familiar binary interpretations.

I believe that, in the main, my consideration has been consistent with Nesher's (1972) view of this ambiguity in her significant analysis of "What does it mean to teach '2 + 3 = 5'?" Admittedly, she prefers to characterize a binary operation as an assignment *rule* (p. 75) rather than as a set of *assignments* (which I prefer for reasons identified in an early section of Part I). But as Nesher has indicated:

"To summarize, in analyzing the phrase '2 + 3' which is a complex name, two main interpretations are found:

"(1) Plus as a binary operator:

$F(a,b)$ where F is '+', $a = 2$ and $b = 3$.

"(2) Plus as a component of a functor:

$F(a)$, where F is '+ 3' and $a = 2$."[6]

"The last two interpretations in regard to the operation sign and its sense are not contradictory, and in fact, since they are a function of one or two arguments, it is more a matter of formulating the function than making a real distinction." (p. 76)

True,--certainly at *our* level of mathematical perception. But I leave as rhetorical for the present the question of whether a "real distinction" exists in the thinking of children, particularly during their embryonic stage(?) of mathematical conceptual development.

However, it does not seem unreasonable to believe that at least some of the contrasts summarized below between binary and unary interpretations of number operations are of consequence in relation to children's thinking.

	Binary	Unary
1. How many operators,--and therefore how many operations,--are involved?	Two: + and -	Infinitely many
2. To how many numbers is any particular operator applied?	Two (ordered pair)	One
3. How many numbers result when a particular operator is applied to a particular number (or pair)?	One	One
4. Within the domains of definition, for every operator does there exist a unique inverse operator?	No	Yes
5. From the standpoint of operations as mappings or sets of correspondences, can the magnitude of every image be compared with the magnitude of its pre-image?	No	Yes

And when we also take into account binary-unary contrasts pertaining to the commutativity concept, we increase the likelihood of dealing with distinctions that are nontrivial in connection with the development of children's thinking about number operations,--in particular, about "addition" and "subtraction."

Resnick & Ford (in press) indicate that "we must understand something about mathematics as the mathematician views it" (p. 4 of typescript). One *mathematics educator's* interpretation of that view as presented in Part I of this paper has focused upon mathematical (as contrasted with Piagetian) conceptualizations of *operations* and some of their properties.

PART II

Some Research Considerations

I have been impressed for some time by the markedly different points of view expressed below, which would spawn markedly different approaches to the development of mathematical skills and abilities among young children.

I

"The objective for mathematics instruction in the elementary grades is familiarity with the [properties/structure of the] real number system and the main ideas of geometry" (p. 31), using the real-number line from the outset in grades K-2, with attention also given at that level to "Symmetry and other transformations leaving geometrical figures invariant" (p. 33) with "possibly the explicit recognition of the group property" therein (p. 34).

II

"I now think that it is fallacy of mathematics curriculum development *for young children* that logical organisation of the subject determines its pedagogical organisation. When a child learns mathematics via firsthand experiences with real things, the reality of the context provides him with all he may need *at that time* to make sense out of what he is learning. . . . I believe that children need a protracted period in which to work with real things and discover mathematical facts. For some children, these may be isolated facts; for others, the facts may point to generalisations. . . .

"There does come a time when a child should bring generalisations together and see that they are linked in logical structures. It is difficult to determine when this should happen. I am convinced from my own observation and from what I know of psychological findings that, although the appropriate time will differ from child to child, we should not begin a serious search for children who are ready for structural organisation of generalisations until they have had four to five years of elementary education behind them. (The fact that one has heard of a mathematician's nephew who could cope with these abstractions when he was seven years old is not a sign that one should build a curriculum designed to bring all seven-year-olds to this level.)" (p. 28)

Position (I) was excerpted from the "official" report of the well known Cambridge Conference on School Mathematics (1963). Position (II) was expressed by the late Max Beberman (1971), erstwhile Director of the University of Illinois Committee on School Mathematics (UICSM). Max's assertions represent a distinct shift from an earlier point of view, and he might have been inclined to express the same feeling that Snoopy did in 1979 (*see page 48 of this manuscript*).

A preponderance of the theoretical frameworks and the research that are of concern to us in this seminar suggest to me a tenor-of-the-times that is much more in tune with the Beberman position than with that of the Cambridge Conference on School Mathematics. And that is very good, I believe.

Some years ago Rappaport (1962) cautioned that "Too much concern must not be centered upon mathematics as a logical subject with too little emphasis on the child as a learner" (p. 69). Several years later Rappaport (1967) took me to task for one of my articles (prompted in large measure by another one of his!) in which he contended that "Weaver gives first priority to logic over and against psychology" (p. 682), somewhat gratuitously adding "although he may not have intended to do so" (p. 682). And just to be certain that I was sufficiently admonished, toward the conclusion of the same paper Rappaport reiterated that "Weaver emphasizes logic at the expense of psychology" (p. 684).

Just to set the record straight: If there were any basis in fact for Rappaport's 1967 contention, then today I too must say: *How embarrassing. I was barking up the wrong tree!* . . .

(You may doubt this after observing a certain degree of fussiness in connection with some of my considerations in Part I of this paper. In any event, I hope that I emphasize neither *at the expense of* the other.)

([illegible] from 03704[illegible])


ARF ARF
© 1978 United Feature Syndicate, Inc.
NOT HERE... OVER THERE!
REALLY?
HOW EMBARRASSING
I WAS BARKING UP THE WRONG TREE!
SCHULZ

Delimiting an Area for Further Investigation

I am delighted when I read Ginsburg's (1979) conviction that

"*A crucial aspect of learning mathematics is learning to perceive.* Children need to learn not only how to execute calculations. They must learn to see how numbers behave, and to detect underlying patterns and regularities" (p. 168),

although I wish we had reached a point where it would no longer be necessary to add that

"This aspect of mathematics education--accurate perception--does not receive sufficient attention" (Ginsburg, 1979, p. 168).

Meaning and *understanding*[7] have not always been welcome or considered necessary or even desirable in the mathematical education of students,--particularly young children. It was my privilege to have worked closely at one time or another with persons such as B. R. Buckingham and W. A. Brownell whose work pioneered an emphasis upon meaning and understanding in elementary mathematics many years ago (Buckingham, 1938; Brownell, 1935; 1937, 1945, 1947).

It was Brownell (1935) who was the "architect" of that which he termed the "meaning theory" of arithmetic instruction, indicating that

This theory makes *meaning*, the fact that children shall see sense in what they learn, the central issue in arithmetic instruction.

"The 'meaning' theory conceives of arithmetic as a closely knit system of understandable ideas, principles, and processes. According to this theory, the test of learning is not mere mechanical facility in 'figuring.' The true test is an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significance (p. 19, italics mine)."

My major professor, although not in the field of mathematics education, contended more generally that

"The attainment of rich meaning and comprehension and understanding is itself one of the major goals of education. It is not merely a means to more fundamental pedagogical goals.

"A rich store of meanings, of comprehensive understandings, and of functioning insights is one of the greatest gifts that the school can bestow on the student (Stephens, 1951, p. 386)."

But even today there are those who do not give things such as meaning and understanding central roles in mathematical learning,--persons who with respect to mathematical learning take a position seemingly akin to that of Bugelski (1964) with respect to learning in general:

"Learning psychologists do not discuss understanding because they have no way of discriminating between understanding and misunderstanding. They are concerned only with with right and wrong answers. . . . In brief, misunderstanding and understanding can occur with exactly the same feeling of assurance or knowledge. If the teacher asks a student if he has 'the idea,' the student can say 'yes' in either case. . . . 'A difference that makes no difference is no difference.' In this sense, there is no difference between understanding and misunderstanding (p. 202)."

"Learning can take place whether or not a student 'understands.' Understanding does not contribute anything but a feeling of satisfaction that can be enjoyed even if the student 'misunderstands' (p. 204)."

You now may be able to sense more clearly why I said "I am delighted when I read Ginsburg's (1979) conviction that . . .;" and why I also am delighted to encounter Greeno's (1977) consideration of "the process of understanding," and to realize (among other instances I might cite) that one of the two principal sections of Resnick & Ford's (in press) forthcoming book deals substantially with "mathematics as conceptual understanding."

And now, coming more to the point:

What area is being delimited for further investigation?

I believe that today Henry Van Engen would say substantially that which he did 30 years ago (Van Engen, 1949):

"The whole object of arithmetic instruction clearly is to help the child devise a system of symbols which, in some sense, is representative of a realm of events . . . with which the child has had direct experience." The symbolized events, which "are predominantly concerned, on elementary levels, with overt acts and images acquired as the result of experiences with the manipulation of objects," "are the primary instruments of knowledge" (pp. 325-326).

Specifically my concern is with the "system of symbols" identified in Table 13 (especially the ~~op~~ *left* column) and with a particular operational interpretation,--a particular operational *meaning*,--associated with that symbol system as it is used with natural or whole numbers, set *N* or set *W*.

- - - - - - - - - - - - - - - -

Insert Table 13 about here

- - - - - - - - - - - - - - - -

Even more specifically, my concern is with a unary-operator change-of-state interpretation of the symbol system as overviewed in Table 14.

- - - - - - - - - - - - - - - -

Insert Table 14 about here

- - - - - - - - - - - - - - - -

Before being more explicit about that which I believe is in need of further investigation, I would like to identify some of the research research reports and theoretical papers that relate in some way to young children and to tasks associated with number "operations." Since a

TABLE 13

Some Types of Simple Number Sentences

Op-left sentence form	Op-right sentence form
Closed sentences	
1. $a + b = c$	1'. $c = a + b$
2. $a - b = c$	2'. $c = a - b$
Open sentences	
1. $a + b = \square$	1'. $\square = a + b$
2. $a + \square = c$	2'. $c = a + \square$
3. $\square + b = c$	3'. $c = \square + b$
4. $a - b = \square$	4'. $\square = a - b$
5. $a - \square = c$	5'. $c = a - \square$
6. $\square - b = c$	6'. $c = \square - b$

TABLE I4

Unary-operator Change-of-state Interpretations for Open Number-sentence Types

Conventional closed-sentence form	Conventional open-sentence form*	Condition (if any)		Change-of-state situation		
		Within N	Within W	Initial state	Operator	Final state
	1. $a + b = n$			a	$+ b$	n
A. $a + b = c$	2. $a + n = c$	$a < c$	$a \leqslant c$	a	∇n	c
	3. $n + b = c$	$b < c$	$b \leqslant c$	n	$+ b$	c
	4. $a - b = n$	$a > b$	$a \geqslant b$	a	$- b$	n
S. $a - b = c$	5. $a - n = c$	$a > c$	$a \geqslant c$	a	∇n	c
	6. $n - b = c$			n	$- b$	c

* Often $\square$ is used in place of n.

N is the set of *natural* numbers; W, the set of *whole* numbers.

forthcoming publication from the Wisconsin R & D Center's Mathematics Work Group[8] will include virtually all of my references among the many that are analyzed and synthesized, I will avoid duplication of that effort by doing little more that alluding to most sources here in Part II of this paper.

Investigations with task stimuli that are exclusively symbolic exemplars of certain sentence types identified in Table 13:

Reports of my own normative investigation of certain task and other variables potentially associated with pupil performance on exemplars of selected open-sentence types (Weaver, 1971, 1972, 1973; Note 1) showed a degree of differential performance with sentence types that suggested some conceptual inadequacies, or whatever. A subsequent categorization of *incorrect* responses that was reported at a much later time (Weaver, Note 2) identified certain kinds of errors as being more commonplace than others; but in no way could there be inferred anything regarding pupils' interpretation of sentences in binary or unary terms, or some (probably garbled) mixture of the two.

It should be noted that for the preceding investigation as well as for others to be identified, the *principal* domain from which number-sentence exemplars have been drawn has been that which we commonly call the "basic addition and subtraction facts." Also, the domain of subjects has been *principally* that of the primary grades.

Findings from use of symbolic exemplars of number-sentence types as stimuli for an entirely different purpose (than Weaver's) have been reported by Groen (1967), Suppes & Groen (1967), Suppes, Hyman, & Jerman (1967), Jerman (1970), Groen & Parkman (1972), Groen & Poll (1973), Rosenthal (1974), Woods, Resnick, & Groen (1975); and by Svenson (1975), Svenson & Broquist (1975), Svenson, Hedenborg, & Lingman (1976). In these investigations linear regression analyses have been applied to performance data in the form of response latencies in an attempt to test (and subsequently refine), the validity of certain *hypothesized* algorithms,--chiefly counting models,--as procedures for solving exemplars of open-sentence types.

Alderman (1978) has reported findings from application of an alternative "tree search" model to the solution of exemplars of "addition" open sentences.

(It may be of significance to note that the "response latency" investigations have been conducted virtually without exception by psychologists rather than mathematics educators. This may have a bearing upon both the intent of such investigations and the interpretation of findings therefrom, along with implications and suggestions for further investigation.)

At this time in this paper there is but one kind of question that I wish to raise in connection with the response-latency investigations: Is there any relation between hypothesized performance algorithms or models and subjects' (experimenter-anticipated or actual) interpretation of stimuli in terms of "operations?" If a binary conceptualization were involved, would this suggest the testing of different hypothesized algorithms or models than if a unary conceptualization were involved (and vice versa)?

It is recognized, I am sure, that in research reports, position papers, etc., not all persons use terms such as *addition, subtraction, operation*, and the like in the same way in which I characterized them in Part I of this paper. This should be kept in mind when interpreting some of the material I shall identify in the next section, where I may refer to "addition," "subtraction," etc. in the sense that a particular investigator does rather than in a strict mathematical sense as a mapping or function.

Other investigations with exemplars of number-sentence types as principal or significant stimuli:

Groen & Resnick (1977) reported two experiments on addition algorithm invention, with five children whose average CA was less than five years as subjects in each experiment.

Grouws's (1974) report of solution methods used by children when solving exemplars of certain open-sentence types gave no hint of binary vs. unary conceptualizations of the operations involved.

Lindvall & Ibarra (Note 3) attributed variation and error in the way in which pupils read open-sentences to different interpretations of "+" or "-" which appear to be associated with binary vs. unary conceptualizations, but were not discussed in such terms explicitly.

Hamrick's (1979) report gave no particular indication of the conceptualization(s) of addition and subtraction for which written-symbol readiness was developed.

Concern for binary vs. unary conceptualizations is implicit (but never explicit that I could find) in reported work from the Project for the Mathematical Development of Children (PMDC) pertaining to the equality relation and closely allied material (Anderson, 1976; Barco, 1977; Behr, Erlwanger, & Nichols, Note 4; Campbell, 1976, 1978; Denmark & others, Note 5; Gerling, 1977; Nichols, Note 6.

Piagetian "reversibility" and its relation to pupil performance on open addition and subtraction sentences was of principal interest in two investigations (Davidson, 1975; Wong, Note 7) and of less interest in another case (Woodward, 1977). In none of these instances was reversibility associated with a unary-operator rather than a binary-operator interpretation of the number sentences involved.

The "missing addend" open-sentence types (in some instances including related verbal problems also) were the particular concern of several investigations: Howlett, 1973; Peck & Jencks, 1976; Gold, 1978, Note 8 and in Case, 1978a, 1978b). In connection with none of these reports have I seen it made explicit that in relation to a unary-operator interpretation, these two forms of missing-addend sentences are conceptually quite different: $a + \square = c$

and $\square + b = c$.

Other investigations, and theoretical papers:

I shall only list a number of references in which principal interest has been in (1) some aspect of "problem solving" as it is associated with addition or subtraction or (2) the development of addition or subtraction concepts per se,--in each instance, with task stimuli that are not chiefly symbolic exemplars of number-sentence types.

Carpenter, Hiebert, & Moser, Note 9, Note 10; Carpenter & Moser, Note 11; Moser, Note 12;

Ginsburg (whose cited references cover much more than the two things just identified) 1975, 1976, 1977b; Allardice, 1977a, 1977b; Brush, 1972, 1978; Brush & Ginsburg, Note 13; Hebbeler, 1977, 1978; Kennedy, 1977; Russell, 1977;

Greeno, 1979, in press; Heller & Greeno, Note 14; Heller, Note 15; Riley & Greeno, Note 16; Riley, Note 17;

Grunau, 1975, 1978;

Kellerhouse, 1974;

Lindvall & Ibarra, Note 18, Note 19; Ibarra & Lindvall, Note 20;

Nesher & Teubal, 1975; Nesher & Katriel, 1977, Note 21;

Rosenthal & Resnick, 1974;

Shores & Underhill, Note 22; Shores, Underhill, Silverman, & Reinauer, Note 23; Harvey, 1976;

Van Engen & Steffe, Note 24; Steffe, 1968, 1970, Note 25, Note 26; LeBlanc, Note 27; Steffe & Johnson, 1971, Note 28; Steffe, Richards, & von Glasersfeld, 1979; Steffe, Spikes, & Hirstein, Note 29; Hirstein, 1978.

Suffice it to say for this paper that many of the preceding references make distinctions that could be associated with binary vs. unary interpretations of number operations, but in no instance did I find that such a distinction was made explicit.

A conviction. Binary *and* unary operations can and should be part of a person's mathematical fund of knowledge, with consideration given to each during the course of systematic instruction within the school context. It is rare to find that done in a school mathematics program (e.g., Comprehensive School Mathematics Program (CSMP), 1977, 1978) in the United States, where the "typical" program is rather procrustean in its treatment of content from a binary-operation standpoint, to the virtual exclusion of unary operations, --an exclusion that I believe is a distinct *dis*advantage when interpreting and working with certain quantitative situations.

But there are programs within the United Kingdom (e.g., Fletcher, 1970, 197) which give explicit attention to unary as well as to binary operations. And if I interpret correctly some of the Soviet work (e.g., Davydov, 1966/1975; Menchinskaya & Moro, 1965/1975), unary operations (at least in essence) have a central role to play in young students' mathematical-development programs.

Change-of-state Situations

Dienes & Golding (1966) have stated that

"A large part of mathematics consists of the study of states and the study of operators which induce these states to change into other states" (p. 35)

Such change-of-state situations,--which by one name or another were of interest in many of the references cited on the preceding page (57) of this manuscript,--seem to me to be particularly suited to interpretation in terms of unary operations and their properties (rather than in terms of binary operations and properties). If systematic intervention within the school setting is to be based upon the quantitative background that many(?) children bring to that setting, "unary addition" and "unary subtraction" concepts and skills within change-of-state contexts very well may be preferred to binary-interpreted situations for initiating instruction pertaining to number operations.

Some relevant evidence? I believe that the work of Gelman (1977, e.g.) and her associates has resulted in findings that give a good indication of the kind of preschoolers' background to which I allude. I interpret the following extensive quotes from Gelman & Gallistel (1978) to be, in the *sense*, if not the language, of unary operations applied to change of state (i.e., state-operator-state) situations:

"Young children use a classification scheme that organizes operations into those that alter number and those that do not alter number." (p. 169)

"The young child's numerical reasoning scheme . . . includes [two] operations that allow the child to deal with transformations that do alter numerosity. The first of these is addition. When young children confront an unexpected increase in numerosity, they postulate the intervention of addition . . . . In other words, they state that something must have been added" (p. 169). . .

"In order to explain unexpected increases in numerosity, the young child says that some set (containing one or more items) has been added to the original array." (p. 169)

"Just as our . . . experiments show that children know the effects of addition, they also provide evidence that young children use another number-altering operation: subtraction." (p. 172)

"The young child regards subtraction as the removal of items from a set." (p. 172)

When "children encountered sets whose numerosity was either more . . . or less . . . than the numerosity they expected" they "reliably indicated the direction of the discrepency and the operation that caused the discrepancy." Furthermore, "the children knew how to eliminate the discrepancy." . . . "When confronted with the discrepancy between an actual numerosity, n, and an expected numerosity, m, they showed that they knew that m could be converted into n by either addition or subtraction. . . . "The children reliably applied the appropriate operation. When m was less than n, they specified addition; when m was greater than n, they specified subtraction. When the difference between n and m was equal to one, the children did more than apply the appropriate operation; they also specified the number to be added or subtracted. This statement, as always, applies only when the numerosities of n and m are both small (less than or equal to four). As the difference between n and m became greater than one, the children reliably indicated that the number to be added or subtracted was greater than one, but they became less precise about the exact value of that number." (p. 173)

"We hesitate to take these results as evidence for granting young children a precise concept of the inverse. Still, much in their behavior warrants the postulation of some principle of reversibility, that is, some principle that leads the child to recognize that addition is what undoes the effect of subtraction [and vice versa ?] and to attempt to alter the arrays in a systematic fashion. What is the simplest principle that explains this repair behavior? We think it is a *principle of solvability*, or the 'you can get there from here' principle." (pp. 175-176)

"The rules that govern the child's numerical reasoning are influenced by what the child regards as belonging to the domain of mental entities that are to be reasoned about numerically. The mental entities to which the child's numerical reasoning principles apply are his representations of numerosity. Because his representations of numerosity derive from a counting procedure, he has no numerical representations corresponding to zero and the negative numbers." (p. 189)

"The young child has a limited solvability principle. He believes that a lesser numerosity may be made equivalent to a greater numerosity by means of the addition operation and that a greater numerosity may be made equivalent to a lesser numerosity by means of the subtraction operation. Embedded in this belief is the belief that addition always increases numerosity and subtraction always decreases numerosity." (p. 189)

[That is precisely the case when dealing with "unary addition" and "unary subtraction" (for proper operands) within the domain of natural numbers, *N*.]

"The child's solvability principle might incorporate the concept of the inverse operation, that is, the concept that subtraction undoes the effect of addition and vice versa. We have no real evidence one way or the other on [whether] the concept of the inverse is implicit in the child's solvability principle. All we really know is that preschoolers believe that differences in numerosity can be eliminated by either removing something from the larger array or adding something to the smaller array. Whether or not the child believes that the numerosity of what must be removed is equivalent to the numerosity of what must be added is a question for further research." (p. 190)

[I believe that some of Brush's and Ginsburg's change-of-state tasks (Brush, 1972, 1978; Brush & Ginsburg, Note 13) are related to this issue. A similar (or identical ?) conceptualization is to be found in the *equalizing* process identified by Romberg (Note 30, p. 163) and incorporated in the *Developing Mathematical Processes* (*DMP*) elementary-school mathematics program.]

Regarding the final point raised by Gelman & Gallistel, Dienes & Golding (1966) have asserted the following (which should be interpreted in terms of "unary addition" and "unary subtraction"):

"If we do an adding of three when we have just done a subtracting of three, we will get back to where we started. Similarly if we do an adding of four followed by a subtracting of four, then we will be back where we started. Teachers are very often not sufficiently aware how far from obvious this is. First of all, it is not immediately obvious that subtraction is the inverse of addition, and secondly that addition is the inverse of subtraction. Subtraction and addition are inverses of one another. These relationships need to be learned, and unless provision is made for it, the learning may not happen." (p. 39)

[Evidence of this at the *symbolic* level was quite clear in connection with one of my own explorations (Weaver, Note 31).]

The difficulty may be due, at least in part, to Dienes' (1964) contention that

"A great deal of confused thinking arises through the lack of realization of the double role of numbers, namely (1) that of describing the quantitative state of a collection and (2) that of the operation of altering such an existing state." (p. 30)

Developing a Particular Meaning for Symbolic Statements

The conceptualizations that have been discussed regarding change-of-state situations are background for the development of a unary-operator change-of-state interpretation of the symbol system overviewed previously in Table 14. In light of an observation made by Gelman & Gallistel and cited earlier, I shall restrict our consideration to the domain of *natural* numbers (N),--and leave it to the reader to make his/her own modifications if the *whole*-number domain (W) were involved instead.

Developing meaning. Van Engen (1949) has contended that

"In any meaningful situation there are always three elements. (1) There is an event, an object, or an action. In general terms, there is a *referent*. (2) There is a *symbol* for the referent. (3) There is an *indi-*

vidual to interpret the symbol as somehow referring to the referent. . . . It is important to remember that the symbol refers to something outside itself. This something may be anything whatsoever, even another symbol, subject only to the condition that in the end it leads to a meaningful act or a mental image." (p. 323)

Figure 15 is intended to convey the sense of Van Engen's contention in relation to the meaning(s) of principal interest in this paper.

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Insert Figure 15 about here

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Regions *C*, *P*, and *S* of Figure 15 suggest kinds of referents than can provide *logical* meaning (Ausubel, 1968) for symbols associated with region *U*, from which an individual derives his/her *idiosyncratic psychological* meaning (Ausubel, 1968).

From the references cited already on manuscript page 57, together with the following, one could cull a variety of potentially suitable (from *un-suitable*) referents for region *U* of Figure 15,--with the understanding that candidates for regions *C* and *P* need not be restricted to ones in which "state" is associated with a collection of discrete entities:

Gibb (1954, 1956), Reckzeh (1956), Van Engen (1955, 1963), Hartung (1959), R. Osborn (1961), Schell & Burns (1962), Williams (1963), Coxford (1965), A. R. Osborne (1966, 1973, 1976), Biggs (1967), Clarkson (1967), O'Brien (1967), Payne (Note 32), Romberg, Fletcher, & Scott (Note 33), Van Wagenen (1973), Van Wagenen, Flora, & Walker (1976), Vest (1968, 1970(a), 1970(b), 1972, 1973, 1974, 1976, 1978), Reys (1971, 1972), Fennema (1972, 1973), Marshall (1976), Sowder (1976), Ashlock (1977), Richards (1979), and Weaver (1979).

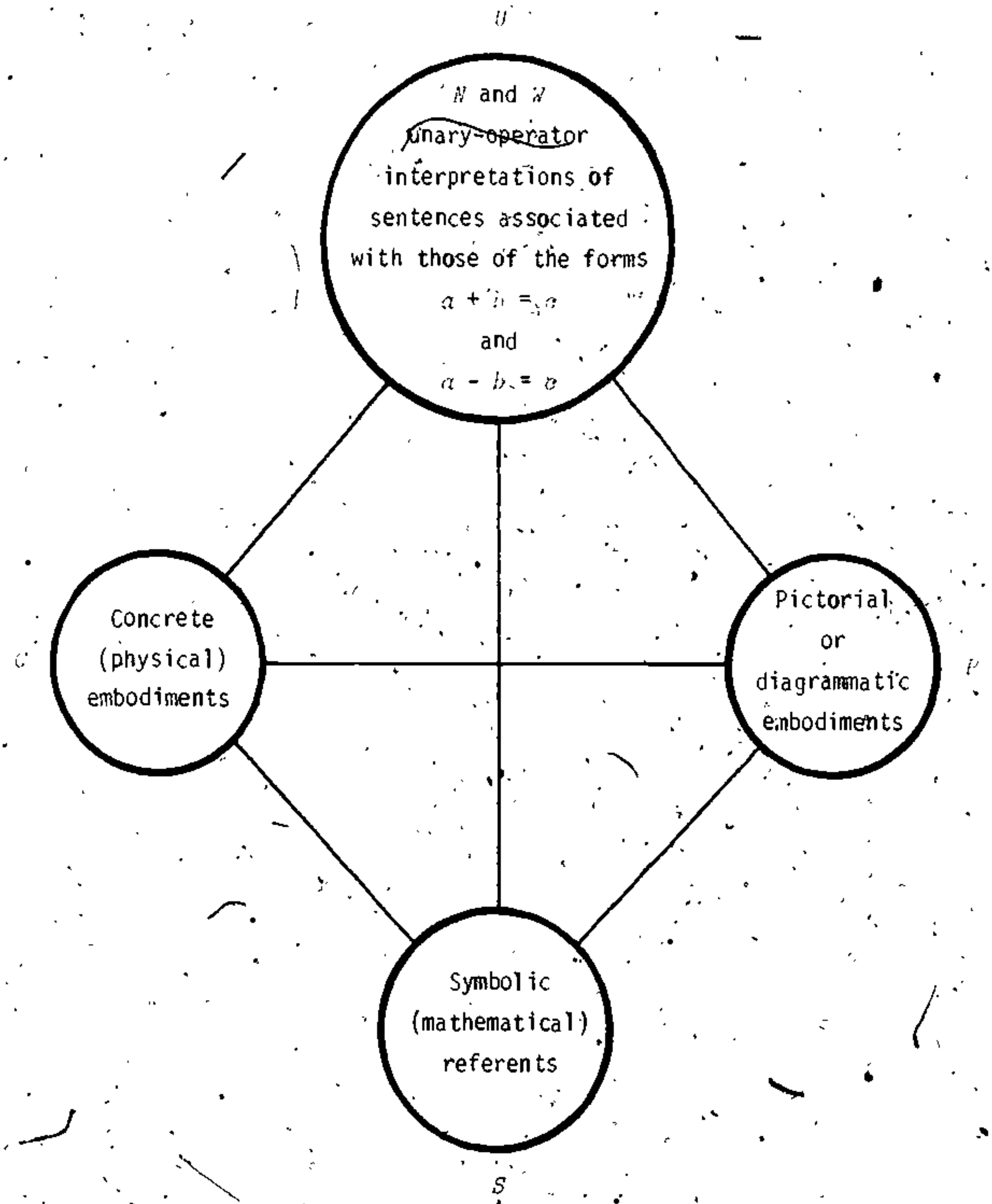

Figure 15. Referents for number sentences.

With young children we undoubtedly are concerned primarily with regions *C* and *P* of Figure 15 as referents rather than with region *S*. (Note, however, a sensible symbolic referent for "$a \underline{-\ b} = c$" is "$c \underline{+\ b} = a$" rather than "$b \underline{+\ c} = a$.").

(It also should be noted that any referents that in connection with statements such as "2 + 5 = 7" and "8 - 1 = 7" interpret "2 + 5" and "8 - 1" and "7" as *different names for the same number* are not suitable for the unary-operator change-of-state interpretations in which we are interested.)

Figure 16 suggests that sentences embedded within region *U* may be associated implicitly or explicitly with suitable situations within region *V*.

Insert Figure 16 about here

Verbally-presented "problem" situations (*V*) conceivably could be related to *U* of Figure 16 at different cognitive levels: for instance, at Avital & Shettleworth's (1968, pp. 6-7) level of *algorithmic thinking*, or at their *open search* level which is more closely associated with Resnick & Glaser's (1976) characterization of a *problem*:

"Psychologists agree that the term 'problem' refers to a situation in which an individual is called upon to perform a task not previously encountered and for which externally provided instructions do not specify completely the mode of solution. The particular task, in other words, is new for the individual, although processes or knowledge already available can be called upon for solution." (p. 209)

Thus, any *U-V* association (Figure 16) may be different for different children. It may be, in fact, that *V* does not function quite as anticipated in the development of meaning(s) within *U*. Grouws (1972), for instance,

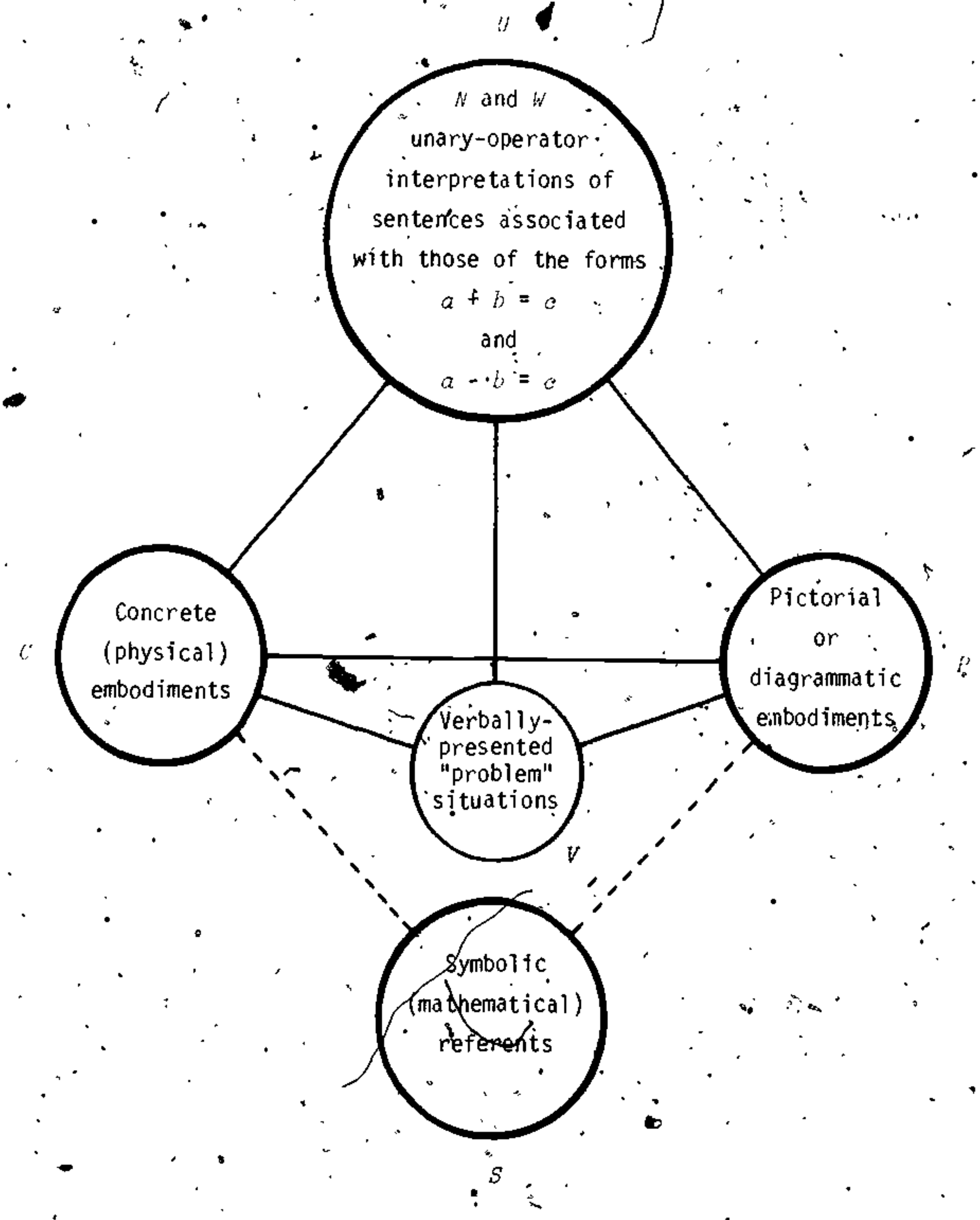

Figure 16. An extension of Figure 15.

reported that explicit association of "word problems" with open sentences to be solved appeared to have no facilitating effect upon pupils' solution performance.

A conjecture. We all have experienced instances in which there seems to be an appreciable gulf or gap (chasm-like at times) between children's comprehension of a mathematical conceptualization and their comprehension of a *symbolic representation* of that conceptualization,--especially when that representation is in conventional mathematical form. It is likely that some *mediating notational form* might be used to advantage at first, leading eventually to comprehension of the ultimate conventional form. Figure 17 is intended to convey such an idea.

Insert Figure 17 about here

The mediating notational form (M) to be suggested is one that may not only contribute to a development of meaning(s) to be associated with U of Figure 17,--the principal concern of this paper,--but also may contribute to pupils' ability to work with V as well.

From among various possibilities (arrow diagrams among them) I suggest the mediating notational form of Figure 18, which is a variation of one used previously in Part I of this paper.

Insert Figure 18 about here

For the counterpart of *open* sentences, the mediating forms would appear as in Figures 19 and 20.

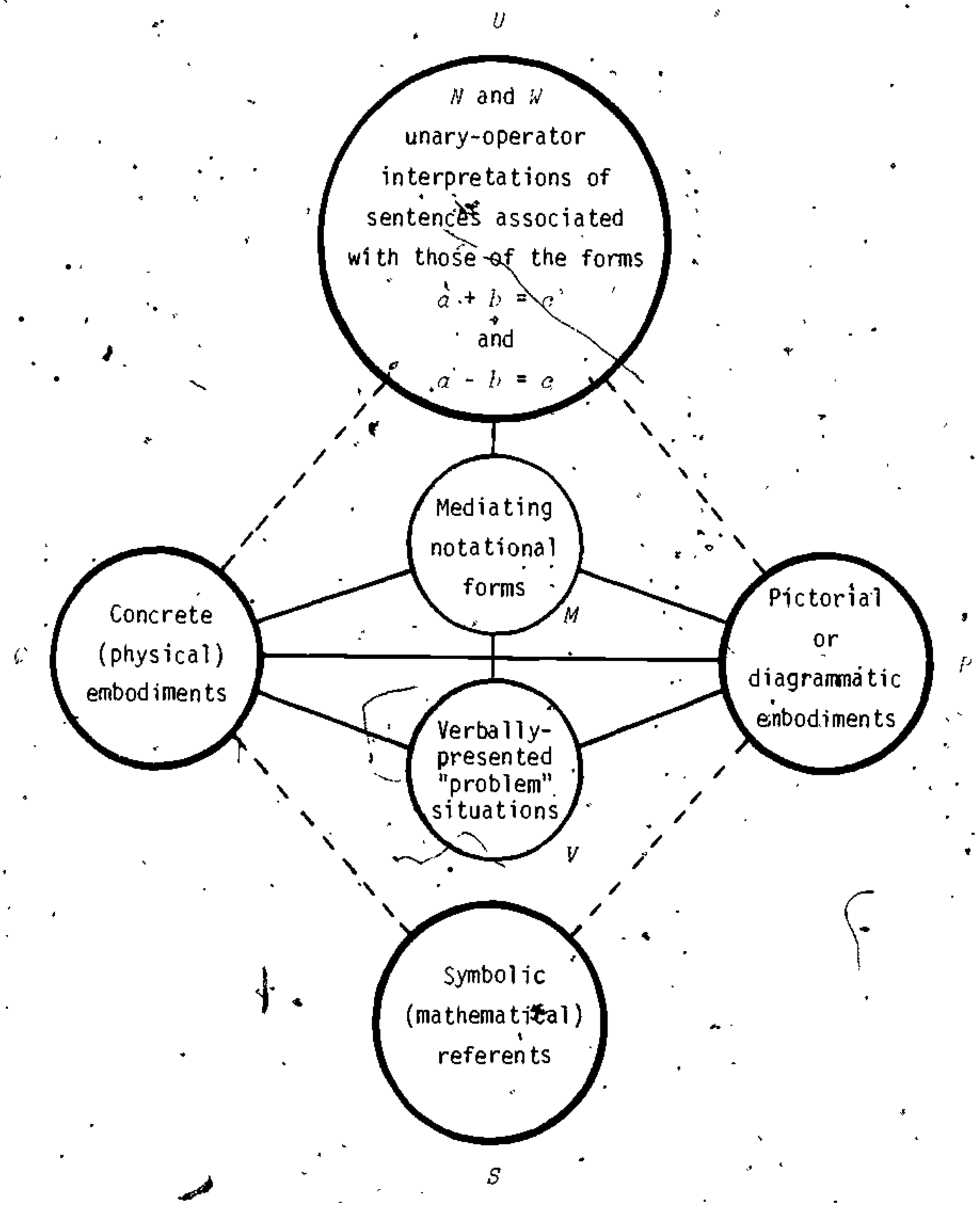

Figure 17. An extension of Figure 16.

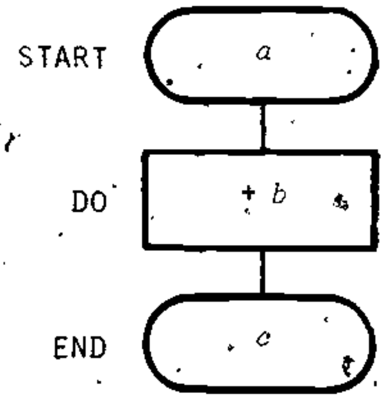

Figure 18.1 Precursor of $a + b = c$

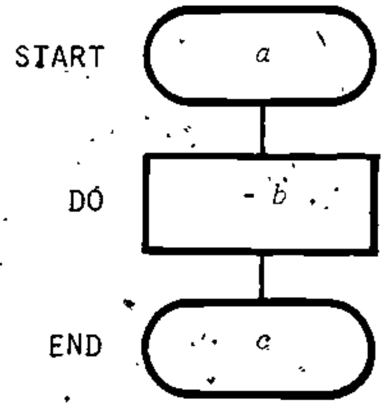

Figure 18.2 $(a > b)$ Precursor of $a - b = c$

Figure 18. Mediating notational form (Domain N)

- - - - - - - - - - - - - - - - - - -

Insert Figures 19 and 20 about here

- - - - - - - - - - - - - - - - - - -

The mediating notational forms provide a convenient systematic way of recording information given (ultimately along with information once missing) in referent situations or verbal problem situations. Forms in no way dictate the nature of such situations (within the state-operator-state context) nor do they in any way dictate strategies that may be used to cope with such situations.

Conceptualizations, relationships, properties, etc. can be "discovered" or whatever from exemplars as recorded with mediating notational forms. In some instances the essence of a property (e.g., the *inverse-operator* property) may be represented by a composite of mediating notational forms, as in Figure 21.

- - - - - - - - - - - - - - - - -

Insert Figure 21 about here

- - - - - - - - - - - - - - - - -

The transition or change to the ultimate conventional form of symbolic notation need not be hurried,--<u>should</u> not be hurried, in fact.

For some things the mediating notational form has a distinct advantage over its ultimate symbolic counterpart. Consider Figures 19.2 and 20.2, for instance: In each, <u>both parts</u> of the operator must be specified,--which is an advantage in building a conceptualization of the nature and use of unary operators.

The mediating notational form also has no troublesome "=" symbol for children to contend with.

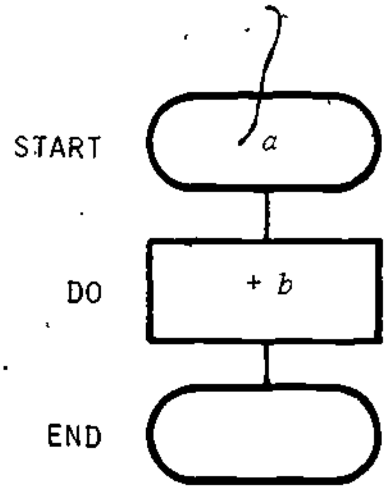

Figure 19.1 Precursor of $a + b = \square$

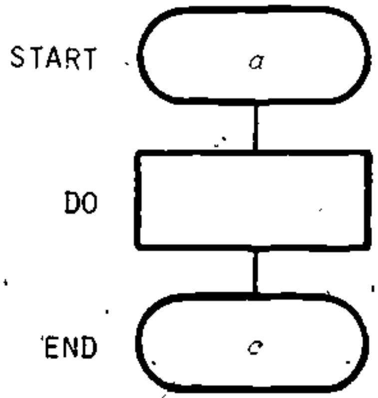

Figure 19.2 $(a < c)$ Precursor of $a + \square = c$

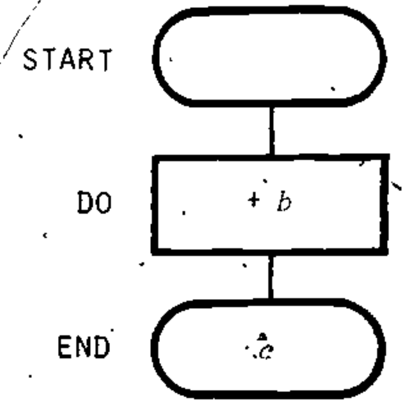

Figure 19.3 $(b < c)$ Precursor of $\square + b = c$

Figure 19. Types of "open" unary-addition situations (Domain N).

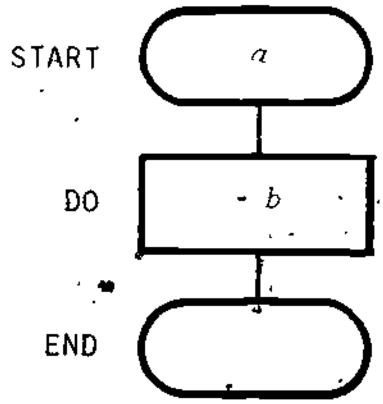

Figure 20.1 ($a > b$) Precursor of $a - b = \square$

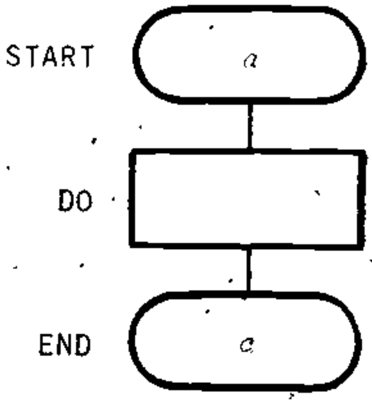

Figure 20.2 ($a > c$) Precursor of $a - \square = c$

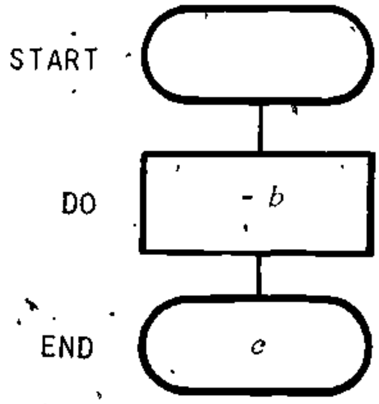

Figure 20.3 Precursor of $\square - b = c$

Figure 20. Types of "open" unary-subtraction situations (Domain N)

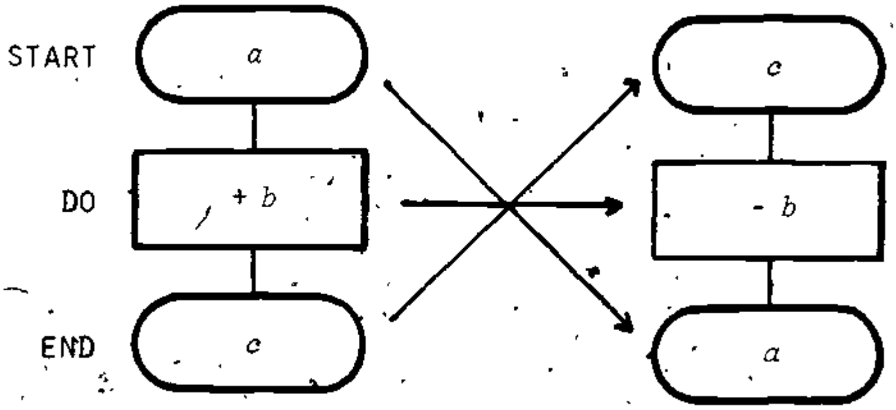

Figure 21.1

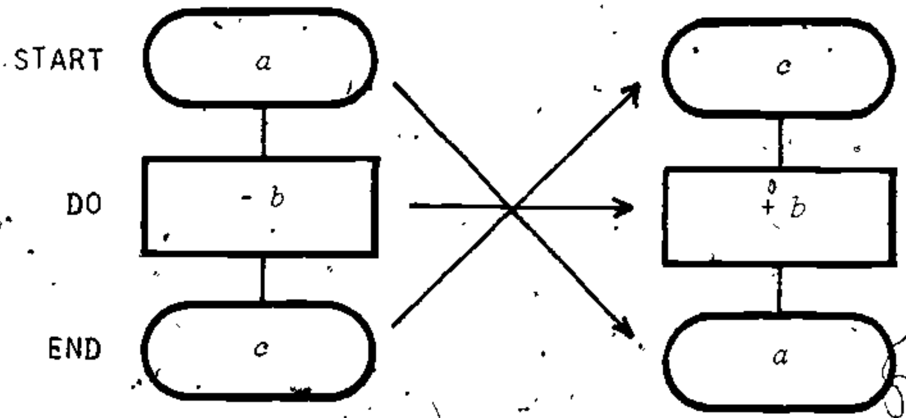

Figure 21.2 ($a > b$)

Figure 21. "$+ b$" and "$- b$" as inverse operators

In Conclusion

I have emphasized one interpretation of symbolic notation within the domain of natural numbers, or whole numbers (and identity operators)

$$a \quad + b \quad = \quad c$$

STATE OPERATOR STATE

$$a \quad - b \quad = \quad c$$

which for change-of-state situations represents that which I believe to be a promising but neglected approach to number operations for young children.

Investigations need to be designed to

(1) develop specific instructional intervention(s) pertaining to the content in question,

(2) examine the feasibility and effectiveness of such intervention(s), and (3) relate that content to other interpretations and situations pertaining to number operations.

With an initial principal focus upon *states* and *operators* rather than upon operations per se, little if any compromise will need to be made with any subsequent mathematical interpretation of operation.

Am I barking up a wrong tree? I don't think so.

Footnotes

[1] If you look at the text in which these sentences appear, you will find that (somewhat to my chagrin) I was a member of the writing team that produced them! The negative reactions that I had 15 years ago (and still have today) to the quoted characterizations simply were overruled by a majority of the writing-team members.

[2] Nothing would be gained here, for instance, by using Norbert Wiener's formal definition of the ordered pair (a,b): $\{\{\{a\},\phi\},\{\{b\}\}\}$ as cited by Buck (1970, p. 255).

[3] Brumfiel (1972) sees no real cause for the concern that persons such as Rappaport (1970) have expressed over a lack of agreement regarding the *names* applied to these two sets. It does behoove a writer (or speaker), however, to make clear the nomenclature being used.

[4] This "general" subset condition applies in the case of W. In the case of N, however, the more restricted condition that B is a *proper* subset of A $(B \subset A)$ must be imposed.

[5] Vest (1969), for instance, has developed a "catalog" of presumably different but isomorphic "models" for addition and subtraction.

[6] To some extent Nesher used different symbolism than I did. Also, where I used the "unary operator" concept, she used the functor concept from category theory.

[7] I am well aware of a distinction between *meaning* and *understanding*, and with discussions of that distinction, such as those by Hendrix (1950) and by Van Engen (1953).

[8] Conceptual Paper No. ____ by Carpenter, Blume, Hiebert, Martin, and Pimm.

[9] It is not uncommon for young children to fail to distinguish in their speaking, etc. between *set* operations (and related language) and *number* operations (and related language). This failure to distinguish between these two markedly different things is evidenced at times among non-children as well.

* Compositions have been considered by Lay (1966), Dienes & Golding (1966), and more recently by Vergnaud & Durand (1976) and Vergnaud (1979).

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(JFW's note: Are they *place-holders* that are "missing?")

Addendum

A revision of the Reference Note 9 document has been released as Tech. Rep. No. 516, Wisconsin Research and Development Center for Individualized Schooling, October 1979.

APPENDIX A

Definition of Subtraction

$$a - b = n \leftrightarrow n + b = a \quad \text{or} \quad a = n + b$$

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$$a - b = n \leftrightarrow b + n = a \quad \text{or} \quad a = b + n$$

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